Mathematical Modelling 3ECE, 2018

FOURIER ANALYSIS & the FOURIER TRANSFORM

(for E&E and Comp. Sys. Engineers)

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ENGSCI 313
MATHEMATICAL MODELLING 3ECE

FOURIER SERIES & THE FOURIER TRANSFORM

LECTURE 1 : FULL-RANGE FOURIER SERIES

LECTURE 1 ~ OVERVIEW

- Fourier & His Theory
  - Sinusoids,
  - Shifting Sinusoids,
  - DC Offset,
  - Linear Superposition
- Full-Range Fourier Series
  - Even/Odd Functions,
  - Integration by Parts
  - Examples

These formulae will be given in tests and exam

MM3 ECE Fourier Series & Fourier Transform Exam Formula

A Fourier Series approximation to function f(t) is given by :

$$S_N(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} A_N.\cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{N} B_N.\sin\left(\frac{2\pi nt}{T}\right)$$

Where,

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt \quad , A_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt \quad , B_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

A Complex Fourier Series approximation to function f(t) is given by :

$$f(t) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{inw_0 t}$$

Where,

$$c_0 = \frac{1}{T}\int_a^b f(t)dt \quad , c_n = \frac{1}{T}\int_a^b f(t)e^{-inw_0 t}dt \ , \qquad w_0 = 2\pi f$$

The Fourier Transform equation is given by :

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt \qquad \text{where, } w = 2\pi f$$

The Inverse Fourier Transform equation is given by :

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{+iwt}dw \qquad \text{where, } w = 2\pi f$$

The Forward Discrete Fourier Transform (DFT) is given by :

$$X[k] = \sum_{n=0}^{N-1} x[n].e^{-i\left(\frac{2\pi K}{N}\right)n} \qquad , K = 0,1,2,3,\ldots,(N-1)$$

The Inverse Discrete Fourier Transform is given by :

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k].e^{+i\left(\frac{2\pi K}{N}\right)n} \qquad , K = 0,1,2,3,\ldots,(N-1)$$

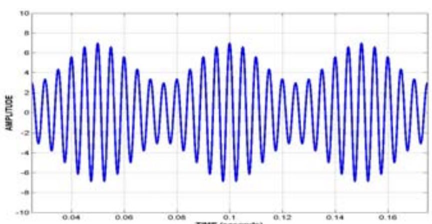
As Engineers, we will encounter signals in the form of :

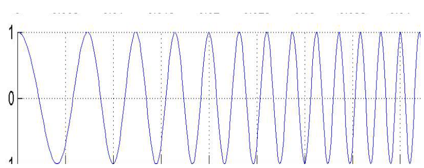
<u>Electromagnetic Radiation ~</u>
Radiowaves (AM & FM Radio, Radar),
Microwaves (Mobile Phone & Military Communications),
Visiblelight (Optics, Lasers).

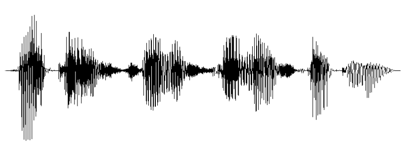
<u>Vibrational Signals~-</u>
Speech, Acoustics, Music, Ultrasound

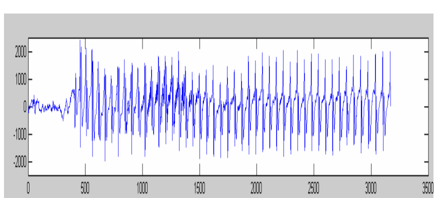
<u>2D & 3D Signals ~</u>
Television, Video and Camera images, Holographic images.

<u>Flow of Data</u> over time can be viewed as a signal –
The Stock Market (Financial Time-series), (EEG)Brain signals, (ECG)Heart signals









In order to understand such signals, one must develop techniques to describe complicated signals in simpler forms. The <u>Field of Signal Processing</u> deals with the development of techniques that can classify, manipulate, filter, synthesise and predict such signals.

Fourier

The field of "Time-Frequency Analysis" allows one to break down complicated signals into simpler functions, known as "Basis Functions".

Fourier Series (FS) was invented by Jean Baptiste Joseph Fourier (1768-1830) during his PhD to help him solve the Heat Equation in a metal plate.

His thesis demonstrated :

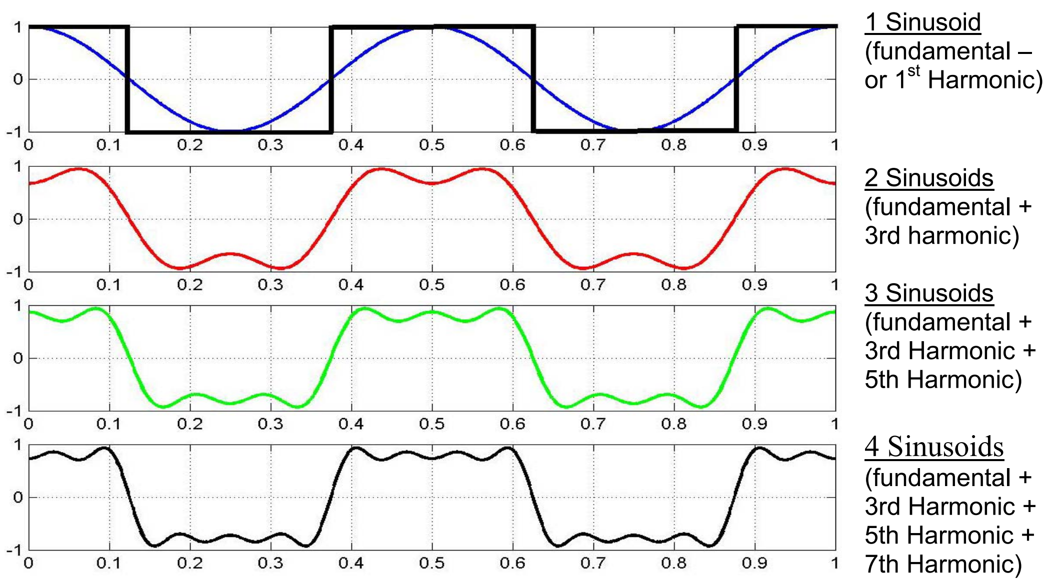
'Any periodic function can be approximated by a Linear Superposition of an infinite series of sinusoids'

Thus, the Basis function used by Fourier is the Sinusoid (sines and cosines).

So being able to understand a simple sinusoid is necessary for us to understand FS.

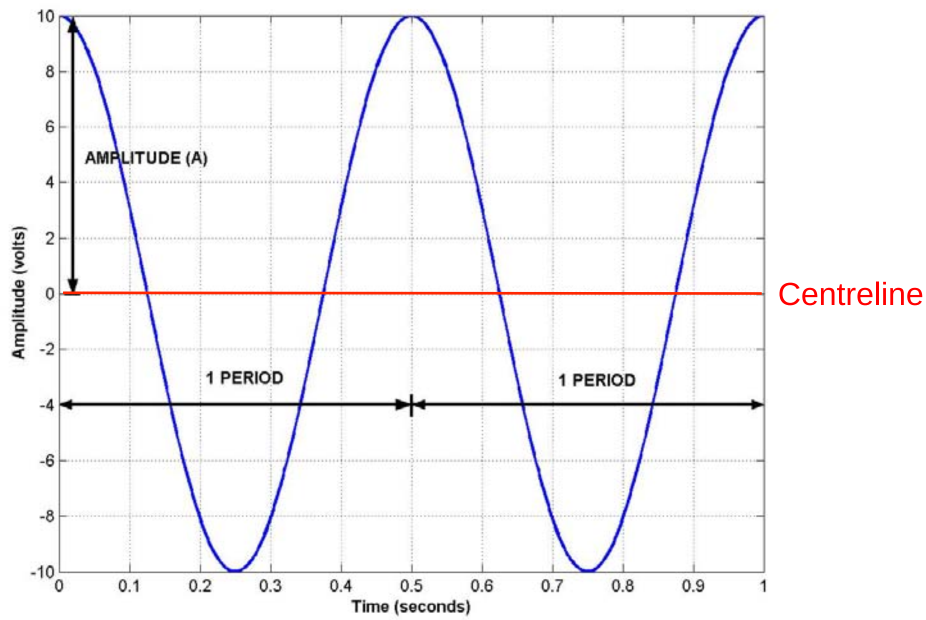
Fourier series & Harmonic related sinusoids

Shows how Fourier's applied his theory to a square wave.



Fourier's approximation to a square wave

Definition of a Continuous Sinusoid



A continuous sinusoid

In general, the height (y) of a sinusoid at any time (t) can be described as :

$$\boxed{y(t) = A.\cos(2\pi.f.t + \varphi) \quad \ldots 1)}$$

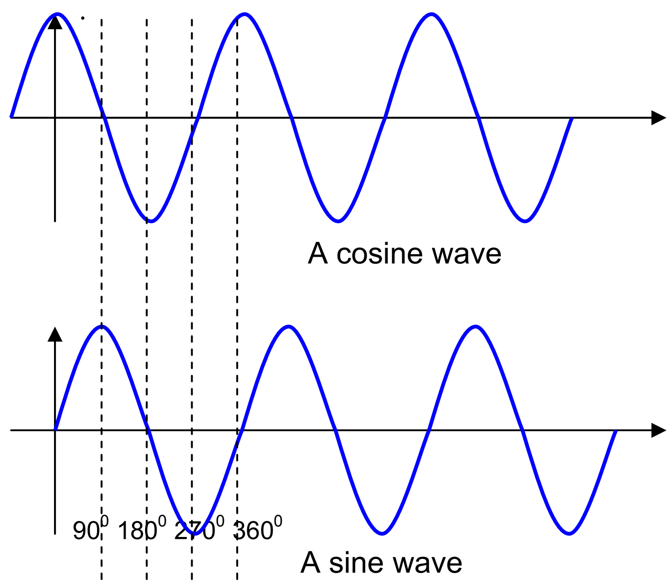
In this case, f = 2 Hz
i.e. T = 1/f = 0.5 s

- $x(t)$ = amplitude at time (t)
- A = maximum amplitude of sinusoid,
- f = frequency = 1/T in Hz, where T = period in secs.
- **φ** = phase-shift, in rads.

$$\boxed{y(t) = A\cos(wt + \varphi) \qquad \text{(Alternative)} \qquad 1.1a)}$$

- **ω** = angular frequency = 2**π**f in rads/sec

Shifting Sinuoids



Observe the diagrams of the sine and cosine wave.

As one can see these are the same wave, except the sine wave is 90^0, or $\pi/2$ radians out of phase from the cosine wave.

Notice how both waves start to repeat a new cycle at 360^0, or 2π radians.

IMPORTANT SINE/COSINE RELATIONSHIP

$$\sin(\theta) = \cos(\theta - \pi/2) \qquad 1.2)$$ sin is cos shifted by 90° to the right

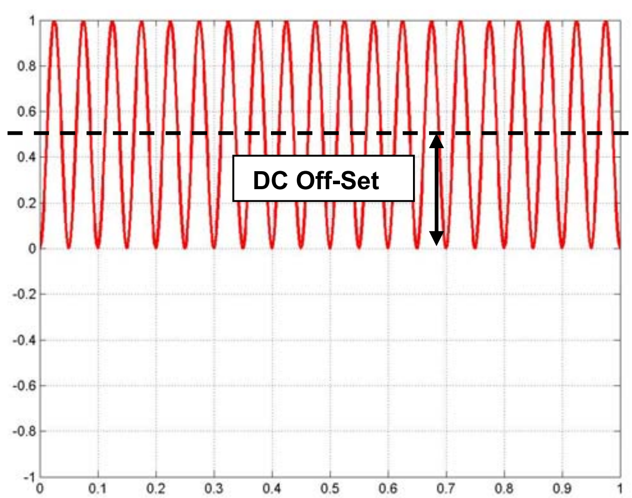
$$\cos(\theta) = \sin(\theta + \pi/2) \qquad 1.3)$$ cos is sin shifted by 90° to the left

SHIFTING SIGN CONVENTION

Thus, in General, if we want to shift the sinusoid to the :

- A Shift to the Right → φ is –ve
- A Shift to the Left → φ is +ve

DC Off-Set



If the whole sinusoid is shifted up or down in the y-direction, then it is said to experience a DC-shift.

This adds a constant to the sinusoid (C_{DC}) which represents the distance from the mean value of the signal from the line x=0.

$$y(t) = A.\cos(2\pi.f.t + \varphi) + C_{DC}$$

Note : For a sinusoid : The mean value is its centreline.

This is not necessarily true for other bases e.g. polynomial and exponential

<u>FULL-RANGE FOURIER SERIES</u>

Fourier Series is a "Time-Frequency" technique that allows us to break down any periodic function into a constituent sum of sinusoids. Also it allows for continuous or discontinuous signals to be included. The function must be convergent (i.e. not infinite at any point) See notes on convergence in lecture 2

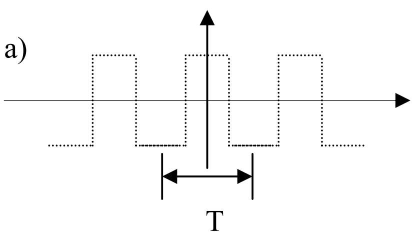
We approximate a complicated periodic signal f(t) by a function $S_N(t)$ which is made up of a series of sinusoids which are harmonically related.

$$S_N(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} A_N.\cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{N} B_N.\sin\left(\frac{2\pi nt}{T}\right) \qquad \ldots \qquad 1.4)$$

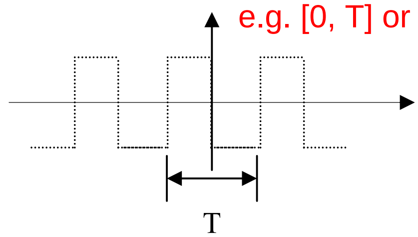
DC offset of signal (average of signal)

Where, (T) is the interval over which f(t) is periodic and <u>centered at the origin.</u>
Any interval of one period will be OK
e.g. [0, T] or [t, t+T]

a)



b)



The variables a_0, A_N and B_N must be determined.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt \qquad \ldots \quad 1.5)$$

Easy to evaluate

$$A_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt \qquad \ldots \quad 1.6)$$

$$B_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt \qquad \ldots \quad 1.7)$$

More difficult to evaluate

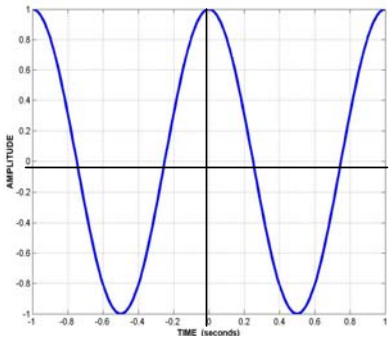
It comes down to evaluating a_0, A_N and B_N.
Once a_0, A_N and B_N have been evaluated we can substitute back into 2) and determine an approximation to the signal $S_N(t)$.

Utilizing the properties of <u>EVEN & ODD functions</u> will help avoid this.

Even & Odd Functions

1^{ST} : Determine if f(t) is an EVEN, ODD or NEITHER

An Even function (like cosine)

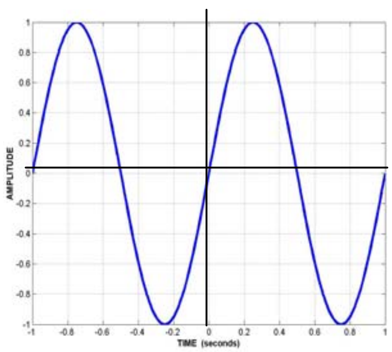


Is a mirror image of itself in the y-axis about zero

$$f(-x) = f(x) \quad \ldots \; 1.8)$$

i.e. cos(-x) = cos(x)

An Odd function (like sine)

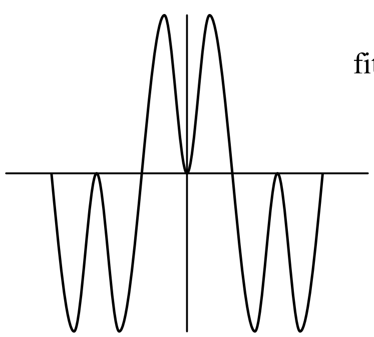


A mirror image of itself when first reflected in the y-axis about zero and then reflected in the x-axis about zero.

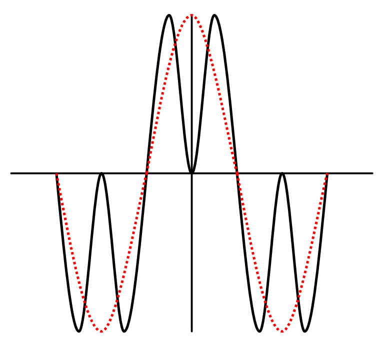
$$f(-x) = -f(x) \quad \ldots \; 1.9)$$

i.e. sin(-x) = -sin(x)

CASE 1 : If f(t) is EVEN



fit f(t) with cosines



This simplifies :

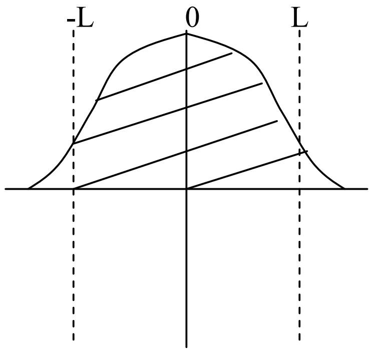
$$A_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt = \int \text{even x even} = \text{even} = \frac{4}{T}\int_0^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$ Since cos is even

$$B_N = 0$$ Since sin is odd

a_0 = 0 (if and only if f(t) is zero mean i.e. no DC shift)

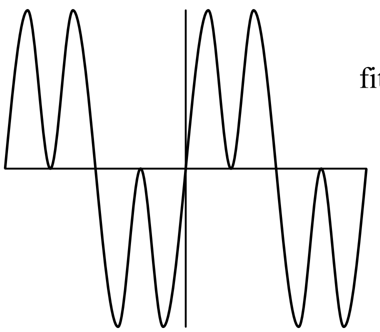
ASIDE : Area under an Even function

An Even function (like cosine) over the limits (-L, L) has twice the area (0, L)

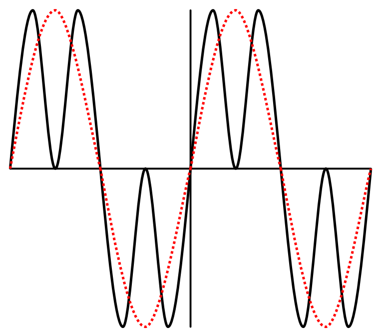


$$\int_{-L}^{L} f(t)_{even}\, dt = 2\int_{0}^{L} f(t)_{even}\, dt \qquad \ldots \quad 1.10)$$

CASE 2 : If f(t) is ODD



fit f(t) with sines



This simplifies :

$a_0 = 0$ if and only if f(t) is zero mean (i.e. There is no DC shift in the signal)

$A_N = 0$ Since cos is even

$$B_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt = \int \text{odd x odd} = \text{even} = \frac{4}{T}\int_{0}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$ Since sin is odd

CASE 3 : If f(t) is NEITHER even or odd

$$A_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt \qquad \ldots \; 1.11)$$

$$B_N = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt \qquad \ldots \; 1.12)$$

Must evaluate both A_N and B_N and a_0 if DC shifted.

Integration by parts (Recap)

Evaluating A_N & B_N requires us to integrate a product (namely integration by parts)
Thus, after simplification evaluate the integrals left using integration by parts

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b vu'\,dx \qquad \ldots\ 1.13)$$

Where v' is used for sin(2πnt/T) or cos(2πnt/T)

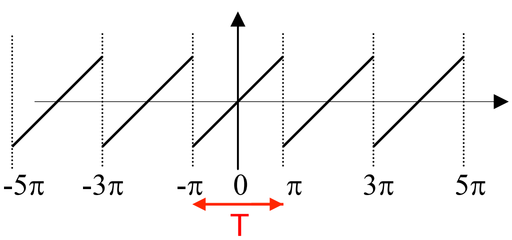
When evaluating the integrals use the following to simplify :

$$\sin(n\pi) = 0 \qquad \ldots\ 1.14)$$

$$\cos(n\pi) = (-1)^n \qquad \ldots\ 1.15)$$

Examples :

i). f(x) is defined as f(x) = x $-\pi \le x \le \pi$ and is said to be periodic



Evaluate A_N, B_N, a_o and determine the F.S. approximation to f(x)

Characterise the signal :
f(x) is Odd (like sine) ⇒ fit with sines ⇒ $\therefore A_N = 0$
No DC shift of the centerline of f(x) ⇒ $\therefore a_o = 0$
Period = T = 2π (note continuous over T)

$\therefore$ Only have to determine B_N

Now : Odd x Odd = Even

$$\int_{-L}^{L} f(t)_{even}\,dt = \quad 2\int_0^L f(t)_{even}\,dt$$

$$B_N = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi nx}{T}\right)dx = \frac{4}{T}\int_0^{T/2} f(x)\sin\left(\frac{2\pi nx}{T}\right)dx$$

$$= \frac{4}{2\pi}\int_0^{2\pi/2} x.\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} x.\sin(nx)dx$$

Now use integration by parts : $u = x, u' = 1, v' = \sin(nx), v = \dfrac{-\cos(nx)}{n}$

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b vu'\,dx$$

u v' u v v u'

$$\therefore \int_0^{\pi} x.\sin(nx)dx = \left[x.\frac{-\cos(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos(nx)}{n}.1dx$$

$$= -\frac{1}{n}\left[x\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)dx$$

Now,
$\sin(n\pi) = 0$
$\cos(n\pi) = (-1)^n$

$$= -\frac{1}{n}\left[\pi\cos(n\pi) - 0.\cos(0)\right] + \frac{1}{n}\left[\frac{\sin(nx)}{n}\right]_0^{\pi}$$

$$= -\frac{\pi(-1)^n}{n} + \frac{1}{n^2}\left[\sin(nx)\right]_0^{\pi} = -\frac{\pi(-1)^n}{n} + \frac{1}{n^2}\left[\sin(n\pi) - \sin(0)\right] = -\frac{\pi(-1)^n}{n}$$

Now F.S. approximation to f(x) is :

$$S_N(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N} A_N.\cos\left(\frac{2\pi nx}{T}\right) + \sum_{n=1}^{N} B_N.\sin\left(\frac{2\pi nx}{T}\right)$$

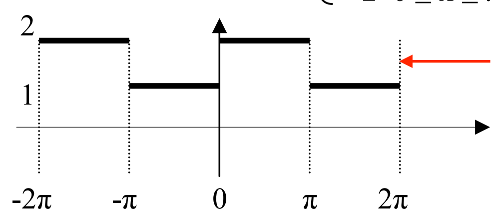
No DC shift →

$$= \frac{0}{2} + \sum_{n=1}^{N}(0).\cos\left(\frac{2\pi nx}{2\pi}\right) + \sum_{n=1}^{N}\left(-\frac{\pi(-1)^n}{n}\right).\sin\left(\frac{2\pi nx}{2\pi}\right) = \sum_{n=1}^{N}\left(-\frac{\pi(-1)^n}{n}\right).\sin(nx)$$

Since f(x) is not even

Note missing minus sign (should be –π)

ii) f(x) is piecewise and defined as f(x) = $\begin{cases} 1 & -\pi \le x \le 0 \\ 2 & 0 \le x \le \pi \end{cases}$. It is also a periodic function.



DC shift is 3/2

Characterise the signal :
f(x) is Odd (like sine) ⇒ fit with sines ⇒ $\therefore A_N = 0$ f(x) is odd after removing DC shift
There is a DC shift of the centerline of f(x) ⇒ ∴calculate DC = $a_o/2$
Period = T = 2π (note discontinuous over T)

∴ Have to determine a_o & B_N : Must split the integral since piecewise over T

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx = \frac{2}{2\pi}\left\{\int_{-\pi}^{0} 1.dx + \int_0^{\pi} 2.dx\right\} = \frac{1}{\pi}\left\{\left[x\right]_{-\pi}^{0} + \left[2x\right]_0^{\pi}\right\} = \frac{1}{\pi}\{\pi + 2\pi\} = 3$$ ← Note: not DC shift

i.e. a_0 is twice the DC shift

Two integrals because f(x) is piecewise over [–π, π]

An = 0 since f(x) is odd

Two integrals because f(x) is piecewise over [–π, π]

$$B_N = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi nx}{T}\right)dx = \frac{2}{2\pi}\left\{\int_{-\pi}^{0} 1.\sin\left(\frac{2\pi nx}{2\pi}\right)dx + \int_{0}^{\pi} 2.\sin\left(\frac{2\pi nx}{2\pi}\right)dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 1.\sin(nx)dx + \int_{0}^{\pi} 2.\sin(nx)dx\right\} = \frac{1}{\pi}\left\{\left[\frac{-\cos(nx)}{n}\right]_{-\pi}^{0} + 2\left[\frac{-\cos(nx)}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{n\pi}\{-1[\cos(0)-\cos(-n\pi)] - 2[\cos(n\pi)-\cos(0)]\}$$

$$= \frac{1}{n\pi}\{-1[1-(-1)^n] - 2[(-1)^n - 1]\}$$

Since cos(nπ) = (–1)^n

$$= \frac{1}{n\pi}\{-1+(-1)^n - 2(-1)^n + 2\} = \frac{1}{n\pi}\{1-(-1)^n\}$$

Now F.S. approximation to f(x) is :

$$S_N(x) \approx \frac{a_0}{2} + \sum_{n=1}^{N} A_N.\cos\left(\frac{2\pi nx}{T}\right) + \sum_{n=1}^{N} B_N.\sin\left(\frac{2\pi nx}{T}\right)$$

$$= \frac{3}{2} + \sum_{n=1}^{N}(0).\cos\left(\frac{2\pi nx}{2\pi}\right) + \sum_{n=1}^{N}\left\{\frac{1}{n\pi}\{1-(-1)^n\}\right\}.\sin\left(\frac{2\pi nx}{2\pi}\right)$$

Since f(x) is odd after removing DC shift

DC shift

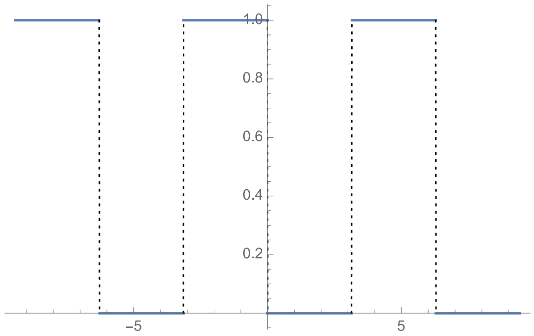
$$\therefore S_N(x) \approx \frac{3}{2} + \sum_{n=1}^{N}\left\{\frac{1}{n\pi}\{1-(-1)^n\}\right\}.\sin(nx)$$

2 Full range Fourier series

The function $f(t)$ is periodic and is defined such that:

$$f(t) = \begin{cases} 1 & -\pi \le t < 0 \\ 0 & 0 \le t < \pi \end{cases} \quad \text{where } f(t + 2n\pi) = f(t)$$

i Sketch the function over the range $[-3\pi, 3\pi]$.



ii Show that for the first 3 terms in a Fourier series approximation to $f(t)$ are:

$$S_N(t) \approx \frac{1}{2} - \frac{2}{\pi}\sin(t) - \frac{2}{3\pi}\sin(3t)$$

Note that

$$S_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

where

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right) dt$$
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right) dt$$

The function has a period of 2π.
The function has a DC shift.
The function is odd after accounting for DC shift, thus $a_n = 0$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\, dt + \int_{0}^{\pi} 0\, dt\right) = 1$$

Thus

$$\frac{a_0}{2} = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right) dt$$
$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \times \cos(nt)\, dt + \int_{0}^{\pi} 0 \times \cos(nt)\, dt\right)$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left(\left[\frac{\sin(nt)}{n}\right]_{-\pi}^{0} + 0\right) \\
&= \frac{1}{n\pi}\big(\sin(0) - \sin(-n\pi)\big) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)\,dt \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\times\sin(nt)\,dt + \int_{0}^{\pi} 0\times\sin(nt)\,dt\right) \\
&= \frac{1}{\pi}\left(\left[\frac{-\cos(nt)}{n}\right]_{-\pi}^{0} + 0\right) \\
&= \frac{1}{n\pi}\big(-\cos(0) + \cos(-n\pi)\big) \\
&= \frac{1}{n\pi}\big(-1 + (-1)^n\big) \\
&= \begin{cases} 0 & n \text{ even} \\ \dfrac{-2}{n\pi} & n \text{ odd} \end{cases}
\end{aligned}
$$

Thus

$$
\begin{aligned}
b_1 &= \frac{-2}{\pi} \\
b_3 &= \frac{-2}{3\pi} \\
S_N(t) &\approx \frac{1}{2} - \frac{2}{\pi}\sin(t) - \frac{2}{3\pi}\sin(3t)
\end{aligned}
$$

iii Using Parseval’s theorem show that the total power in the estimated signal is:

$$\text{power} = \frac{9\pi^2 + 80}{18\pi^2}$$

Parseval's theorem states that the power in a periodic signal is the sum of squares of the Fourier series coefficients.

Thus, for the signal in this question,

$$\text{power} = \frac{(a_0)^2}{2} + \sum_{n=1}^{N}(a_n)^2 + \sum_{n=1}^{N}(b_n)^2$$

iv Hence plot the power spectrum (power versus frequency) of the signal:

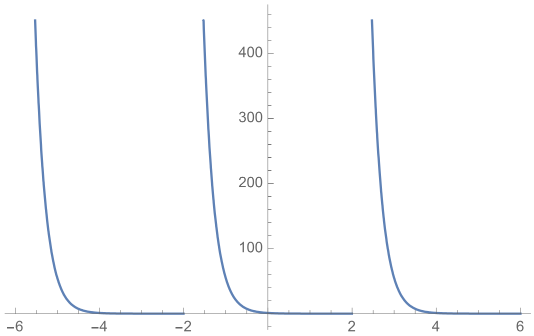
n	0	1	2	3	4
Frequency (Hz)	0	$\frac{1}{2\pi}$	$\frac{1}{\pi}$	$\frac{3}{2\pi}$	$\frac{2}{\pi}$
Frequency (Hz)	0	0.159155	0.31831	0.477465	0.63662
Power	0	$\frac{4}{\pi^2}$	0	$\frac{4}{9\pi^2}$	0
Power	0	0.405285	0	0.045032	0

3 Full range Fourier series

The function $f(x)$ is periodic and is defined such that:

$$f(x) = e^{-4x} \quad -2 \leq x < 2 \quad \text{where } f(x+4) = f(x)$$

i Sketch the function over the range [–6, 6].



ii Calculate the Fourier series coefficients, a_0, a_N and b_N, of $f(x)$

Note that

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi nx}{T}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi nx}{T}\right)$$

where

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi nx}{T}\right) dx$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi nx}{T}\right) dx$$

The function has period $T = 4$.
The function has a DC shift
The function is neither odd nor even after accounting for DC shift, thus a_0, a_N, and b_N, need to be calculated.

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx \\ &= \frac{1}{2}\int_{-2}^{2} e^{-4x} dx \\ &= \frac{1}{2}\left[\frac{e^{-4x}}{-4}\right]_{-2}^{2} \\ &= -\frac{1}{8}(e^{-8} - e^{8}) \\ &= \frac{1}{8}(e^{8} - e^{-8}) \end{aligned}$$

$$= \frac{1}{8}\left(\frac{e^{16}-1}{e^{8}}\right)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi n x}{T}\right)\,dx$$
$$= \frac{1}{2}\int_{-2}^{2} e^{-4x}\cos\left(\frac{\pi n x}{2}\right)\,dx$$

We need to use integration by parts:

$$\int_a^b u\,v'dx = [u\,v]_a^b - \int_a^b u'v\,dx$$

Here, we set

$$u = e^{-4x} \qquad u' = -4e^{-4x}$$
$$v' = \cos\left(\frac{\pi n x}{2}\right) \qquad v = \left(\frac{2}{\pi n}\right)\sin\left(\frac{\pi n x}{2}\right)$$

$$a_n = \frac{1}{2}\left(\left[e^{-4x}\left(\frac{2}{\pi n}\right)\sin\left(\frac{\pi n x}{2}\right)\right]_{-2}^{2} - \int_{-2}^{2} -4e^{-4x}\left(\frac{2}{\pi n}\right)\sin\left(\frac{\pi n x}{2}\right)\,dx\right)$$
$$= \frac{1}{2}\left(\left(\frac{2}{\pi n}\right)\left(e^{-8}\sin(\pi n) - e^{8}\sin(-\pi n)\right) + \left(\frac{8}{\pi n}\right)\int_{-2}^{2} e^{-4x}\sin\left(\frac{\pi n x}{2}\right)\,dx\right)$$
$$= \left(\frac{4}{\pi n}\right)\int_{-2}^{2} e^{-4x}\sin\left(\frac{\pi n x}{2}\right)\,dx$$

Again, we need to use integration by parts:

$$\int_a^b u\,v'dx = [u\,v]_a^b - \int_a^b u'v\,dx$$

Here, we set

$$u = e^{-4x} \qquad u' = -4e^{-4x}$$
$$v' = \sin\left(\frac{\pi n x}{2}\right) \qquad v = \left(\frac{-2}{\pi n}\right)\cos\left(\frac{\pi n x}{2}\right)$$

$$a_n = \left(\frac{4}{\pi n}\right)\left(\left[e^{-4x}\left(\frac{-2}{\pi n}\right)\cos\left(\frac{\pi n x}{2}\right)\right]_{-2}^{2} - \int_{-2}^{2} -4e^{-4x}\left(\frac{-2}{\pi n}\right)\cos\left(\frac{\pi n x}{2}\right)\,dx\right)$$
$$= \left(\frac{4}{\pi n}\right)\left(\left(\frac{-2}{\pi n}\right)\left(e^{-8}\cos(\pi n) - e^{8}\cos(-\pi n)\right) - \left(\frac{16}{\pi n}\right)\frac{1}{2}\int_{-2}^{2} e^{-4x}\cos\left(\frac{\pi n x}{2}\right)\,dx\right)$$

$$= \left(\frac{4}{\pi n}\right)\left(\left(\frac{-2}{\pi n}\right)\left(e^{-8}\,(-1)^n - e^{8}\,(-1)^n\right) - \left(\frac{16}{\pi n}\right)a_n\right)$$
$$= \left(\frac{-8}{\pi^2 n^2}\right)\left(\left(e^{-8}\,(-1)^n - e^{8}\,(-1)^n\right) + 8a_n\right)$$

since $\cos(\pi n) = (-1)^n$.

Because a_n appears on both sides of the equation, we now need to isolate a_n

$$a_n = \left(\frac{-8}{\pi^2 n^2}\right)\left(\left(e^{-8}\,(-1)^n - e^{8}\,(-1)^n\right) + 8a_n\right)$$
$$\left(\frac{-\pi^2 n^2}{8}\right)a_n - 8a_n = \left(e^{-8}\,(-1)^n - e^{8}\,(-1)^n\right)$$
$$a_n\left(\frac{-\pi^2 n^2}{8} - 8\right) = \left(e^{-8} - e^{8}\right)(-1)^n$$
$$a_n\left(\frac{\pi^2 n^2 + 64}{8}\right) = \left(e^{8} - e^{-8}\right)(-1)^n$$
$$a_n = \left(\frac{8}{\pi^2 n^2 + 64}\right)\left(e^{8} - e^{-8}\right)(-1)^n$$
$$a_n = \frac{8\left(e^{8} - e^{-8}\right)(-1)^n}{\pi^2 n^2 + 64}$$

Similarly (you should try this)

$$b_n = \frac{n\pi\left(e^{8} - e^{-8}\right)(-1)^n}{\pi^2 n^2 + 64}$$

Thus, the Fourier series that approximates

$$f(x) = e^{-4x} \quad -2 \le x < 2 \quad \text{where } f(x+4) = f(x)$$

is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n x}{T}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi n x}{T}\right)$$
$$= \frac{1}{16}\left(e^{8} - e^{-8}\right)$$
$$+ \sum_{n=1}^{N} \frac{8\left(e^{8} - e^{-8}\right)(-1)^n}{\pi^2 n^2 + 64} \cos\left(\frac{\pi n x}{2}\right)$$
$$+ \sum_{n=1}^{N} \frac{n\pi\left(e^{8} - e^{-8}\right)(-1)^n}{\pi^2 n^2 + 64} \sin\left(\frac{\pi n x}{2}\right)$$