The Tippe Top (Topsy-Turvy Top)

William A. Pliskin

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The Tippe Top (Topsy-Turvy Top)

WILLIAM A. PLISKIN
Poughkeepsie, New York
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Two fundamentally different explanations have been given in the literature to explain the "unusual" motion of a specially constructed spherical top in which rotation causes the center of gravity of the top to rise, the top turning 180° and finally spinning on its stem like an ordinary top. This independently derived analysis shows that the force due to sliding friction is in such a direction as to result in a torque which causes the angular velocity components to vary in a way which necessitates the rising of the center of gravity. This analysis, which emphasizes the physical picture behind the phenomenon, agrees with two previously published papers that friction plays the primary role. Some experimental evidence is given to support the present analysis.

A SPECIAL toy top whose behavior seems quite unusual has appeared both abroad and in the U. S. This toy, which is referred to as a "tippe top" or "topsy-turvy top," consists of a spherical body and stem with the center of gravity of the top below the center of curvature of the spherical body, as shown in Fig. 1. If the tippe top is given sufficient spin about its axis of symmetry with $\theta=0$ initially it will turn such that θ increases and finally the top will spin on its stem like an ordinary sleeping top.

Two fundamentally different explanations have been written on the unusual behavior of

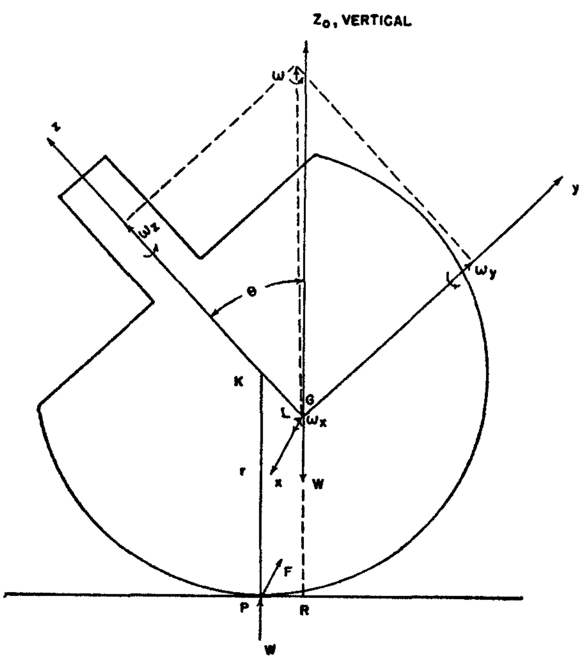

FIG. 1. Diagram of the tippe top.

this top. Synge[1] has maintained that friction seemed to have little effect on the motion of the tippe top and that its motion can be explained on the basis of dynamic instability of the top with no consideration of friction; whereas, both Braams[2] and Hugenholz[3] maintain that the force of friction gives rise to a torque which tends to raise the center of gravity and thus turn the top over. An explanation given by Jacobs[4] is that the friction increases the rate of precession and as a consequence causes the center of gravity to rise. This explanation was given in a letter, and since details on it are lacking no further comment can be made. Because of the fact that these different explanations have been made this note is being written, since in May, 1952, the writer had independently come to the same fundamental conclusion as given in the later publications of both Braams and Hugenholz, and in addition it has come to the attention of the writer that some people are still unaware of the published papers.[5] The detailed analysis is rather similar to those used by both Braams and Hugenholz, but it seems to be more direct and, in addition, puts more emphasis on the physical picture behind the phenomenon. Additional experimental evidence is given in support of the view that the unusual motion is due to friction.

In conjunction with Fig. 1 the following symbols are defined and relationships obtained:

[1] J. L. Synge, Phil. Mag. **43**, 724 (1952).
[2] C. M. Braams, Physica **18**, 503 (1952).
[3] N. M. Hugenholz, Physica **18**, 515 (1952).
[4] J. A. Jacobs, Am. J. Phys. **20**, 517 (1952).
[5] Letters to the Editor, The Johns Hopkins Magazine, February, 1953, p. 3.

G is the center of gravity of the top.
K is the center of curvature.
L is the distance GK.
r is the radius of curvature.
xyz is a moving coordinate system such that the z axis is coincident with the axis of symmetry and the x axis is horizontal.
ω is the angular velocity of the xyz coordinate system and for all practical purposes it is along the vertical since experimental observation of the tippe top easily verifies that $\omega_x = \dot{\theta}$ is small in comparison with ω. Thus we have

$$\omega_y = \omega \sin\theta, \quad \omega_z = \omega \cos\theta. \tag{1}$$

s is the spin of the top about the z axis relative to the moving xyz system. Experimental observations show that it is small in comparison with ω (see Fig. 4).
P is the point of contact of the top with the surface.
R is the projection of G onto the horizontal supporting plane.
F is the force of friction and the angle β between it and the $-x$ direction is negligible for all practical purposes as will be discussed later.
W is the weight of the top.
g is the acceleration due to gravity.
A is the moment of inertia about the x or y axes.
C is the moment of inertia about the z axis.

Before proceeding further it is instructive to point out an optical illusion observed under satisfactory conditions. If one spins the tippe top on a smooth shiny surface with sufficient light in the background it will appear that the top is not touching the supporting surface but that T, the lowest point of the top, is just above the surface and is slightly pointed, as shown in Fig. 2, where the top is shown in two different positions. This is the result of any motion in which the point of contact P makes a circular path due to the rapid rotation of the top. The paths made by R and P are for all practical purposes concentric circles, although not necessarily as drawn in Fig. 2 (or 3).

It will now be shown that for all practical purposes ρ, the radius of the circular path made

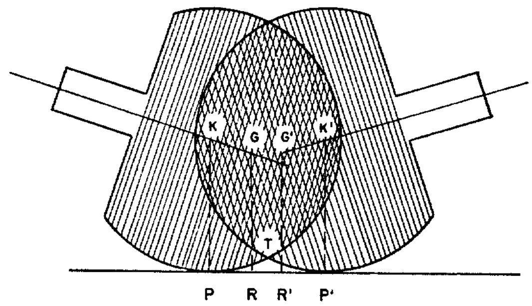

FIG. 2. Optical illusion that the top is not touching the supporting surface.

by R, is negligibly small and therefore one could consider that the top rotates about its center of gravity. It will also be shown that for all practical purposes β is negligibly small and therefore F is in the $-x$ direction. In Fig. 3 any initial horizontal motion given to the tippe top is neglected and thus the concentric circular paths of P and R are shown. In Fig. 3, V_R is the horizontal motion of G, V_P the horizontal motion of P and $V_{P,R}$ is the horizontal motion of P relative to G or R. $V_{P,R}$ is given by

$$V_{P,R} = (L\omega + sr) \sin\theta.$$

The motion of P due to ω_x is neglected since ω_x is comparatively small, as previously mentioned. From Figs. 1 and 3, we have

$$\rho = PR \sin\beta = L \sin\theta \sin\beta,$$
$$V_R = V_{P,R} \sin\beta.$$

The force of friction $F = \mu W$, where μ is the coefficient of sliding friction, balances the centrifugal reaction of G (or R), and therefore

$$V_R = (\mu g \rho)^{\frac{1}{2}},$$

and thus we obtain

$$\sin\beta = \mu g L / (L\omega + sr)^2 \sin\theta,$$
$$\rho = \mu g L^2 / (L\omega + sr)^2.$$

The value of ω is, in general, so large that both β and ρ are relatively small. For example in the case of a home-made top β was found to be less than 5° and ρ less than $0.1L$ in the range $\theta = 30°$ to $\theta = 120°$ (just preceding its rising on its stem).[6] Approximate calculations that were made on

[6] In fact, the 5° value was rather extreme. In the range $\theta = 0°$ to $\theta = 90°$, $\beta < 2°$ and $\rho < 0.03L$.

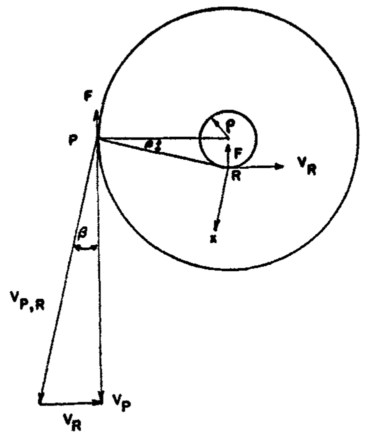

FIG. 3. The horizontal motion of P and R.

some of the mass-produced plastic models also show that β and ρ are relatively small.

Now one can get a simple but crude physical picture of the unusual behavior of the tippe top by referring to Fig. 1. If ω is sufficiently large, which is the case if the top eventually turns over, then ρ and β are small, as has been shown, and one can consider that the body rotates about its center of gravity G resulting in the force of friction F acting in the $-x$ direction, i.e., into the plane of the paper. Thus, if $\theta<\cos^{-1}(L/r)$, the force of friction will result in a positive torque about the y axis tending to increase ω_y and at the same time a negative torque about the z axis tending to decrease ω_z.[7] The over-all effect is a decrease of the ratio ω_z/ω_y, but $\omega_z/\omega_y=\cot\theta$, and thus θ must increase. Provided the moments of inertia are proper, then in the region $\pi/2\geq\theta\leq\cos^{-1}(L/r)$, ω_z could be considered to decrease at a faster rate than ω_y so that $\cot\theta$ decreases and thus θ must increase. Similarly if $\theta>\pi/2$ both ω_y and ω_z must decrease (ω_z becoming more negative); thus Eq. (1) requires a further increase in θ. Thus θ increases until the stem touches the supporting surface. This is only a crude picture and does not explain all the effects but it does serve as a useful introduction to the more detailed equations.

The forces caused by friction and by the weight of the top result in a torque whose three components are $-WL\sin\theta-(\mu W)\times(r-L\cos\theta)\sin\beta$, $\mu W(r\cos\theta-L)\cos\beta$, and $-\mu Wr\sin\theta\cos\beta$. With ω sufficiently large, β is small, and thus for practical purposes $\sin\beta=0$ and $\cos\beta=1$. Since ω_x and ρ are small, one can to a sufficient degree of approximation consider the origin G of the rotating xyz coordinate system as fixed, and thus from Euler's equation[8] and the above torque components we have

$$A\dot{\omega}_x+(C-A)\omega_y\omega_z+Cs\omega_y=-WL\sin\theta, \qquad (2.1)$$

$$A\dot{\omega}_y+(A-C)\omega_z\omega_x-Cs\omega_x=\mu W(r\cos\theta-L), \qquad (2.2)$$

$$C\dot{\omega}_z+C\dot{s}=-\mu Wr\sin\theta. \qquad (2.3)$$

With appropriate changes in coordinates and terms, these equations of motion are the same as those given by Hugenholz. The equations are the same as those for an ordinary symmetric top except for the torque components on the right. In the case of the ordinary top the torque component in Eq. (2.1) is of the opposite sign and the other two components are zero. From these considerations one would expect that if one decreased μ or increased ω sufficiently then motion more similar to that of an ordinary top should be observed. This was tried by Mr. J. E. Mapes by rotating the tippe top at high speeds on a well-lubricated smooth surface. It took the tippe top longer to turn over. In addition definite nutations were observed, whereas on an ordinary surface it nutated very little.

From Eqs. (1), (2.2), and (2.3) we obtain

$$\begin{aligned}\dot{\omega}_y&=\dot{\omega}\sin\theta+\omega(\cos\theta)\omega_x\\&=[\mu W(r\cos\theta-L)\\&\quad-(A-C)\omega_z\omega_x+Cs\omega_x]/A\,; \qquad (3)\end{aligned}$$

$$\dot{\omega}_z=\dot{\omega}\cos\theta-\omega(\sin\theta)\omega_x=-(\mu Wr/C)\sin\theta-\dot{s}. \qquad (4)$$

From these equations, one easily obtains

$$\begin{aligned}[\omega+((A-C)/A)\omega\cos^2\theta-(Cs/A)\cos\theta]\omega_x\\=(\mu W/AC)[C(r\cos\theta-L)\\\times\cos\theta+Ar\sin^2\theta]+\dot{s}\sin\theta. \qquad (5)\end{aligned}$$

Now, from a consideration of the magnitude of the various quantities in Eq. (5), it will be shown that, if ω is sufficiently large, $\omega_x>0$ for the ordinary tippe tops. Since $(A-C)/A<1$ and s is relatively small, therefore the coefficient of ω_x is

[7] This is the torque which could be held responsible for the small value of s in comparison with ω. The tippe top is unlike the ordinary top where the axis of rotation intersects the supporting surface at virtually the same point where the top touches the surface, explaining the negligible frictional torque in the case of an ordinary top.

[8] L. Page, *Introduction to Theoretical Physics* (D. Van Nostrand Company, Inc., New York, 1948), p. 123.

positive. Since s is small, $\dot{s}$ is small and the other term on the left is obviously >0; therefore, $\omega_x=\dot{\theta}>0$. It is thus seen that θ increases until the stem touches the surface, which point then becomes the vertex of an ordinary-type top and, if ω is sufficiently large, the top rises and sleeps like an ordinary top. Further details of the rising on the stem is given by Hugenholz. Equation (5) also shows that ω_x is proportional to μ, the coefficient of sliding friction. This has also been observed experimentally. In fact calculations were made on the home-made top for which the approximate physical quantities were known. The terms involving s and $\dot{s}$ were neglected and ω was determined approximately for various values of θ by use of a stroboscope. The calculated and observed ω_x agreed favorably.

Since $\dot{\omega}_x$ is small, we obtain from Eq. (2.1)[9]

$$s=-WL/C\omega+((A-C)/C)\omega\cos\theta, \tag{6}$$

and the angle θ' for which $s=0$ is given by

$$\cos\theta'=WL/\omega^2(A-C). \tag{7}$$

From the above equations several conclusions can be drawn. If ω is sufficiently large then the direction of the spin relative to the rotating xyz system is dependent on $A-C$. For example, if $A>C$, then s will initially have the same sense as ω and, if $C>A$, then the spin rotation s will be opposite that of ω. Furthermore, s reverses itself on passing θ'. If ω is very large, θ' is nearly 90°. In addition as one decreases ω, θ' decreases (provided $A>C$). In most cases θ' is a few degrees different from 90°. This is in disagreement with the statement made by Jacobs regarding the spin of the top relative to its figure axis when in the horizontal position ($\theta=90°$). These facts were all tested experimentally. Figure 4 gives an illustration of the observed carbon traces after having spun the tops on carbon paper.[10] The broken line is the equator of the top. The arrows shown on each trace indicate the order or sense in which the trace was made so that s is in the sense opposite that given by the trace arrows. In general, in case of all the mass-produced plastic tippe tops which have been tested $A>C$. Two home-made tops were constructed. In one $A>C$ and in the other $C>A$. This last one was specifically constructed this way in order to test the theory. As expected, in case of the second home-made top, s was opposite that found with the other tops.

Now, according to the theory advanced by Synge, the moments of inertia A and B about the x and y axes must at least not be equal in order for the tippe top to be unstable (when rotating with $\theta=0$) and thus turn over. In the present analysis and in the analyses previously reported by Braams and Hugenholz it is assumed that $A=B$. The experimental evidence seems to support that at least A is approximately equal to B. Giving Synge the benefit of the doubt, then some of the tippe tops might possibly meet Case (a) described by him. But in the second home-made top even if A and B are not equal it could only fall under Case (c), which required stability according to Synge. It is not stable.

It must be remembered that Eq. (5) is based

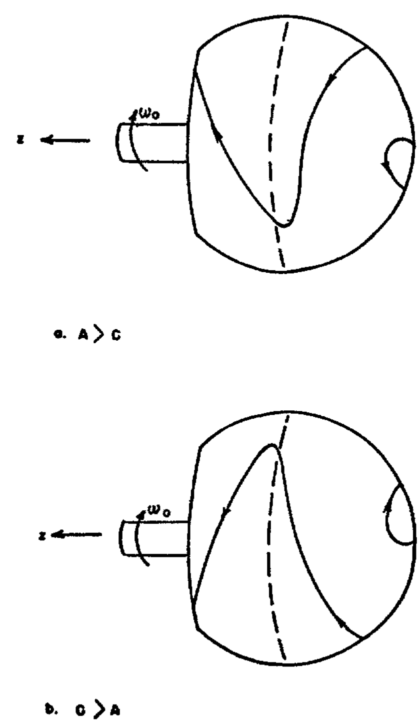

FIG. 4. Reversal in s, the spin relative to the rotating xyz system, as shown by carbon traces. (ω_0 is the direction in which the top was originally rotating when in the vertical position.)

[9] If the term involving $\sin\beta$ were not neglected, then Eq. (6) would be the same as Eq. (4) of Braams.

[10] Very often with the commercial models there is a fine spiral shape superimposed on the over-all form of the trace as shown here. This, as explained by Braams, is due to a small horizontal component of angular momentum.

on a force of friction in the $-x$ direction. Reference to Fig. 3 shows that, if $V_{P,R}=0$, then F vanishes which results in $\omega_x=0$ and thus an equilibrium value for θ. Setting $V_{P,R}=0$, we obtain $(L\omega+sr)\sin\theta=0$ which will occur if $\sin\theta=0$ or if $s=-L\omega/r$. Substituting this latter value of s in Eq. (6) one obtains the same equilibrium values for θ as given by Hugenholz in Eq. (16). Further discussions on whether these equilibrium values are stable or unstable can be found in the papers by Braams and by Hugenholz.[11]

The author wishes to thank Dr. L. C. Roess for the helpful discussions during this study and Mr. James B. Smith for making the drawings.

[11] The stability described by Braams for what he calls a top of the second kind is correct if $e(=-L/r)>0$. If $e<0$ in a top of the second kind, then the stability conditions are just the opposite. The tops discussed here were of the first or second kinds.

Reproductions of Prints, Drawings, and Paintings of Interest in the History of Physics

57. A Contemporary Portrait of Joseph Black

E. C. Watson
California Institute of Technology, Pasadena, California
(Received June 27, 1953)

A photograph of a very fine contemporary portrait of Joseph Black sculptured in wax by the famous modeler, James Tassie, is here reproduced.

A NUMBER of portraits of Joseph Black (1728–1799), the great chemist who laid the foundations of the quantitative science of heat, were reproduced in the fifth article in this series of historical reproductions.[1] However, this article failed to mention the white enamel profile medallion executed by James Tassie. Since the original wax model for this medallion has recently come into my possession, it seems worth while to reproduce a photograph of it in order to call attention to its existence. It is signed "Joseph Black, M.D. 1788" and is mounted on a black background in a contemporary polished walnut frame (6×7 in.). To me, it seems a more delicate composition than the paste medallion cast from it.

James Tassie (1735–1799), the famous gem engraver, executed many profile medallion portraits of his contemporaries, which are of great historic interest as well as artistic value (a collection of more than 150 is now in the Scottish National Portrait Gallery, Edinburgh). They were modeled in wax from life and cast in a hard white enamel paste.

Joseph Black, M.D. 1788.

[1] Am. Phys. Teacher, 7, 123 (1939).