December 2019 - Challenge

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Approximating absolute value.
Find an expression with no more than 15 operations (+, -, *, /) that approximates the absolute value function on the interval [-1,1] with an MSE (mean squared error) of no more than 0.0001.

For example, (x+1)/2 can be computed with only two operations and has an MSE of 1/6.

Bonus '*' for the best approximation (currently 1.803E-7).

Consider the rational function below:

$$f(x) = x\frac{((x+.8)(x+.6)(x+.4)(x+.2)-(-x+.8)(-x+.6)(-x+.4)(-x+.2))}{((x+.8)(x+.6)(x+.4)(x+.2)+(-x+.8)(-x+.6)(-x+.4)(-x+.2))}$$

This simplifies to:

$$f(x) = \frac{4x^4 + .8x^2}{2x^4 + 2.8x^2 + .0768}$$

Using an online integrator such as:
www.desmos.comcalculator

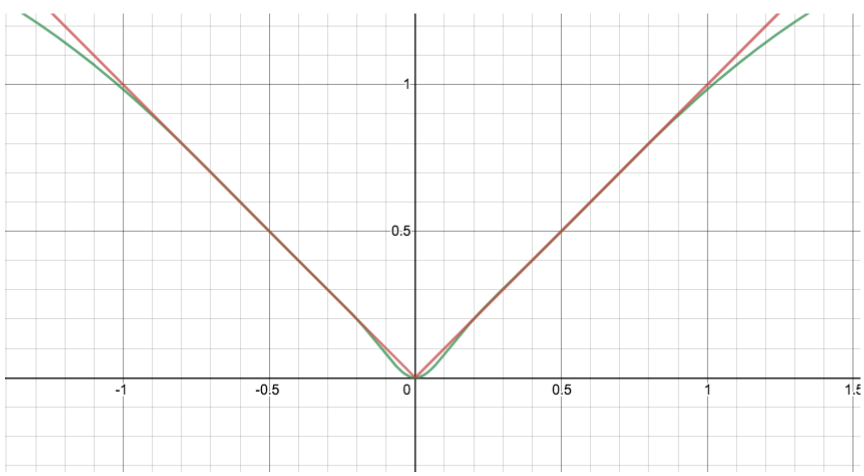
yields a MSE $\quad .5\int_{-1}^{1}\left(f(x) - |x|\right)^2 dx \quad = \quad .00007016 < .0001$

Now

$$f(x) = \frac{x^2(4x^2+.8)}{x^2(2x^2+2.8)+.0768}$$
$$= \frac{x\cdot x\cdot(4\cdot x\cdot x+.8)}{x\cdot x\cdot(2\cdot x\cdot x+2.8)+.0768}$$

Where the last expression can be computed in 5 (+,-,*,/) numerator operations and in 6 (+,-,*,/) denominator operations and in one final numerator/denominator operation for a total of 12 operations. In general, Horner's polynomial evaluation method would predict a total 17 operations for a quotient of two degree-4 polynomials, but in this case due to the missing linear and cubic terms 12 total operations suffice.

A graph of f(x) and the absolute value are included below for comparison:



In researching this problem, various polynomial regression and interpolation schemes were considered, but the desired accuracy was not achieved. Perhaps, higher degree even polynomial solutions exist(?)

For example, the f(x) below obtained from linear regression on 201 equally spaced points

$$f(x) = 0.05224972035590692 + 3.66787073181043x^2 - 12.316776445061928x^4 + 25.07461225868496x^6 - 24.18144181665176x^8 + 8.719159040025703x^{10}$$

has a MSE = 0.00014220708714 and can be computed using Horner's Method in 15 operations.

Probably not germane, the hyperbola h(x) given below has a MSE that can be made arbitrarily small

$$h(x) = \sqrt{x^2 + e^2} \qquad\qquad MSE \rightarrow 0 \; as \; \varepsilon^2 \rightarrow 0$$

NB The function f(x) used in resolving this challenge was suggested by the paper: ***On Rational Interpolation to |x| at the Adjusted Chebyshev Nodes***, by Lev Brutman
Journal of Approximation Theory
Volume 95, Issue 1, October 1998, Pages 146-152
https://www.sciencedirect.com/science/article/pii/S0021904598932063