June 2019 - Challenge

Let's define a *smooth* number as a natural number with prime factors that are single-digit (less than 10).
An example of such a number is 1560674304, which is related (**how?**) to the 108th birthday of IBM.

This month's challenge is to find two sets of smooth numbers whose square root sums are close to each other.
For example, the sets $\{2, 6\}$ and $\{15\}$ yield a distance between their square root sums of less than 0.01 since $\sqrt{2} + \sqrt{6} = 3.86\ldots \approx 3.87\ldots = \sqrt{15}$.

Two sets that meet our criterion even better are $\{2, 10, 15\}$ and $\{6, 36\}$, which yield an even smaller distance between their square root sums:
$\sqrt{2} + \sqrt{10} + \sqrt{15} = 8.44947\ldots \approx 8.44948\ldots = \sqrt{6} + \sqrt{36}$.

Find two sets of smooth numbers that produce a distance d where $0 < d < 10^{-15}$.

Solution:

Let a = (sqrt(2)-1) then since a < 1, a^n approaches 0 as n approaches infinity. Using **Sage** or **Wolfram Alpha** (sqrt(2)-1)^40 is approximately 4.89×10^(-16) < 1E-15. Also a^40 is exactly equal to:

1023286908188737 - 723573111879672 *sqrt(2).

In binary, 1023286908188737 = 11101000101010110010001100101010100111100001000001

Also 723573111879672 = 10100100100001010111111110010011010111011111111000

Now
11101000101010110010001100101010100111100001000001 =

2^49+2^48+2^47+2^45+2^41+2^39+2^37+2^35+2^34+2^31+2^27+2^26+2^23+2^21+2^19+
2^17+2^14+2^13+2^12+2^11+2^6+2^0

and

10100100100001010111111110010011010111011111111000 =

2^49+2^47+2^44+2^41+2^36+2^34+2^32+2^31+2^30+2^29+2^28+2^27+2^26+2^25+2^22+
2^19+2^18+2^16+2^14+2^13+2^12+2^10+2^9+2^8+2^7+2^6+2^5+2^4+2^3

Since the challenge requires the sum of the square root of individual numbers, in each of the two sums we need to double all the exponents so

2^49+2^48+2^47+2^45+2^41+2^39+2^37+2^35+2^34+2^31+2^27+2^26+2^23+2^21+2^19+
2^17+2^14+2^13+2^12+2^11+2^6+2^0

becomes

2^98+2^96+2^94+2^90+2^82+2^78+2^74+2^70+2^68+2^62+2^54+2^52+2^46+2^42+2^38+
2^34+2^28+2^26+2^24+2^22+2^12+2^0

Since the term 723573111879672 has the additional sqrt(2) factor we should double the exponent and **add 1** so

2^49+2^47+2^44+2^41+2^36+2^34+2^32+2^31+2^30+2^29+2^28+2^27+2^26+2^25+2^22+
2^19+2^18+2^16+2^14+2^13+2^12+2^10+2^9+2^8+2^7+2^6+2^5+2^4+2^3

becomes

2^99+2^95+2^89+2^83+2^73+2^69+2^65+2^63+2^61+2^59+2^57+2^55+2^53+2^51+2^45+
2^39+2^37+2^33+2^29+2^27+2^25+2^21+2^19+2^17+2^15+2^13+2^11+2^9+2^7

The first set of smooth numbers is then (*smooth since 2 is the only prime factor*)

{2**98, 2**96, 2**94, 2**90, 2**82, 2**78, 2**74, 2**70, 2**68, 2**62, 2**54, 2**52, 2**46, 2**42, 2**38, 2**34, 2**28, 2**26, 2**24, 2**22, 2**12, 2**0}

And the second set of smooth numbers is (*smooth since 2 is the only prime factor*)

{2**99, 2**95, 2**89, 2**83, 2**73, 2**69, 2**65, 2**63, 2**61, 2**59, 2**57, 2**55, 2**53, 2**51, 2**45, 2**39, 2**37, 2**33, 2**29, 2**27, 2**25, 2**21, 2**19, 2**17, 2**15, 2**13, 2**11, 2**9, 2**7}

(Here ** has replaced ^ for verification in Python)

In Python the sum of the square roots of the first set is 1023286908188737 as expected

In Python the sum of the square roots of the second set is:

1023286908188736.9999999999999995113784843734372871

And the difference of these two is

4.886215156265627129E-16 < 1E-15

For reference, the Python code and output is included below

```
from decimal import *
import math
getcontext().prec = 50

A = [2**98, 2**96, 2**94, 2**90, 2**82, 2**78, 2**74, 2**70, 2**68, 2**62, 2**54, 2**52, 2**46,
2**42, 2**38, 2**34, 2**28, 2**26, 2**24, 2**22, 2**12, 2**0]

sum_a =  Decimal(0)
for i in A:
    sum_a += Decimal(i).sqrt()
print(sum_a)

B = [2**99, 2**95, 2**89, 2**83, 2**73, 2**69, 2**65, 2**63, 2**61, 2**59, 2**57, 2**55, 2**53,
2**51, 2**45, 2**39, 2**37, 2**33, 2**29, 2**27, 2**25, 2**21, 2**19, 2**17, 2**15, 2**13, 2**11,
2**9, 2**7]

sum_b =  Decimal(0)
for i in B:
    sum_b += Decimal(i).sqrt()
print(sum_b)
print(sum_a-sum_b)
```

```
1023286908188737
1023286908188736.9999999999999995113784843734372871
4.886215156265627129E-16
```