

# December 2019 - Challenge

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Approximating absolute value.

Find an expression with no more than 15 operations (+, -, \*, /) that approximates the absolute value function on the interval [-1,1] with an MSE (mean squared error) of no more than 0.0001.

For example,  $(x+1)/2$  can be computed with only two operations and has an MSE of  $1/6$ .

**Bonus '\*'** for the best approximation (currently 1.803E-7).

Consider the rational function below:

$$f(x) = x \frac{((x+.8)(x+.6)(x+.4)(x+.2) - (-x+.8)(-x+.6)(-x+.4)(-x+.2))}{((x+.8)(x+.6)(x+.4)(x+.2) + (-x+.8)(-x+.6)(-x+.4)(-x+.2))}$$

This simplifies to:

$$f(x) = \frac{4x^4 + .8x^2}{2x^4 + 2.8x^2 + .0768}$$

Using an online integrator such as:  
[www.desmos.com/calculator](http://www.desmos.com/calculator)

yields a MSE  $.5 \int_{-1}^1 (f(x) - |x|)^2 dx = .00007016 < .0001$

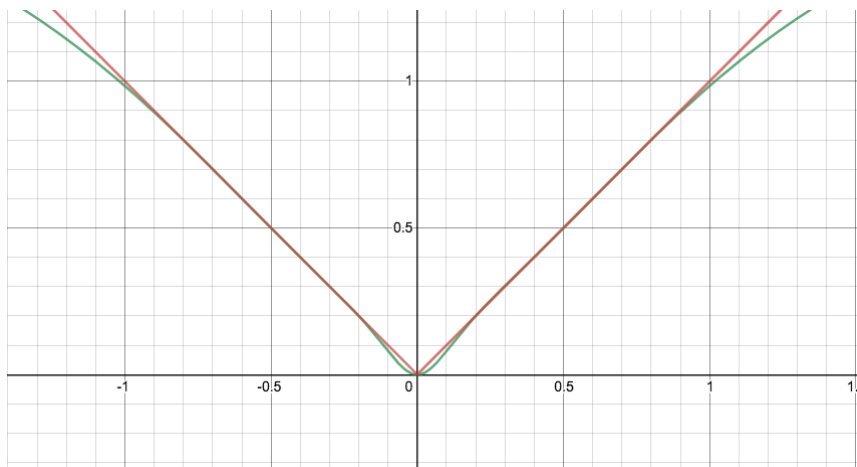
Now

$$f(x) = \frac{x^2(4x^2 + .8)}{x^2(2x^2 + 2.8) + .0768}$$

$$= \frac{x \cdot x \cdot (4 \cdot x \cdot x + .8)}{x \cdot x \cdot (2 \cdot x \cdot x + 2.8) + .0768}$$

Where the last expression can be computed in 5 (+,-,\*,/) numerator operations and in 6 (+,-,\*,/) denominator operations and in one final numerator/denominator operation for a total of 12 operations. In general, Horner's polynomial evaluation method would predict a total 17 operations for a quotient of two degree-4 polynomials, but in this case due to the missing linear and cubic terms 12 total operations suffice.

A graph of  $f(x)$  and the absolute value is included below for comparison:



In researching this problem, various polynomial regression and interpolation schemes were considered, but the desired accuracy was not achieved. Perhaps, higher degree even polynomial solutions exist(?)

For example, the  $f(x)$  below obtained from linear regression on 201 equally spaced points

$$f(x) = 0.05224972035590692 + 3.66787073181043x^2 - 12.316776445061928x^4 + 25.07461225868496x^6 - 24.18144181665176x^8 + 8.719159040025703x^{10}$$

has a  $MSE = 0.00014220708714$  and can be computed using Horner's Method in 15 operations.

Probably not germane, the hyperbola  $h(x)$  given below has a MSE that can be made arbitrarily small

$$h(x) = \sqrt{x^2 + \epsilon^2} \qquad MSE \rightarrow 0 \text{ as } \epsilon^2 \rightarrow 0$$

**NB** The function  $f(x)$  used in resolving this challenge was suggested by the paper: ***On Rational Interpolation to  $|x|$  at the Adjusted Chebyshev Nodes***, by Lev Brutman  
Journal of Approximation Theory  
Volume 95, Issue 1, October 1998, Pages 146-152  
<https://www.sciencedirect.com/science/article/pii/S0021904598932063>