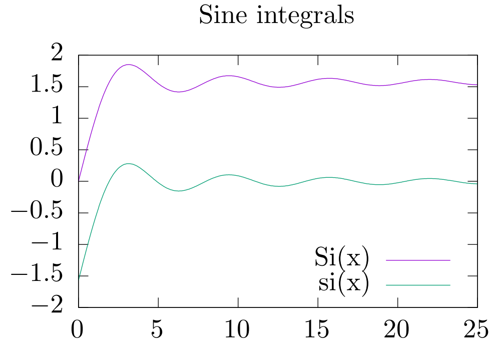
Sine integral

The sine integral is part of the trigonometric integrals.
It is defined in the following two ways

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}x$$
$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\,\mathrm{d}x$$



Both integrals are antiderivatives of the sinc function $\frac{\sin x}{x}$, but pertaining to different limits of the sinc function, as $\mathrm{Si}(x)$ is the antiderivative whose limit is 0 as $x \to 0$, while si is the antiderivative whose value is 0 as $x \to \infty$. The integrals are related to each other by the integral 1, the proof for which can be seen in Appendix A

$$\mathrm{Si}(x) - \mathrm{si}(x) = \int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2} \tag{1}$$

This property is important when implementing the *quad* integration routines, as the infinite integral of the *Sin* function in *C#* takes an enormous amount of time.
Furthermore, for very large arguments ($x > 700$), the solution exhibits sudden noise, uncharacteristic of the actual function. To avoid this, the functions were modified such that $\mathrm{Si}(x > 700) = \frac{\pi}{2}$ and $\mathrm{si}(x > 700) = 0$, as the are already reasonably close to these values at this point.

Appendix A

Starting with equation 1, it is rewritten into a function of the variable a

$$f(a) = \int_0^\infty e^{-at}\frac{\sin t}{t}\,\mathrm{d}t$$

Where the particular integral used is $f(0)$. The, utilizing Feynman's trick, we differentiate with respect to a

$$\tfrac{\mathrm{d}}{\mathrm{d}a} f = \tfrac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty e^{-at}\frac{\sin t}{t}\,\mathrm{d}t = \int_0^\infty \tfrac{\partial}{\partial a} e^{-at}\frac{\sin t}{t}\,\mathrm{d}t = -\int_0^\infty e^{-at}\sin t\,\mathrm{d}t$$

Then rewriting with $\sin t = -\frac{1}{2i}\left(e^{it} - e^{-it}\right)$

$$\begin{aligned}\tfrac{\mathrm{d}f}{\mathrm{d}a} &= -\int_0^\infty e^{-at}\,\frac{e^{it} - e^{-it}}{21}\,\mathrm{d}t = \frac{1}{2i}\int_0^\infty e^{-(a+i)t} - e^{-(a-i)t}\,\mathrm{d}t \\ &= \frac{1}{2i}\left[\frac{1}{a-i}e^{-(a-i)t} - \frac{1}{a+i}e^{-(a+i)t}\right]_0^\infty = -\frac{1}{1+a^2}\end{aligned}$$

Then applying integration

$$f = \int \tfrac{\mathrm{d}f}{\mathrm{d}a}\,\mathrm{d}a = -\int \frac{1}{1+a^2}\,\mathrm{d}a = -\arctan a + C$$

Determining the integration constant C from limit requirements

$$\lim_{a\to\infty} f = 0 = \lim_{a\to\infty}(C - \arctan a) \implies$$
$$C = \lim_{a\to\infty}\arctan a = \frac{\pi}{2}$$

Such that

$$f(a) = \frac{\pi}{2} - \text{arctan} a = \int_0^\infty e^{-at} \frac{\sin t}{t} \, \mathrm{d}t \implies$$
$$\int_0^\infty \frac{\sin t}{t} \, \mathrm{d}t = f(0) = \frac{\pi}{2}$$