CS 476 Assignment 4

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Q5

a)

We have:

$$f(\alpha) = \mathrm{E}(max(L - \alpha, 0))$$

Calculate the expected value using the probability density function $p(l)$. Then we have:

$$\begin{aligned} f(\alpha) &= \int_{\alpha}^{\infty} p(L) \cdot (L - \alpha) dL + \int_{-\infty}^{\alpha} p(L) \cdot 0 dL \\ &= \int_{\alpha}^{\infty} p(L) \cdot (L - \alpha) dL \end{aligned}$$

Take derivative on $f(\alpha)$. Then we have the following as α and $p(L) \cdot (L - \alpha)$ continuously differentiable and using the hint where $a(x) = x$ and $g(x, y) = \int_{\alpha}^{\infty} p(x) \cdot (x - y) dx$.

$$\begin{aligned} \frac{d}{d\alpha} f(\alpha) &= \frac{d}{d\alpha} (\int_{\alpha}^{\infty} p(L) \cdot (L - \alpha) dL) \\ &= -p(\alpha) \cdot (\alpha - \alpha) \cdot \frac{d}{d\alpha} \alpha + \int_{\alpha}^{\infty} \frac{\partial}{\partial \alpha} (p(L) \cdot (L - \alpha)) dL \\ &= 0 + \int_{\alpha}^{\infty} -p(L) dL \\ &= -\int_{\alpha}^{\infty} p(L) dL \\ &= -\int_{\alpha}^{\infty} p(l) dl \end{aligned}$$

Therefore, $f(\alpha)$ is continuously differentiable, and $f'(\alpha) = -\int_{\alpha}^{\infty} p(l) dl$.

b)

From above, we have $f'(\alpha) = -\int_{\alpha}^{\infty} p(l) dl$

As $p(l)$ is a probability distribution function, we have $\int_{-\infty}^{\infty} p(l) dl = 1$. As α increases, then $\int_{\alpha}^{\infty} p(l) dl$ must decreases. As there exist a negative sign on $f'(\alpha)$ to $\int_{\alpha}^{\infty} p(l) dl$, $f'(\alpha)$ is increasing function thus $f''(\alpha)$ is a positive function for all α.

As $f''(\alpha)$ is positive for all α, the function $f(\alpha)$ is convex.

c)

Let the equation inside the optimization problem be $g(\alpha)$. The equation equals the following:

$$g(\alpha) = \alpha + \frac{1}{1-\beta}\mathrm{E}(\max(L-\alpha, 0)) = \alpha + \frac{1}{1-\beta}f(\alpha)$$

Differentiating the equation gives us the following:

$$g'(\alpha) = \frac{d}{d\alpha}(\alpha + \frac{1}{1-\beta}f(\alpha)) = 1 - \frac{1}{1-\beta}\int_{\alpha}^{\infty} p(l)dl$$

From b), we already know that $-\int_{\alpha}^{\infty} p(l)dl$ is an increasing function and 1 is constant, therefore $g'(\alpha)$ is also increasing. Therefore, the optimization problem is also convex.

To find the minimum value of the optimization problem, it suffices to find the solution of $g'(\alpha) = 0$.

$$\begin{aligned} g'(\alpha) &= 0 \\ 1 - \frac{1}{1-\beta}\int_{\alpha}^{\infty} p(l)dl &= 0 \\ \int_{\alpha}^{\infty} p(l)dl &= 1-\beta \end{aligned}$$

As we are working with the loss L with a probability distribution with density function $p(L)$, by the definition of VaR, α^*, the solution of the optimization problem, is the VaR with confidence β.