CS 476 A2

Q3

Jeongseop Yi (Patrick), j22yi

In [ ]:
```
# import libraries
import numpy as np
import pandas as pd
from scipy import stats
import seaborn as sns
import matplotlib.pyplot as plt
```

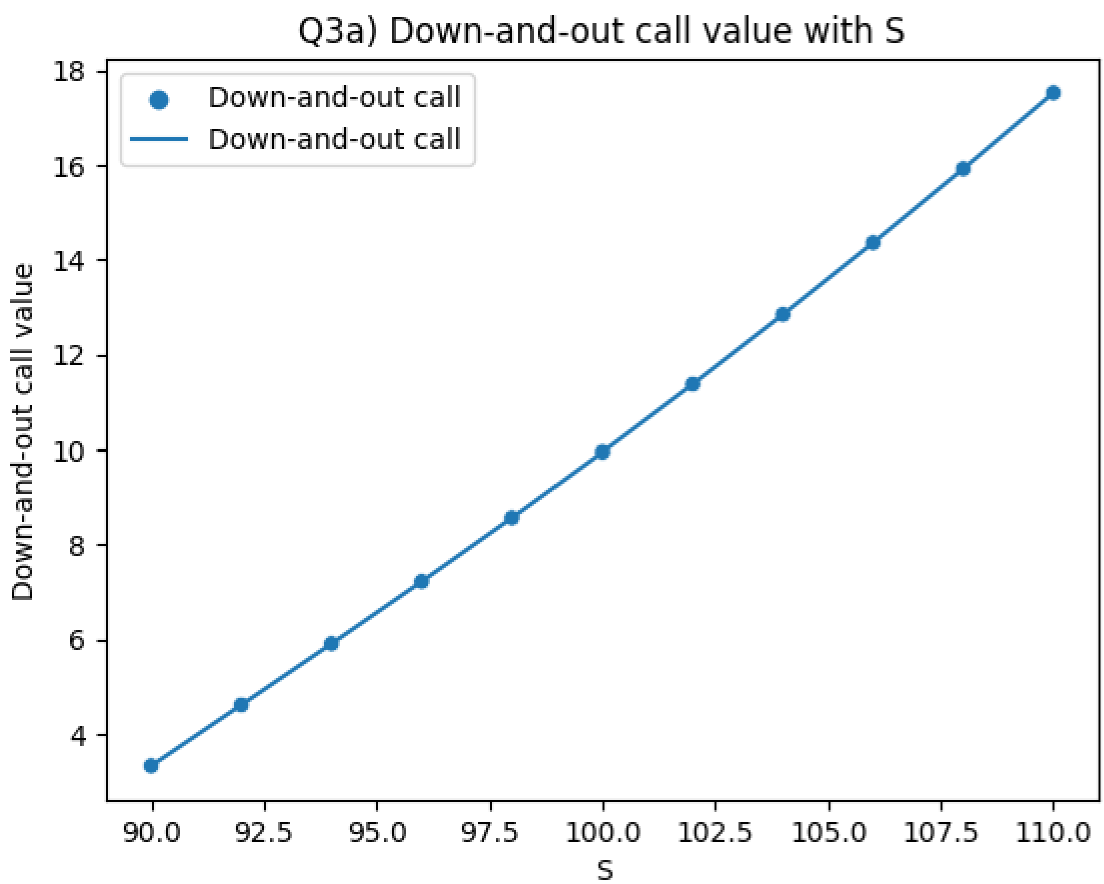
Q3a)

In [ ]:
```
# define functions for down-and-out call
def D1(S, K, r, sigma, T, t):
    return ((np.log(S/K) + (r + sigma**2/2)*(T-t))
            / (sigma*np.sqrt(T-t)))

def D2(S, K, r, sigma, T, t):
    return D1(S, K, r, sigma, T, t) - sigma*np.sqrt(T-t)

def D7(S, K, B, r, sigma, T, t):
    return ((np.log(S*K/(B**2)) - (r - sigma**2/2)*(T-t))
            / (sigma*np.sqrt(T-t)))

def D8(S, K, B, r, sigma, T, t):
    return D7(S, K, B, r, sigma, T, t) - sigma*np.sqrt(T-t)

def down_and_out_call_payout(S, K, B, r, sigma, T, t):
    d1 = D1(S, K, r, sigma, T, t)
    d2 = D2(S, K, r, sigma, T, t)
    d7 = D7(S, K, B, r, sigma, T, t)
    d8 = D8(S, K, B, r, sigma, T, t)

    return (S *
            (stats.norm.cdf(d1) - ((B/S)**(1 + 2 * r / (sigma ** 2)))
             * (1 - stats.norm.cdf(d8))) -
            K * np.exp(-r*(T-t)) *
            (stats.norm.cdf(d2) - ((B/S)**(-1 + 2 * r / (sigma ** 2)))
             * (1-stats.norm.cdf(d7))))
```

In [ ]:
```
# define constants
S0 = 100
K = 100
B = 85
r = 0.05
sigma = 0.2
T = 1
```

```python
# define range of S
S = np.arange(S0 * 0.9, S0 * 1.1, 2)

# calculate the down-and-out call for each S
ret = down_and_out_call_payout(S, K, B, r, sigma, T, 0)
ret = pd.DataFrame(ret, columns=['Down-and-out call'], index=S)
ret.index.name = 'S'
display(ret)

# plot the DataFrame
sns.scatterplot(data=ret)
sns.lineplot(data=ret)
plt.ylabel('Down-and-out call value')
plt.xlabel('S')
plt.title('Q3a) Down-and-out call value with S')
plt.show()
```

	Down-and-out call
S	
90.0	3.327341
92.0	4.614883
94.0	5.908461
96.0	7.222283
98.0	8.566792
100.0	9.949270
102.0	11.374382
104.0	12.844672
106.0	14.361002
108.0	15.922931
110.0	17.529038



Q3b)

In [ ]:
```python
# define functions for cut-off
def down_and_out_cutoff(S, B):
    return np.where(S < B, 0, S)

# define functions for payoff
def down_and_out_payoff(S, K):
    # as the cut-off function takes care of the S_t <= B part,
    # we only need to check S_T > K
    return np.where(S < K, 0, S - K)

# define functions for Monte Carlo simulation
def MC_simulation(S0, B, r, sigma, T, M, Dt, cut_off_func):
    S = np.zeros((M, int(T/Dt)))
    S[:, 0] = S0
    for i in range(1, int(T/Dt)):
        S[:, i] = S[:, i-1] * np.exp((r - sigma**2/2)*Dt +
                                      sigma*np.sqrt(Dt)*np.random.normal(0, 1, M))
        S[:, i] = cut_off_func(S[:, i], B)
    return S

# define function for Monte Carlo pricing
def MC_pricing(S0, K, B, r, sigma, T, M, Dt, cut_off_func, payoff_func):
    S = MC_simulation(S0, B, r, sigma, T, M, Dt, cut_off_func)
    return np.exp(-r*T) * np.mean(payoff_func(S[:, -1], K))
```

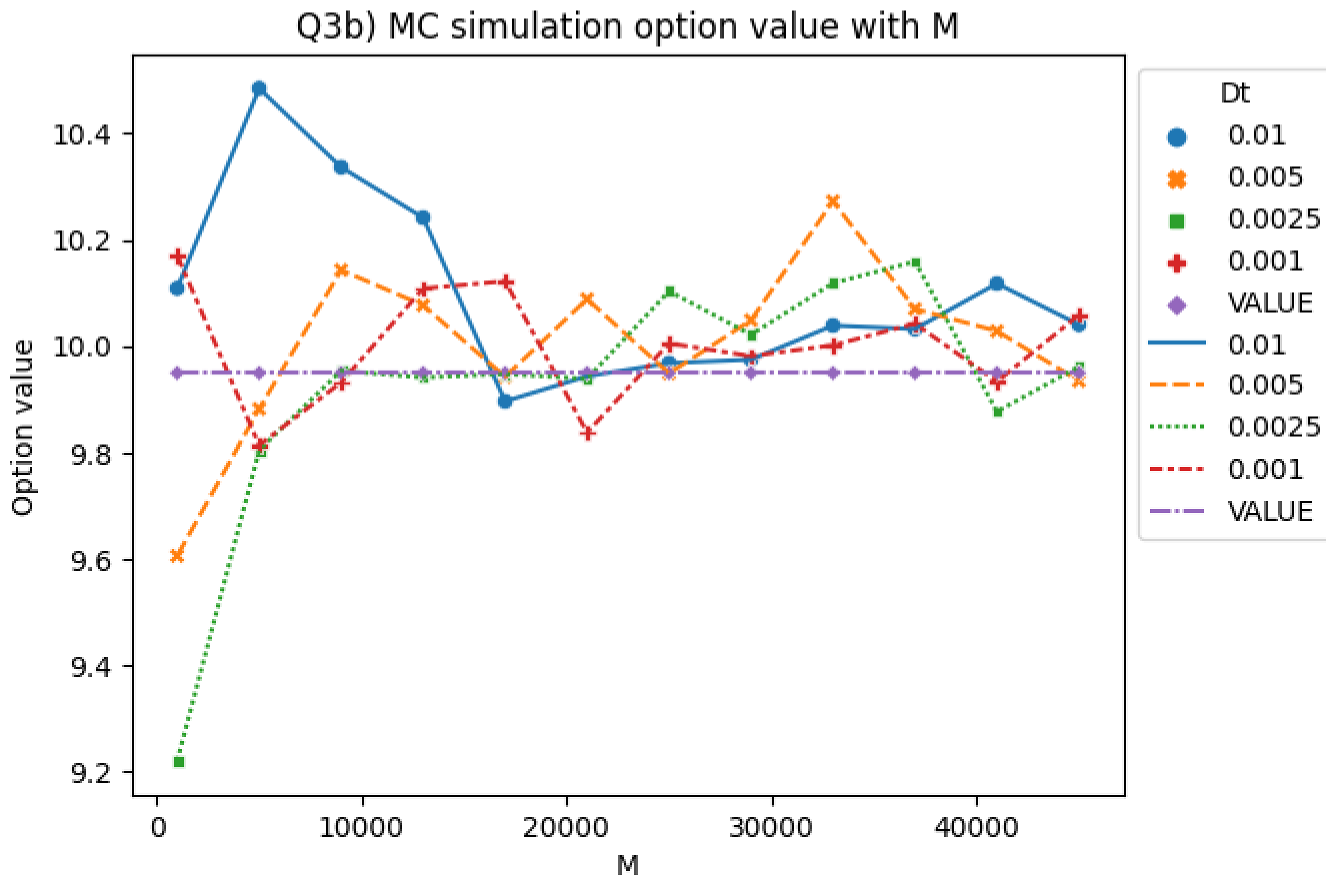
In [ ]:

```
# constants for Monte Carlo simulation
M = np.arange(1000, 45001, 4000)
Dt = [0.01, 0.005, 0.0025, 0.001]

# calculate Monte Carlo pricing using DataFrame
df = pd.DataFrame(index=M, columns=Dt)
for m in M:
    for dt in Dt:
        df.loc[m, dt] = MC_pricing(S0, K, B, r, sigma, T, m, dt,
                                   down_and_out_cutoff, down_and_out_payoff)

df['VALUE'] = down_and_out_call_payout(S0, K, B, r, sigma, T, 0)
```

In [ ]:

```
display(df)
a = sns.scatterplot(data=df)
b = sns.lineplot(data=df)
plt.legend(title="Dt")
sns.move_legend(a, "upper left", bbox_to_anchor=(1, 1))
sns.move_legend(b, "upper left", bbox_to_anchor=(1, 1))
plt.xlabel("M")
plt.ylabel("Option value")
plt.title("Q3b) MC simulation option value with M")
plt.show()
```

	0.01	0.005	0.0025	0.001	VALUE
1000	10.108465	9.606402	9.220892	10.170615	9.94927
5000	10.484565	9.881761	9.804328	9.813585	9.94927
9000	10.336334	10.142847	9.953405	9.931341	9.94927
13000	10.24145	10.076655	9.940652	10.108234	9.94927
17000	9.895963	9.942018	9.94733	10.121978	9.94927
21000	9.943135	10.088282	9.938178	9.837222	9.94927
25000	9.967693	9.948404	10.103164	10.005	9.94927
29000	9.974423	10.04849	10.021375	9.981032	9.94927
33000	10.038559	10.272524	10.118283	10.001381	9.94927
37000	10.031998	10.069738	10.158992	10.042217	9.94927
41000	10.11746	10.028123	9.876753	9.932904	9.94927
45000	10.040213	9.934362	9.961212	10.058364	9.94927



The error in the computed value $\tilde{V}(S(0), 0)$ depends on the time discretization. In theory, Δt can be a infinitely small value close to 0 using limit. However, in computers, the smallest number possible for Δt is always quantized to a value greater than 0. Therefore, there always exist a gap between t_n and $t_{n+1} = t_n + \Delta t$ in computation creating discrepency between the theoretical formula and computed value $\tilde{V}(S(0), 0)$. The discrepency creates the uncertainty (error) to the computation. Therefore, there exists time discretization error for $\tilde{V}(S(0), 0)$.

As we are using randomized parameters, the result in the graph is different for each run. Still, as M increases, the graphs have a tendency to converge to the exact option price computed in a), which is 9.949270. However, Δt does not seem to play a big role on the convergence of the option value compared to M using the above graph.