CS 476/676 Assignment 1

Winter 2023

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Lecture Times:	MW 11:30-12:50	MC 2035
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Due: 11am Feb 3, 2023

1. [(8 marks)] (Imply a Binomial Lattice from Option Prices)

Assume that the ABC stock pays no dividend and is currently priced at $S_0 = \$10$. Assume that, at the expiry time $T > 0$, the stock price goes up to uS_0 with probability $0 < p < 1$ and down to dS_0 with probability $1 - p$. We know that $d < 1 < u$ but do not know d or u. Assume that there is no arbitrage and the interest rate is zero. Consider the following three options with the same expiry T on the ABC stock.

Assume that a European put option with strike price \$9 is priced at \$1 while another European put option with strike price \$8 is priced $\$\frac{1}{3}$.

(a) What is the fair value of a European call option with a strike price of \$7?

(b) How many units of the underlying is required at $t = 0$ to hedge a short position in this call? Explain your answer.

(c) Using the actual probability p, what is the expected option payoff for the European call in (a)? What is wrong with pricing this call option at this expected payoff value? If this European call option is priced at the expected payoff using p which is different from the fair value computed in (a), how can you construct an arbitrage?

2. [(8 marks)] (Lattice Property)

Assume that the price of a stock follows a binomial lattice with price $S_{j+1}^{n+1} = uS_j^n$ with a probability $p > 0$ and $S_j^{n+1} = dS_j^n$ where

$$\begin{aligned} u &= \exp\left[\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t\right] \\ d &= \exp\left[-\sigma\sqrt{\Delta t} + (r - \sigma^2/2)\Delta t\right] \end{aligned} \tag{1}$$

(a) Show that

$$S_j^n = e^{-r\Delta t}(q^* S_{j+1}^{n+1} + (1 - q^*)S_j^{n+1}) \tag{2}$$

where the risk neutral probability

$$q^* = \frac{e^{r\Delta t} - d}{u - d}$$

(b) Show that q^* converges to $\frac{1}{2}$ as $\Delta t \to 0$.

(c) Using the risk neutral probability q^* in (2), as Δt converges to zero, what stochastic differential equation, using standard Brownian motion Z_t, is satisfied by dS_t? Explain.

3. [(10 marks)] (Lattice Property)

Let S_j^n denotes the underlying price at time t_n and binomial lattice node j, for integer $0 \le n \le N$, $0 \le j \le n$. Let $V^{\text{tree}}(S_j^n, K, t_n)$ be the fair value from a binomial tree model for an European straddle which has the payoff

$$\max(K - S_j^N, 0) + \max(S_j^N - K, 0).$$

In addition, assume that volatility is a constant $\sigma > 0$ and the constant risk free rate is $r > 0$. Prove by induction that, for any constant $\lambda > 0$, the following holds

$$V^{\text{tree}}(\lambda S_j^n, \lambda K, t_n) = \lambda V^{\text{tree}}(S_j^n, K, t_n)$$

for integer $0 \le n \le N$, $0 \le j \le n$.

4. [(8 marks)] (European Binomial Option Values)

Assume that the price of a stock follows a N-period binomial lattice with price $S_{j+1}^{n+1} = uS_j^n$ with a probability $p > 0$ and $S_j^{n+1} = dS_j^n$ with probability $1 - p$. Let the constant interest rate be $r > 0$ and the risk neutral probability be

$$q^* \quad = \quad \frac{e^{r\Delta t} - d}{u - d}$$

(a) Using lattice parameters specified, provide expression for all possible stock prices at $T = \Delta t N$.

(b) Let the initial stock price S_0 be given. Under risk neutral binomial lattice (ie with probability q^* for each up-move), what is the probability of reaching at T the stock price which have experienced exactly k up-moves?

(c) Using risk neutral pricing, provide the expression, in terms of T, q^*, K, r, for the fair time $t = 0$ value of a European straddle with T as expiry and strike price K. Justify your answer.

5. [(8 marks)] (Properties of a Standard Brownian Motion)

Assume that a stochastic process S_t satisfies

$$dS_t = \mu dt + \sigma dZ_t,$$

where $\mu > 0$ and $\sigma > 0$ are constants and Z_t is a standard Brownian motion.

(a) What is the stochastic differential equation satisfied by $S(t)^2$? Here $S(t)^2$ denotes the square of $S(t)$.

(b) Show that

$$\int_0^T S(t)dS(t) = \frac{S(T)^2 - S(0)^2}{2} - \left(\frac{T}{2}\right)\sigma^2$$