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c) $= S_t^2 \Big( \big(1+\sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t + \frac{1}{2}\sigma^2\Delta t + o(\Delta t)\big)$
$+ \big(1-\sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t + \frac{1}{2}\sigma^2\Delta t + o(\Delta t)\big)\Big)$
$-2\big(1+r\Delta t + o(\Delta t)\big)$
$-\big(1+(2r-\sigma^2)\Delta t + o(\Delta t)\big) + 1\Big)$

$$= S_t^2\Big(\cancel{2} + (4r-\sigma^2)\Delta t + \sigma^2\Delta t - \cancel{2} - 2r\Delta t - \cancel{1} - (2r-\sigma^2)\Delta t + \cancel{1} + o(\Delta t)\Big)$$

$$= S_t^2\Big(\cancel{4r\Delta t} - \sigma^2\Delta t + \sigma^2\Delta t - \cancel{2r\Delta t} - \cancel{2r\Delta t} + \sigma^2\Delta t + o(\Delta t)\Big)$$

$$= S_t^2\big(\sigma^2\Delta t + o(\Delta t)\big) = S_t^2\sigma^2\Delta t + o(\Delta t)$$

As $\Delta t \to 0$, $o(\Delta t)$ converges to 0 faster than $\Delta t$.

Then $Var(dS_t) = Var(\lim_{\Delta t\to 0} \Delta S_t) = \lim_{\Delta t \to 0} Var(\Delta S_t)$

$$= \lim_{\Delta t\to 0}\big(S_t^2(\sigma^2\Delta t) + o(\Delta t)\big) = S_t^2\sigma^2 dt$$

As $Var(dS_t) = \beta^2 dt = S_t^2\sigma^2 dt = (\sigma S_t)^2 dt$,

$\beta = \sigma S_t \quad (\because \sigma > 0,\ S_t > 0)$

As we now have $\alpha = rS_t$ and $\beta = \sigma S_t$, we have that

$$dS_t = rS_t\,dt + \sigma S_t\,dt$$