CS476 A1 Q1 1

$S_0 = 10 \quad d < 1 < u, \quad r = 0,$

$$S_0 \begin{array}{l} \overset{p}{\nearrow} \; uS_0 \\ \underset{1-p}{\searrow} \; dS_0 \end{array}$$

European put $K=9$: \$1, put $K=8$: \$$\frac{1}{3}$

a) European call $K = \$7$? Get the value using Put & Call Parity

European put $K=9$: $\$1 = p \cdot \max(K - uS_0, 0) + (1-p) \cdot \max(K - dS_0, 0)$

$= p \cdot \max(9 - 10u, 0) + (1-p) \cdot \max(9 - 10d, 0)$

European put $K=8$: $\$\frac{1}{3} = p \cdot \max(K - uS_0, 0) + (1-p) \cdot \max(K - dS_0, 0)$

$= p \cdot \max(8 - 10u, 0) + (1-p) \cdot \max(8 - 10d, 0)$

As $u > 1$, $10u > 10 > 9 > 8$, $\max(8 - 10u, 0) = \max(9 - 10u, 0) = 0$

Then $P_{K=9}$: $\$1 = (1-p) \cdot \max(9 - 10d, 0)$

$P_{K=8}$: $\$\frac{1}{3} = (1-p) \cdot \max(8 - 10d, 0)$

If $8 - 10d \leq 0$, then $\max(8 - 10d, 0) = 0$, a contradiction where $\frac{1}{3} = 0$.

Thus $8 - 10d > 0$, also $9 - 10d > 8 - 10d > 0$.

Then $P_{K=9}$: $\$1 = (1-p) \cdot (9 - 10d)$

$P_{K=8}$: $\$\frac{1}{3} = (1-p) \cdot (8 - 10d)$

$P_{K=9} - P_{K=8}$: $\frac{2}{3} = (1-p)(9 - 10d - 8 + 10d) = (1-p) \cdot 1$

$\therefore p = \frac{1}{3}$, $1 = \frac{2}{3} \cdot (9 - 10d) \Rightarrow 3 = 18 - 20d$

$d = \frac{3}{4}$

a) $d = \frac{3}{4}$ $p = \frac{1}{3}$

$$P_{K=7} = p \cdot \max(K - uS_0, 0) + (1-p) \cdot \max(K - dS_0, 0)$$

$$= p \cdot \max(7 - 10u, 0) + (1-p) \cdot \max(7 - 10d, 0)$$

We know that $10u > 10 > 7$, thus $\max(7 - 10u, 0) = 0$

$$P_{K=7} = (1-p) \cdot \max(7 - 10d, 0) = (1-p) \cdot \max(7 - 7.5, 0) \quad (\because d = \tfrac{3}{4})$$

$$= (1-p) \cdot \max(-0.5, 0)$$

$$= 0$$

We have the value of European put at $K=7$ as 0.

Use Put & Call parity to find the value of European call at $K=7$.

$$C_{K=7} = P_{K=7} + S_0 - K e^{-r\Delta t}$$

$$= 0 + 10 - 7 \cdot 1 = 3 \quad (\because r = 0)$$

$\therefore$ The value of European call at $K=7$: \$3

CS 476 A1Q1 ③

b) We would setup replication equation as follows:

$$\begin{bmatrix}1\\1\end{bmatrix}\eta_0+\begin{bmatrix}uS_0\\dS_0\end{bmatrix}\delta_0=\begin{bmatrix}V_1^u\\V_1^d\end{bmatrix}$$, and $d=\frac{3}{4}$ from a)

Find u using $C_{K=7}=3$.

$$p\cdot\max(10u-7,0)+(1-p)\cdot\max(10d-7,0)$$
$$=\frac{1}{3}\cdot(10u-7)+\frac{2}{3}\cdot\max(7.5-7,0)$$
$$=\frac{1}{3}\cdot(10u-7)+\frac{2}{3}\cdot\frac{1}{2}$$
$$\Rightarrow 9=10u-7+1 \quad\Rightarrow 15=10u\Rightarrow u=1.5$$

Then we have:

$$\begin{bmatrix}1\\1\end{bmatrix}\eta_0+\begin{bmatrix}15\\7.5\end{bmatrix}\delta_0=\begin{bmatrix}15-7\\7.5-7\end{bmatrix}=\begin{bmatrix}8\\0.5\end{bmatrix}$$

Then $7.5\,\delta_0=7.5$, $\delta_0=1$

$\therefore$ 1 unit of the underlying stock is required to hedge a short position in this call. \$-7 of bond is also required to hedge the short (negative means sell/short bond)

δ_0 is defined to be the quantity of the underlying stock of replicating portfolio.

CS 476 A1Q1 ④

c) $r=0$, $d=0.25$ from a), $u=1.5$ from b)

Expected call option payoff $K=7, u=1.5, d=0.25$

$$= p\cdot\max(uS_0-K,0)+(1-p)\cdot\max(dS_0-K,0)$$

$$= p\cdot\max(15-7,0)+(1-p)\cdot\max(7.5-7,0)$$

$$= 8p+(1-p)\cdot 0.5 = 0.5+7.5p$$

Expected put option payoff $K=7, u=1.5, d=0.25$

$$= p\cdot\max(K-uS_0,0)+(1-p)\cdot\max(K-dS_0,0)$$

$$= p\cdot\max(7-15,0)+(1-p)\cdot\max(7-7.5,0) = 0$$

Prove that there are arbitrage opportunities when ① $p<q^*=\frac{1}{3}$ and ② $p>q^*=\frac{1}{3}$ ($q^*=\frac{1}{3}$ is from 1a))

if $p<q^*=\frac{1}{3}$, set-up a portfolio using put-call parity.

At T, we have that $\Pi_T: C_T-P_T-S_T+K=0$ by put-call parity

but at 0, we have that $\Pi_0: C_0-P_0-S_0+K$ ($\because r=0$)

$$= 0.5+7.5p-0-10+7$$

$$= 7.5p-2.5 < 0 \quad (\because p<q^*=\tfrac{1}{3})$$

So we have an arbitrage when $p<q^*$ as $\Pi_0<0$ but $\Pi_T=0$

c) if $p > q^* = \frac{1}{3}$, set up a portfolio using put-call parity.

At T, we have that $\Pi_T : P_T - C_T + S_T - K = 0$, by put-call parity.

Then at 0, we have the same portfolio such that

$$\Pi_0 : P_0 - C_0 + S_0 - K$$
$$= 0 - 0.5 - 7.5p + 10 - 7$$
$$= -7.5p + 2.5 < 0 \quad (\because p > q^* = \tfrac{1}{3})$$

So we have an arbitrage when $\Pi_T = 0$ but $\Pi_0 < 0$ for $p > q^* = \frac{1}{3}$.

As there exist arbitrage oppurtunities for both $p < q^* = \frac{1}{3}$ and $p > q^* = \frac{1}{3}$, we can construct an arbitrage based on the value of $p \neq q^* = \frac{1}{3}$.

CS476 A1Q2 ①

a) $e^{-r\Delta t}\left(q^* S_{j+1}^{n+1} + (1-q^*) S_j^{n+1}\right)$

$= e^{-r\Delta t}\left(q^* \cdot u S_j^n + (1-q^*)\cdot d S_j^n\right)$

$= e^{-r\Delta t}\left(\frac{e^{r\Delta t}-d}{u-d}\cdot u S_j^n + \frac{u \cancel{-d} - e^{r\Delta t} \cancel{+d}}{u-d}\cdot d S_j^n\right)$

$= e^{-r\Delta t}\cdot S_j^n \cdot \left(\frac{e^{r\Delta t}-d}{u-d}\cdot u + \frac{u-e^{r\Delta t}}{u-d}\cdot d\right)$

$= e^{-r\Delta t}\cdot S_j^n \left(\frac{u\cdot e^{r\Delta t} \cancel{-ud} \cancel{+ud} - de^{r\Delta t}}{u-d}\right)$

$= e^{-r\Delta t}\cdot S_j^n \left(\frac{(u-d)\cdot e^{r\Delta t}}{u-d}\right)$

$= e^{-r\Delta t}\cdot e^{r\Delta t}\cdot S_j^n = S_j^n$

$\therefore$ we have $S_j^n = e^{-r\Delta t}\left(q^* S_{j+1}^{n+1} + (1-q^*) S_j^{n+1}\right)$

where $q^* = \frac{e^{r\Delta t}-d}{u-d}$

CS476 A1 Q2 ②

b) $\lim_{\Delta t\to 0} q^* = \lim_{\Delta t\to 0} \frac{e^{r\Delta t}-d}{u-d}$

As $\lim_{\Delta t\to 0} e^{r\Delta t} = 1$, $\lim_{\Delta t\to 0} d = \lim_{\Delta t\to 0} \exp[-\sigma\sqrt{\Delta t} + (r-\sigma^2/2)\Delta t] = e^0 = 1$

$\lim_{\Delta t\to 0} u = \lim_{\Delta t\to 0} \exp[\sigma\sqrt{\Delta t} + (r-\sigma^2/2)\Delta t] = e^0 = 1$

We have $\lim_{\Delta t\to 0} (e^{r\Delta t} - d) = 0$ and $\lim_{\Delta t\to 0} (u-d) = 0$

Use L'Hopital's rule in Δt: $\frac{d}{d\Delta t} u = \left(\frac{-\sigma}{2\sqrt{\Delta t}} + (r-\sigma^2/2)\right) u$

$\frac{d}{d\Delta t} d = \left(\frac{\sigma}{2\sqrt{\Delta t}} + (r-\sigma^2/2)\right) d$

$$\lim_{\Delta t\to 0} \frac{e^{r\Delta t}-d}{u-d} = \lim_{\Delta t\to 0} \frac{r\cdot e^{r\Delta t} - \left(\frac{\sigma}{2\sqrt{\Delta t}} + (r-\sigma^2/2)\right)\cdot d}{\left(\frac{-\sigma}{2\sqrt{\Delta t}} + (r-\sigma^2/2)\right)\cdot u - \left(\frac{\sigma}{2\sqrt{\Delta t}} + (r-\sigma^2/2)\right)\cdot d}$$

$$= \lim_{\Delta t\to 0} \frac{r - \left(\frac{\sigma}{2\sqrt{\Delta t}} + r - \sigma^2/2\right)}{\frac{-\sigma}{2\sqrt{\Delta t}} + r - \sigma^2/2 - \frac{\sigma}{2\sqrt{\Delta t}} - r + \sigma^2/2}$$

$\left(\because \lim_{\Delta t\to 0} e^{\Delta t} = 1,\ \lim_{\Delta t\to 0} d = 1,\ \lim_{\Delta t\to 0} u = 1\right)$

$$= \lim_{\Delta t\to 0} \frac{-\frac{\sigma}{2\sqrt{\Delta t}} + \sigma^2/2}{-\frac{2\sigma}{2\sqrt{\Delta t}}} = \lim_{\Delta t\to 0} \frac{\sigma - \sigma^2\sqrt{\Delta t}}{2\sigma}$$

$$= \lim_{\Delta t\to 0} \frac{\sigma}{2\sigma} - \lim_{\Delta t\to 0} \frac{\sigma^2\sqrt{\Delta t}}{2\sigma} = \frac{1}{2} - 0 = \frac{1}{2}$$

A1Q2 3

C) The stochastic differential equation to prove: $dS_t = 0$

Suppose that $dS_t = \alpha dt + \beta dZ_t$. Then $E(dS_t) = \alpha dt$, $Var(dS_t) = \beta^2 dZ_t$

from course notes.

$$E(\Delta S_t) = E(S_{t+\Delta t} - S_t)$$

$$= q^*(uS_t - S_t) + (1-q^*)(dS_t - S_t)$$

$$= \frac{e^{r\Delta t} - d}{u-d}(uS_t - S_t) + \frac{u - e^{r\Delta t}}{u-d}(dS_t - S_t)$$

$$= \frac{1}{u-d}\Big(e^{r\Delta t} \cdot uS_t - \cancel{e^{r\Delta t}S_t} - \cancel{duS_t} + dS_t$$

$$\cancel{+udS_t} - uS_t - e^{r\Delta t}dS_t + \cancel{e^{r\Delta t}S_t}\Big)$$

$$= \frac{1}{u-d}\left(e^{r\Delta t}uS_t - e^{r\Delta t}dS_t + dS_t - uS_t\right)$$

$$= \frac{1}{u-d}\left(e^{r\Delta t} \cdot S_t(u-d) - S_t(u-d)\right)$$

$$= S_t\left(e^{r\Delta t} - 1\right)$$

when Δt is very small but not 0, then $e^{r\Delta t} = 1$, then $E(\Delta S_t) = 0$
(converges to 0)

Then $E(dS_t) = 0$ regardless of dt.

As $E(dS_t) = \alpha dt = 0$ even for $dt \neq 0$, we have $\alpha = 0$

A1Q2 ④

c) $Var(\Delta S_t) = Var(S_{t+\Delta t} - S_t)$

$$= q^*(uS_t - S_t)^2 + (1-q^*)\cdot(dS_t - S_t)^2$$

$$= \frac{e^{r\Delta t}-d}{u-d}\cdot S_t^2(u^2-2u+1) + \frac{u-e^{r\Delta t}}{u-d}\cdot S_t^2(d^2-2d+1)$$

$$= \frac{S_t^2}{u-d}\Big(e^{r\Delta t}\cdot u^2 - 2\cdot e^{r\Delta t}\cdot u + \cancel{e^{r\Delta t}} - du^2 + \cancel{2du} - d$$
$$+ud^2 - \cancel{2du} + u - e^{r\Delta t}d^2 + 2\cdot e^{r\Delta t}d - \cancel{e^{r\Delta t}}\Big)$$

$$= \frac{S_t^2}{u-d}\Big(e^{r\Delta t}(u^2-d^2) - 2e^{r\Delta t}(u-d)$$
$$-ud(u-d) + (u-d)\Big)$$

$$= S_t^2\left(e^{r\Delta t}(u+d) - 2e^{r\Delta t} - ud + 1\right)$$

$$= S_t^2\left(e^{r\Delta t}(u+d-2) - ud + 1\right)$$

If Δt is very small but not 0, we know that $e^{r\Delta t}=1$, $u=1$, $d=1$
(converges to 0)

Then for very small $\Delta t \neq 0$, $Var(\Delta S_t) = S_t^2(1(2-2)-1+1) = 0$

Similiarly as in a), we have $Var(dS_t)=0$ without having $dt=0$

Then $Var(dS_t) = \beta^2 dZ_t = 0$ even for very small value of $dt \neq 0$, $\beta^2=0, \beta=0$

As we have both $\alpha=0, \beta=0$, $dS_t = 0$

A1 Q3. ①

We prove the property by induction from $n=N$. and $0 \le j \le n$

For $n=N$ and for $0 \le j \le n$, we have the following:

$V^{tree}(\lambda S_j^N, \lambda K, t_N)$

$= \max(\lambda K - \lambda S_j^N, 0) + \max(\lambda S_j^N - \lambda K, 0)$

consider two cases: ① $\lambda S_j^n > \lambda K$, ② $\lambda S_j^n < \lambda K$

① $\lambda S_j^N > \lambda K \Leftrightarrow S_j^N > K \quad (\because \lambda > 0)$

$= \lambda S_j^N - \lambda K = \lambda(S_j^N - K)$

$= \lambda \max(S_j^N - K, 0) + \lambda \max(K - S_j^N, 0) \quad (\because K - S_j^N < 0)$

$= \lambda\left(\max(S_j^N - K, 0) + \max(K - S_j^N, 0)\right)$

$= \lambda V_{tree}(S_j^N, K, t_N)$

② $\lambda S_j^N < \lambda K \Leftrightarrow S_j^N < K \quad (\because \lambda > 0)$

$= \lambda K - \lambda S_j^N = \lambda(K - S_j^N)$ ... from ①

$= \lambda \max(K - S_j^N, 0) + \lambda \max(S_j^N - K, 0) \quad (\because S_j^N - K < 0)$

$= \lambda(\max(K - S_j^N, 0) + \max(S_j^N - K, 0))$

$= \lambda V_{tree}(S_j^N, K, t_N)$

So the property holds for $n=N$ and $0 \le j \le n$. Cont.

Assume the property holds for some $n = i \leq N$ and $0 \leq j \leq n$.

Then we prove for $n = i-1$, (and we limit j to be $0 \leq j \leq n = i-1$)

$V_{tree}(\lambda S_j^{i-1}, \lambda K, t_{i-1}) =$

$e^{-r\Delta t} \cdot \left(q^* \cdot V_{tree}(\lambda S_{j+1}^i, \lambda K, t_i) + (1-q^*) \cdot V_{tree}(\lambda S_j^i, \lambda K, t_i) \right)$

by binomial lattice properties

$= e^{-r\Delta t} \cdot \left(q^* \cdot \lambda V_{tree}(S_{j+1}^i, K, t_i) + (1-q^*) \cdot \lambda V_{tree}(S_j^i, K, t_i) \right)$

by induction hypothesis

$= \lambda \cdot e^{-r\Delta t} \cdot \left(q^* \cdot V_{tree}(S_{j+1}^i, K, t_i) + (1-q^*) \cdot V_{tree}(S_j^i, K, t_i) \right)$

$= \lambda \cdot V_{tree}(S_j^{i-1}, K, t_{i-1})$ by binomial lattice properties

There we have the property that $V_{tree}(\lambda S_j^n, \lambda K, t_n)$

$= \lambda V_{tree}(S_j^n, K, t_n)$ for $0 \leq n \leq N$ and $0 \leq j \leq n$

by induction.

a) All possible stock prices are: $u^i \cdot d^{(N-i)} \cdot S_0$ for $i = 0, \ldots, N$
at time $T = \Delta t N$ and $S_0^0 = S_0$

b) There are ${}_NC_k$ distinct ways that a stock can go up for k times in binomial lattice. The probability for one path is $(q^*)^k \cdot (1-q^*)^{(N-k)}$

Therefore the total prob is ${}_NC_k \cdot (q^*)^k \cdot (1-q^*)^{(N-k)}$ under risk neutral binomial lattice.

c) The expression is as follow:

$$\underbrace{e^{-rT}}_{\text{depreciate}} \cdot \Big(\underbrace{\sum_{i=0}^{N}}_{\text{sum}} \underbrace{{}_NC_i \cdot (q^*)^i \cdot (1-q^*)^{(N-i)}}_{\text{prob}} \cdot \underbrace{\left(\max(K - S_i^N, 0) + \max(S_i^N - K, 0)\right)}_{\text{from Q3}} \Big)$$

where $N = \frac{T}{\Delta t}$. We prove this in three steps

①
We have payoff of a European straddle from 3) for a stock price S_i^N in the binomial lattice as follows: $\max(K - S_i^N, 0) + \max(S_i^N - K, 0)$

for each $i = 0, \ldots, N$ and $N = \frac{T}{\Delta t}$

② From 4b), we have a probability when we end up with the stock price S_i^N, which is ${}_NC_i \cdot (q^*)^i \cdot (1-q^*)^{(N-i)}$.

Thus we have the expected payoff value of a straddle as follows:

$${}_NC_i \cdot (q^*)^i \cdot (1-q^*)^{(N-i)} \cdot \left(\max(K - S_i^N, 0) + \max(S_i^N - K, 0)\right)$$

for each $i = 0, \ldots, N$

As we have $i = 0, \ldots, N$, we need to sum up the expected values for each i

$$\sum_{i=0}^{N} {}_NC_i \cdot (q^*)^i \cdot (1-q^*)^{N-i} \cdot \left(\max(K - S_i^N, 0) + \max(S_i^N - K, 0)\right)$$

A1Q4 ②

c) As this sum of expected payoffs is at time T, we need to deprecate the sum to time 0, so that we get the value at time 0.

Then we have the expression,

$$e^{-rT} \cdot \left(\sum_{i=0}^{N} {}_{N}C_{i} \cdot (q^*)^{i} \cdot (1-q^*)^{(N-i)} \cdot \left(\max(S_i^N - K, 0) + \max(K - S_i^N, 0) \right) \right).$$

where $N = \frac{T}{\Delta t}$.

AIQ5 ①

a) By Ito's process, we have that

$$dY_t = \left[\frac{\partial G}{\partial t} + a\frac{\partial G}{\partial S} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial S^2}\right]dt + b\frac{\partial G}{\partial S}dZ_t$$

and $dS_t = \mu dt + \sigma dZ_t$, where $a = \mu$, $b = \sigma$

Let $Y_t = G(S_t, t)$ and $G(S,t) = S^2$

Then $\frac{\partial G}{\partial t} = 0$, $\frac{\partial G}{\partial S} = 2S$, $\frac{\partial^2 G}{\partial S} = 2$

$$dS_t^2 = [0 + \mu \cdot 2S_t + \tfrac{1}{2}\sigma^2 \cdot 2]\,dt + \sigma \cdot 2S_t\,dZ_t$$

$$= (\sigma^2 + 2\mu S_t)\,dt + 2\sigma S_t\,dZ_t$$

$$\therefore\ dS_t^2 = (\sigma^2 + 2\mu S_t)\,dt + 2\sigma S_t\,dZ_t$$

$$= 2S_t(\mu d_t + \sigma dZ_t) + \sigma^2 dt$$

$$= 2S_t\,dS_t + \sigma^2 dt$$

b) From 5a) we have 2

$$dS_t^2 = 2 \cdot S_t \cdot dS_t + \sigma^2 dt$$

Take integral on both sides

$$S_t^2 = \int (2 S_t \cdot dS_t + \sigma^2 dt)$$

$$= 2 \int S_t \, dS_t + \int \sigma^2 dt = 2 \int S_t \, dS_t + \sigma^2 t + C$$

integration constant

Collect terms to $\int S_t \, dS_t$

Then $\int S_t \, dS_t = \frac{1}{2} \left(S_t^2 - \sigma^2 t \right) + C$

I integration constant

Then $\int_0^T S_t \, dS_t = \int S_t \, dS_t \Big|_0^T = \frac{1}{2} \left(S(T)^2 - \sigma^2 T \right) - \frac{1}{2} \left(S(T)^2 - \sigma^2 \cdot 0 \right)$

$$= \frac{S(T)^2 - S(0)^2}{2} - \frac{T}{2} \sigma^2$$

A1Q2 3

c) We would like to find α, β which satisfy

$$dS_t = \alpha dt + \beta dZ_t.$$

From the coursenotes, we have that $E(dS_t) = \alpha dt$ and $Var(dS_t) = \beta^2 dt$

we will use this property to show that $\alpha = 0$ and $\beta = 0$

we have
$$E(\Delta S) = q^*\left(S_{j+1}^{n+1} - S_j^n\right) + (1-q^*)\left(S_j^{n+1} - S_j^n\right)$$
$$= q^*\left(uS_j^n - S_j^n\right) + (1-q^*)\left(dS_j^n - S_j^n\right)$$
$$= S_j^n\left(q^*(1-u) + (1-q^*)(d-1)\right)$$

By definition, $dS_t = \lim_{\Delta t \to 0} \Delta S_t$. By delta-epsilon theorem, $\Delta t \to 0$, and also $dt \to 0$.

Then we have
$$E(dS_t) = E\left(\lim_{\Delta t \to 0} \Delta S_t\right) = \lim_{\Delta t \to 0} E(\Delta S_t)$$
$$= \lim_{\Delta t \to 0} S_j^n\left(q^*(1-u) + (1-q^*)(d-1)\right)$$
$$= S_j^n \lim_{\Delta t \to 0}\left(q^*(1-u) + (1-q^*)(d-1)\right)$$

From 2b), $\lim_{\Delta t \to 0} q^* = \frac{1}{2}$, and $\lim_{\Delta t \to 0} u = \lim_{\Delta t \to 0} \exp\left[\cancelto{0}{\sigma\sqrt{\Delta t}} + \cancelto{0}{(r - \sigma^2/2)\Delta t}\right] = 1$ and

$$\lim_{\Delta t \to 0} d = \lim_{\Delta t \to 0} \exp\left[-\cancelto{0}{\sigma\sqrt{\Delta t}} + \cancelto{0}{(r - \sigma^2/2)\Delta t}\right] = 1$$

So we have
$$E(dS_t) = S_j^n \lim_{\Delta t \to 0}\left(\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 0\right) = S_j^n \cdot 0 = 0$$

A1Q2 4

c) we have $E(dS_t)=0$ and as $dt\neq 0$, we have $\alpha=0$

We do similarly for $Var(dS_t)$.

we have $Var(\Delta S_t) = q^*\left(S_{j+1}^{n+1}-S_j^n\right)^2+(1-q^*)\left(S_j^{n+1}-S_j^n\right)^2$

$$= q^*(uS_j^n-S_j^n)^2+(1-q^*)(dS_j^n-S_j^n)^2$$

$$= S_j^n\left(q^*(u-1)^2+(1-q^*)(d-1)^2\right)$$

Then $Var(dS_t) = Var\left(\lim_{\Delta t\to 0}\Delta S_t\right) = \lim_{\Delta t\to 0} Var(\Delta S_t)$

$$= \lim_{\Delta t\to 0} S_j^n\left(q^*(u-1)^2+(1-q^*)(d-1)^2\right)$$

$$= S_j^n \lim_{\Delta t\to 0}\left(q^*(u-1)^2+(1-q^*)(d-1)^2\right)$$

$$= S_j^n\cdot\left(\tfrac{1}{2}(0)+\tfrac{1}{2}(0)\right) = S_j^n=0.$$

As $Var(dS_t)=0$ and $dt\neq 0$, so we have $\beta=0$

As we have that $\alpha=0, \beta=0$, we have $dS_t=0=0\cdot dt+0\cdot dZ_t$