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c) Let $dS_t = \alpha\, dt + \beta\, dZ_t$. From course notes, we have

$E(dS_t) = \alpha\, dt$, and $Var(dS_t) = \beta^2\, dt$.

Then we have

$$E(\Delta S_t) = q^*(uS_t - S_t) + (1-q^*)(dS_t - S_t)$$
$$= S_t\left(q^*(u-1) + (1-q^*)(d-1)\right)$$
$$= S_t\left(q^* u - \cancel{q^*} + d - 1 - q^* d \cancel{+q^*}\right)$$
$$= S_t\left(q^*(u-d) + d - 1\right)$$
$$= S_t\left(e^{r\Delta t} - d + d - 1\right)$$
$$= S_t\left(e^{r\Delta t} - 1\right)$$
$$= S_t\left(1 + r\Delta t + o(\Delta t) - 1\right) \quad (\because \text{apply taylor series expansion})$$
$$= S_t\left(r\Delta t + o(\Delta t)\right)$$

As $\Delta t \to 0$, $o(\Delta t)$ term converges to 0 faster than Δt.

Then we have $E(dS_t) = E\left(\lim_{\Delta t \to 0} \Delta S_t\right) = \lim_{\Delta t \to 0} E(\Delta S_t)$

$$= \lim_{\Delta t \to 0} S_t\left(r\Delta t + o(\Delta t)\right) = S_t \cdot r \cdot dt$$

$$\therefore \alpha = r \cdot S_t$$

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c) $\mathrm{Var}(\Delta S_t) = q^*(uS_t - S_t)^2 + (1-q^*)(dS_t - S_t)^2$

$= S_t^2\left(q^*(u^2-2u+1) + (1-q^*)(d^2-2d+1)\right)$

$= S_t^2\left(q^*u^2 - 2q^*u + \cancel{q^*} + d^2 - 2d + 1 - q^*d^2 + 2q^*d \cancel{-q^*}\right)$

$= S_t^2\left(q^*(u^2-d^2-2u+2d) + d^2-2d+1\right)$

$= S_t^2\left(q^*(u-d)(u+d-2) + d^2-2d+1\right)$

$= S_t^2\left((e^{r\Delta t}-d)(u+d-2) + d^2-2d+1\right)$

$= S_t^2\left(e^{r\Delta t}(u+d-2) - ud - d^2 + 2d + d^2 - 2d + 1\right)$

$= S_t^2\left(e^{r\Delta t}(u+d-2) - ud + 1\right)$

$= S_t^2\left(e^{r\Delta t}\cdot\left(e^{\sigma\sqrt{\Delta t}+(r-\sigma^2/2)\Delta t} + e^{-\sigma\sqrt{\Delta t}+(r-\sigma^2/2)\Delta t} - 2\right) - e^{(2r-\sigma^2)\Delta t} + 1\right)$

$= S_t^2\left(e^{\sigma\sqrt{\Delta t}+(2r-\sigma^2/2)\Delta t} + e^{-\sigma\sqrt{\Delta t}+(2r-\sigma^2/2)\Delta t} - 2\cdot e^{r\Delta t} - e^{(2r-\sigma^2)\Delta t} + 1\right)$

Apply taylor series expansion, we have

$$= S_t^2\Big(\left(1+\sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t + \tfrac{\sigma^2}{2}\Delta t + o(\Delta t)\right)$$
$$+\left(1-\sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t + \tfrac{\sigma^2}{2}\Delta t + o(\Delta t)\right)$$
$$-2\left(1+r\Delta t + o(\Delta t)\right)$$
$$-\left(1+(2r-\sigma^2)\Delta t + o(\Delta t)\right) + 1\Big)$$

Cont.

c) $= S_t^2\Big(\big(1+\sigma\sqrt{\Delta t}+(2r-\sigma^2/2)\Delta t+\frac{1}{2}\sigma^2\Delta t+o(\Delta t)\big)$
$+\big(1-\sigma\sqrt{\Delta t}+(2r-\sigma^2/2)\Delta t+\frac{1}{2}\sigma^2\Delta t+o(\Delta t)\big)$
$-2\big(1+r\Delta t+o(\Delta t)\big)$
$-\big(1+(2r-\sigma^2)\Delta t+o(\Delta t)\big)+1\Big)$

$$= S_t^2\Big(\cancel{2}+(4r-\sigma^2)\Delta t+\sigma^2\Delta t - \cancel{2} - 2r\Delta t - \cancel{1} - (2r-\sigma^2)\Delta t + \cancel{1} + o(\Delta t)\Big)$$

$$= S_t^2\Big(\cancel{4r\Delta t}-\sigma^2\Delta t+\sigma^2\Delta t-\cancel{2r\Delta t}-\cancel{2r\Delta t}+\sigma^2\Delta t+o(\Delta t)\Big)$$

$$= S_t^2\Big(\sigma^2\Delta t+o(\Delta t)\Big) = S_t^2\sigma^2\Delta t+o(\Delta t)$$

As $\Delta t\to 0$, $o(\Delta t)$ converges to 0 faster than Δt.

Then $\mathrm{Var}(dS_t)=\mathrm{Var}(\lim_{\Delta t\to 0}\Delta S_t)=\lim_{\Delta t\to 0}\mathrm{Var}(\Delta S_t)$

$$=\lim_{\Delta t\to 0}\Big(S_t^2(\sigma^2\Delta t)+o(\Delta t)\Big) = S_t^2\sigma^2 dt$$

As $\mathrm{Var}(dS_t)=\beta^2 dt = S_t^2\sigma^2 dt = (\sigma S_t)^2 dt$,

$\beta=\sigma S_t \quad (\because \sigma>0,\ S_t>0)$

As we now have $\alpha = rS_t$ and $\beta=\sigma S_t$, we have that

$$dS_t = rS_t\,dt + \sigma S_t\,dt$$

$$S_t^2\left(e^{r\Delta t}(u+d-2)-ud+1\right) \qquad ud=\exp\left((2r-\sigma^2)\Delta t\right)$$

$$e^{\sigma\sqrt{\Delta t}+(2r-\sigma^2/2)\Delta t}+e^{-\sigma\sqrt{\Delta t}+(2r-\sigma^2/2)\Delta t}-2e^{r\Delta t}$$

$$-e^{(2r-\sigma^2)\Delta t}+1$$

$$\left(\cancel{1+\sigma\sqrt{\Delta t}}+(2r-\sigma^2/2)\Delta t\right)$$

$$\cancel{+}\left(\cancel{1-\sigma\sqrt{\Delta t}}+(2r-\sigma^2/2)\Delta t\right)$$

$$\cancel{-2}(1+r\Delta t)$$

$$\cancel{-}\left(\cancel{1}+(2r-\sigma^2)\Delta t\right)\cancel{+1}$$

$$4r-\sigma^2-2r-2r+\sigma^2=\underline{0.}$$