c) Let $dS_t = \alpha\, dt + \beta\, dZ_t$. From course notes, we have

$E(dS_t) = \alpha\, dt$, and $Var(dS_t) = \beta^2\, dt$.

Then we have

$$E(\Delta S_t) = q^*(uS_t - S_t) + (1-q^*)(dS_t - S_t)$$
$$= S_t\left(q^*(u-1) + (1-q^*)(d-1)\right)$$
$$= S_t\left(q^*u - \cancel{q^*} + d - 1 - q^*d \cancel{+q^*}\right)$$
$$= S_t\left(q^*(u-d) + d - 1\right)$$
$$= S_t\left(e^{r\Delta t} - d + d - 1\right)$$
$$= S_t\left(e^{r\Delta t} - 1\right)$$
$$= S_t\left(1 + r\Delta t + o(\Delta t) - 1\right)$$
$$= S_t\left(r\Delta t + o(\Delta t)\right)$$

As $\Delta t \to 0$, $o(\Delta t)$ term converges to 0 faster than Δt.

Then we have $E(dS_t) = E(\lim_{\Delta t \to 0} \Delta S_t) = \lim_{\Delta t \to 0} E(\Delta S_t)$

$$= \lim_{\Delta t \to 0} S_t\left(r\Delta t + o(\Delta t)\right) = S_t \cdot r \cdot dt$$

$\therefore \alpha = r \cdot S_t$

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c) $\mathrm{Var}(\Delta S_t) = q^*(uS_t - S_t)^2 + (1-q^*)(dS_t - S_t)^2$

$$= S_t^2\left(q^*(u^2-2u+1) + (1-q^*)(d^2-2d+1)\right)$$

$$= S_t^2\left(q^*u^2 - 2q^*u + \cancel{q^*} + d^2 - 2d + 1 - q^*d^2 + 2q^*d \cancel{-q^*}\right)$$

$$= S_t^2\left(q^*(u^2-d^2-2u+2d) + d^2-2d+1\right)$$

$$= S_t^2\left(q^*(u-d)(u+d-2) + d^2-2d+1\right)$$

$$= S_t^2\left((e^{r\Delta t}-d)(u+d-2) + d^2-2d+1\right)$$

$$= S_t^2\left(e^{r\Delta t}(u+d-2) - ud - d^2 + 2d + d^2 - 2d + 1\right)$$

$$= S_t^2\left(e^{r\Delta t}(u+d-2) - ud + 1\right)$$

$$= S_t^2\left(e^{r\Delta t}\cdot\left(e^{\sigma\sqrt{\Delta t} + (r-\sigma^2/2)\Delta t} + e^{-\sigma\sqrt{\Delta t} + (r-\sigma^2/2)\Delta t} - 2\right) - e^{(2r-\sigma^2)\Delta t} + 1\right)$$

$$= S_t^2\left(e^{\sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t} + e^{-\sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t} - 2\cdot e^{r\Delta t} - e^{(2r-\sigma^2)\Delta t} + 1\right)$$

Apply taylor series expansion, we have

$$= S_t^2\Big(\left(1 + \sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t + \tfrac{\sigma^2}{2}\Delta t + o(\Delta t)\right)$$
$$+ \left(1 - \sigma\sqrt{\Delta t} + (2r-\sigma^2/2)\Delta t + \tfrac{\sigma^2}{2}\Delta t + o(\Delta t)\right)$$
$$- 2\left(1 + r\Delta t + o(\Delta t)\right)$$
$$- \left(1 + (2r-\sigma^2)\Delta t + o(\Delta t)\right) + 1\Big)$$

Cont.

c) $= S_t^2 \Big(\big(1 + \sigma\sqrt{\Delta t} + (2r - \sigma^2/2)\Delta t + \frac{1}{2}\sigma^2\Delta t + o(\Delta t)\big)$
$+ \big(1 - \sigma\sqrt{\Delta t} + (2r - \sigma^2/2)\Delta t + \frac{1}{2}\sigma^2\Delta t + o(\Delta t)\big)$
$- 2\big(1 + r\Delta t + o(\Delta t)\big)$
$- \big(1 + (2r - \sigma^2)\Delta t + o(\Delta t)\big) + 1 \Big)$

$$= S_t^2 \Big(\cancel{2} + (4r - \sigma^2)\Delta t + \sigma^2\Delta t - \cancel{2} - 2r\Delta t - \cancel{1} - (2r - \sigma^2)\Delta t + \cancel{1} + o(\Delta t) \Big)$$

$$= S_t^2 \Big(\cancel{4r\Delta t} - \sigma^2\Delta t + \sigma^2\Delta t - \cancel{2r\Delta t} - \cancel{2r\Delta t} + \sigma^2\Delta t + o(\Delta t) \Big)$$

$$= S_t^2 \big(\sigma^2\Delta t + o(\Delta t) \big) = S_t^2 \sigma^2 \Delta t + o(\Delta t)$$

As $\Delta t \to 0$, $o(\Delta t)$ converges to 0 faster than Δt.

Then $Var(dS_t) = Var(\lim_{\Delta t \to 0} \Delta S_t) = \lim_{\Delta t \to 0} Var(\Delta S_t)$

$$= \lim_{\Delta t \to 0} S_t^2 \big(\sigma^2\Delta t + o(\Delta t)\big) = S_t^2 \sigma^2 dt$$

As $Var(dS_t) = \beta^2 dt = S_t^2 \sigma^2 dt = (\sigma S_t)^2 dt$,

$\beta = \sigma S_t \quad (\because \sigma > 0,\ S_t > 0)$

As we now have $\alpha = r S_t$ and $\beta = \sigma S_t$, we have that

$$dS_t = r S_t\, dt + \sigma S_t\, dt$$