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c) Let $dS_t = \alpha\, dt + \beta\, dZ_t$. From course notes, we have

$E(dS_t) = \alpha\, dt$, and $Var(dS_t) = \beta^2\, dt$.

Then we have

$$E(\Delta S_t) = q^*(uS_t - S_t) + (1-q^*)(dS_t - S_t)$$
$$= S_t\left(q^*(u-1) + (1-q^*)(d-1)\right)$$
$$= S_t\left(q^* u - \cancel{q^*} + d - 1 - q^* d + \cancel{q^*}\right)$$
$$= S_t\left(q^*(u-d) + d - 1\right)$$
$$= S_t\left(e^{r\Delta t} - d + d - 1\right)$$
$$= S_t\left(e^{r\Delta t} - 1\right)$$
$$= S_t\left(1 + r\Delta t + o(\Delta t) - 1\right) \quad (\because \text{apply taylor series expansion})$$
$$= S_t\left(r\Delta t + o(\Delta t)\right)$$

As $\Delta t \to 0$, $o(\Delta t)$ term converges to 0 faster than Δt.

Then we have $E(dS_t) = E(\lim_{\Delta t \to 0} \Delta S_t) = \lim_{\Delta t \to 0} E(\Delta S_t)$

$$= \lim_{\Delta t \to 0} S_t\left(r\Delta t + o(\Delta t)\right) = S_t \cdot r \cdot dt$$

$\therefore\ \alpha = r \cdot S_t$