CS1200: Intro. to Algorithms and their Limitations	Prof. Salil Vadhan
Sender–Receiver Exercise 5: Reading for Receivers	
Harvard SEAS - Fall 2025	*2025-10-21*

1 Sender–Receiver Exercise: Long Path $\leq$ SAT

If you are the Receiver, you should try to understand the theorem statement and definition in Section 2 below, and review the material on Logic covered in Chapter 17 and in lecture. Your partner sender will communicate the proof of Theorem 2.1.

2 The Result

In Section 17.3 of the textbook, we saw how the (seemingly hard) problem of Graph Coloring can be efficiently reduced to CNF-Satisfiability (SAT). Although SAT also seems to be a hard problem (as we'll formalize in the last part of the course), this allows all the effort put into SAT Solvers to solve many large instances of Graph Coloring in practice.

In this exercise, you'll see a similar reduction for the *Longest Path* problem. Recall that a *path* is a walk with no repeated vertices.

Input: A digraph $G = (V, E)$ and two vertices $s, t \in V$

Output: A *longest path* from s to t in G, if one exists

Computational Problem Longest Path

Actually, it will be more convenient to consider a version where the desired path length is specified in the input.

Input: A digraph $G = (V, E)$, two vertices $s, t \in V$, and a path-length $k \in \mathbb{N}$

Output: A path from s to t in G of length k, if one exists

Computational Problem Long Path

Since a path has no repeated vertices, it suffices to consider $k \leq n$. If we have an efficient algorithm for Long Path, then we can solve Longest Path by trying $k = n, n-1, \ldots, 0$ until we succeed in finding a path. The $k = n$ case is essentially the same as the Hamiltonian Path problem,[1] which is a special case of the notorious Travelling Salesperson Problem (TSP): In TSP, we have a salesperson who wishes to visit n cities (to sell their goods) in the shortest travel time possible. If we model the possible travel between cities as a directed graph, then Hamiltonian Path corresponds to the special case where all pairs u, v of cities either have a

[1] In the Hamiltonian Path problem, we don't specify the start and end vertex; any path of length n suffices. But the two problems can be efficiently reduced to each other (exercise).

travel time of 1 (if $(u, v) \in E$) or a very large travel time (if $(u, v) \notin E$). In such a case, the only way to visit all cities in travel time at most $n-1$ is via a Hamiltonian path.[2]

The reduction from LONG PATH to SAT is given as follows.

Theorem 2.1. LONG PATH *on a digraph with n vertices, m edges, and a path length k reduces to* SAT *in time $O(n^2 k)$.*

Constructing a SAT instance φ from a LONG PATH instance (G, s, t, k).

[2] Often in TSP, it is also required that the salesperson return to their starting city s. If we add edges of travel time 1 from all cities to s, then we see that HAMILTONIAN PATH is also a special case of this variant of TSP.

Converting a satisfying assignment α to φ into a Long Path solution P.

Correctness of the Reduction.

Runtime of the Reduction.