Minimum Cost Spanning Trees: Prim's Algorithm

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Programming, Data Structures and Algorithms using Python

Week 5

Minimum cost spanning tree (MCST)

- Weighted undirected graph, $G = (V, E), W : E \to \mathbb{R}$
  - G assumed to be connected

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  - Start with a smallest weight edge overall
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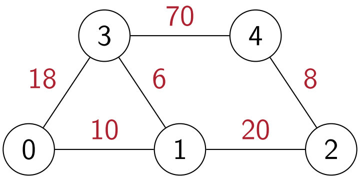

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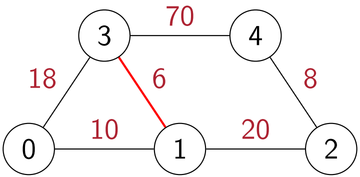

- Start with smallest edge, $(1, 3)$

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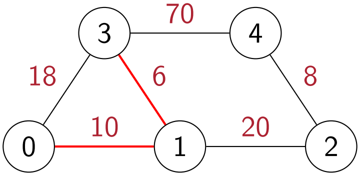

- Start with smallest edge, $(1, 3)$
- Extend the tree with $(1, 0)$

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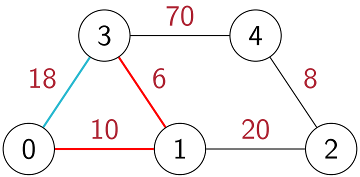

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- Extend the tree with $(1,0)$
- Can't add $(0,3)$, forms a cycle

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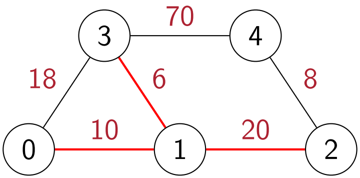

- Start with smallest edge, $(1, 3)$
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- Can't add $(0, 3)$, forms a cycle
- Instead, extend the tree with $(1, 2)$

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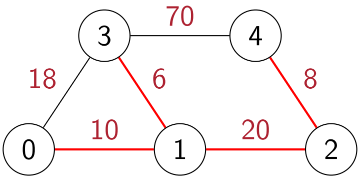

- Start with smallest edge, $(1, 3)$
- Extend the tree with $(1, 0)$
- Can't add $(0, 3)$, forms a cycle
- Instead, extend the tree with $(1, 2)$
- Extend the tree with $(2, 4)$

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- Repeat $n - 2$ times
  - Choose minimum weight edge $f = (u, v)$ such that $u \in TV$, $v \notin TV$
  - Add v to TV, f to TE

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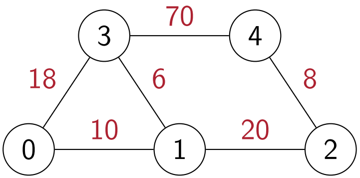

$TV = \emptyset$

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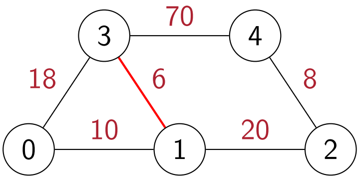

$TV = \{1, 3\}$

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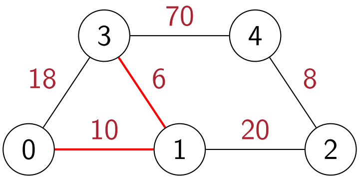

$TV = \{1, 3, 0\}$

$TE = \{(1, 3), (1, 0)\}$

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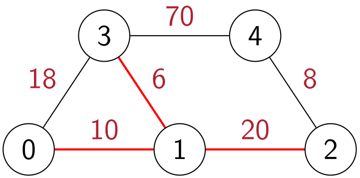

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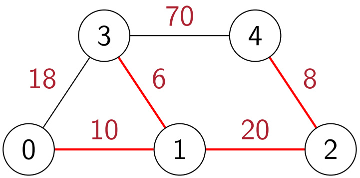

$TV = \{1, 3, 0, 2, 4\}$

$TE = \{(1,3), (1,0), (1,2), (2,4)\}$

Correctness of Prim's algorithm

Minimum Separator Lemma

- Let V be partitioned into two non-empty sets U and $W = V \setminus U$
- Let $e = (u, w)$ be the minimum cost edge with $u \in U$, $w \in W$
- Every MCST must include e

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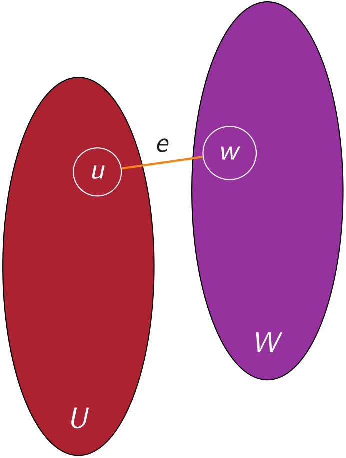

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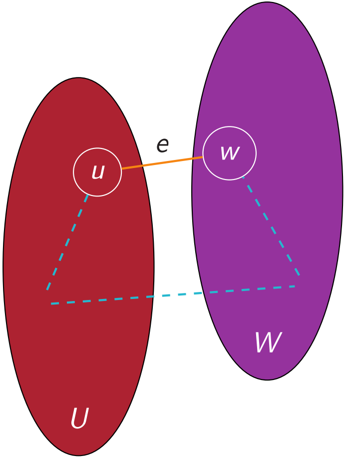

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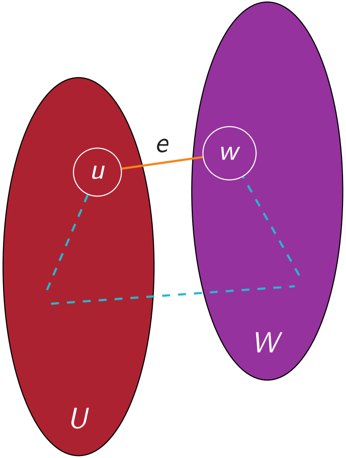

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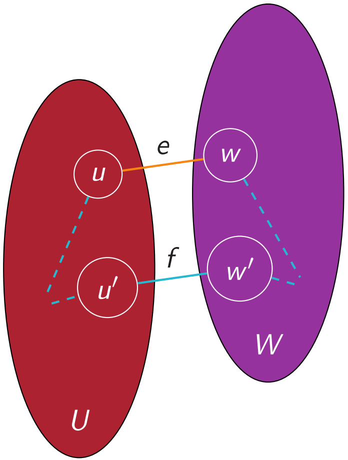

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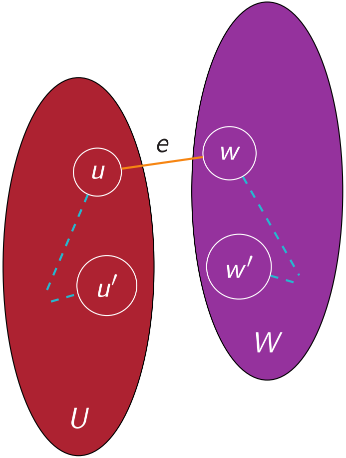

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  - Drop f, add e to get a cheaper spanning tree

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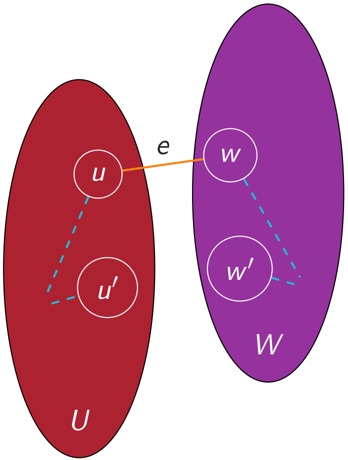

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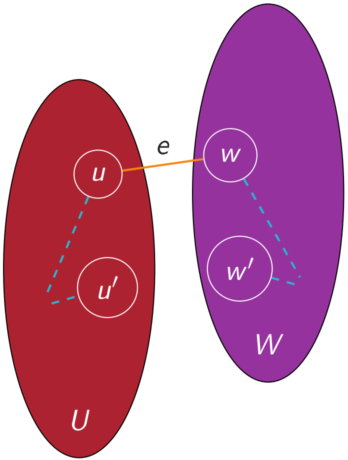
- Assign each edge a unique index from 0 to $m - 1$



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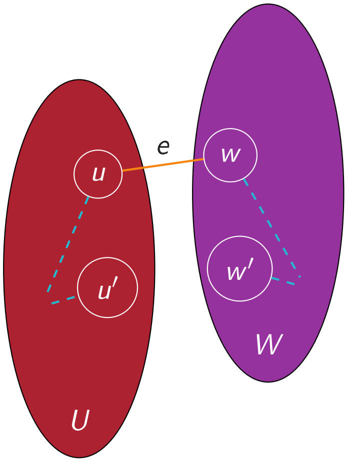
- Define $(e, i) < (f, j)$ if $W(e) < W(j)$ or $W(e) = W(j)$ and $i < j$



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- Algorithm picks smallest edge connecting TV and W, which must belong to every MCST

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- In fact, for any $v \in V$, $\{v\}$ and $V \setminus \{v\}$ form a partition
- The smallest weight edge leaving any vertex must belong to every MCST
- We started with overall minimum cost edge
- Instead, can start at any vertex v, with $TV = \{v\}$ and $TE = \emptyset$
- First iteration will pick minimum cost edge from v

Implementation

- Keep track of
  - `visited[v]` – is `v` in the spanning tree?
  - `distance[v]` – shortest distance from `v` to the tree
  - `TreeEdges` – edges in the current spanning tree

```
def primlist(WList):
  infinity = 1 + max([d for u in WList.keys()
                          for (v,d) in WList[u]])
  (visited,distance,TreeEdges) = ({},{},[])
  for v in WList.keys():
    (visited[v],distance[v]) = (False,infinity)
  visited[0] = True
  for (v,d) in WList[0]:
    distance[v] = d
  for i in WList.keys():
    (mindist,nextv) = (infinity,None)
    for u in WList.keys():
      for (v,d) in WList[u]:
        if visited[u] and (not visited[v]) and d < mindist:
          (mindist,nextv,nexte) = (d,v,(u,v))
    if nextv is None:
      break
    visited[nextv] = True
    TreeEdges.append(nexte)
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v],d)
  return(TreeEdges)
```

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- First add vertex 0 to tree

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    TreeEdges.append(nexte)
    for (v,d) in WList[nextv]:
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        distance[v] = min(distance[v],d)
  return(TreeEdges)
```

Implementation

- Keep track of
  - visited[v] – is v in the spanning tree?
  - distance[v] – shortest distance from v to the tree
  - TreeEdges – edges in the current spanning tree
- Initialize visited[v] to False, distance[v] to infinity
- First add vertex 0 to tree
- Find edge (u,v) leaving the tree where distance[v] is minimum, add it to the tree, update distance[w] of neighbours

```
def primlist(WList):
  infinity = 1 + max([d for u in WList.keys()
                        for (v,d) in WList[u]])
  (visited,distance,TreeEdges) = ({},{},[])
  for v in WList.keys():
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  visited[0] = True
  for (v,d) in WList[0]:
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  for i in WList.keys():
    (mindist,nextv) = (infinity,None)
    for u in WList.keys():
      for (v,d) in WList[u]:
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          (mindist,nextv,nexte) = (d,v,(u,v))
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Complexity

- Initialization takes $(O(n)$

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- Loop to add nodes to the tree runs $O(n)$ times

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- Each iteration takes $O(m)$ time to find a node to add

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- Initialization takes $(O(n)$
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- Overall time is $O(mn)$, which could be $O(n^3)$!

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- Each iteration takes $O(m)$ time to find a node to add
- Overall time is $O(mn)$, which could be $O(n^3)$!
- Can we do better?

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Improved implementation

- For each `v`, keep track of its nearest neighbour in the tree
  - `visited[v]` – is `v` in the spanning tree?
  - `distance[v]` – shortest distance from `v` to the tree
  - `nbr[v]` – nearest neighbour of `v` in tree

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  for i in range(1,len(WList.keys())):
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- Scan all non-tree vertices to find `nextv` with minimum distance
- Then `(nbr[nextv],nextv)` is the tree edge to add
- Update `distance[v]` and `nbr[v]` for all neighbours of `nextv`

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Improved implementation — complexity

- Now the scan to find the next vertex to add is $O(n)$

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- Very similar to Dijkstra's algorithm, except for the update rule for distance

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- Now the scan to find the next vertex to add is $O(n)$
- Very similar to Dijkstra's algorithm, except for the update rule for distance
- Like Dijkstra's algorithm, this is still $O(n^2)$ even for adjacency lists

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Improved implementation — complexity

- Now the scan to find the next vertex to add is $O(n)$
- Very similar to Dijkstra's algorithm, except for the update rule for distance
- Like Dijkstra's algorithm, this is still $O(n^2)$ even for adjacency lists
- With a more clever data structure to extract the minimum, we can do better

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```

Summary

- Prim's algorithm grows an MCST starting with any vertex
- At each step, connect one more vertex to the tree using minimum cost edge from inside the tree to outside the tree
- Correctness follows from Minimum Separator Lemma
- Implementation similar to Dijkstra's algorithms
  - Update rule for distance is different
- Complexity is $O(n^2)$
  - Even with adjacency lists
  - Bottleneck is identifying unvisited vertex with minimum distance
  - Need a better data structure to identify and remove minimum (or maximum) from a collection