Graphs

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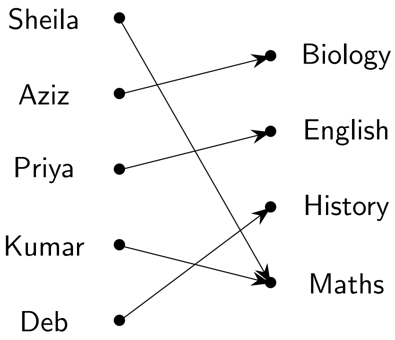
Programming, Data Structures and Algorithms using Python

Week 4

Visualizing relations as graphs

- Teachers and courses
  - T, set of teachers in a college
    C, set of courses being offered
  - $A \subseteq T \times C$ describes the allocation of teachers to courses
  - $A = \{(t, c) \mid (t, c) \in T \times C, t \text{ teaches } c\}$

Teachers and courses



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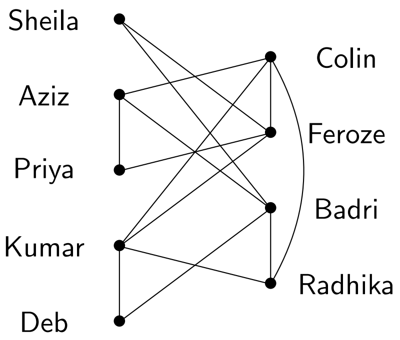
- Friendships
  - P, a set of students
  - $F \subseteq P \times P$ describes which pairs of students are friends
  - $F = \{(p, q) \mid p, q \in P, p \neq q, p \text{ is a friend of } q\}$
  - $(p, q) \in F$ iff $(q, p) \in F$

Friendship



Graphs

- Graph: $G = (V, E)$
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    - One vertex, many vertices
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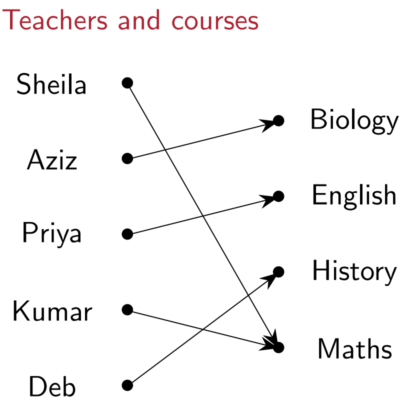
  - The teacher-course graph is directed



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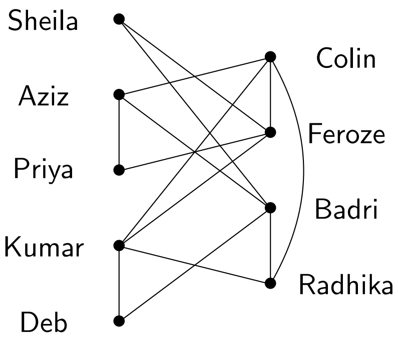
  - The teacher-course graph is directed
- Undirected graph
  - $(v, v') \in E$ iff $(v', v) \in E$
  - Effectively (v, v'), (v', v) are the same edge
  - Friendship graph is undirected

Friendship



Paths

- A path is a sequence of vertices $v_1, v_2, \ldots, v_k$ connected by edges
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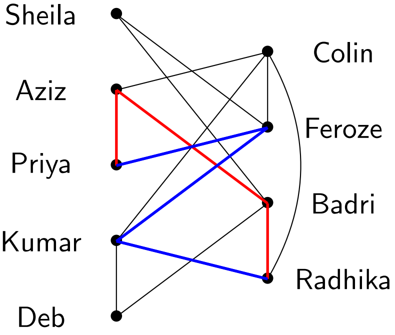
Friendship graph



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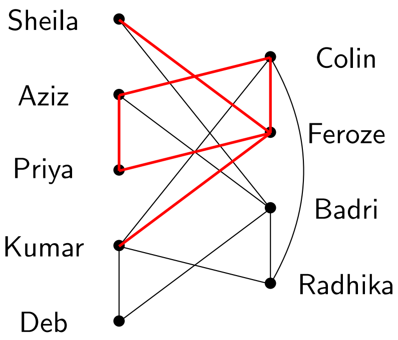
  - For $1 \leq i < k$, $(v_i, v_{i+1}) \in E$
- Normally, a path does not visit a vertex twice
- A sequence that re-visits a vertex is usually called a walk
  - Kumar — Feroze — Colin — Aziz — Priya — Feroze — Sheila

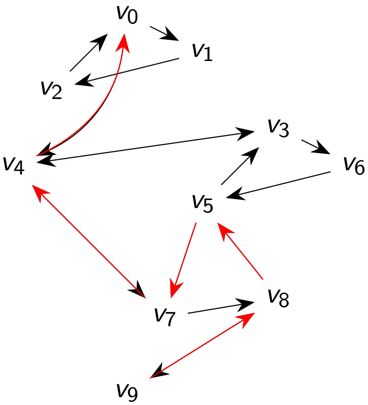
Friendship graph



Reachability

- Paths in directed graphs
- How can I fly from Madurai to Delhi?
  - Find a path from v_9 to v_0
- Vertex v is reachable from vertex u if there is a path from u to v

Airline routes



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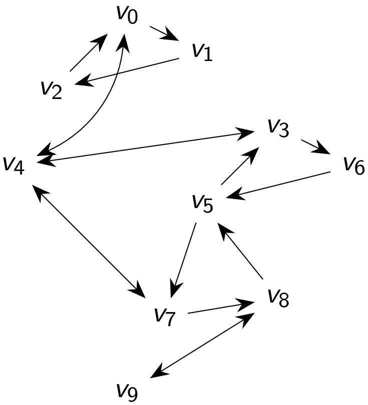

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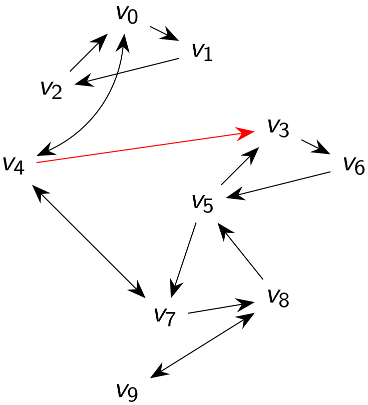

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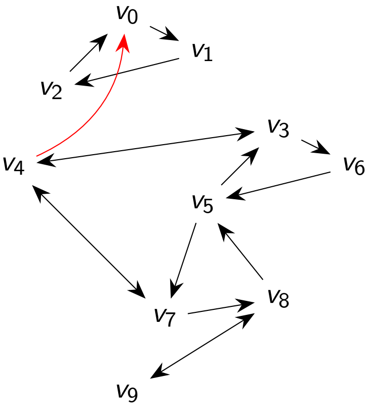

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- Assign each state a colour
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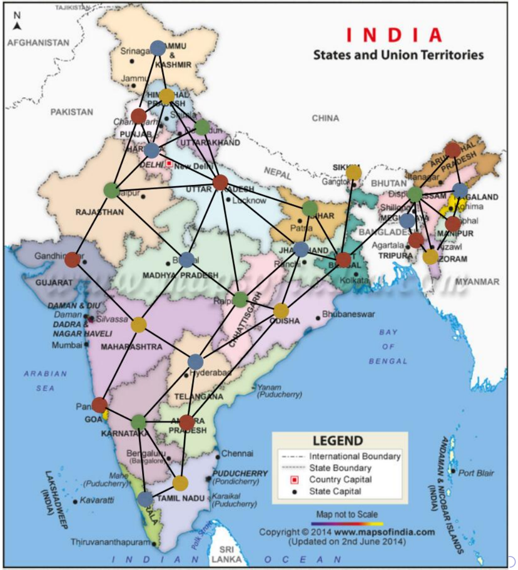

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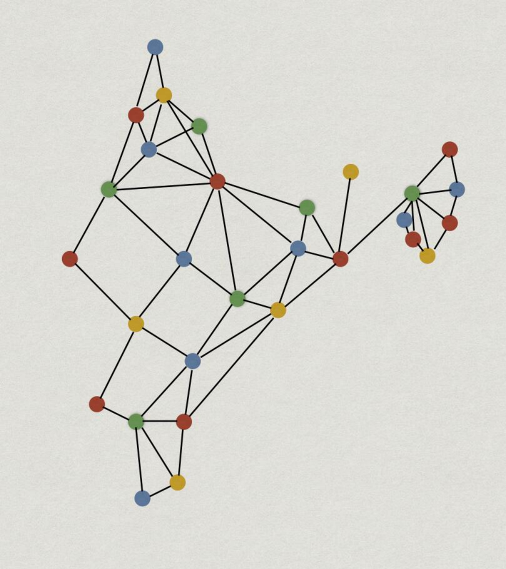

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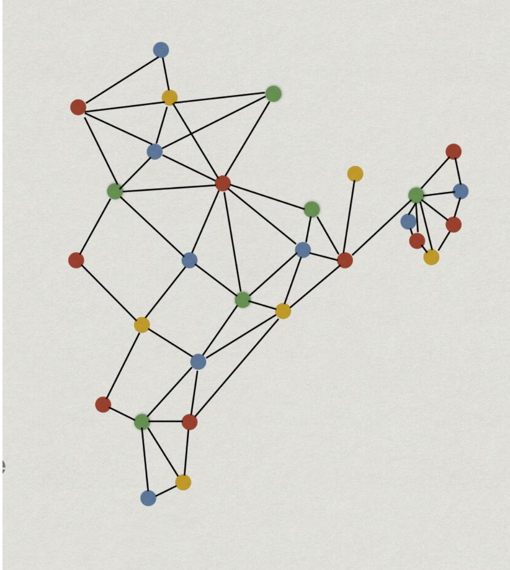
- Abstraction: if we distort the graph, problem is unchanged



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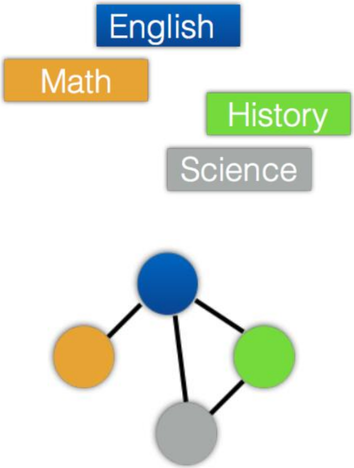

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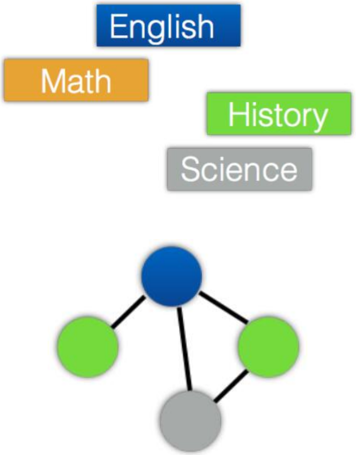

Vertex cover

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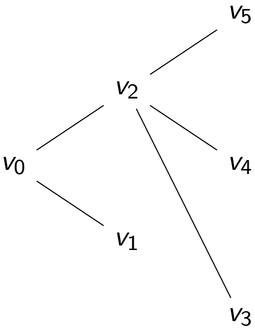

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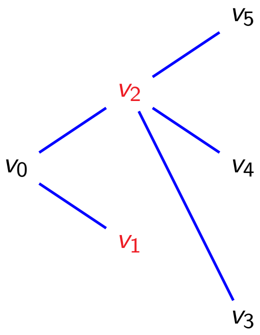
- Vertex cover
  - Marking v covers all edges from v
  - Mark smallest subset of V to cover all edges



Independent set

- A dance school puts up group dances
  - Each dance has a set of dancers
  - Sets of dancers may overlap across dances
- Organizing a cultural programme
  - Each dancer performs at most once
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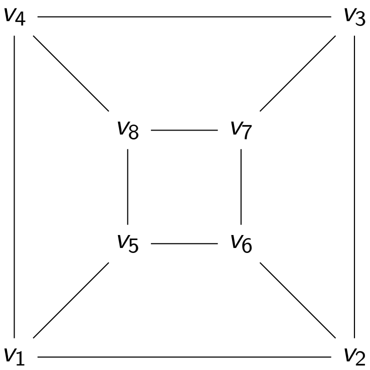

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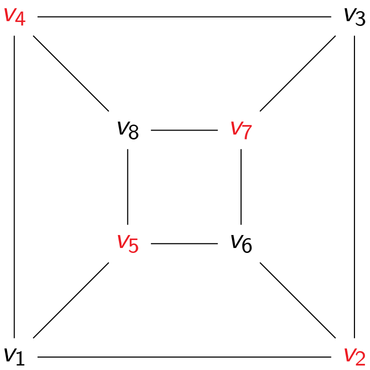
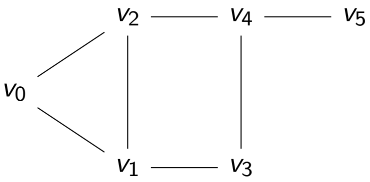
  - Subset of vertices such that no two are connected by an edge



Matching

- Class project can be done by one or two people
  - If two people, they must be friends
- Assume we have a graph describing friendships



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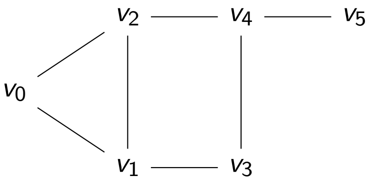

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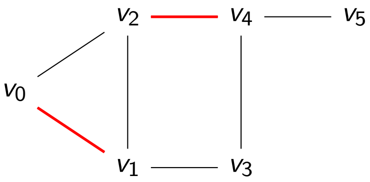

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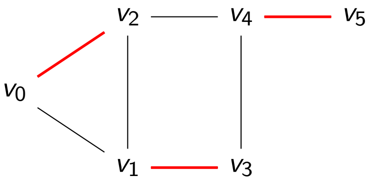
- Is there a perfect matching, covering all vertices?



Summary

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- Reachability: is there a path from u to v?
- Graphs are useful abstract representations for a wide range of problems
- Reachability and connectedness are not the only interesting problems we can solve on graphs
  - Graph colouring
  - Vertex cover
  - Independent set
  - Matching
  - ...