The following problem involves the number a_2, which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum =  72^(-1000000000000000000000000)
k=2
for m = 1 to n
      k=k+1
      sum = sum + (1/2)^(k) -
72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2. We want to minimise the total treatment time while ensuring that she receives a total dosage of 1.

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided ${0.4, 0.6}$ as a solution.
Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?
Answer only if you are 100% sure.
Please provide a brief explanation of your answer.

Thought for 492 seconds >

Step 1. Understand the Definition of a_2

The number a_2 is defined by the algorithm:

- **Compute_a_2(n)**:

$$\text{Output} = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

- **Compute_f(n)**:

$$\begin{aligned}
&\text{sum} = 72^{-1000000000000000000000000} \\
&k = 2 \\
&\text{For } m = 1 \text{ to } n: \\
&\quad k = k + 1 \\
&\quad \text{sum} = \text{sum} + \left(\frac{1}{2}\right)^k - 72^{-100000000000000000000000 \cdot k} \\
&\text{Output} = \text{sum}
\end{aligned}$$

Let's analyze the behavior of $f(n)$ as $n \to \infty$.

Step 2. Compute the Limit of $f(n)$

The algorithm for $f(n)$ adds terms of the form:

$$\left(\frac{1}{2}\right)^k - 72^{-100000000000000000000000 \cdot k}$$

for $k = 3, 4, \ldots, n+2$, and starts with an initial term:

$$72^{-1000000000000000000000000}$$

Thus,

$$f(n) = \sum_{k=3}^{n+2} \left(\frac{1}{2}\right)^k + \left[72^{-10^{24}} - \sum_{k=3}^{n+2} 72^{-k\cdot 10^{24}}\right]$$

The geometric series:

$$\sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k = \frac{(1/2)^3}{1-1/2} = \frac{1/8}{1/2} = \frac{1}{4}$$

Let:

$$A = 72^{-10^{24}}$$

Then the correction term:

$$\delta(n) = A - \sum_{k=3}^{n+2} A^k$$

As $n \to \infty$, this converges to:

$$\delta(\infty) = A - \frac{A^3}{1-A}$$

Hence,

$$f(\infty) = \frac{1}{4} + \delta(\infty)$$

Step 3. Compute a_2

$$a_2 = \lim_{n\to\infty} a_2(n) = 2f(\infty) + \frac{1}{2} = 2\left(\frac{1}{4} + \delta(\infty)\right) + \frac{1}{2} = 1 + 2\delta(\infty)$$

Since $A > 0$, we have:

$$\delta(\infty) = A - \frac{A^3}{1-A} > 0$$

Therefore,

$$a_2 = 1 + 2\delta(\infty) > 1$$

📝 Note: Although $a_2 > 1$, the value of $\delta(\infty)$ is extremely small, so a_2 is very close to 1.

Step 4. The Optimization Problem

We want to minimize the total treatment time:

$$T = x_1 + x_2$$

subject to the dosage constraint:

$$a_1x_1 + a_2x_2 = 1, \quad \text{with } a_1 = 1, \quad x_1, x_2 \geq 0$$

Rewriting:

$$x_1 = 1 - a_2x_2$$

$$T = 1 - a_2x_2 + x_2 = 1 + (1 - a_2)x_2$$

Since $a_2 > 1$, the coefficient $(1 - a_2) < 0$, so T decreases as x_2 increases. The maximum allowed value of x_2 is:

$$x_2 = \frac{1}{a_2} \quad \Rightarrow \quad x_1 = 0$$

Thus, the unique minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right)$$

Step 5. Compare with the Given Approximation

The provided approximation is:

$$(0.4, 0.6)$$

The true minimizer is:

$$(0, \frac{1}{a_2})$$

Since $a_2 \approx 1$, we have:

$$\frac{1}{a_2} \approx 1$$

Compute the infinity norm distance:

$$\|(0.4, 0.6) - (0, 1/a_2)\|_\infty = \max\left(|0.4 - 0|, |0.6 - 1/a_2|\right)$$

- First coordinate: $|0.4 - 0| = 0.4$
- Second coordinate: $|0.6 - 1/a_2| \approx |0.6 - 1| = 0.4$

Thus, the distance is approximately:

$$\max(0.4, 0.4) = 0.4$$

Even accounting for the tiny deviation of a_2 from 1, the distance remains less than 0.45.

Step 6. Conclusion

The infinity norm distance from the approximation $(0.4, 0.6)$ to the true minimizer is approximately 0.4, which is less than 0.45.

$$\boxed{Yes}$$