

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

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"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2 . We want to minimise the total treatment time while ensuring that she receives a total dosage of 1 .

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided $\{0.4, 0.6\}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?

It is revealed that $(0.5, 0.5)$ is a minimiser if $a_2=1$ and that $(0.6, 0.4)$ is at most 0.45 away from a minimiser (in the infinity norm) of the problem above.

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 1296 seconds >

Step 1. Understand the Minimization Problem

Jen receives two therapies with dosage rates:

- $a_1 = 1$ (per second)
- a_2 (per second)

Let the treatment times be x_1 and x_2 . The total dosage is:

$$a_1 x_1 + a_2 x_2 = x_1 + a_2 x_2 = 1$$

The total treatment time is:

$$T = x_1 + x_2$$

Using the constraint $x_1 = 1 - a_2 x_2$, we get:

$$T = 1 - a_2 x_2 + x_2 = 1 + (1 - a_2) x_2$$

- If $a_2 > 1$, then $1 - a_2 < 0$, so T decreases as x_2 increases. The optimal is $x_2 = \frac{1}{a_2}$, $x_1 = 0$.
- If $a_2 < 1$, then $1 - a_2 > 0$, so T increases with x_2 . The optimal is $x_2 = 0$, $x_1 = 1$.
- If $a_2 = 1$, then $T = 1$ for all $x_1 + x_2 = 1$, so all such points are minimizers.

Step 2. What Is a_2 According to the Code?

The code defines:

$$\text{Compute_a_2}(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

$$\text{Compute_f}(n) = 72^{-10^{24}} + \sum_{m=1}^n \left[\left(\frac{1}{2} \right)^{m+2} - 72^{-10^{24} \cdot (m+2)} \right]$$

Let $\delta = 72^{-10^{24}}$, an extremely small positive number. Then:

$$f(n) = \sum_{j=3}^{n+2} \left(\frac{1}{2} \right)^j + \left[\delta - \sum_{j=3}^{n+2} \delta^j \right]$$

$$a_2(n) = 1 - \left(\frac{1}{2} \right)^{n+1} + \left(\frac{1}{4} \right)^n + 2\delta - 2 \sum_{j=3}^{n+2} \delta^j$$

As $n \rightarrow \infty$, the terms $\left(\frac{1}{2} \right)^{n+1}$ and $\left(\frac{1}{4} \right)^n$ vanish, and the sum becomes a geometric series:

$$\sum_{j=3}^{\infty} \delta^j = \frac{\delta^3}{1 - \delta}$$

Thus,

$$a_2 = \lim_{n \rightarrow \infty} a_2(n) = 1 + 2\delta - \frac{2\delta^3}{1 - \delta}$$

Since $\delta > 0$ is very small, $a_2 > 1$.

Step 3. Implications of the Revealed Information

We are told:

- If $a_2 = 1$, then $(0.5, 0.5)$ is a minimizer.
- For the actual a_2 , the point $(0.6, 0.4)$ is within 0.45 (in the infinity norm) of a minimizer.

Since $a_2 > 1$, the unique minimizer is:

$$(x_1, x_2) = \left(0, \frac{1}{a_2}\right)$$

Because $a_2 \approx 1 + 2\delta$, we have:

$$\frac{1}{a_2} \approx 1 - 2\delta \approx 1$$

So the minimizer is very close to $(0, 1)$.

Now consider the point $(0.6, 0.4)$:

- $|0.6 - 0| = 0.6$
- $|0.4 - 1| = 0.6$

So the infinity norm distance is 0.6, which is **not** within 0.45.

But the revealed information says that $(0.6, 0.4)$ is within 0.45 of a minimizer. This implies that the actual a_2 must be ≤ 1 , so that the minimizer is $(1, 0)$, and:

- $|0.6 - 1| = 0.4$
- $|0.4 - 0| = 0.4$

So the distance is 0.4, which is within 0.45.

Thus, the revealed information forces $a_2 \leq 1$.

Step 4. Is $(0.4, 0.6)$ Within 0.45 of a Minimizer?

If $a_2 \leq 1$, the minimizer is $(1, 0)$. Then:

- $|0.4 - 1| = 0.6$
- $|0.6 - 0| = 0.6$

The infinity norm distance is 0.6, which is **greater than** 0.45.

Step 5. Conclusion

The point $(0.4, 0.6)$ is **not** within 0.45 of a minimizer.

<i>No</i>
