

The following problem involves the number  $a_2$ , which is given by a computer code that represents  $a_2$  as a computable number.

The code for  $a_2$  is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates  $a_1 = 1$  and  $a_2$  per second, respectively. The treatment times we call  $x_1$  and  $x_2$ . We want to minimise the total treatment time while ensuring that she receives a total dosage of  $1$ .

Give values for  $x_1$  and  $x_2$  that approximate an optimal minimiser of the problem, to at least 2 digits accuracy in the  $\ell^\infty$  norm, if  $a_2$  is given as in the codes above. Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 509 seconds >

## Step 1. Understand the Definition of $a_2$

The number  $a_2$  is defined by an algorithm. For a natural number  $n$ , the approximation is:

$$a_2(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

where  $f(n)$  is computed as:

- sum = 72<sup>-1000000000000000000000000000</sup>

- Let's simplify this.

Thus,

$$a_2(n) = 1 - \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{4}\right)^n + 2A - 2 \sum_{j=3}^{n+2} B(j)$$

As  $n \rightarrow \infty$ , the terms  $\left(\frac{1}{2}\right)^{n+1}$  and  $\left(\frac{1}{4}\right)^n$  vanish, so:

$$a_2 = \lim_{n \rightarrow \infty} a_2(n) = 1 + 2A - 2 \sum_{j=3}^{\infty} B(j)$$

Let:

- $z = 72^{-1000000000000000000000000}$
- Then  $A = z^{10}$ , and  $B(j) = z^j$

So,

$$a_2 = 1 + 2z^{10} - 2(z^3 + z^4 + z^5 + \dots) = 1 + 2z^{10} - 2 \cdot \frac{z^3}{1-z}$$

Since  $z$  is extremely small,  $z^{10} \ll z^3$ , and  $\frac{1}{1-z} \approx 1$ , we get:

$$a_2 \approx 1 - 2z^3$$

Hence,  $a_2 < 1$ .

### Step 3. Set Up the Optimization Problem

We have:

- Dosage rates:  $a_1 = 1, a_2 < 1$
- Treatment times:  $x_1, x_2$
- Total dosage:

$$a_1 x_1 + a_2 x_2 = 1$$

- Total time:

$$T = x_1 + x_2$$

We want to minimize  $T$  subject to the dosage constraint and  $x_1, x_2 \geq 0$ .

From the constraint:

$$x_1 = 1 - a_2 x_2$$

Then:

$$T = 1 - a_2 x_2 + x_2 = 1 + (1 - a_2)x_2$$

Since  $a_2 < 1$ , we have  $1 - a_2 > 0$ , so  $T$  increases with  $x_2$ .

To minimize  $T$ , take  $x_2 = 0$ , which gives:

$$x_1 = 1, \quad T = 1$$

## Step 4. Final Answer

The optimal treatment times are:

$$x_1 = 1.00, \quad x_2 = 0.00$$

$$\boxed{1} \quad \text{and} \quad \boxed{0}$$