

The following problem involves the number  $a_2$ , which is given by a computer code that represents  $a_2$  as a computable number.

The code for  $a_2$  is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates  $a_1 = 1$  and  $a_2$  per second, respectively. The treatment times we call  $x_1$  and  $x_2$ . We want to minimise the total treatment time while ensuring that she receives a total dosage of  $1$ .

When asked to give values for  $x_1$  and  $x_2$  that approximate an optimal minimiser of the problem someone provided  $\{0.4, 0.6\}$  as a solution. Is this at most  $0.45$  away from a minimiser (in the infinity norm) of the problem, if  $a_2$  is given as in the codes above?

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 394 seconds >

## Step 1. Understand the Definition of $a_2$

The code defines:

- **Compute\_f(n):**
  - Initializes:

$$\text{sum} = 72^{-10^{24}}$$

- For  $m = 1$  to  $n$ , updates:

$$k = k + 1, \quad \text{sum} = \text{sum} + \left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k}$$

- Outputs:

$$f(n) = 72^{-10^{24}} + \sum_{j=3}^{n+2} \left[ \left(\frac{1}{2}\right)^j - 72^{-10^{24} \cdot j} \right]$$

- **Compute\_a\_2(n):**

$$a_2(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

As  $n \rightarrow \infty$ , the geometric series converges:

$$\sum_{j=3}^{\infty} \left(\frac{1}{2}\right)^j = \frac{1}{4}$$

The terms involving powers of 72 are extremely small. Let:

$$A = 72^{10^{24}} \Rightarrow 72^{-10^{24}} = \frac{1}{A}$$

Then:

$$f(n) \rightarrow \frac{1}{4} + (\text{tiny positive number})$$

$$a_2(n) \rightarrow 2 \cdot \left(\frac{1}{4} + \text{tiny}\right) + \frac{1}{2} = 1 + (\text{tiny positive number})$$

Thus,

$$a_2 = \lim_{n \rightarrow \infty} a_2(n) = 1 + \varepsilon, \quad \text{where } \varepsilon > 0 \text{ is extremely small}$$

## Step 2. Formulate the Optimization Problem

Jen receives two therapies:

- Therapy 1: rate  $a_1 = 1$  per second, time  $x_1$
- Therapy 2: rate  $a_2 = 1 + \varepsilon$ , time  $x_2$

Total dosage:

$$a_1x_1 + a_2x_2 = x_1 + (1 + \varepsilon)x_2 = 1$$

Total time:

$$T = x_1 + x_2$$

We want to minimize  $T$  subject to  $x_1, x_2 \geq 0$ .

### Step 3. Find the Optimal Treatment Times

From the constraint:

$$x_1 = 1 - (1 + \varepsilon)x_2$$

Then:

$$T = 1 - (1 + \varepsilon)x_2 + x_2 = 1 - \varepsilon x_2$$

Since  $\varepsilon > 0$ ,  $T$  is minimized when  $x_2$  is maximized.

The maximum value of  $x_2$  occurs when  $x_1 = 0$ :

$$(1 + \varepsilon)x_2 = 1 \quad \Rightarrow \quad x_2 = \frac{1}{1 + \varepsilon}$$

Then:

$$T_{\min} = 1 - \varepsilon \cdot \frac{1}{1 + \varepsilon} = \frac{1}{1 + \varepsilon}$$

So the unique minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{1 + \varepsilon}\right)$$

Since  $\varepsilon$  is extremely small,  $x_2^* \approx 1$ .

### Step 4. Compare with the Proposed Solution

Proposed solution:

$$(x_1, x_2) = (0.4, 0.6)$$

True minimizer:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{1+\varepsilon}\right) \approx (0, 1)$$

Compute the infinity norm distance:

$$d = \max \left\{ |0.4 - 0|, \left| 0.6 - \frac{1}{1+\varepsilon} \right| \right\}$$

Since  $\frac{1}{1+\varepsilon} \approx 1$ , we have:

$$|0.6 - 1| \approx 0.4$$

So:

$$d \approx \max\{0.4, 0.4\} = 0.4$$

Even accounting for the tiny  $\varepsilon$ , the distance remains very close to 0.4.

## Step 5. Conclusion

Since  $0.4 \leq 0.45$ , the proposed solution is within 0.45 (in the infinity norm) of the true minimizer.

Yes