

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2 . We want to minimise the total treatment time while ensuring that she receives a total dosage of 1.

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided $\{0.4, 0.6\}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 508 seconds >

Step 1. Understand the Definition of a_2

The code defines:

- **Compute_a_2(n):**

$$\text{Output} = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

- **Compute_f(n):**

$$\begin{aligned} \text{sum} &= 72^{-10^{24}} \\ k &= 2 \\ \text{For } m &= 1 \text{ to } n : \\ k &= k + 1 \\ \text{sum} &= \text{sum} + \left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k} \\ \text{Output} &= \text{sum} \end{aligned}$$

Let's analyze $f(n)$:

- The initial term is $72^{-10^{24}}$.
- For each $m = 1, 2, \dots, n$, we add:

$$\left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k}$$

where $k = 3, 4, \dots, n + 2$.

So,

$$f(n) = 72^{-10^{24}} + \sum_{k=3}^{n+2} \left[\left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k} \right]$$

As $n \rightarrow \infty$, we get:

$$f(\infty) = \sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k + \left[72^{-10^{24}} - \sum_{k=3}^{\infty} 72^{-10^{24} \cdot k} \right]$$

The geometric series:

$$\sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k = \frac{(1/2)^3}{1 - 1/2} = \frac{1/8}{1/2} = \frac{1}{4}$$

Let:

$$\delta = 72^{-10^{24}} - \sum_{k=3}^{\infty} 72^{-10^{24} \cdot k}$$

Since $72^{-10^{24}}$ is extremely small and the subtracted terms are even smaller, $\delta > 0$. Thus,

$$f(\infty) = \frac{1}{4} + \delta$$

Now compute:

$$a_2 = \lim_{n \rightarrow \infty} \text{Compute_a_2}(n) = 2 \cdot f(\infty) + \frac{1}{2} = 2 \left(\frac{1}{4} + \delta \right) + \frac{1}{2} = 1 + 2\delta$$

So,

$$a_2 = 1 + \varepsilon \quad \text{with } \varepsilon = 2\delta > 0$$

Step 2. The Optimization Problem

We want to minimize total treatment time:

$$T = x_1 + x_2$$

subject to the dosage constraint:

$$1 \cdot x_1 + a_2 \cdot x_2 = 1, \quad x_1, x_2 \geq 0$$

Since $a_2 > 1$, it's more efficient to use therapy 2. Expressing x_1 in terms of x_2 :

$$x_1 = 1 - a_2 x_2 \Rightarrow T = 1 - a_2 x_2 + x_2 = 1 + (1 - a_2)x_2$$

Since $a_2 > 1$, the coefficient $(1 - a_2) < 0$, so T decreases as x_2 increases. The maximum possible x_2 is:

$$x_2 = \frac{1}{a_2} \quad (\text{so that } x_1 = 0)$$

Thus, the optimal solution is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2} \right)$$

Since $a_2 = 1 + \varepsilon$, we have:

$$\frac{1}{a_2} = \frac{1}{1 + \varepsilon} \approx 1 - \varepsilon$$

Step 3. Compare with the Given Approximation

The provided approximation is:

$$(x_1, x_2) = (0.4, 0.6)$$

The optimal solution is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right) \approx (0, 1 - \varepsilon)$$

Compute the infinity norm distance:

$$\|(0.4, 0.6) - (0, 1 - \varepsilon)\|_\infty = \max(|0.4 - 0|, |0.6 - (1 - \varepsilon)|)$$

Since $\varepsilon > 0$ is extremely small:

$$|0.6 - (1 - \varepsilon)| = |0.6 - 1 + \varepsilon| = 0.4 - \varepsilon$$

So,

$$\|(0.4, 0.6) - (0, 1 - \varepsilon)\|_\infty = \max(0.4, 0.4 - \varepsilon) = 0.4$$

Since $0.4 \leq 0.45$, the approximation is within 0.45 of the optimal solution in the infinity norm.

Final Answer

Yes
