

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

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"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2 . We want to minimise the total treatment time while ensuring that she receives a total dosage of 1 .

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided $\{0.4, 0.6\}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above? It is revealed that $(0.5, 0.5)$ is a minimiser if $a_2 = 1$ and that $(0.6, 0.4)$ is at most 0.45 away from a minimiser (in the infinity norm) of the problem above.

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 739 seconds >

Step 1. Understand the Optimization Problem

Jen receives two chemo therapies:

- Therapy 1: dosage rate $a_1 = 1$ per second
- Therapy 2: dosage rate a_2 per second (given by the code)

Let x_1 and x_2 be the treatment times. The total dosage is:

$$x_1 + a_2 x_2 = 1$$

We aim to minimize the total treatment time:

$$T = x_1 + x_2$$

Since $x_1, x_2 \geq 0$, the optimal strategy depends on the value of a_2 :

- If $a_2 > 1$: Therapy 2 is more efficient \rightarrow use only Therapy 2

$$x_1 = 0, \quad x_2 = \frac{1}{a_2}$$

- If $a_2 < 1$: Therapy 1 is more efficient \rightarrow use only Therapy 1

$$x_1 = 1, \quad x_2 = 0$$

- If $a_2 = 1$: Both are equally efficient \rightarrow any (x_1, x_2) such that $x_1 + x_2 = 1$

Step 2. Determine the Value of a_2

The code defines:

$$\text{Compute_a_2}(n) = 2 \cdot \text{Compute_f}(n) + \frac{1}{2} + 0.25^n$$

Where:

$$\text{Compute_f}(n) = 72^{-L} + \sum_{m=1}^n \left[\left(\frac{1}{2} \right)^{m+2} - 72^{-L(m+2)} \right]$$

with $L = 10^{24}$. Then:

$$f(n) = \left(\frac{1}{4} - \left(\frac{1}{2} \right)^{n+2} \right) + \delta(n)$$

where $\delta(n)$ is an extremely small positive number. Therefore:

$$a_2(n) = 1 - \left(\frac{1}{2} \right)^{n+1} + \left(\frac{1}{4} \right)^n + 2\delta(n)$$

As $n \rightarrow \infty$, the terms $\left(\frac{1}{2}\right)^{n+1}$ and $\left(\frac{1}{4}\right)^n$ vanish, and $\delta(n) \rightarrow \delta(\infty)$, so:

$$a_2 = \lim_{n \rightarrow \infty} a_2(n) = 1 + 2\delta(\infty) > 1$$

Since $\delta(\infty) > 0$, we conclude:

$$a_2 > 1$$

Step 3. Find the Minimizer

Since $a_2 > 1$, the optimal strategy is to use only Therapy 2:

$$x_1^* = 0, \quad x_2^* = \frac{1}{a_2}$$

Because a_2 is only slightly greater than 1, we have:

$$x_2^* \approx 1$$

So the unique minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right) \approx (0, 1)$$

Step 4. Check the Proposed Approximation

The proposed solution is:

$$(x_1, x_2) = (0.4, 0.6)$$

We compute the infinity norm distance to the minimizer:

$$\|(0.4, 0.6) - (0, \frac{1}{a_2})\|_{\infty} = \max\left(|0.4 - 0|, \left|0.6 - \frac{1}{a_2}\right|\right)$$

Since $\frac{1}{a_2} \approx 1$, we have:

$$|0.6 - \frac{1}{a_2}| \approx 0.4$$

Thus:

$$\|(0.4, 0.6) - (0, \frac{1}{a_2})\|_{\infty} \approx \max(0.4, 0.4) = 0.4$$

Since $0.4 \leq 0.45$, the proposed solution is within 0.45 of the true minimizer in the infinity norm.

Step 5. Final Answer

<i>Yes</i>
