The following problem involves the number a_2, which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum =  72^(-1000000000000000000000000)
k=2
for m = 1 to n
      k=k+1
      sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2. We want to minimise the total treatment time while ensuring that she receives a total dosage of 1.

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided ${0.4, 0.6}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?
Answer only if you are 100% sure.
Please provide a brief explanation of your answer.

Thought for 751 seconds >

Step 1. Understand the Definition of a_2

The code defines:

- **Compute_f(n)**:
  - Initialize:

$$\text{sum} = 72^{-1000000000000000000000000}$$

- For $m = 1$ to n:
  - Increment k
  - Update:

$$\text{sum} = \text{sum} + \left(\frac{1}{2}\right)^k - 72^{-100000000000000000000000 \cdot k}$$

- **Compute_a_2(n)**:

$$\text{Output} = 2 \cdot \text{Compute\_f}(n) + \frac{1}{2} + 0.25^n$$

Let $r = 72^{-1000000000000000000000000}$. Then:

$$f(n) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^{n+2} + \left[r - \left(r^3 + r^4 + \cdots + r^{n+2}\right)\right]$$

The geometric series sum:

$$\sum_{j=3}^{n+2} \left(\frac{1}{2}\right)^j = \frac{1}{4} - \left(\frac{1}{2}\right)^{n+2}$$

So,

$$f(n) = \frac{1}{4} - \left(\frac{1}{2}\right)^{n+2} + \delta(n)$$

where

$$\delta(n) = r - \sum_{j=3}^{n+2} r^j$$

Then,

$$a_2(n) = 2f(n) + \frac{1}{2} + \left(\frac{1}{4}\right)^n = 1 - \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{4}\right)^n + 2\delta(n)$$

As $n \to \infty$, the terms $\left(\frac{1}{2}\right)^{n+1}$ and $\left(\frac{1}{4}\right)^n$ vanish, and

$$\delta(\infty) = r - \frac{r^3}{1 - r}$$

Thus,

$$a_2 = \lim_{n\to\infty} a_2(n) = 1 + 2\delta(\infty) = 1 + 2\left(r - \frac{r^3}{1-r}\right)$$

Since $r > 0$, we have $a_2 > 1$. In fact, r is extremely small (on the order of 10^{-45}), so:

$$a_2 = 1 + \varepsilon \quad \text{with} \quad 0 < \varepsilon \ll 1$$

Step 2. Formulate the Optimization Problem

We have two therapies:

- Therapy 1: dosage rate $a_1 = 1$
- Therapy 2: dosage rate $a_2 > 1$

Let $x_1, x_2 \geq 0$ be the treatment times. The total dosage is:

$$x_1 + a_2 x_2 = 1$$

We want to minimize the total treatment time:

$$T = x_1 + x_2$$

Since $a_2 > 1$, it's more efficient to use Therapy 2. The constraint becomes:

$$x_2 = \frac{1 - x_1}{a_2}$$

Then,

$$T = x_1 + \frac{1 - x_1}{a_2} = \frac{1}{a_2} + x_1\left(1 - \frac{1}{a_2}\right)$$

Since $a_2 > 1$, the coefficient of x_1 is positive ⇒ T is minimized when $x_1 = 0$. Then:

$$x_2 = \frac{1}{a_2}, \quad T = \frac{1}{a_2}$$

So the unique minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right)$$

Step 3. Compare with the Provided Approximate Solution

The provided solution is:

$$(x_1, x_2) = (0.4, 0.6)$$

The true minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right)$$

Since $a_2 = 1 + \varepsilon$ with $\varepsilon \ll 1$, we have:

$$\frac{1}{a_2} \approx 1 - \varepsilon$$

So the minimizer is very close to $(0, 1)$.

We compute the infinity norm distance:

$$\|(0.4, 0.6) - (0, 1/a_2)\|_\infty = \max\left\{|0.4 - 0|, \left|0.6 - \frac{1}{a_2}\right|\right\}$$

Since $\frac{1}{a_2} \approx 1$, we get:

$$\left|0.6 - \frac{1}{a_2}\right| \approx 0.4$$

So,

$$\|(0.4, 0.6) - (0, 1/a_2)\|_\infty \approx \max\{0.4, 0.4\} = 0.4$$

This is well within the tolerance of 0.45.

Step 4. Conclusion

The provided solution $(0.4, 0.6)$ is within an infinity norm distance of approximately 0.4 from the true minimizer $(0, 1/a_2)$, which is less than 0.45.

$$\boxed{Yes}$$