

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2 . We want to minimise the total treatment time while ensuring that she receives a total dosage of 1 .

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided $\{0.4, 0.6\}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 400 seconds >

Step 1. Understand the Problem

Jen undergoes two chemo therapies with dosage rates:

- $a_1 = 1$ per second
- a_2 per second (given by the provided code)

Let x_1 and x_2 be the treatment times. The total dosage is:

$$a_1 x_1 + a_2 x_2 = x_1 + a_2 x_2 = 1$$

We aim to minimize the total treatment time:

$$T = x_1 + x_2$$

subject to $x_1, x_2 \geq 0$.

Step 2. Determine the True Value of a_2

The code defines:

- **Compute_a_2(n):**

$$\text{Output} = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

- **Compute_f(n):**

$$\text{sum} = 72^{-10^{24}} + \sum_{m=1}^n \left[\left(\frac{1}{2} \right)^{m+2} - 72^{-10^{24} \cdot (m+2)} \right]$$

Let's analyze $f(n)$:

$$f(n) = \sum_{j=3}^{n+2} \left(\frac{1}{2} \right)^j + 72^{-10^{24}} - \sum_{j=3}^{n+2} 72^{-10^{24} \cdot j}$$

As $n \rightarrow \infty$:

- The geometric series converges:

$$\sum_{j=3}^{\infty} \left(\frac{1}{2} \right)^j = \frac{1}{4}$$

- The second sum is extremely small:

$$\sum_{j=3}^{\infty} 72^{-10^{24} \cdot j} \approx 72^{-3 \cdot 10^{24}}$$

- The constant term $72^{-10^{24}}$ is positive and dominates the negative terms.

Thus,

$$f(\infty) \approx \frac{1}{4} + 72^{-10^{24}}$$

Then,

$$a_2 = \lim_{n \rightarrow \infty} \text{Compute_a_2}(n) = 2 \cdot f(\infty) + \frac{1}{2} \approx 2 \cdot \left(\frac{1}{4} + 72^{-10^{24}} \right) + \frac{1}{2} = 1 + 2 \cdot 7$$

So,

$$a_2 = 1 + \varepsilon \quad \text{with} \quad \varepsilon > 0 \quad \text{and extremely small}$$

Step 3. Find the Minimizer of the Treatment Time

We want to minimize:

$$T = x_1 + x_2 \quad \text{subject to} \quad x_1 + a_2 x_2 = 1, \quad x_1, x_2 \geq 0$$

Substitute $x_1 = 1 - a_2 x_2$:

$$T = 1 - a_2 x_2 + x_2 = 1 + (1 - a_2)x_2$$

Since $a_2 > 1$, we have $1 - a_2 < 0$. To minimize T , we want to maximize x_2 , subject to $x_1 \geq 0 \Rightarrow x_2 \leq \frac{1}{a_2}$.

So the optimal solution is:

$$x_1 = 0, \quad x_2 = \frac{1}{a_2}$$

Then the minimal total time is:

$$T_{\min} = \frac{1}{a_2}$$

Since $a_2 = 1 + \varepsilon$, we have:

$$\frac{1}{a_2} = \frac{1}{1 + \varepsilon} \approx 1 - \varepsilon$$

So the true minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2} \right) \approx (0, 1 - \varepsilon)$$

Step 4. Compare with the Provided Approximation

The provided approximation is:

$$(x_1, x_2) = (0.4, 0.6)$$

The true minimizer is approximately $(0, 1)$. The infinity norm distance is:

$$\|(0.4, 0.6) - (0, 1)\|_{\infty} = \max(|0.4 - 0|, |0.6 - 1|) = \max(0.4, 0.4) = 0.4$$

Since $\varepsilon > 0$ is extremely small, the actual distance is slightly less than 0.4, but still:

$$0.4 \leq 0.45$$

Step 5. Conclusion

The infinity norm distance between the provided approximation $(0.4, 0.6)$ and the true minimizer is approximately 0.4, which is less than 0.45.

Yes