The following problem involves the number a_2, which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum =  72^(-1000000000000000000000000)
k=2
for m = 1 to n
        k=k+1
        sum = sum + (1/2)^(k) -
72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2. We want to minimise the total treatment time while ensuring that she receives a total dosage of 1.

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided ${0.4, 0.6}$ as a solution.
Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?
It is revealed that $(0.5,0.5)$ is a minimiser if $a_2=1$ and that $(0.6, 0.4)$ is at most 0.45 away from a minimiser (in the infinity

norm) of the problem above.
Answer only if you are 100% sure.
Please provide a brief explanation of your answer.

Thought for 524 seconds >

Step 1. Understand the Definition of a_2

The code defines:

- **Compute_f(n)**:
  - Initialize:

$$\text{sum} = 72^{-1000000000000000000000000}$$

  - For $m = 1$ to n:
    - $k = k + 1$
    - Update:

$$\text{sum} = \text{sum} + \left(\frac{1}{2}\right)^k - 72^{-100000000000000000000000 \cdot k}$$

  - Output: `sum`
- **Compute_a_2(n)**:
  - Output:

$$2 \cdot \text{Compute\_f}(n) + \frac{1}{2} + 0.25^n$$

Let's analyze the behavior as $n \to \infty$.

Step 2. Compute the Limit of $f(n)$

The loop in `Compute_f(n)` adds terms for $k = 3, 4, \ldots, n + 2$:

$$f(n) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^{n+2} + \delta(n)$$

where

$$\delta(n) = 72^{-10^{24}} - \left[72^{-3\cdot 10^{24}} + 72^{-4\cdot 10^{24}} + \cdots + 72^{-(n+2)\cdot 10^{24}}\right]$$

The geometric series:

$$\sum_{k=3}^{n+2} \left(\frac{1}{2}\right)^k = \frac{1}{4} - \frac{1}{2^{n+2}}$$

So,

$$f(n) = \frac{1}{4} - \frac{1}{2^{n+2}} + \delta(n)$$

As $n \to \infty$, the tail of the geometric series vanishes:

$$f(\infty) = \frac{1}{4} + \delta(\infty)$$

where

$$\delta(\infty) = 72^{-10^{24}} - \frac{72^{-3\cdot 10^{24}}}{1 - 72^{-10^{24}}}$$

Since $72^{-10^{24}}$ is extremely small, $\delta(\infty) > 0$, but negligibly so.

Step 3. Compute a_2

$$a_2 = \lim_{n\to\infty} \text{Compute\_a\_2}(n) = 2 \cdot f(\infty) + \frac{1}{2}$$

$$= 2\left(\frac{1}{4} + \delta(\infty)\right) + \frac{1}{2} = 1 + 2\delta(\infty)$$

Thus,

$$a_2 = 1 + \varepsilon, \quad \text{where } \varepsilon = 2\delta(\infty) > 0 \text{ is extremely small}$$

Step 4. The Optimization Problem

We have:

- Dosage rates: $a_1 = 1$, $a_2 = 1 + \varepsilon$
- Treatment times: $x_1, x_2 \geq 0$
- Total dosage constraint:

$$x_1 + a_2 x_2 = 1$$

- Objective: minimize total time $T = x_1 + x_2$

Substitute $x_1 = 1 - a_2 x_2$:

$$T = 1 - a_2 x_2 + x_2 = 1 - (a_2 - 1)x_2$$

Since $a_2 > 1$, T decreases as x_2 increases. The maximum possible x_2 is when $x_1 = 0$:

$$x_2 = \frac{1}{a_2} = \frac{1}{1+\varepsilon} \approx 1 - \varepsilon$$

So the unique minimizer is:

$$(x_1^*, x_2^*) = (0, 1/a_2) \approx (0, 1)$$

Step 5. Check the Given Approximation

The proposed solution is $(0.4, 0.6)$. The true minimizer is $(0, 1/a_2) \approx (0, 1)$.

Compute the infinity norm distance:

$$\|(0.4, 0.6) - (0, 1)\|_\infty = \max(|0.4 - 0|, |0.6 - 1|) = \max(0.4, 0.4) = 0.4$$

Since $0.4 \leq 0.45$, the approximation is within the required tolerance.

Step 6. Final Answer

$$\boxed{Yes}$$