

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for `a_2` is:

```
"Compute_a_2(n)
```

Input : natural number n

$$\text{Output} = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

end

Compute_f(n)

Input : natural number n

```
sum = 72^(-1000000000000000000000000000)
```

 $k=2$

for $m = 1$ to n

 $k = k + 1$

```
sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
```

end

Output = sum

end"

My friend Jen will undergo two chemo therapies after one another, with dosage rates $\$a_1 = 1\$$ and $\$a_2\$$ per second, respectively. The treatment times we call $\$x_1\$$ and $\$x_2\$$. We want to minimise the total treatment time while ensuring that she receives a total dosage of $\$1\$$.

Give values for x_1 and x_2 that approximate an optimal minimiser of the problem, to at least 2 digits accuracy in the ℓ^∞ norm, if a_2 is given as in the codes above. Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 504 seconds >

Step 1. Understand the Definition of a_2

The number a_2 is defined via a computable algorithm. The code is:

text

Compute_a_2(n)

Input: natural number n

$$\text{Output} = 2 * f(n) + 1/2 + 0.25^n$$

end

Compute_f(n)

Input: natural number n

sum = $72^{-(1000000000000000000000000)}$

k = 2

for m = 1 to n

 k = k + 1

 sum = sum + $(1/2)^k - 72^{-(1000000000000000000000000*k)}$

end

Output = sum

end

Let's analyze this step by step.

Step 2. Analyze $f(n)$

The function $f(n)$ computes:

$$f(n) = 72^{-10^{24}} + \sum_{m=1}^n \left[\left(\frac{1}{2} \right)^k - 72^{-10^{24} \cdot k} \right]$$

where $k = m + 2$, so the sum runs over $k = 3, 4, \dots, n + 2$.

The geometric sum:

$$\sum_{k=3}^{n+2} \left(\frac{1}{2} \right)^k = \frac{1}{8} \cdot \frac{1 - (1/2)^{n+2}}{1 - 1/2} = \frac{1}{4} \left(1 - \left(\frac{1}{2} \right)^{n+2} \right)$$

The tiny terms:

$$T(n) = 72^{-10^{24}} - \left[72^{-3 \cdot 10^{24}} + 72^{-4 \cdot 10^{24}} + \dots + 72^{-(n+2) \cdot 10^{24}} \right]$$

As $n \rightarrow \infty$, the sum of the geometric series of tiny terms is negligible compared to $72^{-10^{24}}$, so:

$$T(\infty) \approx 72^{-10^{24}}$$

Thus,

$$f(\infty) = \frac{1}{4} + T(\infty) > \frac{1}{4}$$

Step 3. Compute a_2

$$a_2 = \lim_{n \rightarrow \infty} \text{Compute_a_2}(n) = 2 \cdot f(\infty) + \frac{1}{2}$$

Since $f(\infty) > \frac{1}{4}$, we have:

$$a_2 > 2 \cdot \frac{1}{4} + \frac{1}{2} = 1$$

Let $\delta = 2 \cdot T(\infty)$, then:

$$a_2 = 1 + \delta, \quad \text{where } \delta > 0$$

The value of δ is extremely small (on the order of $10^{-10^{24}}$), so for all practical purposes:

$$a_2 \approx 1$$

Step 4. Set Up the Optimization Problem

We have:

- Dosage rates: $a_1 = 1, a_2 = 1 + \delta$
- Treatment times: x_1, x_2
- Total dosage: $x_1 + a_2 x_2 = 1$
- Total time: $T = x_1 + x_2$

We want to minimize T subject to:

$$x_1 + (1 + \delta)x_2 = 1, \quad x_1, x_2 \geq 0$$

This is a linear programming problem. The constraint defines a line segment from:

- $(x_1, x_2) = (1, 0)$ with $T = 1$
- $(x_1, x_2) = (0, \frac{1}{1+\delta})$ with $T = \frac{1}{1+\delta} < 1$

Since $\delta > 0$, the minimum occurs at:

$$x_1^* = 0, \quad x_2^* = \frac{1}{1 + \delta}$$

Step 5. Approximate the Optimal Solution

We are asked to approximate the optimal solution to at least 2 digits accuracy in the ℓ^∞ norm.

Since δ is extremely small:

$$x_2^* = \frac{1}{1 + \delta} \approx 1 - \delta \approx 1$$

The error:

$$|x_2 - x_2^*| = \left| 1 - \frac{1}{1 + \delta} \right| = \frac{\delta}{1 + \delta} < \delta$$

Since $\delta \ll 0.01$, the approximation is valid.

Step 6. Final Answer

The optimal treatment times are:

$$x_1 = 0, \quad x_2 = 1$$

$$\boxed{0} \quad \text{and} \quad \boxed{1}$$