

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2 . We want to minimise the total treatment time while ensuring that she receives a total dosage of 1 .

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided $\{0.4, 0.6\}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above? It is revealed that $(0.5, 0.5)$ is a minimiser if $a_2=1$ and that $(0.6, 0.4)$ is at most 0.45 away from a minimiser (in the infinity norm) of the problem above.

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 358 seconds >

Step 1: Understand the Problem

Jen undergoes two chemo therapies with dosage rates:

- $a_1 = 1$
- a_2 (as defined by the given code)

Let x_1 and x_2 be the treatment times. The total dosage must be exactly 1:

$$x_1 + a_2 x_2 = 1$$

We aim to minimize the total treatment time:

$$T = x_1 + x_2$$

Subject to $x_1, x_2 \geq 0$.



Step 2: Find the Optimal Treatment Times

From the constraint:

$$x_1 = 1 - a_2 x_2$$

Substitute into the objective:

$$T = (1 - a_2 x_2) + x_2 = 1 + (1 - a_2)x_2$$

- If $a_2 > 1$, then $(1 - a_2) < 0$, so T decreases as x_2 increases.
- The maximum allowed x_2 is $\frac{1}{a_2}$ (since $x_1 \geq 0$).

Thus, the optimal solution is:

$$x_1 = 0, \quad x_2 = \frac{1}{a_2}$$



Step 3: Determine a_2 from the Code

The code defines:

$$\text{Compute_a_2}(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

Where $f(n)$ is computed as:

- Start with `sum` = $72^{-10^{24}}$
- For $m = 1$ to n , add:

$$\left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k}$$

with $k = m + 2$

As $n \rightarrow \infty$, the geometric series converges:

$$\sum_{j=3}^{\infty} \left(\frac{1}{2}\right)^j = \frac{1}{4}$$

Let $\varepsilon = 72^{-10^{24}}$. Then:

$$f(\infty) = \frac{1}{4} + \varepsilon - \frac{\varepsilon^3}{1 - \varepsilon}$$

So:

$$a_2 = 2 \cdot f(\infty) + \frac{1}{2} = 1 + 2\delta$$

Where $\delta = \varepsilon - \frac{\varepsilon^3}{1 - \varepsilon} > 0$, but extremely small.

Thus, $a_2 > 1$, but very close to 1.

Step 4: Locate the Minimizer

Since $a_2 > 1$, the unique minimizer is:

$$(x_1, x_2) = \left(0, \frac{1}{a_2}\right)$$

Since $a_2 \approx 1$, we have:

$$\frac{1}{a_2} \approx 1$$

So the minimizer is very close to $(0, 1)$.

Step 5: Check the Given Approximation

Someone suggested $(0.4, 0.6)$ as an approximate minimizer.

We compute the infinity-norm distance to the true minimizer $(0, \frac{1}{a_2})$:

$$d = \max \left(|0.4 - 0|, \left| 0.6 - \frac{1}{a_2} \right| \right)$$

Since $\frac{1}{a_2} \approx 1$, we have:

$$\left| 0.6 - \frac{1}{a_2} \right| \approx 0.4$$

So:

$$d \approx \max(0.4, 0.4) = 0.4$$

Since $0.4 \leq 0.45$, the approximation is within the allowed tolerance.



Final Answer

Yes