

The following problem involves the number  $a_2$ , which is given by a computer code that represents  $a_2$  as a computable number.

The code for  $a_2$  is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates  $a_1 = 1$  and  $a_2$  per second, respectively. The treatment times we call  $x_1$  and  $x_2$ . We want to minimise the total treatment time while ensuring that she receives a total dosage of  $1$ .

When asked to give values for  $x_1$  and  $x_2$  that approximate an optimal minimiser of the problem someone provided  $\{0.4, 0.6\}$  as a solution. Is this at most  $0.45$  away from a minimiser (in the infinity norm) of the problem, if  $a_2$  is given as in the codes above?

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 272 seconds >

### **Step 1: Determine the Value of $a_2$**

The code defines:

$$\text{Compute\_a\_2}(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

and

$$\text{Compute\_f}(n) = 72^{-10^{24}} + \sum_{m=1}^n \left[ \left( \frac{1}{2} \right)^k - 72^{-10^{24} \cdot k} \right]$$

where  $k$  starts at 2 and increases by 1 each iteration.

As  $n \rightarrow \infty$ , the geometric series converges:

$$\sum_{j=3}^{\infty} \left( \frac{1}{2} \right)^j = \frac{(1/2)^3}{1 - 1/2} = \frac{1}{8} \cdot 2 = \frac{1}{4}$$

The remaining terms involving powers of 72 are extremely small. Let:

$$\delta = 72^{-10^{24}} - \sum_{j=3}^{\infty} 72^{-10^{24} \cdot j}$$

Then:

$$f(\infty) = \frac{1}{4} + \delta$$

So:

$$a_2 = 2 \cdot \left( \frac{1}{4} + \delta \right) + \frac{1}{2} = 1 + 2\delta$$

Since  $\delta > 0$ , we have:

$$a_2 > 1$$



## Step 2: Formulate the Optimization Problem

We have two therapies with dosage rates:

- $a_1 = 1$
- $a_2 = 1 + 2\delta$

Let  $x_1, x_2 \geq 0$  be the treatment times. The total dosage is:

$$x_1 + a_2 x_2 = 1$$

We want to minimize the total time:

$$T = x_1 + x_2$$

Since  $a_2 > 1$ , it's more efficient to use therapy 2. Set  $x_1 = 0$ , then:

$$a_2 x_2 = 1 \quad \Rightarrow \quad x_2 = \frac{1}{a_2}$$

So the optimal solution is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right)$$

### Step 3: Compute the Distance to the Given Approximation

The approximation provided is:

$$(0.4, 0.6)$$

The true minimizer is:

$$\left(0, \frac{1}{a_2}\right) = \left(0, \frac{1}{1+2\delta}\right)$$

Using the approximation  $\frac{1}{1+2\delta} \approx 1 - 2\delta$ , we get:

$$x_2^* \approx 1 - 2\delta$$

Now compute the infinity norm distance:

- $|0.4 - 0| = 0.4$
- $|0.6 - (1 - 2\delta)| = |0.6 - 1 + 2\delta| = 0.4 - 2\delta$

So:

$$\|(0.4, 0.6) - (0, \frac{1}{a_2})\|_\infty = \max(0.4, 0.4 - 2\delta) = 0.4$$

Since  $\delta$  is extremely small, this distance is **less than** 0.45.

 **Final Answer**

Yes