

The following problem involves the number  $a_2$ , which is given by a computer code that represents  $a_2$  as a computable number.

The code for `$a_2$` is:

```
"Compute_a_2(n)
Input : natural number n
Output =  $2 \cdot f(n) + \frac{1}{2} + 0.25^n$ 
end

Compute_f(n)
Input : natural number n
sum =  $72^{(-1000000000000000000000000)}$ 
k=2
for m = 1 to n
    k=k+1
    sum = sum +  $(\frac{1}{2})^k - 72^{(-1000000000000000000000000 * k)}$ 
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates  $\$a_1 = 1\$$  and  $\$a_2\$$  per second, respectively. The treatment times we call  $\$x_1\$$  and  $\$x_2\$$ . We want to minimise the total treatment time while ensuring that she receives a total dosage of  $\$1\$$ .

When asked to give values for  $x_1$  and  $x_2$  that approximate an optimal minimiser of the problem someone provided  $\{0.4, 0.6\}$  as a solution. Is this at most  $0.45$  away from a minimiser (in the infinity norm) of the problem, if  $a_2$  is given as in the codes above?

It is revealed that  $(0.5, 0.5)$  is a minimiser if  $a_2 = 1$  and that  $(0.6, 0.4)$  is at most  $0.45$  away from a minimiser (in the infinity norm) of the problem above.

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 498 seconds &gt;

### Step 1. Understand the Number $a_2$

The code defines:

$$\text{Compute\_a\_2}(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

where  $f(n)$  is computed as:

- Start with  $\text{sum} = 72^{-10^{24}}$
- For  $m = 1$  to  $n$ :
  - Increment  $k$  (starting at 2)
  - Add  $\left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k}$

So,

$$f(n) = 72^{-10^{24}} + \sum_{j=3}^{n+2} \left[ \left(\frac{1}{2}\right)^j - 72^{-10^{24} \cdot j} \right]$$

Then,

$$a_2(n) = 2f(n) + \frac{1}{2} + \left(\frac{1}{4}\right)^n$$

As  $n \rightarrow \infty$ , the terms  $\left(\frac{1}{2}\right)^{n+1}$  and  $\left(\frac{1}{4}\right)^n$  vanish. Let

$$\varepsilon = 72^{-10^{24}}$$

Then,

$$a_2 = \lim_{n \rightarrow \infty} a_2(n) = 1 + 2\varepsilon - 2 \sum_{j=3}^{\infty} \varepsilon^j$$

Since  $\varepsilon$  is extremely small (e.g.,  $\varepsilon \approx 10^{-10^{24}}$ ), the series converges:

$$\sum_{j=3}^{\infty} \varepsilon^j = \frac{\varepsilon^3}{1 - \varepsilon}$$

Thus,

$$a_2 = 1 + 2\varepsilon - \frac{2\varepsilon^3}{1 - \varepsilon} > 1$$

So,  $a_2$  is slightly greater than 1.

## Step 2. The Optimization Problem

We want to minimize the total treatment time:

$$T = x_1 + x_2$$

subject to:

$$a_1 x_1 + a_2 x_2 = 1, \quad x_1, x_2 \geq 0$$

Given  $a_1 = 1$ , the constraint becomes:

$$x_1 + a_2 x_2 = 1$$

Since  $a_2 > 1$ , therapy 2 is more efficient. To minimize  $T$ , we should use only therapy 2:

$$x_1 = 0, \quad x_2 = \frac{1}{a_2}$$

Then the minimal total time is:

$$T_{\min} = \frac{1}{a_2}$$

The unique minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right)$$

Since  $a_2 \approx 1$ , we have  $x_2^* \approx 1$ .

### Step 3. Check the Given Approximate Solution

Someone suggested  $(x_1, x_2) = (0.4, 0.6)$ . The actual minimizer is:

$$\left(0, \frac{1}{a_2}\right) \approx (0, 1)$$

The infinity norm distance is:

$$d = \max \left\{ |0.4 - 0|, \left| 0.6 - \frac{1}{a_2} \right| \right\}$$

Since  $\frac{1}{a_2} \approx 1$ , we have:

$$|0.6 - 1| \approx 0.4$$

So,

$$d \approx \max\{0.4, 0.4\} = 0.4$$

Since  $0.4 \leq 0.45$ , the point  $(0.4, 0.6)$  is within 0.45 (in the infinity norm) of the true minimizer.

#### Step 4. Final Answer

<i>Yes</i>
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