

The following problem involves the number a_2 , which is given by a computer code that represents a_2 as a computable number.

The code for a_2 is:

```
"Compute_a_2(n)
Input : natural number n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates $a_1 = 1$ and a_2 per second, respectively. The treatment times we call x_1 and x_2 . We want to minimise the total treatment time while ensuring that she receives a total dosage of 1 .

When asked to give values for x_1 and x_2 that approximate an optimal minimiser of the problem someone provided $\{0.4, 0.6\}$ as a solution. Is this at most 0.45 away from a minimiser (in the infinity norm) of the problem, if a_2 is given as in the codes above?

Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 245 seconds >

Step 1. Understand the Definition of a_2

The code defines:

- **Compute_f(n):**
 - Initializes:

$$\text{sum} = 72^{-10^{24}}$$

- For $m = 1$ to n , with initial $k = 2$, updates:

$$k = k + 1, \quad \text{sum} = \text{sum} + \left(\frac{1}{2}\right)^k - 72^{-10^{24} \cdot k}$$

- So,

$$f(n) = 72^{-10^{24}} + \sum_{m=1}^n \left[\left(\frac{1}{2}\right)^{m+2} - 72^{-10^{24}(m+2)} \right]$$

- **Compute_a_2(n):**

$$a_2(n) = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

As $n \rightarrow \infty$, the geometric series converges:

$$\sum_{j=3}^{\infty} \left(\frac{1}{2}\right)^j = \frac{(1/2)^3}{1 - 1/2} = \frac{1}{8} \cdot 2 = \frac{1}{4}$$

The tiny terms involving powers of 72 vanish. Define:

$$\delta = 72^{-10^{24}} - \frac{72^{-3 \cdot 10^{24}}}{1 - 72^{-10^{24}}}$$

Then:

$$f(\infty) = \frac{1}{4} + \delta \quad \Rightarrow \quad a_2 = 2 \left(\frac{1}{4} + \delta \right) + \frac{1}{2} = 1 + 2\delta$$

Since $\delta > 0$, we have:

$$a_2 > 1$$

Step 2. Formulate the Optimization Problem

We have:

- Dosage rates: $a_1 = 1, a_2 > 1$
- Treatment times: $x_1, x_2 \geq 0$
- Total dosage constraint:

$$x_1 + a_2 x_2 = 1$$

- Objective: Minimize total time

$$T = x_1 + x_2$$

From the constraint:

$$x_1 = 1 - a_2 x_2 \Rightarrow T = 1 - a_2 x_2 + x_2 = 1 + (1 - a_2)x_2$$

Since $a_2 > 1$, the coefficient $(1 - a_2) < 0$, so T decreases as x_2 increases.

The maximum possible x_2 occurs when $x_1 = 0$:

$$x_2 = \frac{1}{a_2}$$

Thus, the unique minimizer is:

$$(x_1^*, x_2^*) = \left(0, \frac{1}{a_2}\right)$$

Step 3. Compute the Distance from the Given Point to the Minimizer

Given point: $(0.4, 0.6)$

Minimizer: $(0, \frac{1}{a_2})$

Infinity norm distance:

$$d = \max \left\{ |0.4 - 0|, \left| 0.6 - \frac{1}{a_2} \right| \right\} = \max \left\{ 0.4, \left| 0.6 - \frac{1}{a_2} \right| \right\}$$

Since $a_2 = 1 + 2\delta$, we have:

$$\frac{1}{a_2} = \frac{1}{1 + 2\delta} \approx 1 - 2\delta$$

So:

$$\left| 0.6 - \frac{1}{a_2} \right| \approx |0.6 - (1 - 2\delta)| = 0.4 - 2\delta$$

Since $\delta > 0$, this value is slightly less than 0.4. Therefore:

$$d = \max\{0.4, 0.4 - 2\delta\} = 0.4$$

Step 4. Answer the Question

The infinity norm distance from $(0.4, 0.6)$ to the minimizer is exactly 0.4, which is less than 0.45.

Yes