

The following problem involves the number  $a_2$ , which is given by a computer code that represents  $a_2$  as a computable number.

The code for  $a_2$  is:

```
"Compute_a_2(n)
Optimal Treatment Time Minimization Problem  or n
Output = 2*f(n)+ 1/2 + 0.25^n
end

Compute_f(n)
Input : natural number n
sum = 72^(-1000000000000000000000000)
k=2
for m = 1 to n
    k=k+1
    sum = sum + (1/2)^(k) - 72^(-1000000000000000000000000*k)
end
Output = sum
end"
```

My friend Jen will undergo two chemo therapies after one another, with dosage rates  $a_1 = 1$  and  $a_2$  per second, respectively. The treatment times we call  $x_1$  and  $x_2$ . We want to minimise the total treatment time while ensuring that she receives a total dosage of  $1$ .

Give values for  $x_1$  and  $x_2$  that approximate an optimal minimiser of the problem, to at least 2 digits accuracy in the  $\ell^\infty$  norm, if  $a_2$  is given as in the codes above. Answer only if you are 100% sure.

Please provide a brief explanation of your answer.

Thought for 395 seconds >

## Step 1. Understand the Code and Compute $a_2$

The code defines:

- **Compute\_a\_2(n):**

$$\text{Output} = 2 \cdot f(n) + \frac{1}{2} + 0.25^n$$

- **Compute\_f(n):**

- Initialize:

$$\text{sum} = 72^{-1000000000000000000000000}$$

- Set  $k = 2$
- For  $m = 1$  to  $n$ :
  - $k = k + 1$

Optimal Treatment Time Minimization Problem

Let:

$$N = 1000000000000000000000000 = 10^{24}$$

Then:

$$f(n) = 72^{-N} + \sum_{j=3}^{n+2} \left[ \left( \frac{1}{2} \right)^j - 72^{-N \cdot j} \right]$$

As  $n \rightarrow \infty$ , the geometric series:

$$\sum_{j=3}^{\infty} \left( \frac{1}{2} \right)^j = \frac{(1/2)^3}{1 - 1/2} = \frac{1}{8} \cdot 2 = \frac{1}{4}$$

Let:

$$\delta(\infty) = 72^{-N} - \sum_{j=3}^{\infty} 72^{-N \cdot j}$$

Then:

$$f(\infty) = \frac{1}{4} + \delta(\infty)$$

$$a_2 = 2f(\infty) + \frac{1}{2} = 2 \left( \frac{1}{4} + \delta(\infty) \right) + \frac{1}{2} = 1 + 2\delta(\infty)$$

Since  $\delta(\infty) > 0$ , we conclude:

$$a_2 > 1$$

## Step 2. Formulate the Optimization Problem

We have:

- Dosage rates:  $a_1 = 1, a_2 > 1$
- Treatment times:  $x_1, x_2$
- Total dosage constraint:

$$x_1 + a_2 x_2 = 1$$

Optimal Treatment Time Minimization Problem

$$T = x_1 + x_2$$

Substitute  $x_1 = 1 - a_2 x_2$  into  $T$ :

$$T = (1 - a_2 x_2) + x_2 = 1 + (1 - a_2)x_2$$

Since  $a_2 > 1$ , the coefficient  $(1 - a_2) < 0$ , so  $T$  decreases as  $x_2$  increases.

Constraint:  $x_1 \geq 0 \Rightarrow x_2 \leq \frac{1}{a_2}$

Thus, the optimal solution is:

$$x_1 = 0, \quad x_2 = \frac{1}{a_2}$$

### Step 3. Determine the Numerical Values to the Required Accuracy

We have:

$$a_2 = 1 + 2\delta(\infty)$$

$$x_2 = \frac{1}{a_2} = \frac{1}{1 + 2\delta(\infty)} \approx 1 - 2\delta(\infty)$$

Since  $\delta(\infty)$  is extremely small (on the order of  $72^{-10^{24}}$ ), we have:

$$x_2 \approx 1$$

To two decimal places:

$$x_1 = 0.00, \quad x_2 = 1.00$$

### Final Answer

$0$  and  $1$

Optimal Treatment Time Minimization Problem