Deriving the Sigmoid Kalman Filter for Neural Mass Models

(Dated: August 21, 2015)

Notation

The sigmoid function is given by a normal cdf, $\Phi(x, v_0, \varsigma)$, with mean, v_0 and standard deviation, ς that is defined element-wise in the neural mass model for the vector, $\mathbf{C}\boldsymbol{\xi}$.

$$\Phi(x, v_0, \varsigma) = \frac{1}{\sqrt{2\pi}\varsigma} \int\limits_{-\infty}^{x} \exp\left(-\frac{(z - v_0)^2}{2\varsigma^2}\right) \mathrm{d}z \tag{1}$$

A multivariate Gaussian pdf, $\dot{\phi}(x; \mu, \Sigma)$ is defined for a random variable $x = \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^\top$ with probability density function:

$$\phi(x; \mu, \Sigma) = f(x_1, \dots, x_N) = \frac{1}{(2\pi)^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu)\right) \tag{2}$$

where the mean and covariance are given by

$$\mu = \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_N \end{bmatrix}^\top \tag{3}$$

$$\Sigma = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \dots & \sigma_{1,N} \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_{N,1} & \sigma_{N,2} & \dots & \sigma_{N,N} \end{bmatrix} \tag{4}$$

and $\mu_i = \mathbb{E}[x_i]$, $\sigma_{i,j} = \rho_{ij}\sigma_i\sigma_j$. We also assume Σ is symmetric, pos-def

The multivariate Gaussian can be expressed in the canonical form

$$\phi(x; \nu, \Lambda) = \frac{\exp\left(-\frac{1}{2}\nu^\top \Lambda^{-1} \nu\right)}{(2\pi)^{\frac{N}{2}} |\Lambda|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} x^\top \Lambda x + x^\top \nu\right) \tag{5}$$

where

$$\Lambda = \Sigma^{-1} \tag{6}$$

$$\nu = \Sigma^{-1}\mu \tag{7}$$

Derivation of $E\left[\Phi(x, v_0, \varsigma)\right]$

The derivation is for $\mathbb{E}\left[\Phi(x, v_0, \varsigma)\right]$, where x is univariate Gaussian random variable, i.e. $x \sim \phi(x; \mu, \sigma)$.

$$\mathbb{E}\left[\Phi(x, v_0, \varsigma)\right] = \int_{-\infty}^{\infty} \Phi(x, v_0, \varsigma) \phi(x, \mu, \sigma) \, \mathrm{d}x \tag{8}$$

This is solved by considering two independent normal random variables, X and Y where $X \sim \phi(x;\mu,\sigma)$ and $Y \sim \phi(y;v_0,\varsigma)$. The probability $p(Y \leq X)$ is given by

$$\begin{aligned} p(Y \leq X) &= \int_{-\infty}^{\infty} p(Y \leq X | X = x) p_X(x)\,\mathrm{d}x \qquad (9) \\ &= \int_{-\infty}^{\infty} \Phi(x, v_0, \varsigma)\phi(x;\mu,\sigma)\,\mathrm{d}x \\ &= \mathbb{E}\left[\Phi(x, v_0, \varsigma)\right] \end{aligned}$$

Using the fact that X and Y are independent, so $p(Y \leq X|X = x)$ is given by $\Phi(x, v_0, \varsigma)$. Now we can write

$$p(Y \leq X) = P(Y - X \leq 0) = \Phi(0, \hat{\mu}, \hat{\sigma}) \qquad (10)$$

Since $Y - X$ is also a normal random variable, with mean, $\hat{\mu}$ and variance, $\hat{\sigma}^2$. It is straightforward to see (since X and Y are independent) that

$$\begin{aligned} \hat{\mu} &= \mathbb{E}[Y] - \mathbb{E}[X] \qquad (11) \\ &= v_0 - \mu \\ \hat{\sigma}^2 &= \mathrm{var}(Y) + \mathrm{var}(X) \qquad (12) \\ &= \varsigma^2 + \sigma^2 \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}\left[\Phi(x, v_0, \varsigma)\right] &= \Phi(0, v_0 - \mu, \sqrt{\varsigma^2 + \sigma^2}) \qquad (13) \\ &= \frac{1}{2}\left(\mathrm{erf}\left(\frac{\mu - v_0}{\sqrt{2\left(\varsigma^2 + \sigma^2\right)}}\right) + 1\right) \end{aligned}$$

Derivation of $E\left[x_1\Phi(x_2, v_0, \varsigma)\right]$

Now we have the case where x is bivariate Gaussian, $x \sim \phi(x;\mu,\Sigma)$

$$\mathbb{E}\left[x_1\Phi(x_2, v_0, \varsigma)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\Phi(x_2, v_0, \varsigma)\phi(x,\mu,\Sigma)\,\mathrm{d}x_1\,\mathrm{d}x_2 \qquad (14)$$

This can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 \Phi(x_2, v_0, \varsigma) p(x_1|x_2) p(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \tag{15}$$

where we know $p(x_1|x_2) \sim \phi(x_1, \hat{\mu}(x_2), \hat{\Sigma})$ (i.e a Gaussian distribution with a new mean, which is a function of x_2, and variance, which is independent of x_2) and $p(x_2) \sim \phi(x_2; \mu_2, \sigma_{22})$.

$$\hat{\mu} = \mu_1 + \sigma_{12}\sigma_{22}^{-1}(x_2 - \mu_2) \tag{16}$$

$$\hat{\Sigma}^2 = \sigma_{11} - \sigma_{12}^2\sigma_{22}^{-1} \tag{17}$$

this is a standard formula. Seen also in the Kalman equations

Then we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 \Phi(x_2, v_0, \varsigma) p(x_1|x_2) p(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 = \int_{-\infty}^{\infty} E[p(x_1|x_2)]\Phi(x_2, v_0, \varsigma) p(x_2)\,\mathrm{d}x_2 \tag{18}$$

using the fact that

$$\begin{aligned}\int_{-\infty}^{\infty} x_1 p(x_1|x_2)\,\mathrm{d}x_1 &= E[p(x_1|x_2)] \\ &= \mu_1 + \sigma_{12}\sigma_{22}^{-1}(x_2 - \mu_2)\end{aligned} \tag{19}$$

(see $\hat{\mu}$ above). So we have

$$\begin{aligned}&\mu_1 \int_{-\infty}^{\infty} \Phi(x_2, v_0, \varsigma) p(x_2)\,\mathrm{d}x_2 + \sigma_{12}\sigma_{22}^{-1}\int_{-\infty}^{\infty}(x_2 - \mu_2)\Phi(x_2, v_0, \varsigma)p(x_2)\,\mathrm{d}x_2 \\ &= \mu_1\Phi(0, v_0 - \mu_2, \sqrt{\varsigma^2 + \sigma_{22}}) + \sigma_{12}\int_{-\infty}^{\infty}\frac{x_2 - \mu_2}{\sigma_{22}}\phi(x_2; \mu_2, \sigma_{22})\Phi(x_2, v_0, \varsigma)\,\mathrm{d}x_2 \\ &= \mu_1\Phi(0, v_0 - \mu_2, \sqrt{\varsigma^2 + \sigma_{22}}) + \sigma_{12}\frac{\mathrm{d}}{\mathrm{d}x_2}\int_{-\infty}^{\infty}\phi(x_2; \mu_2, \sigma_{22})\Phi(x_2, v_0, \varsigma)\,\mathrm{d}x_2 \\ &= \mu_1\Phi(0, v_0 - \mu_2, \sqrt{\varsigma^2 + \sigma_{22}}) + \sigma_{12}\frac{\mathrm{d}}{\mathrm{d}x_2}\left[\Phi(0, v_0 - \mu_2, \sqrt{\varsigma^2 + \sigma_{22}})\right] \\ &= \mu_1\Phi(0, v_0 - \mu_2, \sqrt{\varsigma^2 + \sigma_{22}}) + \sigma_{12}\phi(0, v_0 - \mu_2, \sqrt{\varsigma^2 + \sigma_{22}})\end{aligned} \tag{20}$$

Using the previous integral derivation and the derivatives

$$\begin{aligned}\frac{d}{dx}\phi(x; \mu, \sigma) &= -\frac{x - \mu}{\sigma^2}\phi(x; \mu, \sigma) \\ \frac{d}{dx}\Phi(x; \mu, \sigma) &= \phi(x; \mu, \sigma)\end{aligned} \tag{21}$$

When we write out the normal pdf and cdf in full we get to the final solution which is

$$\mathbb{E}\left[f(\mathbf{x})\right] = \frac{\mu_1}{2}\mathrm{erf}\left(\frac{\mu_2 - v_0}{\sqrt{2(\sigma_{22} + \varsigma^2)}}\right) + \frac{\mu_1}{2} + \frac{\sigma_{12}}{\sqrt{2\pi(\sigma_{22} + \varsigma^2)}}\exp\left(-\frac{(\mu_2 - v_0)^2}{2(\sigma_{22} + \varsigma^2)}\right) \tag{22}$$

Derivation of $E\left[\Phi(x_1, v_0, \varsigma)\Phi(x_2, v_0, \varsigma)\right]$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\left(\frac{x_1 - v_0}{\varsigma}\right)\Phi\left(\frac{x_2 - v_0}{\varsigma}\right)\phi(\mathbf{x};\mu,\Sigma)\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{23}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$. We will define two new, independent random variables, z_1 and z_2 that are both normally distributed with mean, v_0 and variance, ς i.e. $z_i \sim \phi(z; v_0, \varsigma)$. Since z_1 and z_2 are also independent of x_1 and x_2 we can write

$$P(z_1 < x_1, z_2 < x_2 | x_1, x_2) = \Phi\left(\frac{x_1 - v_0}{\varsigma}\right)\Phi\left(\frac{x_2 - v_0}{\varsigma}\right) \tag{24}$$

and by law of total probability the unconditional distribution is written

$$\begin{aligned} P(z_1 < x_1, z_2 < x_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(z_1 < x_1, z_2 < x_2 | x_1, x_2)\phi(\mathbf{x};\mu,\Sigma)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= P(z_1 - x_1 < 0, z_2 - x_2 < 0) \\ &= \Phi\left(\frac{0 - \hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\sigma}}}\right) \end{aligned} \tag{25}$$

which is a bivariate Gaussian cdf (i.e evaluated at $\mathbf{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$), with mean $\hat{\boldsymbol{\mu}} = \begin{bmatrix} \hat{\mu}_1 & \hat{\mu}_2 \end{bmatrix}$ and variance $\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{bmatrix}$. These terms can be calculated by recognizing that (since all variables are independent of one another, except for x_1 and x_2)

$$\hat{\mu}_1 = \mathbb{E}[z_1 - x_1] = v_0 - \mu_1 \tag{26}$$

$$\hat{\mu}_2 = \mathbb{E}[z_2 - x_2] = v_0 - \mu_2 \tag{27}$$

$$\hat{\sigma}_{11} = \mathrm{var}(z_1 - x_1) = \varsigma^2 + \sigma_{11} \tag{28}$$

$$\hat{\sigma}_{22} = \mathrm{var}(z_2 - x_2) = \varsigma^2 + \sigma_{22} \tag{29}$$

$$\begin{aligned} \hat{\sigma}_{12} &= \mathrm{cov}(z_1 - x_1, z_2 - x_2) \\ &= \mathrm{cov}(z_1, z_2) + \mathrm{cov}(z_1, x_2) + \mathrm{cov}(x_1, z_2) + \mathrm{cov}(x_1, x_2) \\ &= \mathrm{cov}(x_1, x_2) = \sigma_{12} \end{aligned} \tag{30}$$

Derivation of $E\left[x_1 x_2 \Phi(x_3, v_0, \varsigma)\right]$

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \Phi\left(\frac{x_3 - v_0}{\varsigma}\right)\phi(\mathbf{x};\mu,\Sigma)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \int_{-\infty}^{\infty} x_1 \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_1|x_2,x_3)p(x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 E[p(x_1|x_2,x_3)]\Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3
\end{aligned} \tag{31}$$

Some conditional distributions

$p(x_1|x_2,x_3)$ is Gaussian with mean and variance

$$\begin{aligned}
\hat{\mu}_{1|2,3} = E[p(x_1|x_2,x_3)] &= \mu_1 + \begin{bmatrix}\sigma_{12} \\ \sigma_{13}\end{bmatrix}\begin{bmatrix}\sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33}\end{bmatrix}^{-1}\left(\begin{bmatrix}x_2 \\ x_3\end{bmatrix} - \begin{bmatrix}\mu_2 \\ \mu_3\end{bmatrix}\right) \\
&= \int_{-\infty}^{\infty} x_1 p(x_1|x_2,x_3)\,\mathrm{d}x_1
\end{aligned} \tag{32}$$

$$\begin{aligned}
\hat{\sigma}_{1|2,3} = \mathrm{var}[p(x_1|x_2,x_3)] &= \sigma_{11} - \begin{bmatrix}\sigma_{12} \\ \sigma_{13}\end{bmatrix}^{\top}\begin{bmatrix}\sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33}\end{bmatrix}^{-1}\begin{bmatrix}\sigma_{12} \\ \sigma_{13}\end{bmatrix} \\
&= \int_{-\infty}^{\infty} x_1^2 p(x_1|x_2,x_3)\,\mathrm{d}x_1 - E[p(x_1|x_2,x_3)]^2
\end{aligned} \tag{33}$$

and the solutions written out

$$\hat{\sigma}_{1|2,3} = \sigma_{11}\frac{\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} - 1}{\rho_{23}^2 - 1} \tag{34}$$

$$\hat{\mu}_{1|2,3} = \mu_1 + \frac{\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22}}{\sigma_{22}\sigma_{33} - \sigma_{23}^2}(\mu_3 - x_3) + \frac{\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_{33}}{\sigma_{22}\sigma_{33} - \sigma_{23}^2}(\mu_2 - x_2) \tag{35}$$

$$\hat{\mu}_{2|3} = \mu_2 + \sigma_{23}\sigma_{33}^{-1}(x_3 - \mu_3) \tag{36}$$

$$\hat{\sigma}_{2|3}^2 = \sigma_{22} - \sigma_{23}^2\sigma_{33}^{-1} \tag{37}$$

Then we can write the integral

$$\begin{aligned}
&\mu_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_2|x_3)p(x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&+ \frac{\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22}}{\sigma_{22}\sigma_{33} - \sigma_{23}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu_3 - x_3)x_2\Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_2|x_3)p(x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\
&+ \frac{\sigma_{13}\sigma_{23} - \sigma_{12}\sigma_{33}}{\sigma_{22}\sigma_{33} - \sigma_{23}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu_2 - x_2)x_2\Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_2|x_3)p(x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3
\end{aligned} \tag{38}$$

And solve for the terms separately

NB $\mu^* = v_0 - \mu$ and $\sigma^* = \sqrt{\varsigma^2 + \sigma^2}$

$$\begin{aligned}
&\mu_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_2 \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_2|x_3) p(x_3)\, \mathrm{d}x_2\, \mathrm{d} \\
&= \mu_1 \int_{-\infty}^{\infty} \hat{\mu}_{2|3} \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 \\
&= \mu_1 \mu_2 \Phi\left(\frac{0 - \mu^*}{\sigma^*}\right) + \sigma_{23} \int_{-\infty}^{\infty} \frac{x_3 - \mu_3}{\sigma_{33}} \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 \\
&= \mu_1 \mu_2 \Phi\left(\frac{0 - \mu^*}{\sigma^*}\right) - \sigma_{23} \frac{\mathrm{d}}{\mathrm{d}x_3}\left[\Phi\left(\frac{0 - \mu^*}{\sigma^*}\right)\right] \\
&= \mu_1 \mu_2 \Phi\left(\frac{0 - \mu^*}{\sigma^*}\right) - \sigma_{23} \phi\left(\frac{0 - \mu^*}{\sigma^*}\right)
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\frac{\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22}}{\sigma_{22}\sigma_{33} - \sigma_{23}^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mu_3 - x_3) x_2 \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_2|x_3) p(x_3)\, \mathrm{d}x_2\, \mathrm{d}x_3 \\
&= \mu_2 \sigma_{33} \int_{-\infty}^{\infty} \frac{-(x_3 - \mu_3)}{\sigma_{33}} \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 + \sigma_{23} \int_{-\infty}^{\infty} \frac{x_3 - \mu_3}{\sigma_{33}} (\mu_3 - x_3) \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 \\
&= \mu_2 \sigma_{33} \frac{\mathrm{d}}{\mathrm{d}x_3} \int_{-\infty}^{\infty} \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 - \sigma_{23} \frac{\mathrm{d}}{\mathrm{d}x_3} \int_{-\infty}^{\infty} (\mu_3 - x_3) \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 \\
&= \mu_2 \sigma_{33} \phi\left(\frac{0 - \mu^*}{\sigma^*}\right) - \sigma_{23}\sigma_{33} \frac{\mathrm{d}}{\mathrm{d}x_3} \int_{-\infty}^{\infty} \frac{-(x_3 - \mu_3)}{\sigma_{33}} \Phi\left(\frac{x_3 - v_0}{\varsigma}\right) p(x_3)\, \mathrm{d}x_3 \\
&= \mu_2 \sigma_{33} \phi\left(\frac{0 - \mu^*}{\sigma^*}\right) - \sigma_{23}\sigma_{33} \frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\left[\Phi\left(\frac{0 - \mu^*}{\sigma^*}\right)\right] \\
&= \mu_2 \sigma_{33} \phi\left(\frac{0 - \mu^*}{\sigma^*}\right) - \sigma_{23}\sigma_{33} \frac{\mu^*}{\sigma^{*2}} \phi\left(\frac{0 - \mu^*}{\sigma^*}\right) \\
&= \frac{\sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22}}{\sigma_{22}\sigma_{33} - \sigma_{23}^2} (\mu_2 - \sigma_{23} \frac{\mu^*}{\sigma^{*2}}) \sigma_{33} \phi\left(\frac{0 - \mu^*}{\sigma^*}\right)
\end{aligned} \tag{40}$$

and

$$\frac{\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33}}{\sigma_{22}\sigma_{33}-\sigma_{23}^2} \tag{41}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu_2-x_2)x_2\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_2|x_3)p(x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$=\mu_2\int_{-\infty}^{\infty}(\mu_2+\sigma_{23}\sigma_{33}^{-1}(x_3-\mu_3))\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_3)\,\mathrm{d}x_3-\int_{-\infty}^{\infty}(\hat{\sigma}_{2|3}^2+\hat{\mu}_{2|3}^2)\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_3)\,\mathrm{d}x_3$$
$$=\mu_2^2\Phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\sigma_{23}\phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\int_{-\infty}^{\infty}(\hat{\sigma}_{2|3}^2+\hat{\mu}_{2|3}^2)\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_3)\,\mathrm{d}x_3$$
$$=(\mu_2^2-\sigma_{22}+\sigma_{23}^2\sigma_{33}^{-1})\Phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\sigma_{23}\phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\int_{-\infty}^{\infty}\hat{\mu}_{2|3}^2\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_3)\,\mathrm{d}x_3$$
$$=(\mu_2^2-\sigma_{22}+\sigma_{23}^2\sigma_{33}^{-1})\Phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\sigma_{23}\phi\left(\frac{0-\mu^*}{\sigma^*}\right)$$
$$-\mu_2^2\Phi\left(\frac{0-\mu^*}{\sigma^*}\right)+2\mu_2\sigma_{23}\phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\sigma_{23}^2\frac{\mu^*}{\sigma^{*2}}\phi\left(\frac{0-\mu^*}{\sigma^*}\right)$$
$$=\frac{\sigma_{13}\sigma_{23}-\sigma_{12}\sigma_{33}}{\sigma_{22}\sigma_{33}-\sigma_{23}^2}\left[(\sigma_{23}^2\sigma_{33}^{-1}-\sigma_{22})\Phi\left(\frac{0-\mu^*}{\sigma^*}\right)-\sigma_{23}(1+2\mu_2-\sigma_{23}\frac{\mu^*}{\sigma^{*2}})\phi\left(\frac{0-\mu^*}{\sigma^*}\right)\right]$$

The end result we want is

$$\frac{\mu_1\mu_2+\sigma_{12}}{2}\left(1+\operatorname{erf}\left(\frac{\mu_3-v_0}{\sqrt{2(\varsigma^2+\sigma_{33})}}\right)\right) \tag{42}$$
$$-\left(\frac{\sigma_{13}\sigma_{23}(\mu_3-v_0)}{\sqrt{2\pi}(\varsigma^2+\sigma_{33})^{\frac{3}{2}}}-\frac{\sigma_{23}\mu_w+\sigma_{13}\mu_y}{\sqrt{2\pi(\varsigma^2+\sigma_{33})}}\right)\exp\left(-\frac{(\mu_3-v_0)^2}{2(\varsigma^2+\sigma_{33})}\right)$$

Derivation of $E\left[x_1^2\Phi^2(x_2,v_0,\varsigma)\right]$

The last integral ...

This is the simple case for the diagonal entries only. The generalization to $E\left[x_1^2\Phi(x_2,v_0,\varsigma)\Phi(x_3,v_0,\varsigma)\right]$ is straightforward (I think) but the generalization to $E\left[x_1x_2\Phi(x_3,v_0,\varsigma)\Phi(x_4,v_0,\varsigma)\right]$ perhaps not.

The integral looks like

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^2\Phi\left(\frac{x_2-v_0}{\varsigma}\right)\Phi\left(\frac{x_3-v_0}{\varsigma}\right)\phi(\mathbf{x};\mu,\Sigma)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{43}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1^2\Phi\left(\frac{x_2-v_0}{\varsigma}\right)\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_1|x_2,x_3)p(x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

The integral can be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\hat{\sigma}_{1|2,3}+\hat{\mu}_{1|2,3}^{2})\Phi\left(\frac{x_2-v_0}{\varsigma}\right)\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{44}$$
$$=\hat{\sigma}_{1|2,3}\Phi\left(\frac{0-\boldsymbol{\mu}^*}{\boldsymbol{\sigma}^*}\right)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\mu}_{1|2,3}^{2}\Phi\left(\frac{x_2-v_0}{\varsigma}\right)\Phi\left(\frac{x_3-v_0}{\varsigma}\right)p(x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3$$

with

$$\mathbf{x}=\begin{bmatrix}x_2 & x_3\end{bmatrix}^\top \tag{45}$$
$$\boldsymbol{\mu}^*=\begin{bmatrix}v_0-\mu_2 & v_0-\mu_3\end{bmatrix}^\top \tag{46}$$
$$\boldsymbol{\sigma}^*=\begin{bmatrix}\varsigma^2+\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \varsigma^2+\sigma_{33}\end{bmatrix} \tag{47}$$