Anytime Model Selection for Linear Bandits

Parnian Kassraie, Aldo Pacchiano, Nicolas Emmenegger, Andreas Krause

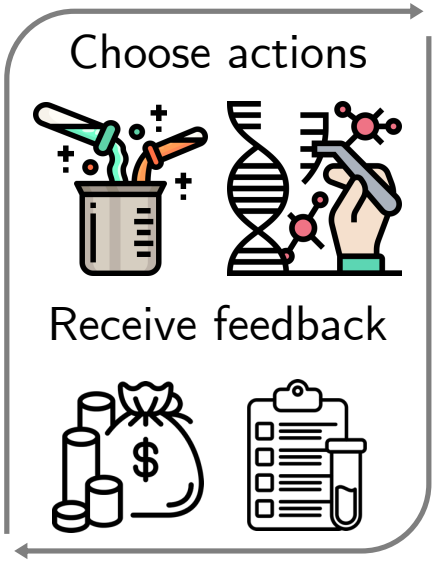



ETH zürich

Online Model Selection

Solving a Bandit/BO problem:

1. Commit to a reward model (a priori)
2. Interact with the environment accordingly



Repeat until happy

(w.r.t. some notion of regret)

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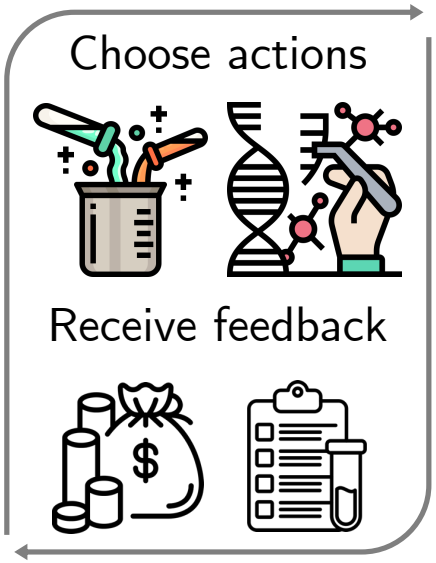
$$M \gg n$$ horizon/stopping time



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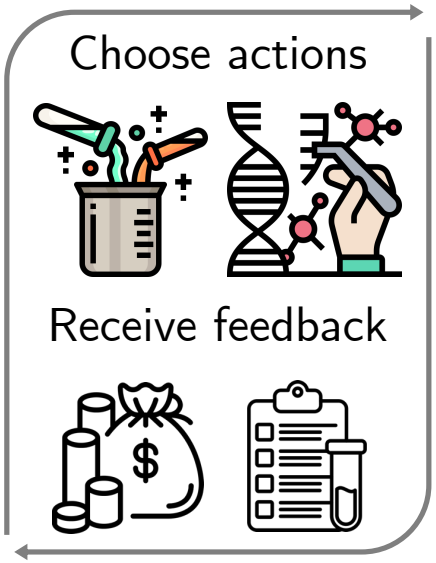

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e.g. in terms of sample efficiency

We can guess based on emprical evidence.

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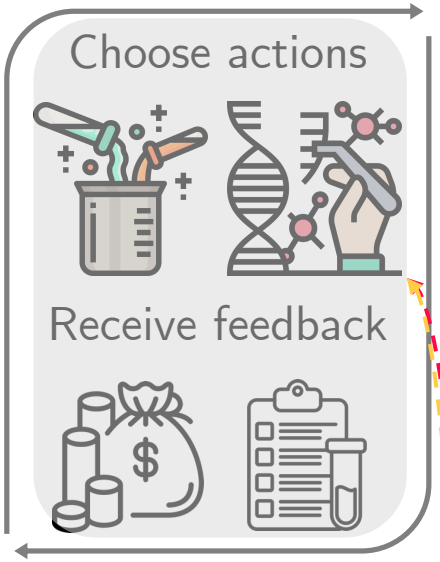

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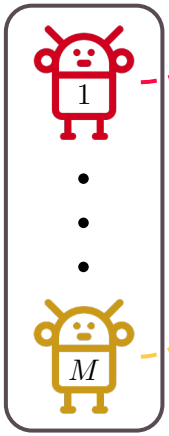
Run all agents in parallel



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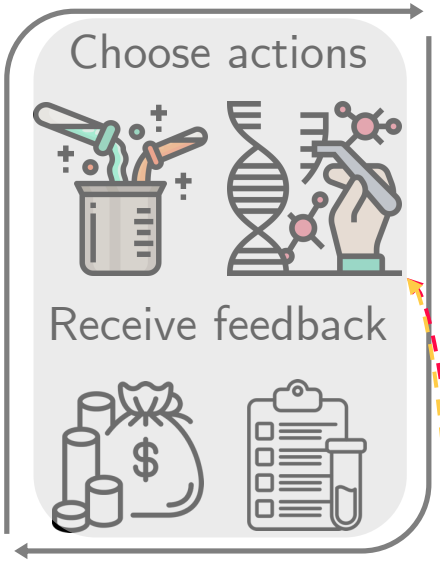

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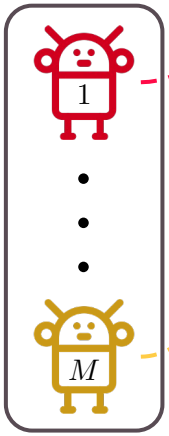

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Statistically expensive

poly$M$

image source: flaticon

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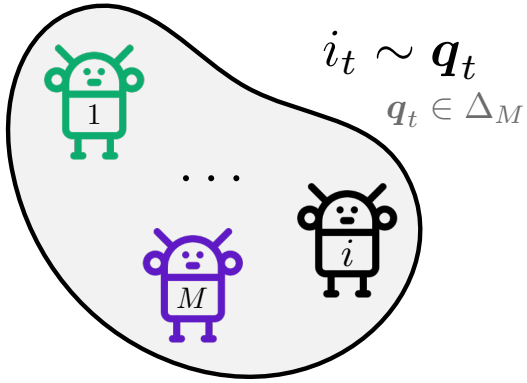
Randomly iterate over the agents and at each step play only one



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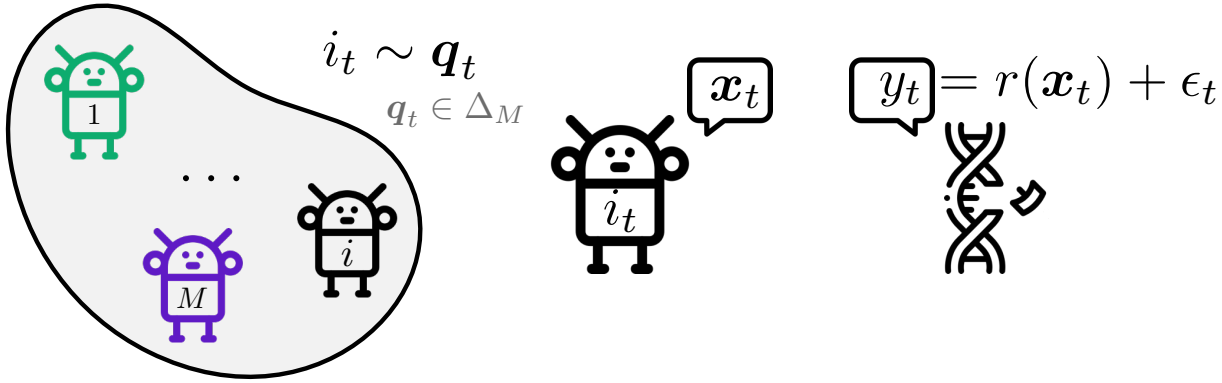

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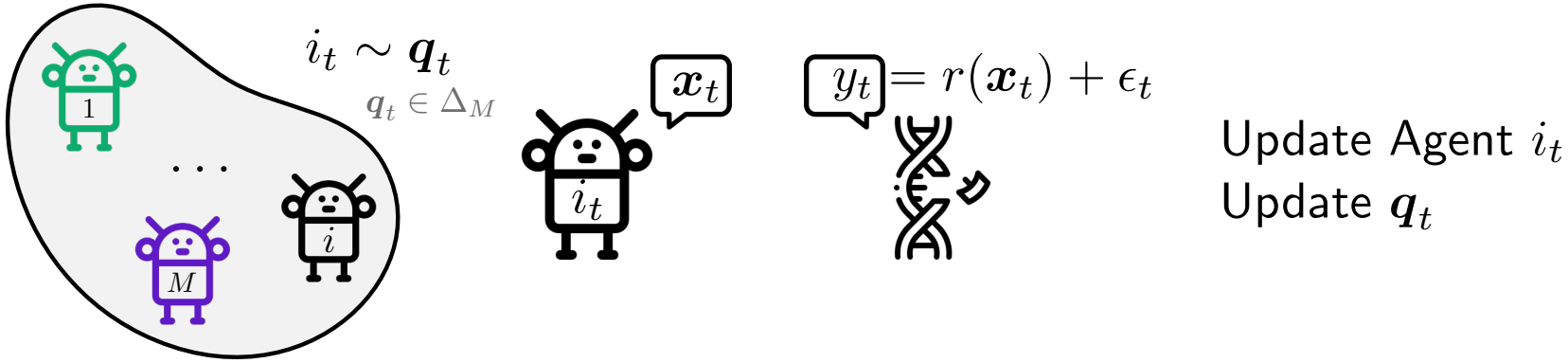

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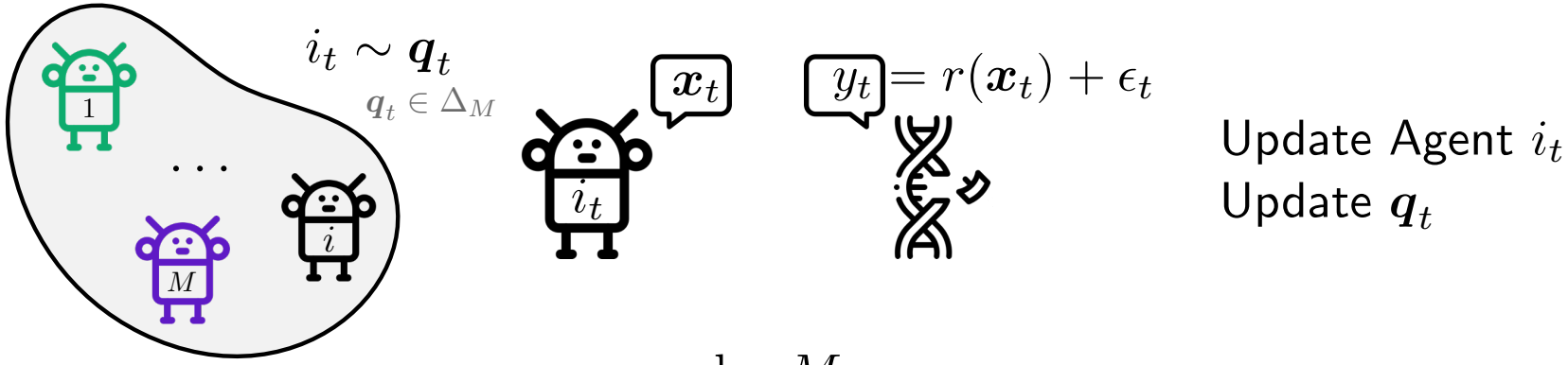

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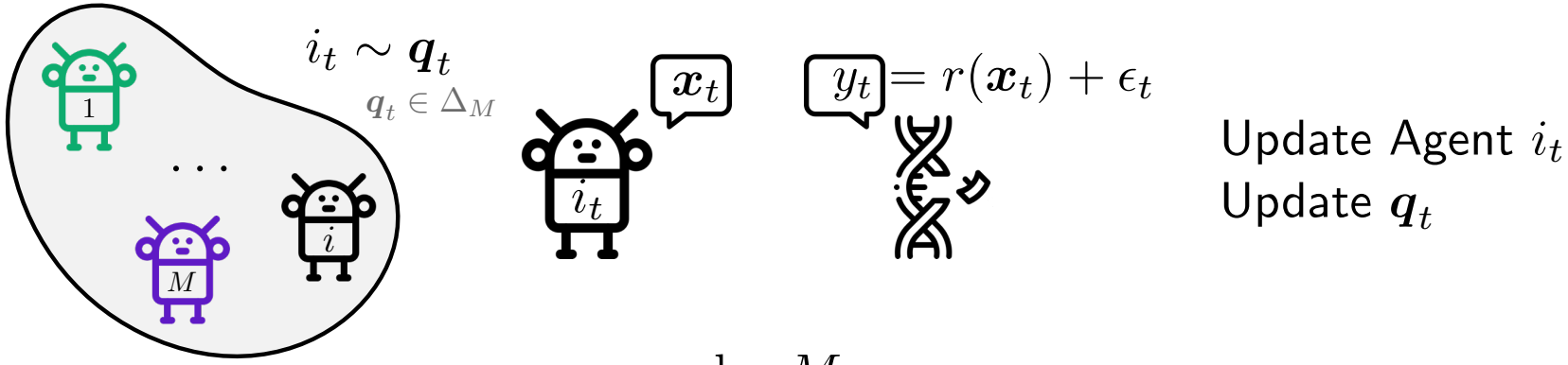

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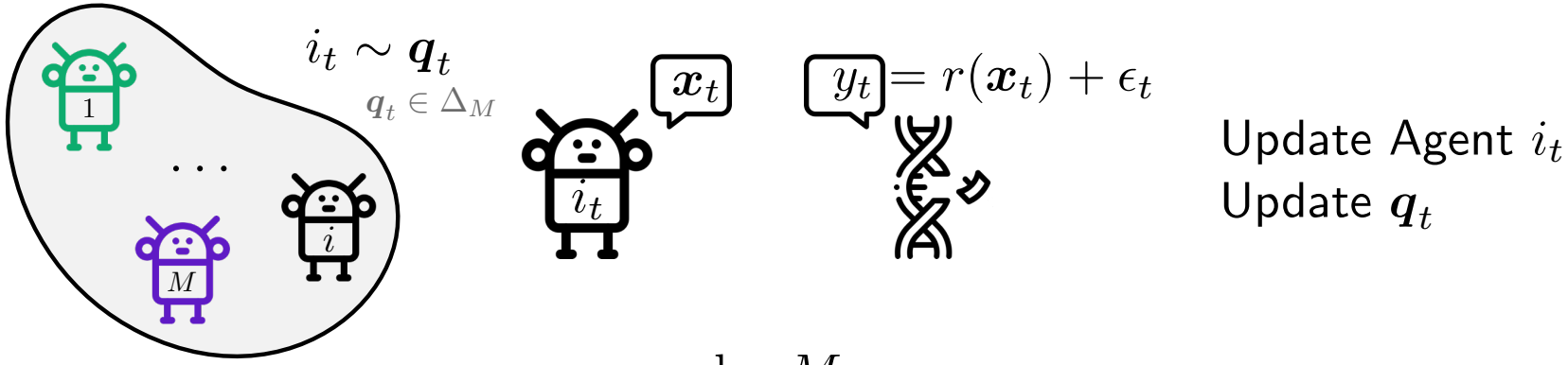

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Play one agent, but update all.
Reward not observed? Estimate it.

Choose your update rule and your estimator **very** carefully.

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Problem Setting

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Environment

i.i.d. zero-mean sub-gaussian noise

$$\forall t \geq 1 \qquad y_t = r(\boldsymbol{x}_t) + \epsilon_t$$

$$\boldsymbol{x}_t \in \mathcal{X} \subset \mathbb{R}^{d_0}$$

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Agent j only uses $\boldsymbol{\phi}_j$ to model the reward

Has action selection policy $p_{t,j} \in \mathcal{M}(\mathcal{X})$

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Goal

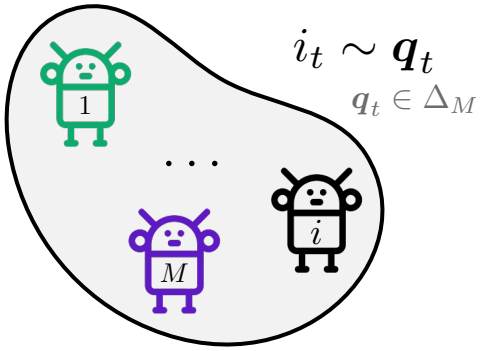
n unknown

$$R(n) = \sum_{t=1}^{n} r(\boldsymbol{x}^*) - r(\boldsymbol{x}_t)$$

– Sublinear

– $\log M$

How to iterate over agents



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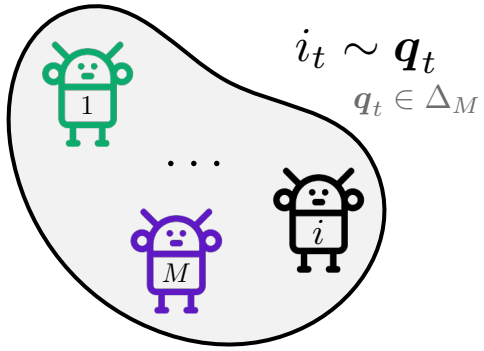

Exponential Weighting

reward obtained by agent j so far

$$q_{t,j} = \frac{\exp(\eta_t \sum_{s=1}^{t-1} r_{s,j})}{\sum_{i=1}^{M} \exp(\eta_t \sum_{s=1}^{t-1} r_{s,i})}$$

sensitivity of updates

Known to yield $\log M$ regret in full-info setting

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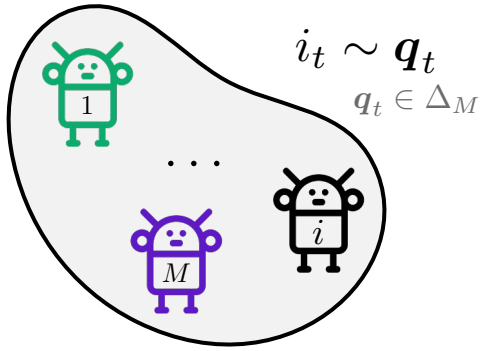

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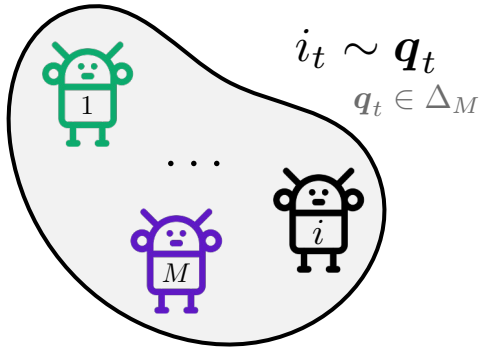

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Classical choices are unbiased high-variance estimators e.g. importance weighted or OLS estimator

Regret will depend on bias and variance of $\hat{\boldsymbol{\theta}}_t$

Ridge/OLS regression oracles are $\sqrt{M} \to \text{poly}M$ regret

How to hallucinate rewards

$$\hat{r}_{t,j} = \mathbb{E}_{\boldsymbol{x} \sim p_{t,j}} \hat{\boldsymbol{\theta}}_t^\top \boldsymbol{\phi}(\boldsymbol{x})$$

Turn lasso into a sparse online regression oracle

$$\hat{\boldsymbol{\theta}}_t = \arg\min \frac{1}{t} ||\boldsymbol{y}_t - \Phi_t \boldsymbol{\theta}||_2^2 + \lambda_t \sum_{j=1}^{M} ||\boldsymbol{\theta}_j||_2$$

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Theorem (Anytime Conf. Seq.)

If for all $t \geq 1$

$$\lambda_t \geq \frac{c_1}{\sqrt{t}} \sqrt{\log(M/\delta) + \sqrt{d\,(\log(M/\delta) + (\log\log d)_+)}}$$

then,

c_1 and c_2 made exact in the paper

$$\mathbb{P}\left(\forall t \geq 1 : \left\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_t\right\|_2 \leq \frac{c_2 \lambda_t}{\kappa^2(\Phi_t, 2)}\right) \geq 1 - \delta$$

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Difference with offline Lasso?

Instead of sub-gaussian concentration,

Design a self-normalized martingale based on $\left|\left|\left(\Phi_t^\top \boldsymbol{\epsilon}_t\right)_j\right|\right|$ — Empirical process error

Apply a "stitched" time uniform boundary [Howard et al. '21]

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Thanks to Aaditya & SAVI friends!

Putting it all together: ALExp

Anytime **Exp**onential weighting algorithm with **L**asso reward estimates

Algorithm 1 ALEXP

Inputs: γ_t, η_t, λ_t for $t \geq 1$

for $t \geq 1$ **do**

Draw $\boldsymbol{x}_t \sim (1-\gamma_t)\sum_{j=1}^{M} q_{t,j} p_{t,j} + \gamma_t \text{Unif}(\mathcal{X})$

Observe $y_t = r(\boldsymbol{x}_t) + \epsilon_t$.

Append history $H_t = H_{t-1} \cup \{(\boldsymbol{x}_t, y_t)\}$.

Update agents $p_{t,j}$ for $j = 1, \ldots, M$.

Calculate $\hat{\boldsymbol{\theta}}_t \leftarrow \text{Lasso}(H_t, \lambda_t)$ and estimate

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prescribed in the paper

Theorem (Regret - Informal)

For appropriate choices of $(\gamma_t, \lambda_t, \eta_t)$,

$$R(n) = \mathcal{O}\left(C(M, \delta, d)\left(\sqrt{n}\log M + n^{3/4}\right)\right)$$

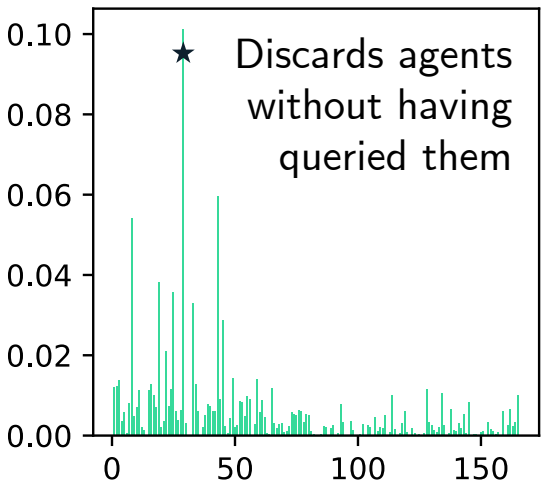
with probability greater than $1-\delta$, *simultaneously for all* $n \geq 1$.

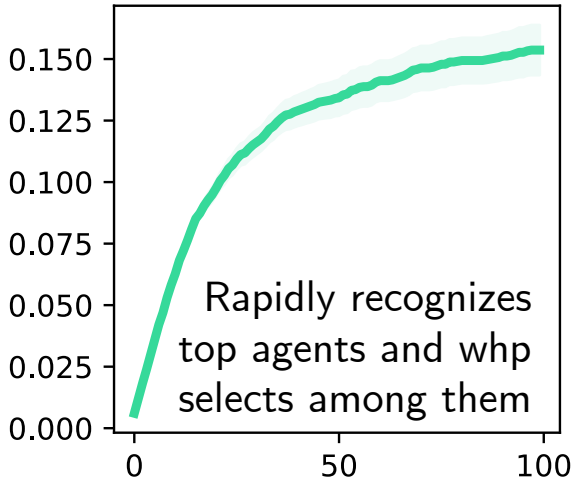
$$C(M, \delta, d) = \mathcal{O}\left(\sqrt{d \log M/\delta + \sqrt{d \log M/\delta}}\right)$$

Synthetic Experiments

data generation & baselines
described in the paper.

How does
ALExp work?





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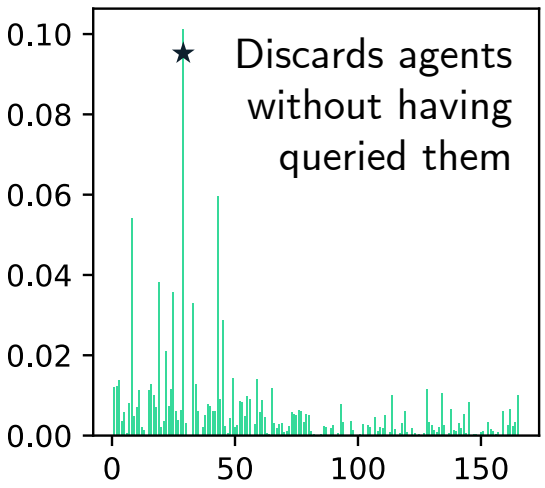

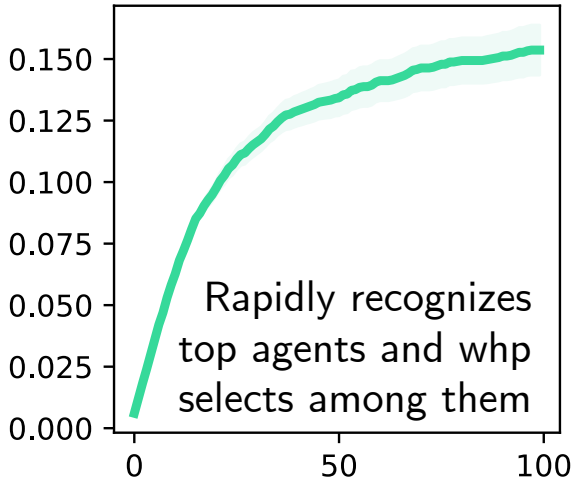

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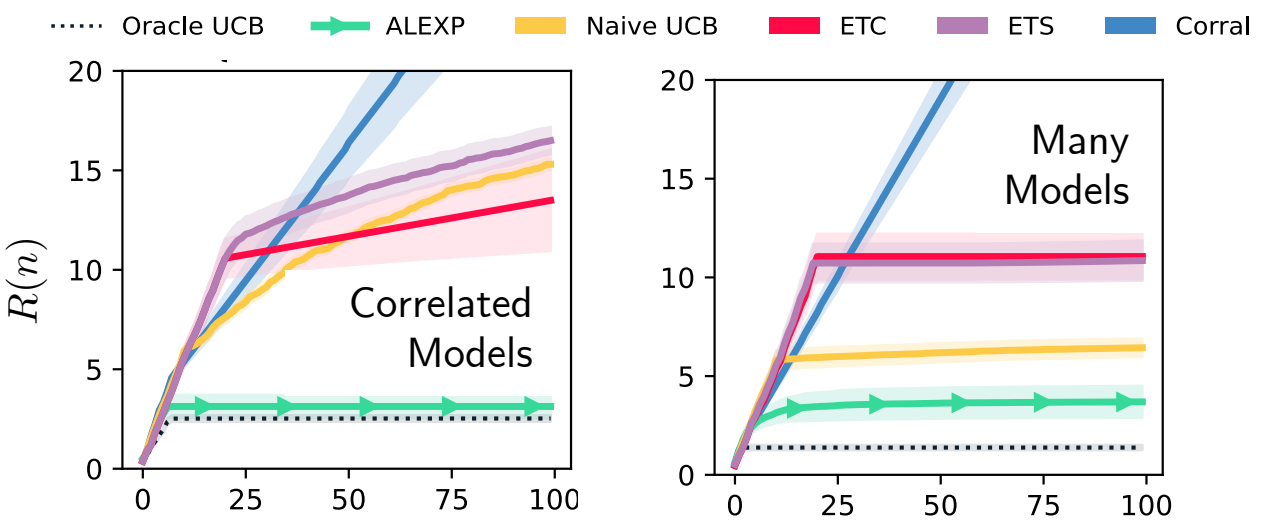

Thank you.