EECS 568 Mobile Robotics Exam

PoKang Chen

1.Probability and Uncertainty Propagation

1.1

For a probability distribution, when encountering a nonlinear function, it should be first linearized. Assume that it is linearized (via Taylor expansion) around point a:

$$\begin{aligned} f(\mathbf{x}) &\approx f(\mathbf{a}) + \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{a}} (\mathbf{x}-\mathbf{a}) \\ &= \left(f(\mathbf{a}) - \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{a}} \mathbf{a} \right) + \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{a}} \mathbf{x} \\ &\triangleq \mathbf{x}_0 + \mathbf{F}\mathbf{x} \end{aligned}$$

During which, it is an affine transform and thus we could propagate a Gaussian through an affine map.

When x is a Gaussian random variable $\mathrm{x} \sim \mathcal{N}(\mu, \Sigma)$ and $\mathrm{y} = \mathrm{Ax} + \mathrm{b}$, then the affine transform would be $\mathbf{y} \sim \mathcal{N}\left(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\top}\right)$.

1.2

Since $z \sim \mathcal{N}\left(\mu_z, \sigma_z^2\right)$, and $y = az + b$ (Affine transform), $y \sim \mathcal{N}\left(A\boldsymbol{\mu} + b, A\boldsymbol{\Sigma}A^{\top}\right)$. For which $\Sigma = \begin{bmatrix} \sigma_1^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n^2 \end{bmatrix}$. The noise model for y is an approximation of the random distributor.

1.3

The model of the measurement is as following:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{\frac{1}{z}\begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}} \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

$$b = AX$$

Thus, the X measurement could be seen as:

$$X \sim \mathcal{N}(\mu_x \ , \ \Sigma_x)$$

Since it is a affine transformation, b could be seen as:

$$b \sim \mathcal{N}(A\mu_x \ , \ A\Sigma_x A^T)$$

2.Probability and Uncertainty Propagation

For the exponential map between so(3), we could expand a exponential map to a Taylor expansion:

$$\exp\left(\phi^{\wedge}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\phi^{\wedge}\right)^{n}$$

We could then get the following equation which means that so(3) is the space extended by rotation matrix in SO(3):

$$\exp\left(\theta \boldsymbol{a}^{\wedge}\right)=\cos\theta \boldsymbol{I}+(1-\cos\theta)\boldsymbol{a}\boldsymbol{a}^{T}+\sin\theta \boldsymbol{a}^{\wedge}$$

And for the exponential map between se(3) and SE(3), it would be like:

$$\exp\left(\boldsymbol{\xi}^{\wedge}\right)=\begin{bmatrix}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\phi^{\wedge}\right)^{n} & \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\phi^{\wedge}\right)^{n}\rho \\ 0^{T} & 1\end{bmatrix}=\begin{bmatrix}R & J\rho \\ 0^{T} & 1\end{bmatrix}$$

J could be gained by ϕ, and ρ could be gained by linear equation. As in the SLAM problem, since $z=Tp+w$ (w is the noise), $e=z-Tp$. We could solve and minimize the following equation:

$$\min_{T} J(T)=\sum_{i=1}^{N}\left\| z_{i}-Tp_{i}\right\|_{2}^{2}$$

Assume a point p is transformed by T(ξ for lie algebra). $\Delta \boldsymbol{T}=\exp\left(\delta\boldsymbol{\xi}^{\wedge}\right)$, we then take derivative of it.

$$\frac{\partial(\boldsymbol{T}\boldsymbol{p})}{\partial\delta\boldsymbol{\xi}}=\lim_{\delta\boldsymbol{\xi}\to\boldsymbol{0}}\frac{\exp\left(\delta\boldsymbol{\xi}^{\wedge}\right)\exp\left(\boldsymbol{\xi}^{\wedge}\right)\boldsymbol{p}-\exp\left(\boldsymbol{\xi}^{\wedge}\right)\boldsymbol{p}}{\delta\boldsymbol{\xi}}=\lim_{\delta\xi\to 0}\frac{\delta\boldsymbol{\xi}^{\wedge}\exp\left(\boldsymbol{\xi}^{\wedge}\right)\boldsymbol{p}}{\delta\boldsymbol{\xi}}=\begin{bmatrix}I & -(\boldsymbol{R}\boldsymbol{p}+\boldsymbol{t})^{\wedge} \\ 0^{T} & 0^{T}\end{bmatrix}$$

3.Normal Random Variables

1. F
2. T
3. F
4. F

4.Estimation and Mapping

1. F
2. T
3. F
4. T
5. T
6. T

5.Bayes Filter

The evidence(Marginal Likelihood) is as following:

$$\Sigma P = 0.7 * 0.3 + 0.25 * 0.6 + 0.05 * 0.1 = 0.365$$

Thus, the posterior probability of getting A, B, and C is:

$$P(A \mid z, A_{prev}) = \frac{0.7 * 0.3}{\Sigma P} = \frac{0.21}{0.365} = 0.575$$
$$P(B \mid z, B_{prev}) = \frac{0.25 * 0.6}{\Sigma P} = \frac{0.15}{0.365} = 0.411$$
$$P(C \mid z, C_{prev}) = \frac{0.05 * 0.1}{\Sigma P} = \frac{0.005}{0.365} = 0.013$$

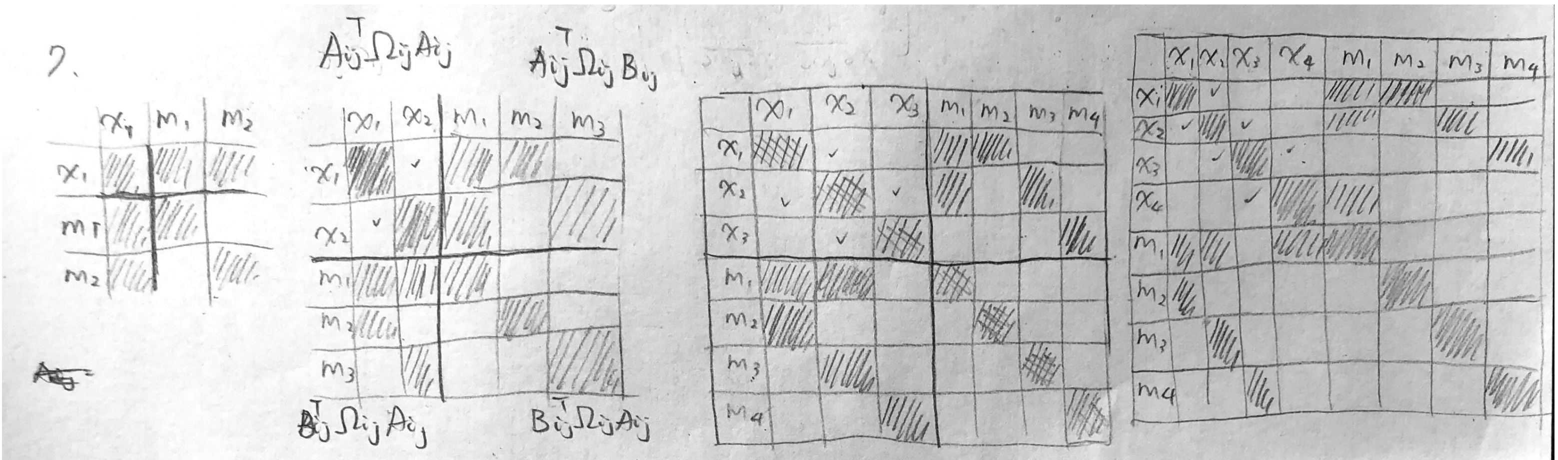
** PS: 57.5% for getting A, Hooray **

6.Cholesky Decomposition

Decomposing 2x2 matrix would be like $A = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$

1. Cholesky, $\begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$, Thus $L = \begin{bmatrix} 2\sqrt{2} & 0 \\ \frac{5\sqrt{2}}{4} & \frac{\sqrt{46}}{4} \end{bmatrix}$
2. Not Cholesky. It is not positive definite matrix.
3. Cholesky. $\begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$, Thus $L = \begin{bmatrix} \sqrt{3} & 0 \\ \frac{-4}{\sqrt{3}} & \frac{\sqrt{38}}{3} \end{bmatrix}$
4. Not Cholesky. It is not symmetric.

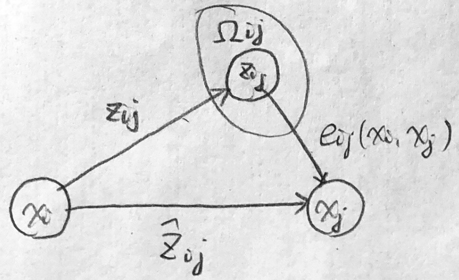
7.Graph SLAM



$$A_{ij} = \frac{\partial e_{ij}(x)}{\partial x_i}$$

$$e_{ij}(x_i, x_j) = \left[z_{ij} - \hat{z}_{ij}(x_i, x_j) \right]$$

$$B_{ij} = \frac{\partial e_{ij}(x)}{\partial x_j}$$



8.Target Tracking

8.1

8.1

STEP 1:

$$K_1 = (\Sigma_0 C_k^T)[C_k \Sigma_0 C_k^T + Q_k]^{-1} = 9 \times (\times 9 \times + 1)^{-1} = \frac{9}{10}$$

$$\mu_1 = (A_k \mu_0^+) + A \cdot (\Sigma_0 C_k^T)(C_k \Sigma_0 C_k^T + Q_k)^{-1} (z_1 - C_k \mu_0) = (20) + \frac{9}{10}(22 - 20) = 21.8$$

$$\Sigma_1 = A_k[\Sigma_0 - K_k C_k \Sigma_0] A_k^T + G_k R_k G_k^T = (9 - \frac{1}{10} \times 9) + 4 = 12.1$$

$$+ 9 \times (1 - \frac{9}{10})$$

STEP 2:

$$\mu_2 = A\mu_1 + A K_2 (z_2 - C_k \mu_1) = 21.8 + 0.92(23 - 21.8) = \underline{22.904}$$

$$\Sigma_2 = A_k[\Sigma_1 - K_2 C_k \Sigma_1] A_k^T + GRG^T = (12.1 - 0.08 \times 12.1) + 4 = \underline{15.132}$$

$$K_2 = \Sigma_1 \times (\Sigma_1 + 1)^{-1} = 12.1 \times (12.1 + 1)^{-1} = 0.92$$

8.2

8.2.

$R = LL^T$ noise $= L \times \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Ax = b$, $\Sigma = L L^T$, $R = L_z L_z^T$

<u>Motion Model</u>

$$A_k = \begin{bmatrix} A_{k-1} & \\ & F^{-1} L_u \end{bmatrix} = \begin{bmatrix} A_{k-1} & \\ & F^{-1} L_u \end{bmatrix}$$

$$b_k = \begin{bmatrix} b_{k-1} \\ G^{-1} L^{-1} \end{bmatrix}$$

<u>Measurement Model:</u>

$$A_k = \begin{bmatrix} A_{k-1} & \\ & H^{-1} L_z \end{bmatrix}$$

$$b_k = \begin{bmatrix} b_{k-1} \\ z^{-1} L_z \end{bmatrix}$$

$$\hat{x} = A \backslash b$$

8.3

For least square estimator of the target trajectory, since it is doing “smoothing”, for which the new information is added to the matrix A and b incrementally. Least square recalculates per step for the whole matrix and the propagation which contains the whole history of the Ax = b motion. As for Kalman Filter, since it is doing ”filtering”, it based on only current state and measurement to propagate.

9.Rigid Body Transformation

9.1

	Equation	Example	Errors	Note
Left Invariant EKF	$Y = X_t b + V$	GPS	$\eta_t^R = \bar{X}_t X_t^{-1} = (\bar{X}_t L)(X_t L)^{-1}$	
Right Invariant EKF	$Y = X_t^{-1} b + V$	Forward Kimematics Landmark	$\eta_t^L = X_t^{-1} \bar{X}_t = (L\bar{X}_t)^{-1}(LX_t)$	Things that are relative

9.2

② $R \in SO(3)$ $\quad so(3) = \left\{ \phi \in R^3, \ \phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in R^{3\times3} \right\}$

$SE(3)$ $\quad se(3) = \left\{ \xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in R^6, \ \rho \in R^3, \ \phi \in so(3), \ \xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} \in R^{4\times4} \right\}$

There are six variables in ξ . Three of them are so(3) for the rotation degree of freedom and three of them are for translation. As for ϕ, there're only three variables for the rotational degree of freedom.

9.3

③ $X(t) \in SE(3)$

$\frac{dX_t}{dt} = X_t u_t^\wedge$, $\quad u_t = vec(\underbrace{w_t}_{\text{angular velocity}}, \underbrace{v_t}_{\text{linear velocity}}) \in R^6$

$f_{u_t}(X_t) = X_t u_t^\wedge$

error dynamics :

$\frac{d}{dt}\eta_t^r = g_{u_t}(\eta^r) = f_{u_t}(\eta^r) - \eta^r f_{u_t}(I) = \eta^r u_t^\wedge - \eta^r u_t^\wedge = 0.$

$\Rightarrow \frac{d}{dt}\xi_t^r = 0.$

$X_t = \begin{bmatrix} R_k & P_k \\ 0 & 1 \end{bmatrix} \in SE(3)$

$\bar{X}_{k+1} = \bar{X}_k e^{u_k^\wedge}$

$\Phi = e^{(A_t^r \Delta t)}$

$P_{k+1} = \Phi P_k \Phi^T + Ad_{\bar{X}_t} Q_d Ad_{\bar{X}_t}^T$

$P_{k+1} = P_k + Ad_{\bar{X}_t} Q_d Ad_{\bar{X}_t}^T$

$Q \approx \Phi Q_t \Phi^T \Delta t = Q_t \Delta t$

9.4

$Y = RX + t$ is the transformed points.

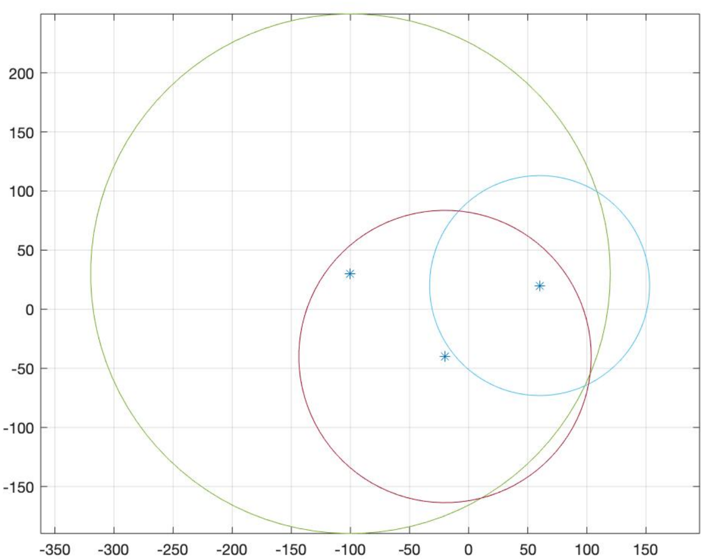
$X = R^T(Y - t) = R^TY - R^Tt$ (Rotation matrix fulfills that $RR^T = I$)

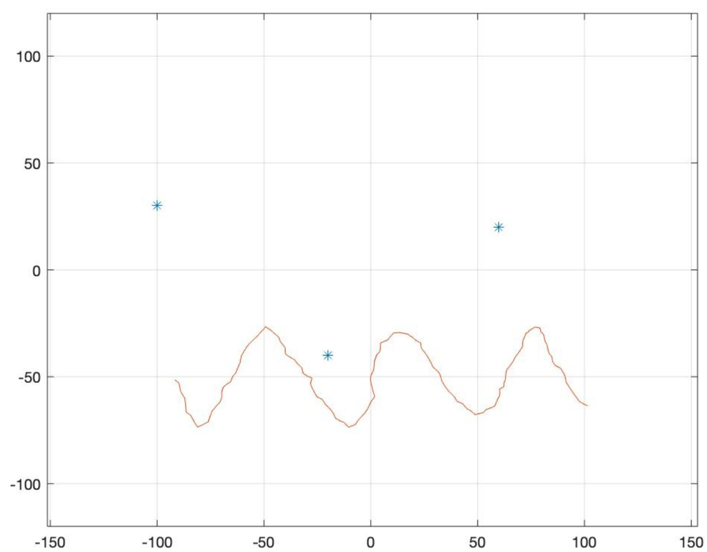
Since rotation matrix is $R^T$and the inverses translation is $-R^Tt$

Thus, $T^{-1} = \begin{bmatrix} R^T & -R^Tt \\ 0 & 1 \end{bmatrix}$.

10.Rigid Body Transformation

Process model: Random Walk. The point of three circles are the initial point.





The right plot indicates the path of the robot.