EECS 568 Mobile Robotics HW1

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Robotics HW 1

Task 1

A.

$$A^T + A = \frac{A^T+A}{2} + \frac{A^T+A}{2}$$

$$E[X] = \int X\, dP = \int_X x p(x) dx$$

$$\therefore E[X+Y+Z] = \int_\Omega (X+Y+Z)\, dP = \int_X x p(x) dx + \int_Y y p(y) dy + \int_Z z p(z) dz$$
$$= E[X] + E[Y] + E[Z]$$

B. If it's valid $\Leftrightarrow x^T A x \geq 0$. positive semi definite. (Using Schur Complement Thm)

By Schur, $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a > 0$ & $\det(M) > 0$. All e-values are non-negative.

(a) $\det(M) > 0$, $8 > 0$ $\therefore$ Valid

(b) $\det(M) < 0$ $\therefore$ Invalid

(c) Not symmetric $\therefore$ Invalid

(d) $\det(M) < 0$ $\therefore$ Invalid

(e) $\det(M) > 0$ $\therefore$ Valid

$$\Sigma_r = \begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} = E\{(X-\mu_x)(X-\mu_x)^T\} = E\{XX^T - \mu_x X^T - X\mu_x^T + \mu_x\mu_x^T\}$$
$$= E\{XX^T\} - E\{\mu_x X^T + X\mu_x^T\} + E\{\mu_x\mu_x^T\}$$
$$= E\{XX^T\} - \mu_x\mu_x^T - \cancel{\mu_x\mu_x^T} + \cancel{\mu_x\mu_x^T}$$

~~Un~~Biased

C.

(a) $\therefore \begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} = E\{XX^T\} - \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ $\therefore E\{XX^T\} = \begin{bmatrix} 9 & 9 \\ 9 & 16 \end{bmatrix} \Rightarrow \sum XX_r^T = \begin{bmatrix} 36 & 36 \\ 36 & 64 \end{bmatrix}$

(b) $\therefore \begin{bmatrix} 8 & 4 \\ 4 & 3 \end{bmatrix} = E\{XX^T\} - \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$ $\therefore E\{XX^T\} = \begin{bmatrix} 12 & 0 \\ 0 & 7 \end{bmatrix} \Rightarrow \sum XX_m^T = \begin{bmatrix} 72 & 0 \\ 0 & 42 \end{bmatrix}$

(c) $\mu_{m+r} = (4\mu_r + 6\mu_m)/10 = \begin{bmatrix} -0.4 \\ 2.4 \end{bmatrix}$

(d) $\Sigma_{r+m} = ?$ $E\{XX^T\} = \Sigma_{r+m} + \mu_{r+m}\mu_{r+m}^T \Rightarrow \Sigma_{r+m} = \sum(XX_r^T + XX_m^T) - \mu_{r+m}\mu_{r+m}^T$

$$= \begin{bmatrix} 108 & 36 \\ 36 & 106 \end{bmatrix} - \begin{bmatrix} 0.16 & -0.96 \\ -0.96 & 5.76 \end{bmatrix} = \begin{bmatrix} 10.64 & 4.56 \\ 4.56 & 4.84 \end{bmatrix}$$

D. (a) By DeMorgan's Law. If A, B are independent $P(A \cap B) = P(A) \cdot P(B)$,

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 - (1 - P(\bar{A}) + 1 - P(\bar{B}) - P(A) \cdot P(B))$$
$$= 1 + P(\bar{A}) - P(\bar{B}) - (1 - P(\bar{A}))(1 - P(\bar{B})) = P(\bar{A}) \cdot P(\bar{B}) \quad \#$$
$$1 - P(\bar{A}) - P(\bar{B}) + P(\bar{A})P(\bar{B})$$

(b) Affine transformation is given by $y = Ax + b$.

$Y = AX + b$, $X \sim N(\mu, \Sigma)$

$\mu_Y = A\mu_x + b$, $\mu_Y = E(Y) = E(Ax + b) = AE(x) + b = A\mu_x + b$

$$\Sigma_{YY} = E[(Y-\mu_Y)(Y-\mu_Y)^T] = E[A(X-\mu_x)(X-\mu_x)^T A^T]$$
$$= A \cdot E[(X-\mu_x)(X-\mu_x)^T] A^T = A\Sigma A^T$$

$Y \sim N(A\mu + b, A\Sigma A^T)$ is still Gaussian since the transformation of mean and cov are all linear

Task 2.

A. $p(C \mid \text{positive}) = ? = \frac{p(\text{positive} \mid C) \cdot p(C)}{p(\text{positive})} = \frac{0.9\% \times 0.01}{0.2 \times 0.99 + 0.9 \times 0.01} = 0.0434$

false positive: 20%   w/ cancer 1%

false negative: 10%   w/o cancer 99%

$= \frac{(1-A) \times 0.01}{B \times 0.99 + (1-A) \times 0.01} = \frac{0.01 - 0.01A}{0.99B - 0.01A + 0.01}$

→ Reducing the false positive rate (B), it'll make $p(C \mid \text{positive})$ bigger.

Sol 1:

B. $p(X=1 \mid u=1) = 0.9$ $p(X=0 \mid u=1) = 0.1$

$p(z=1 \mid X=0) = 0.2$ $p(z=0 \mid X=0) = 0.8$

$p(z=1 \mid X=1) = 0.7$ $p(z=0 \mid X=1) = 0.3$

99

$\overline{Bel} = P(\;\;)(X_0 = 1)$

0.45

0.05

Prediction Step:

$\overline{bel}(X_{t+1} = \text{colored})$

$= p(\text{colored} \mid u_{t+1} = \text{paint}, X_t = \text{blank}) \times bel(X_t = \text{blank}) +$ (0.9, 0.5)

$p(\text{colored} \mid u_{t+1} = \text{paint}, X_t = \text{colored}) \times bel(X_t = \text{colored}) = 0.5 \times 1.9 = 0.95$ (1, 0.5)

✓ $\overline{bel}(X_{t+1} = \text{blank})$

$= p(\text{blank} \mid u_{t+1} = \text{paint}, X_t = \text{blank}) \times bel(X_t = \text{blank})$ ((1−0.9), 0.5)

$p(\text{blank} \mid \text{"} \;\; \text{"}, X_t = \text{colored}) \times bel(X_t = \text{colored}) = 0.05$ (0, 0.5)

Update Step:

$bel(X_{t+1} = \text{colored})$

$= \eta\, p(z = \text{sense\_colored} \mid X_{t+1} = \text{colored}) \cdot \overline{bel}(X_{t+1} = \text{colored}) = \eta \times 0.7 \times 0.95$ (0.7, 0.95)

$bel(X_{t+1} = \text{blank})$

$= \eta\, p(z = \text{sense\_colored} \mid X_{t+1} = \text{blank}) \cdot \boxed{\overline{bel}(X_{t+1} = \text{blank})} = \eta \times 0.2 \times 0.05$ (0.2, 0.05)

$\therefore \eta (0.7 \times 0.95 + 0.2 \times 0.05) = 1$ $\therefore \eta = 1.48$

$\therefore bel(X_{t+1} = \text{blank}) = 1.48 \times 0.01 = 0.0148 \doteq 1.48\%$

Sol 2:

Prediction: $p(X_{t+1} = 0) = p(X_{t+1} = 0 \mid u_{t+1} = 1, X_t = 0) \times p(X_t = 0) +$ ((1−0.9), 0.5)

$p(X_{t+1} = 0 \mid u_{t+1} = 1, X_t = 1) \times p(X_t = 1) = 0.05$ (0, 0.5)

Update: $p(X_{t+1} = 0 \mid z = 1) = \frac{p(z=1 \mid X_{t+1} = 0) \times \boxed{p(X_{t+1} = 0)}}{\Sigma(z = 1 \mid X_{t+1})} = \frac{0.2 \times 0.05}{(0.95 \times 0.7 + 0.2 \times 0.05)}$

$= 1.48\%$

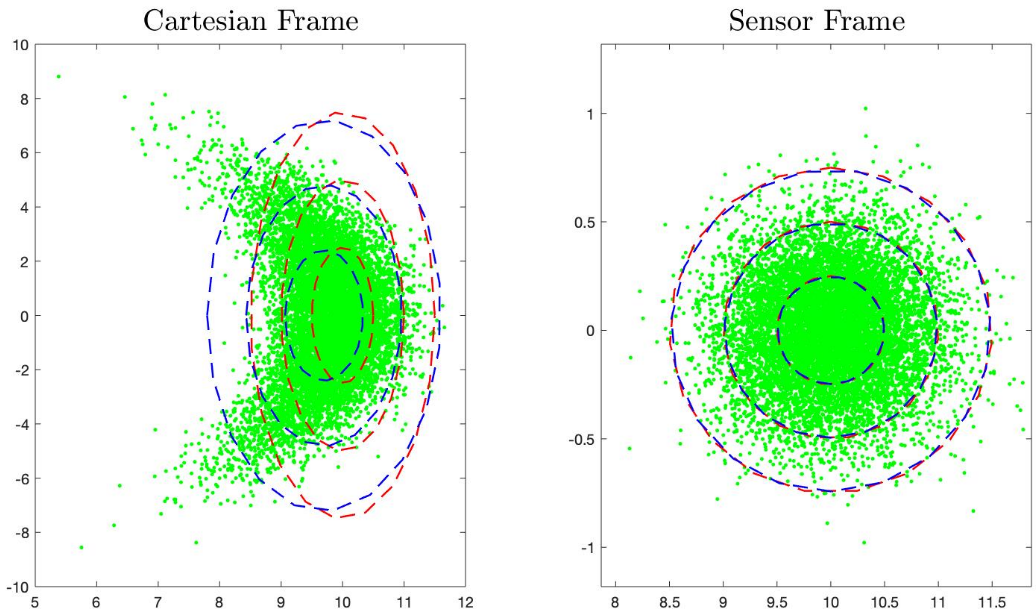
2. C

Place	Probability
0	0.1
1	0.75
2	0.75
3	0.05
4	0.15
5	0.075
6	0.025
7	0.3
8	0.075
9	0.075

Task 3

A, C.

Since the analytical Cartesian frame is the frame being linearized, the red(calculated covariance and mean) and blue(sampled covariance and mean) ellipse wouldn't match. It would lose some information when linearized.



B.

After linearization, we should find out the Jacobian matrix which is as following:

B.

$f(0)$

$$\begin{bmatrix}X\\Y\end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta\end{bmatrix} \xrightarrow{\text{linearized}} \begin{bmatrix}x\\Y\end{bmatrix} = \begin{bmatrix} r\cos\theta\\ r\sin\theta\end{bmatrix}\Bigg|_{\substack{r=\mu_r\\ \theta=\mu_\theta}} + \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}\Bigg|_{\substack{r=\mu_r\\ \theta=\mu_\theta}}\left(\begin{bmatrix}r\\ \theta\end{bmatrix}-\begin{bmatrix}\mu_r\\ \mu_\theta\end{bmatrix}\right)$$

Given that:

$$Cov_{r\theta} = \begin{bmatrix}0.5^2 & 0\\ 0 & 0.25^2\end{bmatrix} \qquad \mu_{r\theta} = \mu.$$

$$\Downarrow J.$$

$$Cov_{xy} = J\cdot\Sigma\, J^T, \qquad \mu_{xy} = J\mu.$$

$$= \begin{bmatrix}0.25 & 0\\ 0 & 6.25\end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$$

D.

By using Mahalanobis distance:

$$D = \sqrt{(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

Count the data point within 1,2,3 σ :

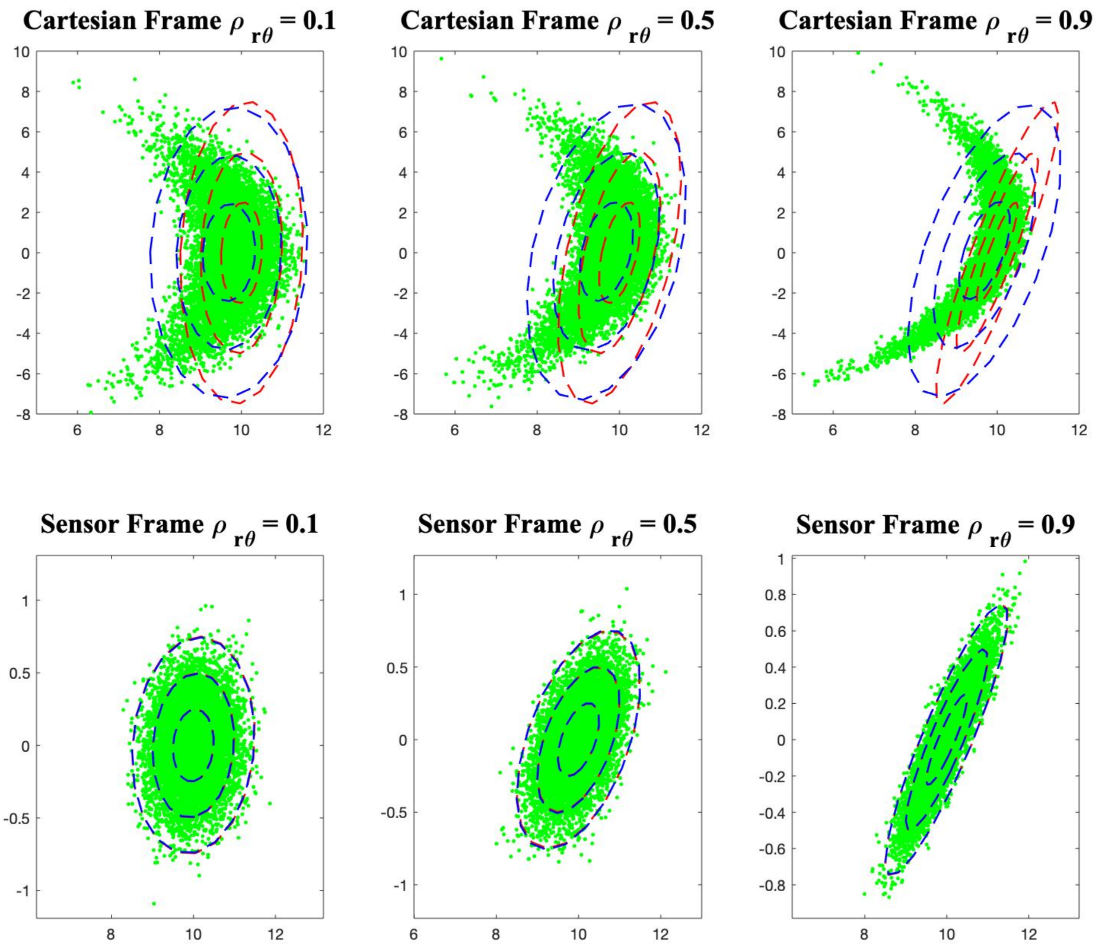
	σ	2σ	3σ
Sensor Frame	39.92%	87.16%	98.92%
Cartesian Frame	38.75%	79.95%	93.62%

E.

	σ	2σ	3σ
Theory	39.35	86.47	98.89
$\sigma_r = 0.5$ $\sigma_\theta = 0.25$	38.75	86.04	98.84
$\sigma_r = 0.000005$ $\sigma_\theta = 0.000025$	39.33	86.45	98.88
$\sigma_r = 0.5$ $\sigma_\theta = 0.00025$	39.40	86.97	98.92
$\sigma_r = 0.00005$ $\sigma_\theta = 0.25$	39.01	86.03	98.80
$\sigma_r = 5000$ $\sigma_\theta = 25000$	38.11	85.88	98.75

When decreasing the variance, the theoretical value and the real counted value would match. It is because if the variance is small, there would be little information to lose when doing linearization.

F.



This question utilized different covariance matrix with $\rho_{r\theta} = 0.1,\ 0.5,\ 0.9$ to plot the sensor frame and cartesian frame. Covariance values were then calculated:

$$\sigma_{r\theta} = \rho_{r\theta} \times \sigma_x \sigma_y \, .$$

Also, we could draw the sample with the following equations:

$$Z = LX + \mu$$

X is just another random value and $\Sigma = LL^T$. By plugging in the L value, we could produce new data point and plot as the new Sensor Frame in the above figure. The data is no longer independent within their two coordinates. When transformed by Jacobian, the sample-based covariance contours are being off with the calculated contours. As long as the correlation value gets bigger, the red and blue contour differs even further. This is because the off-diagonal value of covariance matrix leads to the dependent

Task 4. (MAP) random mean μ, fixed variance: σ^2

A. Only one point:

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)} = f(x|\theta) = L(\theta),$$

To find θ_{MAP}:

→ The distribution of θ (In this case, there's only μ of the data) which is shown by the Gaussian Distribution.

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}}\, f(\theta|x) = \operatorname{argmax} f(x|\theta)\, f(\theta)$$

$$= \operatorname{argmax} \prod_{i=1}^{N}\left[\frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i-\theta}{\sigma_x}\right)^2\right)\right] \cdot \frac{1}{\sigma_0\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\theta-\mu_0}{\sigma_0}\right)^2\right)$$

$$0 = \frac{1}{\sigma_0\sigma_x\cdot 2\pi} \cdot \exp\left[-\frac{1}{2}\left[\sum\left(\frac{x_i-\theta}{\sigma_x}\right)^2 + \left(\frac{\theta-\mu_0}{\sigma_0}\right)^2\right]\right] \times \left[-\left(\frac{x_i-\theta}{\sigma_x^2}\right) - \left(\frac{\theta-\mu_0}{\sigma_0^2}\right)\right] \Rightarrow \sum\left[\frac{x_i-\theta}{\sigma_x^2}\right] = -\frac{\theta-\mu_0}{\sigma_0^2}$$

$$\Rightarrow \frac{\theta-\mu_0}{\sigma_0^2} = -\frac{\sum_{i=1}^{N} x_i}{\sigma_x^2} - \frac{N\theta}{\sigma_x^2} \Rightarrow \theta_{MAP} = \frac{\sigma_x^2\mu_0 + \sigma_0^2\sum_{i=1}^{N} x_i}{\sigma_x^2 + N\sigma_0^2} = \hat{\mu}$$

When $N=1$

$$= \frac{\sigma_x^2\mu_0 + \sigma_0^2\mu_x}{\sigma_x^2+\sigma_0^2} \quad \#$$

B. Because $f(x|\theta)$ and $f(\theta)$ are both Gaussian distribution:

$$f(\theta|x) = \underbrace{f(x|\theta)}_{\text{Likelihood}} \cdot \underbrace{f(\theta)}_{\text{prior}} = \left(\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{x_i-\theta}{\sigma_x}\right)^2}\right)\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\left(\frac{\theta-\mu_0}{\sigma_0}\right)^2}$$

$$\propto \left(\prod_{i=1}^{N} e^{-\frac{1}{2}\left(\frac{x_i-\theta}{\sigma_x}\right)^2}\right) \cdot e^{-\frac{1}{2}\left(\frac{\theta-\mu_0}{\sigma_0}\right)^2} = e^{-\frac{1}{2}\left(\sum_{i=1}^{N}\left(\frac{x_i-\theta}{\sigma_x}\right)^2\right) - \frac{1}{2}\left(\frac{\theta-\mu_0}{\sigma_0}\right)^2}$$

$$= e^{-\frac{\sigma_0^2\left(\sum_{i=1}^{N}(x_i-\theta)^2\right) + \sigma_x^2(\theta-\mu_0)^2}{2\sigma_x^2\sigma_0^2}} = e^{-\frac{1}{2\sigma_x^2\sigma_0^2}\left(\left[\theta\sqrt{N\sigma_0^2+\sigma_x^2} - \frac{\sigma_x^2\mu_0 + \sigma_0^2\sum_{i=1}^{N} x_i}{\sqrt{N\sigma_0^2+\sigma_x^2}}\right]^2\right)} \cancel{e}$$

to fit a Gaussian Model.

$$= e^{-\frac{N\sigma_0^2+\sigma_x^2}{2\sigma_x^2\sigma_0^2}\left(\left[\theta - \frac{\sigma_x^2\mu_0 + \sigma_0^2\sum_{i=1}^{N} x_i}{N\sigma_0^2+\sigma_x^2}\right]^2\right)}$$

$\therefore$ mean $\hat{\mu} = \frac{\sigma_x^2\mu_0 + \sigma_0^2\sum_{i=1}^{N} x_i}{N\sigma_0^2+\sigma_x^2}$, $\sigma = \frac{\sigma_x^2\sigma_0^2}{N\sigma_0^2+\sigma_x^2}$ #

$(N=1)$ $= \frac{\sigma_0^2\mu_x + \sigma_x^2\mu_0}{\sigma_0^2+\sigma_x^2}$, $\sigma = \frac{\sigma_x^2\sigma_0^2}{\sigma_0^2+\sigma_x^2}$

C. Table I

1	2	3	4	5
(10.6608, 0.9998)	(9.7276, 0.4995)	(10.057, 0.333)	(10.45, 0.249)	(10.458, 0.2)
6	7	8	9	10
(10.554, 0.167)	(10.57, 0.143)	(10.46, 0.12)	(10.448, 0.110)	(10.325, 0.1)

Table. II

1	2	3	4	5
(10.7038, 0.6396)	(10.0715, 0.2496)	(9.9894, 0.4618)	(9.5710, 0.4)	(9.7089, 0.3577)

$0.1 < 0.35$

Having more data in sensor I, the variance after ten measurements of table I is smaller than table II with five measurements. There's no "ground truth" of it. So what we discuss is the "precision", which only relates to the value of variance.

$\therefore$ The precision of I is bigger than II #