NA 568 Mobile Robotics: Methods & Algorithms (W2020) Midterm

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This exam is open-book: Calculators may be used only for arithmetic calculations. Write your uniqname at the top of every page.

You are to abide by the University of Michigan/Engineering Honor Code. To receive a grade, please sign below to signify that you have kept the honor code pledge:

> *"I have neither given nor received aid on this examination, nor have I concealed any violations of the Honor Code."*

Name: PoKang Chen

Uniqname: pkchen

Signature: PoKang Chen

Problem 1	___________ out of 10
Problem 2	___________ out of 10
Problem 3	___________ out of 4
Problem 4	___________ out of 6
Problem 5	___________ out of 10
Problem 6	___________ out of 4
Problem 7	___________ out of 10
Problem 8	___________ out of 15
Problem 9	___________ out of 10
Problem 10	___________ out of 21
Total:	___________ **out of 100**

Do not write here.

1 Problem (10 points). Probability and Uncertainty Propagation

Answer the following. Any proofs should be succinctly and clearly written. Include your final copy of the proof.

1.1 (3 points)

Explain how a probability distribution can be propagated through a nonlinear function. Elaborate on the methods and reason why they work.

1.2 (3 points)

A range finder sensor provides noisy measurements $z \sim \mathcal{N}(\mu_z, \sigma_z^2)$. A corrected model for calibrating the sensor is $y = az + b$. Derive the distribution over y. Is the noisy model for y an approximation or exact? Why?

1.3 (4 points)

A projection model from the 3D Cartesian space, $\mathbb{R}^3$, to a 2D image is given by

$$p = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_x \frac{x}{z} + c_x \\ f_y \frac{y}{z} + c_y \end{bmatrix} \in \mathbb{R}^2, \quad \text{and} \quad l = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

where f_x and f_y are focal lengths in x and y directions, respectively, and c_x and c_y are the coordinates of optical center of the image in pixels. Suppose a 3D landmark $l \in \mathbb{R}^3$ is observed via a noisy sensor, i.e., $l \sim \mathcal{N}(\mu_l, \Sigma_l)$. Derive an analytical model for the mean and covariance of p. Show work and provide necessary equations.

2 Problem (10 points). Probability and Uncertainty Propagation

Suppose you are using a Right-Invariant EKF to perform robot localization in 3D and the ground truth is given in coordinates using x, y, z for position and roll, pitch, yaw for the orientation with respect to x, y, and z axes, respectively. The Right-Invariant EKF error and covariance are in the Lie algebra of $\mathrm{SE}(3)$. Derive a model to map the error and its covariance from $\mathfrak{se}(3)$ to $\mathbb{R}^6$ (x, y, z, roll, pitch, yaw). Show work and provide necessary equations for mapping the error and covariance from the Lie algebra to the Cartesian space.

3 Problem (4 points). Normal Random Variables

If X and Y are normal random variables with means μ_x and μ_y and variances σ_x and σ_y, which of the following statements is/are necessarily true.
Answer **T** / **F** / **<blank>** to each question below. Correct answers earn one point; blanks earn zero; incorrect answers earn minus one.

______ $\sigma_{X|Y} = \sigma_X$.

______ $\sigma_{X|Y} <= \sigma_X$.

______ $\sigma_{X|Y} <= \sigma_X$ or $\sigma_{X|Y} >= \sigma_X$ depending on the correlation between X and Y.

______ $\sigma_{X|Y} <= \sigma_X$ and the value of $\sigma_{X|Y}$ does not depend on the cross-covariance between X and Y.

4 Problem (6 points). Estimation and Mapping

Answer **T** / **F** / **<blank>** to each question below. Correct answers earn one point; blanks earn zero; incorrect answers earn minus one.

______ Markov localization is just a different name for the grid-based Bayes filter applied to robot localization.

______ In Kalman filter, the Kalman gain is independent of the state estimate at that time.

______ The state estimate obtained using a non-linear filter is a global minimum.

______ When solving the linear least squares problem the solution is a global minimum.

______ The error dynamics of an Invariant EKF are dependent on the state estimate.

______ The maximum expected utility principle states that the robot should choose the action that maximizes its expected utility, in the current state.

5 Problem (10 points). Bayes Filter

Based on last year's test scores the following is the probability of certain scores.

$$A = 30\% \;\; B = 60\% \;\; C = 10\%$$

At this point in the exam you typically have a good idea of how its going. Your internal barometer tells you that it is going great, this measurement indicates that the following grade distributions apply to you.

$$A = 70\% \;\; B = 25\% \;\; C = 5\%$$

With the prior distribution and internal barometer measurement calculate the posterior probability of getting an A,B or C on the exam.

6 Problem (4 points). Cholesky Decomposition

For each of the matrices below, does the matrix have a unique Cholesky decomposition? If so, compute the *lower triangular* matrix L from the decomposition by hand (show work). If not, state why. Hint: $M = LL^\top$.

$$\begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix}$$ **Cholesky** **No Cholesky** **Reason or L matrix:** ____________________

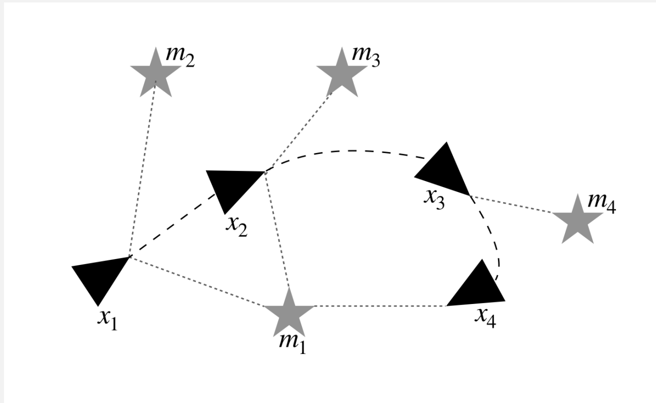
$$\begin{bmatrix} 9 & -10 \\ -10 & 11 \end{bmatrix}$$ **Cholesky** **No Cholesky** **Reason or L matrix:** ____________________

$$\begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix}$$ **Cholesky** **No Cholesky** **Reason or L matrix:** ____________________

$$\begin{bmatrix} 9 & -5 \\ -3 & 5 \end{bmatrix}$$ **Cholesky** **No Cholesky** **Reason or L matrix:** ____________________

7 Problem (10 points). Graph SLAM

Consider a robot moving (with odometry information) and observing landmarks as shown.



7.1 Problem (4 points).

Draw the non-zero block structure of the information matrix at each step (after obtaining measurements).

7.2 Problem (6 points).

Draw the final non-zero block structure of the information matrix for *pose-graph* SLAM of the above example. If the robot did not have odometry information, what would the final pose-graph information matrix look like?

Problem 8 continues ...

8 Problem (15 points). Target Tracking

A target is moving in a 1D plane. The ownship position is known and fixed at the origin, thus the target's position at time step k is x_k. We have no information on the target's trajectory and thus we model the motion as a random walk with variance of 4 m^2. Based on previous measurements, we have an initial guess of the target position as $\mu_0 = 20 \text{ m}$ with variance $\sigma_0^2 = 9 \text{ m}^2$. We then measurement from each of the next two time steps $z_1 = 22 \text{ m}$ and $z_2 = 23 \text{ m}$. Each with variance $1 \ \text{m}^2$.

8.1 Problem (6 points). Extended Kalman Filtering

Use a Kalman filter to estimate the state of the target at time step 2 (μ_2 and σ_2^2).

8.2 Problem (6 points). Linear Least Squares

Write out the linear least squares equations for estimating the target trajectory (smoothing). Fill in the matrices with numbers/symbols where appropriate. Also write the equation to calculate the trajectory, $\hat{x}$ (you do not need to evaluate the numeric solution).

8.3 Problem (3 points).

Explain briefly the difference between trajectories estimated using the Kalman filter and least squares. Do either of these solutions produce a globally optimal result?

9 Problem (10 points). Rigid Body Transformation

9.1 Problem (3 points)

What is the main difference between a right and left Invariant EKF? Which types of measurements should be used for left and right Invariant EKFs, respectively?

9.2 Problem (2 points)

Write the lie algebra of $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$, both in vector and skew symmetric form. Explain the physical meaning behind the dimension of each lie algebra.

9.3 Problem (3 points)

Write out the continuous-time constant velocity motion model for an object moving in $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$, respectively. Assuming a zero-order hold and a sampling time Δt for the input, write an integration rule for each model.

9.4 Problem (2 points)

Show that:

$$\begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

is the inverse of a homogeneous rigid body transformation.

10 Problem (21 points). Coding

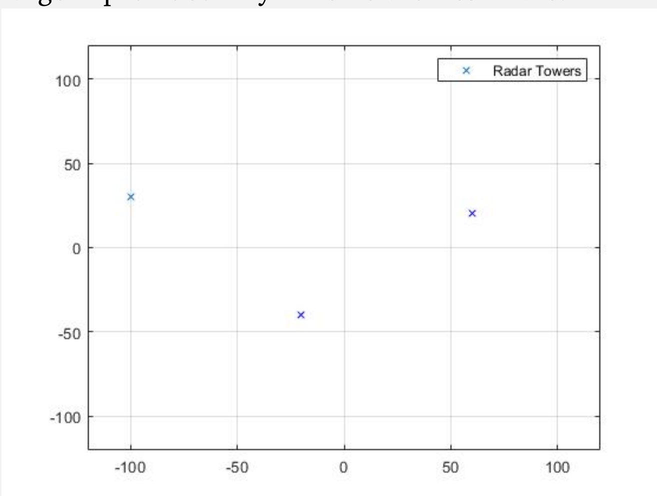
A group of three radar stations provides range measurements to a target in the map. The map limits are x = [-120 120] and y = [-120 120]. The radar stations global x and y locations are as follows,

$$x_1 = -100 \;\; y_1 = 30 \;\; x_2 = 60 \;\; y_2 = 20 \;\; x_3 = -20 \;\; y_3 = -40$$

At each time instance, three measurements are obtained (data found in measurements.mat), these are the range from tower 1, 2 and 3 respectively. Use a particle filter to track the target given the range measurements. The uncertainty in the range measurement is as follows,

$$\sigma^2 = z^{1/3}$$

If the range is less than 25 , the uncertainty $\sigma^2 = 1$. Plot the trajectory of the target provided by the state estimate. Include plot and MATLAB code in submission.



End of Exam.