Steady States Stability of Reaction-Diffusion Equation with periodic source term

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Poincaré's Inequality

Theorem (Poincaré)

For every function f of the Sobolev space $W^{1,2}$ with mean-zero (e.g.$\langle f \rangle = 0$), there exist a constant C such that

$$\|f\|_2 \leq C \|\frac{\mathrm{d}f}{\mathrm{d}x}\|_2$$

Poincaré's inequality for Bloch funcitons

Theorem (Bloch Poincaré)

Let $f(x)$ be an L-periodic mean-zero funcition, $p \in [-\kappa/2, \kappa/2]$. Then,

$$\|\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}px} f(x)\|_2 \geq \left||p| - \kappa\right| \, \|f\|_2$$

PI for Bloch funcitons

We can express f as a fourier series

$$f(x) = \sum_{j=-\infty}^{\infty} a_j \mathrm{e}^{\mathrm{i}\kappa j x} \overset{\text{mean-zero}}{=} \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} a_j \mathrm{e}^{\mathrm{i}\kappa j x}, \qquad \kappa = 2\pi/L.$$

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The L2 norm is given by

$$\|f\|_2^2 = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} |a_j|^2.$$

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Differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}px}f(x) = \mathrm{e}^{\mathrm{i}px}\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty}\mathrm{i}(p+\kappa j)a_j\mathrm{e}^{\mathrm{i}\kappa jx},$$

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with norm

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Reaction-Diffusion equation

A nonlinear reaction-diffusion equation takes the form of

$$\frac{\partial C}{\partial t} = \sigma(x) + N(C) + \frac{\partial^2 C}{\partial x^2}$$

where $N(C)$ is the non-linear term and $\sigma(x)$ is the forcing term, here we consider $\sigma(x)$ to be periodic.

Reaction-Diffusion equation

Suppose we have a periodic steady-state solution $C_0(x)$ such that $\partial_t C_0 = 0$. To determine the stability of the solution, we consider

$$C(x,t) = C_0(x) + \delta C(x,t). \tag{1}$$

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or just

$$\frac{\partial \phi}{\partial t} = s(x)\phi + \frac{\partial^2 \phi}{\partial x^2}. \tag{2}$$

Reaction-Diffusion equation

A basis for the solution space of (2) is given by the set of eigenfunctions

$$\{\psi_{np}(x) = \mathrm{e}^{\mathrm{i}px} u_{np}(x)\}_{n\in\{0,1,2,\ldots\},\ p\in[-\kappa/2,\kappa/2]},$$

where each $u_{np}(x)$ is L-periodic and satisfies an eigenvalue relation

$$-\lambda_{np} u_{np} = \mathcal{D} u_{np}, \qquad \mathcal{D} = \left[s(x) - p^2 + 2\mathrm{i}p\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} \right].$$

Reaction-Diffusion equation

Therefore, the solution can be expressed as

$$\phi(x,t) = \int_{-\kappa/2}^{\kappa/2} \left[\sum_{n=0}^{\infty} \langle \psi_{np}, \phi_0 \rangle \psi_{np}(x) \mathrm{e}^{-\lambda_{np} t} \right] \mathrm{d}p.$$

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For the solution to be bounded for all $t \geq 0$ we require $\lambda_{np} \geq 0$ for all n and p.

Condition for stability

To be able to apply our Poincaré's inequality, we look at the eigenvalue relation

$$-\lambda_{np} u_{np} = \mathcal{D} u_{np}, \qquad \mathcal{D} = \left[s(x) - p^2 + 2\mathrm{i}p \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} \right]. \tag{3}$$

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Taking the inner product gives

$$\lambda_{np} \| u_{np} \|_2^2 = -\langle u_{np}, s u_{np} \rangle + \| \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\mathrm{i}px} u_{np} \|_2^2. \tag{4}$$

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$\lambda_{np} \geq 0$ when $s(x) \leq 0$ for all $x \in [0, L]$. (Trivial case)
Hence, we focus on cases where $s(a) \geq 0$ for some $a \in [0, L]$ but $\langle s \rangle \leq 0$.

Condition for stability

Equation (4) can be rewritten as

$$\lambda_{np}\|\delta u_{np}\|_2^2 = \left[\langle \delta u_{np}, s\delta u_{np}\rangle + \|\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}px}\delta u_{np}\|_2^2\right] + \frac{|\langle \delta u_{np}, \delta s\rangle|}{\lambda_{np} - (|\langle s\rangle| + p^2)}$$

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The solutions of λ is greater than zero if

$$a \geq 0, \qquad \text{and} \qquad b \leq ac. \tag{5}$$

Condition for stability

Condition 1: $a \geq 0$

$$a = -\langle \delta u_{np}, s\delta u_{np} \rangle + \| \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\mathrm{i}px} \delta u_{np} \|_2^2,$$

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Apply Poincaré's inequality for Bloch function

$$a \geq \left[-s_0 + (|p| - \kappa)^2 \right] \| \delta u_{np} \|_2^2, \qquad p \in [-\kappa/2, \kappa/2]. \tag{6}$$

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Hence, a is greater than zero if

$$
s_0 \leq \kappa^2/4.
$$

Condition for stability

Condition 2: $b \leq ac$

$$|\langle u_{np}, \delta s \rangle| \leq \left[-\langle \delta u_{np}, s \delta u_{np} \rangle + \| \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\mathrm{i}px} \delta u_{np} \|_2^2 \right] (|\langle s \rangle + p^2|)$$

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By Cauchy-Schwarz inequality

$$\|\delta u_{np}\|_2^2\|\delta s\|_2^2 \leq \left[-s_0\|\delta u_{np}\|_2^2 + \|\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\mathrm{i}px}\delta u_{np}\|_2^2\right](|\langle s\rangle + p^2|)$$

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$$\|\delta s\|_2^2 \leq \left[-s_0 + (|p| - \kappa)^2\right](|\langle s\rangle + p^2|)$$

Summary

The steady-state solution is stable if

(I) $s(x) \leq 0$ for all $x \in [0, L]$.

(II) $\langle s \rangle \leq 0$ but $s(a) \geq 0$ for some $a \in [0, L]$, then we require two more conditions:

(a) $s_0 \leq \kappa^2/4$

(b) $\|\delta s\|_2^2 \leq \min_{p \in [-\kappa/2, \kappa/2]} \left[-s_0 + (|p| - \kappa)^2\right] (|\langle s \rangle + p^2|)$

Result

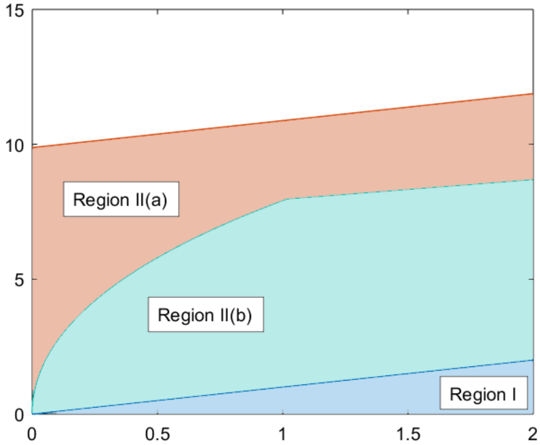

Figure: Region of stability for $N'(C_0) = s(x) = -\alpha + \beta\cos(\kappa x)$.