Where there is amplification, there are sunspots*

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Abstract

We provide a complete analysis of sunspot equilibria in a canonical dynamic economy with financing constraints. Under financial frictions, the wealth distribution modulates the asset ownership distribution. Supposing, in addition, the ownership distribution matters for asset prices, self-fulfilling fluctuations emerge: if agents suddenly become fearful about asset volatility, ensuing fire sales shift allocative efficiency, creating an asset-price decline that justifies the fear. A supplementary complete-market representative-agent example with nominal rigidities illuminates the mechanism. Sunspot equilibria can help resolve several puzzling patterns, permitting arbitrary amounts of volatility, a decoupling of crisis fluctuations from fundamentals, and faster crisis recovery.

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It has by now become commonplace, especially after the 2008 global financial crisis, for macroeconomic models to prominently feature banks, limited participation, imperfect risk-sharing, and other such "financial frictions." Incorporating these features allows macroeconomists to speak meaningfully about financial crises and desirable policy responses. The universal feature of these models is *amplification*: small exogenous shocks to fundamentals can create large endogenous movements in economic observables. Despite the dramatic growth in this literature, there remains a sizeable disconnect between the predictions of such models and actual crisis data. Among other things, the literature has had difficulty reproducing the observed severity and suddenness of economic downturns and asset-price dislocations. One might say that the degree of amplification in the models has been insufficient.[1]

We propose *sunspots* as a potential resolution. Throughout our paper, sunspots refer to non-fundamental fluctuations that only occur because agents expect them. This paper makes four main contributions. First, we show that standard financial friction models are already subject to sunspot volatility, with no further assumptions. In this sense, the very same financial frictions activating amplification also provide scope for sunspots. Second, we provide a near-complete characterization of potential sunspot equilibria, by leveraging some new methodological tools. Third, we demonstrate how sunspot equilibria alleviate some of the empirical shortcomings for this class of models. Fourth, we provide a supplementary complete-market representative-agent model which nevertheless exhibits sunspot fluctuations, to clarify the culprits of the multiplicity we document.

Model and mechanism. We study a simple stripped-down model with financial frictions, similar to Kiyotaki and Moore (1997), Brunnermeier and Sannikov (2014), and many others.[2] There are two types of agents ("experts" and "households") with identical preferences but different levels of productivity when managing capital. Heterogeneous

[1]For example, Gertler and Kiyotaki (2015) and Gertler et al. (2020) attempt to integrate bank runs into a conventional financial accelerator model, in order to capture additional amplification and non-linearity (in particular, the suddenness with which financial systems collapse). As a more fundamental critique, financial accelerator models have a difficult time inducing the financial intermediary leverage and risk concentration needed to generate large amounts of amplification. This can be seen in Di Tella (2017), in which the retirement rate of bankers/experts needs to be calibrated to 115% per year, or in Khorrami (2018), in which the implied entry costs needed to match crisis dynamics of asset prices are on the order of 90% of wealth. Krishnamurthy and Li (2020) and Maxted (2020) build a sentiment process on top of a relatively standard financial accelerator model, in order to help address some of these issues, in particular the onset of crises even in absence of fundamental shocks. The similarity to this paper is that their "behavioral sentiment" is replaced by a rational notion of sentiments.

[2]We work in continuous time, contributing to a burgeoning financial frictions literature. See also He and Krishnamurthy (2012, 2013, 2019), Moreira and Savov (2017), Klimenko et al. (2017), and Caballero and Simsek (2020c). For a survey, see Brunnermeier and Sannikov (2016).

productivity means the identity of capital holders matters for aggregate output. But incomplete markets prevent agents from sharing risks associated to their capital holdings, so optimal capital holdings depend to some degree on individual wealth. There are no other features: no borrowing constraints, no default externalities, and no irrational beliefs. And yet, this basic model can feature a tremendous amount of multiplicity that has been overlooked in the literature.

To understand why, let us examine Figure 1 in detail. The left panel shows the static equilibrium in the capital market. Capital supply is upward-sloping because experts' capital provision (higher κ) raises allocative efficiency, creating higher aggregate capital valuation (higher q). Capital demand is downward-sloping because of risk and financial frictions: for a fixed amount of price volatility $\sigma_q > 0$, experts will only hold more capital if it is cheaper, thereby offering a higher return. The intersection of these two curves determines the capital distribution and price (κ and q), for any fixed wealth distribution and fixed volatility σ_q.

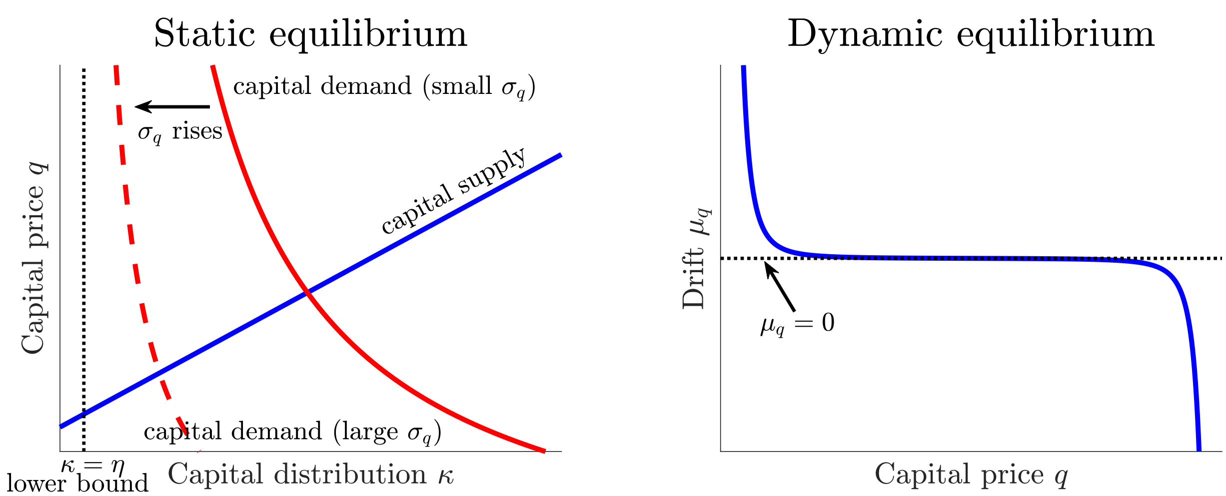

Figure 1: Sunspot mechanism. *Left:* the equilibrium in the capital market at each point in time. The variable κ represents the capital share of "experts" (efficient managers of capital). The capital price is denoted by q. Capital supply represents goods market clearing. Capital demand aggregates agents' portfolio choices. A volatility increase shifts capital demand inward, leaving capital supply fixed. *Right:* the dynamics of capital prices ensure stability, by pushing prices up if they are too low, and vice versa.

Suppose there is a sudden rise in *fear*, which manifests as higher perceived volatility σ_q. Experts, being risk-averse, are less willing to hold capital when volatility is high. This is illustrated as a leftward shift in the capital demand curve from the solid to the dashed line. The new allocation, after this "fire sale," features a less efficient capital allocation, lower asset prices, and higher volatility.

This will only be an equilibrium if it does not lead to explosive paths. While this may seem technical, it is a real concern here: with higher volatility in the new equilibrium, any subsequent fear shocks would have a larger direct impact, further raise volatility, and so on, ad infinitum. Rational forward-looking would rule this out at the beginning and decide to suppress their fear.

In this class of models, explosions are easily prevented because of a fundamental indeterminacy in the dynamic equilibrium. The key observation is that optimal capital holdings are a function of the *risk premium*. Consequently, only the risk premium is pinned down by equilibrium; risky expected returns and riskless rates are not separately determined. This indeterminacy in risky expected returns, hence in expected capital gains, provides a tool with which sunspot equilibria can be engineered. The right panel of Figure 1 plots an example of capital price drift μ_q picked to stabilize prices if they get too low or too high.

If agents understand such a stabilizing force arises in these "extreme states," an entire sunspot-driven sequence of asset price drops can be justified. In our continuous-time setup, the stability restrictions in extreme states are simply boundary conditions, which is both analytically-convenient and emphasizes how mild these restrictions are; many types of dynamics are possible away from these extreme states.

In summary, asset-price fluctuations can be justified solely by agents' coordinated beliefs and active trading in response to those beliefs. A negative sunspot shock that triggers pessimism about asset prices can coordinate fire sales from first-best capital users (experts) to second-best users (households), creating misallocation, lowering capital prices, and thus fulfilling the initial pessimism.

This entire story held fixed the wealth distribution between experts and households. Does this imply that financial frictions are irrelevant? No! Without frictions, experts would always manage all capital, severing the link between risk-sharing and the capital market. One can see a role of the wealth distribution in Figure 1, via the vertical dotted line labelled "$\kappa = \eta$ lower bound." This line simply says that, even if volatility were to rise infinitely, experts would never hold less capital than their wealth share η, as they have a productivity advantage. Experts' wealth share η thus bounds the feasible intensity of fire sales, hence the intensity of sunspot fluctuations.

The resulting equilibria consist of a range of capital prices for each wealth distribution. Implicitly, the literature has collapsed this range in examining q as a function of η, i.e., the capital price is $q(\eta)$ in a Markovian equilibrium. If we tie our hands similarly, we substantially discipline sunspot equilibria but do not eliminate them. In fact, to relate our paper pedagogically to the existing literature, the first half of our paper presents a series of results in this Markovian setting. The second half of our paper dispenses with this assumption and studies the full range of possible sunspot equilibria.

Overview of results. To illustrate our mechanism in a simple way that is pedagogically close to the existing literature, we begin with a Markovian setting, whereby all fluctuations in allocations and prices can be traced to the wealth distribution. Revisit the typical amplification loop: because experts are levered, fundamental shocks shift the wealth distribution, triggering fire-sales, which transmit to output and asset prices, which then feeds back into the wealth distribution, and so on. But we show how this loop opens the door for self-fulfilling fluctuations, even if fundamentals are deterministic. The only difference to typical amplification is that sunspot shocks directly induce trading even before the wealth distribution shifts.

To better understand the coordination mechanism behind our multiplicity, we go on to illustrate how our two initial equilibria of focus – a volatile sunspot equilibrium and a deterministic fundamental equilibrium – in fact bookend an entire continuum of equilibria with intermediate volatility. These equilibria differ only in agents' coordinated beliefs about disaster states, i.e., what capital prices would be in the worst-case scenario. If agents coordinate on an inferior disaster outcome, the economy is more volatile; conversely, more volatile asset prices justify an inferior disaster belief. These equilibria are unified by the insight that the intensity of capital fire sales is only pinned down by agents' asset-price forecasts, which are governed by beliefs about disaster states.

Faced with such a large array of equilibria, we propose a simple equilibrium refinement. If agents can only take finite amounts of leverage, as in environments with limited commitment or enforcement, then disaster beliefs must be consistent with full deleveraging by experts. Even if the leverage bound is arbitrarily loose, this always selects the most volatile equilibrium, the one with the minimal worst-case asset price. That said, this maximal-volatility sunspot equilibrium makes very similar predictions to the typical equilibrium with amplified fundamental shocks, if fundamental volatility is small (formally, our sunspot equilibria in the first half of the paper are simply vanishing-uncertainty limits of fundamental equilibria).

This motivates us in the second half of the paper to study a richer class of self-fulfilling equilibria, all of which survive the aforementioned selection criterion. Mathematically, we dispense with the assumption that equilibria must be Markovian in the wealth distribution. Equivalently, we are expanding the admissible probability distribution of sunspot shocks: whereas sunspot shocks are restricted to be iid in the first half of the paper, we allow arbitrary persistence and heteroskedasticity in the second half. And as a result, the variables governing sunspot dynamics arise as additional state variables in equilibrium, in addition to experts' wealth share. This generalization considerably complicates the analysis but also engenders several new insights.

The overarching methodological lesson from this part of the paper is that equilibrium multiplicity can be vast, only disciplined by a restriction on agents' beliefs about extreme states. We completely characterize sunspot equilibria and use them to address certain shortcomings in the existing literature.

First, arbitrarily large capital price volatility can be justified; the gap between models and data on financial crises is not because the frictions are not powerful enough, but because we are not choosing the equilibrium closest to the data. Second, sunspot-driven crises feature far larger volatility spikes, and thus more closely resemble real-world financial crises, than fundamentals-based recessions. This finding emerges in our equilibrium because high-volatility states are characterized by a decoupling of the economy from real shocks. Third, whereas recoveries from fundamentals-based recessions are too slow, recoveries from sunspot-driven crises may be faster, helping match the countercyclical uncertainty term structure slope in the data.

Finally, we conclude with a complementary model meant to clarify the culprits of the type of multiplicity we document. We study a complete-markets representative-agent economy, which nevertheless can exhibit sunspot volatility due to nominal rigidities and the zero-lower bound. This example features an aggregate demand externality at the zero-lower bound, which operates through a connection between asset prices and output. By contrast, our baseline model features an aggregate supply externality, whereby expert fire sales reduce allocative efficiency – this is also an asset price-output link. Our aim is to analogize the price-output links of these models, to show how financial frictions are not critical *per se*, beyond their role in establishing this link.

Related literature. The theoretical focus on financial frictions and sunspots is not new to this paper. Several studies show how multiplicity emerges through the interaction between asset valuations and borrowing constraints. For instance, bubbles can relax firms' credit constraints, allowing greater investment and thus justifying the existence of the bubble (Farhi and Tirole, 2012; Miao and Wang, 2018; Liu and Wang, 2014). Self-fulfilling credit dynamics can also arise with *unsecured* lending as opposed to collateralized (Gu et al., 2013; Azariadis et al., 2016). Relative to these papers, we study different financial frictions (equity-issuance constraints), offer a complete characterization of sunspot fluctuations, and dispense with features that are auxiliary to our main point.

In a setup close to ours, Mendo (2020) studies self-fulfilled panics that induce collapse of the financial sector, an extreme example of the fluctuations we analyze. Gertler and Kiyotaki (2015) and Gertler et al. (2020) study bank runs in a similar class of models. However, our theory clarifies that the self-fulfilling asset-side fire-sale mechanism begets similar phenomena to traditional liability-side financial panics, bank runs, sudden stops, and the like. When selling assets, investors deleverage, which clarifies our mechanism as a "funding demand" decline rather than the "funding supply" decline that characterizes a run. It is not that investors cannot obtain financing, just that they do not want to. The fact that our results hold even without borrowing constraints or runs illustrates that a much broader class of financial crisis models are subject to sunspots.

We thus believe our study is a minimalist articulation of the mechanism linking financial frictions and sunspots, which we hope adds clarity. This minimalism also highlights which features are absent. We do not rely on many of the traditional multiplicity-inducing assumptions, such as overlapping generations, non-convexities or externalities in technology,[3] asymmetry of information,[4] or multiple assets.[5]

Our equilibrium construction also differs from the literature in a more technical sense. Building on the seminal studies Azariadis (1981) and Cass and Shell (1983), sunspot equilibria are often constructed by essentially randomizing over a multiplicity of deterministic transition paths to a stable steady state. By contrast, the deterministic version of our model features an unstable steady state; critically, the introduction of sunspot volatility flips the stability properties of equilibrium.[6]

Methodologically, we prove such results by employing tools from the "stochastic stability" literature in mathematics (this is the stochastic differential equation analog of Lyapunov stability for ODE systems; see Khasminskii (2011) for a reference). As one might expect from deterministic models, the existence of sunspot equilibria is tied directly to their stability properties. Stochastic stability tools are ideally suited to studying this type of issue, and they scale effortlessly to high-dimensional state spaces.

Outline of paper. Section 1 presents the baseline model. Section 2 seeks to understand self-fulfilling fluctuations in the simplest possible way; readers who are familiar with this literature may anticipate some subset of the results here. Section 3 studies a more complex class of self-fulfilling equilibria that feature several substantive and methodological distinctions. Section 4 presents a complementary model with nominal rigidities. Section 5 concludes. The appendices contain proofs, further details, and model extensions.

[3] For example, see Azariadis and Drazen (1990) for multiplicity under threshold investment behavior. See Farmer and Benhabib (1994) for a multiplicity under increasing returns to scale.

[4] In a macro context, Piketty (1997) and Azariadis and Smith (1998) for self-fulfilling dynamics in the presence of screened/rationed credit. In a finance context, Benhabib and Wang (2015) and Benhabib et al. (2016, 2019) generate sunspot fluctuations in dispersed information models.

[5] Hugonnier (2012), Gârleanu and Panageas (2020), Khorrami and Zentefis (2020), and Zentefis (2020) all build "redistributive" sunspots that shift valuations among multiple positive-net-supply assets.

[6] Peck and Shell (1991), Gottardi and Kajii (1999), and Hens (2000) also obtain sunspot equilibria in models which have a unique fundamental equilibrium, though for different reasons than us.

1 Baseline model

Technology, Preferences, Markets. Time $t \geq 0$ is continuous. There are two goods, a non-durable good (the numeraire, "consumption") and a durable good ("capital") that produces the consumption good. The aggregate supply of capital grows exogenously at constant rate g, i.e., $dK_t = gK_t dt$. Individual capital holdings evolve identically. Both goods are freely tradable, with the relative price of capital denoted by q_t.

There are two types of agents, experts and households, who differ in their production technologies. Experts produce a_e units of the consumption good per unit of capital, whereas households' productivity is $a_h \in (0, a_e)$.

Financial markets consist solely of an instantaneously-maturing, risk-free bond that pays interest rate r_t is in zero net supply. The key financial friction: agents cannot issue equity when managing capital. It is inconsequential that the constraint be this extreme. Partial equity issuance, as long as there is some limit, will generate identical results on sunspot volatility; we return to this issue much later in the paper.[7]

Given the stated assumptions, we can write the dynamic budget constraint of an agent of type j (expert or household) as

$$dn_{j,t} = \Big[(n_{j,t} - q_t k_{j,t})r_t - c_{j,t} + a_j k_{j,t}\Big]dt + d(q_t k_{j,t}), \tag{1}$$

where n_j is the agent's net worth, c_j is consumption, and k_j is capital holdings. The term $d(qk)$ represents the capital and price appreciation that accrues while holding capital.

Experts and households have time-separable logarithmic utility, with discount rates ρ_e and $\rho_h \leq \rho_e$, respectively. Thus, they solve the following maximization problem:

$$\sup_{c_j \geq 0, k_j \geq 0, n_j \geq 0} \mathbb{E}\Big[\int_0^\infty e^{-\rho_j t} \log(c_{j,t}) dt\Big] \tag{2}$$

subject to (1). The constraint $n_{j,t} \geq 0$ is the standard solvency constraint. Everything in this optimization problem is homogeneous in (c, k, n), so we can think of the expert and household as representative agents within their class.

Finally, to guarantee a stationary wealth distribution, we also allow an overlapping generation structure: agents perish idiosyncratically at rate δ; perishing agents are re-

[7] In particular, a partial equity-issuance constraint simply scales the mapping between expert wealth and asset prices. As is well-known, the equilibrium of economies in the class we consider will live in the region where the equity constraint is always-binding. Equity-issuance restrictions, sometimes called "skin-in-the-game" constraints, often arise as the optimal contract in a moral hazard problem, though this micro-foundation is not important for our purposes here.

placed by newborns, who inherit an equal share of perishing wealth; a fraction $\nu \in [0,1]$ of newborns are exogenously designated experts, and $1-\nu$ are households; there are no annuity markets to trade death risk. As the death rate δ affects an agent's lifetime utility, the subjective discount rates ρ_e, ρ_h are assumed inclusive of δ. Finally, to acknowledge the fact that OLG creates intertemporal transfers across agent types, which do not affect alive agents' individual net worth evolution, let N_e and N_h denote aggregate expert and household net worth. The dynamic evolutions of N_e and N_h will mirror (1), with additional terms capturing OLG-related transfers. We reiterate that OLG is unnecessary for our sunspot results and only serves to obtain stationarity in case we set $\rho_e = \rho_h$.

Equilibrium. The definition of competitive equilibrium is standard: (i) taking prices as given, and given an exogenous time-0 allocation of capital and riskless bonds, experts and households solve (2) subject to (1); (ii) the markets for consumption and capital clear at all dates, i.e.,

$$c_{e,t} + c_{h,t} = a_e k_{e,t} + a_h k_{h,t} \tag{3}$$

$$k_{e,t} + k_{h,t} = K_t. \tag{4}$$

The market for riskless bonds clears automatically by Walras' Law.

Despite the fact that all fundamentals of this economy are deterministic, we want to allow the possibility for sunspot volatility in equilibrium. To this end, let Z be a standard Brownian motion, which is extrinsic in the sense that it affects no primitives of the economy. Conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t]. \tag{5}$$

An equilibrium in which $\sigma_q \equiv 0$ is called a *Fundamental Equilibrium* (FE). An equilibrium in which σ_q is not identically zero is called a *Brownian Sunspot Equilibrium* (BSE).

We will show later that any volatility in capital prices is accompanied by output volatility. To benchmark this environment, note that without the equity-issuance friction, the model collapses to a constant-growth path with fully-efficient production and no volatility. With investment added as well, the model becomes a neoclassical growth model. Adding investment is trivial, but we abstract away from it for clarity.[8]

[8] With investment subject to neoclassical convex adjustment costs, capital price q drives Tobin's Q, so sunspot fluctuations in asset prices affect not only current production (through allocative efficiency) but also economic growth (through investment efficiency).

2 Self-fulfilling volatility

2.1 Brownian Sunspot Equilibrium (BSE)

Before determining whether an FE or a BSE exists, we provide a simple characterization of equilibrium that aids much of the future analysis. First, due to all the scalability assumptions embedded in this model, we look for a Markov equilibrium, in which all growing variables scale with aggregate capital K, and experts' wealth share $\eta := N_e/(N_e + N_h) = N_e/qK \in (0, 1)$ serves as the sole non-growing state variable. This is a choice that we will relax later. Conjecture η has dynamics of the following form:

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t}dZ_t, \quad \text{given} \quad \eta_0. \tag{6}$$

Second, given log utility and the scale-invariance of agents' budget sets, optimal consumption satisfies the standard formula $c_j = \rho_j n_j$. Third, let $\kappa := k_e/K$ denote expert's share of capital, which influences total output given heterogeneity in productivity. Substituting optimal consumption, the wealth share η, and the capital share κ, goods market clearing (3) yields an *asset price-output* relation

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h, \tag{PO}$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1-\eta)\rho_h$ is the wealth-weighted average discount rate.

Optimal capital holding by experts and households implies

$$\begin{aligned} &\frac{a_e}{q} + g + \mu_q - r = \sigma_{n_e}\sigma_q \\ &\frac{a_h}{q} + g + \mu_q - r \leq \sigma_{n_h}\sigma_q \quad \text{(with equality if } k_h > 0\text{)}, \end{aligned}$$

where $\sigma_{n_j} := (k_j/n_j)\sigma_q$ denotes the loading of type-j agents on shock dZ. Because all experts (and households) make the same scaled portfolio choices, we have $\sigma_{n_j} = \sigma_{N_j}$, where

$$\sigma_{N_e} = \frac{\kappa}{\eta}\sigma_q \quad \text{and} \quad \sigma_{N_h} = \frac{1-\kappa}{1-\eta}\sigma_q. \tag{7}$$

Using result (7), we can summarize portfolio choices compactly by the *risk balance* condition

$$0 = \min\left[1-\kappa,\, \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}\sigma_q^2\right]. \tag{RB}$$

Either experts manage the entire capital stock ($\kappa = 1$) or the excess return experts obtain over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $(\sigma_{N_e} - \sigma_{N_h})\sigma_q = \frac{\kappa-\eta}{\eta(1-\eta)}\sigma_q^2$. Given logarithmic preferences, σ_{N_j} corresponds to the risk price for type-j agents.

This model incorporates a two-way feedback loop between asset prices and the wealth distribution. Applying Itô's formula to the definition of experts' wealth share η, we obtain $\sigma_\eta = \eta(1-\eta)[\sigma_{N_e} - \sigma_{N_h}]$, which after substituting (7) becomes

$$\sigma_\eta = (\kappa - \eta)\sigma_q. \tag{8}$$

Conversely, in a Markov equilibrium with single state variable η, all stationary variables can be expressed as functions of η, e.g., $q_t = q(\eta_t)$. Then, Itô's formula implies $\sigma_q = \frac{q'}{q}\sigma_\eta$. Combine this with (8) to solve the two-way feedback loop between σ_q and σ_η:

$$\left[1 - (\kappa-\eta)\frac{q'}{q}\right]\sigma_q = 0. \tag{SV}$$

Therefore, there are two possibilities: (i) either $\sigma_q = 0$, which would correspond to FE; or (ii) there might be an equilibrium with $1 = (\kappa - \eta)\frac{q'}{q}$, in which case σ_q and hence σ_η can be non-zero, corresponding to the BSE. For this reason, (SV) can be called a *sunspot volatility* equation.

Finally, one can show that

$$\mu_\eta = \eta(1-\eta)[\rho_h - \rho_e] + \mathbf{1}_{\{\kappa<1\}}(\kappa - 2\eta\kappa + \eta^2)\frac{a_e - a_h}{q} + \delta(\nu - \eta). \tag{9}$$

It turns out, this is enough to determine the equilibrium.

Lemma 1 (Equilibrium Characterization)**.** *An allocation is a Markov equilibrium in η_t if and only if $(\kappa, q, \sigma_q, \sigma_\eta, \mu_\eta)$ are functions of η satisfying (PO), (RB), (SV), and (8)-(9).*

Using this characterization, we can immediately show that an FE exists. As already suggested, this equilibrium corresponds to selecting the solution $\sigma_q = 0$ to equation (SV).

Lemma 2 (Fundamental Equilibrium)**.** *There exists an equilibrium in which experts manage all capital, $\kappa = 1$, and the price of capital $q_t = a_e/\bar{\rho}(\eta_t)$ evolves deterministically.*

But there is also another class of equilibria, the BSEs, which have volatility. To understand this, consider selecting the solution $1 = (\kappa - \eta)\frac{q'}{q}$ to equation (SV), which allows $\sigma_q \neq 0$. Substituting $\kappa < 1$ from the price-output relation (PO), we obtain an ODE for q:

$$q' = \frac{(a_e - a_h)q}{q\bar{\rho} - \eta a_e - (1-\eta)a_h}, \quad \text{if} \quad \kappa < 1. \tag{10}$$

Suppose we solve this first-order ODE subject to the boundary condition $\kappa(0) = 0$, which translates via (PO) to $q(0) = a_h/\rho_h$. We use this boundary condition – which says that experts fully de-lever as their wealth shrinks – in accordance with the literature. Note for now only that the boundary condition selects a particular BSE; we return to this issue in Section 2.2. The ODE (10) is solved on the endogenous region $(0, \eta^*)$ where households manage some capital, i.e., $\eta^* := \inf\{\eta : \kappa(\eta) = 1\} = \inf\{\eta : q(\eta) = a_e/\bar{\rho}(\eta)\}$.[9] Given the solution for (q, κ), capital price volatility is solved from the risk balance equation (RB) as

$$\sigma_q^2 = \frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1. \tag{11}$$

The sign of σ_q is irrelevant, given the symmetry of Brownian shock dZ. Since $\sigma_q \neq 0$, equation (8) confirms the initial conjecture that $\sigma_\eta \neq 0$.

The intuition communicated by the BSE equations above is as follows. If agents believe the sunspot shock dZ can affect asset prices, then the actual arrival of such a shock triggers trading of capital between experts and households. Since experts are more productive than households, capital transfers have real effects and move asset prices. But it does not end there: asset-price fluctuations feed back into the wealth distribution, which initiates another round of capital transfers, and so on. Ultimately, the question is whether there exists an initial belief about asset prices that can be self-justified by this process, and this is tantamount to solving the ODE (10).

It is critical that the aforementioned capital transfers take place. A clear way to see this is to note that sunspot shocks cannot have any effect when $\kappa = 1$ (indeed, price-output relation (PO) implies $q = a_e/\bar{\rho}$, whose derivatives are inconsistent with (SV)). Thus, in states of the world when $\kappa = 1$, equilibrium must feature $\sigma_\eta = \sigma_q = 0$, resembling the FE. Naturally, these states of the world are when experts are sufficiently wealthy, i.e., $\eta_t > \eta^*$. For identical reasons, if all households are so unproductive that

[9] When $\rho_h = \rho_e$, there is a closed form solution for capital price

$$q(\eta) = \frac{1}{\rho}\Big[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2}\Big], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}.$$

they never hold capital ($a_h = -\infty$), or if the capital ownership distribution is irrelevant to aggregate productivity ($a_e = a_h$), no self-fulfilling volatility is possible.

Proposition 1 (Brownian Sunspot Equilibrium)**.** *There exists a BSE with $\kappa(0) = 0$, in which $\sigma_q(\eta) \neq 0$ on $(0, \eta^*)$ and $\sigma_q(\eta) = 0$ on $(\eta^*, 1)$.*

For exposition purposes, we refer to this equilibrium as *the* BSE. Figure 2 displays a numerical example with the capital price q and volatility σ_q as functions of the expert wealth share.

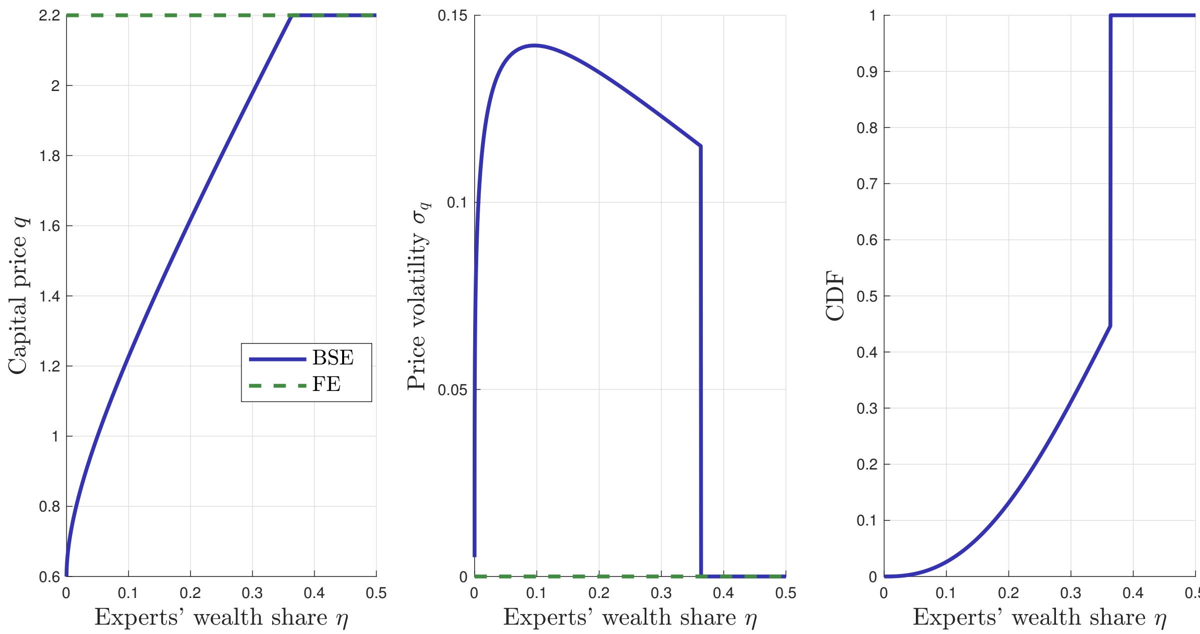

Figure 2: Capital price q, volatility σ_q, and stationary CDF of η. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters (for the CDF): $\nu = 0.1$ and $\delta = 0.04$.

Does sunspot volatility survive in the long run? This question is tied directly to the stationarity properties of $(\eta_t)_{t\geq 0}$, which boils down to equilibrium behavior around $\eta \approx 0$ and $\eta \approx \eta^*$. As we show in Lemma A.1 in Appendix A.2, this economy possesses a stationary distribution on $(0, \eta^*]$ under very mild parameter restrictions. For example, if experts are more impatient than households ($\rho_e > \rho_h$), or the economy has an OLG structure with sufficiently few experts ($\delta > 0$ and $\nu < \eta^*$), then sunspot volatility survives in the long run. The right panel of Figure 2 plots the stationary CDF of η in the BSE.[10]

These results can be generalized in several directions. While we have chosen to model sunspots as Brownian shocks, an extension in Appendix D.1 shows how to solve

[10]Note that $\eta_t = \eta^*$ about 55% of the time in this numerical example, i.e., there is a mass point at η^*. This occurs because of a discontinuity in both the drift μ_η and volatility σ_η at that point.

an equilibrium with Poisson jump sunspots instead. The intuition and analysis of this extension mirror very closely the Brownian model. We have also chosen to examine the log utility model for simplicity, but similar results can be obtained with more general CRRA preferences, as shown in Appendix D.2.

2.2 Inspect the mechanism: beliefs about disaster states

In the BSE, there is a unique viable level of sunspot volatility σ_q at each level of the wealth share η, given by equation (11). This could be seen as restrictive, since agents must somehow coordinate on a particular amount of volatility. In this section, we outline a richer class of sunspot equilibria, which also will help illuminate the mechanism at play. The entire set of sunspot equilibria studied here will be indexed by agents' beliefs about the "tail scenario" in the economy, i.e., what happens when experts are severely undercapitalized.

Mathematically, recall that we previously have assumed $\kappa(0) = 0$; in other words, experts fully deleverage as their wealth vanishes. This turns out to not be necessary. Consider any $\kappa_0 \in (0,1)$ and put $\kappa(0) = \kappa_0$. We will call κ_0 the *disaster belief* in the economy. The sunspot equilibrium is similar to Proposition 1, with the generalization that the boundary condition to the ODE (10) is now $\kappa(0) = \kappa_0$ rather than $\kappa(0) = 0$.[11]

Theorem 1. *For a fixed tail belief $\kappa_0 \in (0,1)$, there exists a Markov sunspot equilibrium, with $\sigma_q(\eta) \neq 0$ on a positive measure subset of $(0,1)$. As $\kappa_0 \to 0$, this equilibrium converges to the BSE. As $\kappa_0 \to 1$, the equilibrium converges to the FE.*

Based on Theorem 1, one can view both the BSE and the FE as outcomes of coordination on experts' deleveraging. If experts never sell any capital, there can be no price volatility, with $\sigma_q = 0$ at all times. If agents expect $\kappa_0 = 0$, which translates to full deleveraging and large capital fire sales, then the BSE prevails. But for any $\kappa_0 \in (0,1)$, an intermediate sunspot equilibrium will prevail, with a self-fulfilling amount of expert deleveraging and associated price dynamics. In this simple way, the boundary condition $\kappa_0 \in [0,1]$ spans an entire range of sunspot equilibria from more to less volatile. An illustration is in Figure 3.[12]

[11] As in footnote 9, there is a closed-form solution when $\rho_h = \rho_e$, which is

$$q(\eta) = \frac{1}{\rho}\Big[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2 + (a_e - a_h)^2\kappa_0^2}\Big], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}(1 - \kappa_0^2).$$

As κ_0 decreases, the slope $q'(\eta)$ increases, consistent with the idea that pessimism about the disaster state raises the sensitivity of equilibrium to sunspot shocks away from disaster. Clearly, this solution converges

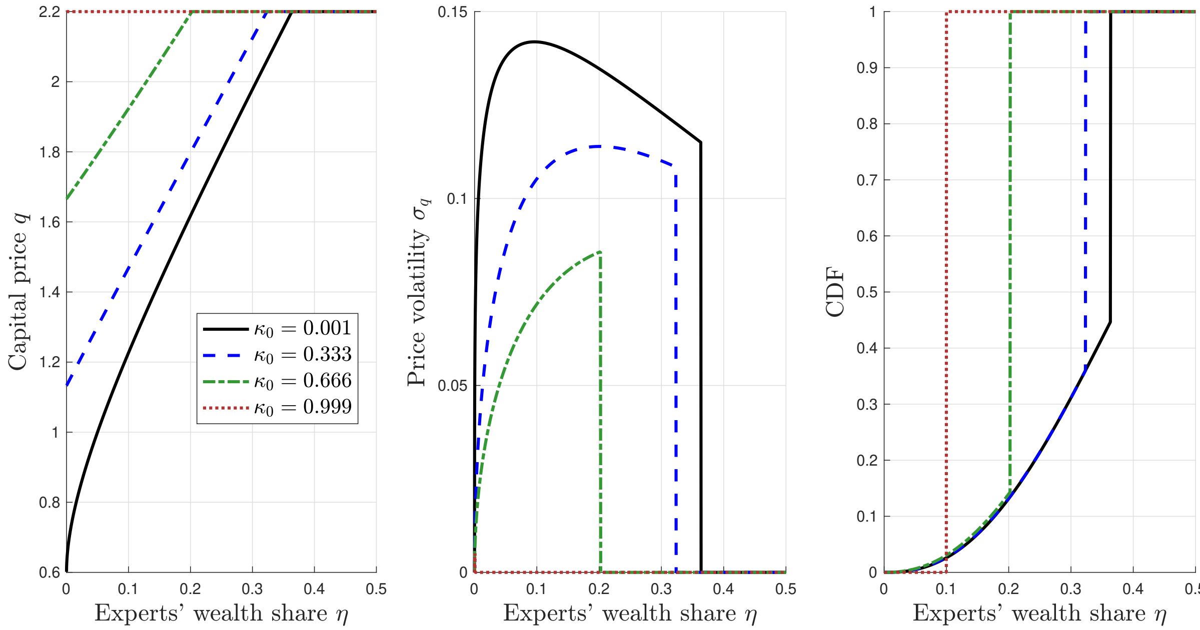

Figure 3: Capital price q, volatility σ_q, and stationary CDFs of η for different levels of disaster belief κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters (for the CDF): $\nu = 0.1$ and $\delta = 0.04$.

This result provides a clear illustration of the central property that the degree of capital fire sales is indeterminate in these models. Intuitively, greater optimism about other experts' ability to retain capital in the tail scenario induces smaller capital fire sales in response to sunspot shocks, which keeps volatility low, asset prices high, and justifies the optimism. What if this optimism stochastically shifts to pessimism, and vice versa? In Appendix D.3, we allow this possibility, further enlarging the space of self-fulfilling equilibria by modeling time-variation in the disaster belief κ_0.

2.3 Limited commitment as equilibrium refinement

Given the wide variety of sunspot equilibria documented so far, which of them are more likely? Here, we add a small limited commitment friction, in the spirit of Gertler and Kiyotaki (2010); it turns out, this will substantially prune the set of possible equilibria.

Suppose capital holders can abscond with a fraction $\lambda^{-1} \in (0, 1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, the capital holder would have net worth $\tilde{n}_{j,t} := \lambda^{-1} q_t k_{j,t}$.

To prevent diversion, bondholders will impose some limitation on borrowing. To see this, note that diversion delivers utility $\log(\tilde{n}_{j,t}) + \xi_t$, where ξ_t is an aggregate process

to the BSE solution in footnote 9 as $\kappa_0 \to 0$, and to the FE solution a_e/ρ as $\kappa_0 \to 1$.

[12]This result is also convenient in some numerical situations. Since the BSE is just the limit of equilibria as $\kappa_0 \to 0$, we can construct an approximate numerical solution with κ_0 very small (but not quite 0).

(independent of the identity j of the diverter). For diversion to be sub-optimal, it must be the case that $\log(\tilde{n}_{j,t}) + \xi_t \leq \log(n_{j,t}) + \xi_t$. As a result, bondholders impose the following leverage constraint to ensure non-diversion is incentive compatible:

$$\frac{q_t k_{j,t}}{n_{j,t}} \leq \lambda. \tag{12}$$

We will study the equilibrium with constraint (12) additionally imposed, and then we will take $\lambda \to \infty$ so that the limited commitment friction is vanishingly small.

Risk balance condition (RB) is now replaced by

$$0 = \min\Big[1-\kappa,\ \lambda\eta - q\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}\sigma_q^2\Big]. \tag{13}$$

The most important feature of equation (13) is that leverage constrained experts ($\lambda\eta = q\kappa$) must hold less than the full capital stock ($\kappa < 1$).

Consider, as before, a Markov equilibrium with experts' wealth share η as the state variable. Condition (13) implies that there exists a threshold $\eta^\lambda := \inf\{\eta : \lambda\eta > q\kappa\}$ below which experts' leverage constraints bind. By combining $\lambda\eta = q\kappa$ with condition (PO) for κ, we obtain an explicit formula for the capital price in this region:

$$q = \frac{1}{2}\Big[\frac{a_h}{\bar{\rho}} + \sqrt{(a_h/\bar{\rho})^2 + 4\lambda\eta(a_e - a_h)/\bar{\rho}}\Big], \quad \text{if} \quad \eta \leq \eta^\lambda. \tag{14}$$

Taking the limit $\eta \to 0$ in equation (14) shows that $q \to a_h/\rho_h$ and thus $\kappa \to 0$. This proves that there is no flexibility for coordination on a worst-case capital price, unlike the leverage-unconstrained economy. The equilibrium is unique along this dimension, coinciding with $\kappa_0 = 0$.

As the limited commitment problem vanishes ($\lambda \to \infty$), the leverage constraint becomes non-binding at all times (formally $\eta^\lambda \to 0$).[13] Hence, the entire equilibrium converges to the BSE of Proposition 1. We collect these results.

Proposition 2. *Among Markov equilibria in η, only the BSE, i.e., the equilibrium with disaster belief $\kappa_0 = 0$, survives a vanishingly-small limited commitment friction.*

In eliminating multiplicity with this refinement, only the most volatile equilibrium survives (recall Figure 3). Why? Intuitively, the leverage constraint gives experts an additional motive to sell capital, which forces coordination on maximal selling in response

[13] This intuitive property can be shown easily by taking $\lambda \to \infty$ in (14). For any fixed $\eta \in (0,1)$, taking this limit implies $q \to \infty$, which is ruled out by price-output relation (PO).

to negative sunspot shocks. Said differently: due to the prospect of violating the leverage constraint, losses incurred from retaining capital when others are selling is larger than losses incurred from selling capital when others are retaining it. This property is reminiscent of "risk dominant" equilibria being selected by strategic uncertainty (Harsanyi and Selten, 1988; Frankel et al., 2003), but the exact modeling is different here.

In Appendix D.4, we study another simple equilibrium refinement: a small amount of idiosyncratic uncertainty. Similar to this limited-commitment extension, idiosyncratic risk dictates full expert deleveraging, i.e., $\kappa(0) = 0$.[14] Otherwise, as experts became poor, they would be sub-optimally holding infinite risk per unit wealth.

2.4 Sunspots as zero-uncertainty limits

In the preceding analysis, economic fundamentals are completely deterministic. With some fundamental uncertainty, equilibrium volatility in this class of models is thought to come from amplification of fundamental shocks. Here, we show that sunspots are just a special case of amplification, in the sense that our sunspot equilibria are limiting equilibria as fundamental shocks vanish.

To demonstrate this formally, we add fundamental uncertainty to capital,

$$dK_t = K_t[gdt + \sigma dZ_t],$$

where Z is a one-dimensional Brownian motion, and $\sigma > 0$ is a constant. Conjecture capital price q and experts' wealth share η follow

$$\begin{aligned} dq_t &= q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t] \\ d\eta_t &= \mu_{\eta,t}dt + \sigma_{\eta,t}dZ_t. \end{aligned}$$

Continue to focus on Markov equilibria, in which the sole state variable will still be η.

A crucial equilibrium condition is

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma. \tag{15}$$

Equation (15) is often interpreted as *amplification*, because $\frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}$ takes the form of a convergent geometric series. In words, a negative fundamental shock reduces experts' wealth share η directly through $(\kappa - \eta)\sigma$, which reduces asset prices through q'/q. This

[14]The one distinction to this limited-commitment extension is that the fundamental equilibrium also survives the addition of idiosyncratic risk, because it also features $\kappa(0) = 0$.

explains the numerator of (15). But the reduction in asset prices has an indirect effect: a one percent drop in capital prices reduces experts' wealth share by $(\kappa - \eta)$, which feeds back into a $(\kappa - \eta)q'/q$ percent further reduction capital prices, which then triggers the loop again. The second-round impact is $[(\kappa - \eta)q'/q]^2$, and so on. This infinite series is convergent if $(\kappa - \eta)q'/q < 1$, such that incremental amplification is reduced in each successive round of the feedback loop.

This contrasts with the BSEs, in which $(\kappa - \eta)q'/q = 1$ (equation (SV)). Intuitively, a BSE has no dampening in successive rounds of the feedback loop, leading to infinite amplification! Infinite amplification is needed, in fact, because there is no fundamental shock to start with: one can think of price volatility as the product of zero fundamental uncertainty multiplied by infinite amplification.

Despite this contrast, BSEs and amplified fundamental equilibria are not so different. As fundamental uncertainty shrinks, amplification rises explosively and equilibria become "sunspot-like".

Proposition 3. *Suppose a Markov equilibrium in η exists for each $\sigma > 0$ small enough, with disaster belief $\kappa_0 \in [0, 1)$. As $\sigma \to 0$, the equilibrium converges to a BSE with disaster belief κ_0.*

This limiting result conveys BSEs as a special case of amplification. There need not be any discontinuity in the nature of equilibria at $\sigma = 0$. Brunnermeier and Sannikov (2014) provide a related limiting result, arguing that asset-price volatility does not vanish as $\sigma \to 0$, also known as the "volatility paradox." Proposition 3 goes further by identifying the limiting equilibria as the BSEs. Related results can be found in Manuelli and Peck (1992) and Bacchetta et al. (2012), in which sunspot equilibria could be seen as limits of fundamental equilibria when fundamental uncertainty vanishes.

Although our paper is primarily about sunspot equilibria, we note in passing that the literature has not entertained the rich set of fundamental equilibria considered here. Even with fundamental shocks, the coordinated disaster belief κ_0 modulates the strength of the core feedback loop of the model in a self-fulfilling way, exactly as in Section 2.2. Appendix C.1 provides several examples of fundamental equilibria with both $\sigma > 0$ and $\kappa_0 > 0$, whereas existing literature has only studied $\kappa_0 = 0$.[15] In fact, self-fulfilling

[15]Brunnermeier and Sannikov (2014) justify $\kappa_0 = 0$ in their online appendix: "because in the event that η_t drops to 0, experts are pushed to the solvency constraint and must liquidate any capital holdings to households." This is technically not needed; as shown in Lemma C.2 of Appendix C.1, the dynamics of η_t will not allow it to ever reach 0, so there is no contradiction to equilibrium with both $\kappa_0 > 0$ and $\sigma > 0$. Although we do not prove an existence result, Appendix C.1 presents several numerical examples. The continuum of fundamental equilibria, indexed by κ_0, may be of independent theoretical interest.

In some sense, the literature has picked the worst possible fundamental equilibrium (minimal-price, maximal-volatility) by imposing $\kappa_0 = 0$. This can be partly justified by the refinement results of Section equilibria are even richer: Appendix C.2 demonstrates fundamental equilibria in which asset prices move oppositely to fundamental shocks! This can only happen through experts' coordination to buy when fundamentals worsen, and vice versa.[16]

2.5 Taking stock of the results so far

The results so far should be taken as simple illustrations of the key forces and mechanisms behind sunspot equilibria. These illustrations are, in several senses, too simple.

First, whereas Section 2.2 offered the promise of a rich variety of sunspot equilibria (depending on coordinated beliefs about the intensity of fire sales, indexed by disaster belief κ_0), the refinement of Section 2.3 singled out the BSE (with $\kappa_0 = 0$). Second, one can view Section 2.4, which identifies sunspot equilibria as zero-uncertainty limits of fundamental equilibria, as an approximate observational equivalence. It says the sunspot equilibria studied thusfar *must* bear a tight resemblance to amplified fundamental equilibria typically studied in this literature.

In fact, the unique equilibrium under our refinement is simply a limiting case of Brunnermeier and Sannikov (2014). If our paper could only generate fluctuations that "look like" those already studied in Brunnermeier and Sannikov (2014), we would not be resolving any of the puzzles of this literature! This is the key reason why we take our first set of results to be mostly illustrative and pedagogical.

To push further, we are motivated to look for sunspot equilibria capable of much richer dynamics. We study a broad class of such equilibria in the next section.

3 Sunspots with richer dynamics

To address the critique that the BSEs "look like" oft-studied fundamental equilibria with amplification, we endeavor here to analyze a richer class of sunspot equilibria. First, we theoretically establish some very general necessary and (almost) sufficient conditions for sunspot equilibria in this class of models (Sections 3.1 and 3.2). Then, we provide some more concrete constructions, all of which survive the particular equilibrium refinement in Section 2.3, and highlight novel substantive insights (Section 3.3).

For this section, we generalize the environment to also include a fundamental shock. This is useful both for realism and for establishing some additional insights. Let $Z :=$

2.3, which carry over to the case with $\sigma > 0$, i.e., only the belief $\kappa_0 = 0$ survives a vanishingly-small limited commitment friction. Despite this choice, the quantitative failures outlined in the introduction persist.

[16] For an initial conjecture of this specific type of indeterminacy, see footnote 16 of Kiyotaki and Moore (1997). We pay homage by including this observation in our own footnote 16.

$(Z^{(1)}, Z^{(2)})$ be a two-dimensional Brownian motion, where $Z^{(1)}$ represents the fundamental shock, and $Z^{(2)}$ represents the sunspot shock. Aggregate capital follows

$$dK_t = K_t\Big[g dt + \sigma\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) \cdot dZ_t\Big]. \tag{16}$$

Conjecture capital prices and the wealth share follow

$$dq_t = q_t\Big[\mu_{q,t} dt + \sigma_{q,t} \cdot dZ_t\Big] \tag{17}$$

$$d\eta_t = \mu_{\eta,t} dt + \sigma_{\eta,t} \cdot dZ_t, \quad \text{given } \eta_0. \tag{18}$$

Whereas only $Z^{(1)}$ affects capital, both $Z^{(1)}$ and $Z^{(2)}$ can potentially impact equilibrium asset prices and the wealth distribution. Since it appears often, denote the shock exposure of capital returns by

$$\sigma_{R,t} := \sigma\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) + \sigma_{q,t}. \tag{19}$$

3.1 What cannot happen: a sunspot impossibility result

Let us first study a Markov equilibrium with state variable η, as in Section 2. In this environment, we obtain the stark result that capital prices must be completely insensitive to the sunspot shock $Z^{(2)}$, in contrast to our previous discovery of sunspot equilibria. To see this, consider the wealth share diffusion $\sigma_\eta = (\kappa - \eta)\sigma_R$, the generalization of (8) since return-on-capital volatility is now $\sigma_R = \sigma\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) + \sigma_q$. As before, capital prices respond to wealth share changes through $\sigma_q = \frac{q'}{q}\sigma_\eta$. Solving this two-way feedback,

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\sigma_q = \big(\begin{smallmatrix}1\\0\end{smallmatrix}\big)(\kappa - \eta)\sigma\frac{q'}{q}. \tag{20}$$

Equation (20) is really two equations stacked. Given $\sigma > 0$, the first equation can only hold if $(\kappa - \eta)\frac{q'}{q} \neq 1$. Staring at the second equation, this immediately proves that $\sigma_q \cdot \big(\begin{smallmatrix}0\\1\end{smallmatrix}\big) = 0$, implying the following lemma.

Lemma 3. *If $\sigma > 0$, any Markov equilibrium in η must be insensitive to sunspot shocks.*

In summary, to obtain richer dynamics, we must search beyond equilibria that revolve solely around experts' wealth share η. If we focus only on Markov equilibria in η, we either obtain sunspot equilibria that look very much like amplified fundamental equilibria (Proposition 3), or we obtain pure fundamental equilibria (Lemma 3).[17]

[17] Another way to state the result of Lemma 3 is that fundamental volatility kills sunspot volatility

3.2 What can happen: general theoretical analysis

In this section, we explore a broader set of equilibria, where η is no longer the sole state variable. To attain maximum theoretical generality, we visit the opposite end of the spectrum in assuming a completely non-Markovian structure. In the subsequent sections, we will study some more concrete applications in which equilibria are defined by a two-dimensional Markov state (η, s), where s is a sunspot variable.

We first define equilibrium. As always, the price-output relation (PO) still holds, linking κ to q and η. The assumption of log utility allows us to explicitly solve agents' optimization problems even without imposing any Markov structure. The resulting portfolio optimality conditions are almost identical to the baseline model, with σ_q replaced by the new return-on-capital diffusion σ_R:

$$\begin{aligned}
&\frac{a_e}{q} + g + \mu_q - r = \frac{\kappa}{\eta}|\sigma_R|^2 \\
&\frac{a_h}{q} + g + \mu_q - r \leq \frac{1-\kappa}{1-\eta}|\sigma_R|^2 \quad \text{(with equality if } \kappa < 1\text{)}.
\end{aligned}$$

Differencing these equations, we obtain the generalization of risk balance condition (RB):

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2\Big]. \tag{RB2}$$

On the other hand, summing agents' portfolio optimality conditions (weighted by κ and $1-\kappa$) yields an equation for the riskless rate:

$$r = \frac{\kappa a_e + (1-\kappa) a_h}{q} + g + \mu_q - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)|\sigma_R|^2. \tag{21}$$

Finally, the dynamics of experts' wealth share are given by

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}|\sigma_R|^2 + \delta(\nu - \eta) \tag{22}$$

$$\sigma_\eta = (\kappa - \eta)\sigma_R. \tag{23}$$

---

in a Markov setting in η. More precisely, *non-traded* fundamental uncertainty eliminates the Markovian sunspot equilibrium: if all agents can frictionlessly access a market to trade claims on the fundamental shock $dZ^{(1)}$, but no such market for the sunspot shock $dZ^{(2)}$ exists, then the possibility of the sunspot equilibrium re-emerges. This sunspot equilibrium is exactly the as the BSE, in the sense that $\binom{0}{1} \cdot \sigma_q$ coincides with the capital price volatility of the BSE.

In Appendix D.4, we provide an extension showing that the presence of idiosyncratic shocks, if they are large enough, also kills sunspot equilibria that are Markov in η. The intuition is similar to Lemma 3, though less stark.

Definition 1. Given $\eta_0 \in (0,1)$, an *equilibrium* consists of adapted processes $(\eta_t, q_t, \kappa_t, r_t)_{t\geq 0}$ such that equations (PO), (RB2), (21), and (22)-(23) hold.

Definition 1 calls for entire time-paths of η and q. Given the evolution equations (17)-(18), this translates to time-paths for the SDE coefficients (σ_η, μ_η) and (σ_q, μ_q), as well as initial value q_0 (since η_0 is given). Because (18) has an initial condition (η_0), while (17) does not (q_0 is unknown), the wealth share η is a state variable, whereas the capital price q should be thought of as a "co-state" (whose evolution equation runs backward).[18]

A sunspot equilibrium is one in which shock $Z^{(2)}$ matters. For a more convenient definition, recall that when $\sigma > 0$, no such equilibrium exists if q is a single-valued function of η (Lemma 3). Thus, the defining characteristic of sunspot equilibria is that different asset prices can prevail for a given wealth distribution. We also extend this notion to $\sigma = 0$ to differentiate this class of equilibria from the BSEs of Section 2.

Definition 2. A *Generalized Brownian Sunspot Equilibrium (GBSE)* is an equilibrium with multiple solutions for q_0, for a positive-measure subset of values $\eta_0 \in (0,1)$.

Remark 1 (Stability and multiplicity: connection to literature)**.** *One benefit of defining GBSEs this way is to connect our results with those of deterministic dynamic economies. For example, recall the neoclassical growth model, in which capital and consumption are the state and co-state variables, respectively, and only one value of initial consumption is consistent with a non-explosive equilibrium. By contrast, OLG versions of the growth model can feature a stable steady state, to which many alternative values of initial consumption would converge (Azariadis, 1981; Cass and Shell, 1983). This literature generates stochastic sunspot equilibria by essentially randomizing over the multiplicity of transition paths. Per this discussion, stability is the critical property enabling sunspots in deterministic dynamical systems.*

GBSEs of our model will also feature a type of stability, whereby many values of the co-state q_0 can all be consistent with non-explosive behavior. But the analogy to deterministic models breaks down in an important sense: Appendix D.4 shows that the deterministic steady state of our class of models is only saddle-path stable, so we cannot *obtain volatility by randomizing over a multiplicity of deterministic transition paths. For the same reason, we cannot hard-wire arbitrary amounts of volatility for any combination (η, q). Rather, as will soon be clear, our model uniquely determines return volatility $|\sigma_R|$ for each (η, q), which is reminiscent of the endogenously-determined sentiment distribution in Benhabib et al. (2015).*

[18]The processes $(\eta_t, q_t)_{t\geq 0}$ thus solve forward-backward stochastic differential equations (FBSDEs). Technically, the backward equation for q should have a terminal condition, which is usually the "transversality condition," but we do not specify any such condition in Definition 1 because q will always be bounded and thus have no impact on transversality. Indeed, equation (PO) implies $q = (\kappa a_e + (1-\kappa)a_h)/\bar{\rho}$, while equation (RB2) implies $\kappa \in (\eta, 1]$ at all times.

If a GBSE supports multiple values of q_0, some force must be keeping the dynamical system $(\eta_t, q_t)_{t\geq 0}$ stable. That force is the drift μ_q, which is indeterminate and can be judiciously chosen.

Why is μ_q indeterminate? Mechanically, notice Definition 1 asks for 7 endogenous objects $(\sigma_\eta, \mu_\eta, \sigma_q, \mu_q, q_0, \kappa, r)$, but it only imposes 5 equilibrium restrictions. The sixth and seventh equilibrium restrictions were previously provided by Itô's formula, but not here.[19] For our GBSEs, the two degrees of freedom will be used to pick q_0 and μ_q.

Economically, indeterminacy stems from agents' trading behavior, which depends on the *risk premium* rather than expected returns and the interest rate separately. Only the spread $\mu_q - r$ is pinned down in equilibrium, as equation (21) shows.

Making a judicious choice for μ_q is straightforward. Because $(\eta_t, q_t)_{t\geq 0}$ evolves in a diffusive fashion, stability criteria conveniently boil down to boundary behavior of the dynamical system. Thus, by imposing certain boundary conditions on μ_q, we ensure a stochastically stable system. We make some mild parameter restrictions and then present the main theoretical results.

Assumption 1. *Parameters satisfy (i)* $0 < \frac{a_h}{\rho_h} < \frac{a_e}{\rho_e} < +\infty$*; (ii)* $\sigma^2 < \rho_e(1 - a_h/a_e)$*; and (iii) either* $0 < \delta\nu < \delta$*, or* $\sigma^2 < \rho_e - \rho_h$*.*

Assumption 1 part (i), only for convenience, makes the very modest assumption that the capital price is higher if experts control 100% of wealth than if households control 100% of wealth. Part (ii), meant to make the problem interesting, ensures experts sometimes hold all capital (i.e., $\kappa = 1$) and sometimes do not ($\kappa < 1$). Part (iii) guarantees experts do not asymptotically hold all wealth.

Theorem 2 (Existence of GBSEs)**.** *Let Assumption 1 hold. Then, there exists a GBSE in which* $(\eta_t, q_t)_{t\geq 0}$ *remains in* $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1-\eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$ *almost-surely and possesses a non-degenerate stationary distribution.*

What can we say about the nature of GBSEs? Most importantly, the amount of multiplicity that can be supported is vast, along three different dimensions. GBSEs allow: (1) an enormous range of prices and volatilities; (2) arbitrary decoupling of volatility from fundamentals; and (3) almost any degree of persistence or mean-reversion in asset prices. We now elaborate on these 3 features in turn.

[19] Indeed, suppose q is a function of η, as in our Markovian BSE of Section 2. Itô's formula then provides us with two additional restrictions: $q\sigma_q = \sigma_\eta q'$ and $q\mu_q = \mu_\eta q' + \frac{1}{2}|\sigma_\eta|^2 q''$. In other words, we have 7 unknown endogenous objects and 7 equilibrium restrictions. Usually, we combine the drift restriction with the other equilibrium conditions to obtain a differential equation that determines the unknown function $q(\eta)$.

First, a large range of asset prices can be supported in a GBSE, for any given wealth distribution. In fact, there is a sense in which "anything goes," in that the set of equilibrium asset prices support a range of return volatility between (near) zero and infinity. The next result states the range of equilibrium prices and volatilities, showing how the GBSE we have constructed satisfies Definition 2 in the maximal possible sense.

Corollary 1 (Anything goes)**.** *Given initial wealth share $\eta_0 \in (0,1)$, let $\mathcal{Q}(\eta_0)$ denote the set of possible GBSE values of q_0, and let $\mathcal{V}(\eta_0)$ denote the associated set of possible GBSE values of return variance $|\sigma_R(\eta_0, q_0)|^2$. Then,*

$$\mathcal{Q}(\eta) = \begin{cases} \left(\frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}, \frac{a_e}{\bar{\rho}(\eta)}\right), & \text{if } \eta < \eta^* := \frac{\rho_h}{\rho_e}\left(\frac{1-a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\right)^{-1}; \\ \left(\frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}, \frac{a_e}{\bar{\rho}(\eta)}\right], & \text{if } \eta \geq \eta^*, \end{cases}$$

and

$$\mathcal{V}(\eta) = \Big(\min\left[\eta\bar{\rho}(\eta)\frac{a_e - a_h}{a_e}, \sigma^2(\bar{\rho}(\eta)/\rho_e)^2\right], +\infty\Big).$$

In particular, if fundamental volatility $\sigma = 0$, then return variance spans any value between 0 and $+\infty$, regardless of the wealth distribution η. Finally, a GBSE can be constructed such that, in the stationary distribution, positive probability is placed on all elements of $\mathcal{Q}(\eta)$ and $\mathcal{V}(\eta)$.

What is the intuition for Corollary 1? The range of admissible volatilities easy to understand as purely a consequence of the range of prices. To see this, refer back to equation (RB2) and combine it with (PO) to obtain the following negative price-variance association:

$$|\sigma_R|^2 = \frac{\eta(1-\eta)(a_e - a_h)^2}{q\bar{\rho}(\eta) - \eta a_e - (1-\eta)a_h}\frac{1}{q}, \quad \text{when } \kappa < 1. \tag{24}$$

For a fixed wealth distribution η, return volatility must rise to justify falling capital prices. As prices fall to their "worst-case" level, $q \to \frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}}$, the capital distribution coincides with the wealth distribution, $\kappa \to \eta$, and volatility rises without bound, $|\sigma_R| \to \infty$. Intuitively, agents with heterogeneous productivities but identical preferences will take identical portfolio positions only if risk is so enormous that it swamps other considerations. That said, equation (24) clearly shows that return volatility is pinned down, given (η, q) together.

What, then, justifies the vast range of admissible asset prices? Intuitively, agents understand that future capital price dynamics will keep things stationary and prevent explosive behavior, so current prices can take arbitrary values. But pause to consider what this requires: beliefs and other endogenous objects must vary independently with asset prices, holding fundamentals and other observables fixed. We usually think of asset prices as a function of the economic state, which is enough to forecast the future. Here, prices are themselves part of the state, and agents should use them explicitly in forecasting. This feature, which is absent in the BSEs of Section 2, is reminiscent of Spear (1989), where prices themselves serve as coordination devices.

Figure 4 plots the admissible set of η and q, along with return volatility $|\sigma_R|$ at each point in the space. For reference, we also place the BSE (with $\sigma = 0$) and the fundamental equilibrium (with $\sigma = 0.1$). These equilibria attain only 10-20% volatility, a tiny amount of what a GBSE can do.

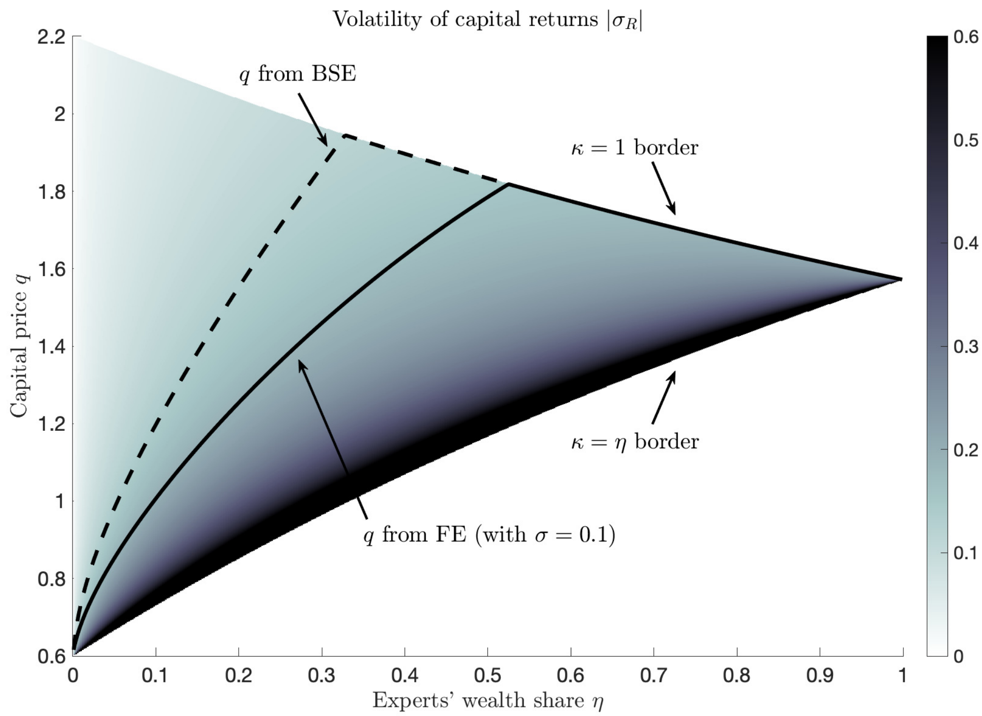

Figure 4: Colormap of capital return volatility $|\sigma_R|$ as a function of (η, q), within the region $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1-\eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Volatility is truncated for aesthetic purposes (because $|\sigma_R| \to \infty$ as $\kappa \to \eta$). For reference, also included are the BSE with the same parameters, and the fundamental equilibrium with $\sigma = 0.1$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$.

The second dimension of indeterminacy is the fraction of volatility stemming from the fundamental and sunspot shocks, $Z^{(1)}$ and $Z^{(2)}$, respectively. The reason: when trading, agents only care about total return variance, not its source.[20] Consequently, asset prices and economic activity can be either closely linked to fundamentals, or completely decoupled from them, and this decoupling can be time-varying in arbitrary ways.

[20] This was somewhat obfuscated earlier when we mentioned that there were 7 endogenous objects and 5 equilibrium restrictions. In fact, accounting for the fact that σ_q and σ_η are 2×1 vectors, there are 9 endogenous objects, but only 6 equilibrium restrictions. Economically, the equation that captures "agents only care about total return variance" is (RB2), which only provides 1 restriction on σ_R.

Nevertheless, the next section presents perhaps the most natural example of a GBSE, in which volatility and fundamentals decouple as total volatility rises.

The third and final dimension of indeterminacy relates to the degree of mean-reversion. As suggested earlier, the proof of Theorem 2 only imposes certain boundary conditions on μ_q, allowing almost arbitrary behavior in the interior of the state space. For example, asset prices could almost always behave like a random walk (corresponding to $\mu_q \approx 0$ in the interior), with just enough mean-reversion in extreme states to keep things stationary. Alternatively, fluctuations could be much more transitory in nature, or anything in between. In the next section, we harness this indeterminacy to address recovery speeds.

3.3 Concrete examples and substantive results

We have just demonstrated that sunspot equilibria, which are endemic to this class of models, in principle can support much richer dynamics. Now, we solve some concrete examples to illustrate several substantive results along these lines.

In contrast to the previous subsection's non-Markovian setting (where q acted as the co-state variable), here we implement our sunspot equilibria with an explicit state variable. Although the equilibria we present here are essentially special cases of the GBSEs in Section 3.2,[21] being more explicit about the sunspot state variable is useful for several reasons. First, the Markov equilibrium construction will be pedagogically more familiar to the literature on sunspots. Second, adding a sunspot state variable brings some clarity, as the sunspot state dynamics can be modeled as locally uncorrelated with fundamental shocks. Third, this setting happens to facilitate building sunspot equilibria that are robust to the limited commitment refinement of Section 2.3.

Let s be a pure sunspot that is irrelevant to economic fundamentals and loads on only the second shock (recall $Z^{(1)}$ affects capital and $Z^{(2)}$ does not):[22]

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \cdot dZ_t, \quad s_t \in \mathcal{S}. \tag{25}$$

The time-varying drift and diffusion of s capture cleanly the non-iid nature of sunspot shocks. By contrast, Section 2 did not include additional state variables governing

[21] In particular, given the results and construction of Theorem 2, one simple and obvious way to build a Markov GBSE of Definition 3 is to just start from a GBSE and then put $s = \psi(\eta, q)$ for some function ψ that maps into $(0, 1)$. Instead, in this section we pursue some more constructive approaches.

[22] This is only for clarity. We have solved examples with sunspots correlated to fundamentals, i.e., with $ds_t = \mu_{s,t}dt + \sigma_{s,t}^{(1)}dZ_t^{(1)} + \sigma_{s,t}^{(2)}dZ_t^{(2)}$. An additional feature that emerges relative to what we show here is that $\sigma_s^{(1)}$ can work to reduce asset price volatility at times, unlike $\sigma_s^{(2)}$. See Appendix D.5 for details.

sunspot dynamics, effectively restricting sunspot shocks to be iid. This is one way to view the core distinction between the BSEs of Section 2 from the GBSEs of this section.

Definition 3. A *Markov GBSE* in state variables $(\eta, s) \in \mathcal{D} := (0,1) \times \mathcal{S}$ consists of functions $(q, \kappa, r, \sigma_\eta, \mu_\eta, \sigma_s, \mu_s) : \mathcal{D} \mapsto \mathbb{R}$, each C^2 almost-everywhere, such that the process $(\eta_t, q(\eta_t, s_t), \kappa(\eta_t, s_t), r(\eta_t, s_t))_{t \geq 0}$ is a GBSE of Definition 2.

To satisfy the GBSE requirement of Definition 3, the equilibrium objects must depend in a non-trivial way on s.

Remark 2 (Endogenous sunspot dynamics)**.** *Note that the statement of Definition 3 allows (σ_s, μ_s) to be endogenous, in the sense that they could depend on the wealth distribution η. Our examples in this section purposefully entertain this endogeneity, partly because we think of this as the more interesting and realistic situation. Why? As discussed in Section 3.2, it is completely sensible for agents in our GBSEs to use asset prices directly in forecasting; in particular, sunspot dynamics (σ_s, μ_s) – which are nothing but belief dynamics – themselves should condition on q. But q will depend on both s and η, implying sunspot dynamics (σ_s, μ_s) depend on η too, through q. That said, we verify in Appendix D.6 that similar types of sunspot equilibria can be constructed with exogenously-specified sunspot dynamics, i.e., (σ_s, μ_s) are only functions of s, not η.*

The equilibrium conditions are derived similarly to previous sections. By applying Itô's formula to $q(\eta, s)$, we obtain the capital price volatility σ_q in terms of σ_η. From equation (23), we also have σ_η in terms of σ_q. Solving this two-way feedback, we obtain

$$\sigma_q = \frac{\binom{1}{0}(\kappa - \eta)\sigma \partial_\eta \log q + \binom{0}{1}\sigma_s \partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q}. \tag{26}$$

Using (26) in (RB2), we obtain the following equation linking capital prices, the capital distribution, and sunspot volatility:

$$0 = \min\Big[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\Big(\frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{(1 - (\kappa - \eta)\partial_\eta \log q)^2}\Big)\Big]. \tag{27}$$

Our strategy to find a Markov GBSE is to guess a capital price function $q(\eta, s)$ and then use equation (27) to "back out" the sunspot volatility σ_s that justifies it. We purposefully perform this construction such that sunspots only increase volatility, to highlight their potential for resolving puzzles.

More specifically, suppose a fundamental equilibrium, where sunspots do not matter, exists with equilibrium capital price q^0. We will think of q^0 as the "best-case" capital price, because despite featuring amplification, q^0 inherits no sunspot volatility. Conversely, think of the capital price q^1 associated to an infinite-volatility equilibrium as the "worst-case" capital price. As $|\sigma_q| \to \infty$, volatility becomes investors' main concern, all investors would hold the same portfolios irrespective of their productivity. Mathematically, this means $\kappa \to \eta$, which identifies the worst-case capital price as $q^1 := \frac{\eta a_e + (1-\eta) a_h}{\bar{\rho}}$ from (PO).

Our strategy is essentially to treat the sunspot variable s as a device to shift continuously between the best-case q^0 and the worst-case q^1. Mathematically, we conjecture a capital price that is approximately a weighted average of q^0 and q^1, with weights s and $1-s$.[23] The novelty of our approach here is to then use equation (27) to solve for sunspot volatility σ_s, which will generically depend on experts' wealth share η. Typically, we are given σ_s exogenously, and we regard (27) as a PDE for q (Appendix D.6 shows that this typical approach also works). In the proposition below, we verify that such a construction is indeed an equilibrium.

Proposition 4. *Let Assumption 1 above hold, and assume a fundamental equilibrium exists for each σ small enough.*[24] *Then, for all $\sigma > 0$ small enough, the exists a Markov GBSE. This equilibrium is stationary in the sense that the paths $(\eta_t, s_t)_{t \geq 0}$ induced by $(\sigma_\eta, \mu_\eta, \sigma_s, \mu_s)$ remain in $\mathcal{D}$ almost-surely and possess a non-degenerate stationary distribution. Finally, this equilibrium is robust to small commitment frictions of the type discussed in Section 2.3.*

Relative to Section 3.2, where we were not concerned with the refinement of Section 2.3, here we intentionally build an equilibrium that is robust to small commitment frictions. All that is required is for $\kappa(0, s) = 0$ for all s. Although we do not prove it, there is a sense in which, among equilibria that are robust to commitment frictions, our construction captures all the relevant sunspot equilibria, being a convex combination of best-case and worst-case scenarios.

Numerical example and substantive properties. We construct a numerical example closely following Proposition 4. Appendix B.3 provides details of the numerics.

[23]In this particular equilibrium, capital prices can never literally achieve the "worst-case" capital price q^1, for two technical reasons, both of which ensure that sunspot volatility stays σ_s bounded: (i) to ensure robustness to the selection criterion of Section 2.3, we need $q(\eta, s)$ to behave like the fundamental solution $q^0(\eta)$ for η close enough to zero, and all s; (ii) we need $q(\eta, s) > q^1(\eta)$, so that $\kappa(\eta, s) > \eta$ for all (η, s). Thus, in the proof of Proposition 4, we actually construct q^1 as a close approximation to the worst-case price, such that (i) and (ii) are satisfied.

[24]This latter assumption is relatively benign, since we know a BSE exists (Proposition 1), and since the BSE corresponds to the limiting economy as $\sigma \to 0$ (Proposition 3). In practice, we can numerically find a fundamental equilibrium for any reasonable value of $\sigma > 0$.

In our example, the sunspot variable s lives in $\mathcal{S} = (0, 1)$, and spans between the fundamental equilibrium (as $s \to 0$) and the worst-case equilibrium (as $s \to 1$). The left panel of Figure 5 shows the capital price function in this construction. Positive sunspot shocks reduce the capital price, independently of experts' wealth share η.

The middle panel of Figure 5 displays capital return volatility, which can be substantially greater than in the fundamental equilibrium. As discussed above, such large volatility is necessary as a self-fulfilling mechanism to justify low capital prices. Implied by capital return volatility is an underlying sunspot shock size σ_s, which is displayed in the right panel of Figure 5. Sunspot dynamics become more volatile both as experts become poor (η shrinks) and as the economy approaches the worst-case equilibrium (s rises). The strong dependence of σ_s on η is the notion of endogenous beliefs that can occur in a GBSE.

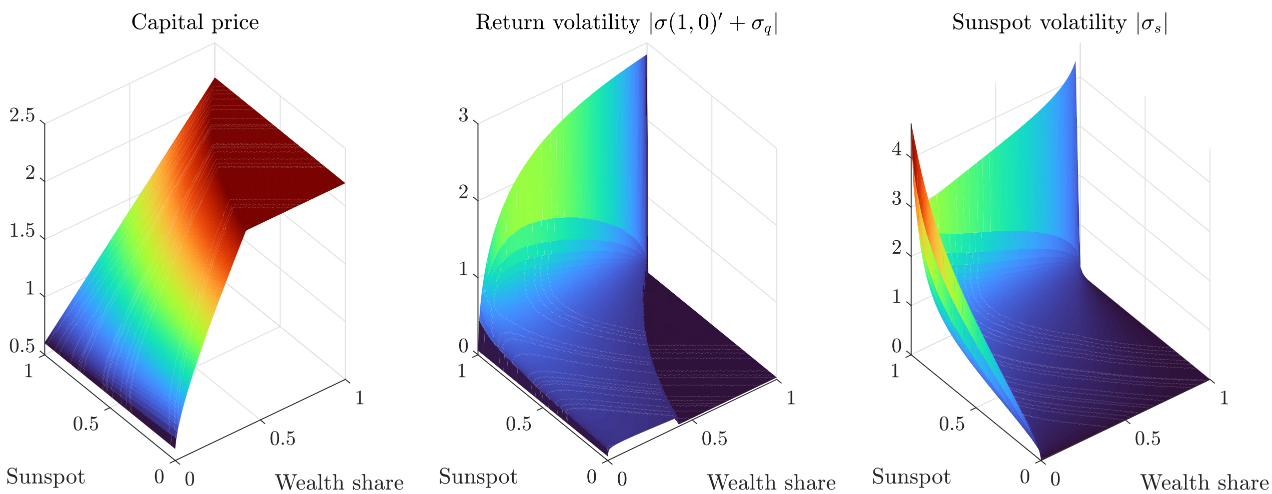

Figure 5: Capital price q, volatility of capital returns $|(1, 0)'\sigma + \sigma_q|$, and sunspot shock volatility σ_s. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

The magnitude of the difference between the sunspot and fundamental equilibrium can be quantified in Figure 6. We plot unconditional cumulative distribution functions for asset prices (left panel) and return volatility (right panel) in both equilibria. We also plot the CDF for $\binom{1}{0} \cdot \sigma_q$, which highlights the contribution to total volatility coming from the fundamental shock $Z^{(1)}$. Relative to the fundamental equilibrium, sensitivity to fundamental shocks $Z^{(1)}$ is reduced in the sunspot equilibrium, with the difference more than made up by sunspot shocks $Z^{(2)}$. In the next two results, we formalize and extend these notions, i.e., that sunspot equilibria can feature much higher volatility (Proposition 5) which is decoupled from fundamentals (Proposition 6).

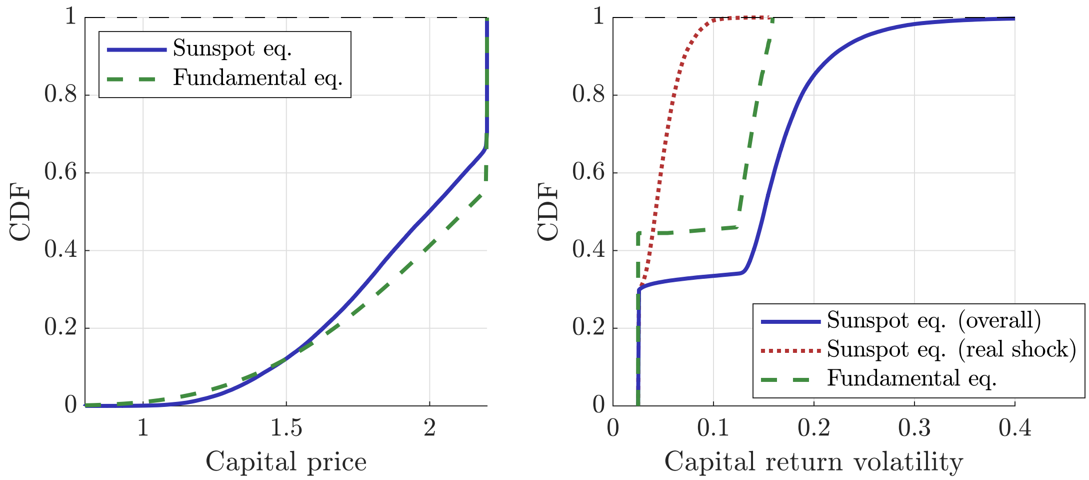

Figure 6: Unconditional CDFs of capital prices and capital return volatility. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = 0.05 + s^{-1.5} - (s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

Proposition 5 (Arbitrary volatility)**.** *Given any target variance $\Sigma^* > 0$ and any parameters satisfying the assumptions of Proposition 4, there exists a Markov GBSE with stationary average return variance exceeding the target, i.e.,* $\mathbb{E}[|\sigma\binom{1}{0} + \sigma_{q,t}|^2] > \Sigma^*$.

Proposition 6 (Volatility decoupling)**.** *The class of Markov GBSEs from Proposition 4 all possess the following monotonicity properties on* $\{(\eta, s) : \kappa < 1,\ \eta > \epsilon,\ \text{some } \epsilon \geq 0\}$:

(i) amplified fundamental volatility $|\binom{1}{0} \cdot \sigma_q|$ *is decreasing in s;*

(ii) sunspot price volatility $|\binom{0}{1} \cdot \sigma_q|$ *is increasing in s;*

(iii) total return volatility $|\binom{1}{0}\sigma + \sigma_q|$ *is increasing in s.*

In the introduction, we suggested the macroeconomics literature has faced difficulties generating high-enough amplification to replicate financial crises; Proposition 5 provides some resolution to that puzzle. The literature has also struggled to identify a "smoking gun" (e.g., TFP shocks, capital efficiency shocks) for financial crises and dramatic price swings; Proposition 6, which shows how asset prices decouple from fundamental shocks at some times but not others, provides helpful insights into this issue.

In the proof of Proposition 5, we leverage an important degree of freedom in Markov GBSEs: besides stability requirements, nothing pins down μ_s at all. Figure 6 specified an ad-hoc sunspot drift μ_s that allowed s_t to visit most of the states between 0 and 1; in other words, for that figure, observed capital prices span most of the range between the best-case capital price q^0 (corresponding to fundamental equilibrium) and the worst-case capital price q^1 (corresponding to an infinite-volatility equilibrium). Alternatively, one can design μ_s to keep s_t forever in the vicinity of the infinite-volatility equilibrium.[25]

The flexibility in μ_s can also help address another puzzle: empirically, recoveries from financial crises are much faster than predictions from models based on bank balance sheets. To see the problem and how sunspots can help, Figure 7 compares impulse responses to large balance-sheet shock (i.e., decline in η) versus a sunspot (i.e., increase in s). The solid lines show the extremely slow speed of recovery from a "balance-sheet recession," with half-lives on the order of 10 years. By contrast, the dashed lines show accelerated recoveries from a "self-fulfilled crisis" driven by high s, with complete recovery in 5 years. Sunspot recovery speeds are closer to empirical evidence.[26]

Moreover, the volatility IRFs in Figure 7 can be read as term structures of uncertainty. The steep downward-slope of the sunspot IRF suggests a plethora of short-term uncertainty, relative to the more modest slope of the balance-sheet IRF. Sunspot-driven recessions are thus more consistent with evidence that uncertainty term structures display strong negative slopes in times of recession (Gormsen, 2020).

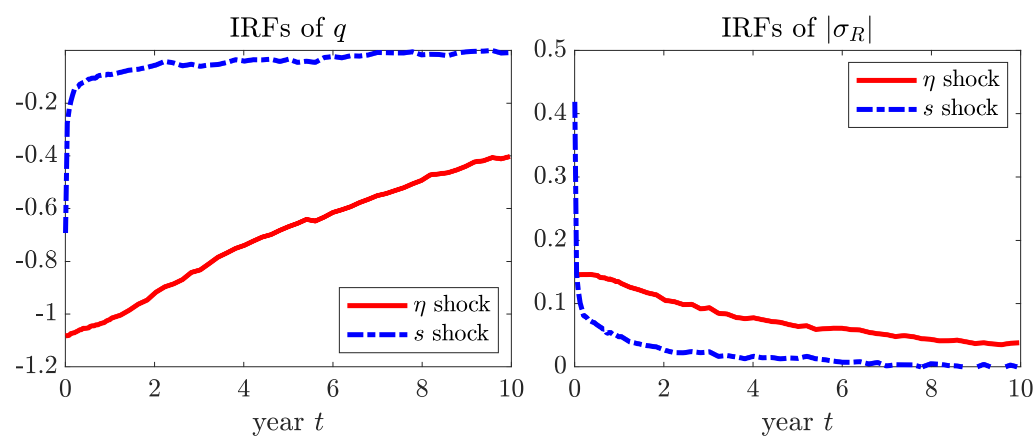

Figure 7: Impulse response functions (IRFs) of capital price q and return volatility $|\sigma_R|$, in response to two shocks. The IRFs labeled "η shock" are responses to a decrease in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.1$, holding s_0 fixed at 0.1. The IRFs labeled "s shock" are responses to an increase in s from $s_{0-} = 0.1$ to $s_0 = 0.9$, holding η_0 fixed at 0.5. Note that η_0 would respond to an "s shock," since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 500 simulations at daily frequency. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = 0.05 + s^{-1.5} - (s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

[25]For example, if we set $\mu_s = -(s_{\max} - s)^{-(1+\beta)} + (s - s_{\min})^{-(1+\beta)}$ for any $\beta > 0$, the stationary distribution of s_t will live in $(s_{\min}, s_{\max})$. This exact claim is demonstrated in the proof of Proposition 4.

[26]For a rough idea of what the data says about crisis recoveries, see Jordà et al. (2013) and Reinhart and Rogoff (2014) for output, and see Muir (2017) and Krishnamurthy and Muir (2020) for credit spreads and stock prices. Across these many measures, and using broad-based international panels, crisis recovery times tend to range from 4-6 years on average.

Remark 3 (What is a sunspot?). *What is the variable s in reality? Mathematically, a sunspot can be anything agents coordinate on. But perhaps it is natural to think of agents coordinating on variables that actually have real effects. We entertain this situation in Appendix D.6, where fundamental volatility is modeled as a function of s. There, we illustrate two main results:*

(a) Depending on coordination on s, equilibrium can either behave more "intuitively" or less. Intuition usually says that prices move inversely to fundamental volatility, but coordination can easily reverse this relationship.

(b) As the contribution of s to fundamental volatility becomes small, equilibrium converges to one where s still matters, i.e., a sunspot equilibrium. Thus, sunspot equilibria are limits of fundamental equilibria, with a vanishingly-small real contribution of s.

Remark 4 (Refinements). *With the availability of a much richer class of sunspot equilibria, all of which are robust to the limited commitment refinement of Section 2.3, one wonders whether some other refinements might provide interesting restrictions on the dynamics.*

In principle, one could employ global-game-type methods with aggregate shocks to help select equilibria (Frankel and Pauzner, 2000); after all, our entire mechanism is predicated on the strategic complementarities experts face when engaging in capital fire-sales. This can substantially refine the set of equilibria (Frankel et al., 2003), but applied use of such methods has been limited to binary action sets (Plantin and Shin, 2008) or highly simplified static settings (Guimaraes and Morris, 2007). Solving and refining a truly dynamic model with strategic complementarities, continuous aggregate states, and continuous actions presents a methodological challenge.

Remark 5 (Learning). *An interesting but difficult exercise would be to study whether sunspot equilibria are robust to adaptive learning. Imagine agents learn about the diffusion $\sigma_{q,t}$ over time – would this process converge to sunspot or fundamental equilibria? For example, such learning could be formalized via "constant gains learning" (Evans and Honkapohja, 2012) that forms beliefs $\hat{\sigma}_{q,t}$ based on exponentially-weighting past observations $(\sigma_{q,s})_{s\leq t}$, e.g.,*

$$\hat{\sigma}_{q,t} = e^{-\beta t} \underbrace{\hat{\sigma}_{q,0}}_{\substack{\textit{initial} \\ \textit{belief}}} + \underbrace{\beta \int_0^t e^{-\beta(t-s)} \sigma_{q,s} ds}_{\textit{weighted-average observations}} .$$

The main obstacle is that, in continuous time, learning about volatility cannot be built on a probability space framework, and we must use a richer setting (Epstein and Ji, 2014).

4 Multiplicity without financial frictions?

In this section, we elaborate on the true prerequisites for sunspot volatility. The framework of Sections 1-3 requires both (i) a link between asset prices and the real economy and (ii) financial frictions. Without (i) a price-output link, capital ownership cannot affect asset prices (e.g., if $a_e = a_h = a$, then $q = a/\bar{\rho}$ independently of κ).[27] Without (ii) financial frictions, risk-sharing is decoupled from capital ownership, and the price-output link becomes trivial (e.g., if experts could issue unlimited equity, they would always manage the entire capital stock, $\kappa = 1$). The fact that the wealth distribution is an endogenous state variable is itself playing no role in supporting multiplicity.[28]

In this section, we clarify this point even further by example. We present a complete-markets economy with nominal rigidities that supports self-fulfilling fluctuations. This economy features a price-output link, but has no financial friction (and the wealth distribution plays no role). We closely follow Caballero and Simsek (2020c), which the reader can consult for more details, but the results on multiplicity are new.

Economic environment. Because of complete markets, it is without loss of generality to assume all agents have the same productivity a.

However, because of nominal rigidities, firms may not always operate at capacity. In particular, let $\chi \leq 1$ denote firms' capital utilization, which will be determined in equilibrium. Aggregate output is χK, where the aggregate capital stock K grows deterministically at rate g. Assume (i) all firms have completely-sticky prices; and (ii) the nominal interest rate is set by monetary policy to implement $\chi = 1$ whenever possible, subject to a zero-lower bound. Assumptions (i) and (ii) imply a zero-lower bound on the real rate, and they imply full utilization whenever the real rate is positive:

$$0 = \min[1 - \chi, r]. \tag{28}$$

[27] This is despite the fact that our model, like Bacchetta et al. (2012), always possesses a negative relationship between asset prices and volatility – see equation (24). Nevertheless, general equilibrium imposes a strong discipline: from (PO), asset prices are given by $a_e/\bar{\rho}$ in the cases $a_h = -\infty$ or $a_e = a_h$. This discipline occurs through the endogenous adjustments by the interest rate. One can only conclude that "self-fulfilling risk panics" of Bacchetta et al. (2012) must be relying on auxiliary features (either OLG or an uncleared bond market).

[28] One sharp way to see this is to consider an endowment economy with heterogeneous preferences or beliefs. Such economies connect the wealth distribution to asset prices, but they generically feature a unique equilibrium (Chan and Kogan, 2002; Bhamra and Uppal, 2014; Borovička, 2020). Even tacking on financial frictions like short-selling and equity-issuance constraints, we are skeptical of sunspot volatility in such a world: if all agents' unconstrained optimal portfolios are positive, then the constraints are effectively non-binding. Markets would be effectively complete and externality-free, ruling out sunspots. It is possible that extreme pessimists would face binding short-sales constraints, but even papers with this feature find unique equilibrium (Scheinkman and Xiong, 2003).

Caballero and Simsek (2020c) discuss a few auxiliary assumptions (lump sum profit taxes and linear capital subsidies) designed to simplify the analysis, which we also implicitly adopt, to ensure the market portfolio dividend equals aggregate output χK.

To prepare for sunspot equilibria, introduce an extrinsic one-dimensional Brownian motion Z that could affect asset prices. In particular, conjecture the capital price q evolves as

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t].$$

Finally, all agents have logarithmic utility with discount rate ρ.

Equilibrium. Agents consume a fraction ρ of their wealth, so goods market clearing can be written as a price-output relation, similar to (PO):

$$\rho q = \chi a. \tag{29}$$

Portfolio choices are unconstrained, and imply the following Euler equation:

$$r = \frac{\chi a}{q} + g + \mu_q - \sigma_q^2. \tag{30}$$

Using (29), the first term on the right-hand-side of (30) equals ρ. Equilibrium involves solutions for processes (q, χ, r) that solve (28)-(30).

There is always an equilibrium with full utilization, $\chi = 1$.[29] Using $\chi = 1$ in equation (29), we obtain $q = a/\rho$, so $\mu_q = \sigma_q = 0$ must hold in this equilibrium. By equation (30), the interest rate is given by $r = \rho + g > 0$, which satisfies (28).

Lemma 4. *The complete-markets model with nominal rigidities always supports an efficient equilibrium with full utilization at all times, $\chi_t = 1$, and zero volatility.*

The efficient equilibrium is "unstable" in the traditional sense. Indeed, suppose $\sigma_q = 0$ so that all dynamics are deterministic, and consider an allocation with $\chi < 1$. By (28), under-utilization implies a binding ZLB. Using $r = 0$ and $\sigma_q = 0$ in (30) implies that $\mu_q = -(\rho + g)$, so q must be converging to zero asymptotically. This instability means, in a deterministic equilibrium, q_t can never take any value other than its efficient steady-state value a/ρ.

[29] If aggregate capital K had additional exogenous fundamental volatility $\sigma > 0$, then the full-utilization equilibrium disappears if σ is large enough, which is the point of Caballero and Simsek (2020c). See also Caballero and Farhi (2018) for a related observation (due to safe asset shortage, which additionally requires a financial friction). Our point, by contrast, is that an inefficient equilibrium exists even when an efficient one does too. This point is most easily expressed under $\sigma = 0$.

Nevertheless, an inefficient volatile equilibrium can emerge. Suppose again $\chi < 1$ but do not impose $\sigma_q = 0$. Using $r = 0$ in (30), we obtain $\mu_q = -(\rho + g) + \sigma_q^2$. Equilibrium places no further restrictions, except that (σ_q, μ_q) must keep $q_t \in (0, a/\rho]$. This is a relatively modest requirement, because the presence of volatility σ_q now adds a risk premium to the price drift, which buoys price q and restores "stability." As σ_q is indeterminate, there are many ways to do this. Appendix D.7 proves the following theorem, and Figure 8 below displays a numerical example.

Theorem 3. *Let $h : \mathbb{R} \mapsto \mathbb{R}_+$ be any Lipschitz continuous function that is strictly positive on $(0, a/\rho)$ and satisfies $h(0) > \rho + g$. The complete-markets model with nominal rigidities always supports an inefficient and volatile equilibrium, in which $\sigma_{q,t}^2 = h(q_t)$ and $\mu_{q,t} = -\rho - g + h(q_t)$ whenever $q_t < a/\rho$. Furthermore, the inefficient equilibrium can be permanent, transitory, or anything in between, i.e., the stationary probability of inefficiency can be any $\pi \in [0, 1]$.*

(i) If $h(a/\rho) = 0$, then inefficiency is permanent: $\pi = 1$.

(ii) If $h(a/\rho) > 0$, inefficiency eventually subsides but can re-emerge: $\pi < 1$.

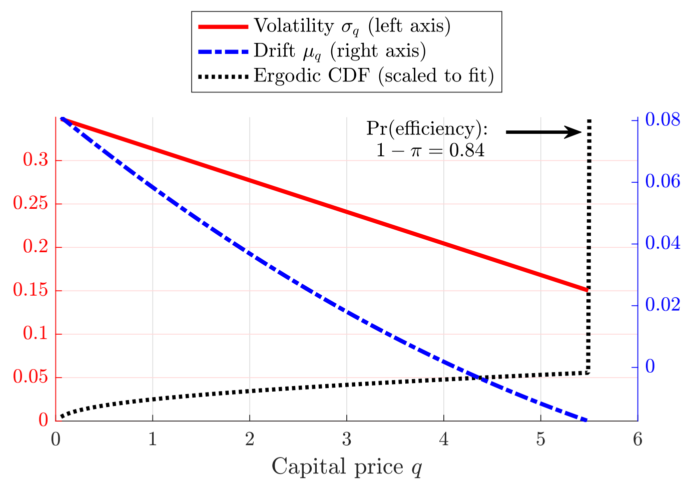

Figure 8: Equilibrium with nominal rigidities. We set volatility $\sigma_q(q) = (1 - \rho q/a)\sqrt{\rho + g} + 0.15$ when $q < a/\rho$, which meets conditions of Theorem 3. To compute the stationary CDF, we specify the dynamics of an auxiliary diffusion x on domain $(0, 1.1a/\rho)$, and put $q = \min[x, a/\rho]$. When $x < a/\rho$, dynamics of x and q match, by definition. When $x > a/\rho$ (i.e., $q = a/\rho$), dynamics of x can be set arbitrarily, and they control how long q stays at a/ρ. The resulting stationary CDF features a mass point of size $1 - \pi = 0.84$ at $q = a/\rho$ (i.e., inefficiency occurs 16% of the time). Parameters: $a = 0.11$, $\rho = 0.02$, $g = 0.02$.

The reader may find it surprising that multiplicity is possible in a representative-agent environment. To understand the logistics, suppose agents are suddenly *fearful*, and they conjecture $\sigma_q > 0$. Is this justified? Fear leads to precautionary savings demand, putting downward pressure on the interest rate. Without a ZLB, r would fall enough to clear bond markets, and agents consume and save the same as before. Thus, there is no impact on goods markets, utilization χ and price q remain fixed, and agents' fear is unsubstantiated. Forward-looking agents can think through this entire hypothetical sequence of events, and they will reject the feeling of fear as irrational.

By contrast, suppose fear and its associated precautionary savings pushes r to the ZLB. Markets only clear if a counteracting force reduces savings, which is why wealth must fall. Due to wealth effects, current consumption also falls, and firms meet their lower demand by operating at less than full capacity in production ($\chi < 1$). Although this process is inefficient, nothing makes this sequence of hypothetical events irrational. Agents' fear will be justified, so long as it does not lead to explosive long-run behavior. Mathematically, non-explosiveness boils down to the requirement that q_t never attain zero, which is precisely what Theorem 3 ensures.

This economy showcases an aggregate demand externality. Because production is demand-determined at the ZLB, a fear-triggered consumption downturn reduces current and future output, hence total wealth, hence other agents' consumption. Conversely, our baseline model features an aggregate supply externality: when fire-selling capital, experts do not account for their impact on production efficiency. And yet this distinction between demand and supply externalities is immaterial to the existence of sunspot equilibria. Commonalities between these frameworks suggest the necessary ingredient is a link between asset prices and the real economy, which can be achieved via financial frictions or via nominal rigidities.[30]

Note that although this economy is stylized, the insights are general: we can add fundamental risk, other state variables, heterogeneous agents (e.g., some hand-to-mouth), and partial price flexibility, and the results will remain qualitatively unchanged.[31]

5 Conclusion

We have shown that macroeconomic models with financial frictions may inherently permit sunspot volatility. In fact, financial frictions are not necessary, as long as a connection exists between asset prices and the real economy – such a link might arise under nomi-

[30] A related interpretation, offered by Benhabib et al. (2020) in extending the model of Bacchetta et al. (2012), is that asset prices should have a direct impact on the stochastic discount factor, which is exactly what happens with a price-output link. Certain OLG specifications, financial frictions, and as we show here, nominal rigidities all connect asset prices to the SDF.

[31] See Caballero and Simsek (2020a,b,c) for some of these extensions. The show that the mechanics of the price-output relation, which is the key here, are unchanged.

nal rigidities, even with complete markets. The types of models we study are extremely common in macroeconomics, so this phenomenon cannot be ignored.

Our paper demonstrates how a fully-rational notion of "sentiments" can be a powerful input into macro-finance dynamics. Unbounded amplification, sharp volatility spikes, and transitory financial crises are among the many interesting possibilities raised by our framework, all robust to simple equilibrium refinements. A deeper analysis of refinements still remains to be done.

While our model is minimalist for theoretical clarity, one wonders whether sunspot equilibria can emerge in more complex medium-scale DSGE models. We see no reason why not. For example, many macro-finance models are log-linearized around some baseline equilibrium point (e.g., the steady state). Nothing about this procedure rules out sunspot equilibria, as we show for our specific model in Appendix B.4.

This suggests a modicum of caution to researchers employing numerical techniques to solve and analyze DSGE models that are built upon the core assumptions in our paper – these procedures implicitly select an equilibrium, without any explicit justification. Designing macro-financial stability policy around an implicit equilibrium selection is unsatisfying, to say the least, and potentially highly problematic.[32]

[32] Many studies in the recent literature on macro dynamics with financing frictions have moved toward policy analysis. See Phelan (2016), Dávila and Korinek (2018), Drechsler et al. (2018), Di Tella (2019), Silva (2017), Elenev et al. (2021), Begenau (2020), Begenau and Landvoigt (2021), and Klimenko et al. (2016).

References

Azariadis, Costas, "Self-fulfilling prophecies," *Journal of Economic Theory*, 1981, *25* (3), 380–396.

__ **and Allan Drazen**, "Threshold externalities in economic development," *The Quarterly Journal of Economics*, 1990, *105* (2), 501–526.

__ **and Bruce Smith**, "Financial intermediation and regime switching in business cycles," *The American Economic Review*, 1998, pp. 516–536.

__ **, Leo Kaas, and Yi Wen**, "Self-fulfilling credit cycles," *The Review of Economic Studies*, 2016, *83* (4), 1364–1405.

Bacchetta, Philippe, Cédric Tille, and Eric Van Wincoop, "Self-fulfilling risk panics," *The American Economic Review*, 2012, *102* (7), 3674–3700.

Begenau, Juliane, "Capital requirements, risk choice, and liquidity provision in a business-cycle model," *Journal of Financial Economics*, 2020, *136* (2), 355–378.

__ **and Tim Landvoigt**, "Financial regulation in a quantitative model of the modern banking system," 2021. Unpublished working paper.

Benhabib, Jess and Pengfei Wang, "Private information and sunspots in sequential asset markets," *Journal of Economic Theory*, 2015, *158*, 558–584.

__ **,** __ **, and Yi Wen**, "Sentiments and aggregate demand fluctuations," *Econometrica*, 2015, *83* (2), 549–585.

__ **, Xuewen Liu, and Pengfei Wang**, "Sentiments, financial markets, and macroeconomic fluctuations," *Journal of Financial Economics*, 2016, *120* (2), 420–443.

__ **,** __ **, and** __ , "Financial markets, the real economy, and self-fulfilling uncertainties," *The Journal of Finance*, 2019, *74* (3), 1503–1557.

__ **,** __ **, and** __ , "Self-Fulfilling Risk Panics: An Expected Utility Framework," 2020. Unpublished working paper.

Bhamra, Harjoat S and Raman Uppal, "Asset prices with heterogeneity in preferences and beliefs," *The Review of Financial Studies*, 2014, *27* (2), 519–580.

Borovička, Jaroslav, "Survival and long-run dynamics with heterogeneous beliefs under recursive preferences," *Journal of Political Economy*, 2020, *128* (1), 206–251.

Brunnermeier, Markus K and Yuliy Sannikov, "A Macroeconomic Model with a Financial Sector," *The American Economic Review*, 2014, *104* (2), 379–421.

__ **and** __ , "Macro, money, and finance: A continuous-time approach," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 1497–1545.

Caballero, Ricardo J and Alp Simsek, "Asset prices and aggregate demand in a "Covid-19" shock: A model of endogenous risk intolerance and LSAPS," 2020. Unpublished working paper.

__ **and** __ , "Monetary Policy with Opinionated Markets," 2020. Unpublished working paper.

__ **and** __ , "A risk-centric model of demand recessions and speculation," *The Quarterly Journal of Economics*, 2020, *135* (3), 1493–1566.

__ **and Emmanuel Farhi**, "The safety trap," *The Review of Economic Studies*, 2018, *85* (1), 223–274.

Cass, David and Karl Shell, "Do sunspots matter?," *Journal of Political Economy*, 1983, *91* (2), 193–227.

Chan, Yeung Lewis and Leonid Kogan, "Catching up with the Joneses: Heterogeneous preferences and the dynamics of asset prices," *Journal of Political Economy*, 2002, *110* (6), 1255–1285.

Dávila, Eduardo and Anton Korinek, "Pecuniary externalities in economies with financial frictions," *The Review of Economic Studies*, 2018, *85* (1), 352–395.

Drechsler, Itamar, Alexi Savov, and Philipp Schnabl, "A model of monetary policy and risk premia," *The Journal of Finance*, 2018, *73* (1), 317–373.

Elenev, Vadim, Tim Landvoigt, and Stijn Van Nieuwerburgh, "A macroeconomic model with financially constrained producers and intermediaries," *Econometrica (forthcoming)*, 2021.

Epstein, Larry G and Shaolin Ji, "Ambiguous volatility, possibility and utility in continuous time," *Journal of Mathematical Economics*, 2014, *50*, 269–282.

Evans, George W and Seppo Honkapohja, *Learning and expectations in macroeconomics*, Princeton University Press, 2012.

Farhi, Emmanuel and Jean Tirole, "Bubbly liquidity," *The Review of Economic Studies*, 2012, *79* (2), 678–706.

Farmer, Roger EA and Jess Benhabib, "Indeterminacy and increasing returns," *Journal of Economic Theory*, 1994, *63*, 19–41.

Frankel, David and Ady Pauzner, "Resolving indeterminacy in dynamic settings: the role of shocks," *The Quarterly Journal of Economics*, 2000, *115* (1), 285–304.

Frankel, David M, Stephen Morris, and Ady Pauzner, "Equilibrium selection in global games with strategic complementarities," *Journal of Economic Theory*, 2003, *108* (1), 1–44.

Gârleanu, Nicolae and Stavros Panageas, "What to Expect when Everyone is Expecting: Self-Fulfilling Expectations and Asset-Pricing Puzzles," *Journal of Financial Economics*, 2020.

Gertler, Mark and Nobuhiro Kiyotaki, "Financial intermediation and credit policy in business cycle analysis," *Handbook of Monetary Economics*, 2010, *3* (3), 547–599.

__ and __ , "Banking, liquidity, and bank runs in an infinite horizon economy," *The American Economic Review*, 2015, *105* (7), 2011–43.

__ , __ , and Andrea Prestipino, "A macroeconomic model with financial panics," *The Review of Economic Studies*, 2020, *87* (1), 240–288.

Gormsen, Niels Joachim, "Time variation of the equity term structure," *Journal of Finance (forthcoming)*, 2020.

Gottardi, Piero and Atsushi Kajii, "The structure of sunspot equilibria: the role of multiplicity," *The Review of Economic Studies*, 1999, *66* (3), 713–732.

Gu, Chao, Fabrizio Mattesini, Cyril Monnet, and Randall Wright, "Endogenous credit cycles," *Journal of Political Economy*, 2013, *121* (5), 940–965.

Guimaraes, Bernardo and Stephen Morris, "Risk and wealth in a model of self-fulfilling currency attacks," *Journal of Monetary Economics*, 2007, *54* (8), 2205–2230.

Harsanyi, John C and Reinhard Selten, "A general theory of equilibrium selection in games," *MIT Press Books*, 1988, *1.*

He, Zhiguo and Arvind Krishnamurthy, "A Model of Capital and Crises," *The Review of Economic Studies*, 2012, *79* (2), 735–777.

— **and** — , "Intermediary Asset Pricing," *The American Economic Review*, 2013, *103* (2), 732–70.

— **and** — , "A macroeconomic framework for quantifying systemic risk," *American Economic Journal: Macroeconomics*, 2019, *11* (4), 1–37.

Hens, Thorsten, "Do sunspots matter when spot market equilibria are unique?," *Econometrica*, 2000, *68* (2), 435–441.

Hugonnier, Julien, "Rational Asset Pricing Bubbles and Portfolio Constraints," *Journal of Economic Theory*, 2012, *147* (6), 2260–2302.

Jordà, Òscar, Moritz Schularick, and Alan M Taylor, "When credit bites back," *Journal of Money, Credit and Banking*, 2013, *45* (s2), 3–28.

Khasminskii, Rafail, *Stochastic stability of differential equations*, Vol. 66, Springer Science & Business Media, 2011.

Khorrami, Paymon, "Entry and slow-moving capital: using asset markets to infer the costs of risk concentration," 2018. Unpublished working paper. Imperial College London.

— **and Alexander Zentefis**, "Arbitrage and Beliefs," 2020. Unpublished working paper, CESifo.

Kiyotaki, Nobuhiro and John Moore, "Credit cycles," *Journal of Political Economy*, 1997, *105* (2), 211–248.

Klimenko, Nataliya, Sebastian Pfeil, and Jean-Charles Rochet, "A simple macroeconomic model with extreme financial frictions," *Journal of Mathematical Economics*, 2017, *68*, 92–102.

— **,** — **,** — **, and Gianni De Nicolò**, "Aggregate bank capital and credit dynamics," 2016. Unpublished working paper. University of Zurich.

Krishnamurthy, Arvind and Tyler Muir, "How credit cycles across a financial crisis," 2020. Unpublished NBER working paper.

— **and Wenhao Li**, "Dissecting mechanisms of financial crises: Intermediation and sentiment," 2020. Unpublished working paper.

Krylov, Nicolai V, "Diffusion on a plane with reflection. Construction of the process," *Siberian Mathematical Journal*, 1969, *10* (2), 244–252.

— , "On weak uniqueness for some diffusions with discontinuous coefficients," *Stochastic Processes and their Applications*, 2004, *113* (1), 37–64.

Liu, Zheng and Pengfei Wang, "Credit constraints and self-fulfilling business cycles," *American Economic Journal: Macroeconomics*, 2014, *6* (1), 32–69.

Manuelli, Rodolfo and James Peck, "Sunspot-like effects of random endowments," *Journal of Economic Dynamics and Control*, 1992, *16* (2), 193–206.

Maxted, Peter, "A macro-finance model with sentiment," 2020. Unpublished working paper. Harvard University.

Mendo, Fernando, "Risky low-volatility environments and the stability paradox," 2020. Unpublished working paper.

Miao, Jianjun and Pengfei Wang, "Asset bubbles and credit constraints," *The American Economic Review*, 2018, *108* (9), 2590–2628.

Moreira, Alan and Alexi Savov, "The macroeconomics of shadow banking," *The Journal of Finance*, 2017, *72* (6), 2381–2432.

Muir, Tyler, "Financial crises and risk premia," *The Quarterly Journal of Economics*, 2017, *132* (2), 765–809.

Peck, James and Karl Shell, "Market uncertainty: correlated and sunspot equilibria in imperfectly competitive economies," *The Review of Economic Studies*, 1991, *58* (5), 1011–1029.

Phelan, Gregory, "Financial intermediation, leverage, and macroeconomic instability," *American Economic Journal: Macroeconomics*, 2016, *8* (4), 199–224.

Piketty, Thomas, "The dynamics of the wealth distribution and the interest rate with credit rationing," *The Review of Economic Studies*, 1997, *64* (2), 173–189.

Plantin, Guillaume and Hyun Song Shin, "Carry trades and speculative dynamics," 2008. NBER working paper.

Reinhart, Carmen M and Kenneth S Rogoff, "Recovery from financial crises: Evidence from 100 episodes," *The American Economic Review*, 2014, *104* (5), 50–55.

Scheinkman, Jose A and Wei Xiong, "Overconfidence and speculative bubbles," *Journal of Political Economy*, 2003, *111* (6), 1183–1220.

Silva, Dejanir, "The Risk Channel of Unconventional Monetary Policy," 2017. Unpublished working paper. University of Illinois.

Spear, Stephen E, "Are sunspots necessary?," *Journal of Political Economy*, 1989, *97* (4), 965–973.

Tella, Sebastian Di, "Uncertainty shocks and balance sheet recessions," *Journal of Political Economy*, 2017, *125* (6), 2038–2081.

—, "Optimal regulation of financial intermediaries," *The American Economic Review*, 2019, *109* (1), 271–313.

Zentefis, Alexander, "Self-fulfilling asset prices," 2020. Unpublished working paper. Yale University, New Haven, CT.

Appendix

A Proofs for Section 2

A.1 Main proofs

Proof of Lemma 1. We omit the proof, because it follows standard arguments, verification of optimality and market clearing, as in Brunnermeier and Sannikov (2014). □

Proof of Lemma 2. Suppose $\kappa = 1$, $q = a_e/\bar{\rho}$, and $\sigma_q = 0$. Both (PO) and (RB) are satisfied. Furthermore, $\sigma_q = 0$ by (SV), which confirms that $\sigma_\eta = 0$ by the Itô equation $\sigma_q = \frac{q'}{q}\sigma_\eta = 0$. Thus, (8) is also satisfied. By Lemma 1, if we set the drift μ_η by (9), this constitutes an equilibrium. □

Proof of Proposition 1. This is a direct consequence of Theorem 1, by taking $\kappa_0 \to 0$. □

Proof of Theorem 1. In the first step, we prove existence of an equilibrium for fixed $\kappa_0 \in (0,1)$. In the second step, we take the limits as $\kappa_0 \to 0$ and $\kappa_0 \to 1$.

Step 1: Existence. Let $F(x,y) := \frac{a_e - a_h}{y\bar{\rho}(x) - xa_e - (1-x)a_h} y$. Fix $\epsilon > 0$. Consider the initial value problem $y' = F(x,y)$, with $y(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h$. As discussed in the text, $y'(0+)$ is bounded, which is enough to ensure that F is bounded and uniformly Lipschitz on $\mathcal{R} := \{(x,y) : 0 < x < 1,\ xa_e + (1-x)a_h < y\bar{\rho}(x)\}$. Thus, the standard Picard-Lidelöf theorem implies that there exists a unique solution q^* to this initial value problem, for $\eta \in (0,b)$, some b. Standard continuation arguments can be used to show that either (i) $b = 1$, (ii) $q^*(\eta)$ is unbounded as $\eta \to b$, or (iii) b satisfies $ba_e + (1-b)a_h = q^*(b)\bar{\rho}(b)$. If case (ii) is true, since $F > 0$ on $\mathcal{R}$, we will in fact have $q^*(b-) = +\infty$. Case (iii) is ruled out by the fact that $F(b-, q^*(b-)) = +\infty$. We are left with cases (i) or (ii).

In case (i), we will set $\eta^* = \inf\{\eta \in (0,1) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that $\eta^* < 1$ in this case: otherwise $q^*(1-)\bar{\rho}(1-) < a_e$, which implies $F(1-, q^*(1-)) < 0$, which by continuity of q^* and F implies an $\eta^\circ \in (0,1)$ such that $\eta^\circ a_e + (1-\eta^\circ)a_h = q^*(\eta^\circ)$, which was just ruled out (case (iii)). In case (ii), we will set $\eta^* = \inf\{\eta \in (0,b) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that we also clearly have $\eta^* < b < 1$ in this case.

Finally, set $q(\eta) = \mathbf{1}_{\eta < \eta^*} q^*(\eta) + \mathbf{1}_{\eta \geq \eta^*} a_e/\bar{\rho}(\eta)$. This function satisfies $q' = F(\eta, q)$ on $(0,\eta^*)$, $q(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h$, and $q(\eta^*) = a_e/\bar{\rho}(\eta)$. Thus, we have found a solution to the capital price satisfying all the desired relations. As discussed in the text, finding such a capital price is enough to prove that a Markov sunspot equilibrium exists.

Since equation (11) implies $\sigma_q^2 > 0$ on $(0,\eta^*)$, in order to establish $\sigma_q(\eta) \neq 0$ on a positive measure subset, it suffices to show that $\eta^* > 0$. But this is automatically implied by the boundary condition $q(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h < a_e/\rho_h$ for $\kappa_0 < 1$, coupled with the continuity of the solution $q(\eta)$.

Step 2: BSE and FE as limiting equilibria. For each initial condition $\kappa(0) = \kappa_0$, let $(q_{\kappa_0}, \eta^*_{\kappa_0})$ be the associated equilibrium capital price and misallocation threshold (at which point households begin purchasing capital).

Define the candidate solution for the BSE, $(q_0, \eta^*_0) := \lim_{\kappa_0 \to 0}(q_{\kappa_0}, \eta^*_{\kappa_0})$. It suffices to show that q_0 satisfies (i) $q_0' = F(\eta, q_0)$ on $(0, \eta^*_0)$, (ii) $q_0(0) = a_h/\rho_h$, and (iii) $q_0(\eta^*_0) = a_e/\bar{\rho}(\eta^*_0)$. Write the integral version of the ODE:

$$q_{\kappa_0}(\eta) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} + \int_0^\eta F(x, q_{\kappa_0}(x))dx.$$

Next, we claim that $q_{\kappa_0}(x)$ is weakly increasing in κ_0, for each x. Indeed, $q_{\kappa_0}(0)$ is strictly increasing in κ_0. By continuity, we may consider $x^* := \inf\{x : q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)\}$ for some $\tilde{\kappa}_0 > \kappa_0$. In that case, since F does not depend on $\tilde{\kappa}_0$ or κ_0, we have $q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)$ for all $x \geq x^*$. This proves $q_{\tilde{\kappa}_0}(x) \geq q_{\kappa_0}(x)$ for all x. Combine this with the fact that $\partial_q F < 0$ to see that $\{F(x, q_{\kappa_0}(x)) : \kappa_0 \in (0,1)\}$ is a sequence which is monotonically (weakly) decreasing in κ_0, for each x. Thus, by the monotone convergence theorem,

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F(x, q_0(x))dx,$$

which proves (i), by differentiating, and (ii), by substituting $\eta = 0$. Similarly,

$$\begin{aligned} q_{\kappa_0}(\eta^*_{\kappa_0}) &= \frac{a_e}{\bar{\rho}(\eta^*_{\kappa_0})} \\ \xrightarrow{\kappa_0 \to 0} q_0(\eta^*_0) &= \frac{a_e}{\bar{\rho}(\eta^*_0)}, \end{aligned}$$

which proves (iii).

Define the candidate solution for the FE, $(q_1, \eta^*_1) := \lim_{\kappa_0 \to 1}(q_{\kappa_0}, \eta^*_{\kappa_0})$. It suffices to show that $\eta^*_1 = 0$, so that $q_1(\eta) = a_e/\bar{\rho}(\eta)$ for all η. Note that $q_{\kappa_0}(0) \to a_e/\rho_h$ as $\kappa_0 \to 1$. By continuity of $(q_{\kappa_0}, \eta^*_{\kappa_0})$ in κ_0, we also have $q_{\kappa_0}(0) \to q_1(0)$ as $\kappa_0 \to 1$. Thus, $q_1(0) = a_e/\rho_h$. By the definition of $\eta^*_1 = \inf\{\eta : q(\eta) = a_e/\bar{\rho}(\eta)\}$, we must have $\eta^*_1 = 0$. □

Proof of Proposition 3. Note that the other equations characterizing equilibrium, beyond (15), are (PO) and

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\Big]. \tag{A.1}$$

Denote the equilibrium solution for $\sigma > 0$ by $(q^{(\sigma)}, \kappa^{(\sigma)})$. Define $q^{(0)} := \lim_{\sigma \to 0} q^{(\sigma)}$ and $\kappa^{(0)} := \lim_{\sigma \to 0} \kappa^{(\sigma)}$. Combine equations (15) and (A.1) and rearrange terms to get

$$\Big(1 - (\kappa^{(\sigma)} - \eta)\frac{(q^{(\sigma)})'}{q^{(\sigma)}}\Big)^2 = \frac{(\kappa^{(\sigma)} - \eta)q^{(\sigma)}}{\eta(1-\eta)(a_e - a_h)}\sigma, \quad \text{if} \quad \kappa^{(\sigma)} < 1. \tag{A.2}$$

Note that this implies $\kappa^{(\sigma)} > \eta$. Furthermore, continuity of $\kappa^{(\sigma)}(\eta)$ and $\kappa_0 = \kappa^{(\sigma)}(0+) < 1$ imply $\kappa^{(\sigma)}(\eta) < 1$ for all η close enough to 0. Using these facts, and writing (A.2) instead as an integral equation, we obtain

$$\frac{q^{(\sigma)}(\eta_2)}{q^{(\sigma)}(\eta_1)} = \exp\Big\{\int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(\sigma)}(x) - x}\Big[1 \pm \sqrt{\frac{(\kappa^{(\sigma)}(x) - x)q^{(\sigma)}(x)}{x(1-x)(a_e - a_h)}\sigma}\Big]dx\Big\}, \quad 0 < \eta_1 < \eta_2,$$

where η_2 is chosen small enough. Because the right-hand-side is continuous in both $q^{(\sigma)}$ and $\kappa^{(\sigma)}$, and both are bounded, taking the limit as $\sigma \to 0$ implies

$$\frac{q^{(0)}(\eta_2)}{q^{(0)}(\eta_1)} = \exp\left\{ \int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(0)}(x) - x} dx \right\}.$$

Differentiate this equation with respect to η_2 to obtain

$$\frac{d}{d\eta} \log q^{(0)} = \frac{1}{\kappa^{(0)} - \eta},$$

for all η small enough. Rearranging this equation delivers the ODE characterizing the BSE, i.e., selecting the solution $(\kappa - \eta)q'/q = 1$ in equation (SV). Since $\kappa^{(\sigma)}(0+) = \kappa_0$ is fixed for all $\sigma > 0$, we also have the desired boundary condition $\kappa^{(0)}(0+) = \kappa_0$, for any $\kappa_0 \in [0, 1)$. Finally, all the other equations of the BSE can be verified by simply taking limits as $\sigma \to 0$. □

Proof of Proposition 2. In the text leading up to the statement of the proposition. □

A.2 Stationarity of sunspot equilibria

Lemma A.1. *In the model with no fundamental volatility, the dynamics prevent η from reaching zero with probability one. Moreover, if one of the following conditions holds*

$$\delta = 0 \quad \text{and} \quad \rho_e > \rho_h \tag{A.3}$$

$$\delta > 0 \quad \text{and} \quad \nu < \eta^* := \inf\{\eta : \kappa(\eta) = 1\} \quad \left(\text{equivalent to } \nu < \frac{1}{2}(1-\kappa_0^2)\frac{a_e - a_h}{a_e} \text{ when } \rho_e = \rho_h\right) \tag{A.4}$$

then expert wealth share $(\eta_t)_{t\geq 0}$ has a non-degenerate stationary distribution on $(0, \eta^*]$, and when $\eta_t \in (\eta^*, 1)$, it follows a deterministic path towards η^*.

Proof of Lemma A.1. We consider the baseline model of Section 2 with disaster belief $\kappa(0+) = \kappa_0 \in [0, 1)$. For reference, we re-state the dynamics of η:

$$\mu_\eta = \delta\nu - (\rho_e - \rho_h + \delta)\,\eta + \frac{a_e - a_h}{q}[\kappa - 2\kappa\eta + \eta^2]\mathbf{1}_{\eta<\eta^*} + (\rho_e - \rho_h)\,\eta^2 \tag{A.5}$$

$$\sigma_\eta^2 = \eta(1-\eta)(\kappa - \eta)\frac{a_e - a_h}{q}\mathbf{1}_{\eta<\eta^*}, \tag{A.6}$$

where equation (A.6) follows from $\sigma_\eta = (\kappa - \eta)\sigma_q$ in (8) and $\sigma_q^2 = \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e - a_h}{q}\mathbf{1}_{\eta<\eta^*}$ in (11). We proceed in 3 steps, examining dynamics of η above η^*, in a neighborhood just below η^*, and in a neighborhood just above 0.

Step 1: Dynamics for $\eta > \eta^*$. Equation (A.6) shows that $\sigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. Thus, η it follows a deterministic path towards η^* if $\mu_\eta(\eta) < 0$ for all $\eta \in [\eta^*, 1)$. Substituting $\kappa = 1$ into (A.5) and using either of parameter restriction (A.3) or (A.4) above delivers the result immediately. Given the deterministic transition toward η^*, we can ignore the sub-interval $(\eta^*, 1)$ in our state space and instead consider only $(0, \eta^*)$.

In general, consider a one-dimensional process $(X_t)_{t\geq 0}$ with $dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t$ that is a regular diffusion on interval $(e_1, e_2) \subset \mathbb{R}$ (i.e., the dynamics of X depend only on X itself, and imply that it reaches every point in (e_1, e_2) with positive probability). Our process $(\eta_t)_{t\geq 0}$ satisfies these conditions for $e_1 = 0$ and $e_2 = \eta^*$.

In such case, we may apply Feller's boundary classification to decide whether boundaries e_1 and e_2 are inaccessible (avoided forever with probability 1) or accessible. To do so, first define $s(y) := \exp(-\int_{x_0}^{y} \frac{2\mu(u)}{\sigma^2(u)} du)$, $m(x) := \frac{2}{s(x)\sigma^2(x)}$, and let ϵ and x_0 be arbitrary numbers within interval (e_1, e_2). Boundary e_1 is inaccessible if and only if

$$I_1 := \int_{e_1}^{\epsilon} m(x)\Big(\int_{e_1}^{x} s(y)dy\Big)dx = +\infty.$$

Boundary e_2 is accessible if and only if

$$I_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{x}^{e_2} s(y)dy\Big)dx < +\infty.$$

We will prove these results in the next two steps.

Step 2: Dynamics near $e_2 = \eta^*$. Compute

$$\mu_\eta(\eta^*-) = \delta(\nu - \eta^*) - \eta^*(1-\eta^*)(\rho_e - \rho_h) + (1-\eta^*)\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}$$
$$\sigma_\eta^2(\eta^*-) = \eta^*(1-\eta^*)^2\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}.$$

Since $\sigma_\eta^2(\eta^*-)$ is bounded away from zero and $\mu_\eta(\eta^*-)$ is finite, it is easy to check that $I_2 < +\infty$, meaning $e_2 = \eta^*$ is an accessible boundary that is hit in finite time with positive probability. Furthermore, we may also show

$$J_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{\epsilon}^{x} s(y)dy\Big)dx < +\infty,$$

which implies $e_2 = \eta^*$ is a regular boundary that must be included in the state space.

We must establish what occurs when η_t hits boundary point $e_2 = \eta^*$. Recall from step 1 that $\mu_\eta(\eta) < 0$ and $\sigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. This implies that η_t can never enter the region $(\eta^*, 1)$ from η^* and that η_t will not stay at point η^* for an infinite amount of time. Consequently, the region $(0, \eta^*]$ is the ergodic set.

Step 3a: General analysis of dynamics near $e_1 = 0$. First, suppose our diffusion satisfied the following near $e_1 = 0$ (the notation $f(x) \sim g(x)$ means $\lim_{x\to 0} f(x)/g(x) = 1$):

$$\sigma^2(x) \sim \phi x^\beta \quad \phi > 0, \quad \beta \geq 0$$
$$\frac{\mu(x)}{\sigma^2(x)} \sim \theta x^{-\alpha}, \quad \alpha \geq 1, \quad \theta > 0.$$

As we will show below in step 3b, this asymptotic description is flexible enough to cover all cases within our model.

If $\alpha = 1$, we have, for x sufficiently small,

$$S_1(x,\theta) := \int_0^x \frac{s(y)}{s(x)}dy = \int_0^x \exp\Big[2\theta(\log(x) - \log(y))\Big]dy \tag{A.7}$$
$$= x^{2\theta}\lim_{z\downarrow 0}\frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta},$$

so letting ϵ be sufficiently small, we obtain

$$I_1 = \int_0^\epsilon \frac{2x^{2\theta-\beta}}{\phi} \lim_{z \downarrow 0} \frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta} dx.$$

If $2\theta \geq 1$ (note that $2\theta = 1$ corresponds to $\frac{z^{1-2\theta}}{1-2\theta}$ being replaced by $\log(z)$ in the expression above), then the interior limit is $+\infty$ for all $x > 0$ and therefore $I_1 = +\infty$. This holds independently of the value of β. If $2\theta < 1$, then

$$I_1 = \int_0^\epsilon \frac{2}{(1-2\theta)\phi} x^{1-\beta} dx = \frac{2}{(1-2\theta)\phi} \Big(\frac{\epsilon^{2-\beta}}{2-\beta} - \lim_{x \downarrow 0} \frac{x^{2-\beta}}{2-\beta} \Big).$$

So, in this case, $I_1 = +\infty$ only if $\beta \geq 2$ (for $\beta = 2$, $\frac{x^{2-\beta}}{2-\beta}$ is replaced by $\log(x)$).

If $\alpha > 1$ instead, we will show that $I_1 = +\infty$ independent of any other parameters. We have

$$S_\alpha(x, \theta) := \int_0^x \frac{s(y)}{s(x)} dy = \int_0^x \exp\Big[\frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})\Big] dy \tag{A.8}$$

The corresponding expression for the case with $\alpha = 1$ is $S_1(x, \theta)$ in (A.7). We showed above that for $\tau < 1/2$, we have $S_1(x, \tau) = +\infty$. Fix such a τ. We now show that $S_\alpha(x, \theta) \geq S_1(x, \tau)$ for all x sufficiently small and all θ.

Fix any $x > 0$, and define $f(y) := 2\tau(\log(x) - \log(y))$ and $g(y) := \frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})$. Since both functions are strictly positive for $y < x$, and since $\lim_{y \to 0} g(y)/f(y) = \lim_{y \to 0} (\theta/\tau) y^{1-\alpha} = +\infty$, there exists $\bar{y} \in (0, x)$ such that $g(y) > f(y)$ for all $y \in (0, \bar{y})$. From this comparison, we conclude $S_\alpha(\bar{y}, \theta) = \int_0^{\bar{y}} \exp(g(y)) dy \geq \int_0^{\bar{y}} \exp(f(y)) dy = S_1(\bar{y}, \tau) = +\infty$. Since this argument is independent of (β, θ, ϕ), this proves that $I_1 = +\infty$ if $\alpha > 1$.

Step 3b: Model-specific analysis of dynamics near $e_1 = 0$. Now, we map our model dynamics into the setup of step 3a. If $\kappa(0+) = \kappa_0 > 0$, then in the limit as $\eta \to 0$, equations (A.5)-(A.6) become

$$\mu_\eta = \delta\nu + \frac{a_e - a_h}{q(0+)} \kappa_0 - \left(\rho_e - \rho_h + \delta + 2\frac{a_e - a_h}{q(0+)} \kappa_0 \right) \eta + o(\eta)$$
$$\sigma_\eta^2 = \frac{a_e - a_h}{q(0+)} \kappa_0 \eta + o(\eta).$$

Hence, in terms of the notation in step 3a, we have $\alpha = 1$, $\beta = 1$ and $\theta = \frac{\delta\nu q(0+)}{\kappa_0(a_e - a_h)} + 1 > \frac{1}{2}$. Thus, η avoids zero with probability one.

If $\kappa(0+) = 0$, we need to know the rate at which $\kappa \to 0$ as $\eta \to 0$. Guess, and verify after, that $\kappa = \varphi\eta^\omega + o(\eta^\omega)$ in the limit as $\eta \to 0$. Differentiating the price-output condition (PO), we have

$$q' = \frac{1}{\bar{\rho}} \left[(a_e - a_h)\kappa' - (\rho_e - \rho_h) q \right]$$

Combining this with the sunspot differential equation for q, equation (SV), we obtain

$$\left[(a_e - a_h)\kappa' - (\rho_e - \rho_h) q \right] (\kappa - \eta) = \bar{\rho} q.$$

Taking the limit as $\eta \to 0$, we have

$$(a_e - a_h) \lim_{\eta \to 0} (\kappa')(\kappa - \eta) = a_h$$

Hence, the guess is verified if $\omega = 1/2$ and $\varphi^2 = 2a_h/(a_e - a_h) > 0$. Substituting this asymptotic behavior into equations (A.5)-(A.6), we have

$$\mu_\eta = \delta\nu + \sqrt{\frac{2(a_e - a_h)}{a_h}}\rho_h \eta^{1/2} + o(\eta^{1/2})$$
$$\sigma_\eta^2 = \sqrt{\frac{2(a_e - a_h)}{a_h}}\rho_h \eta^{3/2} + o(\eta^{3/2}).$$

If $\delta\nu > 0$, then these dynamics match those of step 3a with $\alpha = 3/2$, $\beta = 3/2$ and $\theta > 0$. If $\delta\nu = 0$, then these dynamics match step 3a with $\alpha = 1$, $\beta = 3/2$, and $\theta = 1$. In either case, we have already shown that η cannot reach zero with probability one.

In summary, $(\eta_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution with support $(0, \eta^*]$, the boundary $\{0\}$ is inaccessible, and the boundary η^* is accessible but non-absorbing. $\square$

B Proofs for Section 3

B.1 Main proofs

Proof of Lemma 3. In the text leading up to the statement of the lemma. □

Proof of Theorem 2. The proof proceeds in several steps.

Step 0: Reduce the system. We will start by eliminating $(r, \kappa, \sigma_\eta, \mu_\eta)$ from the system of endogenous objects, given $(\eta, q, \sigma_q, \mu_q)$. First, price-output relation (PO) determines κ as a function of (η, q) and nothing else, given by

$$\kappa(\eta, q) := \frac{q\bar{\rho}(\eta) - a_h}{a_e - a_h}. \tag{B.1}$$

Second, substituting this result for κ, equation (21) fully determines r, given knowledge of $(\eta, q, \sigma_q, \mu_q)$. Third, equations (22)-(23), after plugging in the result for κ, fully determine (σ_η, μ_η), given knowledge of (η, q, σ_q). Thus, given (η, q), it suffices to determine (σ_q, μ_q) from the remaining equilibrium conditions, i.e., namely (RB2).

Step 1: Define perturbed domain. To facilitate analysis, it will be more convenient to analyze a slightly modified system instead of (η, q), and on a slightly perturbed domain.

First, define the following auxiliary functions. Fix $\epsilon \in (0, \frac{a_e - a_h}{\rho_h})$. Let $\beta(\cdot)$ be a strictly increasing, continuously differentiable function such that $\beta(1) = -\beta(0) = \epsilon$, and $\beta(\eta^*_\beta) = 0$, where $\eta^*_\beta \in (\eta^*, 1)$ and

$$\eta^* := \frac{\rho_h}{\rho_e}\Big(\frac{1 - a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big)^{-1}. \tag{B.2}$$

Note that $\eta^* < 1$ by Assumption 1, part (ii). Let $\alpha(\cdot)$ be an increasing, continuously differentiable function such that $\alpha(0) = 0$, $\alpha'(0) \in (0, \infty)$, and $\alpha(1) = \epsilon/2$.

Next, define the following functions,

$$q^H(\eta) := a_e/\bar{\rho}(\eta)$$
$$q^L(\eta) := \bar{a}(\eta)/\bar{\rho}(\eta),$$

where $\bar{a}(\eta) := \eta\rho_e + (1-\eta)\rho_h$. Using (B.1), one notices that q^H corresponds to the capital price when $\kappa = 1$, whereas q^L corresponds to the capital price when $\kappa = \eta$. Put

$$q^H_\beta(\eta) := q^H(\eta) + \beta(\eta)$$
$$q^L_\alpha(\eta) := q^L(\eta) + \alpha(\eta).$$

Using these functions, define the perturbed domain (which is an open set)

$$\mathcal{X} := \Big\{(\eta, x) \,:\, \eta \in (0,1) \quad \text{and} \quad q^L_\alpha(\eta) < x < q^H_\beta(\eta)\Big\}.$$

Note that, boundaries aside, $\mathcal{X}$ will coincide with $\mathcal{D}$ as $\epsilon \to 0$. For reference, the perturbed domain $\mathcal{X}$ is displayed in Figure B.1.

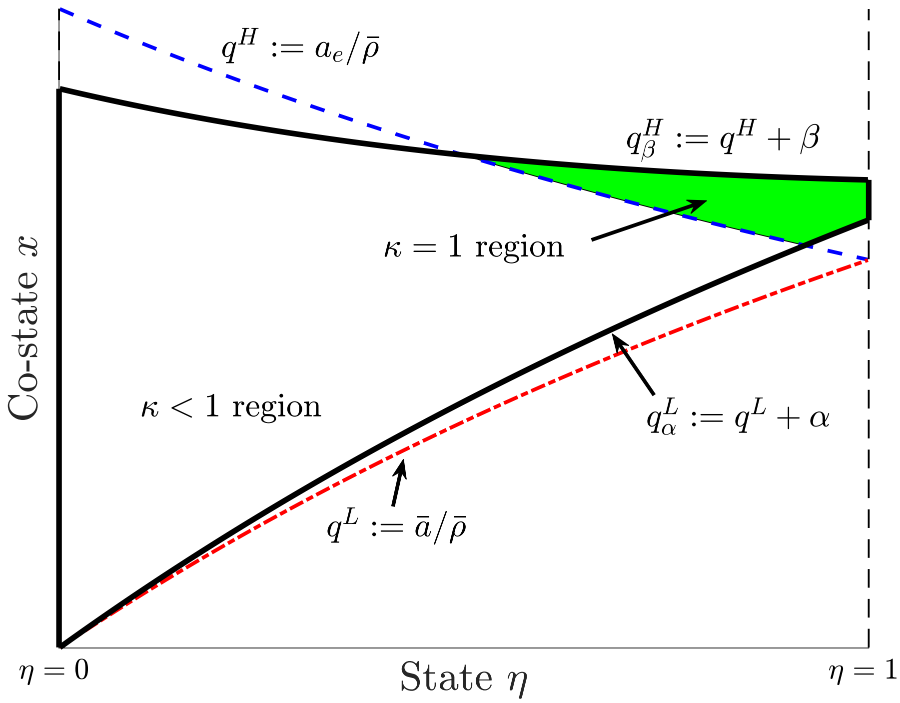

Figure B.1: The perturbed domain $\mathcal{X}$ is shown as the region surrounded by solid black lines. The original domain $\mathcal{D}$ is the region defined by the dashed lines. The perturbation functions α and β are chosen to be linear functions, with $\epsilon = 0.2$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$.

We will define a stochastic process x_t such that the capital price q coincides with x when it lies below q^H, i.e.,

$$q_t = \min\Big[x_t,\, q^H(\eta_t)\Big]. \tag{B.3}$$

By (B.3), we may analyze the dynamical system $(\eta_t, x_t)_{t\geq 0}$ rather than $(\eta_t, q_t)_{t\geq 0}$. Furthermore, to prove the claim that $(\eta_t, q_t)_{t\geq 0}$ remains in $\mathcal{D}$ almost-surely, it suffices to prove $(\eta_t, x_t)_{t\geq 0}$ remains in $\mathcal{X}$ almost-surely (Step 4 below).

Step 2: Construct σ_q so that (RB2) is satisfied. First consider $\{x < q^H(\eta)\}$ so that $q = x$. Note that this case corresponds to $\kappa < 1$. Let $\gamma(\eta, x) : \mathcal{X} \mapsto (0,1)$ be any C^2 function. Put

$$\sigma_q = \begin{bmatrix} \sqrt{\gamma \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e-a_h}{q}} - \sigma \\ \sqrt{(1-\gamma)\frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e-a_h}{q}} \end{bmatrix}, \quad \text{if } x < q^H(\eta). \tag{B.4}$$

Substituting (B.4), one can verify that the second term of condition (RB2) is zero. Importantly, the definitions of q^L_α and q^H_β imply that σ_q is bounded on $\mathcal{X} \cap \{x < q^H(\eta)\}$. Indeed, because of $\alpha'(0) > 0$, the slowest possible rate that $\kappa \to 0$ as $\eta \to 0$ is lower-bounded away from 1, i.e., $\liminf_{\eta\to 0, (\eta,x)\in\mathcal{X}} \kappa/\eta > 1$. And because $\alpha(1) > 0$, we have $\kappa = 1$ for all η near enough to 1; thus η is bounded away from 1 on $\{x < q^H(\eta)\}$.

Next consider $\{x \geq q^H(\eta)\}$ so that $q = q^H(\eta)$. Note that this case corresponds to $\kappa = 1$. Since q is an explicit function of η, we use Itô's formula to compute $\binom{1}{0} \cdot \sigma_q = -\sigma_\eta \bar{\rho}'/\bar{\rho}$, which after substituting equation (23) for σ_η delivers

$$\sigma_q = \begin{bmatrix} -\frac{(1-\eta)(\rho_e-\rho_h)/\bar{\rho}}{1+(1-\eta)(\rho_e-\rho_h)/\bar{\rho}}\sigma \\ 0 \end{bmatrix}, \quad \text{if } x \geq q^H(\eta). \tag{B.5}$$

Substituting (B.5), we obtain $|\sigma_R|^2 = \sigma^2(\bar{\rho}/\rho_e)^2$. Plugging $q = a_e/\bar{\rho}$ into the second term of equation (RB2), we require $|\sigma_R|^2 \leq \eta\bar{\rho}(\eta)(1 - a_h/a_e)$, or equivalently, using the expression for $|\sigma_R|^2$, we obtain

$$x \geq q^H(\eta) \implies \eta \geq \eta^*,$$

where η^* is defined in (B.2). Therefore, for all $\eta < \eta^*$, we insist $x < q^H(\eta)$. As long as $(\eta, x) \in \mathcal{X}$, this will hold, because $q^H_\beta(\eta) < q^H(\eta)$ for all $\eta < \eta^*$, and $x < q^H_\beta(\eta)$ for all η. Therefore, (B.5) will be consistent with (RB2) as long as $(\eta_t, x_t)_{t\geq 0}$ remains in $\mathcal{X}$ almost-surely.

Note that these results are consistent with Lemma 3, since $\binom{0}{1} \cdot \sigma_q = 0$ whenever q is solely a function of η (i.e., when $x \geq q^H(\eta)$).

Note finally that σ_q defined in (B.4)-(B.5) is solely a function of (η, x), so sometimes we will write $\sigma_q(\eta, x)$. Similarly, as mentioned in Step 0, with σ_q in hand, we now have μ_η and σ_η as functions of (η, x) alone.

Step 3: Construct μ_q. Similar to σ_q, separately consider $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. On $\{x \geq q^H(\eta)\}$, since $q = q^H(\eta)$ is an explicit function of η, we set μ_q via Itô's formula. On $\{x < q^H(\eta)\}$, we have no equilibrium considerations restricting μ_q. Thus, we will put $\mu_q = m_q$, where m_q is a function in class $\mathcal{M}$, defined as follows. A function $m : \mathcal{X} \mapsto \mathbb{R}$ is a member of $\mathcal{M}$ if m is C^2 and possesses the following boundary conditions:

$$\inf_{\eta\in(0,1)} \lim_{x\searrow q^L_\alpha(\eta)} \Big(x - q^L_\alpha(\eta)\Big) m(\eta, x) = +\infty \tag{B.6}$$

$$\sup_{\eta\in(0,1)} \lim_{x\nearrow q^H_\beta(\eta)} \Big(q^H_\beta(\eta) - x\Big) m(\eta, x) = -\infty \tag{B.7}$$

$$\text{for any } x \in (q^L_\alpha(0), q^H_\beta(0)), \quad \lim_{\eta\searrow 0} |m(\eta, x)| < +\infty \tag{B.8}$$

$$\text{for any } x \in (q^L_\alpha(1), q^H_\beta(1)), \quad \lim_{\eta\nearrow 1} |m(\eta, x)| < +\infty. \tag{B.9}$$

Collecting these results

$$\mu_q(\eta, x) = \begin{cases} m_q(\eta, x), & \text{if } x < q^H(\eta); \\ \frac{\rho_e-\rho_h}{\bar{\rho}(\eta)^2}[-\bar{\rho}(\eta)\mu_\eta(\eta, x) + |\sigma_\eta(\eta, x)|^2], & \text{if } x \geq q^H(\eta). \end{cases} \tag{B.10}$$

Step 4: Verify stationarity. We demonstrate the time-paths $(\eta_t, x_t)_{t\geq 0}$ remain in $\mathcal{X}$ almost-surely and admit a stationary distribution.

The dynamics of x_t are specified as follows. Denote its diffusion and drift coefficients by $(x\sigma_x, x\mu_x)$, where σ_x and μ_x are functions of (η, x) to be specified shortly. By (B.3), when $q^L_\alpha(\eta) < x < q^H(\eta)$, we must put $\sigma_x = \sigma_q$ and $\mu_x = \mu_q$. Outside of this region, we put σ_x and μ_x to preserve stationarity.

To this end, let $\tilde{\sigma}_x : \mathcal{X} \mapsto \mathbb{R}_+$ be any positive and bounded C^2 function.[33] Put

$$\sigma_x(\eta, x) = \begin{cases} \sigma_q(\eta, x), & \text{if } x < q^H(\eta); \\ \tilde{\sigma}_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Note that σ_x is bounded (recall we have argued σ_q is bounded on $\{x < q^H(\eta)\}$, and $\tilde{\sigma}_x$ is assumed bounded).

Similarly, for the drift, let $m_x : \mathcal{X} \mapsto \mathbb{R}$ be any function in class $\mathcal{M}$, defined above (note: m_x need not coincide with m_q above). Put

$$\mu_x(\eta, x) = \begin{cases} \mu_q(\eta, x), & \text{if } x < q^H(\eta); \\ m_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Thus, μ_x satisfies boundary conditions (B.6)-(B.9) on all boundaries of $\mathcal{X}$.

Corresponding to the SDEs induced by $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$, define the infinitesimal generator $\mathscr{L}$, where

$$\mathscr{L}f = \mu_\eta \partial_\eta f + (x\mu_x)\partial_x f + \frac{1}{2}|\sigma_\eta|^2 \partial_{\eta\eta} f + \frac{1}{2}|x\sigma_x|^2 \partial_{xx} f + x\sigma_x \cdot \sigma_\eta \partial_{\eta x} f$$

for any C^2 function f.

Let $\{\mathcal{X}_n\}_{n\geq 1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{X}$, such that $\cup_{n\geq 1}\mathcal{X}_n = \mathcal{X}$. Note that $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ are bounded on $\mathcal{X}_n$ for each n. Consequently, despite the (potential) discontinuity in $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ at the one-dimensional subset $\{x = q^H(\eta)\}$, there exists a unique weak solution $(\tilde{\eta}^n_t, \tilde{x}^n_t)_{0\leq t\leq \tau_n}$, up to first exit time $\tau_n := \inf\{t : (\eta_t, x_t) \notin \mathcal{X}_n\}$, to the SDEs defined by the infinitesimal generator $\mathscr{L}$. See Krylov (1969, 2004) for weak existence and uniqueness in the presence of drift and diffusion discontinuities.

Letting $\tau := \lim_{n\to\infty} \tau_n$, we thus define $(\eta_t, x_t)_{0\leq t\leq \tau}$ by piecing together $(\tilde{\eta}^n_t, \tilde{x}^n_t)_{0\leq t\leq \tau_n}$ for successive n. In other words, $(\eta_t, x_t) = (\tilde{\eta}^n_t, \tilde{x}^n_t)$ for $0 \leq t \leq \tau_n$, each n. Our goal is to show (a) $\tau = +\infty$ a.s.; and (b) the resulting stochastic process $(\eta_t, x_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution on $\mathcal{X}$. These will be proved if we can obtain a function v satisfying the assumptions (i)-(iii) of Lemma B.1 below.

Define the positive function v by

$$v(\eta, x) := \frac{1}{\eta^{1/2}} + \frac{1}{1-\eta} + \frac{1}{x - q^L_\alpha(\eta)} + \frac{1}{q^H_{\beta,\lambda}(\eta) - x}.$$

Note that v diverges to $+\infty$ at the boundaries of $\mathcal{X}$, so assumption (i) of Lemma B.1 is proved. Next, if assumption (iii) of Lemma B.1 holds (which we will prove below), then there exists N such that $\mathscr{L}v < 0$ on $\mathcal{X}\backslash\mathcal{X}_n$ for all $n > N$. Furthermore, for each given n, $\mathscr{L}v$ is bounded on $\mathcal{X}_n$. Consequently, we can find a constant c large enough such that $\mathscr{L}v \leq cv$ on all of $\mathcal{X}$, which verifies part (ii) of Lemma B.1.

It remains to prove assumption (iii) of Lemma B.1, namely that $\mathscr{L}v \to -\infty$ as $(\eta, x) \to \partial\mathcal{X}$. We will examine the boundaries of $\mathcal{X}$ one-by-one. In the following, we use the notation $g(x) = o(f(x))$ if $g(x)/f(x) \to 0$ as $x \to 0$, and the notation $g(x) = O(f(x))$ if $g(x)/f(x) \to C$ as $x \to 0$, where C is a finite constant.

[33]Note that $\tilde{\sigma}_x$ need not vanish at the boundary of $\mathcal{X}$, but if it does some of the boundary conditions on m_x, to follow, can be relaxed.

As $\eta \to 0$ (and x bounded away from $q^L_\alpha(0)$ and $q^H_\beta(0)$, such that κ is bounded away from 0 and 1, the latter due to the definition of q^H_β), we have

$$\begin{aligned}
\mu_\eta &= \delta\nu + \frac{a_e - a_h}{x}\kappa + \eta[\rho_h - \rho_e - \delta] + o(\eta) \\
\mu_x &= O(1) \\
|\sigma_\eta|^2 &= \eta(\kappa - \eta)\frac{a_e - a_h}{x} + o(\eta) \\
|\sigma_x|^2 &= O(1).
\end{aligned}$$

We used condition (B.8) to obtain μ_x bounded. Thus,

$$\mathscr{L}v = -\frac{1}{2\eta^{3/2}}[\delta\nu + \frac{1}{4}\frac{a_e - a_h}{x}\kappa] + o(\eta^{-3/2}) \to -\infty,$$

irrespective of $\delta\nu > 0$ or $\delta\nu = 0$.

As $\eta \to 1$ (and x bounded away from $q^L_\alpha(1)$ and $q^H_\beta(1)$; note that $\kappa = 1$ at this boundary), we have

$$\begin{aligned}
\mu_\eta &= -\delta(1-\nu) - (\rho_e - \rho_h)(1-\eta) + o(1-\eta) \\
\mu_x &= O(1) \\
|\sigma_\eta|^2 &= (1-\eta)^2\sigma^2 \\
|\sigma_x|^2 &= O(1).
\end{aligned}$$

We used condition (B.9) to obtain μ_x bounded. Thus,

$$\mathscr{L}v = -(1-\eta)^{-2}\delta(1-\nu) - (1-\eta)^{-1}[\rho_e - \rho_h - \sigma^2] + o((1-\eta)^{-1}) \to -\infty,$$

irrespective of $\delta(1-\nu)$, due to Assumption 1, part (iii).

We separately calculate the limit $x \to q^L_\alpha(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$, since $\kappa < 1$ in the first case, and $\kappa = 1$ in the second case. Still, we find that in both cases,

$$\begin{aligned}
\mu_\eta &= O(1) \\
\mu_x &= o((x - q^L_\alpha)^{-1}) \\
|\sigma_\eta|^2 &= O(1) \\
|\sigma_x|^2 &= O(1).
\end{aligned}$$

We used condition (B.6) to obtain the order of μ_x. Thus,

$$\mathscr{L}v = -(x - q^L_\alpha)^{-2}x\mu_x + O((x - q^L_\alpha)^{-3}) \to -\infty.$$

Similarly, we separately calculate the limit $x \to q^H_\beta(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. Again, we find that in both cases,

$$\begin{aligned}
\mu_\eta &= O(1) \\
\mu_x &= (-1) \times o((q^H_\beta - x)^{-1}) \\
|\sigma_\eta|^2 &= O(1) \\
|\sigma_x|^2 &= O(1).
\end{aligned}$$

We used condition (B.7) to obtain the order of μ_x. Thus,

$$\mathscr{L}v = (q^H_\beta - x)^{-2} x\mu_x + O((q^H_\beta - x)^{-3}) \to -\infty.$$

Finally, all the corners of $\mathcal{X}$ can be analyzed in a straightforward way by combining the cases above, with the exception of $(\eta, x) = (0, q^L_\alpha(0)) = (0, a_h/\rho_h)$. Approaching this corner, we must take a particular path of $x \to a_h/\rho_h$ as $\eta \to 0$. Denote this path by $\hat{x}(\eta)$ and denote the asymptotic slope by $\hat{x}'(0) \in (\frac{d}{d\eta} q^L_\alpha(0), +\infty)$, where $\frac{d}{d\eta} q^L_\alpha(0) = [\frac{a_e}{a_h} - \frac{\rho_e}{\rho_h}]\frac{a_h}{\rho_h} + \alpha'(0) > 0$, by Assumption 1, part (i), and the fact that $\alpha'(0) > 0$. Denote the associated path of κ by $\hat{\kappa}(\eta)$ and the corresponding asymptotic slope by $\hat{\kappa}'(0) = \frac{1}{a_e - a_h}[\hat{x}'(0)\rho_h + (\rho_e - \rho_h)a_h/\rho_h]$. Substituting in, we find $\hat{\kappa}'(0) \in (1 + \frac{\alpha'(0)}{a_e - a_h}, +\infty)$. When computing $\mathscr{L}v$, we will take the supremum over all possible paths, meaning over $\hat{x}'(0)$ and $\hat{\kappa}'(0)$. Using similar calculations from the initial $\eta \to 0$ case, but using these paths, we obtain

$$\begin{aligned}
\mu_\eta &= \delta\nu + \eta[\frac{a_e - a_h}{\hat{x}}\hat{\kappa}' + \rho_h - \rho_e - \delta] + o(\eta) \\
\mu_x &= o((\hat{x} - q^L_\alpha)^{-1}) \\
|\sigma_\eta|^2 &= \eta^2[\hat{\kappa}' - 1]\frac{a_e - a_h}{\hat{x}} + o(\eta) \\
|\sigma_x|^2 &= O(1) \\
\sigma_x \cdot \sigma_\eta &= \eta[\frac{a_e - a_h}{\hat{x}} - \sigma(\gamma(\hat{\kappa}' - 1)\frac{a_e - a_h}{\hat{x}})^{1/2}] + o(\eta).
\end{aligned}$$

Since $\hat{x} \geq O(\eta)$ and $\hat{\kappa} \geq O(\eta)$ (in the sense that both could be $+\infty$), we may treat terms like $(\hat{x} - q^L_\alpha)^{-1}$ as smaller than η^{-1}. This identifies the dominant terms as those associated to μ_η, $|\sigma_\eta|^2$, and μ_x. Thus,

$$\begin{aligned}
\mathscr{L}v = &-\frac{1}{2\eta^{3/2}}\delta\nu + \frac{1}{2\eta^{1/2}}[\rho_e - \rho_h + \delta - \frac{a_e - a_h}{\hat{x}} - \frac{a_e - a_h}{\hat{x}}(\hat{\kappa}' - 1)/4] + o(\eta^{-3/2}) \\
&- (\hat{x} - q^L_\alpha)^{-2} x\mu_x + O((\hat{x} - q^L_\alpha)^{-3}) \to -\infty,
\end{aligned}$$

irrespective of $\delta\nu$, because $\rho_e - \rho_h - \frac{a_e - a_h}{a_h/\rho_h} = \rho_h[\rho_e/\rho_h - a_e/a_h] < 0$ by Assumption 1, part (i), and because $\inf\{\hat{\kappa}'(0)\} > 1$.

This completes the verification that $\mathscr{L}v \to -\infty$ as $(\eta, x) \to \partial\mathcal{X}$, which proves stationarity by Lemma B.1. This completes the proof. $\square$

Proof of Corollary 1. Start from the construction of GBSE in Theorem 2, and note that we can make ϵ arbitrarily small such that the boundaries $q^L_\alpha \to \bar{a}/\bar{\rho}$ and $q^H_\beta \to a_e/\bar{\rho}$. In addition, the limit can be taken such that $\eta^*_\beta \to \eta^*$, its minimal possible level. Hence, a GBSE can be constructed such that the set of prices q matches $\mathcal{Q}(\eta)$ arbitrarily closely. The result on return variance comes from using (B.4) when $\kappa < 1$ (i.e., when $\eta < \eta^*$) and using (B.5) when $\kappa = 1$ (i.e., when $\eta \geq \eta^*$ and q is at its upper bound). Using the definition of η^* provides the form of $\mathcal{V}$ with the minimum as the lower bound.

To show that a GBSE can be constructed such that positive probability is placed on all elements of $\mathcal{Q}$ and $\mathcal{V}$, we simply note that a construction exists such that $\sigma_q \neq 0$ on the entirety of $\text{int}(\mathcal{D})$. $\square$

PROOF OF PROPOSITION 4. We proceed by construction. Without loss of generality, let $\mathcal{S} = (0,1)$ so that the domain of the state variables is $\mathcal{D} = (0,1) \times (0,1)$. Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1-\eta)a_h$.

Step 1: Fundamental equilibrium. Let $(\hat{q}^0, \hat{\kappa}^0)$ be the solution to the fundamental equilibrium (which exists by assumption), and let $\eta^0 := \inf\{\eta : \hat{q}^0 \geq a_e/\bar{\rho}\} = \inf\{\eta : \hat{\kappa}^0 \geq 1\}$. By part (v) of Lemma C.1, there exists $\bar{\sigma}_A > 0$ such that, if $\sigma < \bar{\sigma}_A$, then $\eta^0 < 1$. By part (iv) of Lemma C.1, there exists $\bar{\sigma}_B > 0$ such that, if $\sigma < \bar{\sigma}_B$, then $(\hat{q}^0)' > \frac{a_e - a_h}{\bar{\rho}}$ for $\eta \in (0, \eta^0)$. Only to assist with step 9 below, we also denote $\bar{\sigma}_C = \sqrt{\rho_e - \rho_h}\mathbf{1}_{\delta=0} + (+\infty)\mathbf{1}_{\delta>0}$. Assume $\sigma < \min(\bar{\sigma}_A, \bar{\sigma}_B, \bar{\sigma}_C)$. In particular, this implies $\frac{d}{d\eta}[\hat{q}^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$.

Step 2: Two basis functions. We design two "extremal" functions that will assist our construction. First, let φ be a C^2 function with the properties $\varphi(\eta^0) = 0$ and $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})' = \frac{a_e - a_h}{\bar{\rho}}[1 - (1-\eta)\frac{\rho_e - \rho_h}{\bar{\rho}}]$ for all η. Define

$$q^0(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta < \eta^0; \\ \hat{q}^0(\eta) + \varphi(\eta), & \text{if} \quad \eta \geq \eta^0. \end{cases} \tag{B.11}$$

Note that q^0 is C^∞ except at $\eta = \eta^0$, due to part (vi) of Lemma C.1.

To construct the other basis function, fix some $\epsilon \in (0, \eta^0)$, let $\tilde{\epsilon} \in (\epsilon, \eta^0)$, and define a C^∞ (but necessarily non-analytic) function $\beta : (0,1) \mapsto \mathbb{R}_+$ with the following properties

$$\begin{aligned}
\beta(\epsilon) &= q^0(\epsilon) - \bar{a}(\epsilon)/\bar{\rho}(\epsilon) \\
\beta^{(k)}(\epsilon) &= \frac{d^k}{d\eta^k}[q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)]|_{\eta=\epsilon} \quad \text{for each derivative of order } k \geq 1 \\
\beta'(\eta) &< \frac{d}{d\eta}[q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)] \quad \text{for all} \quad \eta > \epsilon \\
\beta(\eta) &= 0 \quad \text{for all} \quad \eta > \tilde{\epsilon}.
\end{aligned}$$

A particular consequence of $\sigma < \bar{\sigma}_B$ in step 1 is $\frac{d}{d\eta}[q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$. A consequence of $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})'$ is $\frac{d}{d\eta}[q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (\eta^0, 1)$. Together, these properties imply such a function β exists. Then, we put

$$q^1(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta \leq \epsilon; \\ \bar{a}(\eta)/\bar{\rho}(\eta) + \beta(\eta), & \text{if} \quad \eta > \epsilon. \end{cases} \tag{B.12}$$

Note that $\eta^1 := \inf\{\eta : q^1 \geq a_e/\bar{\rho}\} = 1$. By the properties of β and φ, note the following slope results:

$$(q^0)' > (q^1)' \quad \text{on} \quad \eta \in (\epsilon, 1) \tag{B.13}$$

$$(q^0)^{(k)}(\epsilon) = (q^1)^{(k)}(\epsilon) \quad \text{for all derivatives of order } k \geq 0. \tag{B.14}$$

Step 3: Useful monotonicity results. Before continuing, we make the following claims:

$$\frac{\bar{a}}{\bar{\rho}} < q^1 = q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for} \quad \eta \in (0, \epsilon); \tag{B.15}$$

$$\frac{\bar{a}}{\bar{\rho}} \leq q^1 < q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for} \quad \eta \in (\epsilon, \eta^0); \tag{B.16}$$

$$\frac{\bar{a}}{\bar{\rho}} = q^1 < \frac{a_e}{\bar{\rho}} < q^0, \quad \text{for} \quad \eta \in (\eta^0, 1). \tag{B.17}$$

All inequalities in relationship (B.15), as well as the third inequality in relationship (B.16), hold by part (ii) of Lemma C.1. The first inequality in relationship (B.16) holds because $\beta \geq 0$, whereas the first equality in relationship (B.17) holds because $\beta = 0$ on that set. The second inequality in relationship (B.16) holds due to (B.13). The second inequality in relationship (B.17) holds by the definition of $\eta^1 = 1$. The second inequality in relationship (B.17) holds since $q^0(\eta^0) = a_e/\bar{\rho}(\eta^0)$ combined with $(q^0 - a_e/\bar{\rho})' > (q^1 - a_e/\bar{\rho})' > 0$, for $\eta > \eta^0$.

Step 4: Construct candidate (q, κ). We proceed to combine our basis functions according to the convex combination

$$\tilde{q}(\eta, s) := (1 - \alpha s) q^0(\eta) + \alpha s q^1(\eta), \quad (\eta, s) \in \mathcal{D} = (0,1) \times \mathcal{S}, \tag{B.18}$$

where $\alpha \in (0,1)$ is fixed. For each $s \in \mathcal{S}$, define

$$\eta^*(s) := \inf\{\eta : \tilde{q}(\eta, s) \geq a_e/\bar{\rho}\}.$$

Note that $\tilde{q}$ is C^2 on $(\eta^0, \eta^1) \times \mathcal{S}$, which implies η^* is C^1. Note that $\eta^*(s)$ is strictly increasing by a combination of (B.16)-(B.17).[34] Put

$$q(\eta, s) := \begin{cases} \tilde{q}(\eta, s), & \text{if} \quad \eta < \eta^*(s); \\ a_e/\bar{\rho}(\eta), & \text{if} \quad \eta \geq \eta^*(s). \end{cases} \quad \text{and} \quad \kappa := \frac{\bar{\rho} q - a_h}{a_e - a_h}.$$

By construction, the pair (q, κ) satisfy equation (PO).

Step 5: Properties of (q, κ). Let $\mathcal{X} := \{(\eta, s) : \eta \in (\epsilon, \eta^*(s)), s \in \mathcal{S}\}$. On this set, we have $\kappa > \eta$, or equivalently $\bar{\rho} q > \bar{a}$, by (B.16)-(B.17). In fact, κ is bounded away from η on $\mathcal{X}$, since $\alpha < 1$ in (B.18). We also have the following derivative conditions on $\mathcal{X}$:

$$\partial_s q = \alpha(q^1 - q^0) < 0 \tag{B.19}$$

$$\partial_\eta q = (1 - \alpha s)(q^0)' + \alpha s (q^1)' > 0 \tag{B.20}$$

$$\partial_\eta q < q/(\kappa - \eta). \tag{B.21}$$

[34] Indeed, (B.16) shows that $\tilde{q}(\eta, s) < a_e/\bar{\rho}(\eta)$ on $(\epsilon, \eta^0) \times \mathcal{S}$, which implies $\eta^*(s) \geq \eta^0$ for all $s \in \mathcal{S}$. Then, use the fact that η^* is C^1 to differentiate $\tilde{q}(\eta^*(s), s) = a_e/\bar{\rho}(\eta^*(s))$ with respect to s, and use the fact that $\partial_s \tilde{q} = q^1 - q^0$, and finally rearrange to obtain

$$(\eta^*)'(s) \Big[\partial_\eta \tilde{q}(\eta^*(s), s) + \frac{a_e}{\bar{\rho}(\eta^*(s))} \frac{\rho_e - \rho_h}{\bar{\rho}(\eta^*(s))} \Big] = q^0(\eta^*(s)) - q^1(\eta^*(s)).$$

If at any point s, we had $(\eta^*)'(s) = 0$, we would necessarily have $q^0(\eta^*(s)) = q^1(\eta^*(s))$. But this contradicts the fact from (B.16)-(B.17) that $q^0 > q^1$ for all $\eta > \epsilon$, since $\eta^*(s) \geq \eta^0 > \epsilon$. Thus, $(\eta^*)'(s) \neq 0$ for all s. We can also rule out $(\eta^*)'(s) < 0$ by the fact that $\eta^*(0+) = \eta^0$ and $\eta^*(s) \geq \eta^0$ for all s. Thus, $(\eta^*)'(s) > 0$ for all s.

Inequality (B.19) holds by (B.16)-(B.17). Inequality (B.20) holds by (B.13) and Assumption 1(ii), which implies $(q^1)' > 0$. Inequality (B.21) is proven as follows. First, note that the function $f(\eta, x) = \frac{(a_e - a_h)x}{\bar{\rho}(\eta)x - \bar{a}(\eta)}$ is strictly decreasing in x on $x > \bar{a}(\eta)/\bar{\rho}(\eta)$. Second, part (i) of Lemma C.1 implies

$$(q^0)' < \frac{(a_e - a_h)q^0}{\bar{\rho}q^0 - \bar{a}} = f(\cdot, q^0).$$

Given (B.13), we thus have $\partial_\eta q < f(\cdot, q^0)$ for any value of s. Finally, since f is decreasing in its second argument, and $q < q^0$ on $\mathcal{X}$, we have $\partial_\eta q < f(\cdot, q)$, which proves the claim.

We remark on one additional smoothness property that holds at $\eta = \epsilon$, due to condition (B.14):

$$\partial_\eta^{(k)} q(\epsilon, s) = (q^0)^{(k)}(\epsilon) \quad \forall s, \quad \text{for all derivatives of order } k \geq 0. \tag{B.22}$$

Step 6: Construct candidate σ_s. Consider solving the following problem.

$$\begin{aligned} \textit{Problem:} \quad & \text{for each } (\eta, s) \in \mathcal{X}, \text{ solve for } y \text{ in the equation} \\ & y(\partial_s \log q)^2 = G, \end{aligned} \tag{B.23}$$

where

$$G := \frac{\eta(1-\eta)}{\kappa - \eta} \frac{a_e - a_h}{q} (1 - (\kappa - \eta)\partial_\eta \log q)^2 - \sigma^2.$$

Note that G is bounded, as κ is bounded away from η (step 5). Checking boundedness of the solution y thus boils down to checking $\partial_s q$ at the boundaries of $\mathcal{X}$. By (B.19), as $s \to 0$ or $s \to 1$, $\partial_s q \not\to 0$, so y remains bounded. To check the result as $\eta \to \epsilon$, we first claim that $\lim_{\eta \searrow \epsilon} \partial_\eta^{(k)} G = 0$ for all derivatives of order $k \geq 0$. This is a consequence of parts (i) and (vi) of Lemma C.1, whereby $\partial_\eta^{(k)} G = 0$ for all $k \geq 0$ on $\eta < \epsilon$, combined with result (B.22). Since we also have $\partial_s q \to 0$, we apply L'Hôpital's rule twice to compute $\lim_{\eta \searrow \epsilon} G/(\partial_s \log q)^2 = 0$, noting both times that $\partial_{s\eta} \log q = \frac{\alpha}{q}[(q^1)' - (q^0)'] < 0$ is non-zero. Therefore, the solution $y = G/(\partial_s \log q)^2$ is bounded on $\mathcal{X}$.

Clearly, $\sqrt{y}$ will be a real number if and only if $G \geq 0$. To prove $G \geq 0$, note that $\lim_{s \searrow 0} G = 0$, meaning it suffices to prove $\partial_s G \geq 0$. Differentiating G, we get

$$\begin{aligned} \frac{\partial_s G}{\eta(1-\eta)} = & -\frac{a_e - a_h}{(\kappa - \eta)q}(1 - (\kappa - \eta)\partial_\eta \log q)\Big[(1 - (\kappa - \eta)\partial_\eta \log q)\Big(\frac{\partial_s \kappa}{\kappa - \eta} + \frac{\partial_s q}{q}\Big) \\ & + 2\frac{(\kappa - \eta)\bar{a}}{\bar{\rho}q - \bar{a}}(\partial_s \log q)(\partial_\eta \log q) + 2(\kappa - \eta)\alpha \frac{(q^1)' - (q^0)'}{q}\Big]. \end{aligned}$$

By properties (B.19)-(B.21), and the fact that $\text{sgn}(\partial_s \kappa) = \text{sgn}(\partial_s q)$, we prove $\partial_s G > 0$ on $\mathcal{X}$. So not only is $\sqrt{y}$ real, it is non-zero.

We set σ_s as follows:

$$\sigma_s(\eta, s) := \begin{cases} \sqrt{y(\eta, s)}, & \text{if} \quad (\eta, s) \in \mathcal{X}; \\ \sqrt{y(\epsilon+, s)} = 0, & \text{if} \quad (\eta, s) \in \{(\eta, s) : \eta \in (0, \epsilon), \, s \in \mathcal{S}\}; \\ \sqrt{y(\eta^*(s)-, s)}, & \text{if} \quad (\eta, s) \in \{(\eta, s) : \eta > \eta^*(s), \, s \in \mathcal{S}\}. \end{cases} \tag{B.24}$$

In passing, we note that we have also shown that $\sigma_s > 0$ on a positive-measure set, as required in a sunspot equilibrium.

Step 7: Verify equation (27) is satisfied. By the construction of σ_s, equation (27) is satisfied on $\mathcal{X}$. On $\{(\eta, s) : \eta \in (0, \epsilon), s \in \mathcal{S}\}$, recall $\partial_s q = 0$, so (27) holds by property (i) of Lemma C.1. On $\{(\eta, s) : \eta > \eta^*(s), s \in \mathcal{S}\}$, recall $\kappa = 1$, so (27) is satisfied if and only if the second term inside the minimum is non-negative. Substituting $\kappa = 1$ and $q = a_e/\bar{\rho}$, hence $\partial_s q = 0$, into this term shows the non-negativity requirement is

$$\sigma^2 \leq \eta\bar{\rho}\frac{a_e - a_h}{a_e}(1 + (1-\eta)\partial_\eta \log \bar{\rho})^2 \quad \text{for} \quad \eta > \eta^*(s), s \in \mathcal{S}. \tag{B.25}$$

On the other hand, property (v) of Lemma C.1, combined with the fact that $\eta^*(s)$ is increasing, imply

$$\eta^*(s) \geq \frac{\rho_h}{\rho_e}\Big(\frac{1 - a_h/a_e}{\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big)^{-1}, \quad \forall s \in \mathcal{S}. \tag{B.26}$$

Straightforward algebra demonstrates that (B.25) and (B.26) are equivalent, proving (27) holds.

Step 8: Finish equilibrium construction. Having determined q, κ, and σ_s, we define μ_η and σ_η by (22)-(23). It remains to determine μ_s. We will pick $\mu_s(\eta, s) = m(\eta, s)$, where m is a C^2 function with the following properties: $\partial_s m < 0$, and for some $0 \leq s^0 < s^1 \leq 1$ thresholds,

$$(\text{if } s^0 > 0) \quad \inf_{\eta \in (0,1)} \lim_{s \searrow s^0} (s - s^0) m(\eta, s) = +\infty \tag{B.27}$$

$$(\text{if } s^0 = 0) \quad \inf_{\eta \in (0,1)} \lim_{s \searrow s^0} m(\eta, s) > 0 \tag{B.28}$$

$$\sup_{\eta \in (0,1)} \lim_{s \nearrow s^1} (s^1 - s) m(\eta, s) = -\infty. \tag{B.29}$$

Step 9: Verify stationarity. Finally, we demonstrate the time-paths $(\eta_t, s_t)_{t\geq 0}$ remain in $\mathcal{D}$ almost-surely and admit a stationary distribution. This step is very similar to Theorem 2. For the purposes of this step, we suppose that $s^0 = 0$ and $s^1 = 1$ in step 8. Corresponding to the SDEs induced by $(\sigma_\eta, \sigma_s, \mu_\eta, \mu_s)$, define the infinitesimal generator $\mathscr{L}$, where

$$\mathscr{L}f = \mu_\eta \partial_\eta f + \mu_s \partial_s f + \frac{1}{2}|\sigma_\eta|^2 \partial_{\eta\eta} f + \frac{1}{2}\sigma_s^2 \partial_{ss} f + \sigma_s \sigma_\eta \cdot \binom{0}{1} \partial_{\eta s} f$$

for any C^2 function f. Let $\{\mathcal{D}_n\}_{n\geq 1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{D}$, such that $\cup_{n\geq 1}\mathcal{D}_n = \mathcal{D}$. Note that $(\sigma_\eta, \sigma_s, \mu_\eta, \mu_s)$ are bounded on $\mathcal{D}_n$ for each n. Furthermore, the diffusions σ_η and σ_s are both bounded away from 0 on $\mathcal{D}_n$ for each n. Consequently, despite the discontinuity in $(\sigma_\eta, \sigma_s, \mu_\eta, \mu_s)$ at $\eta^*(s)$, there exists a unique weak solution $(\tilde{\eta}_t^n, \tilde{s}_t^n)_{0\leq t\leq \tau_n}$, up to first exit time $\tau_n := \inf\{t : (\eta_t, s_t) \notin \mathcal{D}_n\}$, to the SDEs defined by the infinitesimal generator $\mathscr{L}$. See Krylov (1969, 2004) for weak existence and uniqueness in the presence of drift and diffusion discontinuities.

Letting $\tau := \lim_{n\to\infty} \tau_n$, we may thus define $(\eta_t, s_t)_{0\leq t\leq\tau}$ by piecing together $(\tilde{\eta}_t^n, \tilde{s}_t^n)_{0\leq t\leq\tau_n}$ for successive n. In other words, $(\eta_t, s_t) = (\tilde{\eta}_t^n, \tilde{s}_t^n)$ for $0 \leq t \leq \tau_n$, each n. Our goal is to show (a) $\tau = +\infty$ a.s.; and (b) the resulting stochastic process $(\eta_t, s_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution on $\mathcal{D}$. These will be proved if we can obtain a function v satisfying the assumptions (i)-(iii) of Lemma B.1 below.

Define the positive function v by

$$v(\eta, s) := \frac{1}{\eta} + \frac{1}{1-\eta} + \frac{1}{1-s} + \frac{1}{s}.$$

Note that v diverges to $+\infty$ at the boundaries of $\mathcal{D}$, so assumption (i) of Lemma B.1 is proved. Next, if assumption (iii) of Lemma B.1 holds (which we will prove below), then there exists N such that $\mathscr{L}v < 0$ on $\mathcal{D} \backslash \mathcal{D}_n$ for all $n > N$. Furthermore, for each given n, $\mathscr{L}v$ is bounded on $\mathcal{D}_n$. Consequently, we can find a constant c large enough such that $\mathscr{L}v \leq cv$ on all of $\mathcal{D}$.

It remains to prove assumption (iii) of Lemma B.1. We will examine the boundaries of the rectangle $\mathcal{D}$ one-by-one; the corners can be analyzed by combining the cases below. As $\eta \to 0$ (and s bounded away from 0 and 1), we have

$$\begin{aligned}
\mu_\eta &\sim \delta\nu + \sigma^2[(\partial_\eta\kappa)^2 - \partial_\eta\kappa + \rho_h - \rho_e]\eta \\
\mu_s &\sim O(1) \\
|\sigma_\eta|^2 &\sim (\partial_\eta\kappa - 1)^2\sigma^2\eta^2 \\
|\sigma_s|^2 &\sim 0
\end{aligned}$$

Thus, $\mu_\eta\partial_\eta v \sim -\delta\nu/\eta^2 - \sigma^2[(\partial_\eta\kappa)^2 - \partial_\eta\kappa + \rho_h - \rho_e]/\eta$ and $\frac{1}{2}|\sigma_\eta|^2\partial_{\eta\eta}v \sim (\partial_\eta\kappa - 1)^2\sigma^2/\eta$ are the dominant terms in $\mathscr{L}v$ as $\eta \to 0$. If $\delta > 0$, then clearly $\mathscr{L}v \to -\infty$. If $\delta = 0$, then we sum these result to get $\mathscr{L}v \sim \eta^{-1}\sigma^2[-(\partial_\eta\kappa)^2 + \partial_\eta\kappa - \rho_h + \rho_e + (\partial_\eta\kappa - 1)^2] = \eta^{-1}\sigma^2[\rho_e - \rho_h - \partial_\eta\kappa + 1]$. Using the expression for $\partial_\eta\kappa(0+, s)$ from the fundamental equilibrium (part (iii) of Lemma C.1), one can show that this latter expression is negative if $\sigma < 1$; in particular, $\sigma < \min(\bar{\sigma}_A, \bar{\sigma}_B)$ suffices. Thus, $\mathscr{L}v \to -\infty$ as $\eta \to 0$.

As $\eta \to 1$ (and s bounded away from 0 and 1), we have

$$\begin{aligned}
\mu_\eta &\sim -\delta(1-\nu) - (\rho_e - \rho_h)(1-\eta) \\
\mu_s &\sim O(1) \\
|\sigma_\eta|^2 &\sim (1-\eta)^2\sigma^2 \\
|\sigma_s|^2 &\sim O(1)
\end{aligned}$$

Thus, $\mathscr{L}v \sim -\delta(1-\nu)(1-\eta)^{-2} - [\rho_e - \rho_h + \sigma^2](1-\eta)^{-1}$. If $\delta > 0$, then clearly $\mathscr{L}v \to -\infty$. If $\delta = 0$, then $\mathscr{L}v \to -\infty$ if and only if $\rho_e - \rho_h > \sigma^2$, which is ensured by $\sigma < \bar{\sigma}_C$.

As $s \to 1$ (and η bounded away from 0 and 1), we have (the expressions below hold whether η is above or below $\eta^*(s)$ as $s \to 1$)

$$\begin{aligned}
\mu_\eta &\sim O(1) \\
\mu_s &\sim -o((1-s)^{-1}) \\
|\sigma_\eta|^2 &\sim O(1) \\
|\sigma_s|^2 &\sim O(1)
\end{aligned}$$

Note μ_s is "little-oh" of $(1-s)^{-1}$ due to (B.29). Thus, $\mathscr{L}v \sim [-o(\frac{1}{1-s}) + O(\frac{1}{1-s})]O(\frac{1}{(1-s)^2}) \to -\infty$.

As $s \to 0$ (and η bounded away from 0 and 1), we have (the expressions below hold whether η is above or below $\eta^*(s)$ as $s \to 0$)

$$\begin{aligned}
\mu_\eta &\sim O(1) \\
\mu_s &\sim O(1) \\
|\sigma_\eta|^2 &\sim O(1) \\
|\sigma_s|^2 &\sim O(s^2)
\end{aligned}$$

Thus, $\mathscr{L}v \sim [-O(1) + O(s)]O(\frac{1}{s^2}) \to -\infty$. This completes the proof that $\mathscr{L}v \to -\infty$ as $(\eta, s) \to \partial\mathcal{D}$, which completes this step.

Step 10: Robustness to commitment frictions. Consider the limited commitment setting in Section 2.3. In addition to the equilibrium restrictions, we also have the leverage constraint (12). However, for λ large enough, this constraint will never be binding in the sunspot equilibrium we have just constructed. This holds because expert leverage $q\kappa/\eta$ is bounded for all $(\eta, s) \in \mathcal{D}$ as long as $\sigma > 0$. Indeed, it suffices to check the boundary as $\eta \to 0$. By property (iii) of Lemma C.1, a fundamental equilibrium (with $\kappa(0) = 0$) features $\lim_{\eta \to 0} q\kappa/\eta = q(0)\kappa'(0+) = \frac{a_h}{\rho_h} + \frac{a_e - a_h}{\sigma^2} < +\infty$. Since the presently constructed sunspot equilibrium converges to this solution when $\eta \to 0$, we also have bounded leverage in the sunspot equilibrium. $\square$

Proof of Proposition 5. Fix any $\Sigma^* > 0$. The proof is a simple consequence of the fact that σ_q must be unbounded as κ approaches η, which is as q approaches the worst-case price q^1. We fill in the technical details below.

We construct a sequence of equilibria – indexed by $(\alpha, \epsilon, \zeta)$ – as follows. Recall the capital price construction in Proposition 4:

$$q = (1 - \alpha s)q^0 + \alpha s q^1, \quad \text{when} \quad \kappa < 1,$$

where $\alpha < 1$ is a parameter, q^0 is the fundamental equilibrium price, and

$$q^1 = \begin{cases} q^0, & \text{if } \eta < \epsilon; \\ \bar{a}/\bar{\rho} + \beta, & \text{if } \eta \in (\epsilon, \tilde{\epsilon}); \\ \bar{a}/\bar{\rho}, & \text{if } \eta > \tilde{\epsilon}. \end{cases}$$

The function β is a positive mollifier that vanishes uniformly as $\epsilon, \tilde{\epsilon} \to 0$. We set $\tilde{\epsilon} = \epsilon(1 + \epsilon)$. Based on the discussion in the text, we may choose μ_s such that equilibrium concentrates on any particular value of s. Thus, pick μ_s such that $s_t \geq \zeta$ almost-surely. Clearly, the choice of μ_s depends on α and ϵ, but such a choice can always be made for any parameters.

Let $p_{\text{low}} > 0$, $p_{\text{high}} > 0$ be given with $p_{\text{low}} + p_{\text{high}} < 1$. First, note that there exist α^*, ζ^*, and ϵ^* such that $\mathbb{P}[\eta_t \leq \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{low}}$ and $\mathbb{P}[\eta_t \geq 1 - \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{high}}$ for all $\alpha > \alpha^*$, $\zeta > \zeta^*$, and $\epsilon < \epsilon^*$. This is a consequence of the fact that in any stationary distribution, we have $\lim_{x \to 0} \mathbb{P}[\eta_t < x] = \lim_{x \to 1} \mathbb{P}[\eta_t > x] = 0$ and the fact that $\lim_{\alpha \to 1} \lim_{s \to 1} \kappa(\eta, s) < 1$ for all η.

At this point, fix such an $\epsilon < \epsilon^*$. Let a constant $M > 0$ be given satisfying

$$M \leq (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e/\rho_h} \frac{\tilde{\epsilon}(1 - \tilde{\epsilon})}{\Sigma^*}. \tag{B.30}$$

Note that

$$\lim_{\alpha \to 1} \lim_{s \to 1} \sup_{\eta \in (\tilde{\epsilon}, 1 - \tilde{\epsilon})} \Big| q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta) \Big| = 0.$$

Consequently, we may pick $\alpha > \alpha^*$ close enough to 1 and $\zeta > \zeta^*$ close enough to 1 such that

$$\sup_{s \in (\zeta, 1)} \sup_{\eta \in (\tilde{\epsilon}, 1 - \tilde{\epsilon})} \Big| q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta) \Big| \leq M.$$

Finally, using equation (27) and substituting $\kappa < 1$ from (PO), we have $|\sigma\binom{1}{0} + \sigma_q|^2 = \frac{(a_e - a_h)^2}{q} \frac{\eta(1-\eta)}{\bar{\rho} q - \bar{a}}$. Note also that $q \leq a_e/\rho_h$ is an upper bound. Then,

$$\mathbb{E}[|\sigma\tbinom{1}{0} + \sigma_{q,t}|^2] > (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e/\rho_h} \frac{\tilde{\epsilon}(1 - \tilde{\epsilon})}{M}.$$

Using (B.30), we obtain $\mathbb{E}[|\sigma\binom{1}{0} + \sigma_{q,t}|^2] > \Sigma^*$. $\square$

Proof of Proposition 6. Revisiting the proof of Proposition 4, we compute on $\{\kappa < 1\}$ and for each $\eta > \epsilon$,

$$\partial_s[(\kappa - \eta)\partial_\eta \log q] = \alpha\Big[(\kappa - \eta)\frac{(q^1)' - (q^0)'}{q} + \frac{\bar{a}(q^1 - q^0)}{(a_e - a_h)q^2}\partial_\eta q\Big] < 0.$$

The inequality uses (B.13) to say $(q^1)' - (q^0)' < 0$, and (B.16)-(B.17) to say $q^1 - q^0 < 0$, and (B.20) to say $\partial_\eta q > 0$. Therefore, $(1 - (\kappa - \eta)\partial_\eta \log q)^{-1}$ is decreasing in s on $\{\kappa < 1\}$ for each $\eta > \epsilon$. Since q and κ are independent of s on $\{\eta < \epsilon\}$, this proves $(1 - (\kappa - \eta)\partial_\eta \log q)^{-1}$ is weakly decreasing in s on $\{\kappa < 1\}$. Using the fact that $\binom{1}{0} \cdot \sigma_q = \frac{\sigma}{1-(\kappa-\eta)\partial_\eta \log q} - \sigma$, we obtain part (i).

Next, we prove part (iii). From (27), we can solve for return variance $|\sigma_R|^2 := |\sigma\binom{1}{0} + \sigma_q|^2$ on $\{\kappa < 1\}$ to get $|\sigma_R|^2 = \frac{(a_e - a_h)^2}{q}\frac{\eta(1-\eta)}{\bar{\rho}q - \bar{a}}$. Differentiating with respect to s, we obtain

$$\partial_s|\sigma_R|^2 = -\eta(1-\eta)\frac{(a_e - a_h)^2}{q(\bar{\rho}q - \bar{a})}\Big[\frac{1}{q} + \frac{\bar{\rho}}{\bar{\rho}q - \bar{a}}\Big]\partial_s q > 0,$$

since $\partial_s q = \alpha(q^1 - q^0) < 0$ by (B.19).

Finally, part (ii) is a consequence of parts (i) and (iii), because of the identity $|\sigma_R|^2 - [\binom{0}{1} \cdot \sigma_q]^2 = [\sigma + \binom{1}{0} \cdot \sigma_q]^2$. □

B.2 Stochastic stability: a useful lemma

To prove the stationarity claims of Theorem 2 and Proposition 4, we need the following lemma, which is a slight generalization of Theorems 3.5 and 3.7 of Khasminskii (2011), in the sense that weaker conditions are imposed on the coefficients α and β. Indeed, any coefficients (α, β) are permissible as long as they admit existence of a weak solution to the SDE system. See also Remark 3.5 and Corollary 3.1 in Khasminskii (2011) which allow the arguments in $\mathbb{R}^l$ to generalize to any open domain $\mathcal{D}$.

Lemma B.1. *Suppose $(X_t)_{0\leq t\leq \tau}$ is a weak solution to the SDE $dX_t = \beta(X_t)dt + \alpha(X_t)dZ_t$ in an open connected domain $\mathcal{D} \subset \mathbb{R}^l$, where Z is a d-dimensional Brownian motion and $\tau := \inf\{t : X_t \notin \mathcal{D}\}$ is the first exit time from $\mathcal{D}$. Define the infinitesimal generator $\mathscr{L}$ by (for any C^2 function f)*

$$\mathscr{L}f = \sum_{i=1}^n \beta_i \frac{\partial f}{\partial x_i} f + \frac{1}{2}\sum_{i,j=1}^n (\alpha_i \cdot \alpha_j)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Suppose there is a non-negative C^2 function $v : \mathcal{D} \mapsto \mathbb{R}_+$ such that (i) $\liminf_{x\to\partial\mathcal{D}} v(x) = +\infty$; (ii) $\mathscr{L}v \leq cv$ for some constant $c \geq 0$; and (iii) $\limsup_{x\to\partial\mathcal{D}} \mathscr{L}v(x) = -\infty$. Then,

(a) $\tau = +\infty$ almost-surely;

(b) the distribution of X_0 can be chosen such that $(X_t)_{t\geq 0}$ is stationary.

Proof of Lemma B.1. Let $\{\mathcal{D}_n\}_{n\geq 1}$ be an increasing sequence of open sets, whose closures are contained in $\mathcal{D}$, such that $\cup_{n\geq 1}\mathcal{D}_n = \mathcal{D}$. Let $\tau_n := \inf\{t : X_t \notin \mathcal{D}_n\}$, and note that $\tau = \lim_{n\to\infty}\tau_n$ is the monotone limit of these exist times. Define $w(t, x) := v(x)\exp(-ct)$, which satisfies $\mathscr{L}w \leq 0$ by assumption (ii). Using Itô's formula, we have

$$\mathbb{E}[v(X_{\tau_n\wedge t})e^{-c(\tau_n\wedge t)} - v(X_0)] = \mathbb{E}\int_0^{\tau_n\wedge t} \mathscr{L}w(u, X_u)du \leq 0.$$

Since $(\tau_n \wedge t) \le t$ and $v \ge 0$, we obtain

$$\mathbb{E}[v(X_{\tau_n \wedge t})] \le e^{ct}\mathbb{E}[v(X_0)].$$

Because $\mathbb{E}[v(X_{\tau_n \wedge t})] \ge \mathbb{P}[\tau_n \le t] \inf_{x \in \mathcal{D} \setminus \mathcal{D}_n} v(x)$, we thus have

$$\mathbb{P}[\tau_n \le t] \le \frac{e^{ct}\mathbb{E}[v(X_0)]}{\inf_{x \in \mathcal{D} \setminus \mathcal{D}_n} v(x)}.$$

Taking the limit $n \to \infty$, we obtain

$$\mathbb{P}[\tau \le t] \le \frac{e^{ct}\mathbb{E}[v(X_0)]}{\liminf_{x \to \partial \mathcal{D}} v(x)} = 0.$$

Thus, taking $t \to \infty$, we prove (a).

Next, since $\tau = +\infty$ a.s., we may consider $(X_t)_{t \ge 0}$ that is now defined for all time. Using Itô's formula,

$$\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] = \mathbb{E} \int_0^{\tau_n \wedge t} \mathscr{L} v(X_u) du.$$

Note that $\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] \le b_1$ for some constant b_1. Also note that $\sup_{x \in \mathcal{D}} \mathscr{L} v(x) \le b_2$ for some constant b_2, given assumptions (i)-(iii) and the fact that v is C^2. Using these bounds, plus the following obvious inequality

$$\mathscr{L} v(X_u) \le \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} \sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L} v(x) + \sup_{x \in \mathcal{D}} \mathscr{L} v(x),$$

we get

$$-\sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L} v(x) \mathbb{E} \int_0^{\tau_n \wedge t} \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} du \le t b_2 - b_1.$$

Given the proof of (a), we may take the limit $n \to \infty$ (so that $\tau_n \to +\infty$), then apply Fubini's theorem, and then rearrange to obtain

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \le \frac{b_2}{-\sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathscr{L} v(x)}.$$

Taking $k \to \infty$ and using assumption (iii), we obtain

$$\lim_{k \to \infty} \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \le 0.$$

Applying Theorem 3.1 of Khasminskii (2011), there exists a stationary initial distribution for X_0. The process $(X_t)_{t \ge 0}$ augmented with this initial distribution is clearly stationary by definition. □

B.3 Numerical method for Markov GBSE of Section 3.3

Step 0: Let (i, j) index the state space $\mathcal{N} \times \mathcal{S} = (0,1) \times (0,1)$. Let $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$.

Step 1: Construct the candidate capital price function $q(\eta_i, s_j)$.

(a) Find the fundamental solution $q^0(\eta_i)$ to the economy from the ODE in part (i) of Lemma C.1. Our example uses this fundamental solution as an upper bound for the sunspot q function. Calculate also the unbounded fundamental solution $\tilde{q}^0(\eta_i)$, i.e., the solution to the ODE that ignores the restriction $\kappa < 1$.

(b) Construct a lower bound $q^1(\eta_i)$. Let $q^1(\eta_i) = q^0(\eta_i)$ for $i = 1, 2$. This ensures that $(q^1)'(0+) = (q^0)'(0+)$. Set $q^1(\eta_I) = a_e/\rho_e$ and interpolate all other points, i.e., $q^1(\eta_i)$ for $i = 3, \ldots, I-1$. Any interpolation method that delivers a monotonic function should work; we use a linear interpolation.

(c) Set the boundary values by $q(\eta_i, s_1) = q^0(\eta_i)$ and $q(\eta_i, s_J) = q^1(\eta_i)$ for all i. For $j \in \{2, \ldots, J-1\}$, put $q(\eta_i, s_j) = \min\{a_e/\bar{\rho}(\eta_i), (1-s_j)\tilde{q}^0(\eta_i) + s_j q^1(\eta_i)\}$ for all i.

Step 2: Solve for κ by plugging q into (PO).

Step 3: Compute σ_s as follows.

(a) For points where $\kappa < 1$, the variance of capital returns $|\sigma_R|^2$ is recovered exactly from (27) – it is the term in parentheses. If $\partial_s \log q \neq 0$, then we solve for $\sigma_s^2 = |\sigma_R|^2(1 - (\kappa - \eta)(\partial_\eta \log q)^2 - \sigma^2)/(\partial_s \log q)^2$. If $\partial_s \log q = 0$ and $\sigma_R \neq 0$, we must have $\sigma_s \to \infty$ and this is not an equilibrium. This is only the case at $\eta \in \{\eta_1, \eta_2\}$, but as the proof of Proposition 4 shows, this is a non-issue. First, for η_1, equation (27) will hold independently of the value for σ_s. Second, for η_2, the proof shows a mollification ("smoothing out") is possible in a small neighborhood around the point to eliminate this issue. Since we study a finite-difference approximation, we may therefore take the neighborhood smaller than the grid spacing and ignore the issue. Therefore, we may set $\sigma_s(\eta_1, s_j)$ and $\sigma_s(\eta_2, s_j)$ by any extrapolation of $\sigma_s(\eta_i, s_j)$, for $i \geq 3$ and each j.

(b) For points where $\kappa = 1$, we have $\partial_s \log q = 0$, and σ_s can take any value (in our numerical example, we set $\sigma_s(\eta_I, s_j) = 0$ and interpolate the rest of the values).

B.4 Linearization and sunspots

Given the style of the quantitative macroeconomics literature, here we present a version of our model solved by linear approximation. For pedagogical purposes, we work within the model of Section 3.3, where there is a state variable s governing potential sunspot fluctuations, where ds_t only loads on the sunspot shock $Z^{(2)}$ as in equation (25). We could easily re-do this analysis with capital price q directly, as a "co-state" variable as in Section 3.2, rather than using a sunspot state variable. The big-picture point is that there is nothing special about perturbations and approximations that rules out sunspot equilibria in settings resembling ours.

For convenience, we repeat the system of equations characterizing equilibrium:

$$q = \frac{\kappa a_e + (1-\kappa)a_h}{\bar{\rho}} \tag{B.31}$$

$$0 = \min\Big[1-\kappa,\, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\Big(\frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2}\Big)\Big] \tag{B.32}$$

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}\Big(\frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2}\Big) + \delta(\nu - \eta) \tag{B.33}$$

$$\sigma_\eta = (\kappa - \eta)\frac{\binom{1}{0}\sigma + \binom{0}{1}\sigma_s \partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q}. \tag{B.34}$$

The relevant endogenous objects are $(q, \kappa, \sigma_\eta, \mu_\eta, \sigma_s, \mu_s)$. We will linearize our economy around a point where $\kappa \in (\eta, 1)$, which will simplify the second equation. We will also assume $\rho_e = \rho_h = \rho$ to simplify the calculations.

The linearized system, in deviations from an interior benchmark point, is as follows. A variable written x denotes the value at the interior benchmark point, whereas $\hat{x}$ denotes the deviation from that point; specifically, $\hat{x} = B_{x\eta}\hat{\eta} + B_{xs}\hat{s}$. Then, we have for equations (B.31)-(B.32)

$$B_{q\eta}\hat{\eta} + B_{qs}\hat{s} = \frac{a_e - a_h}{\rho}(B_{\kappa\eta}\hat{\eta} + B_{\kappa s}\hat{s}) \tag{B.35}$$

$$\begin{aligned}\frac{a_e - a_h}{q}\Big[1 - 2\hat{\eta} - \frac{\eta(1-\eta)}{q}(B_{q\eta}\hat{\eta} + B_{qs}\hat{s})\Big] &= [(B_{\kappa\eta} - 1)\hat{\eta} + B_{\kappa s}\hat{s}]\Big(\frac{\sigma^2 + (\sigma_s B_{qs}/q)^2}{(1 - (\kappa - \eta)B_{q\eta}/q)^2}\Big) \\ &+ 2(\kappa - \eta)\Big[\Big(\frac{\sigma_s B_{qs}^2/q^2}{(1 - (\kappa - \eta)B_{q\eta}/q)^2}\Big)(B_{\sigma_s\eta}\hat{\eta} + B_{\sigma_s s}\hat{s}) - \Big(\frac{\sigma_s^2 B_{qs}^2/q^3}{(1 - (\kappa - \eta)B_{q\eta}/q)^2}\Big)(B_{q\eta}\hat{\eta} + B_{qs}\hat{s}) \\ &+ \frac{B_{q\eta}}{q}\Big(\frac{\sigma^2 + (\sigma_s B_{qs}/q)^2}{(1 - (\kappa - \eta)B_{q\eta}/q)^3}\Big)\Big(\Big(B_{\kappa\eta} - 1 - \frac{\kappa - \eta}{q}B_{q\eta}\Big)\hat{\eta} + \Big(B_{\kappa s} - \frac{\kappa - \eta}{q}B_{qs}\Big)\hat{s}\Big)\Big].\end{aligned} \tag{B.36}$$

Each of equations (B.35)-(B.36) is really 2 equations: the perturbation must hold for all $(\hat{\eta}, \hat{s})$, so the coefficients on each must separately equate. Thus, we solve (B.35) for $B_{\kappa\eta} = \rho B_{q\eta}/(a_e - a_h)$ and $B_{\kappa s} = \rho B_{qs}/(a_e - a_h)$. Then, we can solve (B.36) for $B_{\sigma_s\eta}$ and $B_{\sigma_s s}$ explicitly, in terms of $(B_{q\eta}, B_{qs})$ and the interior benchmark values $(\eta, q, \kappa, \sigma_s)$.

This procedure shows that there can be sunspot volatility, but it will require non-iid sunspot dynamics. To see this, rearrange the loadings on $\hat{s}$ in equation (B.36) to obtain

$$\begin{aligned}\Big(\frac{\sigma_s B_{qs}}{q}\Big)B_{\sigma_s s} &= \Big(\frac{\sigma_s B_{qs}}{q}\Big)^2 - \Big(\frac{\sigma^2 + (\sigma_s B_{qs}/q)^2}{1 - (\kappa - \eta)B_{q\eta}/q}\Big)\Big(\frac{\rho q}{a_e - a_h} - \kappa - \eta\Big)\frac{B_{q\eta}}{q} \\ &\quad - \frac{(1 - (\kappa - \eta)B_{q\eta}/q)^2}{2(\kappa - \eta)}\Big[\Big(\frac{\sigma^2 + (\sigma_s B_{qs}/q)^2}{(1 - (\kappa - \eta)B_{q\eta}/q)^2}\Big)\frac{\rho q}{a_e - a_h} + \eta(1-\eta)\frac{a_e - a_h}{q}\Big].\end{aligned}$$

This shows that sunspot volatility dynamics are time-varying if and only if capital prices display sunspot volatility, i.e., $B_{\sigma_s s} \neq 0$ if and only if $\sigma_s B_{qs} \neq 0$. But we can say more. Under the linearization, the benchmark sunspot volatility σ_s is determined via equation (B.32) as

$$\frac{a_e - a_h}{q} = \frac{\kappa - \eta}{\eta(1-\eta)}\Big(\frac{\sigma^2 + (\sigma_s B_{qs}/q)^2}{(1 - (\kappa - \eta)B_{q\eta}/q)^2}\Big),$$

which shows that $\sigma_s \neq 0$ if and only if $B_{qs} \neq 0$. Therefore, if we simply conjecture any capital price with the property $B_{qs} \neq 0$, then sunspot dynamics must be time-varying, i.e., $B_{\sigma_s s} \neq 0$.

To see whether the conjecture $B_{qs} \neq 0$ is sustainable, write the state dynamics, which for the drifts take the form

$$\hat{\mu}_\eta = B_{\eta\eta}\hat{\eta} + B_{\eta s}\hat{s} \tag{B.37}$$

$$\hat{\mu}_s = B_{s\eta}\hat{\eta} + B_{ss}\hat{s}, \tag{B.38}$$

where

$$B_{\eta\eta} := -\delta + \frac{a_e - a_h}{q}\Big[(1 - 2\eta)B_{\kappa\eta} - 2(\kappa - \eta) - (\kappa - 2\eta\kappa + \eta^2)\frac{B_{q\eta}}{q}\Big]$$

$$B_{\eta s} := \frac{a_e - a_h}{q}\Big[(1 - 2\eta)B_{\kappa s} - (\kappa - 2\eta\kappa + \eta^2)\frac{B_{qs}}{q}\Big].$$

Note that we do not show the linearization of the volatility equation σ_η, because it affects no other equilibrium object (but note that σ_η at the benchmark point would just be determined by (B.34), with the linearization for q plugged in). The critical property here, which carries over from the model solved globally, is that $B_{s\eta}$ and B_{ss} are not pinned down by any equilibrium restriction. We will choose these objects to ensure stability of our linearized system. Stability boils down to the eigenvalues λ_1, λ_2 of the matrix

$$\mathbf{B} := \begin{bmatrix} B_{\eta\eta} & B_{\eta q} \\ B_{s\eta} & B_{ss} \end{bmatrix}.$$

The eigenvalues solve $\lambda_1 + \lambda_2 = \text{trace}(\mathbf{B}) = B_{\eta\eta} + B_{ss}$ and $\lambda_1 \lambda_2 = \det(\mathbf{B}) = B_{\eta\eta} B_{ss} - B_{\eta s} B_{s\eta}$. By picking $B_{ss} < -B_{\eta\eta}$ and $B_{s\eta} < B_{\eta\eta} B_{ss} / B_{\eta s}$, both eigenvalues will be strictly negative, and the system will be stable. As nothing rules out these choices, sunspot volatility is possible.

C Amplified fundamental equilibria

In this section, we investigate properties of equilibria where sunspot shocks $Z^{(2)}$ are irrelevant. If all equilibrium objects are additionally functions of expert wealth share η, then the key equations describing such equilibria are (PO), (A.1), and (15), restated here for convenience:

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h \tag{C.1}$$

$$0 = \min\Big[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\Big]. \tag{C.2}$$

$$\sigma_q = \frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}\sigma. \tag{C.3}$$

Also, wealth share dynamics are given in (22)-(23), restated here for convenience:

$$\mu_\eta = -\eta(1-\eta)(\rho_e - \rho_h) + \mathbf{1}_{\{\kappa<1\}}(\kappa - 2\kappa\eta + \eta^2)\frac{a_e - a_h}{q} + \delta(\nu - \eta) \tag{C.4}$$

$$\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q). \tag{C.5}$$

We define a fundamental equilibrium as follows.[35]

Definition 4. Given $\eta_0 \in (0,1)$, a *fundamental equilibrium* consists of adapted processes $(\eta_t, q_t, \kappa_t)_{t\geq 0}$ such that equations (C.1), (C.2), and (C.3) hold, and (C.4)-(C.5) describe the dynamics of η_t.

C.1 Properties of the non-sunspot solution with fundamental shocks

We describe here some properties of fundamental equilibria with fundamental volatility $\sigma > 0$.

Lemma C.1. *Assuming it exists, suppose (q,κ) is a fundamental equilibrium in η in the sense of Definition 4. Assume $\kappa(0+) = 0$. Define $\eta^* := \inf\{\eta : \kappa = 1\}$. Then, the following hold:*

(i) $(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}$*, for all* $\eta \in (0,\eta^*)$.

(ii) $\eta a_e + (1-\eta)a_h < \bar{\rho}q < a_e$*, for all* $\eta \in (0,\eta^*)$.

(iii) $\frac{q'(0+)}{q(0+)} = \frac{a_e}{a_h} - \frac{\rho_e}{\rho_h} + \rho_h\big(\frac{a_e - a_h}{\sigma a_h}\big)^2$.

(iv) If σ is sufficiently small, then $q' > \frac{a_e - a_h}{\bar{\rho}}$*, for* $\eta \in (0,\eta^*)$.

(v) If σ is sufficiently small, then $\frac{\rho_h}{\rho_e}\big(\frac{1-a_h/a_e}{\sigma^2} - 1 + \frac{\rho_h}{\rho_e}\big)^{-1} < \eta^* < 1$.

(vi) On $\eta \in (0,\eta^*)$, the solution q is infinitely-differentiable.

Proof of Lemma C.1. Since a fundamental equilibrium is assumed to exist, we make use of equations (C.1) and (C.2). Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1-\eta)a_h$.

(i) Start from equation (C.2), and rearrange to obtain the result, where we have implicitly selected the solution with $1 > (\kappa - \eta)\frac{q'}{q}$.

[35] We omit r_t from the definition, since it can be read off of (21), given other objects, and affects no other equation.

(ii) The first inequality, which is equivalent to $\kappa > \eta$, is a direct implication of equation (C.2). The second inequality, equivalent to $\kappa < 1$, is a restatement of the definition of η^*.

(iii) Start from equation (C.2). Taking the limit $\eta \to 0$, and using $\kappa(0+) = 0$, delivers an equation for $\kappa'(0+)$. Differentiating (C.1), we may then substitute $\kappa'(0+) = \frac{\rho_h q'(0+) + (\rho_e - \rho_h) q(0+)}{a_e - a_h}$. Rearranging, we obtain the desired result.

(iv) By part (iii), there exists $\eta^\circ > 0$ and $\bar{\sigma} > 0$ such that uniformly for all $\sigma < \bar{\sigma}$, we have $q' > \frac{a_e - a_h}{\bar{\rho}}$ on the set $\{\eta < \eta^\circ\}$. On the set $\{\eta^\circ \leq \eta < \eta^*\}$, we know that $\kappa - \eta$ is bounded away from zero, uniformly for all $\sigma < \bar{\sigma}$. Using the expression in part (i), the fact that q is bounded by $a_e / \bar{\rho}$ uniformly for all σ, and the previous fact about $\kappa - \eta = \bar{\rho} q - \bar{a}$, we can write

$$q' = \frac{a_e - a_h}{\bar{\rho} q - \bar{a}} q - o(\sigma), \quad \eta \in (\eta^\circ, \eta^*).$$

Therefore,

$$q' + o(\sigma) = \frac{a_e - a_h}{\bar{\rho} q - \bar{a}} q = \frac{a_e - a_h}{\bar{\rho}} \frac{q}{q - \bar{a}/\bar{\rho}} > \frac{a_e - a_h}{\bar{\rho}}, \quad \eta \in (\eta^\circ, \eta^*),$$

where the last inequality is due to $\bar{\rho} q > \bar{a}$ [part (ii)]. Taking σ is small enough implies the result on (η°, η^*), which we combine with the result on $(0, \eta^\circ)$ to conclude.

(v) Consider the function $\tilde{q} := \bar{a}/\bar{\rho}$, whose derivative is $\tilde{q}' = \frac{a_e - a_h}{\bar{\rho}} - \frac{\bar{a}}{\bar{\rho}} \frac{\rho_e - \rho_h}{\bar{\rho}} < \frac{a_e - a_h}{\bar{\rho}}$. Combining this result with part (iv), we obtain $q' > \tilde{q}'$. If $\tilde{q}$ was the capital price, then equation (C.1) implies the associated capital share $\tilde{\kappa} = \eta$. On the other hand, the fact that $q' > \tilde{q}'$ implies $\kappa' > \tilde{\kappa}' = 1$, which implies $\eta^* < 1$.

Next, consider $\eta \in (\eta^*, 1)$ so that $\kappa = 1$. By equation (C.2), with $q = a_e / \bar{\rho}$, we must have

$$\sigma^2 \leq \eta \bar{\rho} \frac{a_e - a_h}{a_e} \Big(1 + (1 - \eta) \frac{\rho_e - \rho_h}{\bar{\rho}} \Big)^2, \quad \eta \geq \eta^*.$$

This is equivalent to

$$1 \leq \eta \frac{\rho_e}{\rho_h} \Big(\frac{a_e - a_h}{a_e \sigma^2} \rho_e - 1 + \frac{\rho_h}{\rho_e} \Big), \quad \eta \geq \eta^*.$$

Substituting $\eta = \eta^*$, and rearranging, we obtain the first inequality. There is no contradiction with $\eta^* < 1$, due to the assumption that σ can be made small enough.

(vi) Note that $F(\eta, q) := q[\frac{a_e - a_h}{\bar{\rho}(\eta) q - \bar{a}(\eta)} - \sigma(\frac{\eta(1-\eta)(\bar{\rho}(\eta) q - \bar{a}(\eta))}{q})]$ is infinitely differentiable in both arguments on $\{(\eta, q) : \eta \in (0,1), \bar{\rho}(\eta) q > \bar{a}(\eta)\}$. Thus, the result is a simple consequence of differentiating part (i), noting that by part (ii) we have $\bar{\rho}(\eta) q(\eta) > \bar{a}(\eta)$, and then using induction. □

It is not necessary to impose the condition $\kappa(0+) = 0$ to have a fundamental equilibrium. If we let $\kappa_0 \in (0,1)$ be a given "disaster belief" about experts' deleveraging, and we suppose $\kappa(0+) = \kappa_0$, there is no inherent contradiction to equilibrium. Existence of such an equilibrium boils down simply to existence of a solution to a first-order ODE. Thus, a variety of fundamental equilibria could exist, and indeed we provide a numerical example after the following lemma and proof.

Lemma C.2. *A fundamental equilibrium with disaster belief $\kappa_0 \in (0,1)$ exists if the free boundary problem*

$$(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}, \quad \text{on} \quad \eta \in (0,\eta^*), \tag{C.6}$$

$$\text{subject to} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} \quad \text{and} \quad q(\eta^*) = \frac{a_e}{\bar{\rho}(\eta^*)}, \tag{C.7}$$

has a solution.

Proof of Lemma C.2. A fundamental equilibrium in state variable η exists if and only if equations (C.1), (C.2), and (C.3) hold, and if the time-paths $(\eta_t)_{t\geq 0}$ induced by dynamics (σ_η, μ_η) avoid $\eta = 0$ almost-surely. We will demonstrate these conditions.

Suppose (C.6)-(C.7) has a solution (q, η^*) corresponding to $\kappa_0 \in (0,1)$. If there are multiple solutions, we pick the one such that $q(\eta) < a_e/\bar{\rho}(\eta)$ for all $\eta \in (0,\eta^*)$, which is always possible because the boundary conditions (C.7) imply $\bar{\rho}(0)q(0) < \bar{\rho}(\eta^*)q(\eta^*)$. Set $q(\eta) = a_e/\bar{\rho}(\eta)$ for all $\eta \geq \eta^*$. Define $\kappa = \frac{\bar{\rho}q - a_h}{a_e - a_h}$. Note that (C.1) is automatically satisfied. Note that (C.3) is also satisfied automatically, by applying Itô's formula to the solution $q(\eta)$ and using $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$.

We show (C.2) holds separately on $(0,\eta^*)$ and $[\eta^*, 1)$. Using (C.1) and (C.3) in the ODE (C.6) and rearranging, we show that (C.2) holds when $\kappa < 1$. The boundary condition $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$ is equivalent to $\kappa(\eta^*) = 1$, which shows that $\kappa(\eta) < 1$ for all $\eta < \eta^*$. Therefore, we have proven that (C.2) holds on $(0,\eta^*)$.

If $\eta^* = 1$, then we are done verifying (C.2); otherwise, we need to verify (C.2) on $[\eta^*, 1)$. On this set, $\kappa = 1$, so we need to verify

$$\eta\frac{a_e - a_h}{q} \geq (\sigma + \sigma_q)^2 \quad \text{for all} \quad \eta \geq \eta^*. \tag{C.8}$$

First, we show that it suffices to verify this condition exactly at η^*. Indeed, on $(\eta^*, 1)$, we have $\kappa = 1$ and $q = a_e/\bar{\rho}$. Substituting these and (C.3) into (C.8), we obtain

$$\text{(C.8) holds} \Leftrightarrow \Big(\frac{a_e - a_h}{a_e\sigma^2}\rho_e - \frac{\rho_e - \rho_h}{\rho_e}\Big)\eta \geq \frac{\rho_h}{\rho_e} \quad \text{for all} \quad \eta \geq \eta^*.$$

But since the left-hand-side is increasing in η, if it holds at $\eta = \eta^*$, it holds for all $\eta > \eta^*$.

Now, writing (C.8) at η^*, using (C.3) to replace σ_q, and using ODE (C.6) to replace $\eta^*\frac{a_e - a_h}{q(\eta^*)} = \sigma[1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)]^{-1}$, we need to verify

$$\text{(C.8) holds} \Leftrightarrow \frac{\sigma}{1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)} \geq \frac{\sigma}{1 - (1-\eta)q'(\eta^*+)/q(\eta^*)} \Leftrightarrow q'(\eta^*-) \geq q'(\eta^*+).$$

We clearly have $q'(\eta^*-) \geq q'(\eta^*+)$ by the simple fact that $q < a_e/\bar{\rho}$ for $\eta < \eta^*$ and $q = a_e/\bar{\rho}$ for $\eta \geq \eta^*$.

Finally, it remains to very that η_t almost-surely never reaches the boundary 0. Near $\eta = 0$, the dynamics in (C.4)-(C.5) become

$$\mu_\eta(\eta) = \kappa_0\frac{a_e - a_h}{q(0+)} + \delta\nu + o(\eta)$$
$$\sigma_\eta^2(\eta) = \kappa_0\frac{a_e - a_h}{q(0+)}\eta + o(\eta).$$

By the same analysis as in Lemma A.1, the boundary 0 is unattainable [this system corresponds to setting $\alpha = 1$ and $\theta = 1$ in part (3a) of the proof of Lemma A.1]. □

What happens in an equilibrium of Lemma C.2 in which $\kappa_0 > 0$? Behavior at the boundary $\eta = 0$ is substantially different than the $\kappa_0 = 0$ case, because equation (C.2) can only hold there if $\sigma_q \to -\sigma$ as $\eta \to 0$. Capital prices "hedge" fundamental shocks to capital, in a brief region of the state space $(0, \eta^{\text{hedge}})$. Said differently, given the formula (C.3), the fact that $\sigma_q(0+) = -\sigma$ implies $q'(0+) = -\infty$, so that prices rise as experts lose wealth in a region of the state space. The hedging region is exactly what incentivizes experts to take so much leverage (indeed, expert leverage κ/η blows up near 0). For $\eta > \eta^{\text{hedge}}$, this behavior reverses, and the equilibrium behaves very much like the equilibrium with $\kappa_0 = 0$. Overall, there is no inconsistency with equilibrium even though $q' < 0$ in the region $(0, \eta^{\text{hedge}})$.[36]

Figure C.1 displays several examples of equilibria with different choices of $\kappa_0 > 0$. The solid black lines, which are equilibrium outcomes with $\kappa_0 = 0.001$, corresponds approximately to the equilibrium choice made by Brunnermeier and Sannikov (2014). The other curves, with higher disaster beliefs κ_0, are new to the literature. Similar to the the sunspot results of Section 2.2, more optimistic disaster beliefs raise capital prices and reduce capital price volatility.

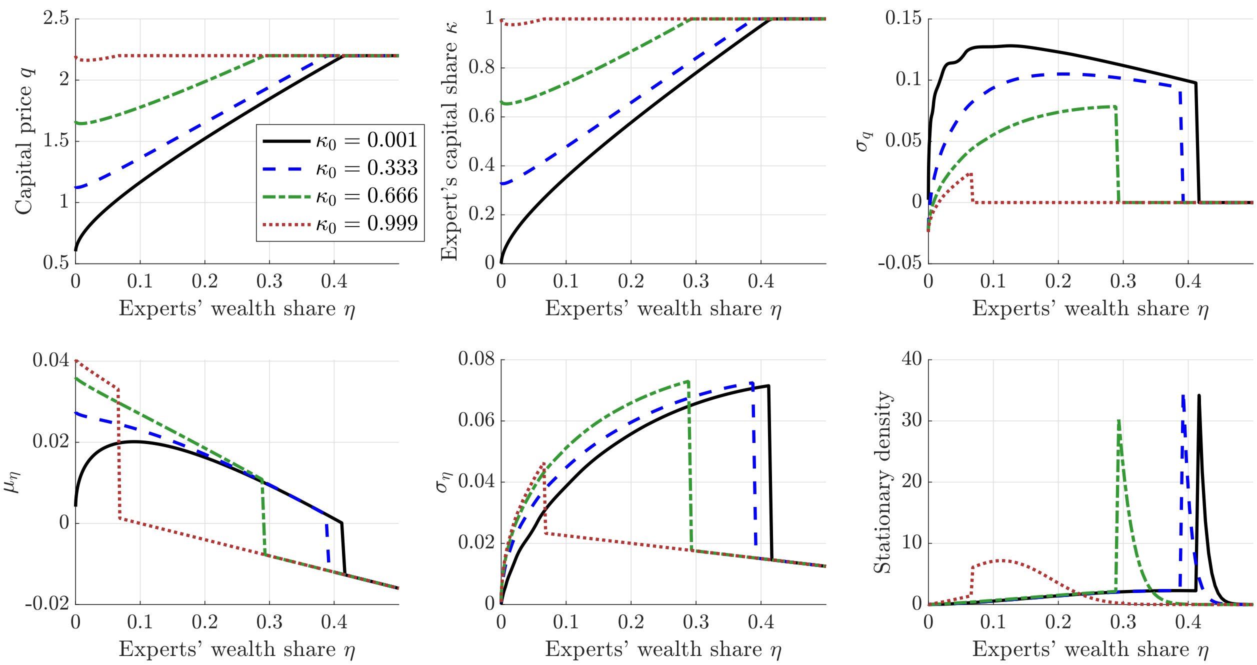

Figure C.1: Fundamental equilibria with different disaster beliefs κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$.

C.2 The "hedging" equilibrium

The equilibria described in Appendix C.1 are "normal" in the sense that a positive exogenous shock increases asset prices and experts' wealth share. But technically, agents do not care about

[36]One may think that $q'(0+) = -\infty$, and more generally that $q' < 0$ in some region of the state space, could imply that κ hits η at some point. However, this cannot happen. Indeed, since $\kappa_0 > 0$, we have that $q(0+) > \tilde{q}(0+)$, where $\tilde{q}(\eta) := ((a_e - a_h)\eta + a_h)/\bar{\rho}$ is the price function consistent with $\kappa = \eta$.

To see this, assume there is an $\hat{\eta} \in (0, 1)$ such that $\kappa(\hat{\eta}) = \hat{\eta}$ (or equivalently, $q(\hat{\eta}) = \tilde{q}(\hat{\eta})$). If there is more than one, consider the minimum among them, so $q(\eta) > \tilde{q}(\eta)$ for all $\eta \in (0, \hat{\eta})$. From the $\tilde{q}(\eta)$ definition, we have $\tilde{q}'(\eta) = (a_e - a_h)/\bar{\rho} - ((a_e - a_h)\hat{\eta} + a_h)(\rho_e - \rho_h)/\bar{\rho}^2 < \infty$, while from (C.6) it must be that $q'(\hat{\eta}-) \to \infty$. This is a contradiction.

the direction prices move when they make their portfolio choices. They only care about risk which is measured in return variance; this can be seen in the optimality condition (C.2) where $(\sigma + \sigma_q)^2$ appears. An immediate implication is that two types of equilibria are possible: the "normal" one has $\sigma + \sigma_q > 0$; an alternative equilibrium has $\sigma + \sigma_q < 0$.

We term this latter equilibrium the "hedging" equilibrium because asset price movements move oppositely to exogenous shocks. In fact, asset price responses are so strong in opposition that experts actually gain in wealth share upon a negative fundamental shock. This can only happen because of coordination: experts and households simply believe negative shocks are good news for asset prices, so they rush to purchase capital, which percolates through equilibrium relationships to justify beliefs about price increases. Such coordination stands in contrast to the normal equilibrium, in which negative shocks beget fire sales that push down asset prices.

Mathematically, we need only solve a slightly different capital price ODE. Whereas ODE (C.6) holds in the normal equilibrium, the hedging equilibrium requires

$$(\bar{\rho} q - \eta a_e - (1-\eta) a_h) \frac{q'}{q} = a_e - a_h + \sigma \sqrt{q \frac{\bar{\rho} q - \eta a_e - (1-\eta) a_h}{\eta(1-\eta)}}, \quad \text{on} \quad \eta \in (0, \eta^*). \tag{C.9}$$

The difference between (C.9) and (C.6) is merely the sign in front of σ, which ensures different signs for σ_q. Finally, note that just like the normal equilibria, hedging equilibria could exist for $\kappa_0 \neq 0$. Figure C.2 compares a normal equilibrium to a hedging equilibrium.

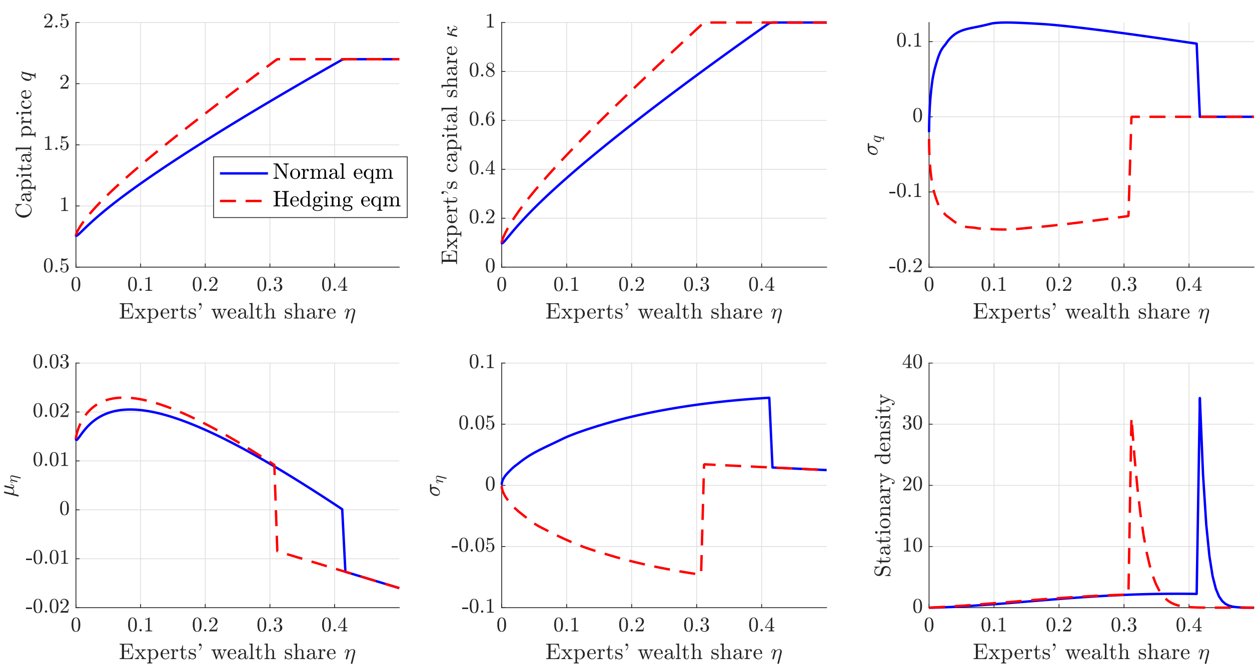

Figure C.2: Two equilibria (normal versus hedging) both with disaster belief $\kappa_0 = 0.1$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$.

D Model extensions

D.1 Poisson Sunspot Equilibrium (PSE)

Rather than model sunspots as Brownian shocks, here we conjecture capital prices can "jump" for non-fundamental reasons. Mathematically, write

$$dq_t = q_{t-}[\mu_{q,t-}dt + \zeta_{q,t-}dJ_t],$$

where J is a Poisson process with intensity λ that does not affect physical capital at all. An equilibrium in which ζ_q is not identically zero will be called the *Poisson Sunspot Equilibrium* (PSE).

In a Markov equilibrium, the sole state variable will still be experts' wealth share η, which follows a jump process

$$d\eta_t = \mu_{\eta,t-}dt + \zeta_{\eta,t-}dJ_t.$$

Note that $\zeta_{\eta,t-} := \eta_t - \eta_{t-}$ by definition. Because agents' portfolios (capital and bonds) are predetermined, we can determine the wealth share jump from the jump in q, with the result being[37]

$$\zeta_\eta = (\kappa - \eta)\frac{\zeta_q}{1+\zeta_q}. \tag{D.1}$$

On the other hand, once the post-jump wealth share is known, the capital price is also known, since η is the sole state variable, i.e., we have $q_t = q(\eta_t)$ for some function q. Thus, if we denote the post-jump wealth share by $\hat{\eta}$,

$$\zeta_q = \frac{q(\hat{\eta}) - q}{q}. \tag{D.2}$$

This is the way to solve the two-way feedback between the wealth distribution and capital prices, similar to the Brownian model. Combining (D.1)-(D.2) yields $\hat{\eta} - \eta = (\kappa - \eta)\frac{q(\hat{\eta})-q}{q(\hat{\eta})}$, which is analogous to the sunspot differential equation of the BSE. Indeed, as $\hat{\eta} \to \eta$, this system converges exactly to $q'/q = (\kappa - \eta)^{-1}$.

Because we do not model bankruptcy procedures, we must also make sure the jump renders experts solvent, meaning $\zeta_{\eta,t-} > -\eta_{t-}$, to preserve the risk-free status of the bond. If solvency cannot be ensured, then no self-fulfilling jump can take place.

Portfolio choices are still relatively simple, because the jump size is locally predictable, i.e., ζ_q is known just before the jump actually occurs. Thus, equations characterizing an equilibrium of this model are given by the following simple lemma.

[37] The derivation is as follows. Let variables with hats, e.g., "$\hat{x}$", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K}\kappa - B$ and $\hat{N}_h = \hat{q}\hat{K}(1-\kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\zeta_\eta = \hat{\eta} - \eta = B[1/(qK) - 1/(\hat{q}\hat{K})] = qK(\kappa - \eta)[1/(qK) - 1/(\hat{q}\hat{K})]$. Using the fact that $\hat{K} = K$ and the definition $\zeta_q := \hat{q}/q - 1$, we arrive at $\zeta_\eta = (\kappa - \eta)[1 - (1+\zeta_q)^{-1}]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump. Conceptually, there is no reason why this needs to be true, but it preserves its risk-free conjecture. If bond prices are allowed to jump at the same time, we would find different expressions.

Lemma D.1 (Equilibrium with Jumps). *An allocation is a Markov equilibrium with jumps only if $(q, \kappa, \hat{\eta}, \zeta_q, \zeta_\eta)$ are functions of $\eta \in (0,1)$ satisfying price-output relation (PO), equations (D.1)-(D.2), and the following:*

$$\hat{\eta} = \eta + (\kappa - \eta)\frac{\zeta_q}{1+\zeta_q} > 0$$

$$0 = \min\Big[1-\kappa,\, \eta(1-\eta)\frac{a_e - a_h}{q} - (\kappa-\eta)\frac{\lambda\zeta_q^2}{(1+\frac{\kappa}{\eta}\zeta_q)(1+\frac{1-\kappa}{1-\eta}\zeta_q)}\Big], \quad \kappa(0) = 0.$$

To show what the PSE looks like, we provide a numerical solution and plot some aspects below. The first panel of figure D.1 displays one simulation of the PSE, also comparing it with a simulation of the BSE. The second panel plots the stationary capital price densities (although note that the BSE "density" in fact has a point mass at $\eta = \eta^*$). Capital prices in the PSE tend to remain at lower levels than in the BSE for our example.

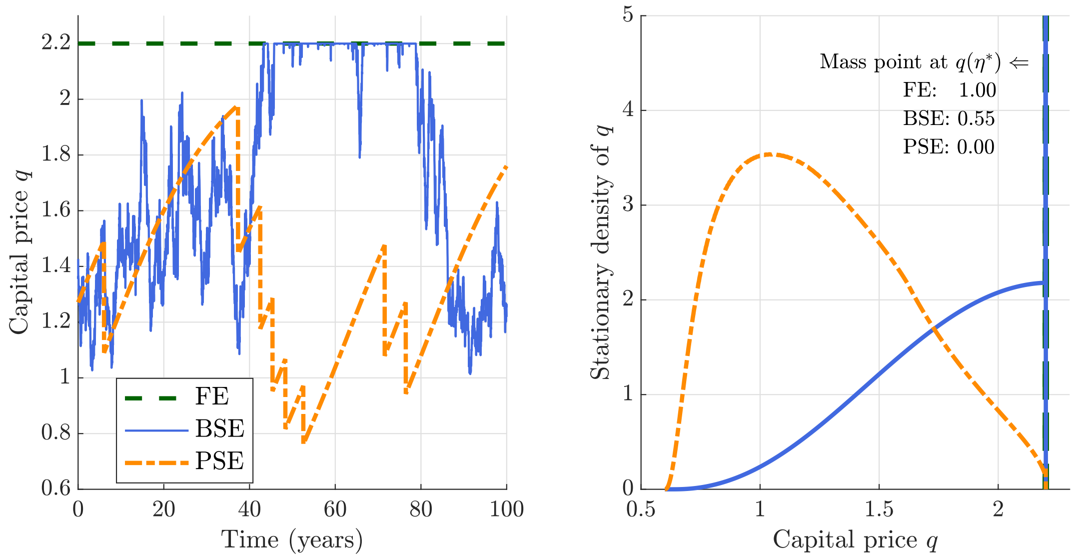

Figure D.1: Time series and stationary density of capital price q in a PSE, BSE, and FE. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters: $\nu = 0.1$ and $\delta = 0.1$.

D.2 General CRRA preferences

We modify the model of Section 1 by generalizing preferences to the CRRA type. In particular, we replace the $\log(c)$ term in utility specification (2) with the flow consumption utility $c^{1-\gamma}/(1-\gamma)$. For simplicity, we consider no OLG structure ($\delta = 0$), but we continue to allow experts' discount rate to exceed households' ($\rho_e \geq \rho_h$). We continue to look for a Markov equilibrium, with experts' wealth share η as the state variable.

Equilibrium. The key equation (SV) still holds, repeated here for convenience:

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\sigma_\eta = 0. \tag{D.3}$$

The sunspot equilibrium is associated with the term in brackets being equal to zero. Unlike with logarithmic preferences, this condition does not pin down $q(\eta)$ function, because we can no longer write $\kappa(q,\eta)$ from the goods market clearing condition: the consumption to wealth ratio is not constant anymore, and depends on agents' value functions.

The value function can be written as $V_i = v_i(\eta)K^{1-\gamma}/(1-\gamma)$ where $v_i(\eta)$ is determined in equilibrium. Then, consumption is $c_i/n_i = (\eta_i q)^{1/\gamma - 1}/v_i^{1/\gamma}$ where η_i corresponds to the wealth share of sector i. Then, goods market clearing becomes

$$q^{1/\gamma}\Big[\Big(\frac{\eta}{v_e}\Big)^{1/\gamma} + \Big(\frac{1-\eta}{v_h}\Big)^{1/\gamma}\Big] = (a_e - a_h)\kappa + a_h. \tag{D.4}$$

Optimal portfolio decisions imply that

$$0 = \min\Big[1-\kappa, \frac{a_e - a_h}{q} - \Big(\frac{v_h'}{v_h} - \frac{v_e'}{v_e} + \frac{1}{\eta(1-\eta)}\Big)(\kappa - \eta)\sigma_q^2\Big]. \tag{D.5}$$

The HJB equation for $i \in \{e,h\}$ has the familiar form $\rho_i V_i = u(c) + \mathbb{E}[\frac{dV_i}{dt}]$, which becomes

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \underbrace{\frac{v_i'}{v_i}\mu_\eta + \frac{1}{2}\frac{v_i''}{v_i}\sigma_\eta^2}_{:=\mu^{v,i}} + (1-\gamma)g. \tag{D.6}$$

The dynamics of η satisfy

$$\sigma_\eta = (\kappa - \eta)\sigma_q \tag{D.7}$$

$$\mu_\eta = \eta(1-\eta)\Big(\varsigma_e\frac{\kappa}{\eta}\sigma_q - \varsigma_h\frac{1-\kappa}{1-\eta}\sigma_q + \frac{c_h}{n_h} - \frac{c_e}{n_e}\Big) - \sigma_\eta\sigma_q \tag{D.8}$$

and agent-specific risk prices satisfy

$$\varsigma_e = -\frac{v_e'}{v_e}\sigma_\eta + \frac{\sigma_\eta}{\eta} + \sigma_q \tag{D.9}$$

$$\varsigma_h = -\frac{v_h'}{v_h}\sigma_\eta - \frac{\sigma_\eta}{1-\eta} + \sigma_q. \tag{D.10}$$

A Markov equilibrium is a set of functions: prices $\{q, \sigma_q, \varsigma_e, \varsigma_h\}$, allocation $\{\kappa\}$, value functions $\{v_h, v_e\}$ and aggregate state dynamics $\{\sigma_\eta, \mu_\eta\}$ that solve the system (D.3)-(D.10).

The fundamental equilibrium corresponds to the solution for (D.3) where $\sigma_\eta = 0$, which implies deterministic economic dynamics. Then, the capital price has no volatility ($\sigma_q = 0$), risk prices are zero ($\varsigma_e = \varsigma_h = 0$), and experts hold the entire capital stock ($\kappa = 1$). The capital price is then solved from (D.4), and the value functions satisfy

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \frac{v_i'}{v_i}\underbrace{\eta(1-\eta)\left(\frac{c_h}{n_h} - \frac{c_e}{n_e}\right)}_{=\mu_\eta} + (1-\gamma)g.$$

Conversely, the sunspot equilibrium corresponds to the solution for (D.3) with $\frac{q'}{q} = (\kappa - \eta)^{-1}$ (and potentially $\sigma_\eta \neq 0$).

Disaster belief. With logarithmic preferences, we proved that any sunspot equilibrium must satisfy $\sigma_q(0+) = 0$. This allowed us, in Section 2.2, to construct sunspot equilibria with $\kappa(0+) = \kappa_0$ for any $\kappa_0 \in [0,1)$. With CRRA preferences, we attempt to construct the same class of equilibria, with $\sigma_q(0+) = 0$ and $\kappa_0 \in (0,1)$.

In order to have a non-degenerate stationary distribution, we have the following requirements. Since $\sigma_\eta(0+) = \kappa_0 \sigma_q(0+) = 0$, the state variable avoids the boundary $\{0\}$ if $\mu_\eta(0+) > 0$. Using (D.5) for $\kappa < 1$, we have[38]

$$\frac{a_e - a_h}{q(0+)} = (\varsigma_e(0+) - \varsigma_h(0+))\sigma_q(0+)$$

which allows us to show that[39]

$$\mu_\eta(0+) = \kappa_0 \frac{a_e - a_h}{q(0+)} > 0.$$

In addition, we need $\mu_\eta(\eta^*+) < 0$ where $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. This requirement should be satisfied for $\rho_e - \rho_h$ sufficiently large.[40]

Numerical solution. We do not provide an existence proof – which involves the existence of a solution to the ODE system – but construct numerical examples. For tractability, the numerical examples are constructed for $\kappa_0 > 0$, which keeps $q'(0+) = q(0+)/\kappa_0$ bounded.[41]

The numerical strategy is the following. Construct a grid $\{\eta_1, \ldots, \eta_N\}$ with limit points arbitrarily close to but bounded away from zero and one. Conjecture value functions $v_h(\eta)$ and $v_e(\eta)$. Impose $\kappa(\eta_1) = \kappa_0$ and use (D.4) to solve for $q(\eta_1)$. At each interior grid point, use $q' = q/(\kappa - \eta)$ and (D.4) to solve for $\kappa(\eta)$ and $q(\eta)$ until $\kappa(\eta^*) = 1$. In this region, recover σ_q from (D.5). For $\eta \in (\eta^*, 1]$ impose $\kappa(\eta) = 1$ and $\sigma_q = 0$, and solve capital price from (D.4). The rest of equilibrium objects are calculated directly from the system above. The guesses of the value functions are updated by augmenting the HJBs (D.6) with a time derivative and moving a small time-step backward, as in Brunnermeier and Sannikov (2016). The procedure terminates when the value functions converge to time-independent functions.

In Figure D.2, we plot the equilibrium objects as functions of η, for different levels of risk aversion γ. In Figure D.3, we make the same plots, for different levels of the disaster belief κ_0. Higher risk aversion (higher γ) or more pessimism about disasters (lower κ_0) generates sunspot equilibria featuring lower capital prices and higher volatility.

D.3 Time-varying disaster beliefs

We can further enlarge the space of self-fulfilling equilibria of Section 2.2 by allowing for time-variation in beliefs about experts' deleveraging in disaster states. This can be thought of as stochastic shifts between equilibria described in Section 2.2, each pertaining to a fixed disaster belief κ_0.

Suppose s_t is a time-varying sunspot that follows the process

$$ds_t = \mu_{s,t} dt + \sigma_{s,t} dZ_t,$$

[38]Note that this implies $\varsigma_e(0+) - \varsigma_h(0+)$ diverges.

[39]This expression also assumes that $\varsigma_h(0+)$ remains bounded. This is a mild assumption since households own all capital.

[40]There is an important distinction between the restriction not to reach $\eta = 0$ and $\mu_\eta(\eta^*+) < 0$. Without the first one, the equilibrium for any $\kappa_0 > 0$ unravels, while without the second one, the equilibrium is still valid, but it has a degenerate stationary distribution at some value $\eta^{ss} > \eta^*$.

[41]With logarithmic utility, we obtain a limiting result in Theorem 1, that as $\kappa_0 \to 0$, the equilibrium converges to the BSE with $\kappa(0) = 0$. With CRRA, we do not prove such a result analytically, but we do observe numerically what looks like convergence as κ_0 becomes small.

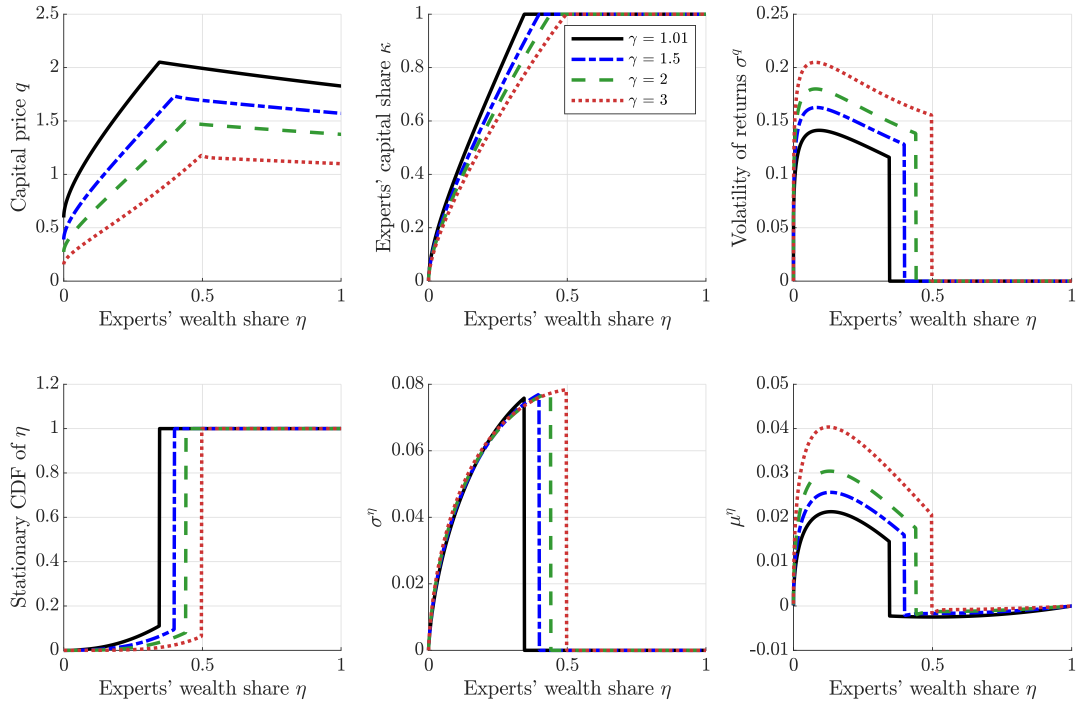

Figure D.2: Sunspot equilibrium for different risk aversion γ. The disaster belief is set to $\kappa_0 = 0.001$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

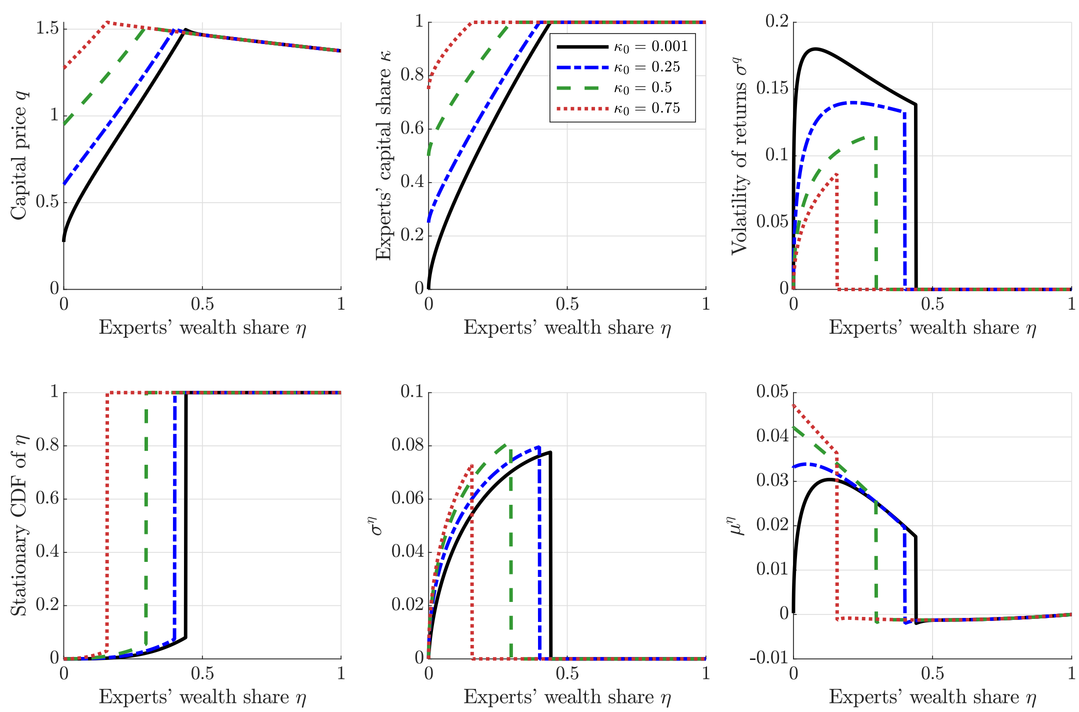

Figure D.3: Sunspot equilibrium for different disaster beliefs κ_0. Risk aversion is set to $\gamma = 2$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

where Z is a one-dimensional Brownian motion. Assume μ_s and σ_s are such that $s_t \in (0,1)$ for all $t \geq 0$ almost-surely. For convenience, assume $\kappa(0,s) = s$, so that the sunspot corresponds to the belief about experts' deleveraging. To solve for a Markov equilibrium in (η, s), we must solve the following equations, derived similarly to Lemma 1:

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h \tag{D.11}$$

$$\sigma_q = \frac{\sigma_s \partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q} \tag{D.12}$$

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\sigma_q^2\right], \quad \kappa(0,s) = s. \tag{D.13}$$

A requirement for a solution to exist is that $\sigma_s \to 0$ as $\eta \to 0$, for any value of s, so that $\sigma_q \to 0$ as $\eta \to 0$.

Note that the central equation (SV) can no longer hold in this model. This is because sunspot shocks move the boundary condition $\kappa(0,s)$, which has a direct effect on the capital price, through the term $\sigma_s \partial_s \log q$. If we had $(\kappa - \eta)\partial_\eta \log q = 1$, this direct effect would be amplified infinitely, and we would have unbounded σ_q in (D.12), which cannot be an equilibrium. Instead, $(\kappa - \eta)\partial_\eta \log q < 1$ is required by equilibrium, and equation (D.12) replaces equation (SV).

Given a prescribed σ_s, the system above becomes a partial differential equation in q. However, similar to Section 3.3, we can instead find a solution in reverse: pre-specifying a candidate equilibrium capital price q, we can obtain the sunspot dynamics (σ_s, μ_s) that justify such a price. The resulting sunspot dynamics will be endogenous.

Here, we implement one example with such endogenous sunspot dynamics which can be solved analytically. We suppose experts and households have the same discount rate, $\rho_e = \rho_h$ (stationarity can still be achieved with the OLG features). Let $\beta \in (0,1)$ be a constant. For each $s \in (0,1)$, solve the ODE

$$\frac{y'}{y} = \frac{\beta(a_e - a_h)}{\bar{\rho}y - \beta(xa_e + (1-x)a_h)}, \quad \eta \in (0, x^*), \quad y(0) = \frac{sa_e + (1-s)a_h}{\rho}, \quad y(x^*) = \frac{a_e}{\rho}. \tag{D.14}$$

The idea of equation (D.14) is to mimic the key BSE differential equation (10), but with a milder slope. Similar to footnote 11, this ODE has a closed-form solution, which is

$$y(x;s) = \frac{1}{\rho}\left[\beta\bar{a}(x) + \sqrt{\beta^2\bar{a}(x)^2 + \bar{a}(s)^2 - 2\beta a_h \bar{a}(s)}\right], \quad \bar{a}(x) := xa_e + (1-x)a_h \tag{D.15}$$

$$x^*(s) = \inf\{x : y(x;s) \geq a_e/\rho\}. \tag{D.16}$$

The solution y is well-defined (i.e., the discriminant is strictly positive), strictly increasing in x, and strictly increasing in s, for $x \in (0,1)$ and $s \in (0,1)$.

Using these results, define our solution as follows. Put $\eta^*(s) = x^*(s)$ and

$$q(\eta, s) = \begin{cases} y(\eta; s), & \text{if} \quad \eta < \eta^*(s); \\ a_e/\rho, & \text{if} \quad \eta \geq \eta^*(s). \end{cases}$$

Set $\kappa(\eta, s)$ from (D.11). Finally, substitute (D.12) into (D.13), and use the result to solve for σ_s, i.e.,

$$\sigma_s = \left(\frac{1 - (\kappa - \eta)\partial_\eta \log q}{\partial_s \log q}\right)\sqrt{\frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}}, \quad \text{on} \quad \{(\eta, s) : \eta < \eta^*(s), \, s \in (0,1)\}. \tag{D.17}$$

Since $\partial_s \log q > 0$ on the set $\{(\eta, s) : \eta < \eta^*(s), \, s \in (0,1)\}$, the solution (D.17) is well-defined. On the complement of this set, we can assume any value for σ_s. Note that μ_s is indeterminate, and we pick it to ensure that the bivariate diffusion $(\eta_t, s_t)_{t \geq 0}$ is stationary, exactly as in the proof of Proposition 4. By construction, the equilibrium conditions (D.11)-(D.13) are all satisfied.

D.4 Idiosyncratic uncertainty

Here, we add idiosyncratic risk to capital. Doing so raises 3 substantive points: (1) small idiosyncratic uncertainty can provide an equilibrium refinement, by selecting equilibria with the property $\lim_{\eta\to 0}\kappa = 0$; (2) large idiosyncratic uncertainty eliminates sunspot equilibria where η is the sole state variable (i.e., where sunspot shocks are iid); (3) idiosyncratic uncertainty allows us to study, in a non-trivial way, the stability properties of the "deterministic steady state" of our model.

Setting. In addition to the model assumptions listed in Section 1, individual capital now evolves as

$$dk_{i,t} = k_{i,t}[gdt + \tilde{\sigma}d\tilde{B}_{i,t}], \tag{D.18}$$

where $(\tilde{B}_i)_{i\in[0,1]}$ is a continuum of independent Brownian motions. Agents with indexes $i \in [0, I]$ are experts, and those with $i \in [I, 1]$ are households. As in Section 1, the aggregate stock of capital $K_t := \int_0^1 k_{i,t}di$ grows deterministically at rate g (no aggregate shocks).

As before, suppose Z is a one-dimensional Brownian motion (a sunspot shock), independent of all $\tilde{B}_i$. Conjecture

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t].$$

We will focus on Markov equilibria in which η is the sole state variable. A *fundamental equilibrium* features $\sigma_q \equiv 0$. A *sunspot equilibrium* features σ_q which is not identically zero.

Small uncertainty as equilibrium refinement. The first result in this environment is that *any* equilibrium (even one with additional state variables beyond η) must feature full deleveraging by experts, as they become poor, simply as a consequence of portfolio optimality. To see this, note that risk balance condition (RB), the combination of expert and household capital FOCs, is now modified to read

$$0 = \min\Big[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}(\tilde{\sigma}^2 + \sigma_q^2)\Big]. \tag{D.19}$$

Note that $a_e - a_h > 0$ and $\tilde{\sigma}^2 + \sigma_q^2 > 0$. Thus, as $\eta \to 0$, we must have $\kappa \to 0$. Since this holds for any arbitrarily small $\tilde{\sigma}$, we conclude that the equilibria with disaster beliefs $\kappa_0 > 0$ (see Section 2.2) are not robust.

Lemma D.2. *Any equilibrium with $\tilde{\sigma} > 0$ has the property $\lim_{\eta\to 0}\kappa = 0$.*

Large uncertainty eliminates iid sunspots. In Section 3.1, we have demonstrated how sunspot equilibria with η as the sole state variable are incompatible with the presence of exogenous aggregate fundamental risk. Here, we show that the conclusion is similar if the exogenous risk is idiosyncratic rather than aggregate.

Even with idiosyncratic risk $\tilde{\sigma}$, one may follow the same analysis as Section 2 to show that equation (10) still determines q if $\sigma_q \neq 0$. In other words, the candidate sunspot equilibrium of this model has a solution (q, κ), both as functions of η, which are independent of the amount of idiosyncratic risk $\tilde{\sigma}$ (i.e., the same as in the BSE). Denote $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$ the boundary point where households begin managing capital. This is also independent of $\tilde{\sigma}$.

Next, use equation (D.19) to solve for σ_q, given the solutions (q, κ). We get

$$\sigma_q^2 = -\tilde{\sigma}^2 + \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1.$$

Since $\sigma_q^2 \geq 0$ is required, an immediate consequence is that $\tilde{\sigma}$ high enough eliminates the existence of any sunspot volatility. We collect these results in the following lemma.

Lemma D.3. *Let (q, κ, η^*) be given by the BSE of Proposition 1. If capital has idiosyncratic risk $\tilde{\sigma}$, and $\tilde{\sigma}^2 \geq \sup_{\eta<\eta^*} \frac{\eta(1-\eta)}{\kappa(\eta)-\eta} \frac{a_e - a_h}{q(\eta)}$, any Markov equilibrium in η requires $\sigma_q = 0$.*

Intuitively, there is a trade-off between endogenous volatility σ_q and exogenous volatility $\tilde{\sigma}$. With higher idiosyncratic volatility $\tilde{\sigma}$, amplification of the aggregate sunspot shock is necessarily reduced. To understand this, consider Merton's optimal capital portfolio when there is only idiosyncratic volatility

$$\frac{k_j}{n_j} = \frac{a_j/q + g - r}{\tilde{\sigma}^2}, \quad j \in \{e, h\}.$$

As $\tilde{\sigma}$ increases the optimal capital demand becomes more inelastic to changes in the capital price q. Thus, for a given shift in the wealth distribution η and change in capital price q, the amount of capital that changes hands between experts and households will be dampened as $\tilde{\sigma}$ increases. But it is exactly such capital purchases/sales which are the key ingredient to our sunspot volatility, allowing price fluctuations to be self-fulfilled. As $\tilde{\sigma}$ increases, this mechanism is weakened, leading to a decrease in σ_q. Eventually, the mechanism is severed altogether because $\sigma_q^2 < 0$ is not possible.

Steady state stability. In an attempt to differentiate ourselves from the literature, here we examine the traditional stability properties of this model. The addition of idiosyncratic risk provides a convenient environment for stability analysis, for the following reason. Stability properties are typically studied around the "steady state" of a deterministic equilibrium. In the baseline model of Section 2 (with $\tilde{\sigma} = 0$), the volatile BSE precludes this, and studying a deterministic equilibrium instead puts us in the FE, which trivially has $\kappa = 1$ always. With idiosyncratic risk, we can study a fundamental equilibrium in which capital prices evolve deterministically, even though $\kappa < 1$ in steady state.

The crucial feature of the BSE, preserved in this model, is that capital prices are determined by a function q such that $q_t = q(\eta_t)$. Supposing that to be true, a steady state is fully characterized by the value $\eta = \eta^{ss}$ such that all non-growing variables are constant over time. This steady state is thus determined by the equation $\dot{\eta} = 0$, where

$$\dot{\eta} = \eta(1-\eta)\Big[\rho_h - \rho_e + \tilde{\sigma}^2\Big((\frac{\kappa}{\eta})^2 - (\frac{1-\kappa}{1-\eta})^2\Big)\Big] + \delta(\nu - \eta).$$

It is straightforward to show that equilibrium features stable state variable dynamics, in the sense that $\frac{\partial \dot{\eta}}{\partial \eta}|_{\eta=\eta^{ss}} < 0$. However, because the "co-state" q is determined explicitly as a function of η, the steady state is not "stable" in the usual sense required by the multiplicity literature. Technically, there is only one stable eigenvalue of the dynamical system (η_t, q_t) near steady state (η^{ss}, q^{ss}).

Proposition D.1. *The steady state of the model with idiosyncratic risk is saddle path stable.*

Proof of Proposition D.1. First, we show that q is a function of η, i.e., $q_t = q(\eta_t)$. Goods market clearing is still characterized by the price-output relation (PO). With idiosyncratic risk, the risk balance condition (RB) is now (D.19). The solution to the system (PO) and (D.19) can be computed explicitly. Indeed, define

$$\eta^* := \sup\{\eta : (a_e - a_h)\eta\bar{\rho}(\eta) = a_e\tilde{\sigma}^2\}.$$

Then, $\kappa = 1$ for all $\eta \in (\eta^*, 1)$. For $\eta \in (0, \eta^*)$, we compute $\kappa < 1$ as the positive root $\tilde{\kappa}$ from

$$0 = (a_e - a_h)\tilde{\kappa}^2 + [a_h - \eta(a_e - a_h)]\tilde{\kappa} - \eta a_h - \frac{\eta(1-\eta)(a_e - a_h)\bar{\rho}(\eta)}{\tilde{\sigma}^2}.$$

After determining κ for all values of η, capital price q can be computed from (PO), as an explicit function of η.

Given $q_t = q(\eta_t)$, the dynamics of q_t are given by $\dot{q}_t = q'(\eta_t)\dot{\eta}_t$, which only depends on η and not q (notice that $\dot{\eta}_t$ also only depends on η and not q). Consequently, the linearized system near steady state takes the form

$$\begin{bmatrix} \dot{\eta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ m_2 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ q \end{bmatrix}$$

for $m_1, m_2 \neq 0$. The eigenvalues of this system are $m_1 < 0$ and 0. □

As a result of Proposition D.1, there is a unique transition path $(\eta_t, q_t)_{t \geq 0}$ to steady state, given an initial condition η_0. In other words, q_0 is pinned down uniquely. Our sunspot equilibria are not constructed by randomizing over a multiplicity of transition paths that arise due to steady state stability, which is the usual approach (Azariadis, 1981; Cass and Shell, 1983). This can be seen in a relatively transparent way by examining Lemma D.3, which shows how sunspot equilibria can exist in this model (if $\tilde{\sigma}$ is small enough), despite the instability of the steady state.

D.5 Correlation between sunspots and fundamentals

What happens if sunspot shocks are correlated with fundamental shocks? To model this, we allow

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}^{(1)}dZ_t^{(1)} + \sigma_{s,t}^{(2)}dZ_t^{(2)}.$$

In Section 3.3, we restricted attention to $\sigma_{s,t}^{(1)} = 0$. Without this assumption, equations (27) and (26) are modified to read:

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\left(\frac{(\sigma + \sigma_s^{(1)}\partial_s \log q)^2 + (\sigma_s^{(2)}\partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2}\right)\right]$$

$$\sigma_q = \frac{\binom{1}{0}(\kappa - \eta)\sigma\partial_\eta \log q + \sigma_s \partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}.$$

The rest of the equilibrium restrictions are identical.

For the present illustration, we additionally assume that $\sigma_{s,t}^{(2)} = 0$, i.e., sunspot shocks *only* load on fundamental shocks. What emerges is the possibility that sunspot shocks "hedge" fundamental shocks: we can have $\sigma_s^{(1)}\partial_s \log q < 0$, which lowers return volatility and raises asset prices. In the extreme, if $\sigma_s^{(1)}\partial_s \log q \to -\sigma$, the economy will converge to the Brownian Sunspot Equilibrium (BSE) of Section 2. At the other end, if $\sigma_s^{(1)}\partial_s \log q \to 0$, the economy resembles the fundamental equilibrium (FE) but with positive fundamental shocks and amplification of those shocks (this FE was q^0 in our baseline construction in Section 3.3). Thus, by constructing our conjectured capital price function as a convex combination of the BSE and the FE, with weight $1-s$ on the BSE and s on the FE, we can ensure that $\sigma_s^{(1)}\partial_s \log q$ endogenously emerges negative. Figure D.4 displays the equilibrium constructed this way.

Figure D.5 displays the distribution of capital prices and return volatility in this sunspot equilibrium, relative to the distributions in the fundamental equilibrium (in which $\sigma_s^{(1)} \equiv 0$). As promised, the presence of $\sigma_s^{(1)}$ allows sunspots to raise asset prices and reduce volatilities.

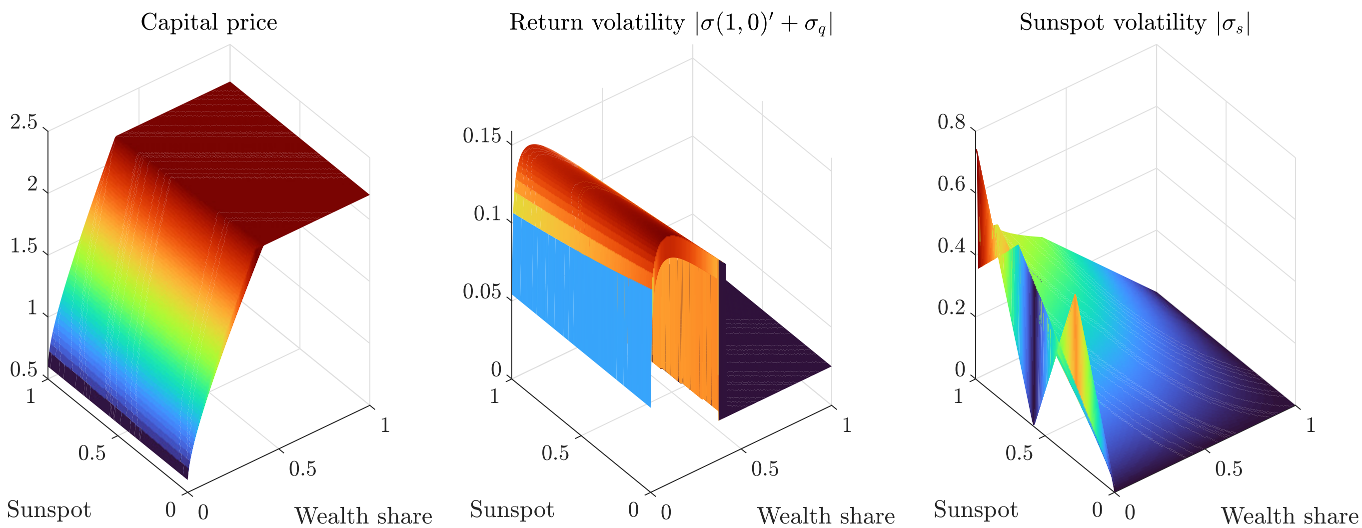

Figure D.4: Capital price q, volatility of capital returns $|(1,0)'\sigma + \sigma_q|$, and sunspot shock volatility $|\sigma_s|$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

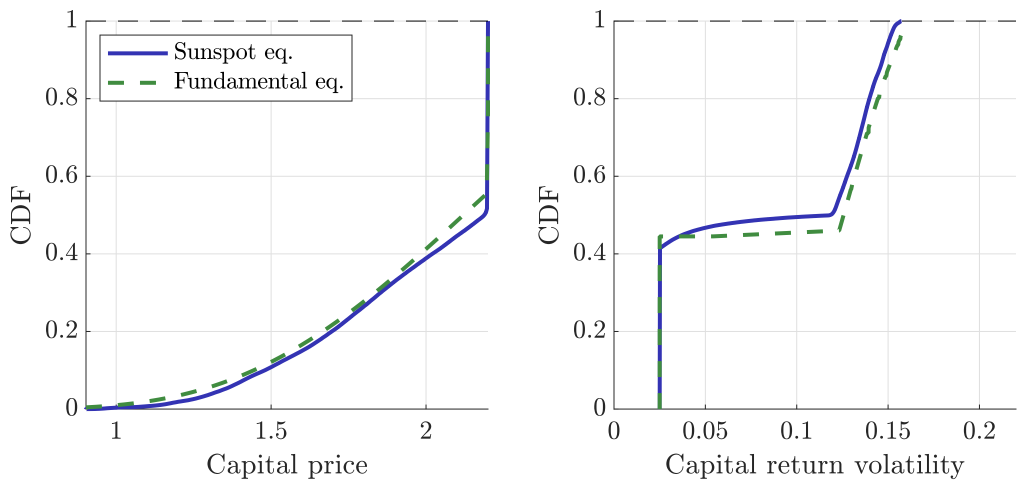

Figure D.5: Unconditional CDFs of capital prices and capital return volatility. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = 0.05 + s^{-1.5} - (s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

D.6 Exogenous sunspot dynamics

In Section 3.3, we solved for a Markov GBSE that featured endogenous sunspot dynamics, i.e., (σ_s, μ_s) could potentially depend on η. Here, we show that sunspot equilibria can be built on top of exogenous sunspot dynamics as well. As we will show, this construction can be naturally viewed as the limit of equilibria in which the variable s has a vanishing contribution to fundamentals. With that in mind, we actually start from a more general setting in which s can impact fundamental volatility, and then we take the limit as this impact becomes vanishingly small.

Consider the following stochastic volatility model:

$$\frac{dK_t}{K_t} = gdt + \sigma\sqrt{1+\omega s_t}dZ_t$$
$$ds_t = \mu_s(s_t)dt + \vartheta\sqrt{1+\omega s_t}dZ_t$$

where $\vartheta > 0$ is an exogenous parameter and $\omega \in \mathbb{R}$ measures the impact of s_t on capital growth volatility. Thus, the diffusion of s_t, namely $\sigma_s(s) := \vartheta\sqrt{1+\omega s}$, is specified exogenously. Also, $\mu_s(s)$ is an exogenous function that is specified to ensure that $s_t \in (s_{\min}, s_{\max})$, for some pre-specified interval satisfying $s_{\min} \geq 0$ and $c s_{\max} > -1$. Such a choice can always be made, e.g., by putting $\mu_s(s) = -(s_{\max} - s)^{-(1+\beta)} + (s - s_{\min})^{-(1+\beta)}$. Note that s_t becomes a sunspot when $\omega = 0$. When $\omega < 0$, the state s_t is an inverse measure of capital's volatility.

For simplicity, we assume there is a single aggregate shock, i.e., Z is a one-dimensional Brownian motion; this can easily be generalized to multiple shocks. Also for simplicity of expressions, we assume here that $\rho_e = \rho_h = \rho$. Then, an equilibrium capital price function $q(\eta, s)$ must satisfy the PDE defined by the following system

$$\rho q = \kappa a_e + (1-\kappa)a_h$$
$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{(\kappa-\eta)(1+\omega s)}{\eta(1-\eta)}\left(\frac{\sigma + \vartheta\partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}\right)^2\right].$$

Technically, the multiplicity arises from the selection of the boundary conditions on $q(\eta, s_{\min})$ and $q(\eta, s_{\max})$, which are not pinned down by any equilibrium restriction.

We perform three exercises. First, we show that there are multiple equilibria for a given set of parameters. We use $\omega < 0$ here, along with $s_{\min} = 0$ and $s_{\max} = 2$. In this case, the "natural" and intuitive solution is for q to increase with s, while volatility decreases. In Figure D.6, we pick a "low" boundary condition for $q(\eta, 0)$ and the solution follows this intuition.[42]

However, agents could equally well coordinate on a "high" boundary condition, which results in the solution of Figure D.7.[43] Notice the capital price and return volatility exhibit a non-monotonicity in s. At low values of s, q is decreasing in s, while return volatility increases. This behavior is made possible by the "coordination component" of the response to changes in s and not by the "fundamental component."

To shed light on this coordination component, our second exercise reverses the direction of the shock, making s positively associated to fundamental volatility. The idea is to compare solutions to models with $\omega > 0$ and $\omega < 0$, but with the same boundary condition $q(\eta, 0)$. The different sign for ω captures the notion that the "natural" solutions to these models should behave oppositely. The common boundary condition $q(\eta, 0)$ captures the notion that coordination is quite flexible here. Comparing Figure D.8 to Figure D.6, both of which have the "low" boundary condition, we see remarkably similar equilibria! In other words, with this much flexibility in coordination on extreme states, s affects equilibrium mainly through coordination and not through its fundamental impact.

The third and final exercise considers the limit $\omega \to 0$. Figure D.9 shows the solution for $\omega = -10^{-6}$, again equipped with the "low" boundary condition for $q(\eta, 0)$. There remains a tremendous

[42]This "low" boundary condition is a weighted average between the solution with infinite volatility and the fundamental equilibrium solution. The fundamental equilibrium, which is the capital price solution that keeps $s = 0$ fixed forever, is discussed in Appendix C. The infinite-volatility solution has $\kappa = \eta$, hence $q = (\eta a_e + (1-\eta)a_h)/\bar{\rho}(\eta)$.

[43]This "high" boundary condition is a weighted average between the BSE of Section 2 (which is a potential solution to the equilibrium with $\sigma = 0$) and the fundamental equilibrium solution.

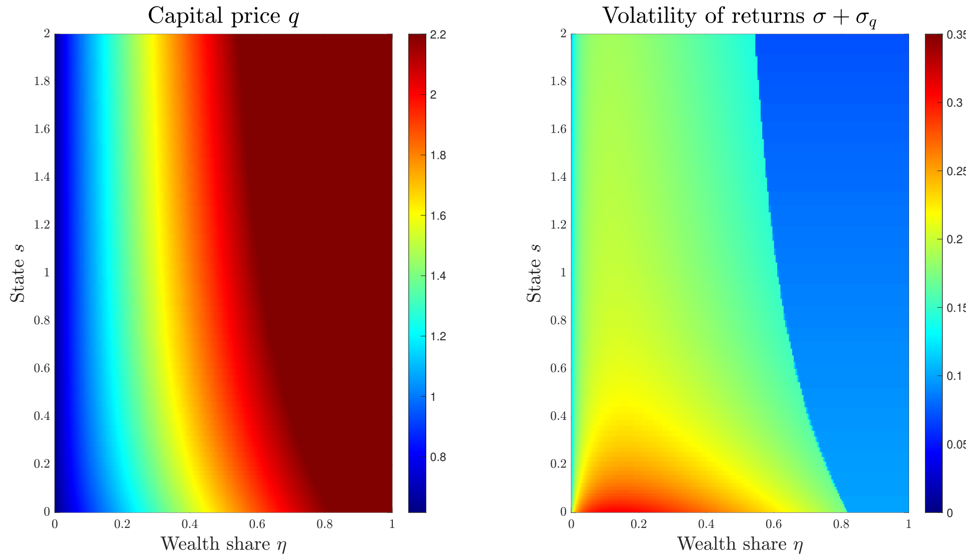

Figure D.6: Equilibrium with $\omega = -0.25$, and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

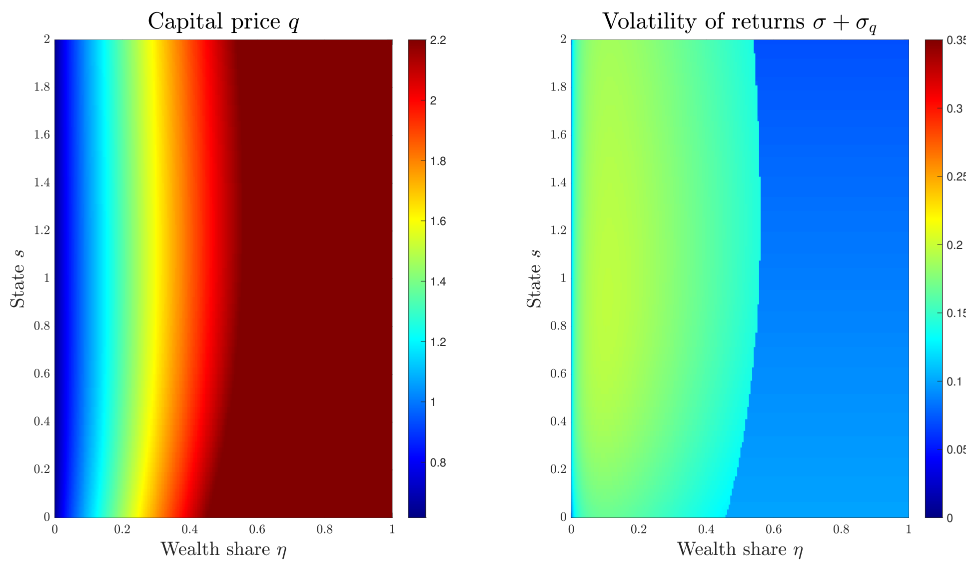

Figure D.7: Equilibrium with $\omega = -0.25$, and the "high" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and a BSE. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

amount of variation in the equilibrium as s varies, illustrating convergence to a sunspot equilibrium. Thus, as promised, we are able to construct sunspot equilibria even if the dynamics (σ_s, μ_s) are specified exogenously. In fact, it appears that the amount of price volatility is relatively insensitive to the real effects s has (i.e., the size of ω), which is reminiscent of the "volatility paradox" of Brunnermeier and Sannikov (2014) but one level deeper. Their paradox is that total volatility is only modestly sensitive to exogenous fundamental volatility; our paradox is that total volatility is only modestly sensitive to the *exogenous impact of s on fundamental volatility*.

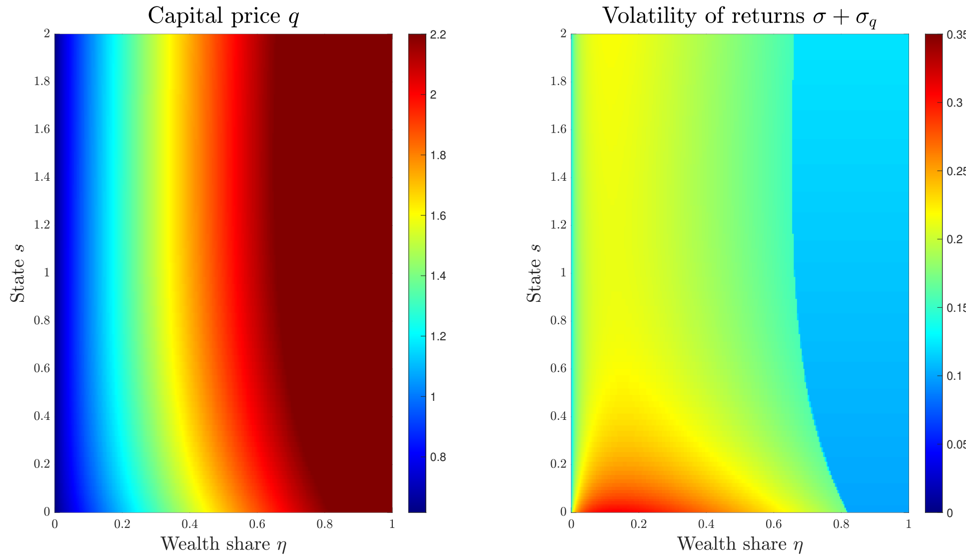

Figure D.8: Equilibrium with reversed fundamental impact of s, specifically with $\omega = 0.25$, and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

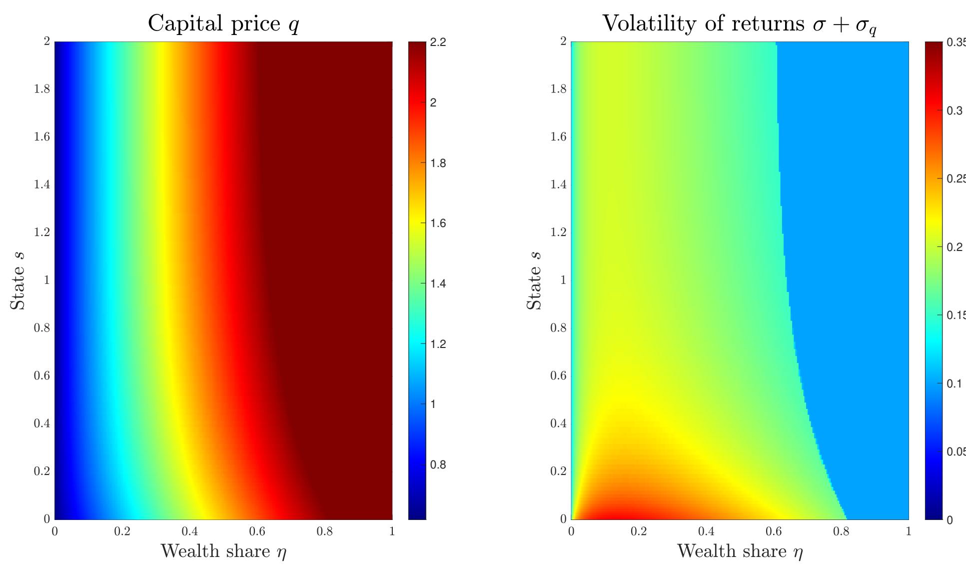

Figure D.9: Equilibrium with near-sunspot $\omega = -10^{-6}$ and the "low" boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s.

D.7 Complete markets and nominal rigidities

Proof of Theorem 3. Consider an auxiliary variable $x_t \in (0, a/\rho + b)$ for some $b > 0$. Write the evolution of x as $dx_t = x_t[\mu_{x,t}dt + \sigma_{x,t}dZ_t]$. We are letting x_t be the state variable in this equilibrium.

Set $q_t = \min[x_t, a/\rho]$, and put $\sigma_{x,t}^2 = h(x_t)$ and $\mu_{x,t} = -\rho - g + h(x_t)$ when $x_t < a/\rho$. Nothing pins down (σ_x, μ_x) when $x_t > a/\rho$, we may simply set them so that x_t never reaches the boundary $a/\rho + b$. Many such choices exist.

Next, based on $h(0) > \rho + g$, one may use the Feller explosion analysis of Lemma A.1 (with $\alpha = 1$ and $\beta = 2$ in the notation of that lemma) to show that x_t never attains the boundary $x = 0$. Alternatively, simply note that x_t behaves like a geometric brownian motion near $x = 0$, with positive drift $\mu_x(0) > 0$ if and only if $h(0) > \rho + g$. Thus, we have proved such an equilibrium exists, but it remains to show any $\pi \in [0, 1]$ is possible.

If $h(a/\rho) = 0$, then the analysis of Lemma A.1 can again be used to show that x_t never attains the point $x = a/\rho$ if started below it. Thus, $\pi = 1$ if $h(a/\rho) = 0$. One similarly shows that $x = a/\rho$ is attainable if $h(a/\rho) > 0$, since then $\sigma_q(x) > 0$ for all $x \in (0, a/\rho]$, whereas $\mu_q(x)$ is bounded. Thus, $\pi < 1$ if $h(a/\rho) > 0$.

To show that $h(a/\rho) > 0$ does not imply $\pi = 0$, one may simply put $\sigma_x(x) = 0$ and $\mu_x(x) < 0$ on $\{x \geq a/\rho\}$. Then, x_t exits the efficient region $\{x \geq a/\rho\}$ in finite time, so efficiency cannot be full probability in the stationary distribution.

Finally, to show that one can construct an equilibrium where inefficiency is completely transitory ($\pi = 0$), consider putting $h(a/\rho) > 0$, so that x_t eventually exceeds a/ρ with probability 1, and putting $\sigma_x(x) = 0$ and $\mu_x(x) > 0$ on $\{x \geq a/\rho\}$ so that x_t never leaves this region. □