

# Arbitrage and Beliefs

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## Abstract

We study a segmented-markets setting in which self-fulfilling volatility can arise. The only requirements for this volatility to come about are (i) asset price movements redistribute wealth across markets (e.g., equities rise as bonds fall) and (ii) a stabilizing force keeps valuation ratios stationary (e.g., cash flow growth rises when valuations rise). We prove that when self-fulfilling volatility exists, arbitrage opportunities must also exist, and vice versa. The tight theoretical connection between price volatility and arbitrage is detectable in currency markets by studying deviations from covered interest parity.

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# 1 Introduction

Arbitrages exist. Empirical research has documented several examples of trades featuring positive profits with zero hold-to-maturity risk. Moving beyond the neoclassical frictionless model of financial markets, a theoretical literature emphasizing *limits to arbitrage* aims to rationalize such trades.<sup>1</sup> More broadly, limits to arbitrage help reconcile high asset returns and extreme price volatility under tame levels of fundamental risk, helping bridge an important gap in financial economics.

In this paper, we argue that these models have not gone far enough. More specifically, we prove that a mild twist on a canonical limits-to-arbitrage setting can feature self-fulfilling volatility which has nothing to do with the risk properties of underlying cash flows. In this sense, we can justify return volatilities even higher than the literature has predicted thus far.

While limits to arbitrage are certainly present to some degree, our reliance on equilibrium multiplicity to generate volatility immediately raises the question of testability: is this excess volatility really present in the data? We address this question in the second part of the paper. We establish a one-to-one theoretical mapping between the amount of self-fulfilling volatility and the amount of arbitrage profits. This mapping is a robust prediction, and we find its presence in currency market data.

To elucidate our theoretical results, we start with a stylized setting with market segmentation (one can also think of “investor habitats”). For example, imagine that a group of investors (call them *A*-types) only trades in market *A*, whereas a different group of investors (call them *B*-types) only trades in market *B*. Without some arbitrageur trading in both markets simultaneously (or a lack of active trading across markets), what determines asset prices? Usually, we appeal to fundamental valuation: each of these local investors (*A* or *B*) knows their risk preferences, so each can look at underlying cash flows, discount them to the present with a risk adjustment, and market prices should adjust to equal this quantity.

Digging into the details helps here. Fundamental valuation normally works because any other price is associated with a violation of a long-run transversality condition. An asset whose price is above fundamental value features a low dividend yield, and valuations must continuously rise without bound to satisfy investors; the opposite occurs if asset prices are below fundamental value. This type of instability is mechanically why transversality fails and why fundamental value prevails.

But fundamental valuation is complicated by various factors in practice, including the

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<sup>1</sup>See related literature section for empirical examples and theoretical antecedents.

endogeneity of cash flows, the appropriate probability distribution to use in forecasting, and the right discount rate to apply. We show that when cash flows (or beliefs about cash flows) are endogenous to valuations in a particular way – namely, they are expected to grow faster when valuations are higher – asset prices may not be uniquely determined.

Instead, the endogeneity of perceived growth sets the stage for self-fulfilling expectations of future price changes. To see why, consider an asset that has a high price-dividend ratio today, meaning its cash flows are assumed to also grow faster. Investors will tolerate future valuation declines because the high cash flow growth is enough to satisfy their required returns. Working in reverse, if investors expect future valuation declines, high prices can be justified today. Unlike the instability discussed above, these examples involve a *stabilizing force* that brings high valuations back to normal levels (and vice versa), ensuring that transversality holds.

In the paper, we also investigate other examples of stabilizing forces beyond the aforementioned valuation-growth endogeneity, to illustrate the robustness of this logic. Layering any such stabilizing force, even if tiny, onto a limits-to-arbitrage model opens the door for self-fulfilling volatility. The basic intuition is that non-fundamental price fluctuations, as long as they disappear “far from steady state,” will not push prices outside of a stable region. By expressing several different examples of stabilizing forces, our aim is to convince the reader that stability is plausible.

To permit self-fulfilling volatility, one additional requirement on beliefs is needed: price movements are *redistributive* across markets. Returning to our example with markets  $A$  and  $B$ , suppose the price of asset  $A$  declines for extrinsic (i.e., non-fundamental) reasons. In reality, this decline could be initiated by some liquidity-based selling that puts downward pressure on prices. Having less wealth after the shock,  $A$ -type investors will want to cut consumption. To do this,  $A$ -types save a bit of the cash flows from asset  $A$  in the bond market. By market clearing,  $B$ -types must borrow this amount, consuming more than the cash flows of asset  $B$ . This consumption plan is only optimal if  $B$ -types believe their wealth has increased, requiring assets  $A$  and  $B$  to experience *equal and opposite* extrinsic shocks. In this sense, market clearing dictates that the beliefs of  $A$ - and  $B$ -types must be negatively correlated. This example also suggests that bond market integration is required for self-fulfilling volatility to arise. This sequence of logic is reflected in Figure 1 below.

We emphasize it is impossible to engender self-fulfilling volatility in a one-asset one-location version of our economy, even if growth is endogenous to valuations. This distinguishes our mechanism from several other studies that build multiplicity through collateral constraints or other financing frictions.

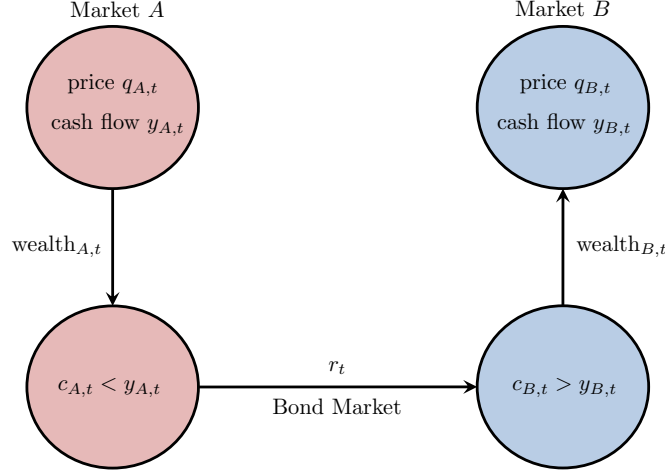


Figure 1: Mechanism of the model. Market  $A$  experiences a negative shock to its asset price  $q_{A,t}$ , without any effect on its cash flow  $y_{A,t}$ . Optimal consumption  $c_{A,t}$  of agent  $A$  wants to fall below the cash flow to reflect lower wealth, which requires saving in the bond market at rate  $r_t$ . By bond market clearing, consumption  $c_{B,t}$  of agent  $B$  must be higher than her local asset cash flow  $y_{B,t}$ . This behavior is only optimal if market  $B$  asset prices  $q_{B,t}$  rise in a manner that offsets the decline in  $q_{A,t}$ .

The general conditions for self-fulfilling volatility are the essence of Theorem 1 of the paper (Propositions 1-3 consider specific examples). These multiplicity results hold with infinitely-lived agents in a bubble-free economy with dynamically-complete, but imperfectly-integrated, financial markets. It is not required that investors be irrational in forecasting growth, but as we show in an extension, such beliefs expand the set of circumstances that can accommodate multiplicity. Besides the fact that markets are not perfectly integrated, there are no other frictions or constraints – agents are always marginal in their local asset markets.

A robust implication that we develop in more detail is the tight connection between volatility and arbitrage (Theorem 2). Going back to Figure 1, if shocks to assets  $A$  and  $B$  are offsetting, one can construct a riskless portfolio containing both. This portfolio must generate *arbitrage profits*. The reason:  $A$ -types demand a risk premium on the extrinsic volatility of asset  $A$ , and similarly for  $B$ , so a portfolio that buys positive amounts of  $A$  and  $B$  will earn more than the riskless rate. In fact, we show that the size of arbitrage profits is directly linked to the amount of self-fulfilling volatility (Proposition 4). This theoretical result is what we ultimately test empirically.

Normally, the presence of large arbitrage profits incentivizes trading by relative-value traders. Enter investment fund  $F$ . If fund  $F$  can trade freely across markets, we return to a neoclassical world without arbitrage and without self-fulfilling volatility. But if fund  $F$  encounters the type of frictions articulated by the limits-to-arbitrage literature

(e.g., margin requirements, search frictions, myopic clients), some arbitrage profits will remain. By extension, self-fulfilling volatility will not be fully eliminated. Quantitatively, the magnitude of limits-to-arbitrage frictions disciplines the magnitude of viable self-fulfilling volatility (Proposition 5 and Corollary 6).

Empirically, we examine the hypothesized link between arbitrage profits and volatility in the context of covered interest parity (CIP) deviations. The 3-month CIP deviation serves as our primary measure of available arbitrage profits. The arbitrage strategy borrows in US dollars (USD) and goes long a synthetic US bond built with a foreign bond and currency swaps (or vice versa if the CIP deviation is negative).

We employ several different proxies for volatility. In one exercise, we compute the weighted-average return volatility on the two legs of the CIP (the US bond and the synthetic US bond). We also examine the longer-maturity counterparts of these legs (10-year bonds as opposed to 3-month bonds), as a way to make a duration-adjustment that brings our volatility magnitude closer to what the model calls for (i.e., the assets in the model are infinitely-lived). Consistent with the model’s prediction, we find very strong positive associations between CIP deviations and each of our volatility proxies: (i) the weighted-average 3-month return volatilities for the CIP components; (ii) the 10-year US bond return volatility; (iii) the 10-year Treasury VIX; and (iv) the 10-year foreign bond return volatility. We cannot determine if volatility is of a self-fulfilling nature or not, which makes our test imperfect, but the strength of the association is striking. We close the paper with a discussion of existing models and their difficulties in matching this new empirical fact.

**Contribution to the literature.** Our paper contributes to two literatures: (1) limits to arbitrage in financial markets; and (2) theories of self-fulfilling dynamics.

The starting point for the limits-to-arbitrage literature is the plethora of observed arbitrage trades<sup>2</sup> and evidence of financial market segmentation<sup>3</sup>. Many papers on the theoretical side of this literature have focused on developing implications of specific micro-foundations – such as margin requirements (Gromb and Vayanos, 2002), myopic performance-based clients (Shleifer and Vishny, 1997), search frictions (Vayanos and

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<sup>2</sup>Examples of such trades include spinoffs (Lamont and Thaler, 2003); “on-the-run / off-the-run” bonds (Krishnamurthy, 2002); convertible bonds (Mitchell, Pedersen and Pulvino, 2007); covered interest parity (Du, Tepper and Verdelhan, 2018; Du, Hébert and Huber, 2019); and Treasury spot and future-implied repo rates (Fleckenstein and Longstaff, 2018).

<sup>3</sup>Identifying segmentation *per se* is difficult, but a few studies have provided evidence in its favor by using cross-sectional data and the pricing of idiosyncratic shocks. This approach has been applied to markets trading catastrophe insurance (Froot and O’Connell, 1999), mortgage-backed securities (Gabaix, Krishnamurthy and Vigneron, 2007), convertible bonds (Mitchell, Pedersen and Pulvino, 2007), and Treasury markets (Hu, Pan and Wang, 2013).

Weill, 2008; Duffie and Strulovici, 2012), or incentive constraints and preference heterogeneity (Biais, Hombert and Weill, 2017). These studies also typically pay detailed attention to the behavior of arbitrageurs.

By contrast, we focus on fundamental traders, relegating the behavior of arbitrageurs to the background. As a benefit, we can analytically develop all pricing implications – including the new insight of multiplicity – more fully. Given the fact that our fundamental traders are fully rational and forward-looking, the excess volatility we showcase is conceptually distinct from that arising from “noise traders”.<sup>4</sup> We also demonstrate the novel connection between the size of arbitrage profits in a financial market and the amount of self-fulfilling volatility in prices. We then empirically verify this relation between volatility and arbitrage profits in currency markets.

International financial markets are a context in which cross-market segmentation is particularly important. As our model studies  $N$  “locations” with their own fundamentals, one can naturally take a global perspective. A recent international finance literature assumes cross-sectional segmentation of local equity or sovereign debt markets (Lustig and Verdelhan, 2019), possibly with some global intermediary participating in all of them (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2017). Our model points out the possibility of self-fulfilling volatility in such a setting.

Our construction of sunspot equilibria shares a similar flavor to seminal studies that build sunspot shocks around a stable steady state. Indeed, the “stabilizing forces” we identify render our deterministic steady state locally stable. We differ from this literature in some of the assumptions we adopt – we require neither overlapping generations<sup>5</sup> nor aggregate increasing returns<sup>6</sup> to induce stability. Our construction of stochastic equilibria is also more general in permitting an arbitrary numbers of markets, several new examples of stabilizing forces, and a broad class of sunspot shocks.

Focusing on asset prices, our paper is close to Gârleanu and Panageas (2020) and Zentefis (2020). As in those models, our multiplicity arises when there are multiple traded assets and some segmentation between them.

In the OLG economy of Gârleanu and Panageas (2020), segmentation arises between physical and human capital because unborn investors have a disproportionate claim to human capital but cannot trade before birth. Multiplicity can arise when extrinsic physical capital shocks are offset by human capital shocks. They interpret this relation as stock market volatility, whereas our equilibrium is better interpreted as self-fulfilling

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<sup>4</sup>See the seminal studies of De Long et al. (1990a,b, 1991). More recently, Vayanos and Vila (2020) investigate a habitat yield-curve model with noise trader demand causing non-fundamental volatility.

<sup>5</sup>See Azariadis (1981), Cass and Shell (1983), and Farmer and Woodford (1997).

<sup>6</sup>See Farmer and Benhabib (1994).

volatility in relative-value trades, or across countries. In fact, their economy features no arbitrage at all times, whereas one of our main contributions is tightly connecting arbitrage profits and volatility.<sup>7</sup>

In [Zentefis \(2020\)](#), leverage constraints can generate self-fulfilling price dynamics. This connection relates to our example with endogenous growth rates, which can be interpreted as the outcome of collateral constraints on investing firms. But by showing that several mechanisms can provide the required “stabilizing force,” and by studying a general  $N$ -asset economy, our goal is to embark on a more general analysis.

The paper is organized as follows. Section 2 presents the model. Section 3 explores when and why the model can have self-fulfilling volatility. Section 4 establishes the central link between the presence of arbitrage opportunities and self-fulfilling volatility. Section 5 empirically examines this key model prediction in the context of currency markets. Section 6 concludes. The Appendix contains the proofs, model extensions, and further analysis.

## 2 Model

**Setup.** The model is set in continuous time with  $t \geq 0$ . The aggregate endowment follows

$$dY_t = g_t Y_t dt,$$

with arbitrary growth rate  $g_t$  that we specify in examples below. We begin with deterministic endowments for theoretical clarity, but we show in [Appendix B.1](#) that our results extend to a setting with aggregate shocks.

The economy features  $N$  locations, which can stand for sectors, industries, countries, or distinct financial markets. The endowment of location  $n$  is given by  $y_{n,t}$ , where

$$dy_{n,t} = y_{n,t} g_{n,t} dt.$$

Note that  $Y_t = \sum_{n=1}^N y_{n,t}$ , so for now, the only restriction on the local growth rate  $g_{n,t}$  is that  $\sum_{n=1}^N y_{n,t} g_{n,t} = g_t Y_t$ . We purposefully leave growth rates otherwise unspecified

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<sup>7</sup>Our construction of a stochastic equilibrium is also related to theirs, by essentially randomizing over a multiplicity of deterministic transition paths. Indeed, one of our examples of a “stabilizing force” generalizes a reduced-form version of their model. There are two other related papers. [Farmer \(2018\)](#) studies capital asset prices in an OLG economy, with nominal government debt (rather than human capital) as the second asset allowing redistribution. [Bacchetta et al. \(2012\)](#) also has an OLG economy, with the uncleared riskless bond market providing the second asset allowing redistribution (to unmodeled foreigners).



because the manner in which they are endogenous to asset prices plays a key role in the type of equilibrium.

Regarding financial markets, each location offers a single positive-net-supply asset that is a claim to  $y_{n,t}$ . The equilibrium price of asset  $n$  is  $q_{n,t}y_{n,t}$ , where  $q_{n,t}$  is the price-dividend ratio. Let us define the endowment share  $\alpha_{n,t} := y_{n,t}/Y_t$  to save notation.

Each location has a different representative agent. This agent can invest only in his or her local asset market and a zero-net-supply short-term bond market that is open to everyone. The equilibrium risk-free rate in the integrated bond market is  $r_t$ .

All agents have rational expectations, infinite lives, logarithmic utility, and discount rate  $\delta > 0$ . Mathematically, their preferences are

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-\delta t} \log(c_{n,t}) dt \right].$$

Clearing of the goods and bond markets is standard:  $\sum_{n=1}^N c_{n,t} = Y_t$  and  $\sum_{n=1}^N q_{n,t}y_{n,t} = Q_t Y_t$ , where  $Q_t$  is the aggregate price-dividend ratio. Because of market incompleteness, the cross-sectional consumption distribution will be a state variable in equilibrium, so we denote  $x_{n,t} := c_{n,t}/Y_t$  the location- $n$  consumption share.

**Extrinsic Shocks.** With market clearing established, we next describe asset prices. In a deterministic economy, any stochastic price changes must inherently originate from agents' self-fulfilling beliefs. To allow for this volatility, we conjecture that the price-dividend ratio of each location's asset follows a stochastic process

$$dq_{n,t} = q_{n,t} \left[ \mu_{n,t}^q dt + \sigma_{n,t}^q d\tilde{Z}_{n,t} \right], \quad (1)$$

where  $\tilde{Z}_{n,t}$  is a one-dimensional Brownian motion. The economy has no intrinsic uncertainty. This shock is therefore *extrinsic*, and it is the source of self-fulfilling asset price volatility, if any exists. Let  $\tilde{Z}_t := (\tilde{Z}_{n,t})_{n=1}^N$  be a vector of all locations' extrinsic shocks.

Economically, the extrinsic  $\tilde{Z}$  shocks arise from sources that we do not explicitly model. Investor sentiment or signals that coordinate beliefs might trigger the self-fulfilling fluctuations, in a manner similar to [Benhabib et al. \(2015\)](#). Heterogeneity in opinions between optimists and pessimists akin to [Scheinkman and Xiong \(2003\)](#) can be another source. Correlated institutional demand shocks as described in [Kojien and Yogo \(2019\)](#) can yet be another driver of the price changes.

We allow the extrinsic shocks in the economy to obey an arbitrary correlation structure. A convenient way to represent this structure uses an  $N$ -dimensional basis of uncorrelated Brownian motions  $Z_t := (Z_{n,t})_{n=1}^N$  and an  $N \times N$  matrix of constants  $M$  that



captures their relation. From these two components, we recast the vector of extrinsic shocks as

$$\tilde{Z}_t = MZ_t. \quad (2)$$

The matrix  $M$  is normalized so that  $\text{diag}[MM'] = (1, \dots, 1)'$ , which preserves  $\tilde{Z}_t$  as a collection of Brownian motions. Substituting equation (2) into equation (1) shows that the self-fulfilling shock to asset  $n$  at time  $t$  is  $\sigma_{n,t}^q M_n dZ_t$ , where  $M_n$  is the  $n$ -th row of  $M$ .<sup>8</sup>

The matrix  $M$  is a crucial parameter of the model. To illustrate its structure, we consider the following examples, which we repeatedly use throughout the text.

**Example 1** (Uncorrelated shocks). Suppose  $M$  is the identity matrix. This structure implies  $\tilde{Z}_t = Z_t$ , which renders all extrinsic shocks uncorrelated.

**Example 2** (Two-by-two redistribution). Suppose  $N = 2$  and let

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

This example presents a setting with two locations and one source of extrinsic uncertainty. The matrix  $M$  puts  $\tilde{Z}_{1,t} = -\tilde{Z}_{2,t}$ , which implies that the self-fulfilling price changes redistribute wealth between the two assets. As one price falls, the other rises.

**Example 3** (General redistribution). This example is the  $N$ -dimensional counterpart to Example 2. Let  $\tilde{M}$  be an  $N \times N$  non-singular matrix. Suppose

$$M = \tilde{M} - \frac{1}{N} \mathbf{1}' \tilde{M} \otimes \mathbf{1}.$$

In this structure, each element of the matrix  $\tilde{M}$  is reduced by the simple average of its columns. This operation makes the column sums of  $M$  equal zero. The key consequence of this design is that  $\mathbf{1}' d\tilde{Z}_t = \mathbf{1}' M dZ_t = 0$  almost-surely. Any other linear combination of  $d\tilde{Z}_t$  does not equal 0. As a result,  $\text{rank}(M) = N - 1$ .

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<sup>8</sup>Although markets are incomplete in the model, they are dynamically complete. The vector  $\tilde{Z}_{n,t} = M_n Z_t$  is generated by  $N$  distinct shocks, but it suffices for agent  $n$  to only trade  $\tilde{Z}_{n,t}$ , which is the shock his local asset loads on. Indeed, if we introduce in each market zero-net-supply Arrow securities spanning  $Z$  that are traded only in market  $n$ , the equilibrium remains unchanged.

### 3 Self-fulfilling volatility

At first glance, readers may be divided about whether non-fundamental volatility is possible in such a model. On one hand, no “arbitrageur” exists to connect market dynamics across locations, so what disciplines local market prices? On the other hand, the presence of unconstrained fundamental investors having the same preferences and deterministic endowments suggests prices should be assigned a common fundamental value. Here, we shed light on this issue, clarifying when non-fundamental price dynamics exist and when they do not.

To develop some understanding of necessary conditions for self-fulfilling volatility, consider the market clearing conditions. Investors with log utility consume  $\delta$  fraction of their wealth, so the aggregate wealth-consumption (price-dividend) ratio is  $Q_t = \delta^{-1}$ . Bond market clearing can then be written as

$$\sum_{n=1}^N \alpha_{n,t} q_{n,t} = \delta^{-1}. \quad (3)$$

Because the aggregate wealth-consumption ratio is constant, if there are any extrinsic shocks to  $q_{n,t}$ , they must be offset by extrinsic shocks to other assets. In this sense, extrinsic shocks (if they affect prices) must be *redistributive* across markets.

**Remark 1** ( $N = 1$  economy). A representative-agent economy ( $N = 1$ ) can never experience self-fulfilling volatility in our model. This holds regardless of the properties of aggregate growth rate  $g_t$ . Multiple segmented locations are critical.

More broadly, redistribution is tied directly to  $\text{rank}(M)$ . By time-differentiating equation (3), we see that the loadings on each of the basis  $dZ_t$  shocks must be zero:

$$\sum_{n=1}^N \alpha_{n,t} q_{n,t} \sigma_{n,t}^q M_n = 0. \quad (4)$$

Write equation (4) as a matrix equation:

$$M' v_t = 0, \quad (5)$$

where  $v_t = (\alpha_{1,t} q_{1,t} \sigma_{1,t}^q, \dots, \alpha_{N,t} q_{N,t} \sigma_{N,t}^q)'$  is the column vector of volatilities. If  $M$  were full rank, the unique solution to (5) would be  $v_t \equiv 0$ . However, the singular situation  $\text{rank}(M) < N$  implies that a non-zero time-invariant solution  $v_t \equiv v^* \neq 0$  exists. If so,

then  $\psi_t v^*$  also solves (5) for any scalar process  $\psi_t$ . Hence, as long as  $M$  is singular – as in Examples 2-3 but not Example 1 – we have a continuum of candidate equilibria.

When are these “candidate equilibria” actual equilibria? It turns out that the only additional requirements are that asset prices be bounded and all consumers survive in the long run, which together ensure that transversality and no-Ponzi conditions hold. Below, we will show how these requirements translate into constraints on model primitives. For now, we summarize this discussion with the following theorem, which collects general necessary conditions that tell us when asset prices can have self-fulfilling volatility. The proof of this result (and all others) is in Appendix A.

**Theorem 1.** *If  $\text{rank}(M) = N$  (full rank), then equilibrium cannot have self-fulfilling volatility:  $(\sigma_{1,t}^q, \dots, \sigma_{N,t}^q) \equiv 0$  for all  $t$ . Conversely, suppose  $\text{rank}(M) < N$  and let  $v^*$  be in the null-space of  $M'$ . Given a process  $\{\psi_t\}_{t \geq 0}$ , an equilibrium can be sustained with volatility  $\alpha_{n,t} q_{n,t} \sigma_{n,t}^q = v_n^* \psi_t$  for all  $n$ , as long as the resulting  $\{(q_{n,t})_{n=1}^N\}_{t \geq 0}$  are bounded, positive processes, and as long as  $\lim_{T \rightarrow \infty} e^{-\delta T} x_{n,T}^{-1} = 0$ , for each  $n$ .*

The remainder of this section sheds light on which model mechanisms preserve the asset price stationarity required by Theorem 1 and which do not.

**Determinacy and instability.** The following is a benchmark case in which equilibrium is unique and non-stochastic.

**Proposition 1.** *Assume constant local growth rates  $g_{n,t} = g$ . Then, equilibrium cannot have self-fulfilling volatility:  $(\sigma_{1,t}^q, \dots, \sigma_{N,t}^q) \equiv 0$  for all  $t$ . All assets have identical constant price-dividend ratios  $q_{n,t} = \delta^{-1}$ .*

Even though no arbitrageur connects locations, Proposition 1 shows that the presence of fundamental traders in each location is enough to pin down asset prices. This equilibrium with  $q_{n,t} = \delta^{-1}$  always exists, but here, it is also unique.

The reason for this strong determinacy is the instability of price-dividend dynamics. To see this, consider the deterministic model with  $(\sigma_{1,t}^q, \dots, \sigma_{N,t}^q) \equiv 0$ . In this case, there is no risk compensation, and all assets must earn the riskless rate. Specifically,

$$\underbrace{\dot{q}_{n,t}/q_{n,t} + g}_{\text{capital gain}} + \underbrace{1/q_{n,t}}_{\substack{\text{dividend} \\ \text{price}}} = r_t. \quad (6)$$

Furthermore, as the equilibrium involves deterministic individual consumption paths, the interest rate is solely determined by time-discounting and economic growth; i.e.,  $r_t = \delta + g$ . Substituting this expression into (6), we have  $\dot{q}_{n,t} = -1 + \delta q_{n,t}$ , which

represents a dynamical system that has a single steady state that is unstable. If the price-dividend ratio is below (above)  $\delta^{-1}$ , then it drifts downwards (upwards) at a pace that accelerates over time, so the required boundedness of Theorem 1 is violated. Adding stochastic shocks does nothing to remedy the core non-stationarity of the system.

**Multiplicity and stability.** For any self-fulfilling volatility to exist, there must be a *stabilizing force* that keeps asset prices stationary. Next, we provide a first example of such a force. Below, we provide several other examples. The multitude of examples is meant to reassure the reader that stability is a natural property in economic settings.

In our first example, we assume local growth is endogenous and increases (sufficiently quickly) with local asset prices. In particular, assume local growth rates satisfy

$$g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}), \quad \text{with } \lambda > \delta^2. \quad (\text{G1})$$

The exact mathematical condition is relatively modest: for a standard discount rate  $\delta = 0.01$ , local growth rates must be at least 0.1% higher than average when local valuations are 10% above average. A valuation-growth connection can arise for many reasons; for example, prices may carry information that modulate real decisions (see [Chen et al., 2007](#); [Bakke and Whited, 2010](#); [Goldstein and Yang, 2017](#) and the review in [Bond et al., 2012](#)).

**Proposition 2.** *Assume local growth rates satisfy (G1). Then, self-fulfilling volatility is possible: there exists a non-zero process  $\{\psi_t\}_{t \geq 0}$  such that an equilibrium exists with  $\alpha_{n,t} q_{n,t} \sigma_{n,t}^q = v_n^* \psi_t$  for all  $n$ , where  $v^*$  is in the null-space of  $M'$ . In terms of the cross-sectional minimums  $\underline{\alpha}_t := \min_n \alpha_{n,t}$ ,  $\underline{x}_t := \min_n x_{n,t}$ , and  $\underline{q}_t := \min_n q_{n,t}$ , the volatility  $\psi_t$  can be any bounded process that satisfies*

(P1)  $\psi_t / \underline{\alpha}_t$  and  $\psi_t / \underline{x}_t$  are bounded;

(P2)  $\psi_t$  vanishes as  $\underline{q}_t$  approaches  $\delta(\epsilon + \lambda^{-1})$  from above, for some  $0 < \epsilon < \delta^{-2} - \lambda^{-1}$ .

The dependence of growth on valuations allows for self-fulfilling expectations of future price changes. For example, if an asset has a high price-dividend ratio today, its cash flows also grow faster, which means investors will tolerate future valuation declines. Working in reverse, investor beliefs about future valuation declines can justify high valuations today. A symmetric intuition applies to low-priced assets that must be expected to appreciate.

Mathematically, the endogeneity of growth rates acts as a *stabilizing force*. To see this clearly, consider again the deterministic model but with these endogenous growth rates.

After substituting  $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$  in place of  $g$  and using the fact that  $r_t = \delta + g$  still, pricing equation (6) is replaced by the Riccati equation

$$\dot{q}_{n,t} = -1 + \delta(1 + \lambda/\delta^2)q_{n,t} - \lambda q_{n,t}^2. \quad (7)$$

Supposing  $\lambda > \delta^2$ , the larger of the two steady-states is the relevant one (with  $q_n = \delta^{-1}$ ). It is locally stable, in the sense that  $\frac{\partial \dot{q}_n}{\partial q_n}|_{q_n=\delta^{-1}} = \delta(1 - \lambda/\delta^2) < 0$ .

In general, when an economy has such a stabilizing force, some amount of self-fulfilling volatility  $\psi_t$  becomes possible. The amount of volatility is only restricted by a requirement that it vanishes when the economy is “far from steady state” so that the stabilizing force enters unabated. This vanishing property is the crux of properties (P1) and (P2). The result provides a strong sense in which the absence of cross-market arbitrage can allow prices of similar assets to move distinctly. Later, by introducing some partial cross-market trading, we will place more restrictions on the volatility  $\psi_t$ .

Our second example of a stabilizing force involves a type of “extrapolative beliefs”, a moniker we justify shortly after stating the results. Suppose local investors become sufficiently optimistic about growth when valuations are high (and vice versa). Specifically, true growth rates are identical ( $g_{n,t} = g$ ) but subjective growth beliefs  $\tilde{g}_{n,t}$  are given by<sup>9</sup>

$$\tilde{g}_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}), \quad \text{with } \lambda > \delta^2. \quad (\text{G2})$$

Clearly, property (G2) mirrors (G1) closely. In fact, the only modification is to replace  $g_{n,t}$  by  $\tilde{g}_{n,t}$  in the asset-pricing equation. Consequently, equation (7) still holds in the deterministic equilibrium, which yields the following.

**Proposition 3.** *Assume subjective beliefs about growth satisfy (G2). Then, self-fulfilling volatility is possible, and in fact, the conclusions of Proposition 2 go through without modification.*

The combination of Propositions 2-3 implies that stabilizing forces provided by endogenous growth rates can be real or merely perceived, or some fraction of each.

To see how formulation (G2) relates to extrapolative beliefs, write perceived expected asset returns  $\tilde{\mu}_{n,t}^R$  in terms of the objective expected returns  $\mu_{n,t}^R := \mu_{n,t}^q + g + 1/q_{n,t}$ :

$$\tilde{\mu}_{n,t}^R = \mu_{n,t}^R + (\tilde{g}_{n,t} - g).$$

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<sup>9</sup>In this deterministic setting, such divergence constitutes a belief that is mutually singular to the true “probability distribution.” We ignore this issue here, but note that we could simply add some additional local endowment risk to eliminate this singularity.

In a stationary equilibrium, the perceived growth differential  $\tilde{g}_{n,t} - g = \lambda(q_{n,t} - \delta^{-1})$  will be a stationary, mean-reverting process that responds positively to positive-return shocks  $d\tilde{Z}_{n,t} > 0$ . This is very similar to [Barberis et al. \(2015\)](#), in which extrapolators' beliefs are the sum of true expected returns and a stationary, mean-zero process that responds positively to return shocks.

To foster a sense of robustness, Appendix B provides two additional settings that produce stabilizing forces, and hence, multiplicity. First, Appendix B.2 shows that under-investment, of the type induced by “debt overhang” ([Hennessy, 2004](#); [DeMarzo et al., 2012](#)), creates the needed stability. The main idea is that potential gains from investment are high relative to actual investment, leaving some surplus on the table. As valuations rise and boost investment, debt overhang problems shrink, and some of this surplus is captured by local investors. The extra returns gained this way compensate for lower dividend yields and ensure stable valuations.

Second, Appendix B.3 analyzes an overlapping generations economy with “creative destruction” along the lines of [Gârleanu and Panageas \(2020\)](#). As long as new producers are sufficiently less disruptive to incumbent producers when asset prices are high, the model has the right stability properties. The main idea in this extension is that alive agents are simultaneously marginal asset pricers and incumbent producers; new firms accrue to newborn cohorts. High valuations reduce dividend yields to investors, but alive cohorts are compensated by a simultaneous preservation of their firms, which removes the need for valuations to continue growing.

Mechanically, the examples we study share a common property: with high valuations, and hence, low dividend yields, investors are compensated somehow. This compensation can take the form of higher growth rates, greater optimism, reduction in under-investment wedges, or reduced creative destruction. It is likely that many other examples of stabilizing forces also exist. By studying several, our point is to emphasize how a wide range of plausible environments all generate a similar type of stability that can undergird self-fulfilling volatility in a segmented markets world.

**The role of the bond market.** Any self-fulfilling equilibria of our model, even going beyond the specific settings of Propositions 2-3, crucially require the bond market. Without the bond market, agent  $n$  only consumes the cash flows from his or her local endowment,  $y_{n,t}$ . As we will show, since this consumption is deterministic, no asset-price volatility can be justified! Even if the local cash flows possess arbitrary fundamental shocks independent of the extrinsic uncertainty, a similar result holds because investors' consumption would be uncorrelated with extrinsic shocks. By contrast, if the bond mar-

ket is open, agent  $n$  can send and receive consumption across locations, with the promise of inter-temporal payback. This opens the door for stochastic individual consumption profiles (i.e.,  $dc_{n,t}$  loads on  $d\tilde{Z}_{n,t}$ ), which then creates a stochastic local pricing kernel (marginal utility loads on  $d\tilde{Z}_{n,t}$ ), and justifies price volatility ( $dq_{n,t}$  loads on  $d\tilde{Z}_{n,t}$ ).

To see the link between price volatility and the pricing kernel, note that any self-fulfilling volatility must be compensated. Agent  $n$  holds exposure to the extrinsic shock  $\tilde{Z}_{n,t}$  through her exposure to  $q_{n,t}$ . Define  $\tilde{\pi}_{n,t}$  as the risk price (Sharpe ratio) associated to this shock and recall the endowment and consumption shares  $\alpha_{n,t} := y_{n,t}/Y_t$  and  $x_{n,t} := c_{n,t}/Y_t$ . Then,

$$\tilde{\pi}_{n,t} = \delta \left( \frac{\alpha_{n,t} q_{n,t}}{x_{n,t}} \right) \sigma_{n,t}^q. \quad (8)$$

Intuitively,  $\alpha_{n,t} Y_t q_{n,t} \sigma_{n,t}^q$  is agent  $n$ 's total exposure to the extrinsic  $\tilde{Z}_{n,t}$  shocks, and  $\delta^{-1} x_{n,t} Y_t$  is the agent's wealth. With log utility, the required compensation for Brownian shocks is the per-unit-of-wealth exposure, so dividing these two quantities gives the risk price  $\tilde{\pi}_{n,t}$ . Formula (8) shows how Sharpe ratios are linked to self-fulfilling volatility:  $\sigma_{n,t}^q > 0$  if and only if  $\tilde{\pi}_{n,t} > 0$  as well.

**Remark 2** ( $N = 1$  economy, redux). Recall Remark 1: self-fulfilling volatility is impossible in a one-location economy. The discussion above provides another intuition. Even if *aggregate* growth rates are endogenous to valuations as in (G1) – e.g., aggregate growth is  $g_t = G(Q_t)$  for some increasing function  $G$  – aggregate consumption growth is still deterministic over a small time-intervals  $dt$ . So, the representative agent demands no risk premium, and by the risk price formula (8), there is zero asset volatility. Our findings, which require multiple segmented markets, are thus distinct from types of multiplicity derived in one-sector economies that feature valuations linked to the real economy. For example, collateral constraints induce a valuation-output link and have often been associated with multiplicity even in a one-location world (e.g., Kiyotaki and Moore, 1997 or Miao and Wang, 2018). Our results should not be confused with these types of models.

**Remark 3** (Small open economy). Although the multiplicity result requires an open and active bond market, it does not require bond market clearing. Consider a “small open economy” in which the asset market for claims to the stream  $\{y_{n,t}\}_{t \geq 0}$  clears for each  $n$ , but the bond market does not. All results go through. Mathematically, since the equilibrium interest rate without extrinsic shocks is constant at  $r_t = \delta + g$ , it plays no role in providing stability. Thus, given any exogenous constant rate  $r$ , and endogenous local growth rates  $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$ , the counterpart to valuation dynamics of



equation (7) is

$$\dot{q}_{n,t} = -1 + (r - g + \lambda\delta^{-1})q_{n,t} - \lambda q_{n,t}^2,$$

which has a stable steady state (the larger of the two) if and only if  $\lambda > \delta^2(1 + \sqrt{\frac{\delta+g-r}{\delta}})$ .

## 4 Arbitrage profits

Having seen that self-fulfilling volatility is possible, we next connect it to arbitrage profits. Section 4.1 shows that the presence of this volatility and the existence of arbitrage opportunities are two sides of the same coin. Section 4.2 demonstrates that limits to arbitrage quantitatively discipline the amount of self-fulfilling price changes.

### 4.1 Volatility implies arbitrages and vice versa

Theorem 1 requires the condition  $\text{rank}(M) < N$  be satisfied for asset prices to have self-fulfilling volatility. To get a sense of what this rank condition means, consider what would happen if a single trader were allowed to participate in all markets. With  $\text{rank}(M) < N$ , there is some asset that this trader can replicate using the other  $N - 1$  assets. But with self-fulfilling volatility, the price of this asset and its replicating portfolio need not move together. In short, this trader would be faced with an *arbitrage opportunity*. In our model, self-fulfilling volatility emerges if and only if arbitrages exist, which provides a more intuitive diagnostic for multiplicity than the rank condition on  $M$ . This link goes beyond the specific “stabilizing forces” raised in Section 3 and applies to any conceivable example with self-fulfilling volatility.

**Theorem 2.** *Self-fulfilling volatility implies an arbitrage. Conversely, if there are arbitrages, then equilibrium must feature self-fulfilling volatility.*

First, consider the second statement of Theorem 2: arbitrages imply self-fulfilling volatility. If there were no volatility, then all assets earn the riskless rate, so there is no way to combine them into a portfolio that outperforms the riskless rate. This no-arbitrage, no-volatility equilibrium is the only one that can emerge when there is no stabilizing force in the economy.

Conversely, arbitrages exist when self-fulfilling volatility emerges. In the proof, we examine the portfolio that puts the amount  $\delta\alpha_{n,t}q_{n,t}$  in each asset  $n = 1, \dots, N$ . By equation (4), the portfolio volatility is identically zero. This is where  $\text{rank}(M) < N$  is critical: even if all assets have positive self-fulfilling volatility, an investor can manufacture a riskless asset from them. The proof shows mathematically why this portfolio pays more

than the riskless rate, but the basic intuition comes from the fact that each local market  $n$  demands a risk premium on its local asset. A portfolio built as a convex combination of components with risk premia must have a premium itself.

Confirming this intuition, this strategy's excess return over the riskless rate  $r_t$  is

$$A_t := \sum_{n=1}^N x_{n,t} \tilde{\pi}_{n,t}^2 > 0, \quad (9)$$

where recall  $\tilde{\pi}_n$  is the local Sharpe ratio. The amount  $A_t$  can be thought of as a measure of *arbitrage profit* in the model. Note that  $A_t$  is already quoted in the standard units used in analysis of arbitrage trades, as the long position  $(\delta \alpha_n q_{n,t})_{1 \leq n \leq N}$  is a return (unit-cost portfolio). In addition, the amount of arbitrage profit  $A_t$  is exactly the difference between the risk-free rate that prevails without self-fulfilling volatility ( $r_t = \delta + g$ ) and the one with self-fulfilling volatility ( $r_t = \delta + g - A_t$ ). One usually reads this term as a precautionary savings term, but based on this discussion,  $A_t$  can also be thought of as the difference between a synthetic bond return and the traded bond return.

The interpretation of the traded and synthetic bond depends on the context. Some common examples are collateralized versus uncollateralized lending (sometimes captured in the TED spread); on-the-run versus off-the-run Treasury bonds; and deviations from covered interest parity (CIP). Measures of this quantity tend to be minimal for much of the time, but can expand to around 3% during financial crisis periods (Fleckenstein and Longstaff, 2018; Du et al., 2018).

The degree of arbitrage profit informs the amount of self-fulfilling volatility. This is because volatility drives location-specific risk prices, which constitute  $A_t$ . The next proposition explains.

**Proposition 4.** *Let  $\text{rank}(M) < N$ . Then, there exists a non-zero vector  $v^*$ , in the null-space of  $M'$ , such that the self-fulfilling volatility  $\psi_t$  of Theorem 1 satisfies*

$$\psi_t = \frac{\delta^{-1} \sqrt{A_t}}{\sqrt{\sum_{n=1}^N x_{n,t} (\frac{v_n^*}{x_{n,t}})^2}} \leq \frac{\delta^{-1} \sqrt{A_t}}{\mathbf{1}' v^*}, \quad (10)$$

where  $A_t$  is the arbitrage profit given in (9). Consequently, the “average return volatility”  $\sigma_t^* := \sum_{n=1}^N \frac{\alpha_{n,t} q_{n,t}}{\sum_{i=1}^N \alpha_{i,t} q_{i,t}} \sigma_{n,t}^q$  satisfies

$$\sigma_t^* = \delta \psi_t \mathbf{1}' v^* \leq \sqrt{A_t}. \quad (11)$$

The average return volatility  $\sigma_t^*$  defined in Proposition 4 is a scale-free summary

statistic for the degree of volatility in the model. The tight link to arbitrage profit,  $\sigma_t^* \leq \sqrt{A_t}$ , is a bonus. To get a sense of magnitudes, consider arbitrage profit that range from  $A_t \in [0, 0.03]$ , consistent with the Treasury evidence of [Fleckenstein and Longstaff \(2018\)](#) and the CIP deviations documented in [Du et al. \(2018\)](#). Then, average return volatilities can range from  $\sigma_t^* \in [0, 17.3\%]$ , a quantitatively-large estimate.

At this point, it should be clear that arbitrages and self-fulfilling volatility are intrinsically linked. A different type of trade that resembles an arbitrage, a so-called *basis trade*, also exists in our model. A basis trade is a long-short strategy designed to capitalize on violations of the law of one price. The covered interest parity (CIP), which we study empirically in Section 5, is one such trade.

**Remark 4** (CIP deviations). Our model can accommodate CIP deviations that we will study empirically. In the reality, one might consider the following basis trade: borrow in Japanese yen, exchange to US dollars in the FX spot market, lend in dollars, and finally convert back into yen via a pre-signed currency futures contract. This trade is a theoretically riskless method to move Japanese yen from today to tomorrow, and it should return the yen risk-free rate. When it does not, CIP fails.

In the model, think of locations as countries, and consider a hypothetical trader that can access all countries' markets. Countries' consumption goods are homogeneous and freely tradable, so the spot and forward exchange rates are always unity.

Next, if one were to construct a "local discount bond" that pays off in the future consumption of the local country, this bond would have price

$$\begin{aligned} b_{n,t \rightarrow T} &= e^{-\delta(T-t)} \mathbb{E}_t \left[ \frac{C_{n,t}}{C_{n,T}} \right] \\ &= e^{-(\delta+g)(T-t)} \mathbb{E}_t \left[ \exp \left( - \int_t^T (-A_u + \frac{1}{2} \tilde{\pi}_{n,u}^2) du - \int_t^T \tilde{\pi}_{n,u} d\tilde{Z}_{n,u} \right) \right] \\ &= e^{-(\delta+g)(T-t)} \tilde{\mathbb{E}}_t^n \left[ \exp \left( \int_t^T A_u du \right) \right], \end{aligned}$$

where  $A_t$  is given in (9) and  $\tilde{\mathbb{E}}^n$  is the local risk-neutral expectation induced by risk prices  $\tilde{\pi}_{n,t}$ . This expression is an expected discounted sum of arbitrage profit, where discounting is performed by the riskless rate that would prevail without arbitrages. With unitary exchange rates, the CIP deviation (in continuously-compounded units) is given by the difference in these yields on these discount bonds; i.e.,

$$\Delta_{t \rightarrow T}^{\text{CIP}(i,j)} := -\frac{1}{T-t} \left( \log b_{j,t \rightarrow T} - \log b_{i,t \rightarrow T} \right) = \frac{1}{T-t} \log \frac{\tilde{\mathbb{E}}_t^i \left[ \exp \left( \int_t^T A_u du \right) \right]}{\tilde{\mathbb{E}}_t^j \left[ \exp \left( \int_t^T A_u du \right) \right]}.$$

Note that  $\Delta_{t \rightarrow T}^{\text{CIP}(i,j)} \neq 0$  if and only if the self-fulfilling equilibrium obtains. Indeed, the risk-neutral measures  $\tilde{\mathbb{E}}^i$  and  $\tilde{\mathbb{E}}^j$  differ in any equilibrium with self-fulfilling volatility, which is when  $A_t > 0$ . Conversely, when  $A_t = 0$  at all times, clearly  $\Delta_{t \rightarrow T}^{\text{CIP}(i,j)} = 0$ .

## 4.2 Cross-market trading limits volatility

Proposition 4 provided a link between volatility and arbitrage profit. Given this connection, impediments to capital mobility and cross-market trading that curb arbitrage profits should curb asset volatility. In this section, we make this argument precise by developing a notion of *limits to arbitrage* and showing how these limits bound the degree of volatility.

Motivated by models like Gromb and Vayanos (2002) and Gârleanu and Pedersen (2011), we assume that cross-sectional risk prices are linked by some amount of relative-value trading. To formalize this notion, we need to examine the location-specific risk prices induced by the basis shocks  $Z_t$ . Recall equation (2) connecting  $\tilde{Z}_t = MZ_t$ . If  $\tilde{\pi}_{n,t}$  is the location- $n$  marginal utility response to  $d\tilde{Z}_{n,t}$ , then

$$\pi_{n,t} := \tilde{\pi}_{n,t} M_n \quad (12)$$

is the marginal utility response to  $dZ_t$ , where  $M_n$  again is the  $n$ th row of  $M$ . Note that  $\tilde{\pi}_{n,t}$  is a scalar, whereas  $\pi_{n,t}$  is a vector.

We make the following reduced-form assumption about these basis risk prices  $\pi_{n,t}$ :

$$\|\pi_{j,t} - \pi_{i,t}\| \leq \Pi_t \quad \forall i \neq j. \quad (13)$$

When  $\Pi_t > 0$ , we say there are *limits to arbitrage*. This terminology is justified by the well-known equivalence between absence of arbitrage and the existence of a stochastic discount factor that prices all assets (and hence, a single risk price vector across all markets,  $\pi_t^* = \pi_{i,t} = \pi_{j,t}$ ).

In microfounded models, the process for  $\Pi_t$  would be linked to fundamental objects like arbitrageur wealth, preferences, constraints, and trading costs. For example, one can think of  $\Pi_t$  as arising from margin constraints and the limited wealth that arbitrageurs can deploy to eliminate risk-price differentials. It is only worth trading if risk-price differentials become sufficiently large. Bounds like (13) pervade most models of limits to arbitrage.<sup>10</sup> Here, we take  $\Pi_t$  as given and do not model the behavior of these arbi-

<sup>10</sup>For instance, Proposition 2' in Appendix B of Gârleanu and Pedersen (2011) explicitly shows how margin constraints lead to a range of viable risk premia.

trageurs, opting instead to characterize equilibrium conditional on partial arbitrage.<sup>11</sup>

Thus far, we have implicitly assumed  $\Pi_t = +\infty$ , which is tantamount to assuming that infinite frictions restrict any arbitrage trades between local markets. What happens when there is only partial, but not full, market segmentation? We have the following link between the degree of market segmentation and the degree of self-fulfilling volatility.

**Proposition 5.** *Let  $0 < \Pi_t < +\infty$  and  $\text{rank}(M) < N$ , Then, there exists a non-zero vector  $v^*$ , in the null-space of  $M'$ , such that the self-fulfilling volatility  $\psi_t$  of Theorem 1 is bounded by*

$$\psi_t \leq \delta^{-1} L_t^{-1} \Pi_t. \quad (14)$$

where  $L_t := \max_{(i,j): i \neq j} \|x_{i,t}^{-1} v_i^* M_i - x_{j,t}^{-1} v_j^* M_j\|$ . The “average return volatility”  $\sigma_t^* = \delta \psi_t \mathbf{1}' v^*$  is bounded by

$$\sigma_t^* \leq \mathbf{1}' v^* L_t^{-1} \Pi_t. \quad (15)$$

Intuitively, with significant limits to arbitrage, there can be large amounts of self-fulfilling volatility because capital is too slow to correct any such price movements. As limits to arbitrage are relaxed, the amount of self-fulfilling volatility gradually vanish. Propositions 4 and 5 are thus similar in that they connect volatility to some quantitative measure of arbitrage efficacy (arbitrage profits and limits to arbitrage, respectively).

Because of the link between volatility  $\psi_t$  and arbitrage profit  $A_t$ , limits to arbitrage as assumed in (13) also puts clear and intuitive bounds on  $A_t$ . This also bounds equilibrium risk prices, akin to Hansen and Jagannathan (1991), even though our limits to arbitrage assumption in (13) is about relative risk prices. The following corollary explains.

**Corollary 6.** *Under the conditions of Proposition 5, risk prices and arbitrage profit are bounded:*

$$\begin{aligned} \|\pi_{n,t}\| &\leq \frac{v_n^*}{x_{n,t}} L_t^{-1} \Pi_t, \\ \sqrt{A_t} &\leq \left( \sum_{n=1}^N x_{n,t} \left( \frac{v_n^*}{x_{n,t}} \right)^2 \right)^{1/2} L_t^{-1} \Pi_t, \end{aligned}$$

where  $L_t := \max_{(i,j): i \neq j} \|x_{i,t}^{-1} v_i^* M_i - x_{j,t}^{-1} v_j^* M_j\|$ .

To get a quantitative sense of the volatility bounds, we calibrate and simulate our model for the stability-inducing endogenous growth rates expressed in Proposition 2;

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<sup>11</sup>We also do not modify any of the market clearing conditions to account for arbitrageur consumption, which can be justified by the idea that infinite trading would occur if  $\|\pi_{j,t} - \pi_{i,t}\| > \Pi_t$  ever occurred, but zero trading is needed otherwise.

i.e.,

$$g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1}) \quad (16)$$

To satisfy the stability requirement  $\lambda > \delta^2$ , we set  $\lambda = \delta^2 + 0.01$ . We study  $N = 10$  locations and set the extrinsic shocks in a similar way as Example 3, which has a zero-sum condition that we referred to as “redistribution:”

$$M = \frac{N}{\sqrt{N(N-1)}} \left[ I_N - \frac{1}{N} \mathbf{1} \otimes \mathbf{1}' \right] \quad (17)$$

$$= \frac{1}{\sqrt{N(N-1)}} \begin{bmatrix} N-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & N-1 & -1 & \cdots & -1 & -1 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & N-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & N-1 \end{bmatrix}.$$

Note that the columns of  $M$  sum to zero and have unit norm. It can easily be verified that  $v^* = \mathbf{1}$  is the unique element, up to scale, in the null-space of  $M$ . To keep things simple, we assume that locations start equally sized ( $\alpha_{n,0} = 1/N$ ), and we initialize the simulation with equally-wealthy locations ( $x_{n,0} = 1/N$  for all  $n$ ). We also set  $\delta = g = 0.02$ .

In terms of the exogenous arbitrage bounds, we set  $\Pi_t = 0.25$  to a time-invariant value. The interpretation is that arbitrageurs are only willing to enter and correct Sharpe ratio differentials greater than 0.25. As will be clear shortly, these limits to arbitrage are quantitatively reasonable.

To simulate  $\{x_{n,t} : t \geq 0\}$ , first note that the dynamics are given by

$$dx_{n,t} = x_{n,t}(1 - x_{n,t}) \left[ \tilde{\pi}_{n,t}^2 - \sum_{i \neq n} \frac{x_{i,t}}{1 - x_{n,t}} \tilde{\pi}_{i,t}^2 \right] dt + x_{n,t} \tilde{\pi}_{n,t} d\tilde{Z}_{n,t}. \quad (18)$$

This is derived by applying Itô's formula to the definition  $x_{n,t} := c_{n,t}/Y_t$ , where the dynamics of  $c_{n,t}$  are given in (27) in the appendix. Because  $\tilde{\pi}_{n,t}$  depends on the self-fulfilling volatility, we assume  $\psi_t$  is always at its upper bound, subject to vanishing when needed.<sup>12</sup>

<sup>12</sup>In particular, Proposition 2 shows that  $\psi_t$  needs to vanish when  $\alpha_t$ ,  $x_t$ , or  $q_t - \delta\lambda^{-1}$  become low enough. We ensure these conditions by capping the ratio of  $\psi_t$  to each of these quantities by 100 in the simulation. In our simulation of  $T = 20$  years, none of these vanishing conditions are ever binding, but they would

The results for average return volatility  $\sigma_t^*$ , and the associated arbitrage profit  $A_t$ , are displayed in Figure 2. The average return volatility  $\sigma_t^*$  fluctuates around 16% (it could be much lower if we assume the economy is not at the upper bound of the volatility bounds). The claim that  $\Pi$  is reasonable can be seen by examining the associated simulated arbitrage profit  $A_t$ , which are around 2.5%, the upper range of measured arbitrage profits in [Fleckenstein and Longstaff \(2018\)](#).

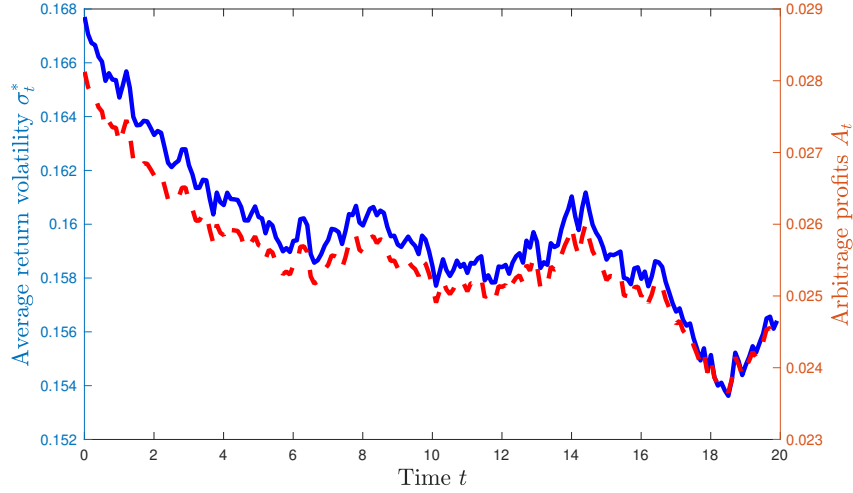


Figure 2: Plotted in solid blue against the left axis are volatility bounds of Proposition 5 from a simulated economy with  $N = 10$  equally-sized locations ( $\alpha_n = 1/N$ ), starting with equal initial wealth ( $x_{n,0} = 1/N$ ), with extrinsic shock matrix  $M$  given in (17), and with endogenous growth rates  $g_{n,t}$  from (16). Plotted in dashed red against the right axis are arbitrage profits  $A_t$  from the simulation. The simulation assumes  $\psi_t$  is always at the upper bound, except when it needs to vanish, i.e., when  $\min_n \alpha_{n,t}$ ,  $\min x_{n,t}$ , or  $\min_n q_{n,t} - \delta\lambda^{-1}$  become close enough to zero (see Proposition 2). We ensure this by capping the ratio of  $\psi_t$  to each of these quantities by 100 in the simulation. Other parameters are described in the text.

These magnitudes can be theoretically verified for the  $M$  in (17) by considering the approximation  $x_{n,t} = 1/N$  for all  $n$ . Then, the bounds simplify to

$$\sigma_t^* = \sqrt{A_t} \leq \sqrt{\frac{N-1}{2N}} \Pi_t.$$

Substituting  $N = 10$ , we obtain  $\sigma_t^* \leq 16.8\%$  and  $A_t \leq 2.8\%$ , which are very close to the ranges displayed in Figure 2.

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in a long enough simulation. In particular, since  $d\alpha_{n,t} = \alpha_{n,t}[g_{n,t} - g]dt$ , some locations' endowments can eventually shrink relative to the aggregate; i.e.,  $\liminf_{T \rightarrow \infty} \alpha_{n,T} = 0$  with positive probability. In such case, self-fulfilling volatility must vanish asymptotically. However, this type of long-run degeneracy is not present if the “stabilizing force” is beliefs (Appendix ??) or creative destruction (Appendix B.3).



## 5 Empirical analysis

We now confront the main model prediction with the data. From Proposition 4, recall

$$\text{average return volatility} = \sigma_t^* \leq \sqrt{A_t} = \text{square-root of arbitrage profits.} \quad (19)$$

Equation (19) predicts that there should be an increasing relationship between the square root of arbitrage profits ( $\sqrt{A_t}$ ) and the value-weighted-average volatilities of assets comprising the arbitrage trade in question ( $\sigma_t^*$ ). We investigate this relationship in the data by studying deviations from covered interest parity (CIP), building on the work of [Du et al. \(2018\)](#).

If we take the model literally, relationship (19) should hold approximately one-for-one: that is, a regression of  $\sigma_t^*$  onto  $\sqrt{A_t}$  should recover a regression coefficient of approximately 1.<sup>13</sup> We do not take the prediction so literally for two key reasons, most of which lead one to expect a regression coefficient below 1, in fact. First, in the presence of additional shocks, a substantial fraction of asset-price volatility will not be related to arbitrage profits. Second, the arbitrage of our model is built with long-lived assets, whereas any practical application uses shorter-dated assets that have lower volatility, through short duration alone. We attempt to address these issues below by using a variety of proxies for  $\sigma_t^*$ , but our choices are not without trade-offs.

**Data and proxies.** As a proxy for arbitrage profits  $A_t$ , we measure the 3-month absolute CIP deviations of the G10 currencies against the USD, and we take a simple average across these currencies. In particular, for currency  $i$  and maturity  $m$ , CIP against the USD says

$$\frac{1}{p_{US,t}^{(m)}} = \frac{s_{US \rightarrow i,t}}{f_{US \rightarrow i,t}^{(m)}} \frac{1}{p_{i,t}^{(m)}}, \quad (20)$$

where  $p_{US,t}^{(m)}$  and  $p_{i,t}^{(m)}$  are the  $m$ -maturity zero-coupon bond prices,  $s_{US \rightarrow i,t}$  denotes the spot exchange rate, and  $f_{US \rightarrow i,t}^{(m)}$  denotes the forward exchange rate. Exactly as in [Du et al. \(2018\)](#), we define the CIP deviation by the annualized incremental foreign bond yield needed to make (20) hold exactly. We take the absolute value of each G10 currency's CIP

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<sup>13</sup>Of course, expression (19) is not an equality. Jensen's inequality was used to obtain this expression. But as long as the Jensen deviation from equality is either small or uncorrelated with the variation in arbitrage profits, a unit regression coefficient should be expected.

deviation to obtain our proxy:

$$\hat{A}_{i,t}^{(m)} := \frac{1}{m} \left| \log \left( p_{i,t}^{(m)} / p_{US,t}^{(m)} \right) + \log \left( f_{US \rightarrow i,t}^{(m)} / s_{US \rightarrow i,t} \right) \right|. \quad (21)$$

For an aggregate time series measure, we take the simple average across currencies:

$$\hat{A}_t^{(m)} := \frac{1}{10} \sum_{i=1}^{10} \hat{A}_{i,t}^{(m)}. \quad (22)$$

All currency and foreign note data used in calculating the CIP deviations are from Bloomberg. We use 3-month forward contracts, which corresponds to setting  $m = 1/4$ . Our choice for  $\hat{A}_t^{(m)}$  matches a few key features of the arbitrage in the model: it represents a relatively short-term trade that generates positive profits over the riskless rate with certainty, if held to maturity. One-month or one-week CIP deviations may match better the fact that  $A_t$  represents profits over the infinitesimal time interval  $[t, t + dt]$ , but these contracts have other idiosyncratic features, as discussed in [Du et al. \(2018\)](#).

An appropriate choice for asset return volatility  $\sigma_t^*$  is more difficult, due to various data limitations and discrepancies between our model and reality. As a baseline, we proxy  $\sigma_t^*$  by the monthly standard deviation of daily log price changes in the 10-year annualized Constant Maturity Treasury note, annualized; i.e.,<sup>14</sup>

$$\hat{\sigma}_t^* := \sqrt{\frac{252}{30} \sum_{u=1}^{30} \left[ \log \left( p_{US,t+u}^{(10)} / p_{US,t+u-1}^{(10)} \right) - \frac{1}{30} \sum_{v=1}^{30} \log \left( p_{US,t+v}^{(10)} / p_{US,t+v-1}^{(10)} \right) \right]^2}. \quad (23)$$

This choice for  $\hat{\sigma}_t^*$  is simple and transparent but has several drawbacks: (a) it is not a very precise estimate of time- $t$  conditional volatility; (b) it includes no volatility information for the synthetic US bonds, which constitute the other leg of the CIP trade; and (c) unlike the model, in which the assets used to construct  $A_t$  correspond to those used to measure  $\sigma_t^*$ , this proxy uses long-maturity bonds instead of the 3-month bonds that make up  $\hat{A}_t^{(1/4)}$ .

To help address concerns (a)-(c), we also consider three alternative measures of  $\hat{\sigma}_t^*$  described briefly below and more extensively in [Appendix C](#).

- (a) For a more real-time measure of conditional volatility, we also examine the CBOE's 10-year Treasury VIX (TYVIX), which is the implied 30-day volatility of CBOT fu-

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<sup>14</sup>These are not holding period returns, as we hold the time-to-maturity constant in calculating the price at day  $t$  and day  $t + 1$ . Without imposing our own model-based interpolation methods, we cannot observe the day- $t + 1$  price of a Treasury that was a 10-year bond on day- $t$ .

tures on 10-year US Treasury Notes. TYVIX applies the CBOE’s VIX methodology to options on 10-year US Treasury Note futures.

- (b) To include information for the foreign leg, we compute the volatility of foreign 10-year notes (converted to USD via spot exchange rates), analogous to (23). We then take the value-weighted-average of this measure and 10-year US note volatility.
- (c) To bring the assets in the volatility construction as close as possible to those used in the arbitrage trade, we examine the value-weighted-average return volatilities of the 3-month US bill and the 3-month synthetic US bill. The prices of these bills are constructed using country-specific IBOR.

**Results.** Figure 3 summarizes the main empirical finding that arbitrage profits and volatility co-move strongly over time. The average absolute CIP deviation is high when the 10-year US Treasury volatility is high. Figure 4 shows that the same result holds disaggregated at the currency level. Appendix C repeats the same figures for our three other proxies of  $\sigma_t^*$  and finds similar results in each case.

To formalize and quantify this link, Table 1, column (1), displays OLS results from a regression of  $\hat{\sigma}_t^*$  on  $[\hat{A}_{i,t}^{(1/4)}]^{1/2}$  for each currency  $i$ . Across all currencies, we document a very strong relationship, both statistically and economically, between 10-year US Treasury note volatility and CIP deviations. Results are strongest for Australia and New Zealand – currencies that play a central role in the carry trade in practice. Surprisingly, despite the caveats outlined at the beginning of this section, the regression coefficients are in the ballpark of 1, in line with the model prediction.

Now, turn to our other proxies for  $\hat{\sigma}_t^*$  at the 10-year maturity in columns (2)-(4). Using instead the 10-year Treasury VIX (column 2) or including the 10-year foreign note (column 3) does not change the empirical message. The Treasury VIX regression coefficients are attenuated a bit, whereas the coefficients including the foreign note are magnified. If we use volatility from the 3-month bonds (column 4), our regressions produce slope estimates approximately 40-50 times lower, in line with the relative durations of a 10-year note and a 3-month bill.

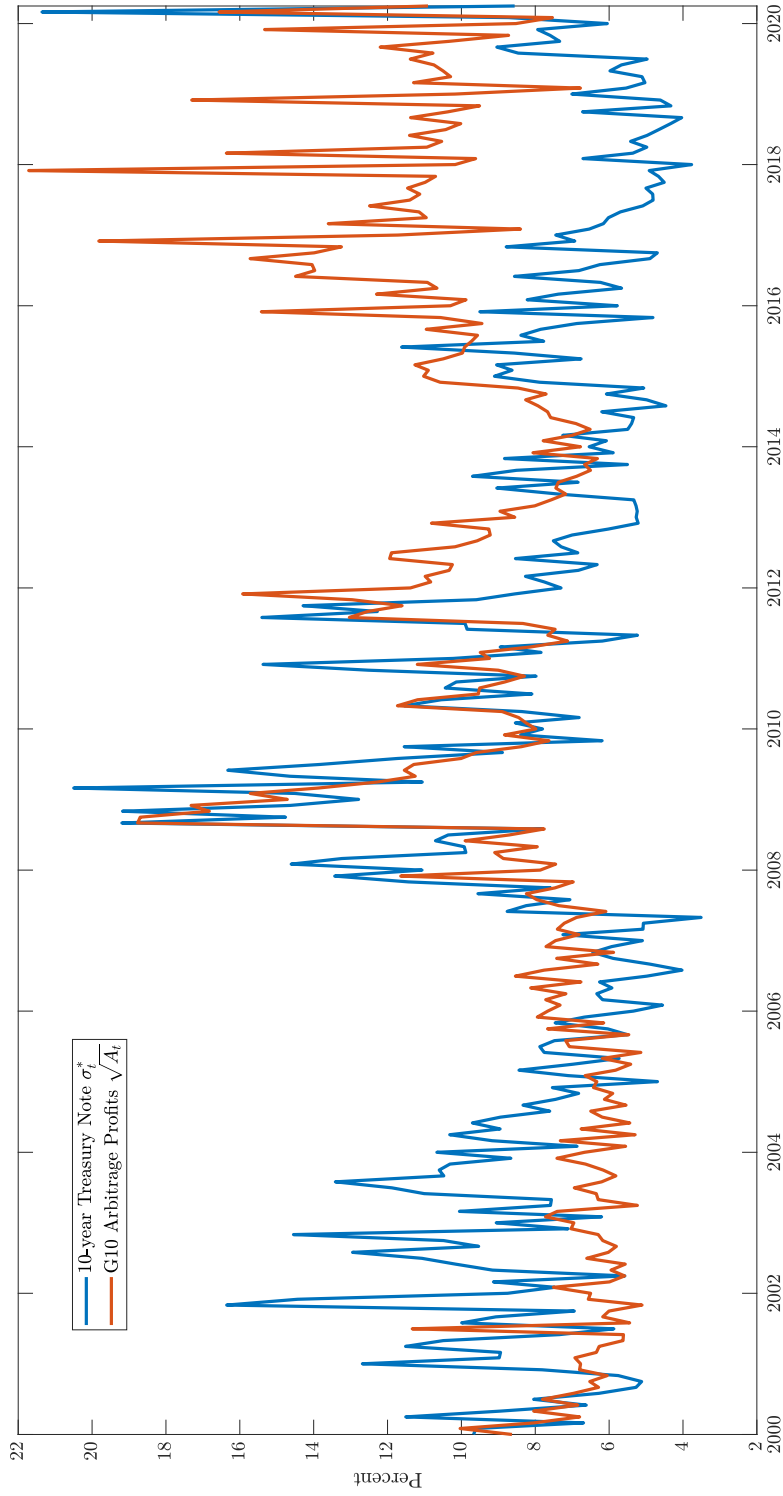


Figure 3: Monthly time-series of proxies for  $A_t$  and  $\sigma_t^*$ . Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is  $\hat{\sigma}_t^*$  from (23), which is constructed using the monthly standard deviation of daily log price changes of a 10-year Constant Maturity US Note. Both measures are annualized. Currency data are from Bloomberg, whereas 10-year Constant Maturity Treasury data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

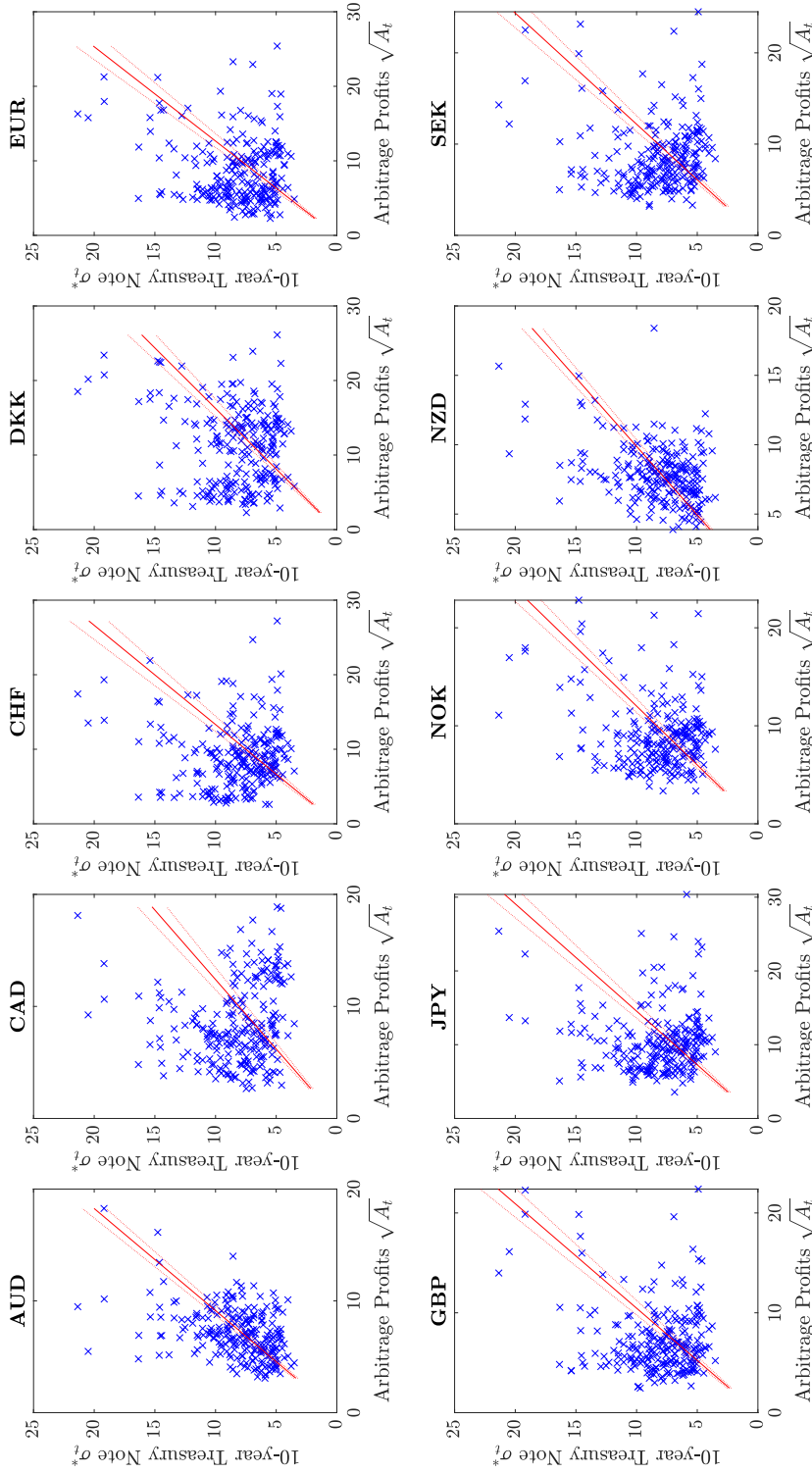


Figure 4: Currency-level OLS regressions of  $\sigma_t^*$  on  $\sqrt{A_t}$  (monthly, no intercept). Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is  $\hat{\sigma}_t^*$  from (23), which is constructed using the monthly standard deviation of daily log price changes of a 10-year Constant Maturity US Note. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency data are from Bloomberg, whereas 10-year Constant Maturity Treasury data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

**Discussion of existing models.** Our empirical results demonstrate a tight link between volatility and arbitrage profits, but they cannot identify volatility as necessarily self-fulfilling. This is an unavoidable limitation. The volatility-arbitrage link might seem consistent with limits to arbitrage in a very general way, going beyond our specific mechanism. To address this possibility, here, we discuss some conventional limits-to-arbitrage models and explain why the volatility-arbitrage prediction is not a straightforward implication of those models.

Value-at-risk (VaR) constraints immediately spring to mind as a direct volatility-arbitrage mechanism ([Adrian and Shin, 2010](#)). Higher-volatility environments can tighten arbitrageurs' constraints, limiting the amount of trading they can do, which makes arbitrage profits higher in equilibrium. Given the relatively high-frequency variation in arbitrage profits seen in Figure 3, a VaR-based explanation requires constraints to respond quickly to changes in volatility, which is not necessarily the case. For example, whereas the Chicago Mercantile Exchange applies variation margin to profits and losses daily, it adjusts its initial margin requirements relatively infrequently. Given the backward-looking methodologies often used in VaR-based constraints, high-frequency spikes in volatility are unlikely to be fully accounted for in initial margins.<sup>15</sup>

Even under fixed margin requirements, [Liu and Longstaff \(2004\)](#) demonstrate a positive correlation between the volatility of a relative-value trade (modeled as a Brownian bridge that is known to start and end at zero) and the terminal price of a logarithmic agent's optimal portfolio that can invest in the trade. Intuitively, high volatility begets large future mispricings (i.e., the Brownian bridge is likely to deviate further from zero), which can be exploited by an arbitrageur from that moment forward. And yet, following this sequence of logic carefully, high volatility leads to high *future* arbitrage profits, different from the contemporaneous relationship we document. Another important omission from their partial equilibrium model is that volatility is not simply an exogenous variable and must come from somewhere in general equilibrium.<sup>16</sup>

A general equilibrium economy that allows arbitrage is the margin-based model of [Gârleanu and Pedersen \(2011\)](#). In their model, holding cash flows fixed, a low-margin claim is more valuable than a high-margin claim. The price discrepancy widens when arbitrageur wealth shrinks, which occurs after negative shocks to the market portfolio (see their Figure 4). However, their model has no noticeable relationship between

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<sup>15</sup>A related alternative, which we cannot rule out, is that arbitrageurs self-impose VaR constraints at either a daily or intra-daily frequency.

<sup>16</sup>[Kondor \(2009\)](#) provides an equilibrium relating endogenous price gaps to arbitrageur capital, but this locally-deterministic model is not ideally-suited to connect these phenomena to volatility.

asset volatilities and arbitrageur wealth.<sup>17</sup> Overall, margin-based asset pricing models do not generally make robust predictions about the relation between arbitrage profits and underlying asset volatilities, primarily because the market-wide shock that shrinks arbitrageur wealth has an ambiguous effect on volatilities.<sup>18</sup>

On the other hand, by allowing a shock directly to non-arbitrageur demands, [Gromb and Vayanos \(2002\)](#) present a general equilibrium margin-based model in which the volatility-arbitrage link could hold, under some assumptions. To understand their mechanism, consider a two-island economy, where the islands' assets share identical cash flows. Local hedgers trade only in their respective islands, whereas an arbitrageur trades in both markets. If one island's local hedgers have a sudden increase in liquidity demands, they sell aggressively in their local market, causing a large discrepancy between their island's asset price and the other island's asset price. The arbitrageur does not fully correct the discrepancy due to a combination of constraints and limited wealth. If the liquidity demand shock is accompanied by an increase in the volatility of future demand shocks, local price volatility would rise as well. A similar equilibrium would emerge if "local hedgers" are replaced in this story by "noise traders" ([Kyle and Xiong, 2001](#)). This example shows that existing models can, in principle, match the volatility-arbitrage relation we identify, but the explanation requires higher liquidity demand volatility at times of large arbitrage profits. In other words, non-arbitrageurs' asset positions must be volatile at these times. We are not aware of any formal empirical evidence for this demand volatility story.

In summary, the existing literature provides some possible mechanisms connecting volatility and arbitrage profits, but the connections are not general, and each comes with caveats. In our view, it is plausible that some piece of the volatility-arbitrage relation that we document is due to self-fulfilling price fluctuations.

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<sup>17</sup>To see this absence in [Gârleanu and Pedersen \(2011\)](#), one must visually compare their Figures 2 and 3, which plot expected returns and Sharpe ratios as a function of the risk-tolerant agent's consumption share  $x$ . Their Figure 2 shows the expected return differential  $\mu_{0.4} - \mu_{0.1}$  between an asset with a 0.4 margin requirement and a derivative with a 0.1 margin requirement, with both assets being claims to the same cash flow. Their Figure 3 shows the Sharpe ratio differential  $SR_{0.4} - SR_{0.1}$  of these assets. As an identity, our model has  $\mu_{0.4} - \mu_{0.1} = \sigma_{0.4}SR_{0.4} - \sigma_{0.1}SR_{0.1} = \sigma_{0.4}(SR_{0.4} - SR_{0.1}) + (\sigma_{0.4} - \sigma_{0.1})SR_{0.1}$ . Because  $\mu_{0.4} - \mu_{0.1}$  and  $SR_{0.4} - SR_{0.1}$  share a similar pattern as a function of  $x$ , one concludes that volatilities  $\sigma_{0.4}$  and  $\sigma_{0.1}$  do not exhibit extreme variation in  $x$ . If anything, it appears that  $\frac{\mu_{0.4} - \mu_{0.1}}{SR_{0.4} - SR_{0.1}}$  is an inverted U-shaped function of  $x$ , consistent with well-known findings of general equilibrium models featuring heterogeneous risk aversions ([Gârleanu and Panageas, 2015](#)).

<sup>18</sup>[Gârleanu and Pedersen \(2011\)](#) do argue (see their footnote 9) that high-margin assets would have higher volatility in equilibrium. So, if the "shock" is to an asset's margin requirement, one would expect an increase in both its volatility and expected excess return (a proxy for arbitrage profits). This link is very similar to VaR constraints, as we discussed earlier. Since margin requirements are somewhat sticky in practice, we are more interested here in the shocks that can lead to higher arbitrage profits, while holding fixed the level of margins.



Currency	Proxy for $\hat{\sigma}_t^*$			
	(1) 10y US Note	(2) TYVIX	(3) 10y US + foreign	(4) 3m US + foreign
aud	1.095 (0.042)	0.813 (0.035)	1.602 (0.071)	0.033 (0.008)
cad	0.806 (0.126)	0.581 (0.103)	0.871 (0.126)	0.019 (0.005)
chf	0.751 (0.068)	0.538 (0.046)	0.812 (0.070)	0.027 (0.005)
dkk	0.616 (0.047)	0.441 (0.035)	0.629 (0.047)	0.021 (0.005)
eur	0.790 (0.076)	0.561 (0.052)	0.941 (0.075)	0.024 (0.006)
gbp	0.956 (0.083)	0.693 (0.058)	1.153 (0.069)	0.031 (0.009)
jpy	0.687 (0.053)	0.514 (0.048)	0.689 (0.053)	0.025 (0.005)
nok	0.833 (0.052)	0.606 (0.043)	0.815 (0.055)	0.026 (0.006)
nzd	1.012 (0.043)	0.755 (0.047)	1.426 (0.066)	0.031 (0.006)
sek	0.821 (0.070)	0.595 (0.051)	0.789 (0.071)	0.026 (0.006)
Observations	244	208	244*	242

Table 1: OLS regressions of volatility  $\hat{\sigma}_t^*$  on arbitrage profits  $\sqrt{\hat{A}_t^{(1/4)}}$  across G10 currencies and using four different volatility proxies  $\hat{\sigma}_t^*$ . Going across the columns, the four volatility proxies are as follows: (1) the monthly volatility of daily log price changes on the 10-year Constant Maturity US Note; (2) the CBOE's 10-year Treasury VIX (TYVIX); (3) the value-weighted average volatilities, computed monthly from daily log price changes, of the 10-year US Treasury Note and the 10-year foreign note, where the latter is adjusted to USD by the spot exchange rate; (4) the value-weighted average volatilities, computed monthly from daily measures of next-two-month holding period returns, on the 3-month US bill and the 3-month synthetic US bill (constructed using a foreign bill, the spot exchange rate, and a forward currency swap). All four measures are annualized. The proxy for arbitrage profits is the currency-specific 3-month absolute CIP deviation as in (21), averaged monthly. Estimated regression coefficients are computed from OLS without an intercept. Standard errors are in parentheses and computed using heteroskedasticity and autocorrelation corrected (HAC) formulas. Currency and foreign note data are from Bloomberg. 10-year Constant Maturity Treasury data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. TYVIX data from the CBOE. Currency data range from Jan. 2000 to Apr. 2020. TYVIX data ranges from Jan 2003 to Apr. 2020. 10-year G10 Note data range from Jan. 2000 to Apr. 2020 for AUD, CAD, DKK, EUR, GBP, NZD, and Jan. 2007 to Apr. 2020 for CHF, NOK, and SEK. \*158 (CHF), 239 (DKK), 243 (JPY), 158 (NOK), 160 (SEK).

## 6 Conclusion

This paper makes two main contributions. First, it provides a general segmented-markets framework that allows self-fulfilling volatility. One criteria for this volatility to arise is a “stabilizing force” that keeps prices stationary, which we formalize through a degree of endogeneity from prices to fundamentals or from prices to beliefs. These stabilizing forces are asset pricing counterparts to the stability conditions provided in seminal papers on sunspots in macroeconomics.

Second, the paper demonstrates a strong connection between the availability of arbitrage profits and the possibility of self-fulfilling volatility. Often, the presence of multiple equilibria and self-fulfilling dynamics are viewed as a nuisance for models, but we provide a theoretical result tightly connecting arbitrage and volatility, thereby providing foundations for a formal test. Empirically, we document an association between available arbitrage profits in foreign exchange markets and the volatility of the underlying instruments in these trades.

Given that levels of asset price volatility often far exceed predictions of many theoretical models, the paper’s mechanism can help bridge a gap in financial economics. For example, consider corporate equity and bond markets. Although equity and bond returns are linked, one cannot construct a riskless portfolio from them in a simple way, unlike for covered interest parity.<sup>19</sup> Still, it is entirely possible, and anecdotally true, that equity investors differ from bond investors and that capital is slow moving, whether due to market segmentation or investor habitats. (Ma 2019 provides evidence in this direction, suggesting that corporate issuances and buybacks act as a mechanism to profit from price differences.) With this in mind, our model suggests some amount of redistributive self-fulfilling volatility should be possible between corporate equity and bond markets. In this sense, our focus on true arbitrages is just for clarity: one can measure accurately the amount of arbitrage profit without having to know investors’ pricing kernels. We think that future research could, through a self-fulfilling mechanism, connect frictions such as market segmentation to “volatility puzzles” in other asset markets beyond those with self-evident arbitrage profits.

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<sup>19</sup>With dynamic trading and knowledge of the underlying shocks that affect both securities, as well as their sensitivities to those shocks, one could obtain a no-arbitrage relation between equities and bonds.

## References

- Adrian, Tobias and Hyun Song Shin**, "Financial intermediaries and monetary economics," *Handbook of Monetary Economics*, 2010, 3, 601–650.
- Azariadis, Costas**, "Self-fulfilling prophecies," *Journal of Economic Theory*, 1981, 25 (3), 380–396.
- Bacchetta, Philippe, Cédric Tille, and Eric Van Wincoop**, "Self-fulfilling risk panics," *American Economic Review*, 2012, 102 (7), 3674–3700.
- Bakke, Tor-Erik and Toni M Whited**, "Which firms follow the market? An analysis of corporate investment decisions," *The Review of Financial Studies*, 2010, 23 (5), 1941–1980.
- Barberis, Nicholas, Robin Greenwood, Lawrence Jin, and Andrei Shleifer**, "X-CAPM: An extrapolative capital asset pricing model," *Journal of Financial Economics*, 2015, 115 (1), 1–24.
- Benhabib, Jess, Pengfei Wang, and Yi Wen**, "Sentiments and aggregate demand fluctuations," *Econometrica*, 2015, 83 (2), 549–585.
- Biais, Bruno, Johan Hombert, and Pierre-Olivier Weill**, "Incentive constrained risk sharing, segmentation, and asset pricing," *Working Paper*, 2017.
- Blanchard, Olivier J**, "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, 1985, 93 (2), 223–47.
- Bond, Philip, Alex Edmans, and Itay Goldstein**, "The real effects of financial markets," *Annual Review of Financial Economics*, 2012, 4 (1), 339–360.
- Cass, David and Karl Shell**, "Do sunspots matter?," *Journal of Political Economy*, 1983, 91 (2), 193–227.
- Chen, Qi, Itay Goldstein, and Wei Jiang**, "Price informativeness and investment sensitivity to stock price," *The Review of Financial Studies*, 2007, 20 (3), 619–650.
- Cvitanić, Jakša and Ioannis Karatzas**, "Convex duality in constrained portfolio optimization," *The Annals of Applied Probability*, 1992, pp. 767–818.
- DeMarzo, Peter M, Michael J Fishman, Zhiguo He, and Neng Wang**, "Dynamic agency and the q theory of investment," *The Journal of Finance*, 2012, 67 (6), 2295–2340.
- Du, Wenxin, Alexander Tepper, and Adrien Verdelhan**, "Deviations from covered interest rate parity," *The Journal of Finance*, 2018, 73 (3), 915–957.
- , **Benjamin M Hébert, and Amy Wang Huber**, "Are Intermediary Constraints Priced?," 2019. Unpublished working paper.
- Duffie, Darrell and Bruno Strulovici**, "Capital Mobility and Asset Pricing," *Econometrica*, 2012, 80 (6), 2469–2509.
- Farmer, Roger EA**, "Pricing assets in a perpetual youth model," *Review of Economic Dynamics*, 2018, 30, 106–124.

- **and Jess Benhabib**, “Indeterminacy and increasing returns,” *Journal of Economic Theory*, 1994, 63, 19–41.
- **and Michael Woodford**, “Self-fulfilling prophecies and the business cycle,” *Macroeconomic Dynamics*, 1997, 1 (4), 740–769.
- Fleckenstein, Matthias and Francis A Longstaff**, “Shadow Funding Costs: Measuring the Cost of Balance Sheet Constraints,” *Working Paper*, 2018.
- Froot, Kenneth A and Paul GJ O’Connell**, “The pricing of US catastrophe reinsurance,” in “The Financing of Catastrophe Risk,” University of Chicago Press, 1999, pp. 195–232.
- Gabaix, Xavier and Matteo Maggiori**, “International liquidity and exchange rate dynamics,” *The Quarterly Journal of Economics*, 2015, 130 (3), 1369–1420.
- , **Arvind Krishnamurthy, and Olivier Vigneron**, “Limits of arbitrage: Theory and evidence from the mortgage-backed securities market,” *The Journal of Finance*, 2007, 62 (2), 557–595.
- Gârleanu, Nicolae and Lasse Heje Pedersen**, “Margin-based asset pricing and deviations from the law of one price,” *The Review of Financial Studies*, 2011, 24 (6), 1980–2022.
- **and Stavros Panageas**, “Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing,” *Journal of Political Economy*, 2015, 123 (3), 670–685.
- **and —** , “What to Expect when Everyone is Expecting: Self-Fulfilling Expectations and Asset-Pricing Puzzles,” *Journal of Financial Economics*, 2020.
- Goldstein, Itay and Liyan Yang**, “Information disclosure in financial markets,” *Annual Review of Financial Economics*, 2017, 9, 101–125.
- Gromb, Denis and Dimitri Vayanos**, “Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs,” *Journal of Financial Economics*, 2002, 66 (2), 361–407.
- Hansen, Lars Peter and Ravi Jagannathan**, “Implications of security market data for models of dynamic economies,” *Journal of Political Economy*, 1991, 99 (2), 225–262.
- Hennessy, Christopher A**, “Tobin’s Q, debt overhang, and investment,” *The Journal of Finance*, 2004, 59 (4), 1717–1742.
- Hu, Grace Xing, Jun Pan, and Jiang Wang**, “Noise as information for illiquidity,” *The Journal of Finance*, 2013, 68 (6), 2341–2382.
- Itskhoki, Oleg and Dmitry Mukhin**, “Exchange rate disconnect in general equilibrium,” 2017. Unpublished working paper.
- Kiyotaki, Nobuhiro and John Moore**, “Credit cycles,” *Journal of Political Economy*, 1997, 105 (2), 211–248.
- Koijen, Ralph SJ and Motohiro Yogo**, “A demand system approach to asset pricing,” *Journal of Political Economy*, 2019, 127 (4), 1475–1515.
- Kondor, Peter**, “Risk in dynamic arbitrage: the price effects of convergence trading,” *The Journal of Finance*, 2009, 64 (2), 631–655.

- Krishnamurthy, Arvind**, "The bond/old-bond spread," *Journal of Financial Economics*, 2002, 66 (2-3), 463–506.
- Kyle, Albert S and Wei Xiong**, "Contagion as a Wealth Effect," *The Journal of Finance*, 2001, 56 (4), 1401–1440.
- Lamont, Owen A and Richard H Thaler**, "Can the market add and subtract? Mispricing in tech stock carve-outs," *Journal of Political Economy*, 2003, 111 (2), 227–268.
- Leland, Hayne E**, "Corporate debt value, bond covenants, and optimal capital structure," *The Journal of Finance*, 1994, 49 (4), 1213–1252.
- Liu, Jun and Francis A Longstaff**, "Losing money on arbitrage: Optimal dynamic portfolio choice in markets with arbitrage opportunities," *The Review of Financial Studies*, 2004, 17 (3), 611–641.
- Long, J Bradford De, Andrei Shleifer, Lawrence H Summers, and Robert J Waldmann**, "Noise trader risk in financial markets," *Journal of Political Economy*, 1990, 98 (4), 703–738.
- , —, —, and —, "Positive feedback investment strategies and destabilizing rational speculation," *the Journal of Finance*, 1990, 45 (2), 379–395.
- , —, —, and —, "The survival of noise traders in financial markets," *Journal of Business*, 1991, pp. 1–19.
- Lustig, Hanno and Adrien Verdelhan**, "Does incomplete spanning in international financial markets help to explain exchange rates?," *American Economic Review*, 2019, 109 (6), 2208–44.
- Ma, Yueran**, "Nonfinancial Firms as Cross-Market Arbitrageurs," *The Journal of Finance*, 2019, 74 (6), 3041–3087.
- Miao, Jianjun and Pengfei Wang**, "Asset bubbles and credit constraints," *The American Economic Review*, 2018, 108 (9), 2590–2628.
- Mitchell, Mark, Lasse Heje Pedersen, and Todd Pulvino**, "Slow moving capital," *The American Economic Review: Papers & Proceedings*, 2007, 97 (2), 215–220.
- Scheinkman, Jose A and Wei Xiong**, "Overconfidence and speculative bubbles," *Journal of Political Economy*, 2003, 111 (6), 1183–1220.
- Shleifer, Andrei and Robert W Vishny**, "The limits of arbitrage," *The Journal of Finance*, 1997, 52 (1), 35–55.
- Vayanos, Dimitri and Jean-Luc Vila**, "A preferred-habitat model of the term structure of interest rates," *Econometrica* (forthcoming), 2020.
- and **Pierre-Olivier Weill**, "A search-based theory of the on-the-run phenomenon," *The Journal of Finance*, 2008, 63 (3), 1361–1398.
- Zentefis, Alexander**, "Self-fulfilling asset prices," 2020. Unpublished working paper. Yale University, New Haven, CT.

# Appendix

## A Proofs for Sections 3 and 4

*Proof of Theorem 1.* To prove the claim, we need to fill in any details that go beyond the discussion following the statement of Theorem 1. There are four brief steps needed to fill in the details.

**Step 1: State prices.** Each location has its own risk price  $\tilde{\pi}_{n,t}$ , which is the marginal utility sensitivity to the  $d\tilde{Z}_{n,t}$  shock. The state price density for location  $n$  is then given by

$$d\tilde{\zeta}_{n,t} = -\tilde{\zeta}_{n,t} \left[ r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} \right]. \quad (24)$$

In these terms, we have the no-arbitrage pricing relation:

$$\mu_{n,t}^q + g_{n,t} + \frac{1}{q_{n,t}} - r_t = \sigma_{n,t}^q \tilde{\pi}_{n,t}, \quad (25)$$

which suffices assuming  $q_{n,t} > 0$ . We can also pose terms with respect to the basis shocks. Let  $\pi_{n,t}$  be the risk price vector pertaining to  $dZ_t$ , which is potentially location-specific because of market segmentation. The link between these two, by substituting (2) into (24), is given in (12).

**Step 2: Optimality.** Log utility agents optimally consume  $\delta$  fraction of their wealth when there are no bubbles. Investor  $n$ 's wealth is given by  $y_{n,t}q_{n,t} + \beta_{n,t}$ , where  $\beta_{n,t}$  is her risk-free bond market position. Let  $\theta_{n,t} := \frac{y_{n,t}q_{n,t}}{y_{n,t}q_{n,t} + \beta_{n,t}}$  be the fraction of wealth this investor puts in the local risky asset. Note that market clearing is imposed automatically in this formula, as the local investor  $n$  holds the entirety of the local asset. Given the dynamic conjecture for asset prices and the consumption-wealth ratio  $\delta$ , each investor then has consumption dynamics:

$$\frac{dc_{n,t}}{c_{n,t}} = \left[ r_t - \delta + \theta_{n,t} \sigma_{n,t}^q \tilde{\pi}_{n,t} \right] dt + \theta_{n,t} \sigma_{n,t}^q d\tilde{Z}_{n,t}. \quad (26)$$

Under these assumptions, optimal portfolio choices are given by the standard mean-variance formula  $\theta_{n,t} \sigma_{n,t}^q = \tilde{\pi}_{n,t}$ . Substituting this portfolio choice into (26), equilibrium consumption dynamics are

$$\frac{dc_{n,t}}{c_{n,t}} = \left[ r_t - \delta + \tilde{\pi}_{n,t}^2 \right] dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t}. \quad (27)$$

From (24) and (27), we obtain  $\tilde{\zeta}_{n,t} c_{n,t} = \tilde{\zeta}_{n,0} c_{n,0} \exp(-\delta t)$ , so that the no-bubble static budget constraint (with wealth defined as  $w_{n,t} := y_{n,t}q_{n,t} + \beta_{n,t}$ )

$$\mathbb{E}_t \left[ \int_0^\infty \frac{\tilde{\zeta}_{n,t+s}}{\tilde{\zeta}_{n,t}} c_{n,t+s} ds \right] = w_{n,t} \quad (28)$$

holds automatically with  $c_{n,t} = \delta w_{n,t}$ . This confirms that the optimal consumption rule and no bubbles are mutually consistent.

**Step 3: Aggregation.** Recall the consumption shares  $x_{n,t} := c_{n,t}/Y_t$  and the endowment shares  $\alpha_{n,t} := y_{n,t}/Y_t$ . Notice that  $\theta_{n,t} = \delta \alpha_{n,t} q_{n,t} / x_{n,t}$ , which, combined with the optimal portfolio choice,

yields equation (8). Time-differentiating the goods market clearing condition  $\sum_{n=1}^N c_{n,t} = Y_t$  and using (27), we have

$$r_t = \delta + g_t - \sum_{n=1}^N x_{n,t} \tilde{\tau}_{n,t}^2 \quad (29)$$

and

$$0 = \sum_{n=1}^N x_{n,t} \tilde{\tau}_{n,t} M_n. \quad (30)$$

Substituting (8) into (30) delivers equation (4). Also, combining the asset-pricing equation (25), which is an equation for  $\mu_n^q$ , with the risk-free rate equation (29), one can show that (3) holds if and only if  $\sum_{n=1}^N \alpha_n q_{n,0} = \delta^{-1}$ , i.e., if an initial restriction holds for prices. In addition, note that consumption share dynamics are obtained by Itô's formula by equation (18).

**Step 4: Transversality and No-Ponzi.** Finally, we require the No-Ponzi condition<sup>20</sup>

$$\lim_{T \rightarrow \infty} \xi_{n,T} \beta_{n,T} = 0, \quad \text{a.s.} \quad (31)$$

Recall that the transversality condition  $\lim_{T \rightarrow \infty} \mathbb{E}_t[\xi_{n,T} w_{n,T}] = 0$  was used to write the static budget constraint (28). Since  $w_{n,T} \geq 0$  by the solvency constraint, this is equivalent to  $\lim_{T \rightarrow \infty} \xi_{n,T} w_{n,T} = 0$ . Then, using the latter version, plus the identity  $w_{n,T} = y_{n,T} q_{n,T} + \beta_{n,T}$ , we have that the No-Ponzi condition (31) is equivalent to  $\lim_{T \rightarrow \infty} \xi_{n,T} y_{n,T} q_{n,T} = 0$ . Finally, since we have assumed  $q_{n,T}$  remains positive, the latter condition is equivalent to the following transversality condition on prices:

$$\lim_{T \rightarrow \infty} \mathbb{E}_t[\xi_{n,T} y_{n,T} q_{n,T}] = 0. \quad (32)$$

We show that (32) is satisfied, hence (31) is satisfied. Recall from the discussion above that  $\xi_{n,T} = \xi_{n,0} c_{n,0} \exp(-\delta t) / c_{n,T}$ , so

$$\xi_{n,T} y_{n,T} q_{n,T} = \xi_{n,0} c_{n,0} e^{-\delta t} \frac{\alpha_{n,T}}{x_{n,T}} q_{n,T}.$$

Note that  $\alpha_{n,T}$  is bounded above by 1. Consequently, under the assumptions of the theorem – i.e.,  $(q_{n,t})_{t \geq 0}$  remains positive and bounded and  $\mathbb{P}[\lim_{T \rightarrow \infty} x_{n,T} > 0] = 1$  – transversality condition (32) holds.  $\square$

**Proof of Proposition 1.** Given the transversality condition (32), we have

$$q_{n,t} = \mathbb{E}_t \left[ \int_t^\infty \frac{\xi_{n,s}}{\xi_{n,t}} \frac{y_{n,s}}{y_{n,t}} ds \right].$$

Using  $g_{n,t} = g$  for all  $(n, t)$  and  $r_t = \delta + g - A_t \leq \delta + g$ , where  $A_t := \sum_{n=1}^N x_{n,t} \tilde{\tau}_{n,t}^2 \geq 0$ , we have

$$q_{n,t} = \int_t^\infty e^{-\delta(s-t)} \tilde{\mathbb{E}}_t^n \left[ \exp\left(\int_t^s A_u du\right) \right] ds \geq \int_t^\infty e^{-\delta(s-t)} ds = \delta^{-1},$$

where  $\tilde{\mathbb{E}}_t^n$  is the location- $n$  risk-neutral expectation, which is mutually absolutely-continuous with respect to  $\mathbb{E}$ . Using the bond-market clearing condition (3), we must have  $q_{n,t} = \delta^{-1}$  for all  $(n, t)$ .  $\square$

<sup>20</sup>Technically, condition (31) should be an inequality “ $\geq$ ” but optimality will impose “ $=$ ” so we immediately write it that way.



**Proof of Proposition 2.** Consider  $g_{n,t} = g + \lambda(q_{n,t} - \delta^{-1})$  with  $\lambda > \delta^2$  and fixed  $\epsilon$  that satisfies  $0 < \epsilon < \delta^{-2} - \lambda^{-1}$ . Supposing  $\text{rank}(M) < N$ , conjecture a stochastic equilibrium exists with  $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = v_n^*\psi_t$  and  $\tilde{\pi}_{n,t} = \delta v_n^*\psi_t/x_{n,t}$  for some process  $\psi_t$ . Substituting these and all other equilibrium objects into the asset-pricing equation (25), we have

$$dq_{n,t} = \left[ -1 + \left( \delta + \lambda\delta^{-1} - \delta^2\psi_t^2 \sum_{i=1}^N \frac{(v_i^*)^2}{x_{i,t}} \right) q_{n,t} - \lambda q_{n,t}^2 + \delta \frac{(v_n^*\psi_t)^2}{\alpha_{n,t}x_{n,t}} \right] dt + \frac{v_n^*}{\alpha_{n,t}} \psi_t M_n dZ_t. \quad (33)$$

We show that if properties (P1) and (P2) are satisfied, then  $q_{n,t}$  remains bounded for all  $n$ . As a preliminary, define

$$D(q) := -1 + (\delta + \lambda\delta^{-1})q - \lambda q^2. \quad (34)$$

When  $\psi_t = 0$ , all local price-dividend ratios follow  $dq_{n,t} = D(q_{n,t})dt$ . Note that  $D(q) = 0$  is a quadratic equation that has two roots:  $\delta^{-1}$  and  $\delta\lambda^{-1}$ . Moreover,  $D(q) > 0$  if and only if  $q \in (\delta\lambda^{-1}, \delta^{-1})$ .

Under property (P2), if  $q_t = \delta(\epsilon + \lambda^{-1})$ , we have  $\psi_t = 0$  and so

$$dq_t = D(\delta(\epsilon + \lambda^{-1}))dt > 0.$$

Note that, under property (P1), the drift and diffusion coefficients of  $q_{n,t}$  are bounded, so  $q_{n,t}$  is almost-surely path-continuous. This proves that the entire path is bounded below: if  $q_{n,0} > \delta(\epsilon + \lambda^{-1})$  for all  $n$ , then  $\{q_{n,t}\}_{t \geq 0} > \delta(\epsilon + \lambda^{-1})$  for each  $n$  almost-surely.

On the other hand, bond market clearing (3), plus this lower bound on valuations, implies an upper bound on the maximal valuation:

$$\bar{q}_t := \max_n q_{n,t} < \underbrace{(\alpha_{\bar{n}_t,t}\delta)^{-1} - (1 - \alpha_{\bar{n}_t,t})\delta(\epsilon + \lambda^{-1})}_{:=b_t}, \quad \text{where } \bar{n}_t := \arg \max_n q_{n,t}.$$

It suffices to show that  $\mathbb{P}[\sup_t b_t < +\infty] = 1$ . As long as  $\alpha_t > 0$ , we always have  $b_t < +\infty$ . As a result, we need only consider the case  $\alpha_t = \alpha_{\bar{n}_t,t}$  (i.e., the location with maximal valuation is the location with minimal endowment share) and suppose  $\alpha_t \searrow 0$ . However, since  $\bar{q}_t > \delta^{-1}$ , we have

$$d\alpha_{\bar{n}_t,t} = \alpha_{\bar{n}_t,t}\lambda[\bar{q}_t - \delta^{-1}]dt > 0,$$

which contradicts  $\alpha_t \searrow 0$ .

In summary,  $\{(q_{n,t})_{n=1}^N : t \geq 0\}$  is positive and bounded almost-surely, so to verify the conditions of Theorem 1, it remains to show that  $\lim_{T \rightarrow \infty} e^{-\delta T} x_{n,T}^{-1} = 0$  a.s. Substituting equilibrium objects into (18), we have

$$dx_{n,t} = \psi_t^2 \delta^2 \left[ (1 - x_{n,t}) \frac{(v_n^*)^2}{x_{n,t}} - x_{n,t} \sum_{i \neq n} \frac{(v_i^*)^2}{x_{i,t}} \right] dt + \psi_t \delta v_n^* M_n dZ_t. \quad (35)$$

Define  $\underline{x}_t := \min_n x_{n,t}$  and  $\underline{n}_t := \arg \min_n x_{n,t}$ . Ignoring local times,  $\underline{x}_t$  follows

$$d\underline{x}_t = \psi_t^2 \delta^2 \left[ \frac{(v_{\underline{n}_t}^*)^2}{\underline{x}_t} - O(\underline{x}_t) \right] dt + \psi_t \delta v_{\underline{n}_t}^* M_{\underline{n}_t} dZ_t.$$

On the set of events  $\{\psi_t > 0\}$ , we have specified in requirement (P1) that  $\psi_t/\underline{x}_t$  be bounded; hence,  $\lim_{T \rightarrow \infty} e^{-\delta T} \mathbf{1}_{\{\psi_T > 0\}} \psi_T \underline{x}_T^{-1} = 0$  a.s., which implies  $\lim_{T \rightarrow \infty} e^{-\delta T} \mathbf{1}_{\{\psi_T > 0\}} \underline{x}_T^{-1} = 0$  a.s. On the complementary event  $\{\psi_t = 0\}$ , it is clear from the evolution equation that  $d\underline{x}_t = 0$ . Thus,  $\lim_{T \rightarrow \infty} e^{-\delta T} \mathbf{1}_{\{\psi_T > 0\}} \underline{x}_T^{-1} = 0$  a.s. Putting these pieces together, we verify  $\lim_{T \rightarrow \infty} e^{-\delta T} \underline{x}_T^{-1} = 0$ . This completes the proof.  $\square$

**Proof of Proposition 3.** The asset-pricing equation is identical to that in Proposition 2, with perceived growth  $\tilde{g}_n$  replacing actual growth  $g_n$ . Thus, the analysis is identical, with the exception that the endowment shares  $\alpha_{n,t}$  are constant over time here, so the condition that  $\psi_t/\underline{\alpha}_t$  be bounded is automatically ensured by the other conditions.  $\square$

**Proof of Theorem 2.** First, assuming the existence of self-fulfilling volatility, let us find a portfolio that has no risk but pays a positive premium over the riskless rate. Consider a portfolio that goes long  $\delta\alpha_{n,t}q_{n,t}$  of each asset  $n = 1, \dots, N$ , which costs 1 by equation (3). As stated in equation (25), each asset  $n$  has expected excess returns that are given by the product of the location- $n$  risk quantity times the risk price:  $\sigma_{n,t}^q \tilde{\pi}_{n,t}$ . Using equation (8) to substitute  $\tilde{\pi}_{n,t}$ , the portfolio excess return is

$$\sum_{n=1}^N x_{n,t} \delta^2 \left( \frac{\alpha_{n,t} q_{n,t}}{x_{n,t}} \right)^2 (\sigma_{n,t}^q)^2 \geq 0,$$

which is strictly positive as long as any self-fulfilling volatility obtains. Using the expression for  $\tilde{\pi}_{n,t}$ , one can easily verify this expression is equivalent to  $A_t$  in (9). At the same time, by equation (4), the portfolio volatility is identically zero. This shows that an arbitrage always emerges if there is self-fulfilling volatility.

Next, the claim that absence of self-fulfilling volatility implies no arbitrage follows from (25), whereby all assets return  $r_t$  when  $\sigma_{n,t}^q = 0$ .  $\square$

**Proof of Proposition 4.** Substituting  $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = \psi_t v_n^*$  from Theorem 1 into location-specific risk prices of (8), and substituting the result into (9), we have

$$A_t = \delta^2 \psi_t^2 \sum_{n=1}^N x_{n,t} \left( \frac{v_n^*}{x_{n,t}} \right)^2$$

By inverting this relationship, the amount of self-fulfilling volatility  $\psi_t$  can be inferred from  $A_t$ , which gives the equality in (10). The upper bound can be obtained by substituting

$$\sum_{n=1}^N x_{n,t} \left( \frac{v_n^*}{x_{n,t}} \right)^2 \geq \left( \sum_{n=1}^N x_{n,t} \frac{v_n^*}{x_{n,t}} \right)^2 = (\mathbf{1}' v^*)^2,$$

which holds by Jensen's inequality. To obtain the equality in (11), substitute (3) into the definition of  $\sigma_t^*$  and use the result from Theorem 1 that  $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = \psi_t v_n^*$ . To obtain the inequality, use (10).  $\square$

**Proof of Proposition 5.** Substitute equation (12) into equation (8) to get

$$\pi_{n,t} = \delta \left( \frac{\alpha_{n,t} q_{n,t}}{x_{n,t}} \right) \sigma_{n,t}^q M_n.$$

Now, use the result of Theorem 1 that  $\alpha_{n,t}q_{n,t}\sigma_{n,t}^q = v_n^* \psi_t$ . Combining these equations, we have

$$\pi_{n,t} = \delta v_n^* \psi_t \frac{M_n}{x_{n,t}}. \quad (36)$$

Assumption (13) is equivalent to

$$\delta\psi_t \max_{(i,j):i \neq j} \left\| \frac{v_i^* M_i}{x_{i,t}} - \frac{v_j^* M_j}{x_{j,t}} \right\| \leq \Pi_t.$$

Solving for  $\psi_t$ , we obtain inequality (14). The bounds for  $\sigma_t^*$  are a direct consequence of (14).  $\square$

**Proof of Corollary 6.** To get the both bounds, begin with the volatility bound (14) of Proposition 5 and use

$$\begin{aligned} \|\pi_{n,t}\| &= \delta\psi_t \frac{v_n^*}{x_{n,t}}, \\ \sqrt{A_t} &= \delta\psi_t \sqrt{\sum_{n=1}^N x_{n,t} \left(\frac{v_n^*}{x_{n,t}}\right)^2}. \end{aligned}$$

The expression for  $\|\pi_{n,t}\|$  comes from taking the norm of equation (36) and using the fact that  $MM'$  has ones on its diagonal (this was a normalization). The expression for  $\sqrt{A_t}$  comes from expression (10) in Proposition 4.  $\square$

# Internet Appendix – not for publication

## B Model extensions

### B.1 Aggregate shocks

Here, we allow for aggregate shocks hitting the endowments. Location-specific endowments now follow

$$dy_{n,t} = y_{n,t}[g_{n,t}dt + \nu dB_t],$$

where  $B_t$  is an aggregate Brownian shock, independent of the extrinsic shocks  $Z_t$  (and by extension  $\tilde{Z}_t$ ). We maintain the restriction  $\sum_{n=1}^N y_{n,t}g_{n,t} = Y_t g_t$ . Thus, the aggregate endowment follows

$$dY_t = Y_t[g_t dt + \nu dB_t].$$

Conjecture that local price-dividend ratios follow

$$dq_{n,t} = q_{n,t}[\mu_{n,t}^q dt + \sigma_{n,t}^q d\tilde{Z}_{n,t} + \varsigma_{n,t}^q dB_t],$$

where  $(\mu_{n,t}^q, \sigma_{n,t}^q, \varsigma_{n,t}^q)$  are determined in equilibrium. We will proceed by making one of two possible assumptions on the tradability of this aggregate shock.

**Assumption 1.** *One of the following holds:*

- (a) *there are no additional markets open beyond those assumed so far;*
- (b) *there is an integrated market in which agents frictionlessly trade a zero-net-supply Arrow security that has a unit loading on  $dB_t$ .*

In both cases of Assumption 1, all previous results on self-fulfilling volatility go through. However, we uncover a surprising nuance: equilibrium is consistent with local assets having nearly arbitrary sensitivities to the aggregate shock.

**Proposition 7.** *With aggregate shocks, the conclusions of Theorem 1 on  $(\sigma_{n,t}^q)_{n=1}^N$  continue to hold without modification. Regarding  $(\varsigma_{n,t}^q)_{n=1}^N$ , we have the following. Let  $(\phi_{n,t})_{n=1}^{N-1}$  be a collection of arbitrary stochastic processes and set  $\phi_{N,t} := -\sum_{n=1}^{N-1} \phi_{n,t}$ . Then, there exists an equilibrium with  $\alpha_{n,t} q_{n,t} \varsigma_{n,t}^q = \phi_{n,t}$  as long as the resulting  $\{(q_{n,t})_{n=1}^N\}_{t \geq 0}$  is a bounded, positive process and  $x_{n,t}$  almost-surely does not converge to zero.*

Before giving a formal proof, we provide the basic sketch of the argument. Because our log agents will still consume  $\delta$  fraction of their wealth in this environment, equilibrium still satisfies equation (3) such that  $\sum_{n=1}^N \alpha_{n,t} q_{n,t} = \delta^{-1}$ . If we time-differentiate this condition as before, matching diffusion terms leads us to

$$\text{(match } dZ_t \text{ terms)} \quad 0 = \sum_{n=1}^N \alpha_{n,t} q_{n,t} \sigma_{n,t}^q M_n, \quad (37)$$

$$\text{(match } dB_t \text{ terms)} \quad 0 = \sum_{n=1}^N \alpha_{n,t} q_{n,t} \varsigma_{n,t}^q. \quad (38)$$

Equation (37) is identical to equation (4), which is why the results of Theorem 1 continue to hold. For equation (38), of course it is possible to have  $\varsigma_{n,t}^q = 0$  for all  $n$ . But we may also set  $(\varsigma_{n,t}^q)_{n=1}^{N-1}$

arbitrarily, so long as  $\zeta_{N,t}^q$  offsets these sensitivities. Thus, the volatilities have a similar redistributive flavor as before.

This is indeed an equilibrium, as long as the induced dynamics of price-dividend ratios are stationary. To this end, we can easily extend Propositions 1 and 2 to this setting with aggregate shocks. With common growth rates  $g_{n,t} = g$ , there will be no multiplicity ( $\sigma_{n,t}^q = \zeta_{n,t}^q = 0$ ), as the only prices consistent with the transversality condition are  $q_{n,t} = \delta^{-1}$ . With growth rates that increase sufficiently quickly in local valuations, we can generate stochastic multiplicity, because all that is required is to have both  $\sigma_{n,t}^q$  and  $\zeta_{n,t}^q$  vanish whenever  $\min_n q_{n,t}$  or  $\min_n \alpha_{n,t}$  become “too small”. We omit the details of these results.<sup>21</sup>

The intuition for self-fulfilling fundamental sensitivities differs depending on whether the shock is hedgable or not. When agents cannot hedge the  $dB_t$  shock, the logic is similar to the baseline model: agents adjust their consumption, through the bond market, to their conjecture about how the local asset co-moves with the fundamental shock. When agents trade Arrow securities on  $dB_t$  in an integrated market, they do not care whether or not their local asset responds to this shock. Enough hedging and risk-sharing will occur in equilibrium such that individual consumptions all have sensitivity  $\nu$  to  $dB_t$ . Under a particular conjecture about  $\zeta_{n,t}^q$ , location- $n$  agents will form a hedging plan to undo this exposure. This is self-fulfilling: as long as asset prices move according to the conjecture, the hedging plan was correct.

**Proof of Proposition 7.** We will nest cases (a) and (b) of Assumption 1 in the following setting. Introduce an Arrow security that pays off  $\eta_{n,t}dt + dB_t$  per unit of time, where  $(\eta_{n,t})_{n=1}^N$  will be determined endogenously. Thus, agent  $n$  faces the state-price density process, modified from (24):

$$d\tilde{\zeta}_{n,t} = -\tilde{\zeta}_{n,t} \left[ r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} + \eta_{n,t} dB_t \right]. \quad (39)$$

Let  $\theta_{n,t}^{\text{agg}}$  be the fraction of wealth a location- $n$  agent invests in the Arrow security, and let  $\theta_{n,t}$  be the fraction of wealth invested in the location-specific capital asset as before. The wealth of agent  $n$  has the following dynamics (i.e., the dynamic budget constraint):

$$\begin{aligned} \frac{dw_{n,t}}{w_{n,t}} = & \left[ r_t - \frac{c_{n,t}}{w_{n,t}} + \theta_{n,t} \sigma_{n,t}^q \tilde{\pi}_{n,t} + \left( \theta_{n,t} (\nu + \zeta_{n,t}^q) + \theta_{n,t}^{\text{agg}} \right) \eta_{n,t} \right] dt \\ & + \theta_{n,t} \sigma_{n,t}^q d\tilde{Z}_{n,t} + \left( \theta_{n,t} (\nu + \zeta_{n,t}^q) + \theta_{n,t}^{\text{agg}} \right) dB_t. \end{aligned} \quad (40)$$

To implement (a), where agents are not allowed to trade the Arrow security, we impose a fictitious market clearing condition  $\theta_{n,t}^{\text{agg}} = 0$  for all  $n$ , which will pin down  $\eta_{n,t}$  such that no trading in the Arrow security occurs. From the results of Cvitanić and Karatzas (1992), this implements the same equilibrium as if we never introduced this fictitious market. To implement (b), in which the Arrow market exists and is integrated, we impose  $\eta_{n,t} = \eta_t$  for all  $n$  and clear the market via  $\sum_{n=1}^N x_{n,t} \theta_{n,t}^{\text{agg}} = 0$ . In both cases, we have the capital market clearing condition  $\theta_{n,t} = y_{n,t} q_{n,t} / w_{n,t}$  as before.

Thus, we may nest cases (a) and (b) by solving unconstrained optimization problems for our investors, augmented with the general state-price density process (39) as long as  $\eta_{n,t}$  is chosen appropriately. Given the state-price density, the pricing condition (25) is replaced by

$$\mu_{n,t}^q + g_{n,t} + \frac{1}{q_{n,t}} + \nu \zeta_{n,t}^q - r_t = \sigma_{n,t}^q \tilde{\pi}_{n,t} + (\nu + \zeta_{n,t}^q) \eta_{n,t}, \quad (41)$$

<sup>21</sup>To prove an analogous result to Proposition 2 formally, it is convenient that all locations have equal exposures  $\nu$  to the aggregate shock, so that  $\alpha_{n,t}$  evolves locally deterministically for all  $n$ .

along with the requirement  $q_{n,t} > 0$ . Because all agents have log utility and effectively solve unconstrained portfolio problems with homogeneous wealth dynamics (40), they all consume  $\delta$  fraction of their wealth; i.e.,  $c_{n,t} = \delta w_{n,t}$ . Then, as  $B_t$  and  $\tilde{Z}_{n,t}$  are independent, optimal consumption dynamics (27) are modified to read

$$\frac{dc_{n,t}}{c_{n,t}} = \left[ r_t - \delta + \tilde{\pi}_{n,t}^2 + \eta_{n,t}^2 \right] dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} + \eta_{n,t} dB_t.$$

Because  $dw_{n,t}/w_{n,t} = dc_{n,t}/c_{n,t}$ , we therefore have

$$\begin{aligned} \tilde{\pi}_{n,t} &= \theta_{n,t} \sigma_{n,t}^q = \frac{\delta \alpha_{n,t} q_{n,t}}{x_{n,t}} \sigma_{n,t}^q, \\ \eta_{n,t} &= \theta_{n,t} (\nu + \varsigma_{n,t}^q) + \theta_{n,t}^{\text{agg}} = \frac{\delta \alpha_{n,t} q_{n,t}}{x_{n,t}} (\nu + \varsigma_{n,t}^q) + \theta_{n,t}^{\text{agg}}. \end{aligned}$$

The first equation is identical to (8).

Now, we aggregate. First, equation (3) still holds, since agents consume  $\delta$  fraction of wealth, and since both the bond market and the Arrow markets are in zero net supply. Next, time-differentiate the goods market clearing condition  $\sum_{n=1}^N c_{n,t} = Y_t$  and match drift and diffusion terms to obtain

$$\begin{aligned} r_t &= \delta + g_t - \sum_{n=1}^N x_{n,t} \tilde{\pi}_{n,t}^2 - \sum_{n=1}^N x_{n,t} \eta_{n,t}^2 \\ 0 &= \sum_{n=1}^N x_{n,t} \tilde{\pi}_{n,t} M_n \\ \nu &= \sum_{n=1}^N x_{n,t} \eta_{n,t}. \end{aligned}$$

Using the expressions for  $\tilde{\pi}_{n,t}$  and  $\eta_{n,t}$  above, along with the condition  $\sum_{n=1}^N x_{n,t} \theta_{n,t}^{\text{agg}} = 0$  (which holds in cases (a) and (b) both), we obtain equations (37)-(38). Thus,  $\sigma_{n,t}^q$  and  $\tilde{\pi}_{n,t}$  are solved exactly as in Theorem 1. Letting  $(\phi_{n,t})_{n=1}^{N-1}$  be arbitrary processes, and putting  $\phi_{N,t} = -\sum_{n=1}^{N-1} \phi_{n,t}$ , we may satisfy (38) by setting  $\varsigma_{n,t}^q$  by  $\phi_{n,t} = \alpha_{n,t} q_{n,t} \varsigma_{n,t}^q$ . As before, this is an equilibrium as long as the transversality condition (32) is satisfied, for which it suffices to show that  $q_{n,t}$  is almost-surely bounded and  $\lim_{T \rightarrow \infty} e^{-\delta T} x_{n,T}^{-1} = 0$  almost-surely.

It remains to solve for  $(\eta_{n,t})_{n=1}^N$ . In case (a), we use  $\theta_{n,t}^{\text{agg}} = 0$  in conjunction with the expression for  $\eta_{n,t}$  above to get  $\eta_{n,t} = \delta \alpha_{n,t} q_{n,t} (\nu + \varsigma_{n,t}^q) / x_{n,t}$ . In case (b), we impose  $\eta_{n,t} = \eta_t$  for all  $n$ , which, after substituting into  $\nu = \sum_{n=1}^N x_{n,t} \eta_{n,t}$ , yields  $\eta_t = \nu$ .  $\square$

## B.2 Debt overhang as a “stabilizing force”

In this section, we sketch an economy where firms face an investment problem, subject to neo-classical adjustment costs and debt-overhang. The result is a version of Q-theory, but with under-investment. Because the predictions of this theory are so well-established, at some points we make reduced-form assumptions to simplify the analysis and illustrate our main points on stability.

**Firms.** There are a continuum of firms in each location  $n$ , each employing a linear technology with productivity  $a$  and capital as the sole input. The evolution of firm-level capital is

$$dk_{n,t}^{(j)} = k_{n,t}^{(j)} [l_{n,t}^{(j)} - \kappa] dt + k_{n,t}^{(j)} \hat{\sigma} d\hat{B}_{n,t}^{(j)},$$

where  $\iota$  is the endogenous investment rate,  $\kappa$  is the exogenous depreciation rate, and  $B^{(j)}$  is an idiosyncratic Brownian shock. The cost of making investment  $\iota k$  is given by  $\Phi(\iota)k$ , where  $\Phi(\cdot)$  is a convex adjustment cost function. Thus, the investment-production block has the standard homogeneity property in capital.

For this section only, we denote by  $q_{n,t}^{(j)}$  the location- $n$  average value of capital to equity, i.e. “average  $Q$ ” (this will not be the same as the price-dividend ratio that is called “ $q$ ” in the main text, because the dividend is output minus investment). Thus, the value of firm  $j$  is given by  $q_{n,t}^{(j)}k_{n,t}^{(j)}$ .

We also assume that all firms have long-term debt outstanding, in fact a perpetuity with a fixed and continuously-paid coupon as in [Leland \(1994\)](#) and its descendent papers, without micro-founding the reasons for why (e.g., debt tax shield), as this is unimportant. Furthermore, to keep things simple, we assume existing firms can never issue new debt. Finally, firms default optimally, subject to some default costs that are proportional to the firm’s capital (these can be redistributed to households to create no deadweight loss). Under these conditions, a typical finding is (see for example [Hennessy, 2004](#), Proposition 2)

$$\tilde{q}_{n,t}^{(j)} := \text{marginal value of capital to equity} < \text{average value of capital to equity} = q_{n,t}^{(j)}.$$

Moreover, essentially by definition of  $\tilde{q}$ , the optimal investment satisfies  $\tilde{q}_{n,t}^{(j)} = \Phi'(\iota_{n,t}^{(j)})$  (see for example [Hennessy, 2004](#), equation 11). Thus, we see that  $q_{n,t}^{(j)} > \Phi'(\iota_{n,t}^{(j)})$ . The lack of equality here measures the deviation from neoclassical  $Q$ -theory.

Despite this deviation, we have the following property. Since  $q_{n,t}^{(j)}$  increases with  $\tilde{q}_{n,t}^{(j)} = \Phi'(\iota_{n,t}^{(j)})$ , and since  $\Phi$  is a convex function, we have  $\iota_{n,t}^{(j)}$  increasing in  $q_{n,t}^{(j)}$ . We will furthermore make the reduced-form assumption that  $\iota_{n,t}^{(j)} = \iota(q_{n,t}^{(j)})$  for some univariate increasing function  $\iota(\cdot)$ . This assumption is quite benign as it is typically satisfied in applications, because  $\tilde{q}_{n,t}^{(j)}$ , hence  $q_{n,t}^{(j)}$ , will typically be monotonic functions of the underlying firm-level state (e.g., leverage ratio).

In summary, we have the following two firm-level properties under debt overhang:

$$q_{n,t}^{(j)} > \Phi'(\iota_{n,t}^{(j)}) \tag{42}$$

$$\iota'(q_{n,t}^{(j)}) > 0. \tag{43}$$

Condition (42) captures the specific debt-overhang mechanism, whereas condition (43) is much more general and applies in almost any investment model. With a more general contractual structure, [DeMarzo et al. \(2012\)](#) also obtains these two results.

**Aggregation.** We will now make two assumptions that are mainly for tractability in aggregation. First, when a firm defaults and exits, it is replaced by another firm with the same identity  $j$  that inherits the defaulting capital stock. We assume this new entrant issues new debt is such that the aggregate location- $n$  value of debt outstanding is always a constant fraction of total location- $n$  capital; i.e., total location- $n$  value of debt is always  $\beta k_{n,t}$ . Alternatively, this proportionality of aggregate debt to capital could be ensured by augmenting the model with a time-varying exogenous exit rate, but allowing new entrants to issue debt in an optimal way. Either way, this set of assumptions implies it suffices to study equity.

Second, we make assumptions to avoid studying the full cross-sectional distribution of firms within a location. We assume that properties (42)–(43) also hold in the aggregate at each location, and we will presume a certain approximate aggregation on investment and investment costs. In



particular, let us define the appropriate aggregates, for capital, average Q, and investment:

$$\begin{aligned} k_{n,t} &:= \int k_{n,t}^{(j)} dj \\ q_{n,t} &:= \frac{1}{k_{n,t}} \int q_{n,t}^{(j)} k_{n,t}^{(j)} dj \\ \iota_{n,t} &:= \frac{1}{k_{n,t}} \int \iota(q_{n,t}^{(j)}) k_{n,t}^{(j)} dj. \end{aligned}$$

As an approximation, we assume the existence of functions  $(\bar{\iota}, \bar{\Phi})$  such that the following hold:

$$\bar{\iota}(q_{n,t}) \approx \int k_{n,t}^{(j)} \iota(q_{n,t}^{(j)}) dj \quad (44)$$

$$k_{n,t} \bar{\Phi}(\bar{\iota}(q_{n,t})) \approx \int k_{n,t}^{(j)} \bar{\Phi}(\bar{\iota}(q_{n,t}^{(j)})) dj. \quad (45)$$

The nature of these approximations is to say that aggregate location- $n$  investment is solely a function of aggregate average Q, rather than the full cross-sectional distribution of average Q's. Furthermore, we assume the following aggregate versions of properties (42)-(43), i.e.,

$$q_{n,t} > \bar{\Phi}'(\bar{\iota}_{n,t}) \quad (46)$$

$$\bar{\iota}'(q_{n,t}) > 0. \quad (47)$$

We conjecture these properties would go through in a full analysis of equilibrium using the cross-sectional distribution of firm size and Q, but this is beyond the scope of this paper. As we make these aggregation approximations, note that we also assume the functions  $(\bar{\iota}, \bar{\Phi})$  are independent of location  $n$ .

**Stability.** Now, we can proceed to study stability. The aggregate portfolio of location- $n$  firms' liabilities (debt plus equity) has value  $(\beta + q_{n,t})k_{n,t}$ , which is a claim to the profits  $\int (a - \Phi(\iota_{n,t}^{(j)})) k_{n,t}^{(j)} dj$ . Based on approximation (45), this aggregate profit can be approximately written  $(a - \bar{\Phi}(\bar{\iota}(q_{n,t})))k_{n,t}$ . Furthermore, the return on this portfolio is deterministic, given that all fundamental shocks are idiosyncratic (hence defaults will be idiosyncratic), and thus the return must equal the riskless bond return  $r_t$  in equilibrium. Thus,  $q_{n,t}$  evolves deterministically, and the (approximate) valuation equation states

$$\frac{a - \bar{\Phi}(\bar{\iota}(q_{n,t}))}{q_{n,t}} + \bar{\iota}(q_{n,t}) - \kappa + \frac{\dot{q}_{n,t}}{q_{n,t}} = r_t. \quad (48)$$

**Lemma 1.** Suppose the number of locations  $N$  is large enough, that approximations (44)-(45) hold, and that properties (46)-(47) hold with sufficient gaps between the left- and right-hand-sides (i.e., under-investment is large enough). Then, the equilibrium of the model with debt overhang is locally-stable.

**Proof of Lemma 1.** We start with approximate valuation equation (48). Time-differentiate  $\dot{q}_{n,t}$  with respect to  $q_{n,t}$  and  $q_{-n,t}$  to obtain

$$\begin{aligned} \frac{d\dot{q}_{n,t}}{dq_{n,t}} &= r_t + \kappa - \bar{\iota}(q_{n,t}) + \bar{\Phi}'(\bar{\iota}(q_{n,t}))\bar{\iota}'(q_{n,t}) - q_{n,t}\bar{\iota}'(q_{n,t}) + q_{n,t}\frac{dr_t}{dq_{n,t}} \\ \frac{d\dot{q}_{n,t}}{dq_{-n,t}} &= q_{n,t}\frac{dr_t}{dq_{-n,t}}. \end{aligned}$$

We will study these equations in the limit  $N \rightarrow \infty$ , which suffices, because the lemma allows us to later make  $N$  large enough.

As  $N \rightarrow \infty$ , one can show that

$$r_t \rightarrow \delta - \kappa + \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{k_{n,t}}{\sum_{i=1}^N k_{i,t}} \bar{l}(q_{n,t}),$$

which has zero derivative with respect to  $q_{i,t}$  for any  $i$ . Substituting this result for  $r_t$ , we obtain  $d\dot{q}_{n,t}/dq_{-n,t} = 0$  and

$$\frac{d\dot{q}_{n,t}}{dq_{n,t}} = \delta + \underbrace{\lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{k_{m,t}}{\sum_{i=1}^N k_{i,t}} \bar{l}(q_{m,t}) - \bar{l}(q_{n,t})}_{=0 \text{ in steady state}} - [q_{n,t} - \bar{\Phi}'(\bar{l}(q_{n,t}))] \bar{l}'(q_{n,t}).$$

The fact that the middle terms net out to zero in steady state is a consequence of the fact that  $dk_{n,t} = k_{n,t}[\bar{l}(q_{n,t}) - \kappa]dt$ , and all locations must experience the same growth rate  $\bar{l}(q_{n,t}) - \kappa$  in steady state. Thus, we will have  $d\dot{q}_{n,t}/dq_{n,t} < 0$ , hence local stability by  $d\dot{q}_{n,t}/dq_{-n,t} = 0$ , if and only if

$$[q_{n,t} - \bar{\Phi}'(\bar{l}(q_{n,t}))] \bar{l}'(q_{n,t}) > \delta.$$

This will be true if properties (46)-(47) hold with sufficient gaps, as assumed.  $\square$

### B.3 Creative destruction as a “stabilizing force”

In this section, we consider another model that allows multiplicity. We show how an overlapping generations (OLG) “perpetual youth” economy – built upon [Blanchard \(1985\)](#) – augmented with a particular type of creative destruction – similar to [Gârleanu and Panageas \(2020\)](#) – creates a stabilizing force upon which extrinsic shocks can be layered. In particular, if new firm creation is more intense when asset valuations are low, the economy possesses a natural stabilizing force. A possible rationale for this feature is that when capital asset valuations are low, they make labor look relatively attractive, which offers a robust outside option for those new entrepreneurs willing to enter. The contribution relative to [Gârleanu and Panageas \(2020\)](#) is to show how this is possible with an arbitrary number of assets (corresponding to the  $N$  locations) whose markets are, in addition, not integrated.

**Cohorts, Endowments, Markets.** In this model, all agents face a constant hazard rate of death  $\beta > 0$ , with all dying agents replaced by newborns (in the same location), so that population size is constant at 1. To keep matters simple, assume all locations have identical constant endowment growth rates and no shocks. That said, the endowment growth of an individual agent differs from the aggregate growth rate; this is the crucial ingredient in this model.

In particular, we assume some amount of *creative destruction*. The endowments of living agents decay at rate  $\kappa_{n,t}$  (obsolescence rate), while newborn agents arrive to the economy with new trees of total size  $\kappa_{n,t} + g$  (or, in per capita units, their individual trees are  $(\kappa_{n,t} + g)/\beta$  in size). Specifically, the time- $t$  endowment accruing to location- $n$  agents born at time  $s \leq t$  is

$$y_{n,t}^{(s)} = y_{n,t}(\kappa_{n,s} + g) \exp \left[ - \int_s^t (\kappa_{n,u} + g) du \right].$$

Note that the aggregate endowment follows

$$dy_{n,t} = d \left( \int_{-\infty}^t y_{n,t}^{(s)} ds \right) = y_{n,t}^{(t)} dt + \int_{-\infty}^t dy_{n,t}^{(s)} ds = \underbrace{y_{n,t}(\kappa_{n,t} + g) dt}_{\text{newborn entry}} - \underbrace{y_{n,t} \kappa_{n,t} dt}_{\text{obsolescence}} = y_{n,t} g dt.$$

For now, we leave  $\kappa_{n,t}$  unspecified, but note that its formulation will be the determinant of whether multiplicity is possible or not.

Agents can only trade in financial markets while alive. In addition to the tradability of claims to local endowments, agents can access a market for annuities that insures their death hazard and provides a stream of  $\beta w_{n,t}^{(s)}$  of income per unit of time, where  $w_{n,t}^{(s)}$  is the wealth of a location- $n$  agent born at time  $s \leq t$ . This assumption is standard in perpetual youth models.

**Solution.** Under these assumptions, one can show that agents consume  $\delta + \beta$  fraction of their wealth, so that the bond market clearing condition (3) is replaced by

$$\sum_{n=1}^N \alpha_n q_{n,t} = (\delta + \beta)^{-1},$$

where  $q_{n,t}$  is the (aggregated across cohorts) location- $n$  valuation ratio. Let  $\xi_{n,t}$  denote the location- $n$  state-price density, which follows

$$d\xi_{n,t} = -\xi_{n,t} \left[ r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} \right].$$

We will continue to assume a bubble-free equilibrium, so that

$$q_{n,t} = \mathbb{E}_t \left[ \int_t^\infty \frac{\xi_{n,\tau}}{\xi_{n,t}} \frac{y_{n,\tau}^{(s)}}{y_{n,t}^{(s)}} d\tau \right] \quad (\text{for any birth-date } s \leq t, \text{ this yields the same answer}).$$

Critically, this valuation does not incorporate wealth gains due to entry of future newborns. The dynamic counterpart of this valuation equation is, for some diffusion coefficient  $\sigma_{n,t}^q$ ,

$$\frac{dq_{n,t}}{q_{n,t}} = \left[ r_t + \kappa_{n,t} - \frac{1}{q_{n,t}} + \sigma_{n,t}^q \tilde{\pi}_{n,t} \right] dt + \sigma_{n,t}^q d\tilde{Z}_{n,t}. \quad (49)$$

The equilibrium riskless rate is obtained as follows. The goods market is integrated across locations, so the market clearing condition is given by

$$Y_t = \sum_{n=1}^N y_{n,t} = \sum_{n=1}^N \int_{-\infty}^t \beta e^{-\beta(t-s)} c_{n,t}^{(s)} ds.$$

Optimal consumption dynamics for alive agents are

$$\frac{dc_{n,t}^{(s)}}{c_{n,t}^{(s)}} = \left[ r_t - \delta + \tilde{\pi}_{n,t}^2 \right] dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t},$$

whereas newborn agents consume

$$\beta c_{n,t}^{(t)} = \underbrace{(\delta + \beta)}_{\text{cons-wealth ratio}} \times \underbrace{(\kappa_{n,t} + g)y_{n,t}q_{n,t}}_{\text{newborn wealth}}.$$

Time-differentiating goods market clearing, and using these results, we obtain

$$r_t = \delta + \beta - \sum_{n=1}^N x_{n,t} \tilde{\pi}_{n,t}^2 - (\delta + \beta) \sum_{n=1}^N \alpha_n q_{n,t} \kappa_{n,t}. \quad (50)$$

**Stability.** To see how the stabilizing force works, it is instructive to once again study the deterministic equilibrium in which extrinsic shocks have no volatility. Substituting (50) into (49) with  $\sigma_{n,t}^q = 0$ , we obtain

$$\dot{q}_{n,t} = \underbrace{-1 + (\delta + \beta)q_{n,t}}_{\text{unstable component}} - \underbrace{\left[ (\delta + \beta) \sum_{i=1}^N \alpha_i q_{i,t} \kappa_{i,t} - \kappa_{n,t} \right] q_{n,t}}_{\text{stabilizing force}} \quad \text{when } \sigma_{i,t}^q = 0 \quad \forall i. \quad (51)$$

The first piece is the unstable component, propelling valuations further and further away from the “steady state” value  $(\delta + \beta)^{-1}$ . The second piece—capturing the relative amount of creative destruction in location  $n$ —is the stabilizing force.

Based on equation (51), we claim that if  $\kappa_{n,t}$  decreases sufficiently rapidly as  $q_{n,t}$  increases, then valuation dynamics are stable. Let  $\kappa_{n,t} = \kappa(q_{n,t})$  for a decreasing function  $\kappa(\cdot)$ . Denote the steady-state mean and sensitivity of this function by  $\bar{\kappa} := \kappa((\delta + \beta)^{-1})$  and  $\lambda := -\kappa'((\delta + \beta)^{-1})$ , respectively. Then, compute

$$\left. \frac{\partial \dot{q}_n}{\partial q_m} \right|_{q_i = (\delta + \beta)^{-1} \forall i} = \begin{cases} \delta + \beta - \lambda(\delta + \beta)^{-1}(1 - \alpha_n) - \alpha_n \bar{\kappa}, & \text{if } m = n; \\ \lambda(\delta + \beta)^{-1} \alpha_m - \alpha_m \bar{\kappa}, & \text{if } m \neq n. \end{cases}$$

Construct the steady-state Jacobian matrix as

$$J := \left[ \left. \frac{\partial \dot{q}_n}{\partial q_m} \right|_{q_i = (\delta + \beta)^{-1} \forall i} \right]_{1 \leq n, m \leq N}. \quad (52)$$

Local stability of the steady-state can be determined by the eigenvalues of  $J$ . By the Gershgorin circle theorem, all of these eigenvalues will have strictly negative real parts if  $J$  has negative diagonal elements and is diagonally dominant. This is easily guaranteed by making  $\bar{\kappa}$  and  $\lambda$  large enough, meaning the amount of creative destruction and its sensitivity to prices are both large enough. The result is summarized in the following lemma, with the proof omitted.

**Lemma 2.** *Assume  $\bar{\kappa} > \delta + \beta$  and  $\lambda > (\delta + \beta)\bar{\kappa}$ . Then, all eigenvalues of  $J$  have strictly negative real parts. Consequently, the equilibrium of the creative destruction model is locally stable.*

## C Other empirical proxies for volatility

Recall our baseline choice for  $\hat{\sigma}_t^*$  in (23) uses only the monthly volatility of daily price changes for the 10-year US Treasury Note. This measure is simple and transparent but has several drawbacks: (a) it is not a very precise estimate of time- $t$  conditional volatility; (b) it includes no volatility information for the synthetic US notes, which constitute the other leg of the CIP trade; and (c) unlike the model, in which the assets used to construct  $A_t$  correspond to those used to measure  $\sigma_t^*$ , this proxy uses long-maturity notes instead of the 3-month bills that make up  $\hat{A}_t^{(1/4)}$ .

To help address concerns (a)-(c), we also consider three alternative measures of  $\hat{\sigma}_t^*$ .

- (a) For a more real-time measure of conditional volatility, we examine the CBOE's 10-year Treasury VIX (TYVIX), which is the implied 30-day volatility of CBOT futures on 10-year US Treasury Notes. The disadvantage of asset-implied volatility is that it corresponds to risk-neutral volatility and may be a biased estimate of actual volatility.
- (b) To include information for the foreign leg, we also compute a foreign volatility analogue to (23). In particular, we compute the volatility of the 10-year constant maturity foreign note's daily price changes, measured in USD (i.e., the bond prices are adjusted by the spot exchange rate each day). We then take a value-weighted average of the 10-year Constant Maturity US Treasury Note volatility and this foreign note volatility, which delivers a country-specific volatility measure. The disadvantage of this construction is that it introduces additional variation by using the spot exchange rate to convert future prices to dollars, rather than the forward exchange rate as in the CIP trade.
- (c) To bring the assets in the volatility construction as close as possible to those used in the arbitrage trade, we examine the value-weighted average return volatilities of the 3-month US Treasury bill and the 3-month synthetic US bill. The disadvantages of using this proxy are twofold. First, because we have no way of interpolating the forward exchange rate curve between the 3-month and 1-month forward rates, we construct 2-month holding period returns on both bills. The prices of these bills are constructed using country-specific IBOR. This holding period is very long relative to the maturity of the bill, and so our estimate of a conditional volatility is likely to be highly imprecise. Second, mainly due to their short durations, 3-month bill volatilities are mechanically much smaller than those of 10-year notes (approximately 40 times smaller, as their relative durations suggests). Whereas the values for  $\sigma_t^*$  and  $\sqrt{A_t}$  are roughly on the same scale in equation (19)—which comes from a setting with infinitely-lived assets—3-month bill volatility is not on a scale comparable to the 3-month CIP deviation.

We repeat our analysis with these proxies. Aggregate time series plots associating volatility to arbitrage profits are below in Figures 5, 6, and 7. Disaggregated analysis at the currency level are below in Figures 8, 9, and 10.

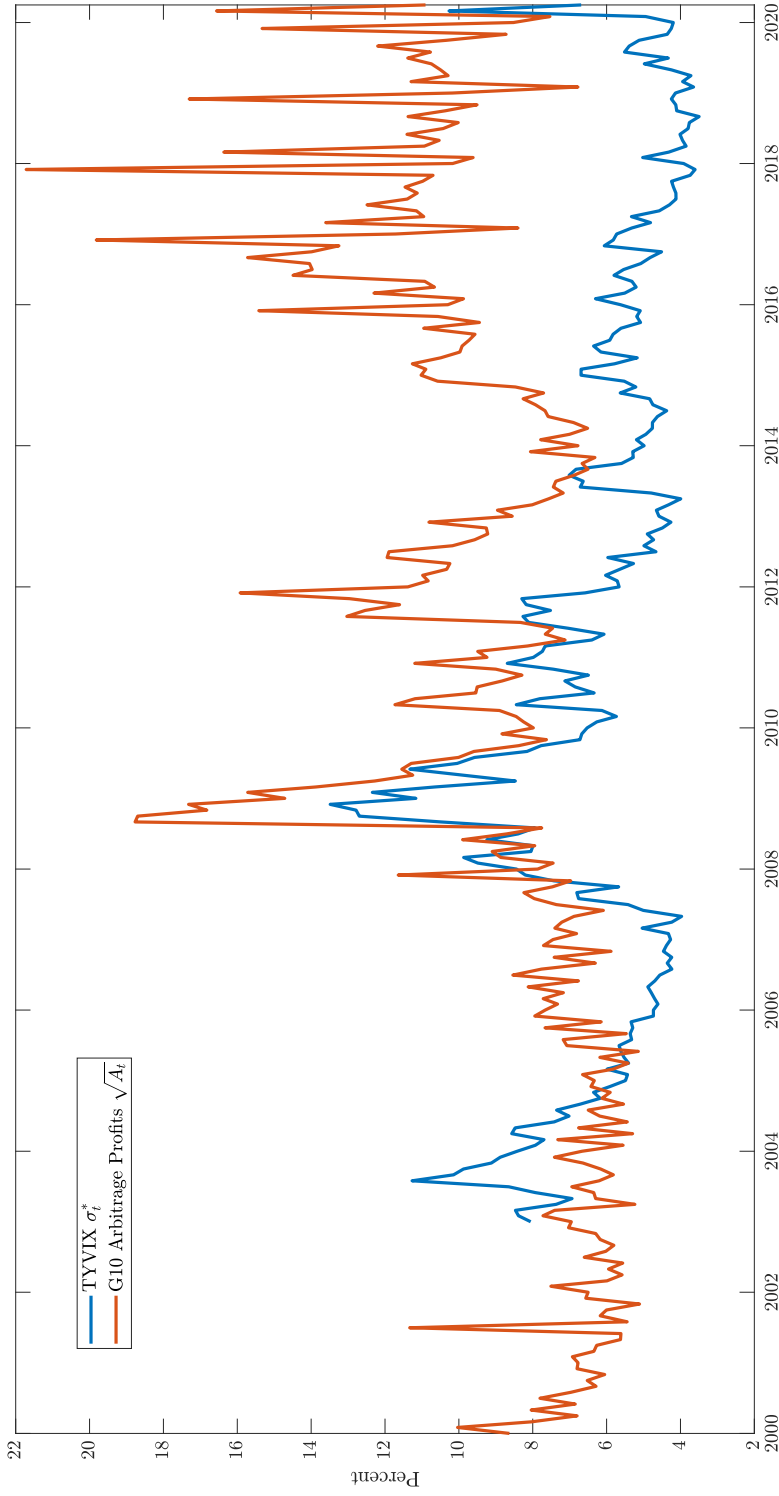


Figure 5: Monthly time-series of proxies for  $A_t$  and  $\sigma_t^*$ . Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is constructed using the monthly average of TYVIX, which is the implied volatility of CBOE options on 10-year US Treasury Note futures. Both measures are annualized. Currency data are from Bloomberg and range from Jan. 2000 to Apr. 2020. TYVIX data are from the CBOE and range from Jan. 2003 to Apr. 2020.

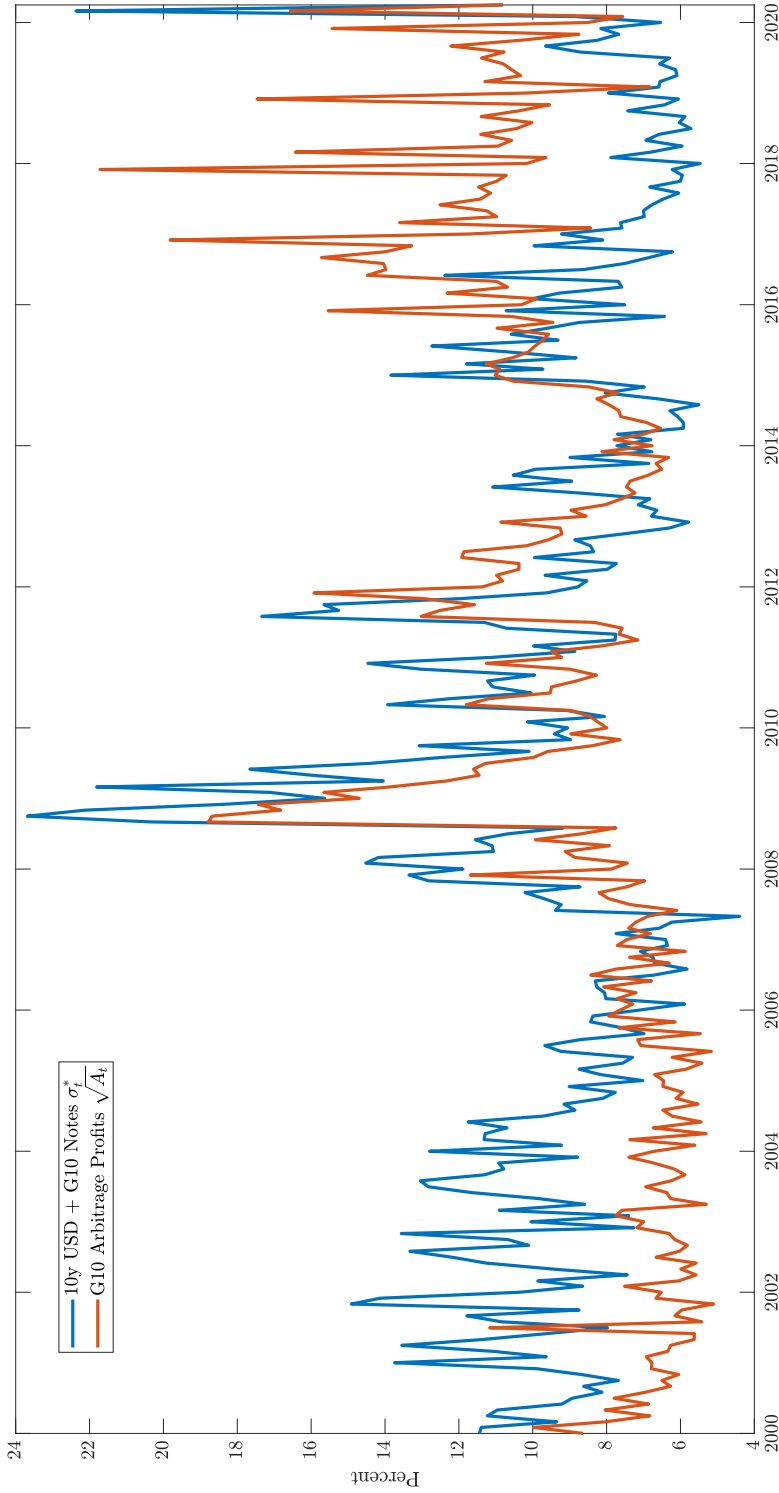


Figure 6: Monthly time-series of proxies for  $A_t$  and  $\sigma_t^*$ . Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is constructed using the value-weighted average volatilities of daily log price changes on (i) the 10-year Constant Maturity US Treasury Note; and (ii) one of the G10's 10-year constant maturity note, adjusted by the respective spot exchange rate to be measured in USD. Volatility  $\sigma_t^*$  is calculated as a within-month standard deviation of these daily price changes, then (simple) averaged across the G10 currencies. Both measures are annualized. Currency and foreign notes data are from Bloomberg, whereas 10-year US Treasury Note data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

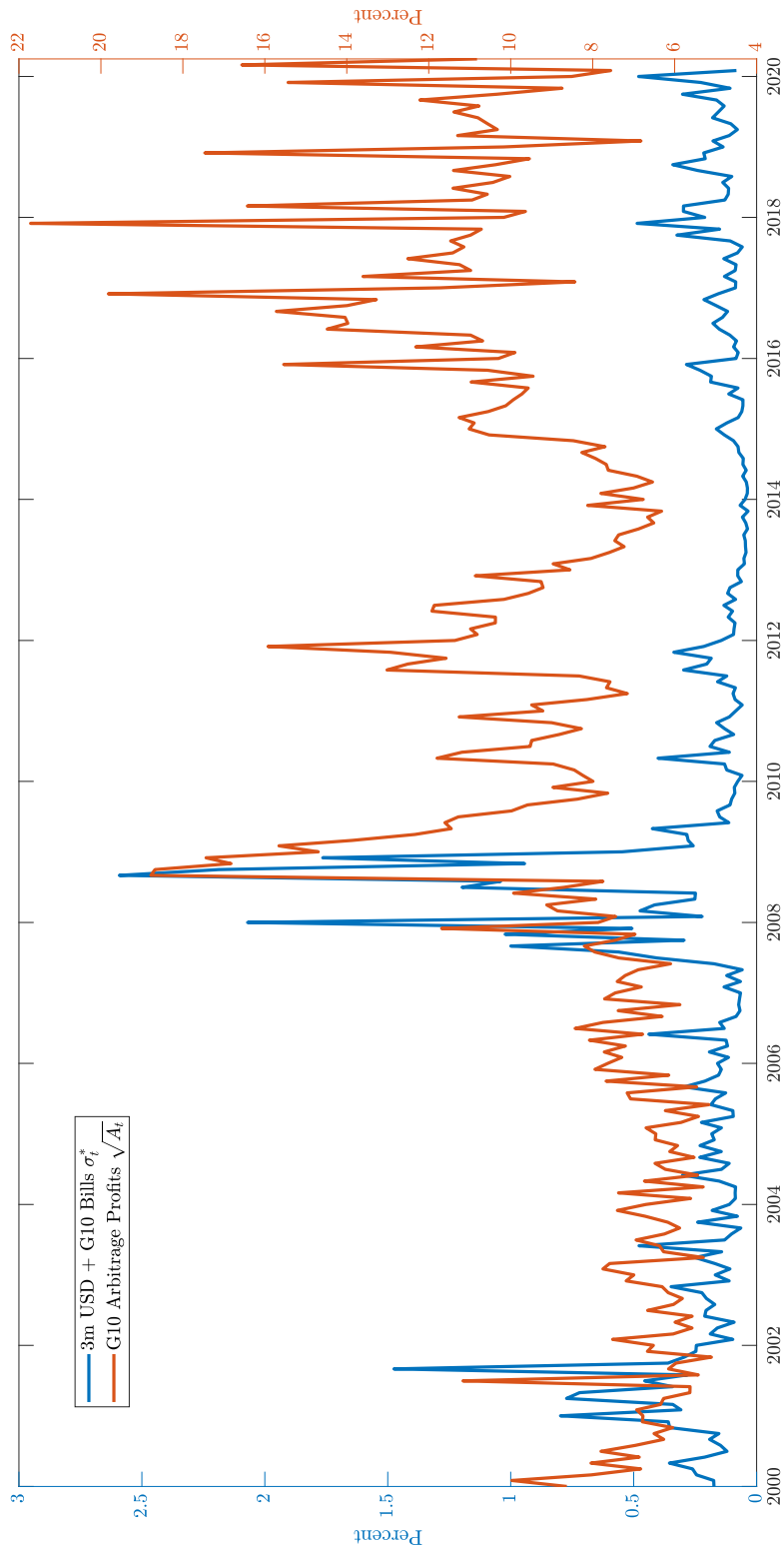


Figure 7: Monthly time-series of proxies for  $A_t$  and  $\sigma_t^*$ . Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly; then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is constructed using the value-weighted average volatilities of next-2-month returns on (i) 3-month US bills; (ii) 3-month synthetic US bills, built using foreign bills, spot and forward exchange rates. The 2-month returns are measured at a daily frequency; then volatility is calculated as a within-month standard deviation of these daily measures, then these measures are (simple) averaged across the G10 currencies. Both measures are annualized. Currency, foreign bills, and US bills data are from Bloomberg. Data range from Jan. 2000 to Apr. 2020.



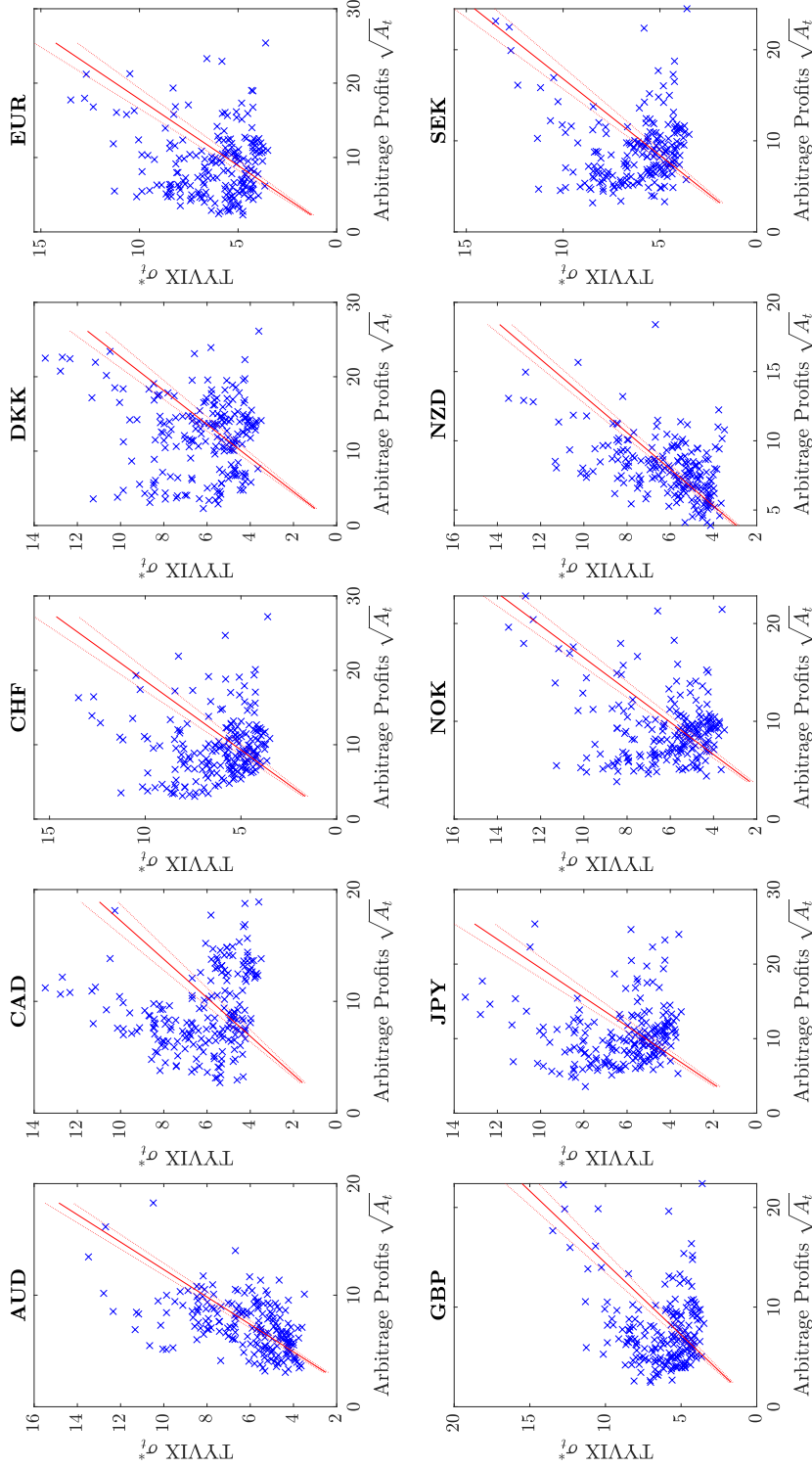


Figure 8: Currency-level OLS regressions of  $\sigma_t^*$  on  $\sqrt{A_t}$  (monthly, no intercept). Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly; then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is constructed using the monthly average of TYVIX, which is the implied volatility of CBOE options on 10-year US Treasury Note futures. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency data are from Bloomberg and TYVIX data are from the CBOE. Data range from Jan. 2003 to Apr. 2020.

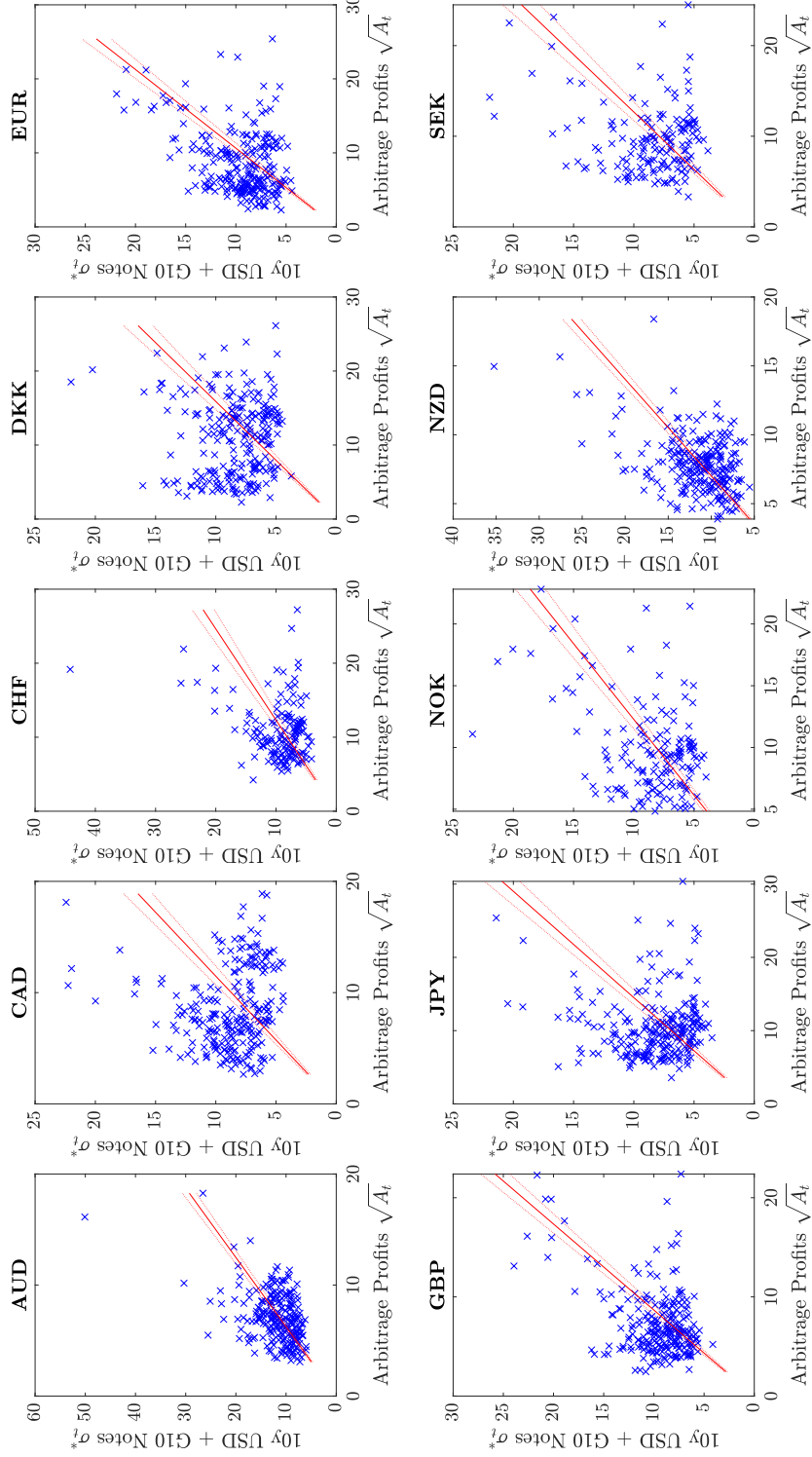


Figure 9: Currency-level OLS regressions of  $\sigma_t^*$  on  $\sqrt{A_t}$  (monthly, no intercept). Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is constructed using the value-weighted average volatilities of daily log price changes on (i) the 10-year Constant Maturity US Treasury Note; and (ii) one of the G10's 10-year constant maturity note, adjusted by the respective spot exchange rate to be measured in USD. Volatility  $\sigma_t^*$  is calculated as a within-month standard deviation of these daily price changes, then (simple) averaged across the G10 currencies. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency and foreign notes data are from Bloomberg, whereas 10-year US Treasury Note data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

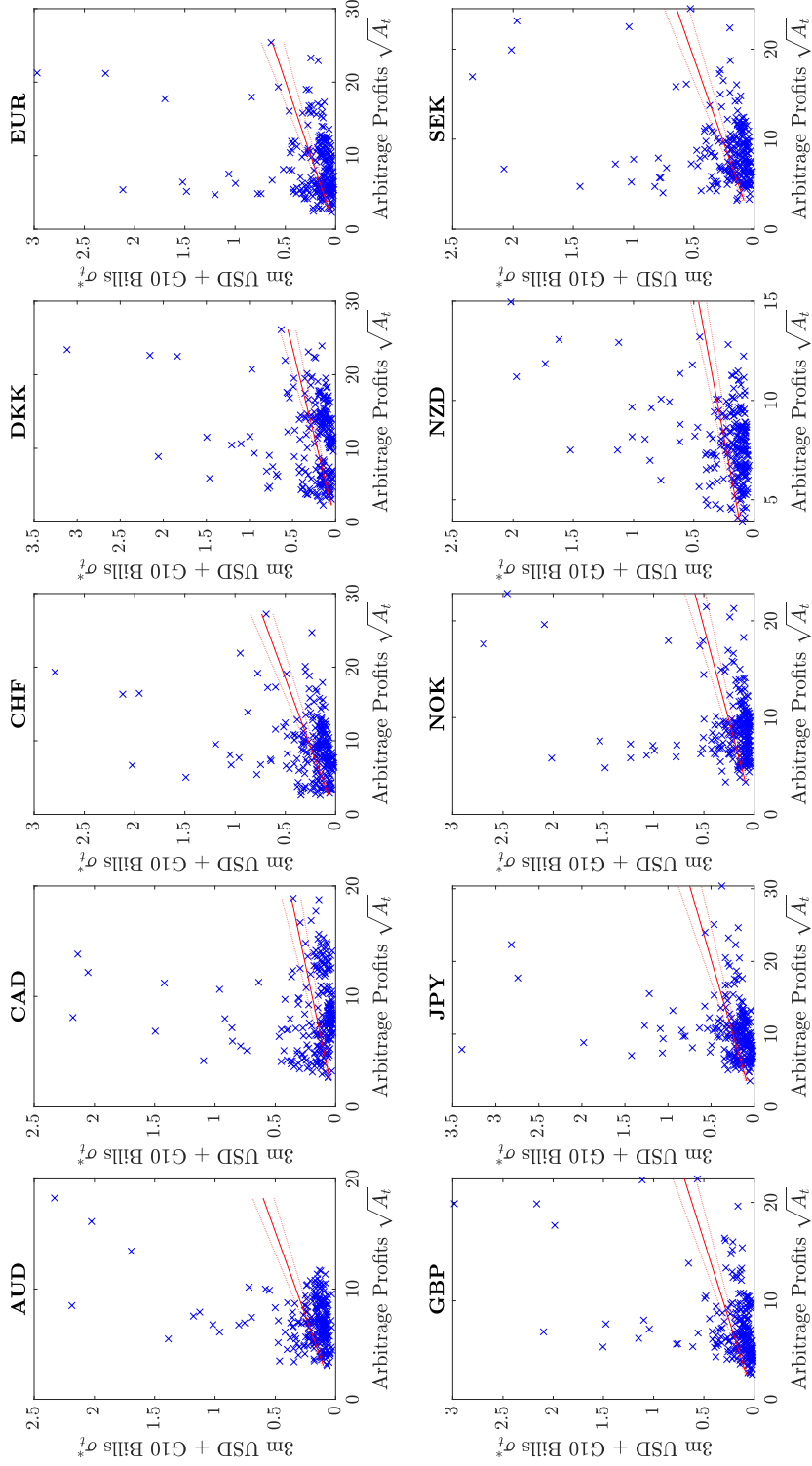


Figure 10: Currency-level OLS regressions of  $\sigma_t^*$  on  $\sqrt{A_t}$  (monthly, no intercept). Proxy for  $A_t$  is  $\hat{A}_t^{(1/4)}$  from (22), which is constructed using 3-month absolute CIP deviations (against the USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for  $\sigma_t^*$  is constructed using the value-weighted average volatilities of next-2-month returns on (i) 3-month US bills; (ii) 3-month synthetic US bills, built using foreign bills, spot and forward exchange rates. The 2-month returns are measured at a daily frequency; volatility is calculated as a within-month standard deviation of these daily measures, and then these measures are (simple) averaged across the G10 currencies. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency, foreign bills, and US bills data are from Bloomberg. Data range from Jan. 2000 to Apr. 2020.