Arbitrage and Beliefs

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Abstract

At times when arbitrage opportunities exist, asset markets are susceptible to self-fulfilling volatility. Conversely, when investor beliefs of future price changes lead to their own fulfillment, arbitrage opportunities must exist. The only requirement is that such beliefs involve asset price movements redistributing wealth across markets (e.g., equities rise as bonds fall). The size of arbitrage profits and speed of capital mobility quantitatively discipline the amount of self-fulfilling fluctuations. The tight theoretical connection between price volatility and the presence of arbitrage is detectable in currency markets by studying deviations from covered interest parity.

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1 Introduction

Arbitrages exist. There are many documented examples of trades featuring positive profits with zero hold-to-maturity risk. Moving beyond the neoclassical frictionless model of financial markets, a theoretical literature emphasizing *limits to arbitrage* aims to rationalize such trades.[1]

We prove that a canonical limits-to-arbitrage setting features self-fulfilling volatility. Fundamental payoffs (dividends) are deterministic, meaning any asset-price volatility is excess volatility. Our result holds with infinitely-lived rational agents in an externality-free, bubble-free economy with dynamically-complete, but imperfectly-integrated, financial markets. Besides the fact that markets are not perfectly integrated, there are no other frictions or constraints – agents are always marginal in their local asset markets.

The mechanism is simple. In our segmented-markets world, agent A only trades asset A, whereas agent B only trades asset B. Suppose the price of asset A declines randomly, for extrinsic reasons. Having less wealth after the shock, agent A will want to cut consumption. To do this, agent A saves a bit of the cash flows from asset A in the bond market. But by market clearing, agent B must be borrowing this amount, consuming more than the cash flows of asset B. This consumption plan is only optimal if agent B believes his wealth has increased, requiring assets A and B to experience *equal and opposite* extrinsic shocks. This sequence of logic is reflected in figure 1 below. As long as the bond market is integrated and this redistributive condition holds, nothing in a segmented-markets world rules out arbitrarily large extrinsic shocks. The conditions for boundless self-fulfilling volatility are the essence of Theorem 1 of our paper.

But if shocks to assets A and B are offsetting, one can construct a riskless portfolio containing both. This portfolio must generate *arbitrage profits*. The reason: agent A demands a risk premium on the extrinsic volatility of asset A, and similarly for B, so a portfolio that buys positive amounts of A and B will earn more than the riskless rate. Theorem 2 demonstrates how arbitrage profits and self-fulfilling volatility are two sides of the same coin. In fact, a natural and correct conjecture is that the size of arbitrage profits are directly linked to the amount of self-fulfilling volatility (Proposition 1).

Normally, the presence of large arbitrage profits incentivizes trading by relative-value traders. Enter investment fund F. If fund F can trade freely across markets, we return to a neoclassical world without arbitrage and without self-fulfilling volatility. But if fund F encounters the type of frictions articulated by the limits-to-arbitrage literature (e.g., margin requirements, search frictions, myopic performance-based investors), some

[1] See related literature section for empirical examples and theoretical antecedants.

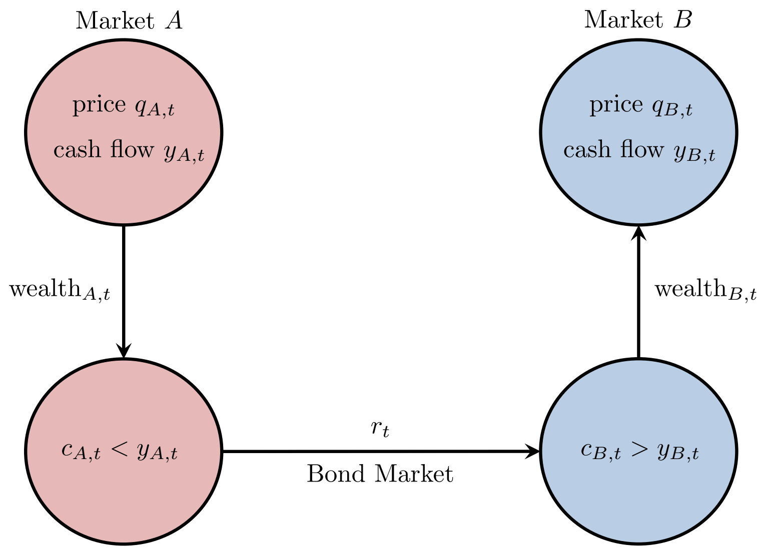

Figure 1: Mechanism of the model. Market A experiences a negative shock to its asset price $q_{A,t}$, without any effect on its cash flow $y_{A,t}$. Optimal consumption $c_{A,t}$ of agent A wants to fall below the cash flow to reflect lower wealth, which requires saving in the bond market at rate r_t. By bond market clearing, consumption $c_{B,t}$ of agent B must be higher than her local asset cash flow $y_{B,t}$. This is only optimal if market B asset prices $q_{B,t}$ rise in a manner that offsets the decline in $q_{A,t}$.

arbitrage profits will be left on the table. By extension, self-fulfilling volatility will not be fully eliminated. Quantitatively, the magnitude of limits-to-arbitrage frictions disciplines the magnitude of viable self-fulfilling volatility (Proposition 2 and Corollary 3).

These theoretical results on the existence of self-fulfilling volatility continue to hold in a variety of alternative environments. We add fundamental aggregate shocks (Proposition 4) and fundamental idiosyncratic shocks (Proposition 5). This extension allows us to avoid claiming that all asset-price volatility is extrinsic. We can also modify the model to suit the international finance literature: agents have "home bias" over their local goods and are restricted to trading their local equity (Propositions 6, 7, 8).

Empirically, we examine the hypothesized link between arbitrage profits and volatility in the context of covered interest parity (CIP) deviations. The 3-month CIP deviation serves as our primary measure of available arbitrage profits. The arbitrage strategy borrows in USD and goes long a synthetic US bond built with a foreign bond and currency swaps (or vice versa if the CIP deviation is negative). Volatility is proxied by the weighted-average return volatility on these two legs (the US bond and the synthetic US bond). We obviously cannot determine if volatility is of a self-fulfilling nature or not, which makes our test imperfect. Still, we detect a strong association between the CIP deviation and the weighted-average volatility of the two legs.

In the paper, we sometimes use the longer-maturity counterparts of these legs (10-

year bonds as opposed to 3-month bonds). This is a way to make a duration-adjustment that brings our volatility magnitude closer to what the model calls for. We find very strong positive associations between CIP deviations and the 10-year Treasury VIX, 10-year US bond return volatility, and foreign bond return volatility.

Related literature. Our paper is most closely related to the theoretical literature on limits to arbitrage. See Shleifer and Vishny (1997) for a micro-foundation for mis-pricing due to "performance-based arbitrage." See Gromb and Vayanos (2002) for a margin-based analysis of price deviations in multiple identical markets. See Garleanu and Pedersen (2011) for dynamic asset pricing model with margin constraints. See Duffie and Strulovici (2012) for a model in which slow-moving capital arises due to search frictions. See Vayanos and Weill (2008) for a search friction application to the "on-the-run / off-the-run" bond phenomenon. See Duffie (2010) and Gromb and Vayanos (2010) for further reviews of the literature and existing mechanisms.

Empirically, there are a plethora of documented arbitrage trades: examples include spinoffs (Lamont and Thaler, 2003); "on-the-run / off-the-run" bonds (Krishnamurthy, 2002); covered interest parity (Du, Tepper and Verdelhan, 2018); Treasury spot and future repo rates (Fleckenstein and Longstaff, 2018).

These empirical examples are analyzed for clarity, in the sense that matching cash flows leaves only frictions to explain price differences. Although they may be more difficult to identify empirically, similar frictions may pervade other markets. For example, Hu, Pan and Wang (2013) suggest that hedge fund capital modulates the closeness of the yield curve to no-arbitrage models. The idea is that there is some market segmentation between Treasuries of different maturities, perhaps because investors have maturity-specific "habitats" in which they like to focus. In such a world, if arbitrageur capital is somewhat limited, prices may deviate from the no-arbitrage benchmark. This may even be true in riskier markets: Ma (2019) suggests that corporate bond and equity markets may be partially segmented, with corporate issuances and buybacks acting as a mechanism to profit from price differences.

Our paper also relates to, but is distinct from, the literature on self-fulfilling dynamics. First, most models of self-fulfilling fluctuations have at their core either overlapping generations (with the resulting possibility of bubbles)[2] or aggregate increasing returns[3], of which we have neither. Moreover, without belief shocks, we have a unique steady state which is locally unstable, ruling out traditional forms of indeterminacy. Typical

[2]See Azariadis (1981), Cass and Shell (1983), and Farmer and Woodford (1997) [originally published in 1984] for early models with two-period lifetimes.

[3]See Farmer and Benhabib (1994).

self-fulfilling stochastic equilibria build "sunspot shocks" around a locally-stable steady state (essentially lotteries on the multiplicity of deterministic transition paths).[4]

Focusing on asset prices, our paper is closer to Hugonnier (2012), Gârleanu and Panageas (2019), and Zentefis (2020). As in those models, our multiplicity arises when there are multiple traded assets and some limits to arbitrage between them. Multiple assets is crucial in the sense that shocks to one asset class must be offset by the others in order to keep aggregate wealth smooth. Where we depart from these three papers is our notion of limits to arbitrage, which drives a distinction in our results and interpretation.

Hugonnier (2012) generates multiplicity from the presence of an aggregate bubble, which arises in stockholder-bondholder economies, and can be sub-divided arbitrarily to redistribute wealth between asset classes.[5] Our markets are segmented cross-sectionally, so we do not require such a bubble to obtain indeterminacy.

More related is the no-bubble OLG economy of Gârleanu and Panageas (2019), which also has cross-sectionally segmented markets, in the sense that unborns have a disproportionate claim to human capital but cannot trade before birth. Multiplicity arises through wealth redistribution across overlapping generations, as extrinsic stock market shocks are offset by human capital shocks. They interpret this as a volatile aggregate stock market, whereas our equilibrium is better interpreted as self-fulfilling volatility in relative-value trades (e.g., basis trades).[6]

Finally, Zentefis (2020) demonstrates that leverage constraints can generate interesting self-fulfilling price dynamics in multi-asset models. Leverage cycles and boom-bust dynamics become natural outcomes in that model. Our contribution is to isolate the role of market segmentation.

The paper is organized as follows. Section 2 presents the model. Section 3 analyzes the central link between the presence of arbitrage opportunities and self-fulfilling volatility. Section 4 presents model extensions. Section 5 studies the key model prediction in the context of currency markets. Section 6 concludes. The Appendix contains the proofs and further analysis.

[4] Benhabib and Farmer (1999) reviews this class of models in macroeconomics. Farmer (2016) discusses the intellectual history and compares to newer models in which a continuum of steady states arises.

[5] That said, the necessity of an aggregate bubble in limited participation economies is fragile, in the sense that *any* amount of entry by non-participants, no matter how tiny, eliminates the bubble (Khorrami, 2018) and would thus eliminate this multi-sector indeterminacy.

[6] Gârleanu and Panageas (2019) also generate volatility in the traditional way, by randomizing over a multiplicity of deterministic transition paths, similar to papers cited above and different to us. There are two other related papers. Farmer (2018) studies capital asset prices in an OLG economy, with nominal government debt (rather than human capital) as the second asset allowing redistribution. Bacchetta et al. (2012) also has an OLG economy, with the uncleared riskless bond market providing the second asset allowing redistribution (to unmodeled foreigners).

2 Model

Setup. The model is set in continuous time with $t \geq 0$. For simplicity, we assume here that the setting is deterministic, though below we show that our analysis applies to a general environment with both aggregate and idiosyncratic shocks. An aggregate endowment follows

$$dC_t = gC_t dt.$$

The economy features N locations, where each one can stand for a sector, industry, country, or distinct financial market. Each location offers a single tradeable asset in positive net supply. The cash flows for asset $n \in \{1, \ldots, N\}$ are $\alpha_n C_t$, where $\alpha_n > 0$ are constant fractions and $\sum_{n=1}^{N} \alpha_n = 1$. The equilibrium price of asset n is $q_{n,t}\alpha_n C_t$, where $q_{n,t}$ is the price-dividend ratio.

Each location has a different representative agent. Each agent can invest only in his or her local asset market and a zero-net-supply short-term bond market that is open to everyone. Later we show the effects of both fully and partially opening trade across financial markets. The equilibrium risk-free rate in the integrated bond market is r_t.

All agents have rational expectations, infinite lives, logarithmic utility, and discount rate $\delta > 0$. Mathematically, their preferences are

$$\mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\log(c_{n,t})dt\Big].$$

Clearing of the goods and bond markets is standard: $\sum_{n=1}^{N} c_{n,t} = C_t$ and $\sum_{n=1}^{N} q_{n,t}\alpha_n C_t = Q_t C_t$, where Q_t is the aggregate price-dividend ratio.

Extrinsic Shocks. With market clearing established, we next describe asset prices. In a deterministic economy, any stochastic price changes must inherently originate from agents' self-fulfilling beliefs. To allow for this volatility, we conjecture that the price-dividend ratio of each location's asset follows a stochastic process

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t}dt + \sigma^q_{n,t}d\tilde{Z}_{n,t}\Big], \tag{1}$$

where $\tilde{Z}_{n,t}$ is a one-dimensional Brownian motion. The economy has no intrinsic uncertainty. This shock is therefore *extrinsic*, and it is the source of any possible self-fulfilling asset price volatility. Let $\tilde{Z}_t := \big(\tilde{Z}_{n,t}\big)_{n=1}^{N}$ be a vector of all locations' extrinsic shocks.

Economically, the extrinsic $\tilde{Z}$ shocks arise from sources that we do not explicitly model. Investor sentiment or signals that coordinate beliefs might trigger the self-fulfilling fluctuations, in a manner similar to Benhabib et al. (2015). Heterogeneity in opinions between optimists and pessimists akin to Scheinkman and Xiong (2003) can be another source. Correlated institutional demand shocks as described in Koijen and Yogo (2019) can yet be another driver of the price changes.

We allow the extrinsic shocks in the economy to obey an arbitrary correlation structure. A convenient way to represent this structure uses an N-dimensional basis of uncorrelated Brownian motions $Z_t := (Z_{n,t})_{n=1}^{N}$ and an $N \times N$ matrix of constants M that captures their relation. From these two components, we recast the vector of extrinsic shocks as

$$\tilde{Z}_t = MZ_t. \tag{2}$$

The matrix M is normalized so that $\operatorname{diag}[MM'] = (1, \ldots, 1)'$, which preserves $\tilde{Z}_t$ as a collection of Brownian motions. Substituting equation (2) into equation (1) shows that the self-fulfilling shock to asset n at time t is $\sigma^q_{n,t} M_n dZ_t$, where M_n is the n-th row of M.[7]

The matrix M is a crucial parameter of the model. To illustrate its structure, we consider the following examples, which we repeatedly use throughout the text.

Example 1 (Uncorrelated shocks)**.** Suppose M is the identity matrix. This structure implies $\tilde{Z}_t = Z_t$, which renders all extrinsic shocks uncorrelated.

Example 2 (Two-by-two redistribution)**.** Suppose $N = 2$ and let

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

This example presents a setting with two locations and one source of extrinsic uncertainty. The matrix M puts $\tilde{Z}_{1,t} = -\tilde{Z}_{2,t}$, which implies that the self-fulfilling price changes redistribute wealth between the two assets. As one price falls, the other rises.

Example 3 (General redistribution)**.** This example is the N-dimensional counterpart to example 2. Let $\tilde{M}$ be an $N \times N$ non-singular matrix. Suppose

$$M = \tilde{M} - \frac{1}{N}\mathbf{1}'\tilde{M} \otimes \mathbf{1}.$$

[7] Although markets are incomplete in the model, they are dynamically complete. The vector $\tilde{Z}_{n,t} = M_n Z_t$ is generated by N distinct shocks, but it suffices for agent n to only trade $\tilde{Z}_{n,t}$, which is the shock his local asset loads on. Indeed, if we introduce in each market zero-net-supply Arrow securities spanning Z that are traded only in market n, the equilibrium remains unchanged.

In this structure, each element of the matrix $\tilde{M}$ is reduced by the simple average of its columns. This operation makes the column sums of M equal zero. The key consequence of this design is that $\mathbf{1}'d\tilde{Z}_t = \mathbf{1}'MdZ_t = 0$ almost-surely. Any other linear combination of $d\tilde{Z}_t$ does not equal 0. As a result, $\text{rank}(M) = N - 1$.

3 Self-fulfilling volatility

Here we present the core theoretical results. Section 3.1 gives the conditions when self-fulfilling asset price volatility emerges. Section 3.2 shows that the presence of this volatility and the existence of arbitrage opportunities are two sides of the same coin. Section 3.3 demonstrates that limits to arbitrage quantitatively discipline the amount of self-fulfilling price changes.

3.1 Conditions for self-fulfilling volatility

The following theorem characterizes when self-fulfilling volatility emerges and when it does not. A detailed explanation comes thereafter. The proof is in Appendix A.

Theorem 1. *Let $N \geq 2$. If $\text{rank}(M) = N$ (full rank), then equilibrium cannot have self-fulfilling volatility: $(\sigma^q_{1,t}, \ldots, \sigma^q_{N,t}) \equiv 0$ for all t. Conversely, if $\text{rank}(M) < N$, then there can be arbitrary self-fulfilling volatility: for any non-negative process $\{\psi_t\}_{t\geq 0}$ adapted to $\{Z_t\}_{t\geq 0}$, an equilibrium can be sustained with $\alpha_n q_{n,t}\sigma^q_{n,t} = v^*_n \psi_t$ for all n, where v^* is in the null-space of M'.*

Note that the self-fulfilling volatility in Theorem 1 is essentially arbitrary. Indeed, there are no restrictions on the process ψ_t. Volatility may be a non-Markovian process, and it may vanish for a period of time before re-emerging spontaneously. This is unlike much of the rational bubble and multiplicity literature (Santos and Woodford, 1997).[8] Later, by introducing some partial cross-market arbitrage behavior, we will place explicit restrictions on what the volatility process ψ_t can be.

Under what conditions can such self-fulfilling volatility emerge? Theorem 1 says that some linear dependence in the extrinsic shocks $\tilde{Z}_t$ is required. Extrinsic shocks cannot be uncorrelated as in example 1. The intuition is as follows. Individuals with log utility consume δ fraction of their wealth, so the aggregate wealth-consumption (price-

[8]This differs from resale premia that are termed "bubbles" in Scheinkman and Xiong (2003).

dividend) ratio is $Q_t = \delta^{-1}$. Bond market clearing can then be written as

$$\sum_{n=1}^{N} \alpha_n q_{n,t} = \delta^{-1}. \tag{3}$$

Because the aggregate wealth-consumption ratio is constant, extrinsic shocks to $q_{n,t}$ can occur, but they must be offset by extrinsic shocks to other assets. Offsetting is just another way to say "linear dependence" mathematically, or "redistribution" economically. The requirement for such offsetting shows clearly why multiple assets are needed.

But what does this offsetting have to do with rank(M)? By time-differentiating equation (3), we see that the loadings on each of the basis dZ_t shocks must be zero:

$$\sum_{n=1}^{N} \alpha_n q_{n,t} \sigma_{n,t}^{q} M_n = 0. \tag{4}$$

Write equation (4) as a matrix equation

$$M' v_t = 0, \tag{5}$$

where $v_t = (\alpha_1 q_{1,t} \sigma_{1,t}^{q}, \ldots, \alpha_N q_{N,t} \sigma_{N,t}^{q})'$ is the column vector of volatilities. Note that $\alpha_n > 0$ and $q_{n,t} > 0$. Therefore, it is straightforward to reach the conclusion of Theorem 1. If M were full rank, the unique solution to (5) would be $v_t \equiv 0$. However, the singular situation rank$(M) < N$ implies a non-zero time-invariant solution $v_t \equiv v^* \neq 0$ exists. If so, then $\psi_t v^*$ also solves (5) for any non-negative scalar process ψ_t. Hence, we have a continuum of equilibria indexed by $\{\psi_t : t \geq 0\}$.

This is essentially a proof sketch for Theorem 1, but all additional technical details are contained in Appendix A. To build further understanding, we now remark on several features of the equilibrium in Theorem 1.

First, our self-fulfilling volatility is different from the extant multiplicity and sunspots literature (Azariadis, 1981; Cass and Shell, 1983; Farmer and Woodford, 1997). In the extant literature, self-fulfilling stochastic equilibria are built using "sunspot shocks" around a locally-stable deterministic steady state, which is possible because of the multiplicity of potential transition paths to that steady state. To differentiate ourselves, we now demonstrate how our model has a unique steady state which is unstable.

Remark 1 (Unstable steady state)**.** Consider any non-singular matrix M (e.g., $M = I_N$ as in example 1), which implies zero self-fulfilling volatility by Theorem 1. Consequently, there is no risk compensation, and all assets must earn the riskless rate, i.e.,

$$\underbrace{\dot{q}_{n,t}/q_{n,t} + g}_{\text{capital gain}} + \underbrace{1/q_{n,t}}_{\frac{\text{div}}{\text{price}}} = r_t. \tag{6}$$

Furthermore, as the unique equilibrium involves deterministic individual consumption paths, the interest rate is solely determined by time-discounting and economic growth, i.e., $r_t = \delta + g$ (see equation (36) in Appendix A). Substituting this into (6), we have $\dot{q}_{n,t} = -1 + \delta q_{n,t}$, a dynamical system that has a single steady state which is unstable. By (3), price-dividend ratios are constant at $q_{n,t} = \delta^{-1}$ for all n, t. There is no opportunity to add sunspot shocks in this model.

Second, any self-fulfilling volatility is compensated. Agent n holds exposure to the extrinsic shock $\tilde{Z}_{n,t}$ through his exposure to $q_{n,t}$. If we define $\tilde{\pi}_{n,t}$ as the risk price (or Sharpe ratio) associated to this shock, and define the consumption shares $x_{n,t} := c_{n,t}/C_t$, then

$$\tilde{\pi}_{n,t} = \delta\Big(\frac{\alpha_n q_{n,t}}{x_{n,t}}\Big)\sigma^q_{n,t}. \tag{7}$$

Intuitively, $\alpha_n q_{n,t} C_t \sigma^q_{n,t}$ is the total exposure to $\tilde{Z}_{n,t}$ shocks, and $\delta^{-1} x_{n,t} C_t$ is the wealth of agent n, who bears these shocks. With log utility, the required compensation for Brownian shocks is the per-unit-of-wealth exposure, so dividing these two reveals the risk price. Formula (7) also shows, in conjunction with Theorem 1, that Sharpe ratios are linked to self-fulfilling volatility; in fact, $\tilde{\pi}_{n,t}$ is proportional to $\delta\psi_t/x_{n,t}$.

Third, the mechanical logistics for our main result crucially require the bond market. Without the bond market, agent n only consumes the cash flows from his local asset, $\alpha_n C_t$. Since this consumption is deterministic, no asset-price volatility can be justified! If the bond market is open, agent n can send and receive consumption across locations, with the promise of inter-temporal payback. This opens the door for stochastic individual consumption profiles ($dc_{n,t}$ can load on $d\tilde{Z}_{n,t}$), which then creates a stochastic local pricing kernel (marginal utility loads on $d\tilde{Z}_{n,t}$), and finally justifies price volatility ($dq_{n,t}$ loads on $d\tilde{Z}_{n,t}$).

Finally, as mentioned above, the self-fulfilling volatility process ψ_t of Theorem 1 need not be Markov. But can it be Markov? In Appendix C, we argue self-fulfilling Markov equilibria cannot simply depend on "fundamental" variables like the wealth distribution or consumption shares, but they can exist if we introduce "sunspot" variables that coordinate beliefs. We make heavy use of example 2 to illustrate these points explicitly.

3.2 Volatility implies arbitrages and vice versa

Theorem 1 requires the condition $\text{rank}(M) < N$ in order to have self-fulfilling volatility. To get a sense of what this rank condition means, consider what would happen if a single trader was allowed to participate in all markets. With $\text{rank}(M) < N$, there is some asset that this trader can replicate using the other $N-1$ assets. But with self-fulfilling volatility, the price of this asset and its replicating portfolio need not move together. In short, this trader would be faced with an *arbitrage opportunity*. In our model, self-fulfilling volatility emerges if and only if arbitrages exist, which provides a more intuitive diagnostic for multiplicity than the rank condition on M.

Theorem 2. *Self-fulfilling volatility implies an arbitrage. Conversely, if there are arbitrages, then equilibrium must feature self-fulfilling volatility.*

First, consider the second statement of Theorem 2, that arbitrages imply self-fulfilling volatility. If there were no volatility (i.e., if $\text{rank}(M) = N$ or $\psi_t = 0$), then all assets earn the riskless rate (see remark 1), so there is no way to combine them into a portfolio that outperforms the riskless rate.

Conversely, arbitrages exist when self-fulfilling volatility emerges. In the proof in Appendix A, we examine the portfolio that puts $\delta\alpha_n q_{n,t}$ in each asset $n = 1, \dots, N$. By equation (4), the portfolio volatility is identically zero. This is where $\text{rank}(M) < N$ is critical: even if all assets have positive self-fulfilling volatility, we can manufacture a riskless asset from them. The proof shows mathematically why this portfolio pays more than the riskless rate, but the basic intuition comes from the fact that each local market n demands a risk premium on its local asset. A portfolio built as a convex combination of components with risk premia must have a premium itself.

Confirming this intuition, this strategy's excess return over the riskless rate r_t is

$$A_t := \sum_{n=1}^{N} x_{n,t} \tilde{\pi}_{n,t}^2 > 0. \tag{8}$$

This can be thought of as a measure of *arbitrage profit* in this model. Note that A_t is already quoted in the standard units used in analysis of arbitrage trades, because the long position $(\delta\alpha_n q_{n,t})_{1 \le n \le N}$ is a return (unit-cost portfolio). In addition, the amount of arbitrage profit A_t is exactly the difference between the risk-free rate that prevails without self-fulfilling volatility ($r_t = \delta + g$) and the one with self-fulfilling volatility ($r_t = \delta + g - A_t$). One usually reads this term as a precautionary savings term, but based on this discussion, A_t can also be thought of as the difference between a synthetic bond return and the traded bond return.

The interpretation of the traded and synthetic bond depends on the context. Some common examples are collateralized versus uncollateralized lending (sometimes captured in the TED spread); on-the-run versus off-the-run Treasury bonds; and deviations from covered interest parity (CIP). Measures of this quantity tend to be minimal for much of the time, but can expand to around 3% during financial crisis periods (Fleckenstein and Longstaff, 2018; Du et al., 2018).

The degree of arbitrage profit informs the amount of self-fulfilling volatility. This is because volatility drives location-specific risk prices, which constitute A_t.

Proposition 1. *Let rank$(M) < N$. Then, there exists a non-zero vector v^*, in the null-space of M', such that the self-fulfilling volatility ψ_t of Theorem 1 satisfies*

$$\psi_t = \frac{\delta^{-1}\sqrt{A_t}}{\sqrt{\sum_{n=1}^N x_{n,t}(\frac{v_n^*}{x_{n,t}})^2}} \leq \frac{\delta^{-1}\sqrt{A_t}}{\mathbf{1}'v^*}, \tag{9}$$

where A_t is the arbitrage profit given in (8). Consequently, the "average return volatility" $\sigma_t^* := \sum_{n=1}^N \frac{\alpha_n q_{n,t}}{\sum_{i=1}^N \alpha_i q_{i,t}} \sigma_{n,t}^q$ satisfies

$$\sigma_t^* = \delta \psi_t \mathbf{1}'v^* \leq \sqrt{A_t}. \tag{10}$$

The average return volatility σ_t^* defined in Proposition 1 is a scale-free summary statistic for the degree of volatility in our model. The tight link to arbitrage profits, $\sigma_t^* \leq \sqrt{A_t}$, is a bonus. To get a sense of magnitudes, consider arbitrage profits that range from $A_t \in [0, 0.03]$, consistent with the Treasury evidence of Fleckenstein and Longstaff (2018) and the CIP deviations documented in Du et al. (2018). Then, average return volatilities can range from $\sigma_t^* \in [0, 17.3\%]$, a quantitatively-large estimate.

At this point, it should be clear that arbitrages and self-fulfilling volatility are intrinsically linked. A different type of trade that resembles an arbitrage, a so-called *basis trade*, also exists in our model. A basis trade is a long-short strategy designed to capitalize on violations of the law of one price, i.e., price discrepancies between two assets with identical cash flows.

Example 4 (Simplest basis trade)**.** The simplest basis trade in our model takes any two locations (i, j) and applies an appropriate long-short strategy that yields zero future cash flows. Imagine a trader can access both markets. She buys $1/\alpha_i C_t$ of location-i assets and sells $1/\alpha_j C_t$ of location-j assets, which has future cash flows

$$\frac{\alpha_i C_{t+u}}{\alpha_i C_t} - \frac{\alpha_j C_{t+u}}{\alpha_j C_t} = 0.$$

The price of this portfolio is $q_{i,t} - q_{j,t}$, which is non-zero as long as either market i or j features self-fulfilling volatility. Thus, in a sense, this is an arbitrage trade. That said, the profits of this strategy (per unit of time) are

$$q_{i,t}\sigma^q_{i,t}(\tilde{\pi}_{i,t}dt + d\tilde{Z}_{i,t}) - q_{j,t}\sigma^q_{j,t}(\tilde{\pi}_{j,t}dt + d\tilde{Z}_{j,t}),$$

which can become arbitrarily negative as time progresses, given self-fulfilling volatility. This is the generic well-understood reason why trading frictions and portfolio constraints can generate limits to arbitrage and allow such trades to remain present in the market. In the next section, we formalize limits to arbitrage.

Example 5 (CIP deviations)**.** Our model can accommodate CIP deviations that we will study empirically. In the real world, one might consider the following basis trade: borrow in Japanese yen, exchange to US dollars in the FX spot market, lend in dollars, and finally convert back into yen via a pre-signed currency futures contract. This trade is a theoretically riskless method to move Japanese yen from today to tomorrow, and it should return the yen risk-free rate. When it does not, CIP fails.

In our model, think of locations as countries, and consider a hypothetical trader that can access all countries' markets. Countries' consumption goods are homogeneous and freely tradable, so the spot and forward exchange rates are always unity.

Next, if one were to construct a "local discount bond" that pays off in the future consumption of the local country, this bond would have price

$$\begin{aligned} b_{n,t\to T} &= e^{-\delta(T-t)}\mathbb{E}_t\Big[\frac{c_{n,t}}{c_{n,T}}\Big] \\ &= e^{-(\delta+g)(T-t)}\mathbb{E}_t\Big[\exp\Big(-\int_t^T(-A_u + \frac{1}{2}\tilde{\pi}^2_{n,u})du - \int_t^T \tilde{\pi}_{n,u}d\tilde{Z}_{n,u}\Big)\Big] \\ &= e^{-(\delta+g)(T-t)}\tilde{\mathbb{E}}^n_t\Big[\exp\Big(\int_t^T A_u du\Big)\Big], \end{aligned}$$

where A_t is given in (8) and $\tilde{\mathbb{E}}^n$ is the local risk-neutral expectation induced by risk prices $\tilde{\pi}_{n,t}$. This is expression is an expected discounted sum of arbitrage profits, where discounting is performed by the riskless rate that would prevail without arbitrages. With unitary exchange rates, the CIP deviation (in continuously-compounded units) is given by the difference in these yields on these discount bonds, i.e.,

$$\Delta_{t\to T}^{\text{CIP}(i,j)} := -\frac{1}{T-t}\Big(\log b_{j,t\to T} - \log b_{i,t\to T}\Big) = \frac{1}{T-t}\log\frac{\tilde{\mathbb{E}}_t^i\big[\exp\big(\int_t^T A_u du\big)\big]}{\tilde{\mathbb{E}}_t^j\big[\exp\big(\int_t^T A_u du\big)\big]}.$$

Note that $\Delta_{t\to T}^{\text{CIP}(i,j)} \neq 0$ if and only if the self-fulfilling equilibrium obtains. Indeed, the risk-neutral measures $\tilde{\mathbb{E}}^i$ and $\tilde{\mathbb{E}}^j$ are different in any equilibrium with self-fulfilling volatility, which is when $A_t > 0$. Conversely, when $A_t = 0$ forever, clearly $\Delta_{t\to T}^{\text{CIP}(i,j)} = 0$.

In section 4.2, we present an alternative setting from international finance: N countries produce differentiated goods, with investors exhibiting consumption home bias. In that model, which has time-varying exchange rates, we return to this CIP example.

3.3 Cross-market trading limits volatility

Proposition 1 provided a link between volatility and a measure of arbitrage profits. Given this connection, impediments to capital mobility and cross-market trading, which work to limit arbitrage profits, should limit asset volatility. In this section, we make this argument precise by developing a notion of *limits to arbitrage* and showing how it bounds the degree of volatility.

Motivated by models like Gromb and Vayanos (2002) and Garleanu and Pedersen (2011), we assume that cross-sectional risk prices are linked by some amount of relative-value trading going on in the background. To formalize this notion, we need to examine the location-specific risk prices induced on the basis shocks Z_t. Recall equation (2) connecting $\tilde{Z}_t = MZ_t$. If $\tilde{\pi}_{n,t}$ is the location-n marginal utility response to $d\tilde{Z}_{n,t}$, then

$$\pi_{n,t} := \tilde{\pi}_{n,t} M_n. \tag{11}$$

is the marginal utility response to dZ_t, where M_n is the nth row of M. Note that $\tilde{\pi}_{n,t}$ is a scalar, while $\pi_{n,t}$ is a vector.

We make the following reduced-form assumption about these basis risk prices $\pi_{n,t}$:

$$\underline{\kappa}_t \leq \|\pi_{j,t} - \pi_{i,t}\| \leq \bar{\kappa}_t \quad \forall i \neq j. \tag{12}$$

When $\bar{\kappa}_t > 0$ or $\underline{\kappa}_t > 0$, we say there are *limits to arbitrage*. This terminology is justified by the well-known equivalence between absence of arbitrage and the existence of a stochastic discount factor that prices all assets (hence a single risk price vector across all markets, $\pi_t^* = \pi_{i,t} = \pi_{j,t}$).

In microfounded models, the processes for $(\underline{\kappa}_t, \bar{\kappa}_t)$ would be linked to fundamental objects like arbitrageur wealth, preferences, constraints, and trading costs. For example, one can think of $\underline{\kappa}_t$ arising due to trading speed frictions or transactions costs limiting arbitrageurs' ability to completely eliminate risk-price differentials. On the other hand, one can think of $\bar{\kappa}_t$ arising due to margin constraints and the limited wealth that arbitrageurs can thus deploy in eliminating risk-price differentials. Bounds like (12) pervade most models of limits to arbitrage.[9] Here, we take $(\underline{\kappa}_t, \bar{\kappa}_t)$ as given and do not model the behavior of these arbitrageurs, opting instead to characterize equilibrium conditional on partial arbitrage.[10]

Up to now, we have been implicitly assuming $\bar{\kappa}_t = +\infty$, which is tantamount to infinitely-frictional arbitrage behavior. What happens when there is some partial amount of market segmentation? We have the following link between the degree of market segmentation and the degree of self-fulfilling volatility. The proof is in Appendix A.

Proposition 2. *Let $0 \leq \underline{\kappa}_t < \bar{\kappa}_t < +\infty$ and $\text{rank}(M) < N$. Then, there exists a non-zero vector v^*, in the null-space of M', such that the self-fulfilling volatility ψ_t of Theorem 1 is bounded by*

$$\delta^{-1} \ell_t^{-1} \underline{\kappa}_t \leq \psi_t \leq \delta^{-1} L_t^{-1} \bar{\kappa}_t. \tag{13}$$

where $\ell_t := \min_{(i,j):i \neq j} \left\| x_{i,t}^{-1} v_i^* M_i - x_{j,t}^{-1} v_j^* M_j \right\|$ and $L_t := \max_{(i,j):i \neq j} \left\| x_{i,t}^{-1} v_i^* M_i - x_{j,t}^{-1} v_j^* M_j \right\|$. The "average return volatility" $\sigma_t^* = \delta \psi_t \mathbf{1}' v^*$ is bounded by

$$\mathbf{1}' v^* \ell_t^{-1} \underline{\kappa}_t \leq \sigma_t^* \leq \mathbf{1}' v^* L_t^{-1} \bar{\kappa}_t. \tag{14}$$

If this interval is trivial, i.e., if $\ell_t \bar{\kappa}_t < L_t \underline{\kappa}_t$, then $\psi_t = 0$ is the only equilibrium.

Intuitively, with large limits-to-arbitrage, there can be large amounts of self-fulfilling volatility, because capital is too slow to correct any such price movements. As limits to arbitrage are relaxed, the amount of self-fulfilling volatility must vanish. Propositions 1 and 2 are thus similar in that they connect volatility to some quantitative measure of arbitrage efficacy (arbitrage profits and limits to arbitrage, respectively). Because of the link between volatility ψ_t and arbitrage profits A_t, limits-to-arbitrage as assumed in (12) also puts clear and intuitive bounds on A_t. This also bounds equilibrium risk prices, like

[9] For instance, Proposition 2' in Appendix B of Garleanu and Pedersen (2011) explicitly shows how margin constraints lead to a range of viable risk premia.

[10] We also do not modify any of the market clearing conditions to account for arbitrageur consumption, which can be justified by the idea that infinite trading would occur if $\|\pi_{j,t} - \pi_{i,t}\| > \bar{\kappa}_t$ ever occurred, but zero trading is needed otherwise.

Hansen and Jagannathan (1991), even though our primitive limits-to-arbitrage assumption in (12) is about relative risk prices.

Corollary 3. *Under the conditions of Proposition 2, risk prices and arbitrage profits are bounded:*

$$\frac{v_n^*}{x_{n,t}}\ell_t^{-1}\underline{\kappa}_t \leq \|\pi_{n,t}\| \leq \frac{v_n^*}{x_{n,t}}L_t\bar{\kappa}_t$$
$$\Big(\sum_{n=1}^{N} x_{n,t}(\frac{v_n^*}{x_{n,t}})^2\Big)^{1/2}\ell_t^{-1}\underline{\kappa}_t \leq \sqrt{A_t} \leq \Big(\sum_{n=1}^{N} x_{n,t}(\frac{v_n^*}{x_{n,t}})^2\Big)^{1/2}L_t^{-1}\bar{\kappa}_t.$$

To get a quantitative sense of the volatility bounds, we calibrate and simulate our model for an economy with $N = 10$ locations. We set the extrinsic shocks in a similar way as example 3, which has a zero-sum condition that we referred to as "redistribution":

$$M = \frac{N}{\sqrt{N(N-1)}}\Big[I_N - \frac{1}{N}\mathbf{1}\otimes\mathbf{1}'\Big] \tag{15}$$
$$= \frac{1}{\sqrt{N(N-1)}}\begin{bmatrix} N-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & N-1 & -1 & \cdots & -1 & -1 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & N-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & N-1 \end{bmatrix}.$$

Note that the columns of M sum to zero and have unit norm. It can easily be verified that $v^* = \mathbf{1}$ is the unique element, up to scale, in the null-space of M. To keep things simple, we assume equally-sized locations ($\alpha_n = 1/N$) and initialize the simulation with equally-wealthy locations ($x_{n,0} = 1/N$ for all n). We also set $\delta = 0.02$.

In terms of the exogenous arbitrage bounds, we set $\underline{\kappa}_t = 0.05$ and $\bar{\kappa}_t = 0.25$ to time-invariant values. The interpretation of these values is as follows. Various trading impediments prevent arbitrageurs from correcting Sharpe ratio differentials to less than 0.05; on the other hand, arbitrageurs are only willing to enter and correct Sharpe ratio differentials greater than 0.25. As will be clear shortly, these limits to arbitrage are quantitatively reasonable.

To simulate $\{x_{n,t} : t \geq 0\}$ over time, first note that the analytical dynamics are given by

$$dx_{n,t} = x_{n,t}(1 - x_{n,t})\Big[\tilde{\pi}_{n,t}^2 - \sum_{i\neq n}\frac{x_{i,t}}{1 - x_{n,t}}\tilde{\pi}_{i,t}^2\Big]dt + x_{n,t}\tilde{\pi}_{n,t}d\tilde{Z}_{n,t}. \tag{16}$$

This is derived by applying Itô's formula to the definition $x_{n,t} := c_{n,t}/C_t$, where the dynamics of $c_{n,t}$ are given in (34) in the appendix. Because $\tilde{\pi}_{n,t}$ depends on the self-fulfilling volatility, we make one of three assumptions in our simulations, which results in three time-paths of the economy: (i) ψ_t is always at its lower bound; (ii) ψ_t is always at its upper bound; (iii) ψ_t is at the mid-point between the two bounds. Thus, we obtain three sets of volatility time series.

The results for average return volatility σ_t^*, and the associated arbitrage profits A_t, are displayed in figure 2. The average return volatility σ_t^* can range from 3% to 17%, depending on where the economy lies within the bounds. The claim that $(\underline{\kappa}, \bar{\kappa})$ are reasonable can be seen by examining the associated simulated arbitrage profits A_t, which range from miniscule (left panel, ψ_t assumed at lower bound) to almost 3%, the upper range of measured arbitrage profits (right panel, ψ_t assumed at upper bound).

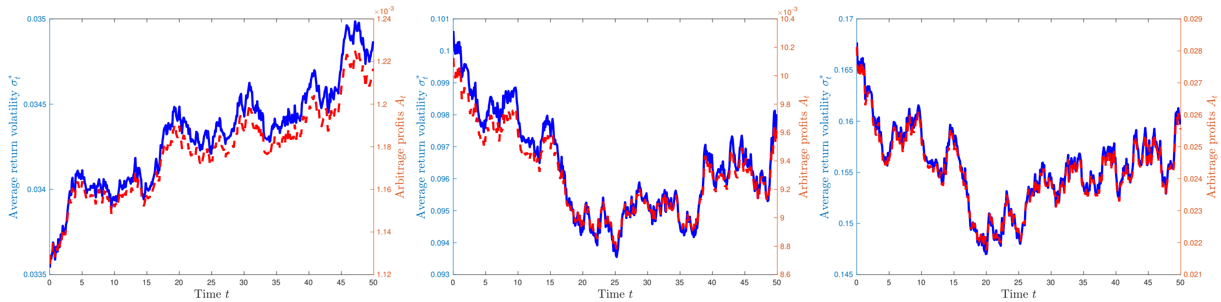

Figure 2: Plotted in solid blue against the left axes are volatility bounds from Proposition 2, from a simulated economy with $N = 10$ equally-sized locations ($\alpha_n = 1/N$) starting with equal initial wealth ($x_{n,0} = 1/N$) and with extrinsic shock matrix M given in (15). Plotted in dashed red against the right axes are arbitrage profits A_t from the simulation. The three panels correspond to assuming ψ_t is always at the (i) lower bound, (ii) mid-point between the bounds, and (iii) upper bound, respectively. Other parameters are described in the text.

These magnitudes can be theoretically verified for the M in (15) by considering the approximation $x_{n,t} = 1/N$ for all n. Then, the bounds simplify to

$$\sqrt{\frac{N-1}{2N}}\underline{\kappa}_t \leq \sigma_t^* = \sqrt{A_t} \leq \sqrt{\frac{N-1}{2N}}\bar{\kappa}_t.$$

Substituting $N = 10$, we obtain $3.4\% \leq \sigma_t^* \leq 16.8\%$ and $0.1\% \leq A_t \leq 2.8\%$, very close to the ranges displayed in figure 2.

4 Extensions

Below, we present two types of model extensions. In section 4.1, we introduce fundamental shocks, both aggregate and idiosyncratic. In section 4.2, we modify the setting to mimic leading international finance models, with countries producing differentiated goods and consumers exhibiting home bias in consumption. In these extensions, our main results continue to hold, with some additional interesting nuances.

4.1 Fundamental shocks

For theoretical clarity, our baseline model only has extrinsic shocks and deterministic cash flows. One wonders how the analysis changes when cash flows have fundamental shocks as well. In this section, we address this question first for aggregate shocks, then for idiosyncratic shocks.

Aggregate shocks. Suppose the aggregate endowment now follows

$$dC_t = C_t\Big[g dt + \nu dB_t\Big],$$

where B_t is an aggregate Brownian shock, independent of the extrinsic shocks Z_t (and by extension $\tilde{Z}_t$). Location-specific endowments are still given by $\alpha_n C_t$, so all endowments have exposure ν to aggregate shocks (this is only for simplicity). Conjecture that local price-dividend ratios now follow

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t} dt + \sigma^q_{n,t} d\tilde{Z}_{n,t} + \varsigma^q_{n,t} dB_t\Big],$$

where $(\mu^q_{n,t}, \sigma^q_{n,t}, \varsigma^q_{n,t})$ are all determined in equilibrium.

We will proceed by making one of two possible assumptions on the tradability of this aggregate shock. Either (a) there are no additional markets open beyond those assumed so far; or (b) there is an integrated market in which agents frictionlessly trade a zero-net-supply Arrow security that has a unit loading on dB_t. In both cases, all previous results on self-fulfilling volatility go through. However, we uncover a surprising nuance: equilibrium is consistent with local assets having nearly arbitrary sensitivities to the aggregate shock.

Proposition 4. *With aggregate shocks, the conclusions of Theorem 1 on $(\sigma^q_{n,t})_{n=1}^N$ continue to hold without modification. Regarding $(\varsigma^q_{n,t})_{n=1}^N$, we have the following. Let $(\phi_{n,t})_{n=1}^{N-1}$ be a collection of arbitrary stochastic processes, adapted to $\{Z_t : t \geq 0\}$, and set $\phi_{N,t} := -\sum_{n=1}^{N-1} \phi_{n,t}$. Then, there exists an equilibrium with $\alpha_n q_{n,t} \varsigma^q_{n,t} = \phi_{n,t}$ for $n = 1, \ldots, N$.*

The basic sketch of the argument is as follows. Because our log agents will still consume δ fraction of their wealth in this environment, equilibrium still satisfies equation

(3), that $\sum_{n=1}^{N} \alpha_n q_{n,t} = \delta^{-1}$. If we time-differentiate this condition as before, matching diffusion terms leads us to

$$\text{(match } dZ_t \text{ terms)} \quad 0 = \sum_{n=1}^{N} \alpha_n q_{n,t} \sigma_{n,t}^{q} M_n \tag{17}$$

$$\text{(match } dB_t \text{ terms)} \quad 0 = \sum_{n=1}^{N} \alpha_n q_{n,t} \varsigma_{n,t}^{q}. \tag{18}$$

The first equation is identical to equation (4), which is why the results of Theorem 1 continue to hold. For the second equation, of course it is possible to have $\varsigma_{n,t}^{q} = 0$ for all n. But we may also set $(\varsigma_{n,t}^{q})_{n=1}^{N-1}$ arbitrarily, so long as $\varsigma_{N,t}^{q}$ offsets these sensitivities. Thus, the volatilities have a similar redistributive flavor as before. Even if the loadings on the extrinsic shocks are all zero ($\sigma_{n,t}^{q} = 0$), we may still have non-zero loadings on the fundamental shock ($\varsigma_{n,t}^{q} \neq 0$). In this sense, there is a separable procedure to solve for equilibrium loadings on the extrinsic and fundamental shocks. As we prove in Appendix B, this is indeed an equilibrium, and there are no further restrictions.

The intuition for self-fulfilling fundamental sensitivities differs depending on whether the shock is hedgable or not. When agents cannot hedge the dB_t shock, the logic is similar to the baseline model: agents adjust their consumption, through the bond market, to their conjecture about how the local asset co-moves with the fundamental shock. When agents trade Arrow securities on dB_t in an integrated market, they do not care whether or not their local asset responds to this shock. Enough hedging and risk-sharing will occur in equilibrium such that individual consumptions all have sensitivity ν to dB_t. Under a particular conjecture about $\varsigma_{n,t}^{q}$, location-n agents will form a hedging plan in order to undo this exposure. This is self-fulfilling: as long as asset prices move according to the conjecture, the hedging plan was correct.

Idiosyncratic shocks. We now add fundamental, non-tradable idiosyncratic shocks to local endowments. Let

$$dy_{n,t} = y_{n,t} \Big[g dt + \hat{\nu} d\hat{B}_{n,t} \Big], \tag{19}$$

where $(\hat{B}_{n,t})_{n=1}^{N}$ are N independent Brownian motions, also independent of extrinsic shocks $\tilde{Z}$. Aggregate consumption follows

$$dC_t = C_t \Big[g dt + \hat{\nu} \sum_{n=1}^{N} \alpha_{n,t} d\hat{B}_{n,t} \Big],$$

where $\alpha_{n,t} := y_{n,t}/C_t$ is the location-n endowment share. Note the stochastic term in dC_t coming from non-aggregation of these idiosyncratic shocks. However, by Proposition 4, this aggregate stochastic term will be irrelevant to our results. We do not introduce any additional markets, which means $\hat{B}_{n,t}$ shocks cannot be hedged by agents. Conjecture that

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t}dt + \sigma^q_{n,t}d\tilde{Z}_{n,t} + \varsigma^q_{n,t}dB_{n,t} + \hat{\varsigma}^q_{n,t}d\hat{B}_{n,t}\Big],$$

where

$$dB_{n,t} := (\sum_{i\neq n}\alpha^2_{i,t})^{-1/2}\sum_{j\neq n}\alpha_{j,t}d\hat{B}_{j,t}$$

is a Brownian motion and $(\mu^q_{n,t}, \sigma^q_{n,t}, \varsigma^q_{n,t}, \hat{\varsigma}^q_{n,t})$ are determined endogenously. We verify that our results on self-fulfilling equilibria continue to go through in this setting. The proof is in Appendix B.

Proposition 5. *With non-tradable local endowment shocks as in (19), the conclusions of Theorem 1 on $(\sigma^q_{n,t})_{n=1}^N$ continue to hold without modification.*

Given Proposition 5, we can now offer a reinterpretation of our results as multiplicity arising from the familiar Bewley-Huggett-Aiyagari type model. Because that model is such a workhorse model, it may be surprising to think that it could have self-fulfilling equilibria, but our results suggest this is a possibility.

Remark 2 (Reinterpretation as Bewley-Huggett-Aiyagari)**.** Because location-n agents are constrained to hold the entire stock of their local capital asset, and cannot hedge either $d\tilde{Z}_{n,t}$ or $d\hat{B}_{n,t}$ shocks in any market, one can think of the claim to $\{y_{n,t} : t \geq 0\}$ as entirely non-tradable. From this perspective, one can think of the endowment $y_{n,t}$ as a labor endowment and the "price-dividend" ratio $q_{n,t}$ as a perceived human wealth-income ratio. The only traded financial assets are riskless bonds, with holdings denoted by $\beta_{n,t}$ and following

$$d\beta_{n,t} = \Big[r_t\beta_{n,t} - c_{n,t} + y_{n,t}\Big]dt.$$

A model of this type is equivalent to a Huggett (1993) economy (see also Bewley, 1986, and Aiyagari, 1994) without borrowing constraints. Despite this analogy to such a workhorse model, self-fulfilling volatility results persist in the sense that location-n consumption can move for non-fundamental reasons. In particular, agents form self-fulfilling beliefs about the dynamics of their non-traded human wealth, which causes them to adjust their consumption and self-confirms their conjectured human capital dynamics. Self-fulfilling volatility in Bewley-Huggett-Aiyagari models is an interesting exploration for future research.

4.2 Locations as countries

In order to compare our theoretical results more closely to our empirical findings on the covered interest parity, we now modify the model to engage with canonical international finance models. Locations $n \in \{1, \dots, N\}$ are to be thought of as countries. Each country produces its own unique good in quantity $y_{n,t}$, where

$$dy_{n,t} = gy_{n,t}dt.$$

This is the endowment received by agent n.

Agents have preferences over all goods, not just their endowed good, but they display a "home bias" towards their local good. Mathematically, agent n lives in country n and maximizes

$$\mathbb{E}_0\Big[\int_0^\infty e^{-\delta t}\Big(\omega\log(c_{n,n,t}) + \frac{1-\omega}{N-1}\sum_{i\neq n}\log(c_{n,i,t})\Big)dt\Big], \quad \text{where} \quad 1 > \omega > \frac{1}{N}. \tag{20}$$

Note that $c_{n,i,t}$ is the consumption of good i by agent n. The assumption $\omega > 1/N$ captures home bias. All N consumption goods markets are competitive and integrated, with market clearing condition $\sum_{i=1}^N c_{i,n,t} = y_{n,t}$ for all n. Let $p_{n,t}$ denote the traded price of good n. We set good 1 to the numeraire, so $p_{1,t} = 1$. Think of country 1 as the "domestic country" in the parlance of international macro-finance.

As in our baseline model, the riskless bond market is integrated at interest rate r_t in units of good 1, whereas the claims to the cash flow stream $\{y_{n,t}\}_{t\geq 0}$ are traded only in country n, at price $q_{n,t}p_{n,t}y_{n,t}$. As before, conjecture

$$dq_{n,t} = q_{n,t}\Big[\mu^q_{n,t}dt + \sigma^q_{n,t}d\tilde{Z}_{n,t}\Big].$$

In this model, agents still spend δ fraction of their wealth in consumption, which is why the weighted-average of price-dividend ratios is still δ^{-1}:

$$\sum_{n=1}^N \alpha_{n,t}q_{n,t} = \delta^{-1}, \quad \text{where} \quad \alpha_{n,t} := \frac{p_{n,t}y_{n,t}}{\sum_{i=1}^N p_{i,t}y_{i,t}}. \tag{21}$$

Equation (21) suggests that redistributive self-fulfilling volatility is still possible in this model, in the sense that variation in $\alpha_{n,t}q_{n,t}$ is offset by variation in all the other assets. What is different in (21), relative to (3), is the fact that weights $\alpha_{n,t}$ are time-varying and endogenous. Furthermore, the aggregate expenditure $\sum_{i=1}^N p_{i,t}y_{i,t}$ loads on extrinsic shocks, a feature which acts to anchor all non-fundamental asset price volatility. This alters the argument for self-fulfilling volatility and renders it significantly more tedious to prove. We will need the following additional assumption.[11]

Assumption 1. *Let $N \geq 2$ and $\text{rank}(M) < N$. Assume M and $v^* \neq 0$ in the null-space of M' satisfy one of: (i) $M_1 \equiv 0$; (ii) $v_1^* = 0$; or (iii) $M_n = \pm M_1$ for all n such that $v_n^* \neq 0$.*

Parts (i) and (ii) will ultimately imply that country-1 investors have no consumption growth volatility, which eliminates any self-fulfilling volatility on the numeraire asset. Part (iii) can be visualized in the two-country case: one country's extrinsic shock is exactly the opposite of the other, as in example 2. We now have the following analogs of Theorems 1-2, demonstrating when self-fulfilling volatility exists. Proofs are in Appendix B.

Proposition 6. *Consider $N \geq 2$ countries producing differentiated goods and consumers exhibiting home bias. Under Assumption 1, self-fulfilling volatility is arbitrary: for any non-negative process $\{\psi_t\}_{t\geq 0}$ adapted to $\{Z_t\}_{t\geq 0}$, an equilibrium can be sustained with $\alpha_{n,t}q_{n,t}\sigma^q_{n,t} = H(x_{1,t}, x_{n,t}, q_{n,t})v_n^*\psi_t$ for some function $H \neq 0$ and vector $v^* \neq 0$ in the null-space of M'. Conversely, if Assumption 1 fails, equilibrium has no self-fulfilling volatility: $\sigma^q_{n,t} \equiv 0$ for all n.*

Proposition 7. *Consider $N \geq 2$ countries producing differentiated goods and consumers exhibiting home bias. Self-fulfilling volatility arises if and only if arbitrages exist.*

Similar to the baseline model, in the process of constructing an arbitrage for Proposition 7, we discover that its profit in excess of the riskless rate is proportional to a weighted-average of squared risk prices

$$A_t^{(\omega)} := \sum_{n=1}^{N} \frac{\omega_n x_{n,t}}{\sum_{i=1}^{N} \omega_i x_{i,t}} \|\pi_{n,t}\|^2, \quad \text{where} \quad \omega_n := \omega \mathbf{1}_{n=1} + \frac{1-\omega}{N-1}\mathbf{1}_{n\neq 1}. \tag{22}$$

Furthermore, exactly as in the baseline model arbitrage profit $A_t^{(\omega)}$ is the difference between the prevailing risk-free rate ($r_t = \delta + g - A_t^{(\omega)}$) and that which would prevail without self-fulfilling volatility ($r_t = \delta + g$).[12] Given this measure of arbitrage profit, we also have the analog to Proposition 1.

[11]Obviously, part (i) of Assumption 1 violates the aforementioned normalization $\text{diag}[MM'] = \mathbf{1}$. In this case, we only maintain the normalization $\text{diag}[MM']_n = 1$ for $n > 1$. In addition, one can easily verify that it is without loss of generality to set $v_1^* = 0$ if $M_1 \equiv 0$, so one may combine parts (i) and (ii).

[12]This statement can be verified as follows. Start with equation (53) in the appendix, then substitute equations (61)-(62), and finally simplify. Along the way, make use of the fact that $\alpha_{1,t} = \sum_{i=1}^{N} \omega_i x_{i,t}$, given formula (48) and $\sum_{n=1}^{N} x_{n,t} = 1$.

Proposition 8. *Let Assumption 1 hold. Then, there exists a non-zero vector v^*, in the null-space of M', such that the self-fulfilling volatility ψ_t of Proposition 6 satisfies*

$$\psi_t = \Big(\sum_{n=1}^{N} \frac{\omega_n x_{n,t}}{\sum_{i=1}^N \omega_i x_{i,t}} \big(\frac{v_n^*}{\omega_n x_{n,t}}\big)^2 \Big)^{-1/2} \sqrt{A_t^{(\omega)}} \leq \frac{1}{\mathbf{1}' v^*} \sqrt{\alpha_{1,t} A_t^{(\omega)}}, \tag{23}$$

where $A_t^{(\omega)}$ is the arbitrage profit given in (22). Consequently, if $\sigma_{n,t}^R$ is the loading of the asset-n return on basis shocks dZ_t, the "average return volatility" $\sigma_t^* := \sum_{n=1}^N \frac{\alpha_{n,t} q_{n,t}}{\sum_{i=1}^N \alpha_i q_{i,t}} \|\sigma_{n,t}^R\|$ satisfies

$$\sigma_t^* = \Big(\sum_{n=1}^{N} \frac{v_n^*}{\omega_n} \Big) \psi_t \leq \Big(\sum_{n=1}^{N} \frac{v_n^*}{\mathbf{1}' v^*} \omega_n^{-1} \Big) \sqrt{\alpha_{1,t} A_t^{(\omega)}}. \tag{24}$$

Example 5 (CIP deviations, continued)**.** Finally, we may revisit example 5 in this more familiar international finance framework. The spot (real) exchange rate is determined by the ratio of goods prices, so currency n costs $s_{n,t} = p_{n,t}$ in units of currency 1, which is the numeraire. When self-fulfilling volatility is present, exchange rates become stochastic, and arbitrages arise (Proposition 7). This makes computation of a "forward exchange rate" $f_{n,t\to T}$ non-trivial. Indeed, there is no forward exchange market in this model.

We address this with the following arbitrage-inspired method. Suppose $b_{n,t\to T}$ denotes the discount bond of country n. CIP says that

$$f_{n,t\to T} = \frac{b_{n,t\to T}}{b_{1,t\to T}} s_{n,t}. \tag{25}$$

One measure of CIP deviations is the discrepancy between the right-hand-side of (25) computed two ways: (i) using the SDF of a country-1 investor, $\xi_{1,t}$; (ii) using the SDF of a country-n investor, $\xi_{n,t}$. Note that the discount bond price of country n, computed using an arbitrary SDF ξ_t is given by $b_{n,t\to T} = \mathbb{E}_t[\frac{\xi_T s_{n,T}}{\xi_t s_{n,t}}]$. Thus, in continuously-compounded units, our proxy for CIP deviations is

$$\begin{aligned}
\Delta_{t\to T}^{\text{CIP}(i,j)} &:= -\frac{1}{T-t} \Big(\log \mathbb{E}_t\Big[\frac{\xi_{n,T} s_{n,T}}{\xi_{n,t}}\Big] - \log \mathbb{E}_t\Big[\frac{\xi_{n,T}}{\xi_{n,t}}\Big] - \log \mathbb{E}_t\Big[\frac{\xi_{1,T} s_{n,T}}{\xi_{1,t}}\Big] + \log \mathbb{E}_t\Big[\frac{\xi_{1,T}}{\xi_{1,t}}\Big] \Big) \\
&= -\frac{1}{T-t} \Big(\underbrace{\log \frac{\tilde{\mathbb{E}}_t^1[\exp(\int_t^T A_u^{(\omega)} du)]}{\tilde{\mathbb{E}}_t^n[\exp(\int_t^T A_u^{(\omega)} du)]}}_{\text{same term from the baseline model}} - \underbrace{\log \frac{\tilde{\mathbb{E}}_t^1[\exp(\int_t^T A_u^{(\omega)} du) s_{n,T}]}{\tilde{\mathbb{E}}_t^n[\exp(\int_t^T A_u^{(\omega)} du) s_{n,T}]}}_{\text{new term due to stochastic exchange rates}} \Big),
\end{aligned}$$

where $\tilde{\mathbb{E}}_t^n$ is the risk-neutral measure of country n. Note that, because of the necessary presence of arbitrages, the country-specific risk-neutral measures will not coincide when self-fulfilling volatility arises. In addition, we will have $A_t^{(\omega)} > 0$. Thus, CIP deviations are necessarily present with self-fulfilling volatility.

5 Empirical analysis

We now confront the main model prediction with the data. From Proposition 1, recall

$$\text{average return volatility } = \sigma_t^* \leq \sqrt{A_t} = \text{ square-root of arbitrage profits.} \tag{26}$$

Equation (26) predicts that there should be an increasing relationship between the square root of arbitrage profits ($\sqrt{A_t}$) and the value-weighted-average volatilities of assets comprising the arbitrage trade in question (σ_t^*). Building on Du et al. (2018), we investigate deviations from covered interest parity (CIP).

If we take the model very literally, relationship (26) should hold approximately one-for-one: that is, a regression of σ_t^* onto $\sqrt{A_t}$ should recover a regression coefficient of approximately 1.[13] We do not take the prediction so literally for two key reasons, most of which lead one to expect a regression coefficient below 1, in fact. First, in the presence of additional shocks, a substantial fraction of asset-price volatility will not be related to arbitrage profits.[14] Second, the arbitrage of our model is built with long-lived assets, whereas any practical application uses shorter-dated assets that have lower volatility, through short duration alone. We attempt to partially address this below, with a variety of proxies for σ_t^*, but our choices are not without trade-offs.

Data and proxies. As a proxy for arbitrage profits A_t, we measure the 3-month absolute CIP deviations of the G10 currencies against the USD, and take a simple average across these currencies. In particular, for currency i and maturity m, CIP against the US says

$$\frac{1}{p_{US,t}^{(m)}} = \frac{s_{US\to i,t}}{f_{US\to i,t}^{(m)}} \frac{1}{p_{i,t}^{(m)}}, \tag{27}$$

where $p_{US,t}^{(m)}$ and $p_{i,t}^{(m)}$ are the m-maturity zero-coupon bond prices, $s_{US\to i,t}$ denotes the spot exchange rate, and $f_{US\to i,t}^{(m)}$ denotes the forward exchange rate. Exactly as in Du et

[13] Of course, expression (26) is not an equality, because Jensen's inequality was used to obtain this expression. But as long as the Jensen deviation from equality is uncorrelated with the variation in arbitrage profits, a unit regression coefficient should be expected.

[14] See section 4.1 for fundamental aggregate and idiosyncratic shocks. That said, if these fundamental shocks are orthogonal to the extrinsic shocks that lead to self-fulfilling volatility, then the regression coefficient of volatility on root arbitrage profits would be unaffected.

al. (2018), we define the CIP deviation by the annualized incremental foreign bond yield needed to make (27) hold exactly. We take the absolute value of each G10 currency's CIP deviation to obtain our proxy

$$\hat{A}_{i,t}^{(m)} := \frac{1}{m}\left| \log\left(p_{i,t}^{(m)} \big/ p_{US,t}^{(m)}\right) + \log\left(f_{US\to i,t}^{(m)} \big/ s_{US\to i,t}\right)\right|. \tag{28}$$

For an aggregate time series measure, we take the simple average across currencies:

$$\hat{A}_t^{(m)} := \frac{1}{10}\sum_{i=1}^{10} \hat{A}_{i,t}^{(m)}. \tag{29}$$

Using 3-month contracts corresponds to setting $m = 1/4$. Our choice for $\hat{A}_t^{(m)}$ matches a few key features of the arbitrage in the model: it represents a relatively short-term trade that generates positive profits over the riskless rate for sure, if held to maturity. One-month or one-week CIP deviations may match better the fact that A_t represents profits over the infinitesimal time interval $[t, t + dt]$, but these contracts have other idiosyncratic features, as discussed in Du et al. (2018).

An appropriate choice for asset return volatility σ_t^* is more difficult, due to various data limitations and discrepancies between our simple model and the real world. As a baseline, we proxy σ_t^* by the monthly standard deviation of daily log price changes in the 10-year Constant Maturity Treasury note, annualized, i.e.,[15]

$$\hat{\sigma}_t^* := \sqrt{\frac{252}{30}\sum_{u=1}^{30}\left[\log\left(p_{US,t+u}^{(10)} \big/ p_{US,t+u-1}^{(10)}\right) - \frac{1}{30}\sum_{v=1}^{30}\log\left(p_{US,t+v}^{(10)} \big/ p_{US,t+v-1}^{(10)}\right)\right]^2}. \tag{30}$$

This choice for $\hat{\sigma}_t^*$ is simple and transparent but has several drawbacks: (a) it is not a very precise estimate of time-t conditional volatility; (b) it includes no volatility information for the synthetic US bonds, which constitute the other leg of the CIP trade; and (c) unlike the model, in which the assets used to construct A_t correspond to those used to measure σ_t^*, this proxy uses long-maturity bonds instead of the 3-month bonds comprising $\hat{A}_t^{(1/4)}$.

To help address concerns (a)-(c), we also consider three alternative measures of $\hat{\sigma}_t^*$ described briefly below and more extensively in Appendix D.

(a) For a more real-time measure of conditional volatility, we also examine the CBOE's 10-year Treasury VIX (TYVIX), which is the implied 30-day volatility of CBOT fu-

[15] These are not holding period returns, because we hold the time-to-maturity constant in calculating price at day t and day $t + 1$. Without imposing our own model-based interpolation methods, we cannot observe the day-$t + 1$ price of a Treasury that was a 10-year bond on day-t.

tures on 10-year US Treasury Notes. TYVIX applies the CBOE's VIX methodology to options on 10-year US Treasury Note futures.

(b) To include information for the foreign leg, we compute the volatility of foreign 10-year notes (converted to USD via spot exchange rates), analogous to (30). We then take the value-weighted-average of this measure and 10-year US note volatility.

(c) To bring the assets in the volatility construction as close as possible to those used in the arbitrage trade, we examine the value-weighted-average return volatilities of the 3-month US bill and the 3-month synthetic US bill. The prices of these bills are constructed using country-specific IBOR.

Results. Figure 3 summarizes the main empirical finding that arbitrage profits and volatility co-move strongly over time. The average absolute CIP deviation is high when the 10-year US Treasury volatility is high. Figure 4 shows the same result holds disaggregated at the currency level. Appendix D repeats the same figures for our three other proxies of σ_t^* and finds similar results in each case.

To formalize and quantify this link, table 1, column (1), displays OLS results from a regression of $\hat{\sigma}_t^*$ on $[\hat{A}_{i,t}^{(1/4)}]^{1/2}$ for each currency i. Across all currencies, we document a very strong relationship, both statistically and economically, between 10-year US Treasury note volatility and CIP deviations. Results are strongest for Australia and New Zealand, currencies that play a central role in the carry trade, in practice. Amazingly, despite the caveats outlined at the beginning of this section, the regression coefficients are in the ballpark of 1, in line with the model prediction.

Now, turn to our other proxies for $\hat{\sigma}_t^*$ at the 10-year maturity in columns (2)-(4). Using instead the 10-year Treasury VIX (column 2) or including the 10-year foreign note (column 3) does not change the empirical message. The Treasury VIX regression coefficients are attenuated a bit, whereas the coefficients including the foreign note are magnified. If we use volatility from the 3-month bonds (column 4), our regressions produce slope estimates approximately 40-50 times lower, in line with the relative durations of a 10-year note and a 3-month bill.

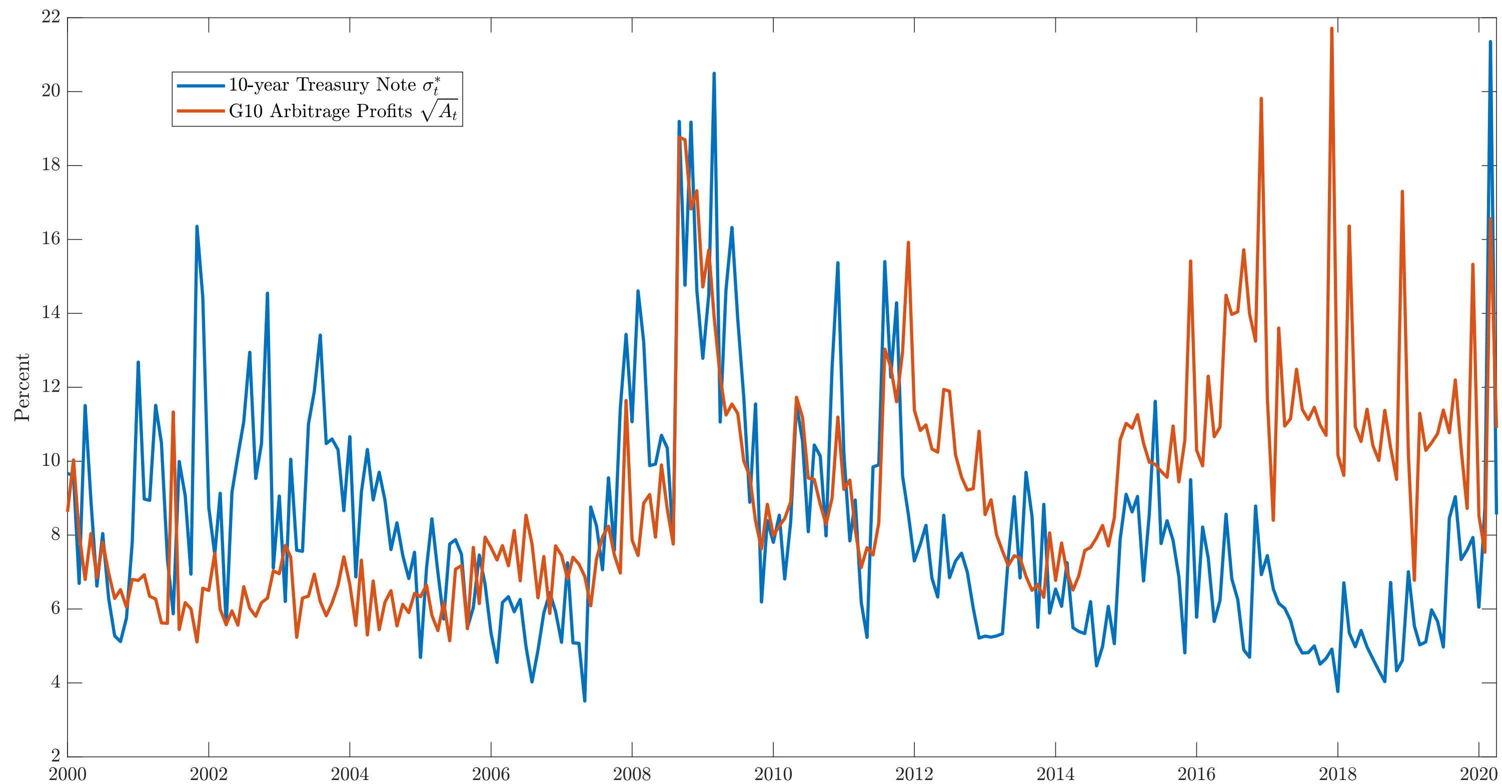

Figure 3: Monthly time-series of proxies for A_t and σ_t^*. Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is $\hat{\sigma}_t^*$ from (30), which is constructed using the monthly standard deviation of daily log price changes of a 10-year Constant Maturity US Note. Both measures are annualized. Currency data are from Bloomberg, whereas 10-year Constant Maturity Treasury data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

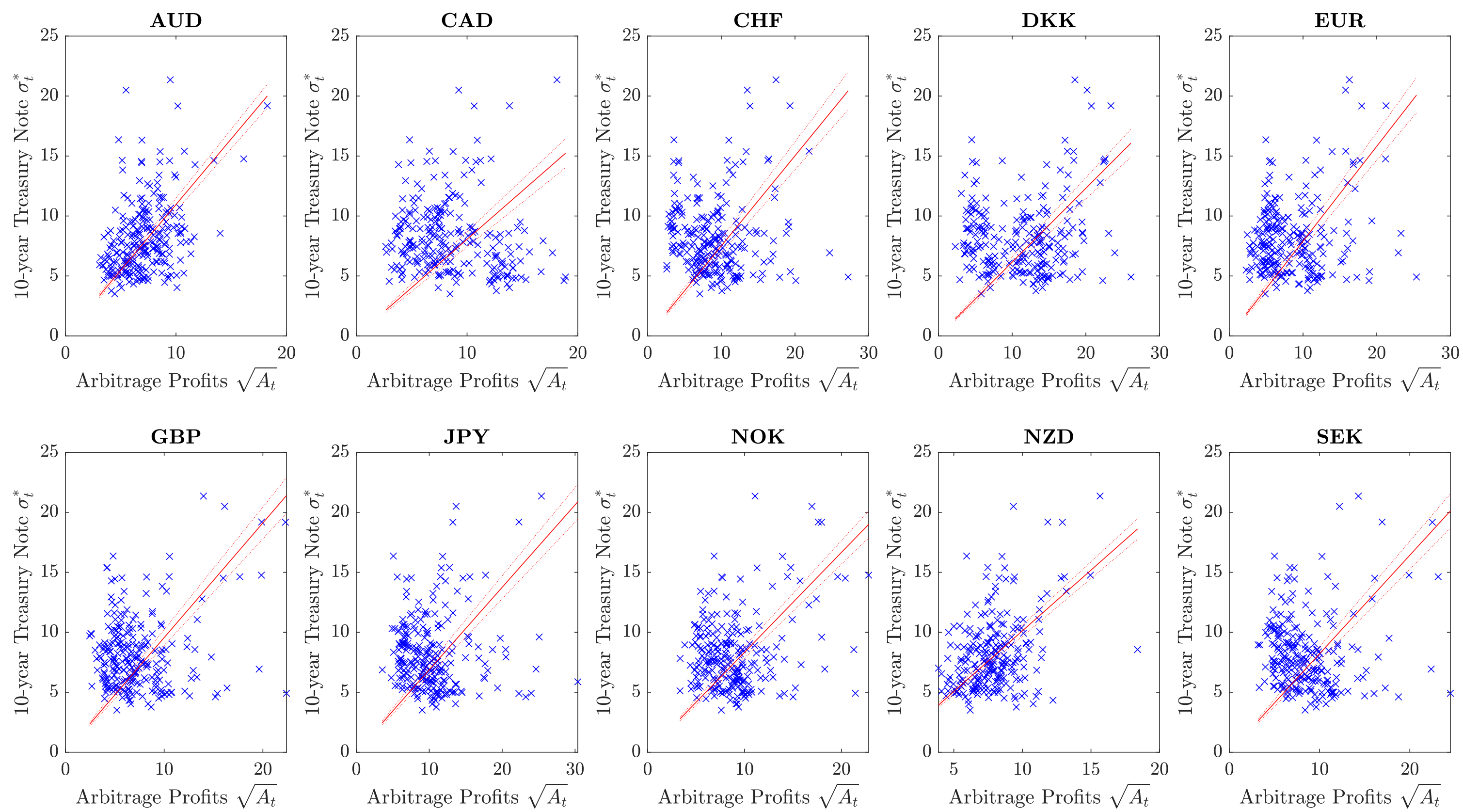

Figure 4: Currency-level OLS regressions of σ_t^* on $\sqrt{A_t}$ (monthly, no intercept). Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is $\hat{\sigma}_t^*$ from (30), which is constructed using the monthly standard deviation of daily log price changes of a 10-year Constant Maturity US Note. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency data are from Bloomberg, whereas 10-year Constant Maturity Treasury data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

	Proxy for $\hat{\sigma}_t^*$			
Currency	(1) 10y US Note	(2) TYVIX	(3) 10y US + foreign	(4) 3m US + foreign
aud	1.095 (0.042)	0.813 (0.035)	1.602 (0.071)	0.033 (0.008)
cad	0.806 (0.126)	0.581 (0.103)	0.871 (0.126)	0.019 (0.005)
chf	0.751 (0.068)	0.538 (0.046)	0.812 (0.070)	0.027 (0.005)
dkk	0.616 (0.047)	0.441 (0.035)	0.629 (0.047)	0.021 (0.005)
eur	0.790 (0.076)	0.561 (0.052)	0.941 (0.075)	0.024 (0.006)
gbp	0.956 (0.083)	0.693 (0.058)	1.153 (0.069)	0.031 (0.009)
jpy	0.687 (0.053)	0.514 (0.048)	0.689 (0.053)	0.025 (0.005)
nok	0.833 (0.052)	0.606 (0.043)	0.815 (0.055)	0.026 (0.006)
nzd	1.012 (0.043)	0.755 (0.047)	1.426 (0.066)	0.031 (0.006)
sek	0.821 (0.070)	0.595 (0.051)	0.789 (0.071)	0.026 (0.006)
N	244	208	244*	242

Table 1: OLS regressions of volatility $\hat{\sigma}_t^*$ on arbitrage profits $\sqrt{\hat{A}_t^{(1/4)}}$ across G10 currencies and using four different volatility proxies $\hat{\sigma}_t^*$. Going across the columns, the four volatility proxies are as follows: (1) the monthly volatility of daily log price changes on the 10-year Constant Maturity US Note; (2) the CBOE's 10-year Treasury VIX (TYVIX); (3) the value-weighted average volatilities, computed monthly from daily log price changes, of the 10-year US Treasury Note and the 10-year foreign note, where the latter is adjusted to USD by the spot exchange rate; (4) the value-weighted average volatilities, computed monthly from daily measures of next-two-month holding period returns, on the 3-month US bill and the 3-month synthetic US bill (constructed using a foreign bill, the spot exchange rate, and a forward currency swap). All four measures are annualized. The proxy for arbitrage profits is the currency-specific 3-month absolute CIP deviation as in (28), averaged monthly. Estimated regression coefficients are computed from OLS without an intercept. Standard errors are in parentheses and computed using heteroskedasticity and autocorrelation corrected (HAC) formulas. Currency and foreign note data are from Bloomberg. 10-year Constant Maturity Treasury data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. TYVIX data from the CBOE. Currency data range from Jan. 2000 to Apr. 2020. TYVIX data ranges from Jan 2003 to Apr. 2020. 10-year G10 Note data range from Jan. 2000 to Apr. 2020 for AUD, CAD, DKK, EUR, GBP, NZD, and Jan. 2007 to Apr. 2020 for CHF, NOK, and SEK. *158 (CHF), 239 (DKK), 243 (JPY), 158 (NOK), 160 (SEK).

6 Conclusion

We have demonstrated, in a canonical limits-to-arbitrage framework, the strong connection between the availability of arbitrage profits and the possibility of self-fulfilling volatility. Empirically, we have documented an association between available arbitrage profits in foreign exchange markets and volatility of the underlying instruments in these trades.

Often, the presence of multiple equilibria and self-fulfilling dynamics are viewed as a nuisance for theoretical models. But given that levels of asset-price volatility often far exceed predictions of many theoretical models, our mechanism can help bridge a gap in financial economics.

For example, consider corporate equity and bond markets. Although equity and bond returns are linked, one cannot construct a riskless portfolio from them in a simple way, unlike for covered interest parity.[16] Still, it is entirely possible, and anecdotally true, that equity investors differ from bond investors and that capital is slow moving, whether due to market segmentation or investor habitats. With this in mind, our model suggests some amount of redistributive self-fulfilling volatility should be possible between corporate equity and bond markets. In this sense, our focus on true arbitrages is just for clarity: one can measure the amount of arbitrage profit (and correlate it to volatility) without having to know investors' pricing kernels. We think that future research could, through a self-fulfilling mechanism, connect frictions such as market segmentation to "volatility puzzles" in other asset markets beyond those with self-evident arbitrage profits.

[16]With dynamic trading, knowledge of the underlying shocks that affect both securities, as well as their sensitivities to those shocks, one could obtain a no-arbitrage relation between equities and bonds.

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Appendix

A Proofs for Section 3

Proof of Theorem 1. To prove the claim, we need to fill in any details that go beyond the discussion following the statement of Theorem 1. There are three brief steps needed to fill in the details.

Step 1: State prices. Each location has its own risk price $\tilde{\pi}_{n,t}$, which is the marginal utility sensitivity to the $d\tilde{Z}_{n,t}$ shock. The state price density for location n is then given by

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t}\Big]. \tag{31}$$

In these terms, we have the no-arbitrage pricing relation

$$\mu^q_{n,t} + g + \frac{1}{q_{n,t}} - r_t = \sigma^q_{n,t}\tilde{\pi}_{n,t}. \tag{32}$$

We can also pose things in terms of the basis shocks. Let $\pi_{n,t}$ be the risk price vector pertaining to dZ_t, which is potentially location-specific because of market segmentation. The link between these two, by substituting equation (2) into (31), is given in equation (11).

Step 2: Optimality. Log agents optimally consume δ fraction of their wealth when there are no bubbles. Investor n wealth is given by $\alpha_n C_t q_{n,t} + \beta_{n,t}$ where $\beta_{n,t}$ is their risk-free bond market position. Let $\theta_{n,t} := \frac{\alpha_n C_t q_{n,t}}{\alpha_n C_t q_{n,t} + \beta_{n,t}}$ be the fraction of wealth this investor puts in the local risky asset. Note that market clearing is imposed automatically in this formula, as the local investor n holds the entirety of the local asset. Given the dynamic conjecture for asset prices, and the consumption-wealth ratio δ, each investor then has consumption dynamics

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta + \theta_{n,t}\sigma^q_{n,t}\tilde{\pi}_{n,t}\Big]dt + \theta_{n,t}\sigma^q_{n,t} d\tilde{Z}_{n,t}. \tag{33}$$

Under these assumptions, optimal portfolio choices are given by the standard mean-variance formula $\theta_{n,t}\sigma^q_{n,t} = \tilde{\pi}_{n,t}$. Substituting this portfolio choice into (33), equilibrium consumption dynamics are

$$\frac{dc_{n,t}}{c_{n,t}} = \Big[r_t - \delta + \tilde{\pi}^2_{n,t}\Big]dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t}. \tag{34}$$

From (31) and (34), we obtain $\xi_{n,t}c_{n,t} = \xi_{n,0}c_{n,0}\exp(-\delta t)$, so that the no-bubble static budget constraint (with wealth defined as $w_{n,t} := \alpha_n q_{n,t} C_t + \beta_{n,t}$)

$$\mathbb{E}_t\Big[\int_0^\infty \frac{\xi_{n,t+s}}{\xi_{n,t}} c_{n,t+s} ds\Big] = w_{n,t} \tag{35}$$

holds automatically with $c_{n,t} = \delta w_{n,t}$. This confirms that the optimal consumption rule and no bubbles are mutually consistent.

Step 3: Aggregation. Define the consumption shares $x_{n,t} := c_{n,t}/C_t$. Notice that $\theta_{n,t} = \delta\alpha_n q_{n,t}/x_{n,t}$, which, combined with optimal portfolio choice, yields equation (7). Time-differentiating the goods market clearing condition $\sum_{n=1}^N c_{n,t} = C_t$ and using (34), we have

$$r_t = \delta + g - \sum_{n=1}^N x_{n,t}\tilde{\pi}^2_{n,t} \tag{36}$$

and

$$0 = \sum_{n=1}^{N} x_{n,t} \tilde{\pi}_{n,t} M_n. \tag{37}$$

Substituting (7) into (37) delivers equation (4). Also, combining the asset-pricing equation (32), which is an equation for μ_n^q, with the risk-free rate equation (36), one can show that (3) holds if and only if $\sum_{n=1}^{N} \alpha_n q_{n,0} = \delta^{-1}$, i.e., if an initial condition holds on prices. Finally, note that consumption share dynamics are obtained by Itô's formula by equation (16). This completes the list of equilibrium restrictions. □

***Proof of Theorem* 2.** First, assuming the existence of self-fulfilling volatility, let us find a portfolio that has no risk but pays a positive premium over the riskless rate. Consider a portfolio that goes long $\delta \alpha_n q_{n,t}$ of each asset $n = 1, \ldots, N$, which costs 1 by equation (3). As stated in equation (32), each asset n has expected excess returns that are given by the product of the location-n risk quantity times the risk price: $\sigma^q_{n,t} \tilde{\pi}_{n,t}$. Using equation (7) to substitute $\tilde{\pi}_{n,t}$, the portfolio excess return is

$$\sum_{n=1}^{N} x_{n,t} \delta^2 \left(\frac{\alpha_n q_{n,t}}{x_{n,t}} \right)^2 (\sigma^q_{n,t})^2 \geq 0,$$

which is strictly positive as long as any self-fulfilling volatility obtains. Using the expression for $\tilde{\pi}_{n,t}$, one can easily verify this expression is equivalent to A_t in (8). At the same time, by equation (4), the portfolio volatility is identically zero. This shows that an arbitrage always emerges if there is self-fulfilling volatility.

Next, the claim that absence of self-fulfilling volatility implies no arbitrage follows from (32), whereby all assets return r_t when $\sigma^q_{n,t} = 0$. □

***Proof of Proposition* 1.** Substituting $\alpha_n q_{n,t} \sigma^q_{n,t} = \psi_t v_n^*$ from Theorem 1 into location-specific risk prices of (7), and substituting the result into (8), we have

$$A_t = \delta^2 \psi_t^2 \sum_{n=1}^{N} x_{n,t} \left(\frac{v_n^*}{x_{n,t}} \right)^2$$

By inverting this relationship, the amount of self-fulfilling volatility ψ_t can be inferred from A_t, which gives the equality in (9). The upper bound can be obtained by substituting

$$\sum_{n=1}^{N} x_{n,t} \left(\frac{v_n^*}{x_{n,t}} \right)^2 \geq \left(\sum_{n=1}^{N} x_{n,t} \frac{v_n^*}{x_{n,t}} \right)^2 = (\mathbf{1}' v^*)^2,$$

which holds by Jensen's inequality. To obtain the equality in (10), substitute (3) into the definition of σ_t^* and use the result from Theorem 1 that $\alpha_n q_{n,t} \sigma^q_{n,t} = \psi_t v_n^*$. To obtain the inequality, use (9). □

Proof of Proposition 2. Substitute equation (11) into equation (7) to get

$$\pi_{n,t} = \delta\Big(\frac{\alpha_n q_{n,t}}{x_{n,t}}\Big)\sigma^q_{n,t} M_n.$$

Now, use the result of Theorem 1 that $\alpha_n q_{n,t}\sigma^q_{n,t} = v^*_n \psi_t$. Combining these equations, we have

$$\pi_{n,t} = \delta v^*_n \psi_t \frac{M_n}{x_{n,t}}. \tag{38}$$

Assumption (12) is equivalent to

$$\delta\psi_t \min_{(i,j):i\neq j} \Big\| \frac{v^*_i M_i}{x_{i,t}} - \frac{v^*_j M_j}{x_{j,t}} \Big\| \geq \underline{\kappa}_t \quad \text{and} \quad \delta\psi_t \max_{(i,j):i\neq j} \Big\| \frac{v^*_i M_i}{x_{i,t}} - \frac{v^*_j M_j}{x_{j,t}} \Big\| \leq \bar{\kappa}_t.$$

Solving for ψ_t, we obtain inequality (13). The bounds for σ^*_t are a direct consequence of (13). □

Proof of Corollary 3. To get the both bounds, begin with the volatility bound (13) of Proposition 2 and use

$$\|\pi_{n,t}\| = \delta\psi_t \frac{v^*_n}{x_{n,t}}$$
$$\sqrt{A_t} = \delta\psi_t \sqrt{\sum_{n=1}^{N} x_{n,t}(\frac{v^*_n}{x_{n,t}})^2}.$$

The expression for $\|\pi_{n,t}\|$ comes from taking the norm of equation (38) and using the fact that MM' has ones on its diagonal (this was a normalization). The expression for $\sqrt{A_t}$ comes from expression (9) in Proposition 1. □

B Proofs for model extensions in Section 4

Proof of Proposition 4. Recall we have one of the two assumptions on the tradability of dB_t:

(a) There are no additional markets open beyond those assumed so far;

(b) There is an integrated market in which agents frictionlessly trade a zero-net-supply Arrow security that has a unit loading on dB_t.

We will nest cases (a) and (b) in the following setting. Introduce an Arrow security that pays off $\eta_{n,t}dt + dB_t$ per unit of time, where $(\eta_{n,t})_{n=1}^N$ will be determined endogenously. Thus, agent n faces the state-price density process, modified from (31):

$$d\xi_{n,t} = -\xi_{n,t}\Big[r_t dt + \tilde{\pi}_{n,t} d\tilde{Z}_{n,t} + \eta_{n,t} dB_t\Big]. \tag{39}$$

Let $\theta^{\text{agg}}_{n,t}$ be the fraction of wealth a location-n agent invests in the Arrow security, and let $\theta_{n,t}$ be the fraction of wealth invested in the location-specific capital asset as before. The wealth of agent n has the following dynamics (dynamic budget constraint)

$$\begin{aligned}\frac{dw_{n,t}}{w_{n,t}} = &\Big[r_t - \frac{c_{n,t}}{w_{n,t}} + \theta_{n,t}\sigma^q_{n,t}\tilde{\pi}_{n,t} + \Big(\theta_{n,t}(\nu + \varsigma^q_{n,t}) + \theta^{\text{agg}}_{n,t}\Big)\eta_{n,t}\Big]dt \\ &+ \theta_{n,t}\sigma^q_{n,t} d\tilde{Z}_{n,t} + \Big(\theta_{n,t}(\nu + \varsigma^q_{n,t}) + \theta^{\text{agg}}_{n,t}\Big) dB_t.\end{aligned} \tag{40}$$

To implement (a), where agents are not allowed to trade the Arrow security, we impose a fictitious market clearing condition $\theta_{n,t}^{\text{agg}} = 0$ for all n, which will pin down $\eta_{n,t}$ such that no trading in the Arrow security occurs. From the results of Cvitanić and Karatzas (1992), this implements the same equilibrium as if we never introduced this fictitious market. To implement (b), in which the Arrow market exists and is integrated, we impose $\eta_{n,t} = \eta_t$ for all n and clear the market via $\sum_{n=1}^{N} x_{n,t}\theta_{n,t}^{\text{agg}} = 0$. In both cases, we have the capital market clearing condition $\theta_{n,t} = \alpha_n q_{n,t} C_t / w_{n,t}$ as before.

Thus, we may nest cases (a) and (b) by solving unconstrained optimization problems for our investors, augmented with the general state-price density process (39) as long as $\eta_{n,t}$ is chosen appropriately. Given the state-price density, the pricing condition (32) is replaced by

$$\mu_{n,t}^{q} + g + \frac{1}{q_{n,t}} + \nu\varsigma_{n,t}^{q} - r_t = \sigma_{n,t}^{q}\tilde{\pi}_{n,t} + (\nu + \varsigma_{n,t}^{q})\eta_{n,t}.$$

Because all agents have log utility and effectively solve unconstrained portfolio problems with homogeneous wealth dynamics (40), they all consume δ fraction of their wealth, i.e., $c_{n,t} = \delta w_{n,t}$. Then, as B_t and $\tilde{Z}_{n,t}$ are independent, optimal consumption dynamics (34) are modified to read

$$\frac{dc_{n,t}}{c_{n,t}} = \left[r_t - \delta + \tilde{\pi}_{n,t}^2 + \eta_{n,t}^2\right]dt + \tilde{\pi}_{n,t}d\tilde{Z}_{n,t} + \eta_{n,t}dB_t.$$

Because $dw_{n,t}/w_{n,t} = dc_{n,t}/c_{n,t}$, we therefore have

$$\begin{aligned}
\tilde{\pi}_{n,t} &= \theta_{n,t}\sigma_{n,t}^{q} = \frac{\delta\alpha_n q_{n,t}}{x_{n,t}}\sigma_{n,t}^{q} \\
\eta_{n,t} &= \theta_{n,t}(\nu + \varsigma_{n,t}^{q}) + \theta_{n,t}^{\text{agg}} = \frac{\delta\alpha_n q_{n,t}}{x_{n,t}}(\nu + \varsigma_{n,t}^{q}) + \theta_{n,t}^{\text{agg}}.
\end{aligned}$$

The first equation is identical to (7).

Now, we aggregate. First, equation (3) still holds, since agents consume δ fraction of wealth, and since both the bond market and the Arrow markets are in zero net supply. Next, time-differentiate the goods market clearing condition $\sum_{n=1}^{N} c_{n,t} = C_t$ and match drift and diffusion terms to obtain

$$\begin{aligned}
r_t &= \delta + g - \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}^2 - \sum_{n=1}^{N} x_{n,t}\eta_{n,t}^2 \\
0 &= \sum_{n=1}^{N} x_{n,t}\tilde{\pi}_{n,t}M_n \\
\nu &= \sum_{n=1}^{N} x_{n,t}\eta_{n,t}.
\end{aligned}$$

Using the expressions for $\tilde{\pi}_{n,t}$ and $\eta_{n,t}$ above, along with the condition $\sum_{n=1}^{N} x_{n,t}\theta_{n,t}^{\text{agg}} = 0$ (which holds in cases (a) and (b) both), we obtain

$$\begin{aligned}
0 &= \sum_{n=1}^{N} \alpha_n q_{n,t}\sigma_{n,t}^{q}M_n \\
\nu &= \sum_{n=1}^{N} \delta\alpha_n q_{n,t}(\nu + \varsigma_{n,t}^{q}).
\end{aligned}$$

The first equation is (17). Thus, $\sigma^q_{n,t}$ and $\tilde{\pi}_{n,t}$ are solved exactly as in Theorem 1. The second equation, after using equation (3), yields (18). Letting $(\phi_{n,t})_{n=1}^{N-1}$ be arbitrary processes, and putting $\phi_{N,t} = -\sum_{n=1}^{N-1} \phi_{n,t}$, we may satisfy (18) by setting $\varsigma^q_{n,t}$ by $\phi_{n,t} = \alpha_n q_{n,t} \varsigma^q_{n,t}$.

It remains to solve for $(\eta_{n,t})_{n=1}^N$. In case (a), we use $\theta^{\text{agg}}_{n,t} = 0$ in conjunction with the expression for $\eta_{n,t}$ above to get $\eta_{n,t} = \frac{\delta \alpha_n q_{n,t}}{x_{n,t}}(\nu + \varsigma^q_{n,t})$. In case (b), we impose $\eta_{n,t} = \eta_t$ for all n, which after substituting into $\nu = \sum_{n=1}^N x_{n,t}\eta_{n,t}$ yields $\eta_t = \nu$. □

Proof of Proposition 5. Similar to the proof of Proposition 4, we introduce fictitious location-i-specific markets for trading $dB_{i,t}$ and $d\hat{B}_{i,t}$ and determine appropriate risk prices $\eta_{i,t}$ and $\hat{\eta}_{i,t}$ such that no trading occurs in these markets. Because the arguments are very similar to Proposition 4, we merely state the results. With this assumption, the location-specific state-price density is

$$d\xi_{i,t} = -\xi_{i,t}\Big[r_t dt + \tilde{\pi}_{i,t} d\tilde{Z}_{i,t} + \eta_{i,t} dB_t + \hat{\eta}_{i,t} d\hat{B}_{i,t}\Big].$$

Pricing equation (32) is replaced by

$$\mu^q_{i,t} + g + \frac{1}{q_{i,t}} + \hat{\nu}\hat{\varsigma}^q_{i,t} - r_t = \sigma^q_{i,t}\tilde{\pi}_{i,t} + \varsigma^q_{i,t}\eta_{i,t} + (\hat{\nu} + \hat{\varsigma}^q_{i,t})\hat{\eta}_{i,t}.$$

Optimal consumption is δ fraction of wealth, and consumption dynamics are given by

$$\frac{dc_{i,t}}{c_{i,t}} = \Big[r_t - \delta + \tilde{\pi}^2_{i,t} + \eta^2_{i,t} + \hat{\eta}^2_{i,t}\Big]dt + \tilde{\pi}_{i,t} d\tilde{Z}_{i,t} + \eta_{i,t} dB_{i,t} + \hat{\eta}_{i,t} d\hat{B}_{i,t}.$$

Since agent i needs to hold his local capital asset, which is valued at $\frac{\delta \alpha_{i,t} q_{i,t}}{x_{i,t}}$ fraction of his wealth (where $x_{i,t} := c_{i,t}/C_t$ is the consumption share), we require

$$\begin{aligned}
\tilde{\pi}_{i,t} &= \frac{\delta \alpha_{i,t} q_{i,t}}{x_{i,t}} \sigma^q_{i,t} \\
\eta_{i,t} &= \frac{\delta \alpha_{i,t} q_{i,t}}{x_{i,t}} \varsigma^q_{i,t} \\
\hat{\eta}_{i,t} &= \frac{\delta \alpha_{i,t} q_{i,t}}{x_{i,t}} (\hat{\nu} + \hat{\varsigma}^q_{i,t}).
\end{aligned}$$

Time-differentiating the goods market clearing condition $\sum_{i=1}^N c_{i,t} = C_t$ and matching drift and diffusion terms, we obtain

$$\begin{aligned}
r_t &= \delta + g - \sum_{i=1}^N x_{i,t}\Big(\tilde{\pi}^2_{i,t} + \eta^2_{i,t} + \hat{\eta}^2_{i,t}\Big) \\
0 &= \sum_{i=1}^N x_{i,t}\tilde{\pi}_{i,t} M_i \\
\hat{\nu}\alpha_{j,t} &= x_{j,t}\hat{\eta}_{j,t} + \alpha_{j,t}\sum_{i \neq j} x_{i,t}\eta_{i,t}\Big(\sum_{k \neq i}\alpha^2_{k,t}\Big)^{-1/2} \quad \forall j.
\end{aligned}$$

Substituting the expression for $\tilde{\pi}_{i,t}$, we obtain (4) once again. Thus, all the arguments from Theorem 1 continue to hold. This argument holds regardless of the equilibrium values of $(\varsigma^q_{i,t})_{i=1}^N$ and $(\hat{\varsigma}^q_{i,t})_{i=1}^N$, which are the remaining undetermined variables. This completes the proof. □

Proof of Proposition 6. First, define total expenditures $Y_t := \sum_{i=1}^N p_{i,t} y_{i,t}$. By goods market clearing in each market plus the definition of expenditures $e_{n,t} := \sum_{i=1}^N p_{i,t} c_{n,i,t}$, we have $Y_t = \sum_{i=1}^N e_{i,t}$. Also define the consumption and endowment (value) shares

$$x_{n,t} := \frac{e_{n,t}}{Y_t} \quad \text{and} \quad \alpha_{n,t} := \frac{p_{n,t} y_{n,t}}{Y_t}. \tag{41}$$

Note that the endowment share $\alpha_{n,t}$ is now time-varying and endogenous. We will characterize all equilibrium objects, except for self-fulfilling variables, in terms of $(x_{n,t})_{n=1}^N$, so we write their dynamics as

$$dx_{n,t} = \mu_{n,t}^x dt + \sigma_{n,t}^x \cdot dZ_t,$$

where $(\mu_{n,t}^x, \sigma_{n,t}^x)$ will be determined endogenously. Note that we are using the basis shocks dZ_t above, and indeed, the entire proof will be more straightforward if the analysis is done using the basis shocks rather than the extrinsic shocks $d\tilde{Z}_t = M dZ_t$.

Conjecture country n faces a state-price density

$$d\xi_{n,t} = -\xi_{n,t}[r_t dt + \pi_{n,t} dZ_t].$$

This implies the risk price on extrinsic shock $d\tilde{Z}_{n,t}$ is given by $\tilde{\pi}_{n,t}$, where $\pi_{n,t} = \tilde{\pi}_{n,t} M_n$, consistent with equation (11). Defining wealth $w_{n,t}$ and bond market position $\beta_{n,t}$, the static budget constraint can be written

$$w_{n,t} := \beta_{n,t} + q_{n,t} p_{n,t} y_{n,t} = \mathbb{E}_t\Big[\int_t^\infty \frac{\xi_{n,s}}{\xi_{n,t}}\Big(\sum_{i=1}^N p_{i,s} c_{n,i,s}\Big) ds\Big]. \tag{42}$$

Agent n maximizes (20) subject to (42). Optimal consumption satisfies

$$e^{-\delta t} \omega \frac{1}{c_{n,n,t}} = \xi_{n,t} p_{n,t} \tag{43}$$

$$e^{-\delta t} \frac{1-\omega}{N-1} \frac{1}{c_{n,i,t}} = \xi_{n,t} p_{i,t}, \quad i \neq n. \tag{44}$$

Dividing optimality conditions (43)-(44), we obtain the usual Cobb-Douglas static expenditure rule:

$$p_{n,t} c_{n,n,t} = \omega e_{n,t} \tag{45}$$

$$p_{i,t} c_{n,i,t} = \frac{1-\omega}{N-1} e_{n,t}, \quad i \neq n. \tag{46}$$

Substituting (43)-(44) back into (42), we obtain the usual log utility dynamic expenditure rule $e_{n,t} = \delta w_{n,t}$. Log utility portfolio choice says that optimal wealth exposure to dZ_t is equal to $w_{n,t} \pi_{n,t}$. Putting these results together, we obtain expenditure dynamics

$$\frac{de_{n,t}}{e_{n,t}} = \Big[r_t - \delta + \|\pi_{n,t}\|^2\Big] dt + \pi_{n,t} dZ_t. \tag{47}$$

Next, we aggregate.[17] Start with location-n goods market clearing $\sum_{i=1}^N c_{i,n,t} = y_{n,t}$, multiply by price $p_{n,t}$, substitute (45)-(46), and then divide by Y_t to obtain

$$\alpha_{n,t} = x_{n,t} \omega + (1 - x_{n,t}) \Big(\frac{1-\omega}{N-1}\Big). \tag{48}$$

[17] To obtain equation (21), sum agents' wealth, apply bond market clearing $\sum_{n=1}^N \beta_{n,t} = 0$, and use the fact that all agents spend δ fraction of their wealth.

From equation (48), we can solve for all the prices $(p_{n,t})_{n=1}^{N}$ given Y_t. The special price is $p_{1,t} = 1$ (good 1 is the numeraire), so that equation (48) can be used with $n = 1$ to solve for Y_t. Indeed, combining (41) with (48) with $p_{1,t} = 1$, we find

$$Y_t = \frac{y_{1,t}}{x_{1,t}\omega + (1 - x_{1,t})(\frac{1-\omega}{N-1})}. \tag{49}$$

Time-differentiating equation (49), we can write

$$dY_t = Y_t[g - \Omega_{0,t} + \|\Omega_{1,t}\|^2]dt + Y_t\Omega_{1,t} \cdot dZ_t, \tag{50}$$

where

$$\Omega_{0,t} := \frac{\omega - \frac{1-\omega}{N-1}}{x_{1,t}\omega + (1 - x_{1,t})(\frac{1-\omega}{N-1})}\mu^x_{1,t} \tag{51}$$

$$\Omega_{1,t} := -\frac{\omega - \frac{1-\omega}{N-1}}{x_{1,t}\omega + (1 - x_{1,t})(\frac{1-\omega}{N-1})}\sigma^x_{1,t} \tag{52}$$

Time-differentiating the aggregate expenditure equation $Y_t = \sum_{n=1}^{N} e_{n,t}$ with (47) and (50), we get an equation for the risk-free rate,

$$r_t = g + \delta - \sum_{n=1}^{N} x_{n,t}\|\pi_{n,t}\|^2 - \Omega_{0,t} + \|\Omega_{1,t}\|^2, \tag{53}$$

and an equation for the risk price vector,

$$\sum_{n=1}^{N} x_{n,t}\pi'_{n,t} = \Omega_{1,t}. \tag{54}$$

Note that the local asset has value $\alpha_{n,t}q_{n,t}Y_t$, so its return process can be written

$$\begin{aligned}
dR_{n,t} &= \mu^R_{n,t}dt + \sigma^R_{n,t} \cdot dZ_t \qquad (55)\\
\mu^R_{n,t} &:= \mu^q_{n,t} + \frac{1}{q_{n,t}} + g - \Omega_{0,t} + \|\Omega_{1,t}\|^2 + \frac{(\omega - \frac{1-\omega}{N-1})\mu^x_{n,t}}{\alpha_{n,t}} \\
&\quad + \sigma^q_{n,t}M_n\Big(\Omega_{1,t} + \frac{(\omega - \frac{1-\omega}{N-1})\sigma^x_{n,t}}{\alpha_{n,t}}\Big) + \Big(\omega - \frac{1-\omega}{N-1}\Big)\Omega_{1,t} \cdot \sigma^x_{n,t} \\
\sigma^R_{n,t} &:= \sigma^q_{n,t}M'_n + \Omega_{1,t} + \frac{(\omega - \frac{1-\omega}{N-1})\sigma^x_{n,t}}{\alpha_{n,t}}.
\end{aligned}$$

Since $\theta_{n,t} = \delta\alpha_{n,t}q_{n,t}/x_{n,t}$ is the equilibrium portfolio share of representative agent n in his country's local asset (a scalar), and this agent optimally sets $\theta_{n,t}\sigma^R_{n,t} = \pi_{n,t}$, we obtain

$$\pi'_{n,t} = \frac{\delta\alpha_{n,t}q_{n,t}}{x_{n,t}}\Big[\sigma^q_{n,t}M'_n + \Omega_{1,t} + \Big(\omega - \frac{1-\omega}{N-1}\Big)\frac{\sigma^x_{n,t}}{\alpha_{n,t}}\Big]. \tag{56}$$

which is the modified version of equation (7).[18] In addition, we have the local no-arbitrage pricing relation for local asset n:

$$\mu^R_{n,t} - r_t = \pi_{n,t}\sigma^R_{n,t}, \tag{57}$$

[18]Note that, for the numeraire $n = 1$, this equation reads $\tilde{\pi}_{1,t} = \delta\alpha_{1,t}q_{1,t}\sigma^q_{1,t}/x_{1,t}$, exactly as equation (7).

which, given (55), can be viewed as an equation for $\mu^q_{n,t}$.

All equilibrium quantities and prices will be solved in terms of the consumption shares $(x_{n,t})_{n=1}^N$ and their dynamics, so at this point we use Itô's formula to time-differentiate the definition of $x_{n,t} = \frac{e_{n,t}}{\sum_{i=1}^N e_{i,t}}$, using (47). The result is

$$dx_{n,t} = \underbrace{x_{n,t}\Big[\|\pi_{n,t}\|^2 - \sum_{i=1}^N x_{i,t}\|\pi_{i,t}\|^2 + \|\Omega_{1,t}\|^2 - \pi_{n,t}\Omega_{1,t}\Big]}_{=\mu^x_{n,t}} dt + \underbrace{x_{n,t}\Big[\pi'_{n,t} - \Omega_{1,t}\Big]}_{=\sigma^x_{n,t}} \cdot dZ_t. \tag{58}$$

Combining (51) and (52) with (58) for $n = 1$, we obtain four equations in the four unknowns $(\Omega_0, \Omega_1, \mu^x_1, \sigma^x_1)$, which has the unique solution (in terms of risk prices)

$$\sigma^x_{1,t} = \frac{N-1}{1-\omega}\alpha_{1,t}x_{1,t}\pi'_{1,t} \tag{59}$$

$$\mu^x_{1,t} = x_{1,t}\Big[\Big(1 - \frac{N-1}{1-\omega}\alpha_{1,t} + \Big(\frac{N-1}{1-\omega}\Big)^2\alpha^2_{1,t}\Big)\|\pi_{1,t}\|^2 - \sum_{i=1}^N x_{i,t}\|\pi_{i,t}\|^2\Big] \tag{60}$$

$$\Omega_{1,t} = -\frac{N-1}{1-\omega}\alpha_{1,t}\Big(\omega - \frac{1-\omega}{N-1}\Big)\frac{x_{1,t}}{\alpha_{1,t}}\pi'_{1,t} \tag{61}$$

$$\Omega_{0,t} = \Big(\omega - \frac{1-\omega}{N-1}\Big)\frac{x_{1,t}}{\alpha_{1,t}}\Big[\Big(1 - \frac{N-1}{1-\omega}\alpha_{1,t} + \Big(\frac{N-1}{1-\omega}\Big)^2\alpha^2_{1,t}\Big)\|\pi_{1,t}\|^2 - \sum_{i=1}^N x_{i,t}\|\pi_{i,t}\|^2\Big]. \tag{62}$$

After this, we can substitute (61) back into (58) for $n \neq 1$ to obtain $(\mu^x_{n,t}, \sigma^x_{n,t})_{n\neq 1}$ completely in terms of the risk prices and consumption shares. In addition, (61)-(62) can be substituted back into (53) to obtain r_t in terms of risk prices and consumption shares. Similarly, substituting (61), (59), and (58) into (56) and rearranging and simplifying, we obtain

$$\sigma^q_{n,t}M_n = \frac{x_{n,t}}{\delta\alpha_{n,t}q_{n,t}}\pi_{n,t} - \Big[\frac{x_{n,t}(\omega - \frac{1-\omega}{N-1})}{x_{n,t}\omega + (1-x_{n,t})\frac{1-\omega}{N-1}}\Big]\pi_{n,t} + \Big[\frac{x_{1,t}(\omega - \frac{1-\omega}{N-1})}{x_{1,t}\omega + (1-x_{1,t})\frac{1-\omega}{N-1}}\Big]\pi_{1,t}, \tag{63}$$

which is an equation for $\sigma^q_{n,t}M_n$ in terms of risk prices, consumption shares, and price-dividend ratios $(q_{n,t})_{n=1}^N$.

Finally, it remains to solve for risk prices $(\pi_{n,t})_{n=1}^N$. Substitute (61) into (54) to obtain

$$\omega x_{1,t}\pi_{1,t} + \frac{1-\omega}{N-1}\sum_{n=2}^N x_{n,t}\pi_{n,t} = 0. \tag{64}$$

Transform risk prices back into the extrinsic shock risk prices via $\pi_{n,t} = \tilde{\pi}_{n,t}M_n$, and write equation (66) as a matrix equation:

$$M'v_t = 0, \tag{65}$$

where $v_t = (\omega x_{1,t}\tilde{\pi}_{1,t}, \frac{1-\omega}{N-1}x_{2,t}\tilde{\pi}_{2,t}, \ldots, \frac{1-\omega}{N-1}x_{N,t}\tilde{\pi}_{N,t})'$. Therefore, under Assumption 1, there exists a non-zero time-invariant solution $v_t = v^* \neq 0$ to (65). Let v^* denote such a solution. It is clear that $\psi_t v^*$ is also a solution for any scalar process $\psi_t > 0$. Thus, a valid extrinsic risk price is

$$\tilde{\pi}_{n,t} = \begin{cases} \omega^{-1}x^{-1}_{n,t}v^*_n\psi_t, & \text{if } n = 1; \\ (1-\omega)^{-1}(N-1)x^{-1}_{n,t}v^*_n\psi_t, & \text{if } n \neq 1. \end{cases} \tag{66}$$

Consequently, $\pi_{n,t} = \tilde{\pi}_{n,t} M_n$ is also only determined up to scale.

Substitute $\pi_{n,t} = \tilde{\pi}_{n,t} M_n$ back into (63), and use Assumption 1 to write the result in the general form:

$$\sigma^q_{n,t} = \frac{x_{n,t}\tilde{\pi}_{n,t}}{\delta \alpha_{n,t} q_{n,t}} - \frac{x_{n,t}(\omega - \frac{1-\omega}{N-1})\tilde{\pi}_{n,t}}{x_{n,t}\omega + (1-x_{n,t})\frac{1-\omega}{N-1}} + \frac{x_{1,t}(\omega - \frac{1-\omega}{N-1})\tilde{\pi}_{1,t}}{x_{1,t}\omega + (1-x_{1,t})\frac{1-\omega}{N-1}}\big(\mathbf{1}_{M_1=M_n} - \mathbf{1}_{M_1=-M_n}\big). \tag{67}$$

Because of the homogeneity of the right-hand-side of (67) in ψ_t, the left-hand-side is also only determined up to scale.

Combined with Assumption 1, we can now prove that $\alpha_{n,t} q_{n,t} \sigma^q_{n,t} = H(x_{1,t}, x_{n,t}, q_{n,t}) v_n^* \psi_t$ for some function H. Indeed, under part (i) or (ii) of Assumption 1, the final term of (67) vanishes (recall these parts say $v_1^* = 0$ or $M_1 \equiv 0$). Given formula (66) for $\tilde{\pi}_n$, and formula (48) for α_n, we have existence of H under parts (i) or (ii), given by

$$\text{[parts (i) or (ii)]} \quad H(x_1, x_n, q_n) = \frac{N-1}{1-\omega}\Big[\delta^{-1} - q_n\Big(\omega - \frac{1-\omega}{N-1}\Big)\Big]. \tag{68}$$

On the other hand, under part (iii) of Assumption 1, along with equation (66), we may replace $(\mathbf{1}_{M_1=M_n} - \mathbf{1}_{M_1=-M_n})\tilde{\pi}_{1,t} = -\omega^{-1} x_{1,t}^{-1} v_n^* \psi_t$ in the final term of equation (67). This is because if $M_1 = M_n$ (respectively, $M_1 = -M_n$), then a valid solution to $M' v^* = 0$ involves $v_1^* = -v_n^*$ (respectively, $v_1^* = v_n^*$). Thus, using (48), we have

$$\begin{aligned} \text{[part (iii)]} \quad H(x_1, x_n, q_n) = {} & \frac{N-1}{1-\omega}\Big[\delta^{-1} - q_n\Big(\omega - \frac{1-\omega}{N-1}\Big)\Big] \\ & - \frac{x_n\omega + (1-x_n)(\omega - \frac{1-\omega}{N-1})}{x_1\omega + (1-x_1)(\omega - \frac{1-\omega}{N-1})} q_n \Big(\omega - \frac{1-\omega}{N-1}\Big)\frac{1}{\omega}. \end{aligned} \tag{69}$$

The only remaining detail to prove is $H \neq 0$, which we do in a case-by-case method.

Under either part (i) or (ii) of Assumption 1, equation (68) shows that $H = 0$ requires a time-invariant q_n given by

$$q_n = \delta^{-1}(\omega - \frac{1-\omega}{N-1})^{-1}. \tag{70}$$

In addition, parts (i) or (ii) or Assumption 1 also imply $\sigma_1^x = \Omega_1 = 0$, $\Omega_0 = -(\omega - \frac{1-\omega}{N-1}) A x_1/\alpha_1$, and $r = g + \delta - A - \Omega_0$, where A is defined by (8). Substitute (70) into the pricing formula (57) – along with $\mu_n^q = \sigma_n^q = 0$ (because q_n is time-invariant), $\mu_n^x = x_n[\tilde{\pi}_n^2 - A]$, $\sigma_n^x = x_n \tilde{\pi}_n M_n'$, and the results for Ω_0, Ω_1, r – to obtain

$$\delta(1-\omega)\frac{N}{N-1} = A\Big[1 - \Big(\omega - \frac{1-\omega}{N-1}\Big)\frac{x_n}{\alpha_n}\Big].$$

But this is impossible unless the equilibrium is non-stochastic, including $x_{n,t}$, which contradicts $\tilde{\pi}_n \neq 0$ (and hence $\sigma_n^x \neq 0$). Thus, $H \neq 0$ is required under parts (i) or (ii) of Assumption 1.

On the other hand, under part (iii) of Assumption 1, equation (69) shows that $H = 0$ requires

$$\frac{1}{\delta q_n} = \Big(\omega - \frac{1-\omega}{N-1}\Big)\Big(1 + \frac{1-\omega}{\omega(N-1)}\frac{\alpha_n}{\alpha_1}\Big).$$

Since q_n is locally non-stochastic when $H = 0$, this equation implies α_n/α_1 is locally non-stochastic. Equating the Brownian terms of $\log(\alpha_n)$ and $\log(\alpha_1)$, using equation (48) in conjunction with formulas (58), (59), and (61), we obtain

$$\frac{x_n}{\alpha_n}\Big[\pi_n' + \frac{N-1}{1-\omega} x_1 \Big(\omega - \frac{1-\omega}{N-1}\Big)\pi_1'\Big] = \frac{x_1}{\alpha_1}\frac{N-1}{1-\omega}\alpha_1 \pi_1'. \tag{71}$$

Using $\pi_n = \tilde{\pi}_n M_n$, equation (66) for $\tilde{\pi}_n$, and our assumption that $M_1 = M_n$ and $v_1^* = -v_n^*$ (or $M_1 = -M_n$ and $v_1^* = v_n^*$), we have $\pi_1 = -\frac{1-\omega}{\omega(N-1)}\frac{x_n}{x_1}\pi_n$. Making this substitution into equation (71), and dropping $\pi_n \neq 0$ and $x_n > 0$, we have $\alpha_n = -\omega + x_n(\omega - \frac{1-\omega}{N-1})$, which contradicts (48) for any $x_n \in [0,1]$. Thus, $H \neq 0$ is required under part (iii) of Assumption 1.

Finally, we prove that Assumption 1 is necessary for self-fulfilling volatility. If M is full rank, then the only solution to (65) is $v^* = 0$, which directly implies $\sigma_n^q = 0$ for all n. Thus, let $\text{rank}(M) < N$, take solution $v^* \neq 0$ to (65), but suppose none of parts (i), (ii), nor (iii) of Assumption 1 hold. In this case, the result of equation (67) for $\tilde{\pi}_{n,t}$ still holds, along with $\pi_n = \tilde{\pi}_n M_n$. Substitute these results into equation (63) to obtain

$$\sigma_{n,t}^q M_n = \frac{N-1}{1-\omega}\Big(\frac{1}{\delta\alpha_{n,t}q_{n,t}} - \frac{\omega - \frac{1-\omega}{N-1}}{\alpha_{n,t}}\Big)v_n^* M_n \psi_t + \frac{1}{\omega}\Big(\frac{\omega - \frac{1-\omega}{N-1}}{\alpha_{1,t}}\Big)v_1^* M_1 \psi_t. \tag{72}$$

Consider $n \neq 1$ such that $v_n^* \neq 0$. Then, note that $v_1^* \neq 0$ and $M_1 \neq \pm M_n$, which immediately implies the only solution to (72), which constitutes at least two linearly independent equations, is $\sigma_{n,t}^q = \psi_t = 0$. Thus, as long as there exists any $n \neq 1$ with $v_n^* \neq 0$, we are done. Suppose, leading to contradiction, that $v^* = (1, 0, \ldots, 0)'$. If so, then equation (65) implies $M_1 \equiv 0$, which contradicts the supposition that Assumption 1 fails. □

Proof of Proposition 7. Given self-fulfilling volatility, we construct an arbitrage as follows. Define

$$\omega_n := \begin{cases} \omega, & \text{if } n = 1; \\ \frac{1-\omega}{N-1}, & \text{if } n \neq 1. \end{cases}$$

Consider a portfolio that goes long $\delta\omega_n\alpha_{n,t}q_{n,t}$ of each asset $n = 1, \ldots, N$. Using pricing equation (57), along with the formula for return volatility in (55) and the formula linking risk prices to volatility in (56), the expected excess return of this portfolio is

$$\sum_{n=1}^{N} \frac{\delta\omega_n\alpha_{n,t}q_{n,t}}{\sum_{i=1}^N \delta\omega_i\alpha_{i,t}q_{i,t}}\pi_{n,t}\sigma_{n,t}^R = \frac{1}{\sum_{i=1}^N \delta\omega_i\alpha_{i,t}q_{i,t}}\Big(\sum_{n=1}^N \omega_n x_{n,t}\|\pi_{n,t}\|^2\Big) \geq 0,$$

which is strictly positive as long as any self-fulfilling volatility obtains. At the same time, the return volatility of this portfolio is given by

$$\sum_{n=1}^{N} \frac{\delta\omega_n\alpha_{n,t}q_{n,t}}{\sum_{i=1}^N \delta\omega_i\alpha_{i,t}q_{i,t}}\sigma_{n,t}^R = \frac{1}{\sum_{i=1}^N \delta\omega_i\alpha_{i,t}q_{i,t}}\Big(\sum_{n=1}^N \omega_n x_{n,t}\pi_{n,t}'\Big) = 0,$$

where the equality to zero comes from (66).

Conversely, if $\sigma_{n,t}^q = 0$ for all n, then equation (63) implies $\pi_{n,t} = 0$ for all n. Hence, all assets earn the riskless rate by (57), implying no arbitrage. □

Proof of Proposition 8. To obtain the equality in (23), use (22) and substitute $\|\pi_{n,t}\| = |\tilde{\pi}_{n,t}| = \frac{v_n^*}{\omega_n x_{n,t}}\psi_t$ (see equation (66)). This holds since $\pi_{n,t} = \tilde{\pi}_{n,t}M_n$ and $\|M_n\| = 1$ for all $n \neq 1$, and since we may without loss of generality pick $v_1^* = 0$ when $M_1 = 0$. To obtain the inequality in (23), apply Jensen's inequality to show $\sum_{n=1}^N \frac{\omega_n x_{n,t}}{\sum_{i=1}^N \omega_i x_{i,t}}(\frac{v_n^*}{\omega_n x_{n,t}})^2 \geq (\sum_{i=1}^N \omega_i x_{i,t})^{-2}(\sum_{n=1}^N v_n^*)^2$ and then use $\alpha_{1,t} = \sum_{i=1}^N \omega_i x_{i,t}$.

To obtain the equality in (24), use the definition of σ_t^*, along with (21), (55), and again $\|\pi_{n,t}\| = \frac{v_n^*}{\omega_n x_{n,t}}\psi_t$. To obtain the inequality in (24), apply (23). □

C Markov equilibria and sunspots

As mentioned in the text, the self-fulfilling volatility process ψ_t of Theorem 1 need not be Markov. But can it be Markov? In this section, we argue self-fulfilling Markov equilibria cannot simply depend on "fundamental" variables like the wealth distribution, but they can exist if we introduce "sunspot" variables that coordinate beliefs.

If we restricted ourselves to look for volatility $\psi_t = \Psi(x_{1,t}, \dots, x_{N,t})$ that were some function of the consumption shares, then the entire equilibrium would be characterized by the vector $x_t := (x_{1,t}, \dots, x_{N,t})$. This can be seen from the dynamics of these consumption shares in (16) Since ψ_t and hence $\tilde{\pi}_{n,t}$ would solely be functions of consumption shares, x_t would be a sufficient statistic or state variable for the economy. But as the following remark suggests and subsequent example shows explicitly, these type of Markov equilibria are too restrictive for our purposes, ruling out interesting self-fulfilling phenomena.

Remark 3 (Markov equilibrium without "sunspots")**.** Conjecture that $x_t := (x_{1,t}, \dots, x_{N,t})$ is a sufficient statistic (state variable) describing all equilibrium objects. In that case, by the usual application of Itô's formula to $q_{n,t} = q_n(x_t)$, the entire equilibrium is characterized by a system of PDEs for $(q_n)_{n=1}^N$. Of particular interest is the "Itô condition"

$$\sigma_{n,t}^q M_n = \sum_{i=1}^N \sigma_{i,t}^x M_i \frac{\partial \log q_n(x_t)}{\partial x_{i,t}},$$

so that volatilities are tied to consumption shares. In such a Markov equilibrium, we could use (7) and (16) to substitute $\sigma_{i,t}^x = \delta \alpha_i q_{i,t} \sigma_{i,t}^q$ into the "Itô condition" to obtain[19]

$$M' \Big[D_t^{-1} - \delta G_t \Big] \mathrm{diag}(v_t) = 0_{N \times N},$$

where $D_t := \mathrm{diag}[(\alpha_n q_{n,t})_{n=1}^N]$, $G_t := [\frac{\partial \log q_n(x_t)}{\partial x_t'}]_{1 \le n \le N}$, and $v_t = (\alpha_1 q_{1,t} \sigma_{1,t}^q, \dots, \alpha_N q_{N,t} \sigma_{N,t}^q)'$ is the unknown column vector of volatilities. Suppose $\mathrm{rank}(M) = N - 1$ so that our equilibrium has potential for self-fulfilling volatility. Then, either $v_t \equiv 0$, or there exists an $N \times 1$ vector $v^* \neq 0$, in the null-space of M' and unique up to scale, such that

$$\Big[D_t^{-1} - \delta G_t \Big] \mathrm{diag}(v_t) = v^* \otimes \mathbf{1}'.$$

But this constitutes N^2 equations for the N unknowns in v_t, which will not generically be solvable. In summary, this Markov equilibrium will generically have $v_t \equiv 0$.

To more clearly see the effect of restricting ourselves to such a Markov equilibrium depending only on consumption shares, let us return to example 2, which allows an explicit solution. Whereas a non-Markov equilibrium can have arbitrary volatility, a Markov equilibrium cannot have any volatility at all!

Example 2 (Two-by-two redistribution, continued)**.** Equations (4) and (7) imply the following three equations in the four unknowns $(\tilde{\pi}_1, \tilde{\pi}_2, \sigma_1^q, \sigma_2^q)$:

$$\begin{aligned} \tilde{\pi}_{1,t} &= \delta \alpha_1 q_{1,t} \sigma_{1,t}^q / x_{1,t} \\ \tilde{\pi}_{2,t} &= \delta \alpha_2 q_{2,t} \sigma_{2,t}^q / x_{2,t} \\ x_{1,t} \tilde{\pi}_{1,t} &= x_{2,t} \tilde{\pi}_{2,t}. \end{aligned}$$

[19] This matrix equation is derived by plugging in $\sigma_{i,t}^x = \delta \alpha_i q_{i,t} \sigma_{i,t}^q$ into the "Itô condition" and then stacking the equations for $n = 1, \dots, N$.

The general solution to this system is, for some $\psi_t \geq 0$,

$$\sigma^q_{1,t} = \psi_t / \alpha_1 q_{1,t} \quad \text{and} \quad \sigma^q_{2,t} = \psi_t / \alpha_2 q_{2,t}$$
$$\tilde{\pi}_{1,t} = \delta \psi_t / x_{1,t} \quad \text{and} \quad \tilde{\pi}_{2,t} = \delta \psi_t / x_{2,t}.$$

One solution to this system is to take $\psi_t \equiv 0$, in which case $\tilde{\pi}_{1,t} \equiv \tilde{\pi}_{2,t} \equiv \sigma^q_{1,t} \equiv \sigma^q_{2,t} \equiv 0$. In such an equilibrium, the same analysis as in remark 1 shows that $q_{1,t} \equiv q_{2,t} \equiv \delta^{-1}$ for all times. But this is not the only solution: there are a continuum of stochastic equilibria that are indexed by the asset volatility process $\{\psi_t : t \geq 0\}$. Theorem 1 places no further restrictions on ψ_t.

In a Markov equilibrium, it suffices to keep track of $x_t = x_{1,t}$ as the unique state variable, since $x_{2,t} = 1 - x_t$. Let $q_{n,t} = q_n(x_t)$ for functions q_1 and q_2 and let $\psi_t = \Psi(x_t)$ for some function Ψ. We can use Itô's formula to get $\sigma^q_{1,t} = \sigma^x_t q_1'(x_t)/q_1(x_t)$ and $\sigma^q_{2,t} = -\sigma^x_t q_2'(x_t)/q_2(x_t)$, where $\sigma^x_t = x_t \tilde{\pi}_{1,t} = \delta \psi_t$. So we have the Markov restrictions (after substituting the results above)

$$\Psi(x_t) = \delta \Psi(x_t) \alpha_1 q_1'(x_t) \quad \text{and} \quad \Psi(x_t) = -\delta \Psi(x_t) \alpha_2 q_2'(x_t).$$

For these equations to hold, there are two possibilities: we must either have (i) $\Psi(\cdot) \equiv 0$ or (ii) functions (q_1, q_2) satisfy

$$q_1(x) = \bar{q}_1 + \frac{x}{\delta \alpha_1} \quad \text{and} \quad q_2(x) = \bar{q}_2 + \frac{1-x}{\delta \alpha_2}.$$

Because of aggregation equation (3), we also require $\alpha_1 \bar{q}_1 + \alpha_2 \bar{q}_2 = 0$, so $\bar{q}_1 = \bar{q}_2 = 0$ (otherwise, $\bar{q}_1$ and $\bar{q}_2$ have opposite signs, so that either $q_1(0)$ or $q_2(1)$ is negative, which violates optimality of local agents' portfolios). This proves that any positive-volatility Markov equilibrium must have equilibrium prices

$$q_{n,t} = \frac{x_{n,t}}{\delta \alpha_n}.$$

But in equilibrium, there is an additional restriction on $q_{n,t}$, namely the pricing conditions

$$\mu^q_{n,t} + g + \frac{1}{q_{n,t}} - r_t = \sigma^q_{n,t} \tilde{\pi}_{n,t}, \tag{73}$$

which says that investors earn a fair risk premium on their investments. Using Itô's formula to get $\mu^q_n = \mu^x q_n'(x)/q_n(x)$ and substituting the results above, one can show that (73) translates to the requirement that $x_t = \alpha_1$ for all t. This contradicts the conjecture that x_t is a diffusion process, which implies $\psi_t = 0$ is the only Markov equilibrium.

Does this rule out Markov equilibria? No, as long as we enrich our state space with additional variables ("sunspots") that act as coordinating devices.

Example 2 (Two-by-two redistribution, continued)**.** Introduce a sunspot variable s_t that follows an exogenous Itô process (i.e., μ^s, σ^s are pre-specified):

$$ds_t = \mu^s(s_t, x_t)dt + \sigma^s(s_t, x_t)dZ^{(1)}_t.$$

We only include a loading on the first component of Z_t in order to be consistent with the M of this example (recall, M sets $\tilde{Z}_{1,t} = Z^{(1)}_t$ and $\tilde{Z}_{2,t} = -Z^{(1)}_t$). Note that (s_t, x_t) jointly form a Markov diffusion and can be considered the state variables for equilibrium, as long as volatility depends only on these variables: $\psi_t = \Psi(s_t, x_t)$ for some function Ψ to be determined.

Using Itô's formula, we now have the following connection between price-dividend volatility and state volatility:

$$q_1\sigma_1^q = \sigma^x\partial_x q_1 + \sigma^s\partial_s q_1 \quad \text{and} \quad -q_2\sigma_2^q = \sigma^x\partial_x q_2 + \sigma_t^s\partial_s q_2.$$

We keep only the first of these equations, because the second is redundant given (3). Substituting $\sigma_1^q = \Psi/\alpha_1 q_1$ and $\sigma^x = \delta\Psi$, we obtain

$$\Psi(s,x) = \sigma^s(s,x)\frac{\alpha_1\partial_s q_1(s,x)}{1-\delta\alpha_1\partial_x q_1(s,x)}. \tag{74}$$

Next, use Itô's formula to plug $q_1\mu_1^q = [\mu^x\partial_x + \mu^s\partial_s + \frac{1}{2}(\sigma^x)^2\partial_{xx} + \frac{1}{2}(\sigma^s)^2\partial_{ss} + \sigma^x\sigma^s\partial_{xs}]q_1$ in the pricing equation (73), and then substitute the other equilibrium objects to obtain

$$\mathcal{L}q_1 = \delta q_1 - \frac{\delta^2\Psi^2}{x(1-x)}q_1 + \frac{\delta\Psi^2}{\alpha_1 x} - 1, \tag{75}$$

where $\mathcal{L}$ is a differential operator defined by

$$\mathcal{L} := \frac{\delta^2\Psi^2(1-2x)}{x(1-x)}\partial_x + \mu^s\partial_s + \frac{1}{2}\delta^2\Psi^2\partial_{xx} + \frac{1}{2}(\sigma^s)^2\partial_{ss} + (\delta\Psi)(\sigma^s)\partial_{xs}$$

and Ψ is given by (74). Equation (75) is a PDE for q_1. We need only solve the q_1-PDE and may determine q_2 from (3) that automatically satisfies its pricing equation (73). Assuming existence of a solution to this PDE, we have a self-fulfilling Markov equilibrium with sunspots.

D Other empirical proxies for volatility

Recall our baseline choice for $\hat{\sigma}_t^*$ in (30) uses only the monthly volatility of daily price changes for the 10-year US Treasury Note. This measure is simple and transparent but has several drawbacks: (a) it is not a very precise estimate of time-t conditional volatility; (b) it includes no volatility information for the synthetic US notes, which constitute the other leg of the CIP trade; (c) and unlike the model, in which the assets used to construct A_t correspond to those used to measure σ_t^*, this proxy uses long-maturity notes instead of the 3-month bills comprising $\hat{A}_t^{(1/4)}$.

To help address concerns (a)-(c), we also consider three alternative measures of $\hat{\sigma}_t^*$.

(a) For a more real-time measure of conditional volatility, we also examine the CBOE's 10-year Treasury VIX (TYVIX), which is the implied 30-day volatility of CBOT futures on 10-year US Treasury Notes. The downside of asset-implied volatility is that it corresponds to risk-neutral volatility and may be a biased estimate of actual volatility.

(b) To include information for the foreign leg, we also compute a foreign volatility analogue to (30). In particular, we compute the volatility of the 10-year constant maturity foreign note's daily price changes, measured in USD (i.e., the bond prices are adjusted by the spot exchange rate each day). We then take a value-weighted average of the 10-year Constant Maturity US Treasury Note volatility and this foreign note volatility, which delivers a country-specific volatility measure. The downside of this construction is that it introduces additional variation by using the spot exchange rate to convert future prices to dollars, rather than the forward exchange rate as in the CIP trade.

(c) To bring the assets in the volatility construction as close as possible to those used in the arbitrage trade, we examine the value-weighted average return volatilities of the 3-month US Treasury bill and the 3-month synthetic US bill. The downsides of using this proxy are twofold. First, because we have no way of interpolating the forward exchange rate curve between the 3-month and 1-month forward rates, we construct 2-month holding period returns on both bills. The prices of these bills are constructed using country-specific IBOR. This is very long relative to the maturity of the bill, so our estimate of a conditional volatility is likely to be highly imprecise. Second, mainly due to their short durations, 3-month bill volatilities are mechanically much smaller than those of 10-year notes (approximately 40 times smaller, as their relative durations suggests). Whereas σ_t^* and $\sqrt{A_t}$ are on approximately the same scale in equation (26), which comes from a model with infinitely-lived assets, 3-month bill volatility is not on a scale comparable to the 3-month CIP deviation.

We repeat our analysis with these proxies. Aggregate time series plots associating volatility to arbitrage profits are below in figures 5, 6, and 7. Disaggregated analysis at the currency level are below in figures 8, 9, and 10.

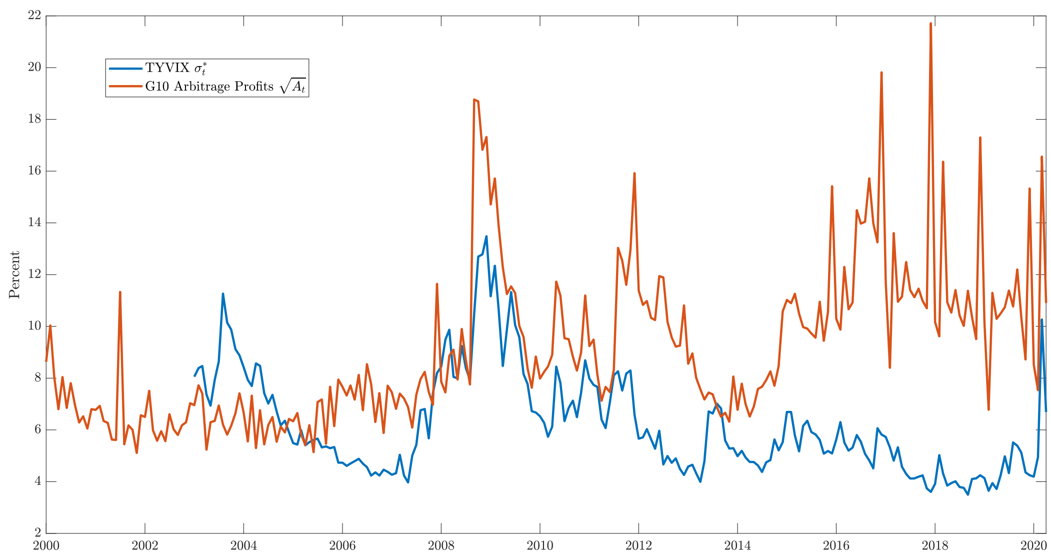

Figure 5: Monthly time-series of proxies for A_t and σ_t^*. Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is constructed using the monthly average of TYVIX, which is the implied volatility of CBOE options on 10-year US Treasury Note futures. Both measures are annualized. Currency data are from Bloomberg and range from Jan. 2000 to Apr. 2020. TYVIX data are from the CBOE and range from Jan. 2003 to Apr. 2020.

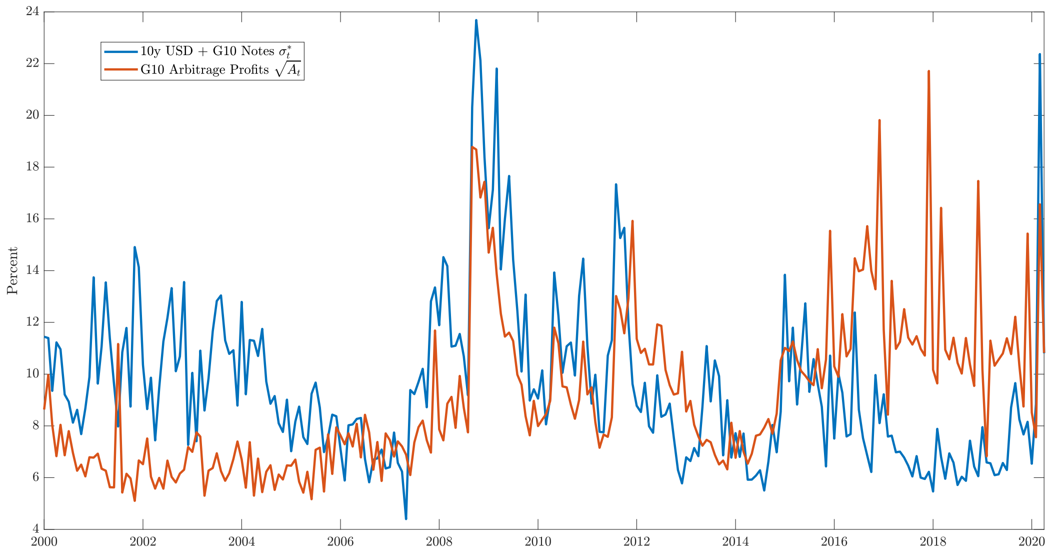

Figure 6: Monthly time-series of proxies for A_t and σ_t^*. Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is constructed using the value-weighted average volatilities of daily log price changes on (i) the 10-year Constant Maturity US Treasury Note; and (ii) one of the G10's 10-year constant maturity note, adjusted by the respective spot exchange rate to be measured in USD. Volatility σ_t^* is calculated as a within-month standard deviation of these daily price changes, then (simple) averaged across the G10 currencies. Both measures are annualized. Currency and foreign notes data are from Bloomberg, whereas 10-year US Treasury Note data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

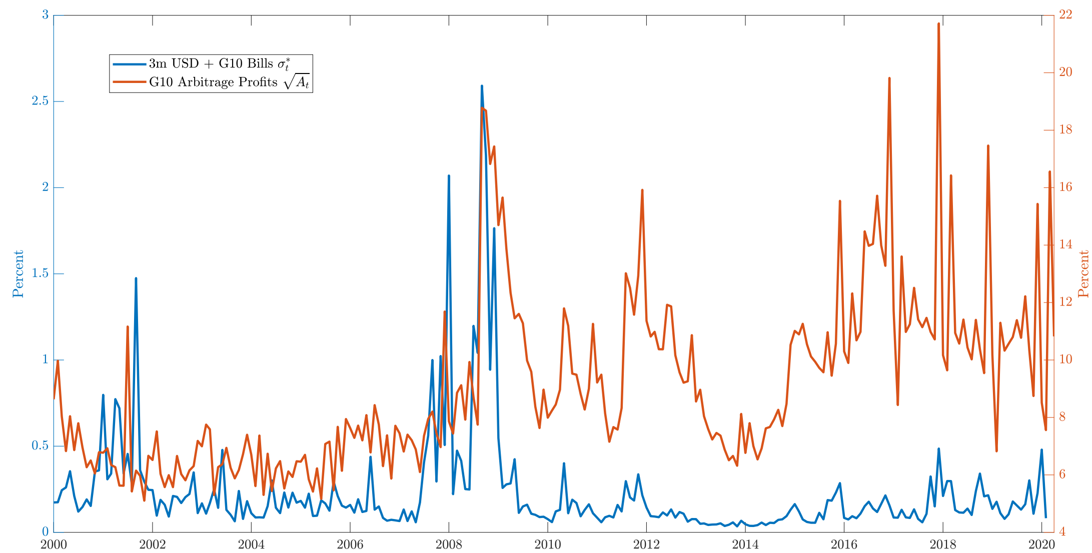

Figure 7: Monthly time-series of proxies for A_t and σ_t^*. Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is constructed using the value-weighted average volatilities of next-2-month returns on (i) 3-month US bills; (ii) 3-month synthetic US bills, built using foreign bills, spot and forward exchange rates. The 2-month returns are measured at a daily frequency, then volatility is calculated as a within-month standard deviation of these daily measures, then these measures are (simple) averaged across the G10 currencies. Both measures are annualized. Currency, foreign bills, and US bills data are from Bloomberg. Data range from Jan. 2000 to Apr. 2020.

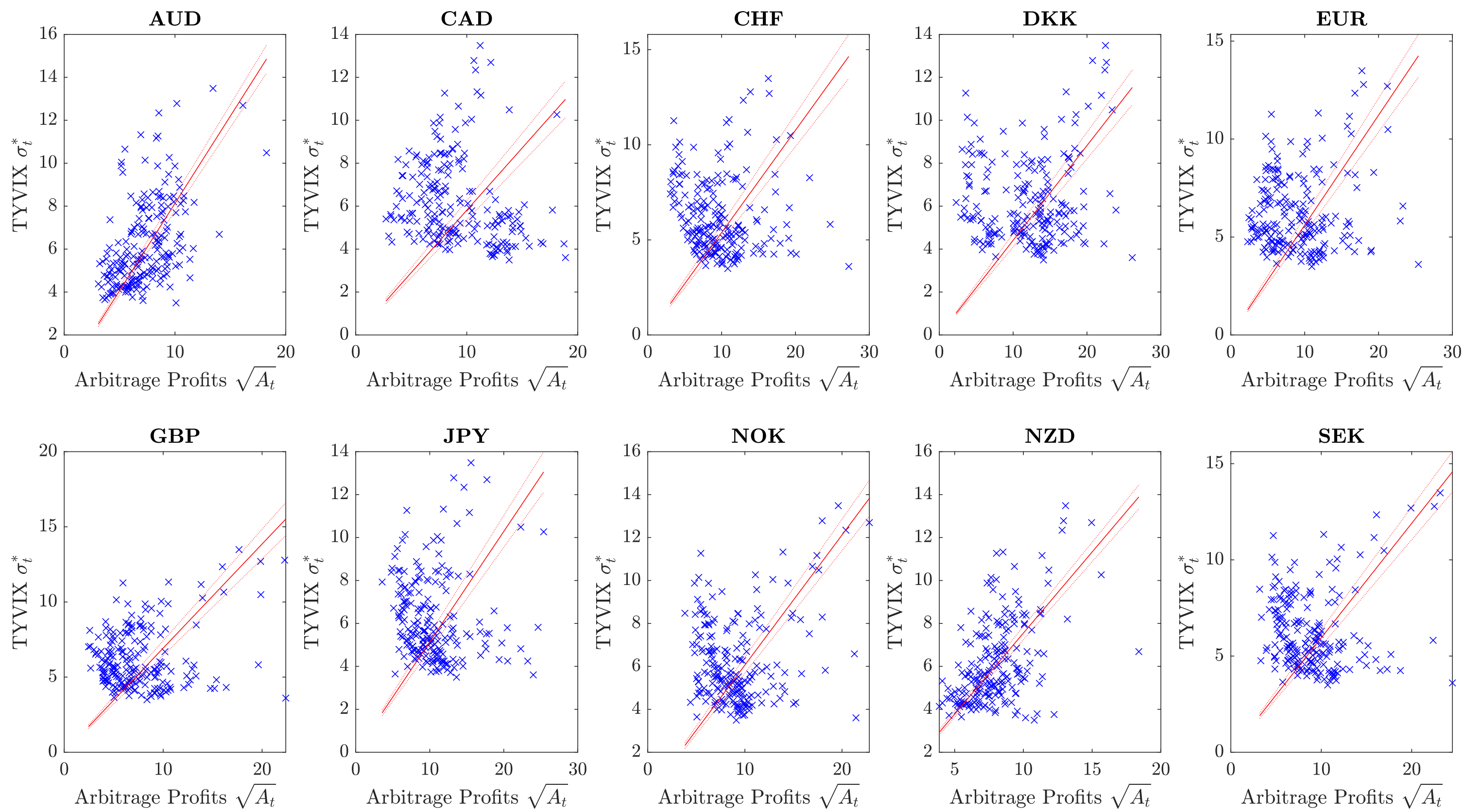

Figure 8: Currency-level OLS regressions of σ_t^* on $\sqrt{A_t}$ (monthly, no intercept). Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is constructed using the monthly average of TYVIX, which is the implied volatility of CBOE options on 10-year US Treasury Note futures. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency data are from Bloomberg and TYVIX data are from the CBOE. Data range from Jan. 2003 to Apr. 2020.

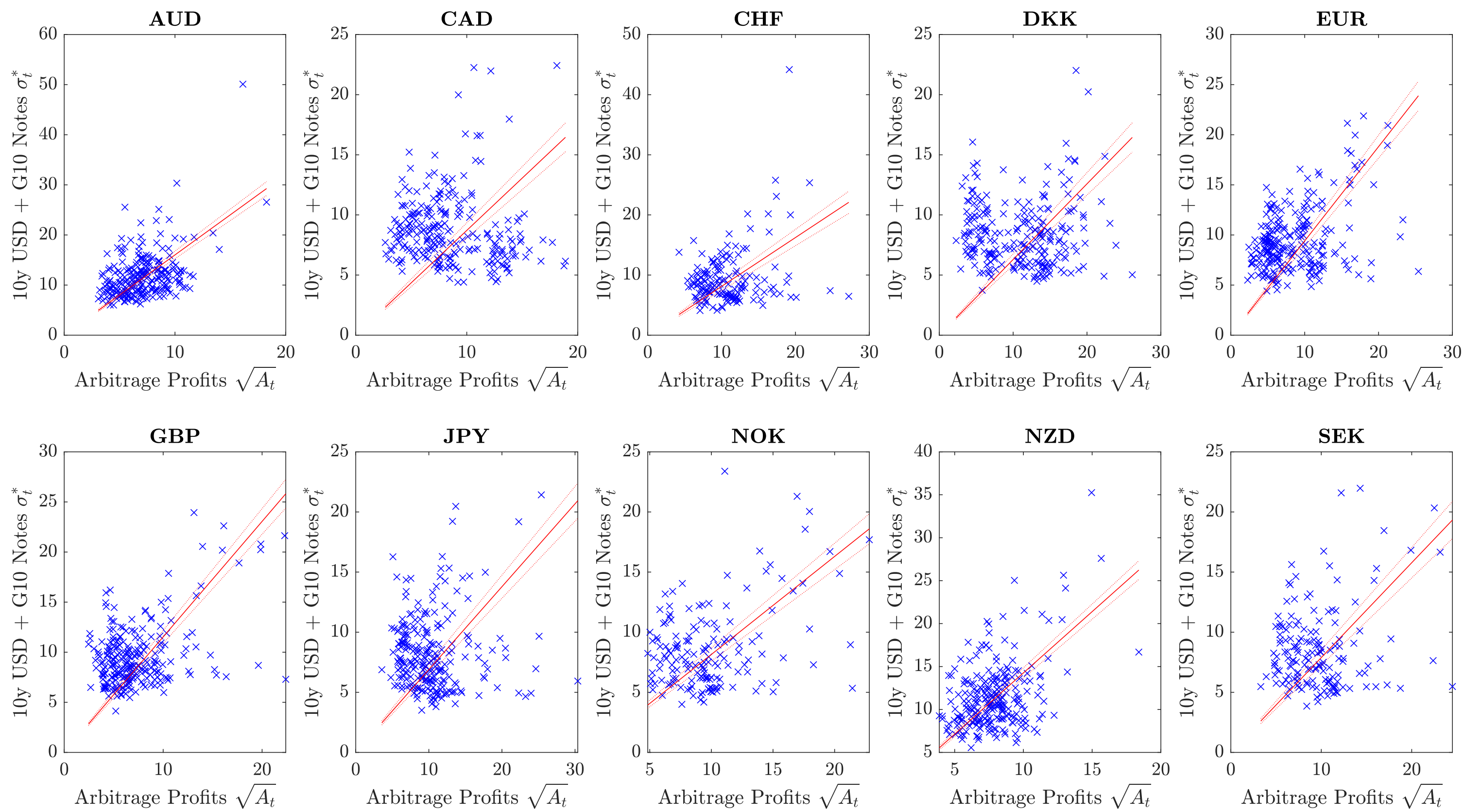

Figure 9: Currency-level OLS regressions of σ_t^* on $\sqrt{A_t}$ (monthly, no intercept). Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is constructed using the value-weighted average volatilities of daily log price changes on (i) the 10-year Constant Maturity US Treasury Note; and (ii) one of the G10's 10-year constant maturity note, adjusted by the respective spot exchange rate to be measured in USD. Volatility σ_t^* is calculated as a within-month standard deviation of these daily price changes, then (simple) averaged across the G10 currencies. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency and foreign notes data are from Bloomberg, whereas 10-year US Treasury Note data are from the Board of Governors of the Federal Reserve System, retrieved from FRED. Data range from Jan. 2000 to Apr. 2020.

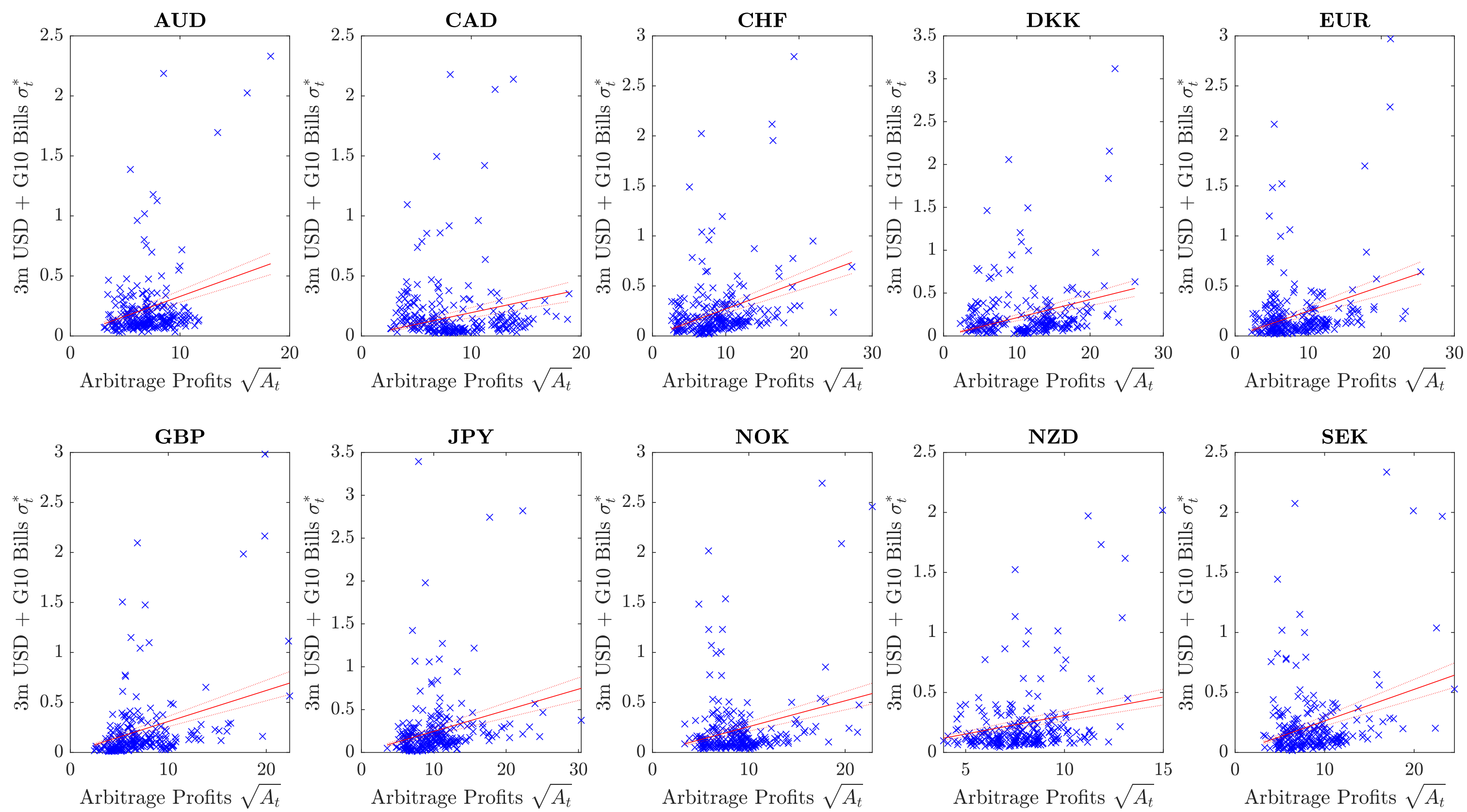

Figure 10: Currency-level OLS regressions of σ_t^* on $\sqrt{A_t}$ (monthly, no intercept). Proxy for A_t is $\hat{A}_t^{(1/4)}$ from (29), which is constructed using 3-month absolute CIP deviations (against USD), measured daily, then averaged monthly, then (simple) averaged across the G10 currencies. Proxy for σ_t^* is constructed using the value-weighted average volatilities of next-2-month returns on (i) 3-month US bills; (ii) 3-month synthetic US bills, built using foreign bills, spot and forward exchange rates. The 2-month returns are measured at a daily frequency, volatility is calculated as a within-month standard deviation of these daily measures, and then these measures are (simple) averaged across the G10 currencies. Both measures are annualized. Regression lines, and 95% confidence intervals (using HAC standard errors), are also displayed. Currency, foreign bills, and US bills data are from Bloomberg. Data range from Jan. 2000 to Apr. 2020.