Fear and Volatility at the Zero Lower Bound*

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Abstract

We uncover new sentiment-driven equilibria in a canonical dyanmic economy with nominal rigidities. Self-fulfilling fluctuations emerge because output is demand-determined at the zero lower bound. Higher uncertainty lowers asset prices, reducing wealth, aggregate demand, and production, which then justifies the higher uncertainty. The multiplicity is unrelated to self-fulfilling beliefs about inflation. The model helps explain high endogenous volatility at the zero lower bound, and it suggests a simple rationale for unconventional monetary policies.

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1 Introduction

Many macroeconomists share the view that constraints on monetary policy can exacerbate recessions and crises. For example, at the zero lower bound (ZLB), accommodative interest rate policy is ineffective by definition; policymakers have to explore unconventional, often thought to be less powerful, methods to stimulate the economy. Another prominent view, albeit less widespread, is that shifts in beliefs, especially uncertainty, cause recessions and crises. These views are connected: pessimism and uncertainty tend to induce saving in safe stores of value, depressing interest rates and propelling the economy towards the ZLB.

In this note, we uncover a new type of self-fulfilling volatility that epitomizes these views. We study a standard New Keynesian model: markets are complete, the representative agent is fully rational, but prices are sticky and monetary policy is constrained by the ZLB. Despite this type of model being widely used, the equilibria we study have gone unnoticed.

The intuition for self-fulfilling volatility is as follows. Consider a sudden wave of *fear*, meaning agents perceive higher volatility going forward. Fearful agents engage in precautionary savings, putting downward pressure on the interest rate. If the fear is sufficiently strong, the economy is pushed to the ZLB. The interest rate can no longer clear the bond market, so instead aggregate wealth falls. The resulting drop in aggregate demand also lowers output, since production is demand-determined at the ZLB.

Agents' beliefs will be justified, so long as they lead to stable long-run behavior. It turns out that fear-induced volatility, which raises risk premia, is precisely what generates stability. Indeed, in such a recession, required returns must be satisfied by an expected future appreciation in asset prices (since asset dividends are low after the output drop), and this positive drift pushes the economy back towards recovery in expectation.

In summary, our theory predicts self-fulfilled uncertainty-driven recessions that monetary policy has little power to prevent or tame. This process is inefficient.

Nothing about our equilibrium design relies on inflation dynamics. In fact, to differentiate ourselves starkly from the literature on self-fulfilling inflation at the ZLB (Benhabib et al., 2001a,b), we study a stylized setting where prices are fully rigid, meaning inflation is always equal to zero. Our volatility is a real phenomenon.

Our results are also distinct from the literature on monetary policy and asset price bubbles (Galí, 2014; Allen et al., 2018; Miao et al., 2019; Dong et al., 2020; Asriyan et al., 2021). The model we present does not admit bubbles, output is always weakly below potential, and optimal monetary policy has a clear directive to maximize output. Nevertheless, it would be interesting for future research to explore how rational bubbles interact with sentiment-driven volatility at the ZLB.

More closely related is Benigno and Fornaro (2018), in which pessimistic beliefs about growth can be self-fulfilling at the ZLB. Their mechanism of stagnation, which operates through R&D investment, likely operates on a longer time scale than ours, which is more about short-term volatility and temporary recessions.

A key methodological novelty of our sunspot equilibria is that they emerge even though the model admits a unique non-sunspot equilibrium. For example, the models cited above build stochastic equilibria on top of a stable steady state, which is the classic approach (Azariadis, 1981; Cass and Shell, 1983). By contrast, our non-sunspot equilibrium is an unstable steady state; the introduction of volatility itself is what delivers stability to the stochastic equilibrium. This type of stochastic stability boils down conveniently to boundary conditions in continuous time, which is why we adopt a continuous-time articulation of the New Keynesian setting, due to Caballero and Simsek (2020c).

This entire class of New Keynesian models features a well-known aggregate demand externality at the ZLB: privately lower demand reduces output and wealth, which induces others to cut demand as well. Ultimately, the externality manifests as a connection between asset prices and output efficiency. See Khorrami and Mendo (2021) for a different setting in which multiple equilibria also arise due to a price-output link. There, an aggregate supply externality operates through fire sales that reduce allocative efficiency. This comparison suggests that the distinction between demand and supply externalities is immaterial to the existence of sunspot equilibria; what really matters is a price-output link, which can be achieved via financial frictions or via nominal rigidities.[1]

2 Model

We present a complete-markets economy with nominal rigidities that supports self-fulfilling fluctuations. The setup is a simplified version of Caballero and Simsek (2020c), which the reader can consult for additional details.

Sunspot shocks. Our baseline model features no fundamental uncertainty in preferences or technologies. Nevertheless, we want to allow the possibility that economic objects evolve stochastically due to coordinated behavior. To do this, we introduce a standard

[1] A related interpretation, offered by Benhabib et al. (2020) in extending the model of Bacchetta et al. (2012), is that asset prices should have a direct impact on the stochastic discount factor, which is exactly what happens with a price-output link. Certain OLG specifications, financial frictions, and (as we show here) nominal rigidities all connect asset prices to the SDF.

Brownian motion Z that is extrinsic to all economic primitives. All random processes will be adapted to Z.[2]

Preferences. The representative agent has rational expectations and time-separable logarithmic utility with discount rate ρ:

$$\sup_{C \geq 0} \mathbb{E}\Big[\int_0^\infty e^{-\rho t} \log(C_t) dt\Big]. \tag{1}$$

Technology. There are two goods, a non-durable good (the numéraire, "consumption") and a durable good ("capital") that produces the consumption good. The aggregate supply of capital grows deterministically as

$$\dot{K}_t = gK_t, \tag{2}$$

where g is an exogenous constant. For simplicity, there is no investment in the model.

The relative price of capital, denoted by q_t, is determined in equilibrium. Since capital is the only positive net supply asset in the economy, aggregate wealth is $q_t K_t$. Conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t} dt + \sigma_{q,t} dZ_t]. \tag{3}$$

The term σ_q measures sunspot volatility that only exists because agents believe in it.

Producers employ capital in a linear production technology with productivity A. The assumption of a single productivity level is without loss of generality because of complete financial markets.[3] Producers' prices are fully rigid, which is a convenient assumption that also allows us to distinguish our results from the self-fulfilling inflation volatility that can occur in New Keynesian models (Benhabib et al., 2001a,b). Here, inflation will always be equal to zero. As a result, note that the real riskless interest rate r_t is equal to the nominal rate, which is set by monetary policy.

The appendix of Caballero and Simsek (2020c) discusses a few auxiliary assumptions (lump sum profit taxes and linear capital subsidies, which we also implicitly adopt)

[2]In the background, the Brownian motion Z exists on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, assumed to be equipped with all the "usual conditions." All equalities and inequalities involving random variables are understood to hold almost-everywhere and/or almost-surely.

[3]Indeed, in a heterogeneous-productivity world, all physical resources would be distributed among the most productive agents, who would then issue financial claims, so the economy would look as if there were a single productivity level.

designed to simplify the analysis, namely to ensure the market portfolio dividend equals aggregate output.

Monetary policy. Although the economy's potential output is AK_t, firms may not always operate at capacity because of nominal rigidities. We assume that monetary policy aims to achieve full utilization whenever possible, but they are subject to the ZLB $r_t \geq 0$.

In particular, let $\chi_t \leq 1$ denote firms' capital utilization, which will be determined in equilibrium. Aggregate output is $\chi_t AK_t$. Monetary authorities set the nominal rate (hence the real rate) to implement $\chi_t = 1$ whenever possible, subject to the ZLB. Under this rule, full utilization prevails whenever the real rate is positive, and inefficient utilization must arise at the ZLB:

$$0 = \min[1 - \chi_t, r_t]. \tag{4}$$

In the rest of the paper, we simply assume the central bank acts using a policy rule that implements (4). In the appendix, we show that within the class of equilibria we study, (4) is actually the outcome of optimal discretionary monetary policy (i.e., monetary policy without commitment to future policies).

Equilibrium definition. Log utility agents consume a fraction ρ of their wealth, and aggregate wealth is $q_t K_t$, so goods market clearing can be written as

$$\rho q = \chi A. \tag{5}$$

We can think of (5) as a link between asset prices (q) and output efficiency (χ), which is critical for our mechanism. We also define $q^* := A/\rho$, which is the efficient capital valuation.

Consumption and portfolio choices are unconstrained and imply the Euler equation:

$$r = \frac{\chi A}{q} + g + \mu_q - \sigma_q^2. \tag{6}$$

Note that $\chi A/q + g + \mu_q$ is the expected return-on-capital, and σ_q^2 is the risk premium in the economy. Using (5), we can substitute $\chi A/q$ with ρ.

Finally, we will restrict attention to allocations satisfying

$$\lim_{t\to\infty} q_t > 0 \quad \text{almost-surely}. \tag{7}$$

Although (7) is not strictly speaking a necessary condition, we impose it to make the problem interesting.[4]

Definition 1. An *equilibrium* consists of processes $(q_t, \chi_t, r_t)_{t\geq 0}$ such that equations (4)-(6) hold for all $t \geq 0$ and (7) holds. A *fundamental equilibrium* features $\sigma_q \equiv 0$. A *sunspot equilibrium* features $\sigma_q \neq 0$.

3 Equilibria

3.1 Fundamental equilibrium

There is always an equilibrium with full utilization, $\chi = 1$. Using $\chi = 1$ in equation (5), we obtain $q = q^*$, so $\mu_q = \sigma_q = 0$ must hold in this equilibrium. By equation (6), the interest rate is given by $r = \rho + g > 0$, which satisfies (4). Thus, the efficient equilibrium is a fundamental equilibrium.

On the other hand, any fundamental equilibrium must be efficient. Indeed, suppose $\sigma_q = 0$ so that all dynamics are deterministic, and consider an allocation with $\chi < 1$. By (4), under-utilization implies a binding ZLB. Using $r = 0$ and $\sigma_q = 0$ in (6) implies that $\mu_q = -(\rho + g)$, so q must be converging to zero asymptotically, in violation of (7). In other words, the efficient equilibrium is "unstable." This instability means, in a fundamental equilibrium, q_t can never take any value other than its efficient steady-state value q^*. Thus, we have proved the following.

Proposition 1. *A unique fundamental equilibrium exists and coincides with the efficient equilibrium featuring full utilization at all times, $\chi_t = 1$.*

3.2 Sunspot equilibrium

Despite the fact that the unique fundamental equilibrium is an "unstable" steady state, an inefficient volatile equilibrium can emerge. This differs from most of the sunspot literature, which typically builds volatile equilibria around a stable steady state.

Suppose again $\chi < 1$, so that $r = 0$, but do not impose $\sigma_q = 0$. Using $r = 0$ in (6), we obtain $\mu_q = -(\rho + g) + \sigma_q^2$. Equilibrium places no further restrictions, except that (σ_q, μ_q) must keep $q_t \in (0, q^*]$ to satisfy (7). This is a relatively modest requirement, because the

[4] Of course, we clearly require $q_t > 0$ for all t almost-surely (hence $C_t > 0$), so that utility is well-defined and not equal to $-\infty$. However, condition (7) implies—and is thus stronger than—$q_t > 0$ for all t. Also note that (7) does not follow from the transversality condition $\lim_{T\to\infty} \mathbb{E}[e^{-\rho T} C_T^{-1} q_T K_T] = 0$, which holds trivially given optimal consumption is $C_T = \rho q_T K_T$. Nevertheless, we impose (7) to emphasize that none of our results stem from the existence of equilibria with asymptotically vanishing wealth.

presence of volatility σ_q now adds a risk premium to the price drift, which buoys price q and restores "stability." As σ_q is indeterminate, there are many ways to do this. The following theorem is our main result.

Theorem 1. *Let $v : \mathbb{R} \mapsto \mathbb{R}_+$ be any Lipschitz continuous function that is strictly positive on $(0, q^*)$ and satisfies $v(0) > \rho + g$. A sunspot equilibrium exists, in which $\sigma_{q,t}^2 = v(q_t)$ and $\mu_{q,t} = -\rho - g + v(q_t)$ whenever $q_t < q^*$. Furthermore, the inefficiency in this equilibrium can be permanent, transitory, or anything in between, i.e., the stationary probability of inefficiency can be any $\pi \in [0, 1]$.*

(i) If $v(q^*) = 0$, then inefficiency is permanent: $\pi = 1$.

(ii) If $v(q^*) > 0$, inefficiency eventually subsides but can re-emerge: $\pi < 1$.

Proof of Theorem 1. Consider an auxiliary variable $x_t \in (0, q^* + b)$ for some $b > 0$. Write the evolution of x as $dx_t = x_t[\mu_{x,t}dt + \sigma_{x,t}dZ_t]$. We are letting x_t be the state variable in this equilibrium. Set $q_t = \min[x_t, q^*]$, and put $\sigma_{x,t}^2 = v(x_t)$ and $\mu_{x,t} = -\rho - g + v(x_t)$ when $x_t < q^*$. Nothing pins down (σ_x, μ_x) when $x_t > q^*$, we may simply set them so that x_t never reaches the boundary $q^* + b$. Many such choices exist (e.g., σ_x vanishes as $x \to q^* + b$ while μ_x remains strictly negative).

To prove such an equilibrium exists, it remains to show that $(x_t)_{t \geq 0}$, hence $(q_t)_{t \geq 0}$, almost-surely never attains the boundary $\{0\}$. Heuristically, given $v(0)$ is positive and bounded, x_t behaves like a geometric Brownian motion near $x = 0$, with positive drift $\mu_x(0) > 0$ if and only if $v(0) > \rho + g$. Such a geometric Brownian motion almost-surely never attains the boundary $\{0\}$, and furthermore does not concentrate probability near $\{0\}$ asymptotically. A rigorous proof of this claim, using Feller's boundary classification for diffusions, is provided by Lemma 2 in the appendix.

It remains to show any $\pi \in [0, 1]$ is possible. If $v(q^*) = 0$, then $\mu_x(q^*-) < 0$, so that x_t never attains the point $x = q^*$ if started below it. Thus, $\pi = 1$ if $v(q^*) = 0$. One similarly shows that $x = q^*$ is attainable if $v(q^*) > 0$, since then $\sigma_x(x) > 0$ for all $x \in (0, q^*]$, whereas $\mu_x(x)$ is bounded. Thus, $\pi < 1$ if $v(q^*) > 0$. Furthermore, since (σ_x, μ_x) are not pinned down when $x_t > q^*$, appropriate choices of these dynamics can deliver any value of $\mathbb{P}[x_t \geq q^*] = 1 - \pi$.

Finally, to show that one can construct an equilibrium where inefficiency is completely transitory ($\pi = 0$), consider putting $v(q^*) > 0$, so that x_t eventually exceeds q^* with probability 1, and putting $\sigma_x(x) = 0$ and $\mu_x(x) > 0$ on $\{x \geq q^*\}$ so that x_t never leaves this region. $\square$

Theorem 1 proves the existence of an entire class of equilibria. All that is required is $\lim_{q \to 0} \sigma_q^2(q) > \rho + g$, but aside from this boundary condition, essentially arbitrary levels of volatility are feasible at any price $q > 0$. The class of equilibria is even broader than this, however, since when $q = q^*$ nothing pins down the speed at which the economy re-enters the inefficient region. For this reason, the probability of inefficiency π can be anything. Of course, the possibility of permanent inefficiency is unrealistic and tied to the fact that goods prices are completely rigid in this stylized model. Given that prices eventually adjust, a more appropriate reading of the result says short-run volatility and inefficiency can be highly transitory or somewhat more persistent, with the maximal degree of persistence likely related to the degree of price stickiness.

Figure 1 below displays a numerical example of an equilibrium from Theorem 1. In this example, the economy features inefficiency 16% of the time. While there, the amount of non-fundamental volatility can be large, on the order of 15-30%. Asset prices can drop a significant amount, as seen in the long left tail of the stationary CDF.

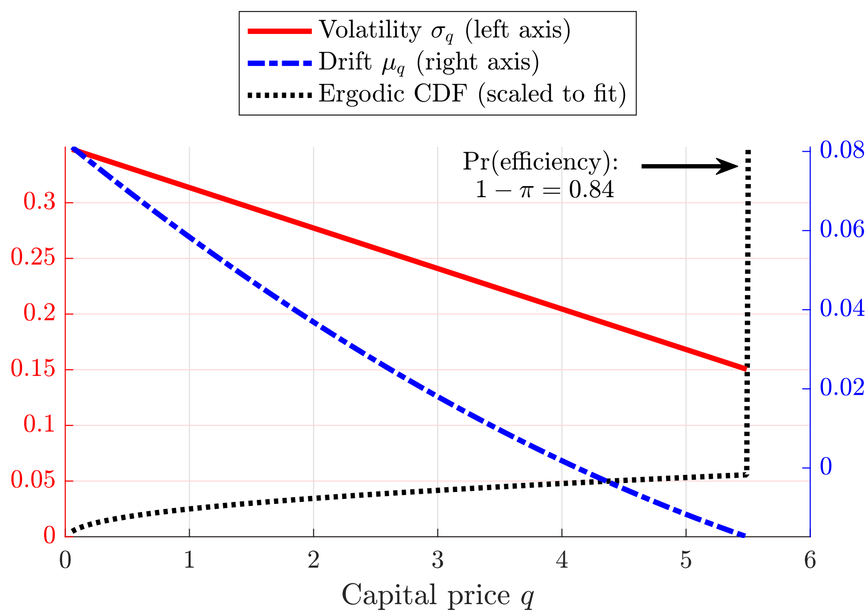

Figure 1: Equilibrium with nominal rigidities. We set volatility $\sigma_q(q) = (1 - q/q^*)\sqrt{\rho + g} + 0.15$ when $q < q^*$, which meets conditions of Theorem 1. To compute the stationary CDF, we specify the dynamics of an auxiliary diffusion x on domain $(0, 1.1q^*)$, and put $q = \min[x, q^*]$. When $x < q^*$, dynamics of x and q match, by definition. When $x > q^*$ (i.e., $q = q^*$), dynamics of x can be set arbitrarily, and they control how long q stays at q^*. The resulting stationary CDF features a mass point of size $1 - \pi = 0.84$ at $q = q^*$ (i.e., inefficiency occurs 16% of the time). Parameters: $A = 0.11$, $\rho = 0.02$, $g = 0.02$.

To understand the logistics of this sunspot equilibrium, suppose agents are suddenly *fearful*, and they conjecture $\sigma_q > 0$. Is this justified? Fear leads to a precautionary savings motive, putting downward pressure on the "natural interest rate." Without a ZLB, the central bank has the power to lower r enough to match the natural rate and clear bond markets, with agents consuming and saving as before. Goods markets would be unaffected, utilization χ and price q would remain fixed, and agents' fear would be unsubstantiated. Forward-looking agents can think through this entire hypothetical sequence of events, and they will reject the feeling of fear as irrational. And as a result, the central bank would not actually have to do anything.

By contrast, suppose a ZLB exists. If fear and its associated precautionary savings pushes the natural rate below zero, the central bank cannot lower r enough to clear bond markets. Markets only clear if a counteracting force reduces savings, which is why wealth must fall. Due to wealth effects, current consumption also falls, and firms meet their lower demand by operating at less than full capacity in production ($\chi < 1$). Although this process is inefficient, nothing makes this sequence of hypothetical events irrational. Agents' fear will be justified, so long as it does not lead to unstable long-run behavior. Mathematically, stability boils down to the requirement that q_t never attain zero, which is precisely what the condition $v(0) > \rho + g$ in Theorem 1 ensures.

This entire discussion is conditional on a monetary policy rule that implements (4), which recall corresponds to optimal discretionary policy (see Lemma 1 of the appendix). It is easy to see that sunspot equilibria are eliminated if the central bank has full commitment power to future policies. For example, the bank can threaten to raise the policy rate to arbitrarily high values when q dips below q^* (i.e., $r \to +\infty$ when $q < q^*$). From equation (6), this would create large upward price pressure (i.e., $\mu_q \to +\infty$) and thus quickly restore efficiency. Knowing this policy threat exists, households will never reduce demand and fear for volatility, and efficiency will always prevail. With this off-equilibrium threat in place, the observed interest rate will always be the efficient natural rate $r = r^* := \rho + g$.[5]

Without commitment, the central bank will not want to ever implement the threat $r \to +\infty$. More specifically, a discretionary central bank finds it optimal to reduce the interest rate whenever demand is below potential. Perversely, the policy response under discretion goes in the opposite direction of the optimal full-commitment response: high r raises μ_q and reduces length of time that $q < q^*$, but the short-run discretionary incentive is to reduce r.

Motivated by the stark distinction between discretionary and full-commitment policy, the next section generalizes our results by also allowing for fundamental risk. High

[5]Despite this observation, a central bank pegging the nominal rate at $r = r^* := \rho + g$ will not eliminate the inefficient equilibrium. Indeed, omitting equation (4) and instead using the peg rule ($r = \rho + g$), equation (6) says that $\mu_q = \sigma_q^2$. As long as $\sigma_q(0) > 0$, we will have $\mu_q(0) > 0$ so that $q \not\to 0$; hence, a volatile equilibrium can exist generically under the peg. This example shows that it is crucial for the central bank to have the right off-equilibrium threats at its disposal.

enough fundamental risk temporarily eliminates the efficient equilibrium, even for a sophisticated central bank with commitment power. Such a central bank faces the more subtle task of using policy to select between two alternative inefficient equilibria, one with and one without sunspot volatility.

While this economy above is stylized, the insights are general to the extent that the link between asset prices and output efficiency, captured in equation (5), is not severed. We can add other state variables, heterogeneous agents (e.g., some hand-to-mouth), or partial price flexibility, and the results will remain qualitatively unchanged.[6]

3.3 Adding fundamental uncertainty

In this section, we add fundamental risk to verify the robustness of our results. Some additional insights emerge. First, the efficient equilibrium is eliminated with high enough fundamental risk, suggesting that even more sophisticated monetary policy (e.g., with some commitment power) cannot necessarily eliminate self-fulfilling volatility in truly adverse states. Second, the source of volatility becomes indeterminate: fundamental shocks could be amplified or volatility could be attached to sunspot shocks.

Suppose aggregate capital has some fundamental risk s. Assume additionally that at a random time $\tau \sim \exp(\lambda)$, this fundamental risk reverts to zero permanently. At that time, we will suppose the economy transitions into the efficient equilibrium forever after (i.e., $\chi_t = 1$, $q_t = q^*$, and $r_t = \rho + g$ for all $t \geq \tau$). Prior to time τ, the capital evolution equation (2) is modified to

$$dK_t = K_t[gdt + sdB_t], \quad \text{for} \quad t < \tau, \tag{2'}$$

where B is a standard Brownian motion independent of Z. We will assume that s^2 is sufficiently high, so that the efficient equilibrium ceases to exist, but not so high as to prevent any equilibrium.[7]

[6] See Caballero and Simsek (2020a,b,c) for some of these extensions.

[7] If $s^2 > \rho + g + \lambda$, no equilibrium could exist. A proof sketch of this argument is as follows. Without endogenous volatility ($\sigma_q = 0$), the drift of q would be $\mu_q = -(\rho + g) + s^2 - \lambda(q^* - q)/q^* > -(\rho + g + \lambda) + s^2 > 0$ in this situation, so that q_t would eventually attain q^*, if unabated. With endogenous volatility ($\sigma_q \neq 0$), q_t would eventually attain q^* simply due to shocks. But since $s^2 > \rho + g$, the efficient equilibrium cannot be supported, so another force must arise to prevent q_t from ever attaining q^*. In particular, there must be a predictable negative movement in q_t, which cannot be absolutely continuous with respect to time, either at or before hitting q^* (for example, a reflecting boundary at $q^* - \epsilon$). In such case, no-arbitrage requires that the riskless bond have a singular return equal to $r_t dt - dL_t$, where L is the singular process keeping $q_t \leq q^*$ (see Karatzas and Shreve (1998), Appendix B). The ZLB disallows this riskless bond return, and thus no equilibrium can exist.

Assumption 1. *Assume $\rho + g < s^2 < \rho + g + \lambda$.*

In this extension, volatility can either be connected to fundamentals, with possible amplification so that endogenous fluctuations are greater than the fundamental shock, or related to sentiments. Mathematically, prior to the transition to efficiency, the dynamics of q in (3) are now modified to read

$$dq_t = q_t\Big[\mu_{q,t}dt + \sigma_{q,t} \cdot \Big(\begin{smallmatrix} dB_t \\ dZ_t \end{smallmatrix}\Big)\Big], \quad \text{for} \quad t < \tau. \tag{3'}$$

We continue to assume monetary policy attempts to implement (4). The price-output link (5) also still holds in this setting, but the asset-pricing equation (6) is modified to read

$$r = \rho + g + \mu_q + s\big(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\big) \cdot \sigma_q - \big|s\big(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\big) + \sigma_q\big|^2 + \lambda \frac{q^* - q}{q^*}. \tag{6'}$$

Similar to the baseline model, an equilibrium satisfies (4), (5), (6′), and $q_t > 0$ for all t. The fundamental equilibrium, described below, is a simplified version of the one studied in Caballero and Simsek (2020c).

Proposition 2 (Fundamental equilibrium)**.** *Under Assumption 1, there exists an equilibrium with $\mu_q = 0$ and $\sigma_q = 0$ at all times. When exogenous volatility is high (i.e., when $t < \tau$), this equilibrium features $\chi < 1$, $r = 0$, and $q = \bar{q} := q^*(\rho + g + \lambda - s^2)/\lambda$. Among equilibria having $\mu_q = 0$ and $\sigma_q = 0$, this equilibrium is unique.*

Proof of Proposition 2. Plug in $r = 0$, $\mu_q = 0$, and $\sigma_q = 0$ into (6′) to solve uniquely for $q = \bar{q}$ under volatility s. Note that $\bar{q} < q^*$, so $\chi < 1$ and thus (4) holds. Uniqueness within the class of equilibria having $\mu_q = 0$ and $\sigma_q = 0$ can be established by using $s^2 > \rho + g$ and equation (6′) to show that $\chi = 1$ is impossible. □

As in the baseline model, there are also sunspot equilibria in this setting, with the following properties. First, at the ZLB, these equilibria can feature excess volatility, and this level of price volatility is essentially arbitrary (i.e., σ_q is essentially arbitrary). Second, when efficiency fails, an arbitrary fraction of total return variance $|s\big(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\big) + \sigma_q|^2$ can be connected to fundamental shocks versus sentiment shocks. The reasoning for this latter indeterminacy is that agents only care about total capital return variance when trading capital; the source of the shocks is irrelevant. Finally, the inefficiency in the sunspot equilibrium is worse than the fundamental equilibrium of Proposition 2, in the sense that q and χ are always lower. One can think of this situation as a *volatility trap*: beliefs about endogenous volatility will produce stability in the equilibrium dynamics, which is enough to keep the endogenous volatility around until exogenous risk disappears. We formalize this discussion in the following theorem, a generalization of Theorem 1. In this theorem, the function v corresponds to total return variance and f to the variance share associated to the fundamental shock.

Theorem 2 (Sunspot equilibria)**.** *Let Assumption 1 hold. Let $f, v : \mathbb{R} \mapsto \mathbb{R}_+$ be any two Lipschitz continuous functions satisfying $v \geq 0$, $0 \leq f \leq 1$, and boundary conditions*

(i) $v(0) > \rho + g + \lambda + s\sqrt{f(0)v(0)}$;

(ii) $v(\frac{\bar{q}}{M}) = s^2$ *and* $f(\frac{\bar{q}}{M}) = 1$, *where* $\bar{q} := q^*(\rho + g + \lambda - s^2)/\lambda$ *and* $M > 1$ *is any number.*

An equilibrium exists with $r_t = 0$, $\chi_t < 1$, and volatility

$$\sigma_{q,t} = \left[\frac{\sqrt{f(q_t)v(q_t)} - s}{\sqrt{(1 - f(q_t))v(q_t)}} \right], \quad for \quad t < \tau.$$

In this equilibrium, q_t is always strictly below the fundamental equilibrium price $\bar{q}$, for $t < \tau$.

Proof of Theorem 2. Substitute all the proposed equilibrium objects into (6′) to find the price drift

$$\mu_{q,t} = s^2 - (\rho + g + s\sqrt{f(q_t)v(q_t)}) + v(q_t) - \lambda(1 - q_t/q^*), \quad \text{for} \quad t < \tau.$$

These dynamics must prevent q_t from ever reaching zero, which is guaranteed by condition (i). Indeed, this condition implies that $\lim_{q \to 0} \mu_q(q; s) > 0$, so that q_t behaves locally near zero as a geometric Brownian motion with positive drift (the formal argument is identical to that of Lemma 2, which was used in the proof of Theorem 1).

The dynamics also must prevent q_t from ever reaching $\bar{q}$, which is guaranteed by condition (ii). Indeed, this condition implies that $\lim_{q \to \bar{q}/M} \sigma_q(q; s) = 0$ as well as $\lim_{q \to \bar{q}/M} \mu_q(q; s) = s^2 - \rho - g - \lambda + \lambda \frac{\bar{q}}{Mq^*}$. By plugging in $\bar{q}$, we see that the drift expression is negative at this boundary for any $M > 1$. Together with the vanishing volatility (and the Lipschitz continuity assumption on σ_q), this implies that q_t cannot reach $\bar{q}/M$ (hence it cannot reach $\bar{q} > \bar{q}/M$).

As a result of these dynamics, any initial price $q_0 \in (0, \frac{\bar{q}}{M})$ is consistent with equilibrium. Thus, q_t is always below $\bar{q}$ for $t < \tau$, as desired. Since $\bar{q} < q^*$, this also verifies that $\chi < 1$ and hence $r = 0$ by equation (4). □

4 Conclusion

We have shown that macroeconomies with nominal rigidities—New Keynesian models—may inherently permit sunspot volatility. The volatility we document is distinct from inflation volatility and self-fulfilling beliefs about inflation; in particular, inflation is always zero in our model.

Discretionary monetary policy will not eliminate our volatile equilibrium, even when an efficient one exists. Full-commitment monetary policy may not be able eliminate sunspot volatility in situations where efficient equilibria do not exist. For example, in this paper, the efficient equilibrium ceases to exist if exogenous risk is high enough; the central bank then has to think about the more subtle problem of implementing policy to select between inefficient equilibria with more or less excess volatility.

Although we do not formally pursue results along these lines, the sentiment-driven equilibria we analyze suggest a rationale for central bankers to pursue unconventional monetary policies when rates are stuck at the ZLB. For example, quantitative easing (e.g., asset purchases) could raise wealth and aggregate demand, which refines the set of possible equilibria. Speeches might also matter, even with no particular policy attached, by inspiring confidence and "calming the market," which may eliminate fear-based equilibria.

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Appendix:
Fear and Volatility at the Zero Lower Bound

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Lemma 1. *Optimal discretionary monetary policy—which maximizes (1) subject to $r_t \geq 0$, $\chi_t \leq 1$, optimal household and firm decisions, and its own future decisions—implements (4).*

PROOF OF LEMMA 1. Optimal discretionary monetary policy seeks to pick a r_t to maximize (1), subject to (2), (3), (5), (6), $\chi_t \leq 1$, the ZLB $r_t \geq 0$, and subject to its own future decisions.

We will discretize the problem to time intervals of length Δ and later take $\Delta \to 0$. Noting that $C_t = \rho q_t K_t$, and using the fact that ρ and the time-path of K_t are exogenous, the time-t household utility is proportional to

$$\begin{aligned}
&\mathbb{E}_t\Big[\int_0^\infty \rho e^{-\rho s}\log(q_{t+s})ds\Big] \\
&\quad\approx \rho\log(q_t)\Delta + \mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s}\log(q_{t+s})ds\Big] \\
&\quad\approx -\rho\Delta\mathbb{E}_t[\log(q_{t+\Delta}) - \log(q_t)] + \underbrace{\mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s}\log(q_{t+s})ds\Big] + \rho\Delta\mathbb{E}_t[\log(q_{t+\Delta})]}_{\text{terms taken as given by discretionary central bank}}.
\end{aligned}$$

The term with brackets underneath is taken as given by the time-t discretionary central bank, because it involves expectations of future variables that the future central bank can influence.

Thus, taking $\Delta \to 0$, the time-t central bank solves

$$\min_{r_t \geq 0} \mathbb{E}_t[d\log(q_t)]$$

subject to the constraints

$$\begin{aligned}
r_t &= \rho + g + \mu_{q,t} - \sigma_{q,t}^2 \\
\mu_{q,t} &= 0 \quad \text{if} \quad q_t = q^* \\
\sigma_{q,t} &= 0 \quad \text{if} \quad q_t = q^*.
\end{aligned}$$

Note that $\sigma_{q,t}$ is independent of policy when $q_t \neq q^*$. There are two cases. If $q_t = q^*$, then the constraints imply that $r_t = \rho + g$. If $q_t \neq q^*$, we may substitute the dynamics of $\log(q_t)$ (using Itô's lemma and replacing μ_q from the first constraint) to re-write the problem as

$$\min_{r_t\geq 0}[r_t-\rho-g+\frac{1}{2}\sigma_{q,t}^2].$$

Since σ_q is taken as given, the optimal solution is $r_t=0$. Thus, the discretionary central bank optimally sets

$$r_t=(\rho+g)\mathbf{1}_{\{q_t=q^*\}}=(\rho+g)\mathbf{1}_{\{\chi_t=1\}}.$$

In other words, the complementary slackness condition $(1-\chi_t)r_t=0$ holds, which together with $r_t\geq 0$ implies (4). $\square$

Lemma 2 (Boundary classification)**.** *Let $(x_t)_{t\geq 0}$ be a one-dimensional diffusion satisfying $dx_t=[-(\rho+g)+v(x_t)]dt+\sqrt{v(x_t)}dZ_t$ with $v(\cdot)$ strictly positive on $(0,q^*)$. Then, if $v(0)>\rho+g$, the boundary $\{0\}$ is almost-surely never attained by x_t.*

Proof of Lemma 2. Note that $(x_t)_{t\geq 0}$ is a regular one-dimensional diffusion on interval $(0,q^*)$, i.e., the dynamics of x_t depend only on x_t itself, and imply that it reaches every point in $(0,q^*)$ with positive probability. In such case, we may apply Feller's boundary classification (e.g., Karatzas and Shreve (1991), Section 5.5) to decide whether boundary $\{0\}$ is inaccessible (avoided forever with probability 1) or accessible. To do so, first define the scale function and speed measure $s(y):=\exp(-\int_{x_0}^{y}\frac{2\mu_x(u)}{x\sigma_x^2(u)}du)$ and $m(x):=\frac{2}{s(x)\sigma_x^2(x)x^2}$. Let ϵ and x_0 be arbitrary numbers within interval $(0,q^*)$. According to the Feller classification, boundary $\{0\}$ is inaccessible if and only if

$$I:=\int_0^{\epsilon}m(x)\Big(\int_0^x s(y)dy\Big)dx=+\infty.$$

Substituting $\sigma_x^2=v$ and $\mu_x=-(\rho+g)+v$, and using the fact that v is strictly positive at and near zero, we see that $I=+\infty$ if and only if

$$I^*:=\int_0^{\epsilon}\frac{1}{x^2}\int_0^x\exp\Big[2\int_y^x\frac{v(u)-\rho-g}{uv(u)}du\Big]dydx=+\infty.$$

Note that $v(0)>\rho+g$ implies, by continuity, that if ϵ is small enough, there exists $\delta>0$ such that $v(u)\geq(1+\delta)(\rho+g)$ for all $u<\epsilon$. [In evaluating I and I^*, it suffices to consider such ϵ small enough, because $x\mu_x$ is bounded and $x^2\sigma_x^2$ is bounded above and away from zero for $x\geq\epsilon>0$.] Using this fact, the integral inside the exponential is

$$\int_y^x\frac{v(u)-\rho-g}{uv(u)}du\geq\int_y^x\Big(\frac{1}{u}-\frac{\rho+g}{(1+\delta)(\rho+g)u}\Big)du=\frac{\delta}{1+\delta}\log(\frac{x}{y}).$$

Then,

$$I^* \geq \int_0^\epsilon \frac{1}{x^2} \int_0^x (\frac{x}{y})^{\frac{2\delta}{1+\delta}} dy dx = \int_0^\epsilon x^{-\frac{2}{1+\delta}} \int_0^x y^{-\frac{2\delta}{1+\delta}} dy dx.$$

If $\delta \geq 1$, then we are finished, because the inner integral equals $+\infty$. If $\delta < 1$, then

$$I^* \geq (1 - \frac{2\delta}{1+\delta})^{-1} \int_0^\epsilon x^{-\frac{2}{1+\delta}} x^{1-\frac{2\delta}{1+\delta}} dx = (1 - \frac{2\delta}{1+\delta})^{-1} \int_0^\epsilon x^{-1} dx = +\infty.$$

This completes the proof. $\square$