Comparative Valuation Dynamics in Production Economies:

Long-run Uncertainty, Heterogeneity, and Market Frictions*

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1 Introduction

Beware the person of one ~~book~~ model. Thomas Aquinas (almost)

An under-appreciated task in the study of dynamic macroeconomics is model comparison. This is especially true for models requiring numerical methods to solve and analyze. While journals seemingly embrace publications that target specific models, there is much to be gained by looking formally across models.

One strategy for making comparisons across models is to nest models within a common framework in which each model of interest is a special case. At this juncture, we could turn things over to a statistician to test which model within this nesting best fits the data. This strategy makes the most sense when we could plausibly view one of the models within the family as being "correctly specified," given data. But in many cases, we see models as providing valuable insights even when they are not designed to fit some agreed upon list of favorite facts. As we explore nonlinear models more fully, this nesting-testing approach becomes all the more challenging. But even for examples when linearized approximations work well, the fitting all of some predesignated facts can lead to black box outcomes when driven by the simplistic ambitions of "full" empirical success. Models end up with multiple pieces often clouding the ability to isolate and understand better particular economic mechanisms.

In this paper, we develop a framework and diagnostic tools for comparing and contrasting dynamic macroeconomic models. The models that interest us require special attention relative to most dynamic stochastic equilibrium models because of the important role played by nonlinearity in the implied dynamic evolution. This nonlinearity has notable implications for both economic and financial market outcomes. Given these ambitions, our analysis is explicitly numerical and not limited to "paper and pencil" style analyses. It is necessary that we solve such models using global solution methods as the competitive equilibrium is typically characterized by a set of highly nonlinear second-order elliptic partial differential equations. Moreover, even with the option of numerical solutions, we find it revealing to explore and compare highly stylized models featuring particular economic mechanisms. In accompanying notebooks and user-friendly software, we propose and explore quantitative methods that expose salient features of the macroeconomic and valuation dynamics of the models we investigate. This essay provides illustrations of possible computations.

While we explore three different classes of a models, a common feature in all of them is a long-run process altering investment opportunities. Our technologies can be viewed as production-based specifications inclusive of long-run risk. Analogous to Bansal and Yaron (2004), we capture this risk with a continuous-time version of a first-order autoregressive process. The process is meant to be a simple proxy for uncertainty of such phenomenon as secular stagnation, technological progress or other forms of long-term uncertainty.

The first class of models is in some sense approximately linear. While including stochastic growth following in the footsteps of Lucas and Prescott (1971) and Brock and Mirman (1972), these models include single investor type and a single capital stock with a long-run risk contribution to the investment opportunities. While we provide some sensitivity analyses that are of interest in their own right, understanding this class of models sets the stage for our subsequent investigations.

The second class of models considers specifications with two capital stocks differentially exposed to macroeconomic shocks. Capital movements are sluggish in the sense that there are adjustment costs in both capital technologies. This class of models extend those of Eberly and Wang (2009) and Eberly and Wang (2011). We investigate the consequences of heterogeneous technological exposure to long-run risk in conjunction with motives for diversification. Including production in which the two capital stocks are not perfect substitutes adds an additional economic channel with interesting nonlinear impacts.

The third class of models, motivated in part by financial crises like 2008, considers two heterogeneous investor types. These agents can differ in skill, preferences, or contractual and regulatory constraints. Dynamic trading between these heterogeneous investors induces potentially dramatic economic and financial market outcomes in some states of the world, especially those in which constraints are binding. Our exercise is motivated by a substantial literature with a variety of different modeling ingredients. These include, for instance, the models in Basak and Cuoco (1998), He and Krishnamurthy (2011), Brunnermeier and Sannikov (2014), and Gârleanu and Panageas (2015). Recently, several papers have exposed a more complex representation of the role of financial intermediation than that captured by the stylized models we consider here. It is not our aim in this essay to survey this literature. The models we consider, however, do have mechanisms that are of interest to expose that enhance our understanding of nonlinear linkages between financial markets and the macroeconomy, even if they miss some of the actual complexities that limit financial intermediaries or other such specialists.

2 Investor Preferences

In this essay we use a continuous-time specification of a Kreps and Porteus (1978) utility recursion as in Duffie and Epstein (1992) in connection with an information structure generated expressed in terms of a vector standard Brownian motion $B \stackrel{\text{def}}{=} \{B_t : t \geqslant 0\}$ of dimension d. Thus we are imposing "local normality". While shocks are normally distributed, we entertain nonlinear transition mechanisms that permit endogenously determined variables to possess transition probabilities and stationary distributions that are not even approximately normal. In this section, we provide a heuristic link between the continuous-time and discrete-time representation of preferences since the discrete-time formulation has been used extensively in the quantitative asset pricing literature. The local normality does allow for some simplicity when we study continuous-time limiting economies. We do not ask the reader to be knowledgeable of the subtleties associated with the continuous-time mathematics.

2.1 Discrete-time

Continuation values provide a convenient way to specify recursive preferences. With this is in mind, let $V \stackrel{\text{def}}{=} \{V_t : t \geqslant 0\}$ be the continuation utility process where V_t is a date-t utility index that summarizes current and future prospective contributions to preferences. In discrete time with a time interval ϵ, we use two CES, homogeneous of degree one recursions to represent the evolution of continuation values:

$$\begin{aligned} V_t &= \left[[1 - \exp(-\delta\epsilon)] \, (C_t)^{1-\rho} + \exp(-\delta\epsilon) \mathbb{R}(V_{t+\epsilon} \mid \mathfrak{F}_t)^{1-\rho} \right]^{\frac{1}{1-\rho}} \\ \mathbb{R}(V_{t+\epsilon} \mid \mathfrak{F}_t) &= \left(\mathbb{E} \left[(V_{t+\epsilon})^{1-\gamma} \mid \mathfrak{F}_t \right] \right)^{\frac{1}{1-\gamma}} \end{aligned} \tag{1}$$

where $\mathfrak{F}_t$ is the time-t information set. Notice that the second equation computes a certainty equivalent with parameter γ. If the continuation utility $V_{t+\epsilon}$ is known at t, then γ has no impact on the recursion since $\mathbb{R}(V_{t+\epsilon} \mid \mathfrak{F}_t) = V_{t+\epsilon}$ implying that this contribution is indeed an adjustment for risk. Taking the two equations together, this is a forward looking-recursion whereby we start with a terminal specification of the continuation utility and work backwards. We consider infinite horizon counterparts in our computations. Notice that this recursive specification is governed by three underlying parameters:

i) δ – the subjective discount rate;

ii) ρ – the inverse of the intertemporal elasticity of substitution (the "IES");

iii) γ – the risk aversion.

In some later examples, we will have two investor types with possibly heterogenous specifications of the preference parameters (δ, ρ, γ).

Two benchmark special cases of these preferences are: $\rho = \gamma$ and $\rho = 1$. When $\rho = \gamma$, this utility recursion defines preferences that are equivalent to those implied by discounted, time-separable, power utility. Specifically, when $\gamma = \rho$, by solving the recursions forward, it follows that:

$$V_t = \left(\mathbb{E} \left[\frac{1}{1 - \exp(-\delta\epsilon)} \sum_{j=0}^{\infty} \exp(-\delta j \epsilon) \left(C_{t+j\epsilon}\right)^{1-\gamma} \mid \mathfrak{F}_t \right] \right)^{\frac{1}{1-\gamma}}, \quad \text{if} \quad \rho = \gamma. \tag{2}$$

Imposing $\rho = 1$ implies a unitary IES, and the limiting recursion has a Cobb-Douglas representation:

$$V_t = (C_t)^{[1-\exp(-\delta\epsilon)]} \left[\mathbb{R}(V_{t+\epsilon} \mid \mathfrak{F}_t)\right]^{\exp(-\delta\epsilon)}, \quad \text{if} \quad \rho = 1.$$

Continuation values are only defined up to increasing transformations. Numerical and conceptual convenience lead us to use $\widehat{V}_t = \log V_t$. (We will always use the notation "$\widehat{X}$" to designate the logarithm of a variable X.) The logarithmic counterparts to the underlying recursions are given by:

$$\begin{aligned}
\widehat{V}_t &= \frac{1}{1-\rho} \log \left[[1 - \exp(-\delta\epsilon)] \left(C_t\right)^{1-\rho} + \exp(-\delta\epsilon) \exp\left[(1-\rho) \widehat{\mathbb{R}}(\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t) \right] \right] \\
\widehat{\mathbb{R}}\left(\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t\right) &= \frac{1}{1-\gamma} \log \left(\mathbb{E}\left[\exp[(1-\gamma)\widehat{V}_{t+\epsilon}] \mid \mathfrak{F}_t \right] \right).
\end{aligned} \tag{3}$$

For this representation, $\rho = \gamma = 1$ is a relevant benchmark whereby the recursions become:

$$\begin{aligned}
\widehat{V}_t &= [1 - \exp(-\delta\epsilon)] \log C_t + \exp(-\delta\epsilon) \widehat{\mathbb{R}}(\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t) \\
\widehat{\mathbb{R}}(\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t) &= \mathbb{E}[\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t],
\end{aligned} \tag{4}$$

which has discounted logarithmic utility scaled by $[1 - \exp(-\delta\epsilon)]$ as the solution.

2.2 Robustness to model misspecification

The recursive utility representation (1) can also be interpreted through the lens of robust control theory. To begin with, consider a positive random variable $L_{t+\epsilon}$ with unit conditional expectation — a convenient mathematical device pertinent to models of subjective beliefs that are distinct from those implied by the data generating process:

$$\mathbb{E}\left(L_{t+\epsilon} \mid \mathfrak{F}_t\right) = 1.$$

Think of $L_{t+\epsilon}$ as a relative density (likelihood ratio) that alters the transition probability from t to $t+\epsilon$. To obtain the implied subjective conditional expectations, multiply the next-period random variables by $L_{t+\epsilon}$ prior to forming the conditional expectations. For instance, the implied subjective expectation of next period's continuation value is $\mathbb{E}(L_{t+\epsilon}\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t)$.

While a subjective belief specification allows for departures from a "rational expectations" assumption that investors know the data generating process, we use the modeling approach differently. Suppose that the investor has a benchmark model of the transition probabilities without full confidence in that specification. This skepticism is expressed by entertaining other models, with a particular interest in ones that are "statistically close" to the benchmark model. This approach has antecedents in the robust control literature.[1] Formally, solve

$$\min_{\substack{L_{t+\epsilon}\geqslant 0 \\ \mathbb{E}(L_{t+\epsilon}|\mathfrak{F}_t)=1}} \mathbb{E}\left(L_{t+\epsilon}\widehat{V}_{t+\epsilon} \mid \mathfrak{F}_t\right) + \xi\mathbb{E}\left(L_{t+\epsilon}\log L_{t+\epsilon} \mid \mathfrak{F}_t\right) = -\xi \log \mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+\epsilon}\right) \mid \mathfrak{F}_t\right], \quad (5)$$

which is familiar from applied probability theory. This minimization problem investigates the expected utility consequences of altering the probability distribution subject to a conditional relative entropy penalty used as a Kullback-Leibler measure of statistical divergence. The parameter ξ penalizes the search over alternative probabilities. Setting $\xi = \infty$ implements expected logarithmic utility. Small values of the penalty imply a large aversion to uncertainty about the transition probabilities.

The minimizing solution to problem (5) is:

$$L^*_{t+\epsilon} = \frac{\exp\left(-\frac{1}{\xi}\widehat{V}_{t+\epsilon}\right)}{\mathbb{E}\left[\exp\left(-\frac{1}{\xi}\widehat{V}_{t+\epsilon}\right)|\mathfrak{F}_t\right]} \quad (6)$$

[1]See, for instance, Jacobson (1973), Whittle (1981), James (1992), and Petersen et al. (2000).

provided that the denominator is well defined. This formulation gives an example of what Maccheroni et al. (2006) call variational preferences designed to confront broader notions of uncertainty other than risk. The implied minimizer is of interest for the reasons articulated by the robust Bayesian, Good (1952), as a way to assess plausibility. Moreover, the implied measure of statistical divergence is revealing as a measure of statistical challenges implicit in the choice of the penalty parameter ξ.

This construction is an alternative interpretation for the large risk aversion often imposed in recursive utility models. The mathematical equivalence can be seen by letting $\xi = \frac{1}{\gamma - 1}$. The economic interpretation, however, is very different as is the assessment of what are plausible calibrations.

2.3 Continuous-time limit

To depict the continuous-time counterpart to equation (1), suppose that the continuation utility evolves as:[2]

$$d\widehat{V}_t = \hat{\mu}_{v,t} dt + \sigma_{v,t} \cdot dB_t.$$

where $\hat{\mu}_{v,t}$ is the local mean and $|\sigma_{v,t}|^2$ is local variance. In positing this evolution we are using local normality induced by the Brownian increments to deduce the local normality of the continuation utility increments.

The limiting version of recursion (1) gives the following restriction on $(\hat{\mu}_{v,t}, |\sigma_{v,t}|^2)$:

$$0 = \left(\frac{\delta}{1-\rho} \right) \left[(C_t/V_t)^{1-\rho} - 1 \right] + \hat{\mu}_{v,t} + \left(\frac{1-\gamma}{2} \right) |\sigma_{v,t}|^2. \tag{7}$$

For the unitary IES case ($\rho = 1$), equation (7) becomes:

$$0 = \delta \left(\widehat{C}_t - \widehat{V}_t \right) + \hat{\mu}_{v,t} + \left(\frac{1-\gamma}{2} \right) |\sigma_{v,t}|^2 \tag{8}$$

Equations (7)-(8) provide an expression for the local mean $\hat{\mu}_{v,t}$ as a function of $\widehat{C}_t - \widehat{V}_t$ and the local variance $|\sigma_{v,t}|^2$.[3]

[2]Starting with V instead of $\widehat{V}$, we would write $dV_t = V_t[\mu_{v,t} dt + \sigma_{v,t} \cdot dB_t]$ where $\hat{\mu}_{v,t} = \mu_{v,t} - \frac{1}{2}|\sigma_{v,t}|^2$.

[3]We find this representation to be both pedagogically revealing with a direct heuristic link to familiar discrete-time specifications. Continuation values are only well defined up to a strictly increasing transformation as emphasized by Duffie and Epstein (1992). For mathematical reasons, often a different ordinally equivalent representation, $(V_t)^{1-\gamma}/(1-\gamma)$, is used in many papers constructed to remove the volatility contribution to the recursion.

Consider once again the robust interpretation of our recursive preferences and the minimization problem (5). This problem has a simplified version in the case of a Brownian motion information structure. Let L be a positive martingale or likelihood ratio used to induce an alternative probability distribution. From the Girsanov Theorem, under the probability measure induced by L, the process B becomes a Brownian motion with a drift $H \stackrel{\text{def}}{=} \{H_t : t \geqslant 0\}$. Locally, the Brownian increment dB_t inherits a drift $H_t dt$. The evolution of L thus takes the form

$$dL_t = L_t H_t \cdot dB_t$$

and in logarithms:

$$d\widehat{L}_t = -\frac{1}{2}|H_t|^2 dt + H_t \cdot dB_t$$

where for convenience we normalize $L_0 = 1$ or equivalently $\widehat{L}_0 = 0$. Under the implied change of probability measure, the drift of $\widehat{L}$ is $-\frac{1}{2}|H_t|^2$, which is a local measure of Kullback-Leibler divergence or relative entropy. The continuous-time recursive formulation of (5) then becomes

$$\min_{H_t} \hat{\mu}_{v,t} + \sigma_{v,t} H_t + \frac{\xi}{2}|H_t|^2.$$

The minimizing H_t is

$$H_t^* = -\frac{1}{\xi}{\sigma_{v,t}}' \tag{9}$$

with a minimized objective given by:

$$\widehat{\mu}_{v,t} - \frac{1}{2\xi}|\sigma_{v,t}|^2.$$

Comparing this result to the limiting recursion (7), and consistent with our discrete time discussion of section 2.2, the parameter γ can be viewed as a form of uncertainty aversion, instead of a measure of risk aversion, implemented by formula: $\gamma - 1 = 1/\xi$.

2.4 Stochastic discount factor process

We deduce a representation for the shadow stochastic discount factor (SDF) process in discrete and continuous time. For economies with a single agent type, this shadow SDF provides a convenient representation of equilibrium asset prices. In heterogeneous agent economies with financing frictions, the shadow SDFs are typically not equalized across agents types but they can be used to represent commonly traded assets. Moreover, their differences reflect the absence of full risk sharing induced by market frictions.

Think of the SDF process S as providing a way to depict shadow prices over any investment horizon. In particular, $S_{t+\epsilon}/S_t$ in conjunction with the transition probabilities associated with an underlying probability measure give date-t prices for a payoff at date $t+\epsilon$. Deduce the shadow SDF process by computing the intertemporal marginal rate of substitution across different possible realized states in the future. By differentiating through the utility recursion, the evolution over a period of length ϵ, expressed in logarithms, is

$$\widehat{S}_{t+\epsilon}-\widehat{S}_t = -\epsilon\delta-\rho\left(\widehat{C}_{t+\epsilon}-\widehat{C}_t\right)+(1-\gamma)\left[\widehat{V}_{t+\epsilon}-\widehat{\mathbb{R}}(\widehat{V}_{t+\epsilon}\mid\mathfrak{F}_t)\right]+(\rho-1)\left[\widehat{V}_{t+\epsilon}-\widehat{\mathbb{R}}(\widehat{V}_{t+\epsilon}\mid\mathfrak{F}_t)\right].$$

Of particular interest, the term $(1-\gamma)[\widehat{V}_{t+\epsilon}-\widehat{\mathbb{R}}(\widehat{V}_{t+\epsilon}\mid\mathfrak{F}_t)]$ adjusts for risk or robustness. Its exponential has conditional expectation equal to unity and is equal to the minimizer $L^*_{t+\epsilon}$ in (6). Thus, this particular contribution to the SDF induces a change in the probability distribution motivated explicitly by robustness considerations. More generally, the difference between $\widehat{V}_{t+\epsilon}$ and its certainty equivalent $\widehat{R}_t$ is forward looking and depends on the decision maker's perspective of the future. This contribution vanishes when $\gamma=\rho$. When $\rho=1$, only the change in measure contribution is forward looking.

Consider next the local evolution of the SDF. Write:

$$dS_t = -r_t S_t dt - S_t\pi_t\cdot dB_t$$

With this representation, r_t is the instantaneous risk-free rate and π_t is the vector of local prices of exposure to the Brownian increment dB_t, also called "risk prices". Similarly, write the local consumption evolutions as:

$$d\widehat{C}_t = \hat{\mu}_{c,t}dt + \sigma_{c,t}\cdot dB_t.$$

Then, in terms of the dynamics of $\widehat{C}$ and $\widehat{V}$ (above), we have the following riskless rate and risk prices

$$\begin{aligned} r_t &= \delta + \rho\hat{\mu}_{c,t} - \frac{1}{2}|\pi_t|^2 + \frac{(\gamma-1)(\gamma-\rho)}{2}|\sigma_{v,t}|^2 \\ \pi_t &= \rho\sigma_{c,t} + (\gamma-\rho)\sigma_{v,t}. \end{aligned}$$

3 Local measures of exposures and prices

In all the models we consider, the logarithms of several quantities of interest will grow or decay stochastically over time with increments that are stationary Markov processes. Let M be such a process and $\widehat{M}$ its logarithm. Restrict the process $\widehat{M}$ to display linear, stochastic growth or decay. Write

$$\widehat{M}_{t+\epsilon} - \widehat{M}_t = \epsilon \hat{\mu}_m(X_t) + \sigma_m(X_t) \cdot (B_{t+\epsilon} - B_t) \tag{10}$$

where X is an asymptotically stationary Markov process. Examples of such $\widehat{M}$ processes in our models are the log SDF $\widehat{S}$ and log consumption $\widehat{C}$.

3.1 Shock elasticities

Shock elasticities are constructed using local changes in the exposure to shocks. For instance, consider a shock, $B_\epsilon - B_0$ that is distributed as a multivariate standard normal. We introduce a parameterized family of random variables $H_\epsilon(\mathsf{r})$ where

$$\log H_\epsilon(\mathsf{r}) = \mathsf{r}\nu(X_0) \cdot (B_\epsilon - B_0) - \frac{\mathsf{r}^2}{2}\epsilon|\nu(X_0)|^2.$$

where we normalize the row vector ν so that $\mathbb{E}\left[|\nu(X_0)|^2\right] = 1$. In our applications, ν is state independent and selects one of the components of $B_\epsilon - B_0$. Notice that $H_\epsilon(\mathsf{r})$ is positive and has conditional expectation equal to one. Consider:

$$\frac{d}{d\mathsf{r}} \log \mathbb{E}\left[\left(\frac{M_t}{M_0}\right) H_\epsilon(\mathsf{r}) \mid X_0\right]\Bigg|_{\mathsf{r}=0} = \frac{\nu(X_0) \cdot \mathbb{E}\left[\left(\frac{M_t}{M_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\mathbb{E}\left[\left(\frac{M_t}{M_0}\right) \mid X_0\right]}. \tag{11}$$

We refer the outcome as a shock elasticity because we differentiate a logarithm with respect to an argument $H_\epsilon(\mathsf{r})$ which is equal to one at $\mathsf{r} = 0$. This elasticity depends on the state X_0 and horizon t.

In formula (11), notice that the essential input is:

$$\frac{\mathbb{E}\left[\left(\frac{M_t}{M_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\mathbb{E}\left[\left(\frac{M_t}{M_0}\right) \mid X_0\right]}, \tag{12}$$

which is a "distorted expectation" of the shock $B_\epsilon - B_0$. When scaled by $\frac{1}{\epsilon}$, this has well defined diffusion limit as $\epsilon \downarrow 0$. This limit has been characterized in the stochastic process literature as "Malliavin derivative" and can be computed numerically in a straightforward way for Markovian economies. See Borovička et al. (2014) for further discussion.

The scaling by H_ϵ in formula 11 (or its continuous time limit) has two distinct interpretations depending on the application:

i) it changes the distribution of B_ϵ by giving it a conditional mean $\epsilon \mathsf{r}\nu(X_0)$

ii) it changes the exposure of $\widehat{M}_t - \widehat{M}_0$, and hence M_t/M_0, to the shock $B_\epsilon - B_0$ through the addition of $\mathsf{r}\nu(X_0) \cdot (B_\epsilon - B_0)$.

The first of these interpretations provides a distributional version of an impulse response function. It matches exactly for the linear, log-normal model, in which case X is a multivariate, Gaussian vector autoregression, μ is affine in x, and ν and σ_m are vectors of constants. Once we include nonlinearities, the state x can matter along with the time horizon t. See Gallant et al. (1993) and Koop et al. (1996) for related constructs of nonlinear impulse responses. For intertemporal asset pricing applications, the second interpretation will help us understand shock elasticities as implied compensations for changes in the exposures. We discuss this asset pricing application next.

3.2 Compensations for exposure to uncertainty

Let $\widehat{Y}$ denote the logarithm of a cash flow process, and let $\widehat{S}$ denote the equilibrium log SDF process both of which have stochastic evolutions of the form (10). Compute:

i) exposure elasticity
$$\frac{\nu(X_0) \cdot \mathbb{E}\left[\left(\frac{Y_t}{Y_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\epsilon \mathbb{E}\left[\left(\frac{Y_t}{Y_0}\right) \mid X_0\right]};$$

ii) value elasticity
$$\frac{\nu(X_0) \cdot \mathbb{E}\left[\left(\frac{S_t Y_t}{S_0 Y_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\epsilon \mathbb{E}\left[\left(\frac{S_t Y_t}{S_0 Y_0}\right) \mid X_0\right]};$$

iii) price elasticity (exposure minus value)

$$\frac{\nu(X_0) \cdot \mathbb{E}\left[\left(\frac{Y_t}{Y_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\epsilon \mathbb{E}\left[\left(\frac{Y_t}{Y_0}\right) \mid X_0\right]} - \frac{\nu(X_0) \cdot \mathbb{E}\left[\left(\frac{S_t Y_t}{S_0 Y_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\epsilon \mathbb{E}\left[\left(\frac{S_t Y_t}{S_0 Y_0}\right) \mid X_0\right]};$$

These all have well defined continuous-time limits as $\epsilon \downarrow 0$. As mentioned above, one can interpret the price elasticity as the expected excess return required for a marginal increase in risk exposure to Y.

There is one additional calculation that is also of interest. Suppose that $M = \exp(\widehat{M})$ is a martingale, L. This is of interest when entertain beliefs that differ from the data generating process and deducing the value contributions. From the Law of Iterated Expectations,

$$\frac{\nu(X_0) \cdot \mathbb{E}\left[\left(\frac{L_t}{L_0}\right)(B_\epsilon - B_0) \mid X_0\right]}{\epsilon \mathbb{E}\left[\left(\frac{L_t}{L_0}\right) \mid X_0\right]} = \left(\frac{1}{\epsilon}\right) \nu(X_0) \cdot \mathbb{E}\left[\left(\frac{L_\epsilon}{L_0}\right)(B_\epsilon - B_0) \mid X_0\right],$$

and does not depend on the horizon t. In this circumstance (and perhaps others as well), we find it revealing to change the date of the Brownian increment by reporting the small ϵ limit of

$$\frac{1}{\epsilon} \nu(X_0) \cdot \mathbb{E}\left[\left(\frac{L_t}{L_0}\right)(B_t - B_{t-\epsilon}) \mid X_0\right] \tag{13}$$

as a term structure of "uncertainty prices." These prices will be horizon dependent.

4 An initial benchmark economy

For pedagogical purposes, we begin our exposition by focusing on a "representative household" with recursive preferences in a complete-market production economy featuring long-run-risk shocks. We may view the economy as a production-based counterpart to that in the seminal paper by Bansal and Yaron (2004). In part we share a similar ambition to that of Jermann (1998) in describing a production-based model with asset pricing, but we also use this class of models as a benchmark for model classes that include heterogeneous capital or heterogeneous investors. We follow Bansal and Yaron (2004) by focusing on recursive utility in contrast to Jermann (1998), who features habit persistence preferences.

Since our benchmark model features complete markets, we study the planner problem to characterize equilibrium quantities and prices in the economy. A decentralized version of the model allows for a rich set of assets local spanning of the Brownian increments along with a riskless security. These local prices are embedded in the stochastic discount factor evolution.

Even for a models with a single capital stock, the introduction of capital turns out to be important relative to endowment economies when we change preference parameters. Much of the asset pricing literature features endowment economies in which changes in the intertemporal elasticity of substitution (IES) has only pricing impact. As we will illustrate, in a production economy changing the IES has a substantial impact on the investment/capital ratio and hence growth in the underlying economy.

4.1 Exogenous stochastic inputs

We presume that there are two underlying exogenous processes that evolve as solutions to stochastic differential equations

$$dZ_t^1 = -\beta_1 Z_t^1 dt + \sqrt{Z_t^2}\sigma_1 \cdot dB_t \tag{14}$$

$$dZ_t^2 = -\beta_2 (Z_t^2 - 1) dt + \sqrt{Z_t^2}\sigma_2 \cdot dB_t \tag{15}$$

where $\beta_1 > 0$, $\beta_2 > \frac{1}{2}|\sigma_2|^2$. In addition, σ_1, σ_2, are d-dimensional vectors of real numbers. The Z^1 process governs the conditional mean of the stochastic component to technology growth and the process Z^2 captures the exogenous component to aggregate stochastic volatility. Notice that $\sqrt{Z^2}$ scales the Brownian increment to both of the processes. The local variance of the exogenous technology shifter is $Z_t^2|\sigma_1|^2$, and the local variance for the stochastic volatility process is $Z_t^2|\sigma_2|^2$.

The stochastic variance process Z^2 is a special case of a Feller square root process. The exogenous stochastic technology growth process, Z^1, is a continuous-time version of an autoregression with innovations that are conditionally heteroskedastic. The autoregressive coefficients for discrete-time counterparts are $\exp(-\beta_1)$, $\exp(-\beta_2)$. Values of β_1 and β_2 that are close to zero imply a large amount of persistence. The unconditional mean of Z^1 is normalized to be zero, and the unconditional mean of Z^2 is normalized to be one. In what follows, we let

$$Z_t \stackrel{\text{def}}{=} \begin{bmatrix} Z_t^1 \\ Z_t^2 \end{bmatrix}.$$

and

$$\mu_z(Z_t) \stackrel{\text{def}}{=} \begin{bmatrix} -\beta_1 Z_t^1 \\ -\beta_2(Z_t^2 - 1) \end{bmatrix} \qquad \sigma_z \stackrel{\text{def}}{=} \sqrt{Z_t^2} \begin{bmatrix} {\sigma_1}' \\ {\sigma_2}' \end{bmatrix}.$$

4.2 Technology

We use a so-called AK technology with adjustment costs to represent production.[4] Let K_t be the stock of capital, I_t the investment rate, and C_t the consumption rate at date t. The technology consists of two equations: an output and a capital evolution equation. Output is constrained by:

$$C_t + I_t = \alpha K_t, \tag{16}$$

where α is a fixed productivity parameter. Our capital accumulation equation features aggregate shocks as follows:

$$dK_t = K_t \left[\Phi\left(\frac{I_t}{K_t}\right) + \beta_k Z_t^1 - \eta_k \right] dt + K_t \sqrt{Z_t^2} \sigma_k \cdot dB_t, \tag{17}$$

where η_k embeds an adjustment for depreciation and σ_k is a $d \times 1$ vector quantifying the importance of the Brownian motion in generating stochastic returns to investment. The function Φ, called the installation function by Hayashi (1982), is an increasing and concave function. A leading example of Φ in our essay is

$$\Phi(i) = \frac{1}{\phi} \log\left(1 + \phi i\right). \tag{18}$$

where i is a stand in for a realization of the investment-capital ratio. The small i quadratic approximation is:

$$\Phi(i) \approx i - \frac{\phi}{2} i^2$$

We note this relationship since quadratic specifications are often imposed in the investment literature.

By design, the technology is homogeneous of degree one in investment, capital and consumption. This model has stochastic shocks that i) alter the physical returns to investment; ii) shift the conditional mean of that investment; and iii) shift the aggregate volatility of the technology. For such a stylized model, capital should be interpreted very broadly and potentially should include human, organizational, and intangible contributions. The shock

[4]See, e.g., Cox et al. (1985), Merton (1973), Jones and Manuelli (1990) and Brock and Magill (1979).

to physical returns to investment is sometimes referred to as "capital quality shock" or a "technology shock."[5]

4.3 Value function

Given the homogeneity properties of both preferences and technology, the value function scales linearly with the capital stock. It will be most convenient to work with the logarithm of the value function, which we posit takes the following form:

$$\widehat{V}_t = \widehat{K}_t + \upsilon(Z_t). \tag{19}$$

We combine the evolutions of $\upsilon(Z_t)$ and $\widehat{K}_t$ to deduce a Hamilton-Jacobi-Bellman equation for the function υ:

$$\begin{aligned} 0 = \max_{c+i=\alpha} \Bigg\{ \left(\frac{\delta}{1-\rho}\right) \left(c^{1-\rho} \exp\left[(\rho-1)\upsilon\right] - 1\right) + \Phi(i) + \beta_k z_1 - \eta_k - \frac{1}{2} z_2 |\sigma_k|^2 \\ + \mu_z \cdot \frac{\partial \upsilon}{\partial z} + \frac{z_2}{2} \text{trace} \left\{ {\sigma_z}' \frac{\partial^2 \upsilon}{\partial z \partial z'} \sigma_z \right\} + \frac{(1-\gamma) z_2}{2} \left| \sigma_k + {\sigma_z}' \frac{\partial \upsilon}{\partial z} \right|^2 \Bigg\}, \end{aligned} \tag{20}$$

where c is the consumption-to-capital ratio and i is the investment-to-capital ratio. The first-order condition for the optimal consumption-capital ratio, c^*, is:

$$\delta \left[c^*(z)\right]^{-\rho} \exp\left[(\rho-1)\upsilon(z)\right] = \Phi' \left[\alpha - c^*(z)\right]. \tag{21}$$

Capital provides the sole source of wealth in this economy. Total wealth is given by the continuation value divided by the marginal utility of consumption, evaluated at equilibrium outcomes:[6]

$$\frac{1}{\delta} \left[c^*(z)\right]^{\rho} \exp[(1-\rho)\upsilon(z)] k.$$

The implied price of capital is given by $Q_t = q(Z_t)$ where

$$q(z) = \frac{1}{\delta} \left[c^*(z)\right]^{\rho} \exp[(1-\rho)\upsilon(z)] = \frac{1}{\Phi' \left[\alpha - c^*(z)\right]} = 1 + \phi i^*(z). \tag{22}$$

[5] Our model is isomorphic to an AK model where productivity (instead of capital K_t) is being hit by Brownian shocks, and in which adjustment costs also scale up and down with such shock.

[6] The two recursions in (1) are both homogeneous of degree one. From an infinite-dimensional version of Euler's Theorem, the continuation value divided by the marginal utility of consumption is the current period shadow price of current and future consumption which equals wealth in equilibrium.

The instantaneous capital return in this economy has an exposure to the vector, dB_t, of Brownian increments given by

$$\sqrt{Z_t^2}\sigma_k + \sqrt{Z_t^2}\frac{\partial \ln q}{\partial z'}(Z_t)\sigma_z$$

where the first term captures the exposure of capital to the Brownian increments and the second one reflects the exposure of valuation to these same increments.

4.4 Example economies

In contrast to the other economies that we study, this economy can be well approximated by log-quadratic approximations. We use this as a benchmark to the study of economies that are more explicitly nonlinear. We imagine a family of economies indexed by $(\rho, \gamma, \delta, \alpha)$. Of course other parameter sensitivity could also be explored. Our use of a production economy provides a revealing contrast to the familiar Lucas (1978) endowment economy.

In consumption-based models with endowment specifications, the preference parameter ρ has a substantial impact on the risk-free rate. In models with production, like the ones we explore here, changing ρ while holding other parameters of preferences and technology fixed, has a substantial impact on production and savings. Table 1 gives parameter values that we hold fixed in these computations, and Table 2 reports the steady state investment- and consumption-to-output ratios along with the steady state growth rate. The IES has a dramatic impact on all these average macroeconomic aggregates.

η_k	ϕ	β_k	β_1	β_2	σ_k	σ_1	σ_2
.04	8	.01	.056	.145	$\begin{bmatrix} .0095 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} .022 & .050 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & .26 \end{bmatrix}$

Table 1: Parameter values that we hold fixed for the one-capital model. The numbers for β_1, σ_k and σ_1 come from Hansen and Sargent (2021), and the numbers for β_2 and σ_2 come from Schorfheide et al. (2018). In both cases, we use the medians of their econometric evidence for post war data. When Schorfheide et al. (2018) fit to a longer time series including the depression, their counterpart to σ_2 is about double.

ρ	0.67	1	1.5
consumption-output ratio	0.012	0.175	0.279
investment-output ratio	0.988	0.825	0.721
steady state growth rate	0.028	0.019	0.013

Table 2: Steady states for alternative specifications of ρ for $\alpha = .092$ and $\delta = .01$. These are computed by setting shock variances to zero.

To diminish this impact, we change the productivity parameter α to pin down a common growth rate in consumption. Table 3 reports the results. There is still a noticeable impact of ρ on investment- and consumption-to-output ratios, but not nearly as dramatic. The subjective discount rate also impacts these steady states by increasing the consumption-to-output ratios as also seen by Table 3.

	$\delta = .01$		
ρ	0.67	1	1.5
consumption-output ratio	0.071	0.175	0.296
investment-output ratio	0.929	0.825	0.704
productivity (α)	0.082	0.092	0.108
growth rate	0.019	0.019	0.019
	$\delta = .015$		
ρ	0.67	1	1.5
consumption-output ratio	0.155	0.242	0.346
investment-output ratio	0.845	0.758	0.654
productivity (α)	0.090	0.100	0.116
growth rate	0.019	0.019	0.019

Table 3: Steady states adjusting the productivity parameter α to match a fixed the growth rate. These are computed by setting the shock variances to zero.

We next consider shock exposure and shock price elasticities. We focus on the growth-rate shock. The capital evolution shock is also quantitatively important. In contrast, the impact of the stochastic volatility shock is quantitatively small.[7] Stochastic volatility does induce state dependence in the other shock elasticities as we will illustrate.

[7]The quantitative magnitudes could be amplified by pushing the mean reversion parameter β_2 even closer to zero, as is done in calibrations of asset pricing models.

Consider the shock exposure elasticity, or equivalently the local impulse response function, for the investment-to-output ratio. Since output is proportional to capital, formula (22) implies these are also approximately the elasticities for the price of capital (which is affine in the investment-to-capital ratio). As Figure 1 shows, the responses to a growth rate shock are positive when $\rho < 1$ and negative when $\rho > 1$. The elasticities are only modestly sensitive to changing the risk aversion parameter γ, while they increase notably when the subjective discount rate δ is increased.

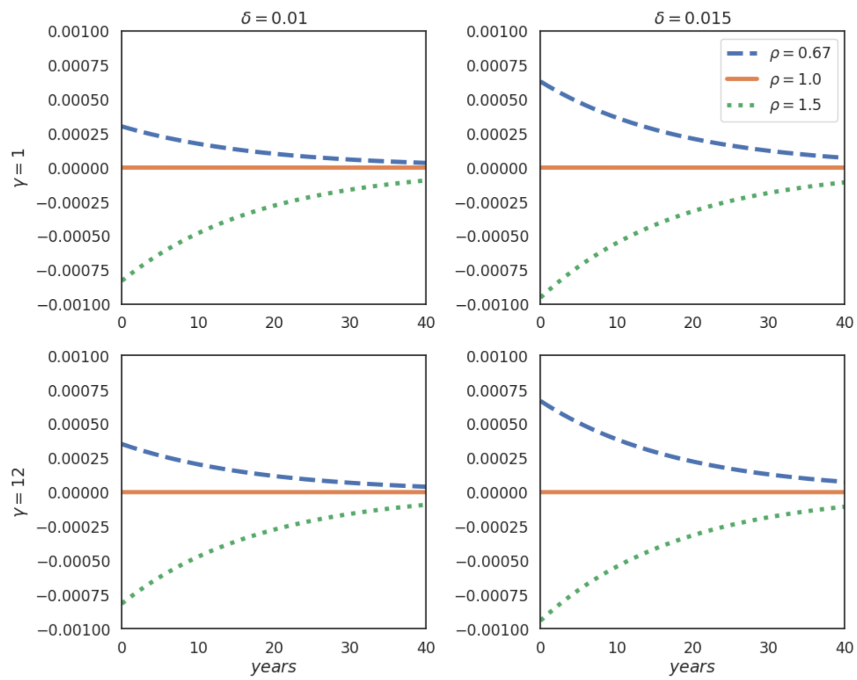

Figure 1: Investment-output ratio exposure elasticities to a growth-rate shock. The elasticities are initialized by setting the stochastic growth rate state to zero.

Finally, we consider both the shock exposure and price elasticities of consumption in Figure 2. The consumption elasticity to a growth rate shock builds over time, as expected given investment adjustment costs. The $\rho = 1$ elasticities imitate those of an endowment economy like the Bansal and Yaron (2004) economy (without stochastic volatility). The risk aversion parameter γ has very little impact on these exposure elasticities, in contrast to the price elasticities. As revealed by Figure 2, the shock price elasticities are very sensitive, as expected, to the choice of γ. Recall the robustness interpretation of recursive utility, where misspecification concerns contribute a martingale component to valuation. This component comes to dominate as γ becomes larger and this leads to relatively flat shock price elasticity trajectory.

Figure 3 shows how the elasticities depend on the initial level of volatility. The key takeaway is that stochastic volatility provides exogenous fluctuations in risk pricing, in contrast to some of the more endogenous mechanisms that we explore going forward. In addition, as is well understood, a shock to exogenous volatility itself is priced under these preferences, as shown via its shock price elasticity in the right panel.

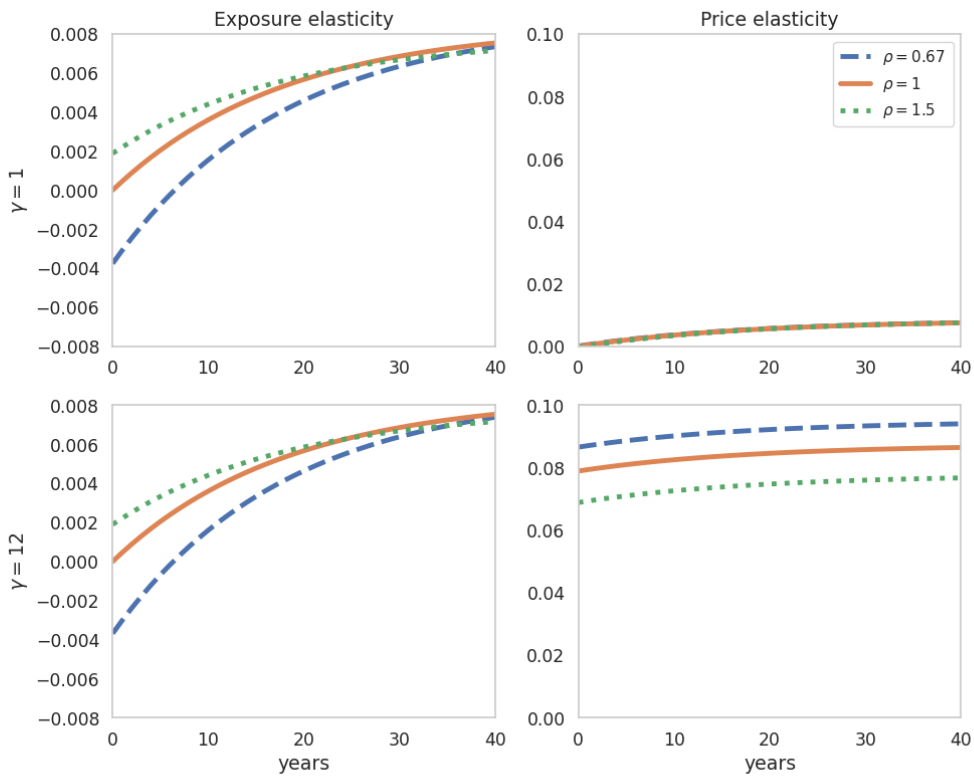

Figure 2: Exposure and price elasticities for the growth rate shock. Perturbations are relative to the equilibrium consumption process. The growth and volatility states are set to their medians.

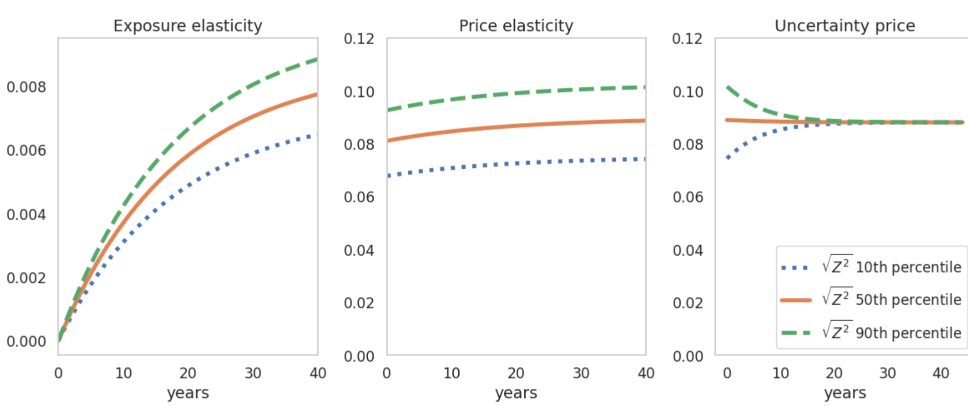

Figure 3: Exposure and price elasticities for $\gamma = 12$, $\rho = 1$, and for alternative volatility quantiles.

4.5 Endogenous fluctuations in valuation

Here we illustrate an endogenous channel induced by ambiguity aversion by building on ideas from Chen and Epstein (2002), Hansen (2007), Andrei et al. (2019), and, in particular, Hansen and Sargent (2021). As we will show, this adds a form of state dependence in valuation. For this illustration we focus exclusively on the case in which $\rho = 1$. To feature the endogenous of fluctuations in valuation, we abstract from exogenous stochastic volatility in this subsection (by setting $\sigma_2 = 0$).

We follow Hansen and Sargent (2022) by considering both model ambiguity and potential model misspecification. Recall that recursive utility provides a direct link to the latter, an approach that we continue to use here. For model ambiguity, we proceed differently. Given a parameterized family of models, the investor is unsure how much weight should be given to each. For a Bayesian decision maker, this would be addressed with subjective inputs in the form of a prior. Our investor is unsure which such prior to impose. Formally, we using a framework for diffusion processes that is consistent with Chen and Epstein (2002) to entertain a rich family of what Hansen and Sargent (2022) refer to as "structured" models.

In our application we start with four-dimensional space of unknown parameters in the drifts of capital K and the growth rate Z^1. We modify the evolution of Z^1 to be:

$$dZ^1_t = \psi_1 - \beta_1 Z^1_t + \sigma_1 \cdot dB_t$$

where the parameter ψ_1, which we have taken to be zero so far, allows for a shift in the local drift dynamics that does not scale with Z^1. In the long-term, $\psi_1 \neq 0$ could induce a nonzero unconditional mean in Z^1 process. The unknown parameters are $\eta_k, \beta_k, \psi_1, \beta_1$. Recall that η_k governs depreciation and β_k the exposure to long-term growth rate uncertainty. Our investors take uncertainty in these parameters as a starting point, but they entertain a so-called time varying parameter perspective without imposing a prior on the form of the time variation. Instead the parameters are constrained to be in an ambiguity set using a recursive measure of relative entropy or Kullback-Leibler divergence as described in Hansen and Sargent (2021). Figure 4 plots two-dimensional projections of the ambiguity set.[8] In effect, this approach entertains misspecification relative to a benchmark in a much more structured way than that embedded in the robust interpretation of Kreps and Porteus

[8]We constructed these sets using, in the notation of Hansen and Sargent (2021), $q = 0.3$ with $\rho_1 = 0$ and $\rho_2 = \frac{q^2}{2|\sigma_1|^2}$.

(1978) utility.

In this recursive formulation of ambiguity aversion, investors minimize the expected value function increment over the four dimensional set of parameter values, instant-by-instant. The minimizer over this four-dimensional set will reside somewhere on the boundary and its location will depend on the realized growth-rate state, z^1. The problem is made tractable in part because the minimization problem is quadratic. We also include potential model misspecification in the same manner as described previously.

We illustrate the nonlinear outcome by reporting the implied uncertainty-adjusted (minimizing) drift for the long-run growth process in Figure 5. The downward slope of the line in the baseline models governs the pull towards zero in the conditional mean dynamics for Z^1. The dashed and dot-dashed curves are the uncertainty-adjusted nonlinear counterparts. The dot-dashed curve includes misspecification concerns in addition to parameter ambiguity. Observe that these curves are flatter for negative growth rates and steeper for positive growth rates. This is to be expected because investors fear persistence when growth is sluggish and the lack of persistence when growth is brisk. This outcome emerges in the computations in part because of how the minimizing choice of β_1 over the ambiguity set displayed in Figure 4 of depends on z^1. The investor is exploring the other three parameters as well, and outcome of minimization also impacts Figure 5 and a counterpart for drift specification for capital.

While the one-capital model without ambiguity concerns can be approximately solved using log-quadratic approximation, the model with ambiguity requires a global alternative to capture the potential nonlinearities that are entertained by the decision maker.

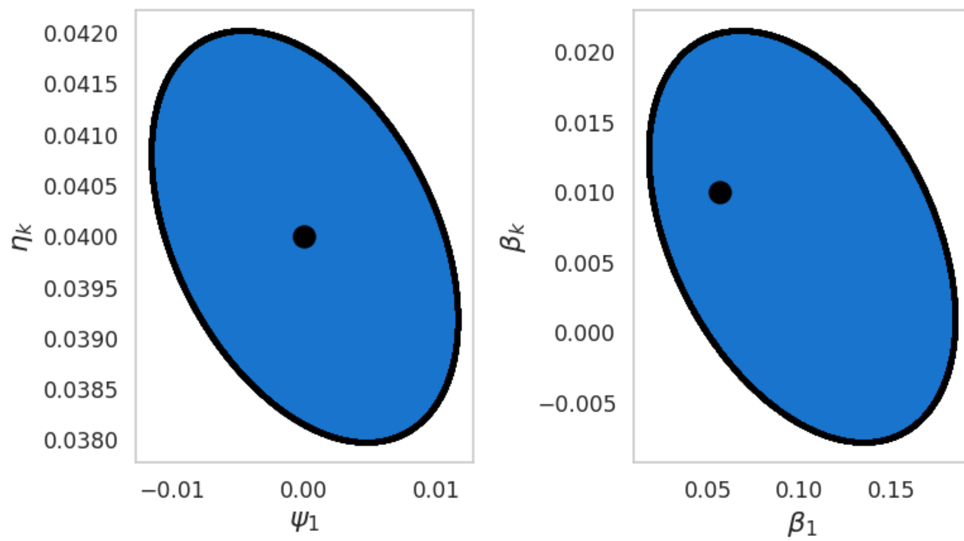

Figure 4: Ambiguity parameter sets constrained by a flow measure of relative entropy developed in Hansen and Sargent (2021). The plot on the left informs us about the assumed ambiguity in the depreciation parameter (α) and a constant term (ψ_1) in the evolution equation for Z^1. The plot on the right gives the assumed ambiguity in the slope coefficients β_k for the state Z_t^1 in the capital evolution and the persistence parameter in the state evolution (β_1). The baseline parameters are recorded as dots in the two figures.

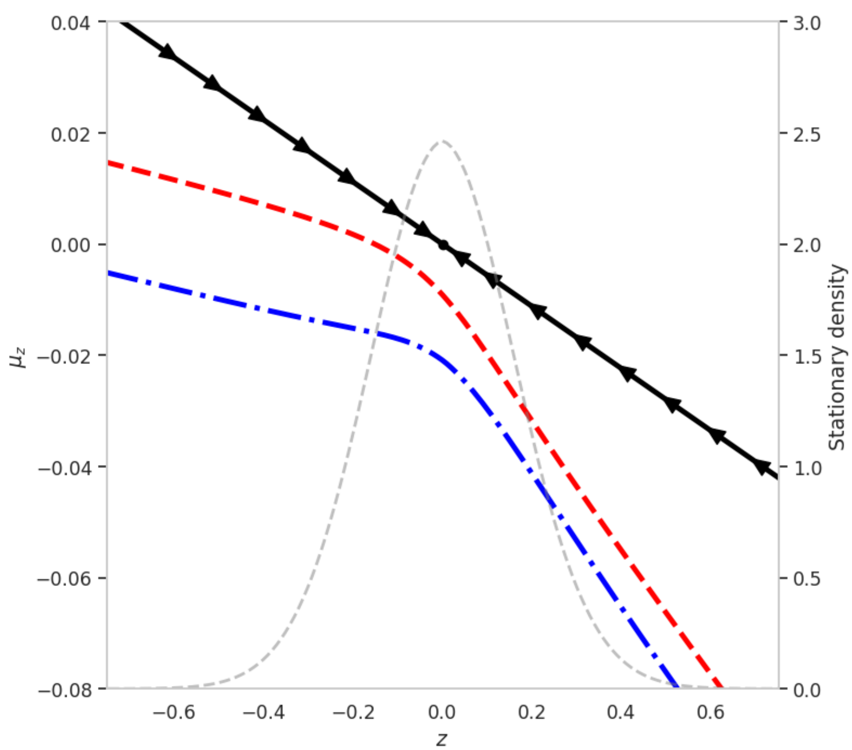

Figure 5: Distorted growth rate drift and stationary density for Z^1. Black solid: baseline model; red dashed: $\gamma = 1$; blue dot-dashed: $\gamma = 12$; and gray dashed: Z^1 stationary density

The two forms of uncertainty aversion we consider introduce a composite martingale component to valuation. We explore its properties by looking at the implied uncertainty price elasticities using the formula (13). The results are reported in Figure 6. As we have shown, increasing γ is equivalent to enhancing overall concerns about model misspecification. This increases the uncertainty prices. We represent state dependence by exploring not only the median, but also the 10th and 90th percentiles. While the 90th percentile prices start higher than the others, this gets reversed as we go out to longer horizons. This reflects the decrease in persistence in the uncertainty-adjusted probability measure for relatively high realized values of the growth state Z_t^1.

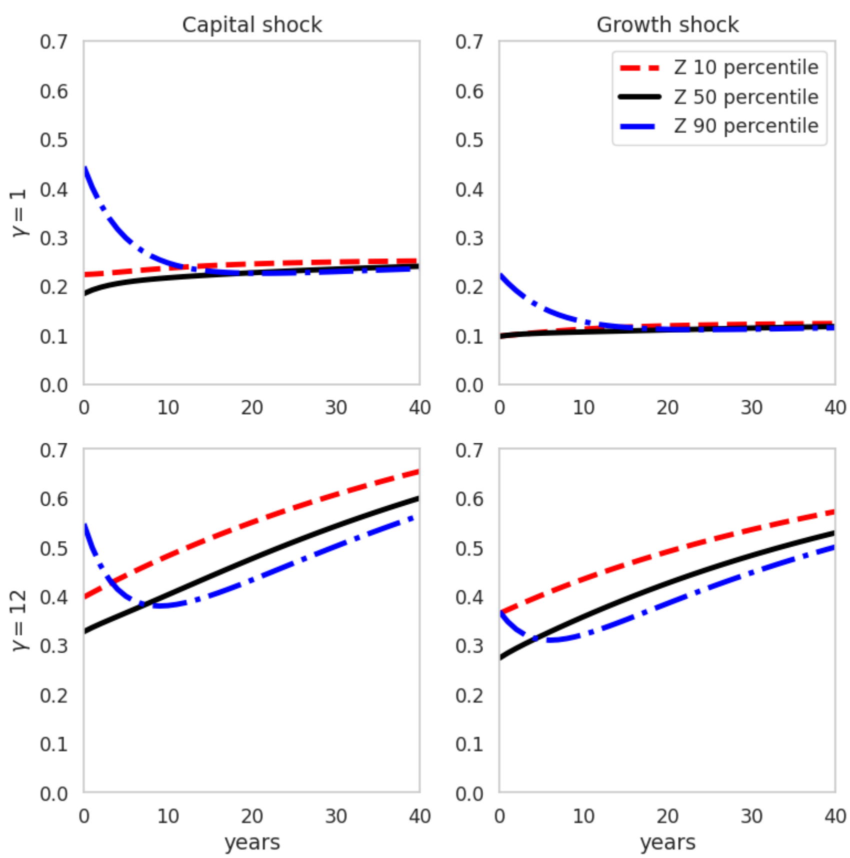

Figure 6: Shock price elasticities for the martingale contribution induced by uncertainty aversion. Black solid: median of the Z^1 stationary distribution; red dashed: .1 decile; and dot-dashed: .9 decile.

In summary, we get movements in asset values induced by changes in what models are most concerning within the constrained ambiguity set. This fluctuation is induced in large part by uncertainty in the persistence of the process Z^1. In low growth states investors are concerned about being "stuck in a rut" whereas in good times they worry that brisk growth will end soon. This type of mechanism was noted in Hansen (2007) in a distinct but related modeling framework. That paper uses a different specification of ambiguity aversion and entertains explicit learning. In the example here, learning is off the table because of potential time or state variation in parameters. Relatedly, learning about persistence was also featured in Andrei et al. (2019) as mechanism for fluctuations over time in valuation.

5 Sluggish heterogeneous capital stocks

We now explore two capital models with growth rate uncertainty. Precursors of these models are the multiple tree models of Cochrane et al. (2008) and Martin (2013). These models do not entertain capital movements from one production source to another. Here we follow Eberly and Wang (2009), Eberly and Wang (2012), Hansen et al. (2020), and Kozak (2022) by allowing capital mobility subject to adjustment costs. In this sense, capital movements are sluggish. We extend the capital evolution in Eberly and Wang (2009), Eberly and Wang (2012), and Kozak (2022) by introducing exposures to an exogenously specified growth rate uncertainty consistent with our previous examples, similar to Hansen et al. (2020). We allow for the exposure to this uncertainty to be heterogeneous.

Formally, consider a family of models with two capital stocks and adjustment costs.

$$dK_t^j = K_t^j \left[\Phi^j \left(\frac{I_t^j}{K_t^j} \right) + \beta_k^j Z_t^1 - \eta^j \right] dt + K_t^j \sqrt{Z_t^2} \sigma_k^j \cdot dB_t,$$

for $j = 1, 2$. Suppose that the output equation is now

$$C_t + I_t^1 + I_t^2 = \alpha K_t^a$$

where aggregate capital is a CES aggregator of the two capital stocks:

$$K_t^a = \left[(1-\zeta) \left(K_t^1 \right)^{(1-\tau)} + \zeta \left(K_t^2 \right)^{(1-\tau)} \right]^{\frac{1}{1-\tau}}$$

for $0 \leqslant \zeta < 1$ and $\tau \geqslant 0$. For characterization and computation, we form two state variables: one is $\widehat{Y}_t = \log(K_t^2/K_t^1)$ and the other is $\widehat{K}_t^a$. For this class of models, the value function has the separable form:

$$\widehat{V}_t = \widehat{K}_t^a + \upsilon(\widehat{Y}_t, Z_t).$$

Eberly and Wang (2009), Eberly and Wang (2012), Hansen et al. (2020) and Kozak (2022) feature the case in which the two capital stocks are perfect substitutes ($\tau = 0, \zeta = .5$). In the illustrations that follow, we also impose this restriction. Our computational software allows for production curvature among the two capital stocks and opens the door to an even richer collection of examples. With perfect substitutability, the deterministic limit of this model has multiple steady states. This makes locally linear-quadratic approximations inoperative. Even with production curvature, local methods can be unreliable. Global solutions' approaches are necessary for this class of examples.

Parameters common across the two capitals				
η_k	ϕ	$\alpha,\ \rho$	β_1	σ_1
.04	8	$\alpha = .164,\ .184,\ .216$	.056	$\begin{bmatrix} .0156 & .0156 & .050 & 0 \end{bmatrix}$
		$\rho = .67,\ 1,\ 1.5$		

symmetric	asymmetric	capital volatilities
$\beta_k^1 = .01$	$\beta_k^1 = 0$	$\sigma_k^1 = \begin{bmatrix} .0126 & 0 & 0 & 0 \end{bmatrix}$
$\beta_k^2 = .01$	$\beta_k^2 = .02$	$\sigma_k^2 = \begin{bmatrix} 0 & .0126 & 0 & 0 \end{bmatrix}$

Table 4: Parameter values for the two capital model. For the two capital model, we include a separate capital shock for each technology. The coefficients on the two capital stocks are given by the first two entries of the σ's. We doubled α for the two capital because K_t^a is the average capital stock for each of the three specifications of ρ. We scaled down the first two entries of σ_1 in order that the overall instantaneous standard deviation $|\sigma_1|$ remains the same as for the one capital model. We follow Hansen et al. (2020) and scale up σ_k^i for $i = 1, 2$ in order that consumption volatility remain about the same as for the one-capital model. The specification "symmetric" presumes symmetric exposure to growth uncertainty, while the specification "asymmetric" presumes that only the second capital is exposed to growth uncertainty.

We consider two different specification of exposures. One specification is "symmetric." While each capital stock has its own shock, the relative importance of long-term uncertainty to each K^j is the same. The other specification is "asymmetric." The first capital stock is not exposed to long-run uncertainty while the second one is. Table 4 gives parameter values supporting figures that follow along with explanations. For these economies we abstract from stochastic volatility and parameter ambiguity.

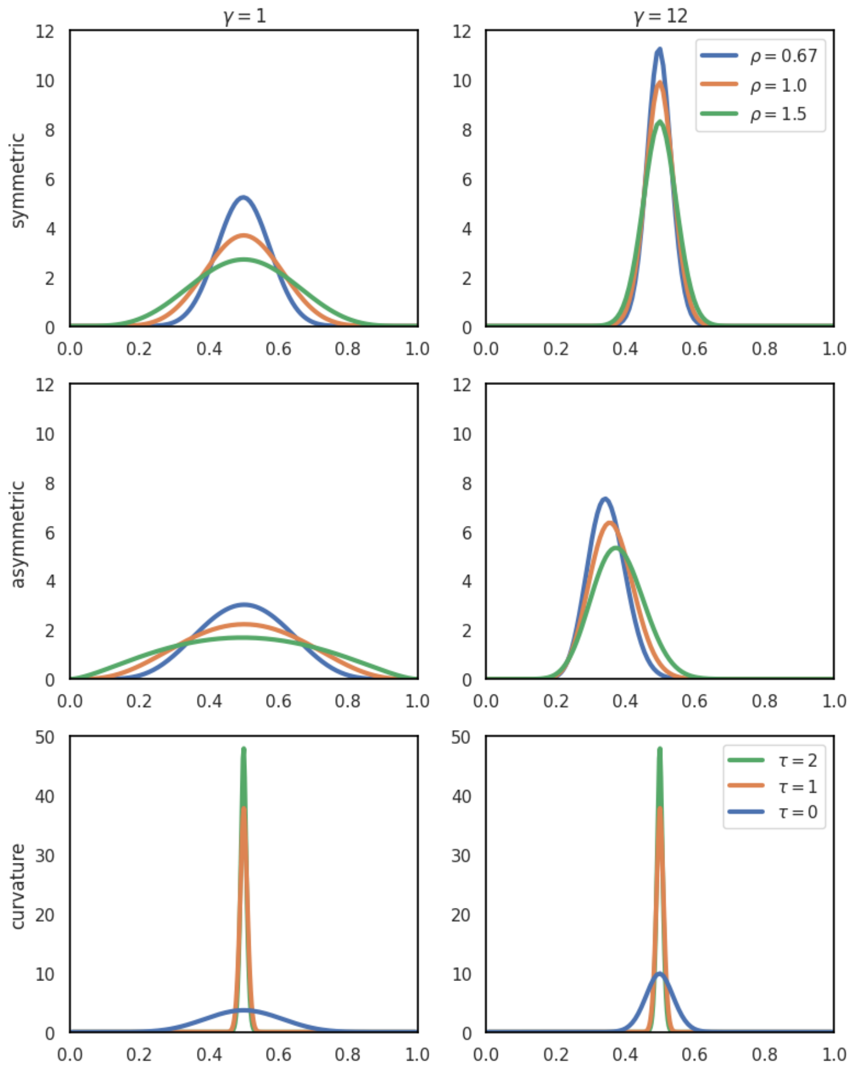

Figure 7: Stationary densities for the second capital stock as share of total capital. For the "asymmetric" row, only the second capital stock is exposed to growth-rate uncertainty. Finally, for the "curvature" row, the $\tau = 1$ specification assumes a unitary substitution elasticity across the two types of capital, and the $\tau = 2$ specification assumes a substitution elasticity equal to $1/2$. The results in the third row impose $\rho = 1$ and the same exposure to long-term uncertainty for both capital stocks.

We start by reporting stationary densities in Figure 7 for the fraction of the capital that is allocated to the second technology. Consider first the case of symmetric exposures. We see some sensitivity to the IES with the plots for $\rho = .67$ being more peaked. As Eberly and Wang (2012) emphasize, increasing risk aversion through changing γ (or increasing the concern for misspecification) makes diversification all the more attractive giving rise to densities that are much more sharply peaked. It is noteworthy that when $\gamma = 1$, the asymmetric parameterization flattens out the allocation densities. But arguably more interesting is that for $\gamma = 12$ the second capital stock becomes much less attractive and even more so as we decrease ρ. The mode of the density is now centered near .2 instead of .5 as investors seek to avoid exposure to long-term uncertainty. For the model specifications discussed so far, the two capital stocks are perfect substitutes in the production of output.

So far, the only heterogeneity in the capital stock is in the exposure to shocks and long-term uncertainty. We next illustrate the impact of production function curvature by making the elasticity of substitution across the two types of capital one ($\tau = 1$) and one-half ($\tau = 2$). See the third row of Figure 7. This decrease in elatisticity of substitution in production makes the stationary densities more peaked. This is to be expected given the more central role both capital stocks in the production of output. We include this computation as an illustration only, as there are alternative substantive motivations for multiple capital stocks with differential impacts on production. For example, intangible, organizational or human capital contribute to production in arguably distinct ways. While incorporation of these components could lead to even richer models, the force on display in Figure 7 will still be present.[9]

Figure 8 plots the shock elasticity or local impulse responses for the aggregate investment-to-capital ratio. We only depict these for $\gamma = 12$ as the $\gamma = 1$ responses are very similar. The elasticities for the symmetric case are very similar to those we computed for the one-capital model. In contrast, for the asymmetric case the responses are more muted consistent with the flatter densities reported in Figure 7. Figure 9 depicts the shock price elasticities for the growth shock. We report only the case in which $\gamma = 12$ as the $\gamma = 1$ results are unsurprisingly small. The price elasticities are very flat reflecting a dominant martingale component to the SDF. Recall we used robustness concerns to model misspecification as an important contributor to this martingale. The magnitude of the growth-rate shock price elasticities are very close to those we reported for the single-capital model. In the asymmetric case, the prices are significantly smaller because capital is reallocated to reduce the exposure to growth rate uncertainty.

[9] See Crouzet et al. (2022) for a recent discussion of modeling and measuring intangible capital.

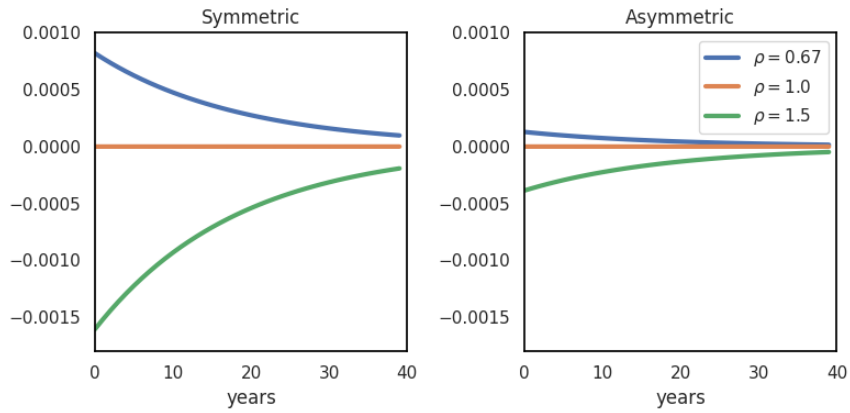

Figure 8: Investment-output ratio exposure elasticities for $\gamma = 12$. The reported elasticities condition on the medians of the state variables.

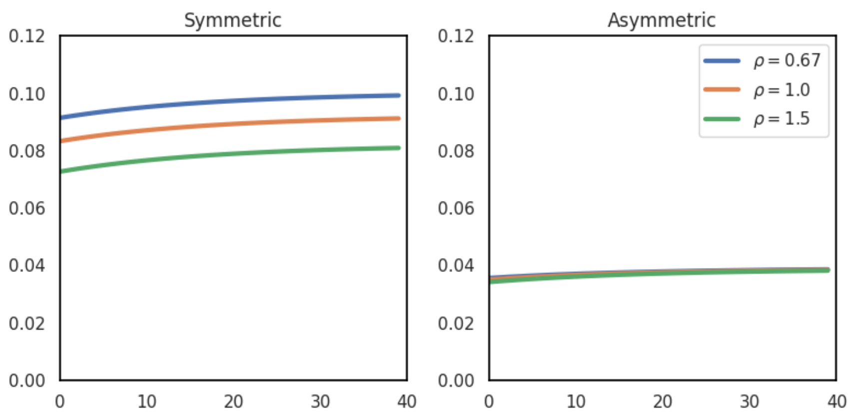

Figure 9: Consumption price elasticities for $\gamma = 12$. The reported elasticities condition on the medians of the state variables.

6 Heterogeneous agents and financial frictions

We now explore a different form of heterogeneity. We alter our one-capital baseline model in Section 4 to include (ex-ante) agent heterogeneity and financial frictions. Agents will be heterogeneous in both their preferences, productivities, and financial market access. We think of the baseline economy as one in which multiple economic agents have homogeneous preferences and homogeneous access to the production technology. In this case, consumption and wealth are proportional over time, making aggregation immediate. This simple aggregation will not be true in the class of economies that we explore in this section. With various forms of market impediments, we can no longer focus on the planner problem as has been true in our previous examples. Instead we study a competitive equilibrium in which wealth heterogeneity matters. As in our previous economies, we entertain the possibility of growth-rate uncertainty in the production technology. We feature model comparisons within a conveniently nested class of models.

6.1 Environment, equilibrium, and solution overview

There are two agent types in the economy: "experts" and "households", indexed by e and h, respectively. Both agents have recursive preferences, but their preference parameters (δ, γ, ρ) can differ. There is a single capital accumulation technology, but the productivity of this capital stock may differ in the hands of each of the agents, with $\alpha_e \geqslant \alpha_h$. Ownership of the capital trades freely amongst agents, with price Q_t that follows endogenous diffusive dynamics. This price is the analogue of the Q_t described previously for the one-agent economy (see equation (22)).

Several financial instruments also trade: risk-free short term debt at an interest rate r_t, and various financial claims exposed to aggregate risk: (a) derivatives contracts traded amongst households at vector π_t^h per unit of Brownian increment risk exposure and similarly among experts at a vector of prices π_t^e; and (b) equity contracts issued by experts with payoff proportional to the return on capital they hold. In some of our economies, experts face a financial restriction: they must remain exposed to at least a fraction $\underline{\chi}$ of the total capital they hold. Experts therefore cannot issue unlimited equity nor can they trade freely in hedging contracts.

Let N_t^j be the date-t net worth of type-j agent for $j = h, e$. Then,

$$\frac{dN_t^j}{N_t^j} = \left(\mu_{n,t}^j - C_t^j/N_t^j\right) dt + \sigma_{n,t}^j \cdot dB_t, \tag{23}$$

where the local mean $\mu_{n,t}^j$ net of consumption and the shock exposure vector $\sigma_{n,t}^j$ are equal to

$$\mu_{n,t}^j = r_t + \frac{Q_t K_t^j}{N_t^j}\left[\mu_{R,t}^j - r_t\right] + \theta_t^j \cdot \pi_t \qquad\qquad \sigma_{n,t}^j = \frac{Q_t K_t^j}{N_t^j}\sigma_{R,t} + \theta_t^j,$$

and where K_t^j and θ_t^j denote the chosen capital and hedging positions held by the type-j agent. A hedging position θ_t^j implies an exposure $N_t^j \theta_t^j \cdot dB_t$ to Brownian risk. As capital is also exposed to Brownian risk, $\sigma_{n,t}^j$ reflects both exposures. When households and experts have different productivities, the expected excess return on capital $\mu^j_{R,t} - r_t$ is type-specific. The risk exposure vector, $\sigma_{R,t}$, for capital is common for households and experts and has a direct contribution from capital accumulation technology and a market value contribution from the price of capital.

Market incompleteness is encoded via a constraint on the hedging vector θ^e_t on experts. While households are unconstrained, experts have restrictions on their exposure to aggregate risk. Suppose experts choose θ^e_t to reduce their exposure to capital risk by a fraction χ_t. To achieve this reduction,

$$\theta^e_t = (\chi_t - 1)\frac{Q_t K^e_t}{N^e_t}\sigma_{R,t}.$$

Imposing a so-called "skin-in-the-game constraint": $\chi_t \geqslant \underline{\chi}$ restricts the ability of the intermediaries to hedge their risk to the capital that they own:

$$\theta^e_t \in \left\{(\chi_t - 1)\frac{Q_t K^e_t}{N^e_t}\sigma_{R,t} \quad : \quad \chi_t \geqslant \underline{\chi}\right\}, \tag{24}$$

For the purposes of making model comparisons, the structure just described embeds three types of heterogeneity. First, there is preference heterogeneity. In addition to heterogeneous subjective discounting, we allow for $\gamma_h > \gamma_e$, which can reflect either an enhanced aversion to risk on the part of households or less confidence in the under probability model. Second, we allow for experts to use capital in a more productive than households ($\alpha_e \geqslant \alpha_h$.) Finally, we entertain a skin-in-the-game type restriction on experts limiting how much of the capital risk exposure they can offset in financial markets as captured by (24). These alternative forms of heterogeneity open the door to making revealing comparisons across alternative model specifications.[10]

Our definition of a competitive equilibrium is standard: it is a set of price processes (Q, π, π, r) and allocation processes $(C^e, C^h, N^e, N^h, K^e, K^h, \chi, \theta^e, \theta^h)$, such that agents

[10]For some our specifications, we alter the utility recursion (8) to be

$$0 = \left(\frac{\delta + \lambda}{1-\rho}\right)\left[(C_t/V_t)^{1-\rho} - 1\right] + \hat{\mu}_{v,t} + \left(\frac{1-\gamma}{2}\right)|\sigma_{v,t}|^2$$

for $\lambda \geqslant 0$. In addition, we follow Gârleanu and Panageas (2015) by introducing an adjustment to the net worth evolution involving λ to ensure a stationary wealth distribution. Gârleanu and Panageas (2015) interpret λ as a exponentially distributed death probability with decay rate λ induced by a Poisson process with a constant arrival rate. This rationale precludes a bequest motive, but upon death net-worth is redistributed to newborn agents. The death risk is treated differently in the preference than the other risks. See their Appendix D for an elaboration.

solve their constrained optimization problems, taking price processes as given, and all markets — the goods market, the market for capital, and the market for aggregate risk — clear. By Walras' law, the risk-free debt market will also clear.

We look for a Markovian equilibrium in which the wealth distribution, the aggregate stock of capital, as well as the driving processes Z^1, Z^2 are state variables. Given the homogeneity properties of our model, (i) the wealth distribution can be summarized by the experts' wealth share $W_t \stackrel{\text{def}}{=} N_t^e / \left(N_t^e + N_t^h\right)$, and (ii) all growing processes scale with K_t, which means that $X_t{}' \stackrel{\text{def}}{=} (W_t, Z_t^1, Z_t^2)$ can serve as a state vector for our economy. While (Z^1, Z^2) are specified exogenously, the wealth share W evolves endogenously.

The log continuation value of each type-j agent takes the additively separable form, analogous to the value function for benchmark economy given by (19):

$$\widehat{V}_t^j = \widehat{N}_t^j + \upsilon^j(X_t)$$

where $\widehat{N}^j = \log N^j$. We construct a Hamilton-Jacobi-Bellman equation analogous to that given in (20) for the social planner in the benchmark economy. The homogeneity properties of our model allow us to derive agents' optimal consumption and portfolio choices as a function of υ^j. For instance, the optimal consumption-wealth ratio for each agent type is

$$c^j(x) = \delta^{1/\rho} \exp\left[(1 - 1/\rho)\upsilon^j(x)\right],$$

and their portfolio choice solves a familiar problem that includes both a mean-variance and a hedging component:

$$\max\left\{ \underbrace{\mu_n^j - \frac{1}{2}\gamma|\sigma_n^j|^2}_{\text{mean-variance}} + \underbrace{(1-\gamma)\sigma_n^j{}'\sigma_x \frac{\partial \upsilon^j}{\partial x}}_{\text{hedging}} \right\}, \tag{25}$$

subject to the portfolio constraints where μ_n^j and σ_n^j depend on the portfolio weight vector θ^j. The outcome of this portfolio problem is a set of Euler equations (when constraints are non-binding) and inequalities (when constraints are binding). For instance, households will hold strictly positive amounts of capital if and only if their expected excess return $\mu_{R,t}^h - r_t$ is sufficiently high to match the market compensation they could otherwise obtain. Similarly, experts have an incentive to issue as much equity as possible (and their financial constraint will then bind) when their expected return on capital $\mu_{R,t}^e - r_t$ is greater than the market compensation $\pi_t \cdot \sigma_{R,t}$ they need to pay. Their issuance constraint does not bind otherwise.

Since experts are more productive than households, it is efficient for them to hold all the capital in the economy and exhaust their equity-issuance capacity. In fact, one can show that whenever households hold positive amounts of capital, experts' equity issuance constraint must bind.

The consumption and portfolio choice of the various agent types leads to endogenous dynamics for the experts' wealth share W_t; its drift rate depends on the consumption-to-wealth ratio of households relative to that of experts, on experts' leverage and their expected excess return on capital relative to its required market compensation and finally the differential aggregate risk exposure between household and experts. The diffusion coefficient of W_t only depends on this latter force. The wealth share dynamics depend on asset prices, which themselves depend on wealth share dynamics — generating a two-way feedback loop that amplifies capital return volatility (Brunnermeier and Sannikov, 2014).

The remainder of this section explores this heterogeneous agent model in a series of "model comparisons," offering some general takeaways. We begin by discussing outcomes that both unite and distinguish these models (Section 6.2). We next highlight new ways in which auxiliary shocks to growth and volatility interact with heterogeneity and financial frictions (Section 6.3). Finally, we provide some discussion of the extant literature.

6.2 Properties of heterogeneous-agent models

We begin by highlighting some general properties of our heterogeneous-agent economies. First, many of the models we consider are united by the fact that they generate high and counter-cyclical risk prices. Second, among these models, we distinguish those that allow "deleveraging" from those that do not—this demarcation represents a significant divide in the nature of model dynamics. Third, we further illustrate the difference between the type of "smooth deleveraging" associated to frictionless models versus the rapid deleveraging in models with frictions. To simplify our exposition, in this section we shut down all auxiliary shocks besides the TFP shock (i.e., set $\sigma_1 = \sigma_2 = 0$), so that W_t is the unique state variable.

Risk prices and the wealth distribution. In heterogeneous-agent models, amplified and time-varying risk premia emerge naturally from a risk concentration mechanism. To see this, focus on the risk prices π_e of a log utility expert ($\gamma_e = \rho_e = 1$), which are

$$\pi_e = \frac{\chi\kappa}{w}\sigma_R, \quad \text{where} \quad \begin{cases} \chi \overset{\text{def}}{=} \text{equity retained by experts} \\ \kappa \overset{\text{def}}{=} \text{experts' share of capital.} \end{cases} \tag{26}$$

Experts each hold $\chi\kappa$ fraction of the aggregate risk in the economy, through their capital holdings and equity retention. Because they only hold wealth share w, $\chi\kappa/w$ is their exposure per unit of wealth. In this section, experts are either more productive or more risk-tolerant than households, which leads them to take leverage $\chi\kappa > w$ and amplifies their risk prices. Other things equal, expert leverage raises risk prices π_e above those in the corresponding representative-agent economy.

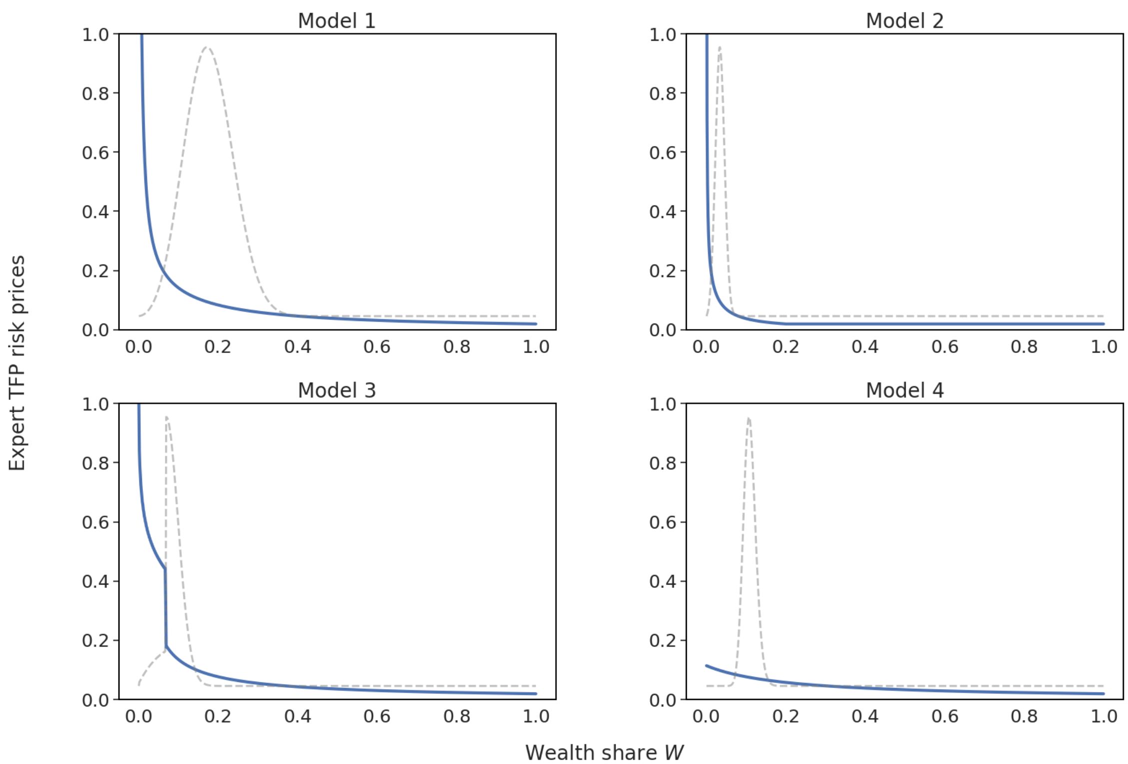

Figure 10: Expert risk prices π_e in four illustrative models. Model 1 $\underline{\chi} = 1$, $\alpha_h = -\infty$, and $\gamma_e = \gamma_h = 2$ (Basak and Cuoco, 1998). Model 2 has $\underline{\chi} = 0.2$, $\alpha_h = -\infty$, and $\gamma_e = \gamma_h = 2$ (He and Krishnamurthy, 2013). Model 3 has $\underline{\chi} = 1$, $\alpha_h = 0.075 < \alpha_e$, and $\gamma_e = \gamma_h = 2$ (Brunnermeier and Sannikov, 2014). Model 4 has $\underline{\chi} = 0$, and heterogeneous risk aversions $\gamma_e = 2 < \gamma_h = 12$ (Gârleanu and Panageas, 2015). All other parameters are from the calibration in Table 1, with the Z state variables shut down. In addition, we pick subjective discount rates and the OLG structure to keep the stationary distributions in reasonable locations. Models 1-3 assume $\delta_e = 0.015 > \delta_h = 0.01$; Model 4 assumes $\delta_e = \delta_h = 0.01$. Models 3-4 assume an OLG structure with death rate $\lambda = 0.02$ and newborn expert population share 0.01 and 0.10, respectively.

Countercyclical risk prices emerge naturally from shifts in the wealth distribution, even without auxiliary shocks or recursive preferences.To visualize this pervasive feature, we plot expert risk prices for four different types of heterogeneous-agent models are displayed in Figure 10, all of which show π_e decreasing in w. These four economies roughly correspond to the models of Basak and Cuoco (1998), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), and Gârleanu and Panageas (2015).

Deleveraging mechanics. Models at the top and bottom of Figure 10 differ in important ways. In the top row, as $w \to 0$ risk prices explode to infinity; the bottom row features bounded risk prices. The discrepancy is fundamentally tied to the amount of *deleveraging*. Deleveraging refers to experts' decrease in their share of aggregate risk exposure $\chi\kappa$ as their net worth share w declines. We classify models into one of two categories:

ND ("No Deleveraging"): As $w \to 0$, $\chi\kappa \to$ constant $C \in (0,1]$.

D ("Deleveraging"): As $w \to 0$, $\chi\kappa/w \to$ constant $C \in (1,\infty)$.

In models of class **ND**, such as Basak and Cuoco (1998) and He and Krishnamurthy (2013), experts hold a positive fraction of aggregate risk, even as they hold a vanishing amount of wealth. We capture this in our framework via $\alpha_h = -\infty$ (infinitely-unproductive households) and $\underline{\chi} > 0$ (limited equity issuance). In models of class **D**, such as Brunnermeier and Sannikov (2014) or Gârleanu and Panageas (2015), experts hold a vanishing amount of aggregate risk, since they deleverage as their wealth declines. In our framework, this will naturally occur if either α_h is large enough or $\underline{\chi} = 0$.

Since experts in a **ND** economy asymptotically hold unbounded amount of risk per unit of wealth, they demand unbounded risk prices $\pi_e \to +\infty$ as $w \to 0$ (see top panels of Figure 10). High risk prices allow experts to earn high profits and naturally recapitalize their balance sheets, which manifests itself via the drift in the wealth share, i.e., $\mu_w \to +\infty$ as $w \to 0$. Instead, the volatility σ_w stays bounded, since in the limit experts are too poor to substantially influence capital price volatility. This phenomenon — the drift becoming infinitely large relative to the volatility — gives rise to what researchers have called *fast recoveries*.

By contrast, experts in a **D** economy can delever and require bounded risk prices π_e (see bottom panels of Figure 10). This translates into state dynamics with $\mu_w \to 0$ and $\sigma_w \to 0$ as $w \to 0$. In some sense, as experts become poor and delever, the economy converges to a representative-agent economy solely populated by households. Unlike class **ND**, since experts do not recapitalize their balance sheets with unbounded profits, models of class **D** feature *slow recoveries*.

In the appendix, we formalize the distinction between **ND** and **D** economies by analyzing the tail of the ergodic wealth distribution. We prove that models in class **D** generally feature more mass in low-w states, reflecting slower recoveries. We also provide a formula for the tail thickness, in terms of the various parameters, which guides us on which features tend to make recoveries slower or faster.

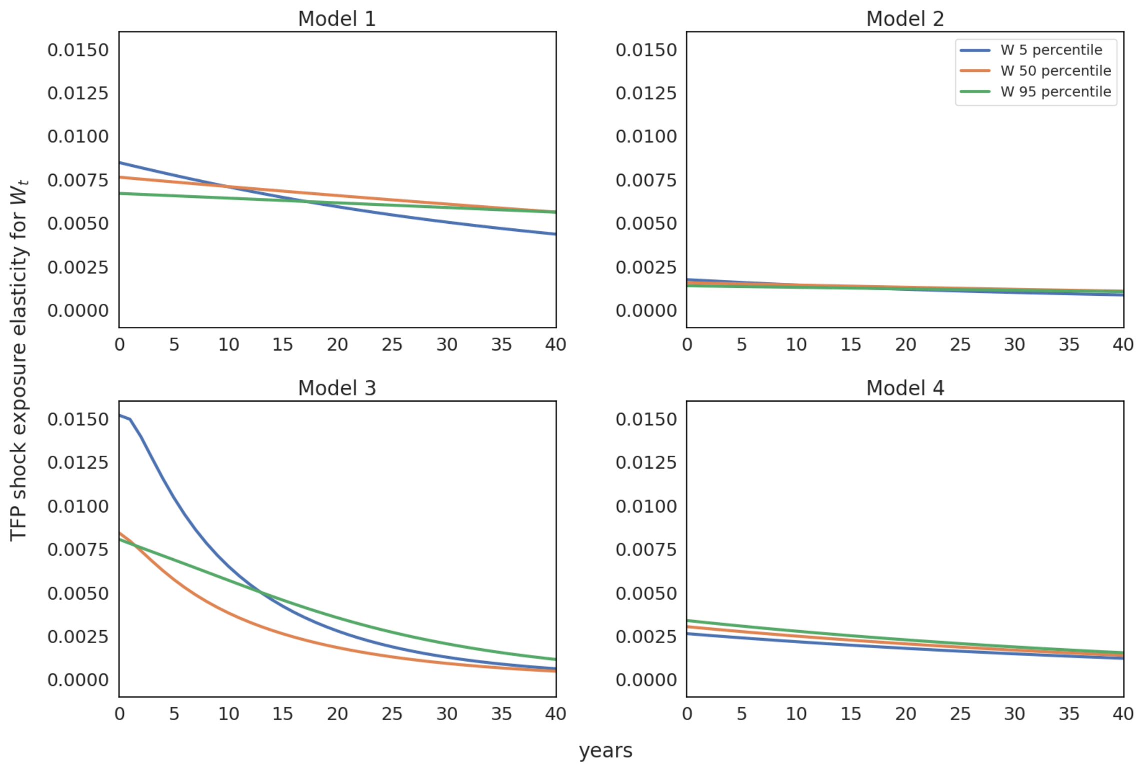

Figure 11: Shock elasticities of expert wealth share W_t to a TFP shock in four illustrative economies. The models and parameters are the same as in Figure 10.

Since expert wealth share W_t is the unique state variable in the example economies of this section, recoveries can be understood through the impulse response function of W_t: a steep impulse response indicates fast recovery after a TFP shock. Figure 11 displays these impulse responses (calculated via shock exposure elasticities) for the same 4 economies as in Figure 10. Recoveries from crisis are fast in class-**ND** economies (top two panels), understood via the steep impulse responses at low wealth percentiles. Interestingly, recoveries are also fast in a class-**D** economy resembling Brunnermeier and Sannikov (2014) (bottom left panel)—this is due to parameter choices that prevent mass from building up at low values of w, as we will demonstrate shortly. By contrast, recoveries from crisis are slow in a class-**D** economy resembling Gârleanu and Panageas (2015), understood via the relatively flatter responses at low wealth percentiles.

In summary, economies that do not allow deleveraging in crisis (class **ND**) allow faster recoveries than economies with some deleveraging (class **D**). The difference in recovery speed is encoded in the left tail of the stationary wealth distribution.

Frictions versus frictionless economies. It may be surprising that economies can be classified into these two groups, irrespective of auxiliary features. In particular, class **D** features models with and without financial frictions. As we will show, an important difference between these models is whether deleveraging occurs smoothly or not.

We compare two models: experts could be more productive managers of capital or they could simply be more risk-tolerant. For expositional clarity, we name these models **F** (for "frictions") and **RA** (for "risk aversion"):

F: $\underline{\chi} = 1$, $-\infty < \alpha_h < \alpha_e$, $\gamma_e = \gamma_h$

RA: $\underline{\chi} = 0$, $-\infty < \alpha_h = \alpha_e$, $\gamma_e < \gamma_h$.

The models closest to this comparison are Brunnermeier and Sannikov (2014) (**F**) versus Gârleanu and Panageas (2015) (**RA**). As we have already shown, these models share several properties, like countercyclical risk prices and the possibility of thicker lower tails for the wealth distribution.

In both economies, experts hold a disproportionate fraction of aggregate risk, either because they are more productive at doing so, or because they are more tolerant of it. Households in economy **F** will enter the market and directly manage capital only after expert wealth has deteriorated sufficiently. At that point, households' expected excess returns on capital management are high enough to compensate them for its inherent risks. Households in economy **RA**, instead, will always hold a non-zero fraction of the aggregate risk.[11] This distinction is shown in Figure 12, which plots the endogenous distribution of

[11] Both models feature regions where $\kappa \in (0,1)$, but the heterogeneous productivity model showcases a large part of the state space where experts hold the entire capital stock. To understand this, consider the Merton portfolio $\sigma_{n_j} = \frac{\mu_{R,j} - r}{\gamma_j |\sigma_R|}$ for agent j. With higher productivity, experts obtain a discretely larger expected return on capital than households ($\mu_{R,e} > \mu_{R,h}$), so it is possible for households' desired capital holdings to be negative ($K_h^* < 0$), leaving them on their no-shorting constraint. If experts and households faced the same returns ($\mu_{R,e} = \mu_{R,h}$) but their risk aversions differed ($\gamma_e < \gamma_h$), experts and households would both hold positive quantities of capital but at different scales ($K_e^* > K_h^* > 0$).

capital risk $\chi\kappa$ as a function of state variable w.[12] Furthermore, the risk distribution $\chi\kappa$ features smoother dynamics in economy **RA** relative to economy **F**.

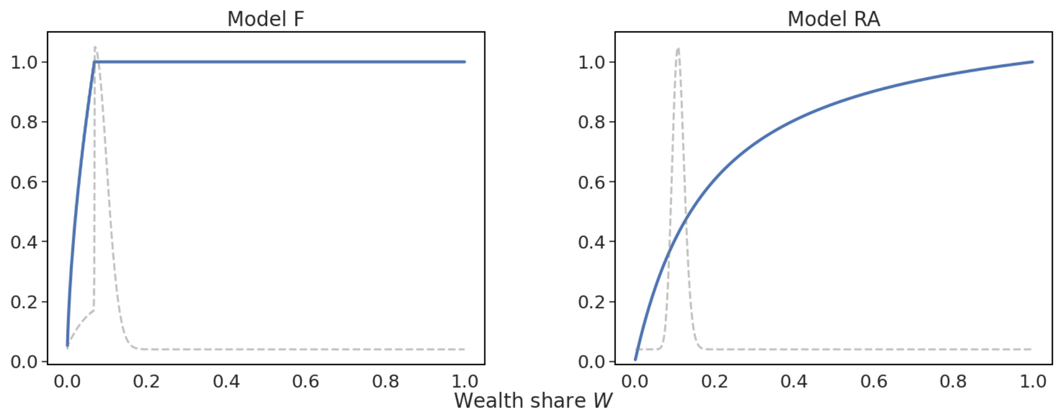

Figure 12: Experts' risk share $\chi\kappa$ in two illustrative models with deleveraging. The two models correspond to Models 3 and 4 in Figure 10.

Reflecting these smoother risk-sharing dynamics, risk compensations and asset prices are "smoother" in economy **RA** relative to economy **F**. Look back at the bottom panels of Figure 10, which are exactly these two models. Economy **F** features a discontinuous jump in expert risk prices at the moment households begin participating, whereas economy **RA** features more gradual risk price dynamics.

In this sense, frictional economies like **F** feature greater crisis-type nonlinearities than frictionless economies like **RA**. Sudden deleveraging and a jump in risk compensation in **F** contrasts with smooth deleveraging and smooth risk compensation in **RA**.

Occasionally-binding constraints? Next, we show how preference heterogeneity fundamentally determines the binding/non-binding nature of constraints. Given the nesting structure of our model, this is simple to investigate by interacting preference heterogeneity with different values of the equity-retention constraint. Assume that experts face the equity-issuance friction $\chi_t \geqslant \underline{\chi} \in (0, 1)$. One might expect such constraints to be occasionally-binding, as in He and Krishnamurthy (2013). Surprisingly, when experts and households have identical risk aversions, this is generally not the case.

Indeed, the appendix shows that, when $\gamma_e = \gamma_h$, experts' optimal risk retention χ takes

[12] In model **RA**, because productivities are equal, households can equivalently hold capital or experts' outside equity. Experts' equity-retention χ and experts' capital share κ only appear multiplicatively as $\chi\kappa$ in equilibrium equations.

the form

$$\chi = \max(\underline{\chi}, w). \tag{27}$$

Furthermore, we prove that a large variety of parameter constellations imply $W_t \leqslant \underline{\chi}$ almost-surely in the ergodic distribution, so that χ is at its constrained level $\underline{\chi}$. Formula (27) and the conclusion that $\chi_t = \underline{\chi}$ almost-surely are very generally true as long as risk aversions are homogeneous.[13]

The connection between risk-aversion heterogeneity and binding nature of constraints is displayed in Figure 13. When agents have identical levels of risk-aversion (left panel), the stationary distribution lives completely in the region where $\chi = \underline{\chi}$. When households are more risk-averse than experts (right panel), this is no longer true. The equity constraint can be occasionally-binding or even almost always-binding, as this particular example shows. This phenomenon arises because risk-tolerant experts retain more risk than their wealth (i.e. $\chi > w$), so the unconstrained region remains "stochastic" (in the sense that $\sigma_w \neq 0$ even when $\chi > \underline{\chi}$). This is *de facto* what occurs in the occasionally-binding equilibrium of He and Krishnamurthy (2013): their "Parameter Assumption 1" forces households to always invest a fixed positive fraction of their wealth in risk-free assets, which makes them act more risk-averse than experts.

Does this distinction matter? Figure 14 shows how the equity constraint binding or not strongly affects the nonlinearity in experts' TFP risk prices.

On one hand, our results here provide a partial justification for the procedure, performed by a majority of DSGE models with financial frictions, that consists in log-linearizing equilibrium equations assuming constraints are always binding. On the other hand, this exercise illustrates that some models with occasionally-binding risk-sharing constraints may be standing on, perhaps hidden, assumptions about risk aversion heterogeneity.

[13] Our result is analogous to Corollary 1 in Gârleanu and Panageas (2015)—generalized to an economy with additional exogenous state variables ($\hat{x}$), financial frictions ($\underline{\chi} > 0$), and productivity heterogeneity ($\alpha_e \geqslant \alpha_h$)—in which a homogeneous risk aversion economy has deterministic wealth share evolution. The key intuition is that agents with identical risk aversion make the same portfolio choices in the absence of financial frictions, and this is robust to heterogeneity in the IES $1/\rho$ and the discount rate δ. Thus, when experts and households are both unconstrained, their wealth exposure to aggregate shocks σ_{n_j} is identical, so Brownian shocks do not alter the relative expert-household net worth. This feature allows us to characterize binding constraints based on the drift μ_w only at the point where the constraint "just binds", i.e., $w = \underline{\chi}$. As long as $\sup_{\hat{x}} \mu_w(\underline{\chi}, \hat{x}) < 0$, where $\hat{x}$ is the set of exogenous state variables, then equilibrium features an always-binding equity constraint.

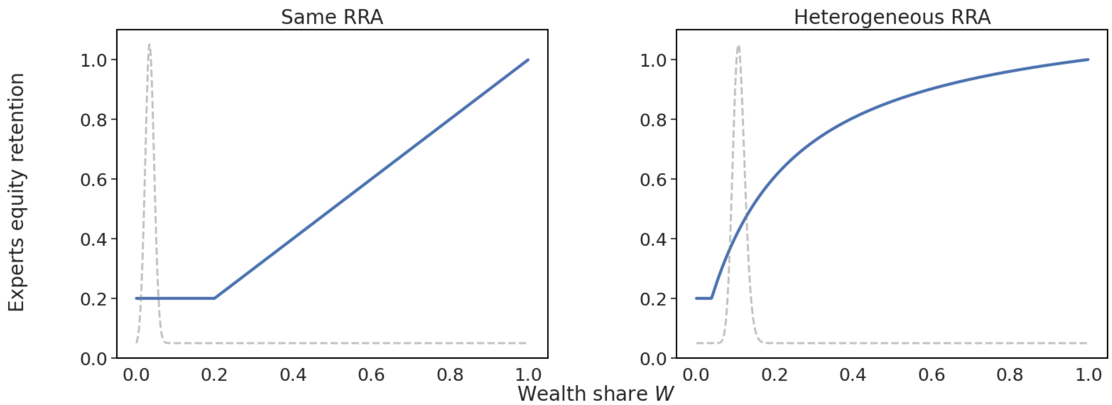

Figure 13: Expert equity retention χ in two models resembling He and Krishnamurthy (2013). The left panel has identical preferences ($\gamma_e = \gamma_h = 2$; this is Model 2 in Figure 10); the right panel has more-risk-tolerant experts ($\gamma_e = 2 < \gamma_h = 12$; this is a blend of Models 2 and 4 in Figure 10).

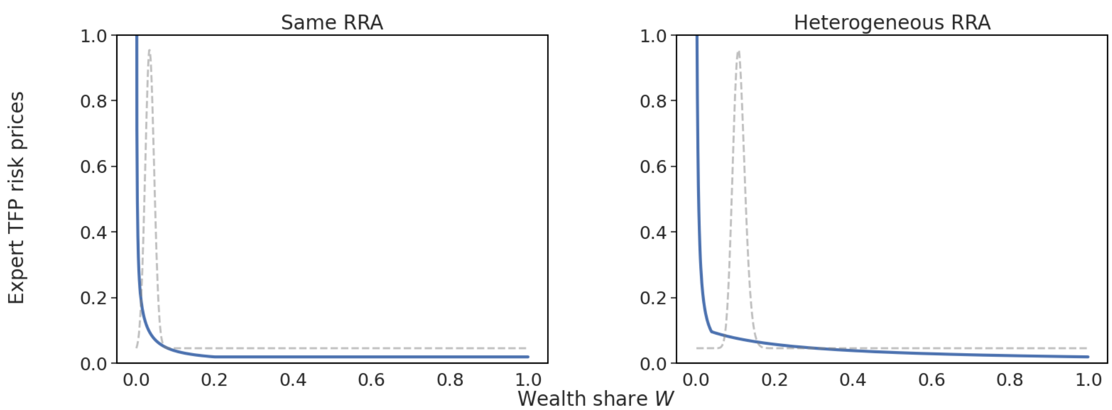

Figure 14: Expert TFP risk prices π_e in the same two economies as Figure 13

6.3 Frictions and auxiliary shocks

We reintroduce shocks to the TFP growth rate Z_t^1 and TFP variance Z_t^2. How do heterogeneous agents perceive these auxiliary shocks? How do these shocks magnify or dampen the effects of financial frictions?

Figure 15 compares experts' growth risk prices in three models with frictions (these are Models 1-3 in Figure 10; roughly speaking, these correspond to Basak and Cuoco (1998), He and Krishnamurthy (2013), and Brunnermeier and Sannikov (2014)) and one frictionless representative agent model. Surprisingly, growth risk prices are identical in all models. (Although not shown, this equalization holds for all levels of the wealth share

w, and households' growth risk prices are also equalized to experts'.) This irrelevance of frictions on growth risk pricing is formalized in the appendix, which proves that when agents share identical risk aversions, unitary IES, and the shocks to K, Z^1, and Z^2 are mutually uncorrelated, we have that growth risk prices are independent of W.[14]

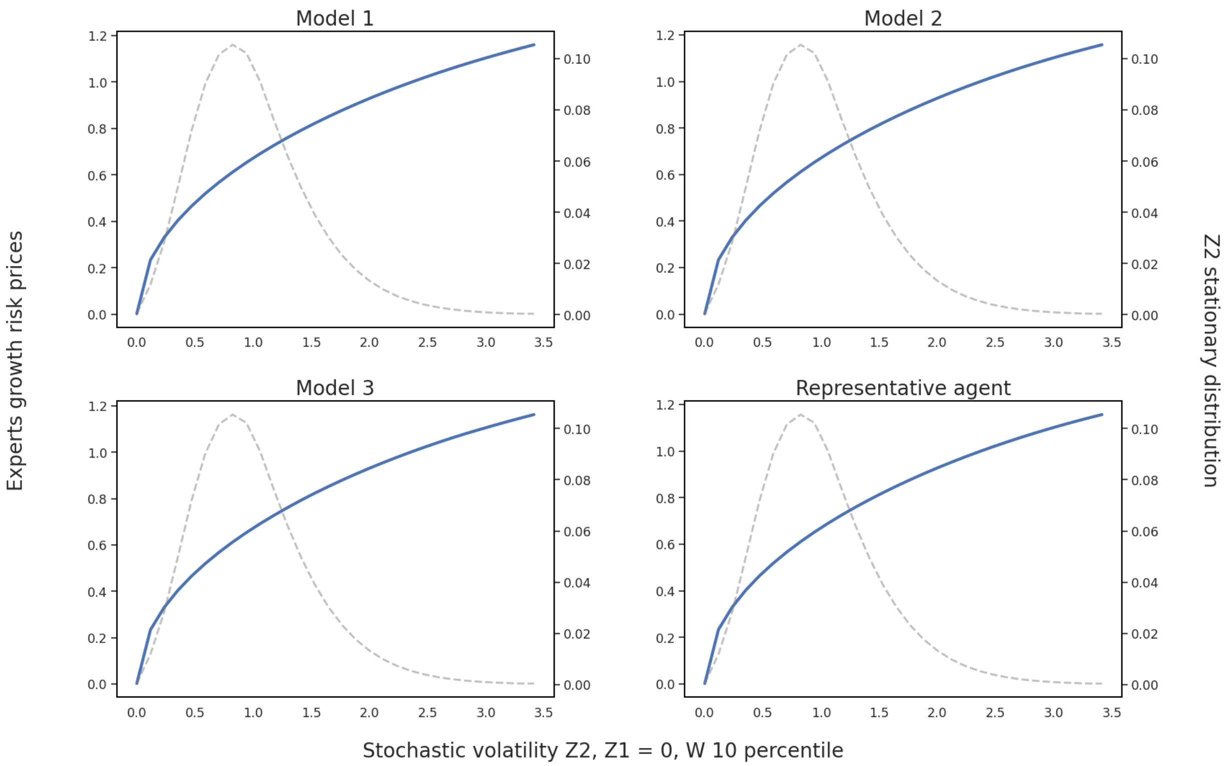

Figure 15: Expert risk prices π_e for the growth shock, in the three-dimensional versions of Models 1-3 described in Figure 10, along with a representative agent model in the fourth panel. Stationary densities for Z_t^2 are in the background. For the plots, $Z_t^1 = 0$ and W_t is equal to its 10th percentile. All other parameters are from the calibration in Table 1, with the Z state variables reintroduced.

Intuitively, under unitary IES, exogenous growth shocks do not directly affect the capital stock or the price of capital in the short-run, so their only risk is to agents' investment opportunity sets, in particular the risk-free rate. Since all agents have equal unrestricted access to the risk-free rate, this risk is perfectly shared between experts and households.

By contrast, volatility risk pricing is altered by frictions. Figure 16 shows how, in models with frictions, experts' volatility risk prices are generally non-monotonic in the level of volatility. Moderate levels of volatility are feared most by experts. One explanation is that a scenario with very high volatility increases experts' risk compensation so much

[14]The sharp result of Figure ?? is more-or-less similar numerically even with non-unitary IES.

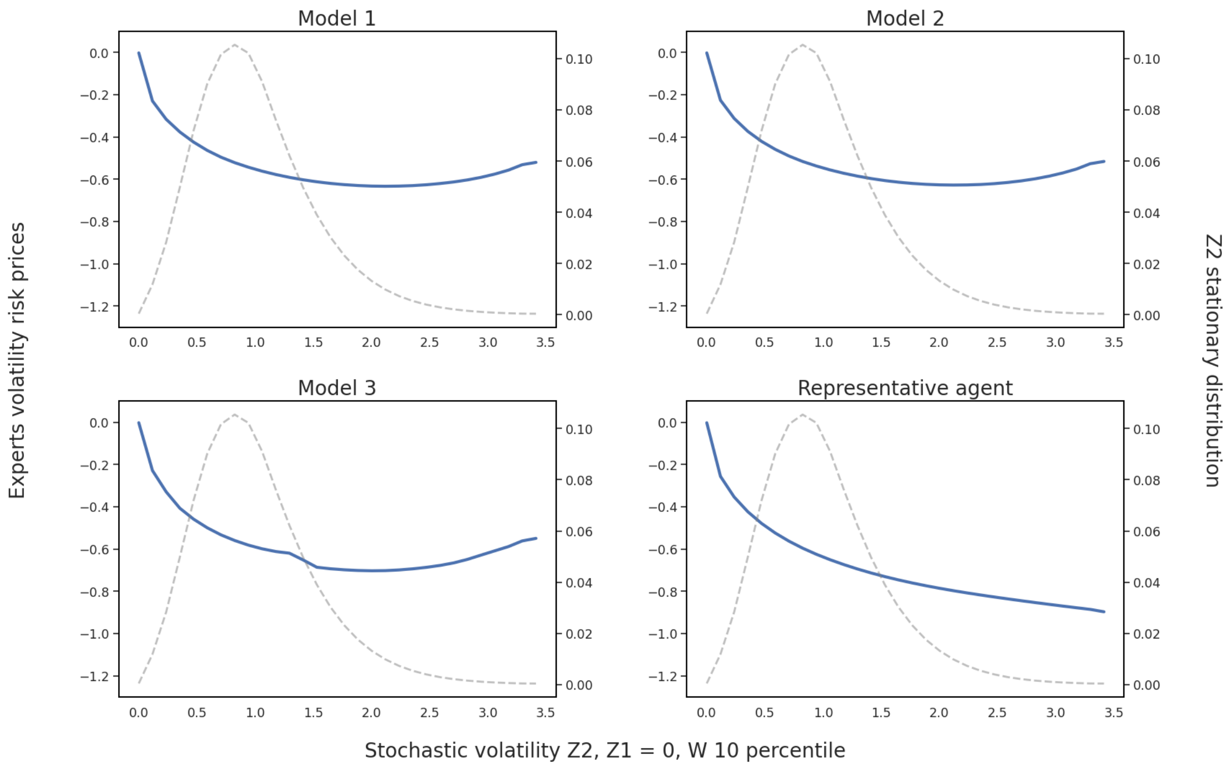

Figure 16: Expert risk prices π_e for the volatility shock, in the three-dimensional versions of Models 1-3 described in Figure 10, along with a representative agent model in the fourth panel. Stationary densities for Z_t^2 are in the background. For the plots, $Z_t^1 = 0$ and W_t is equal to its 10th percentile. All other parameters are from the calibration in Table 1, with the Z state variables reintroduced.

that that it is less feared. In some model calibrations, we have even found that models with frictions can generate positive volatility risk prices when their relative wealth is low enough, implying experts become fond of volatility shocks.[15] Given the intriguing influence of frictions on volatility risk pricing, we think it will be productive for future research to investigate volatility shocks in more detail.

Quantitatively, the biggest impact of introducing the auxiliary shocks to Z is found in TFP risk pricing and the stationary wealth distribution. Figure 17 shows experts' TFP risk prices for these same four models. These risk prices are significantly higher in general than those depicted in Figure 10 for in the one-dimensional versions of these models.

[15] Although we have not proven it, we conjecture this occurs because experts' TFP risk price is strongly decreasing and convex in w (recall Figure 10). Jensen's inequality, coupled with this convexity, then suggests that a positive volatility shock benefits experts. This effect can outweigh the standard volatility aversion inherent in these preferences. Opening a volatility market might be counterproductive in this situation, as experts might be led to speculate when w is low, and households would gladly take the other side of this trade given they continue to have negative volatility risk prices in those calibrations.

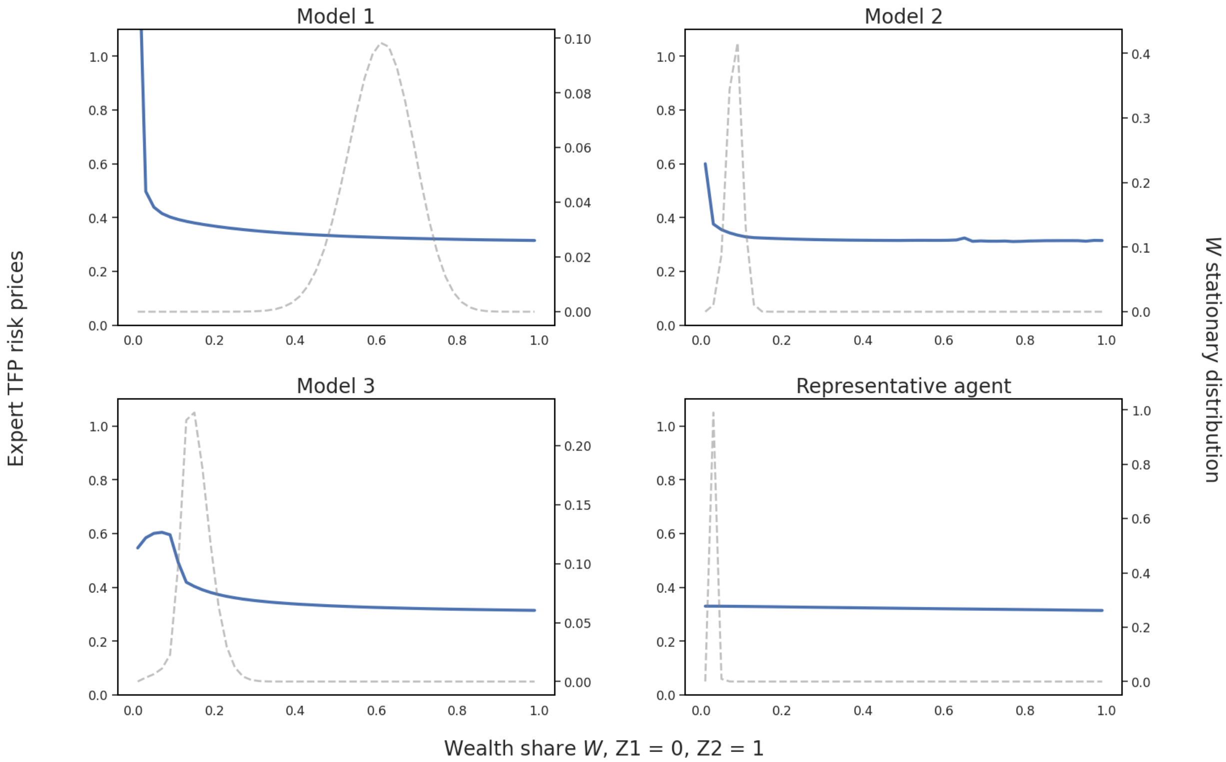

Figure 17: Expert risk prices π_e for the TFP shock, in the three-dimensional versions of Models 1-3 described in Figure 10, along with a representative agent model in the fourth panel. Stationary densities for W_t are in the background. For the plots, $Z_t^1 = 0$ and $Z_t^2 = 1$. All other parameters are from the calibration in Table 1, with the Z state variables reintroduced.

Moreover, the locations and dispersions in the stationary distributions are dramatically affected. This illustrates how matching larger magnitudes of risk pricing in "normal times" when constraints are less binding can produce a tension by reducing the probability of severe nonlinear crises.

6.4 Discussion of related literature

The models we have explored in this section highlight the role of ex-ante agent heterogeneity and risk-sharing. The literature studying this class of models is voluminous, and we do not attempt to survey all of it here. However, we will comment briefly on which existing mechanisms we have covered and which we have not, along with what we see as the challenges for future research in this area.

As mentioned above, the models closest to ours include Basak and Cuoco (1998), He and Krishnamurthy (2011, 2013, 2019), Brunnermeier and Sannikov (2014, 2016), and Gârleanu and Panageas (2015). All of these are models where pricing dynamics become interesting either because risk-sharing is constrained or because of the trading dynamics induced by attempts to share risks. Our framework essentially nests these models, pairing them with a more setup with long-run uncertainty in the macroeconomic growth.

These core frameworks have been extended to think about a variety of substantive issues. While our framework does not nest these extensions, we collect some of them here to illustrate the wide range of possibilities: capital requirements and leverage restrictions (Phelan, 2016, Klimenko et al., 2016); margin constraints (Gromb and Vayanos, 2002, Garleanu and Pedersen, 2011); shadow banking (Moreira and Savov, 2017); liquidity premia and monetary policy (Drechsler et al., 2018); unconventional monetary policy (Silva, 2016); international capital flows (Brunnermeier and Sannikov, 2015); the link between idiosyncratic and aggregate risk-sharing (Di Tella, 2017, 2019); financial innovation driven boom-bust cycles (Khorrami, 2020); and entry into the intermediation sector (Haddad, 2014, Khorrami, 2021). While we work in continuous time, related issues have been explored in discrete-time frameworks (Gertler and Karadi, 2011, Gertler and Kiyotaki, 2010, Mendoza, 2010, Bianchi, 2011, Gertler and Kiyotaki, 2015, Christiano et al., 2014).

While this class of models is rich enough to have have some interesting insights, there are reasons to expand their scope. First, financial crises are often more sudden and extreme than the models we explore here would predict. Second, large booms in credit and asset prices have some predictive power for a subsequent bust and financial crisis. Modeling additional amplification mechanisms like bank runs is one way to generate more realistically extreme crises (Mendo, 2018, Krishnamurthy and Li, 2021). Modeling investor "sentiment," both via non-rational beliefs (Maxted, 2023, Krishnamurthy and Li, 2021) and rational fear (Khorrami and Mendo, 2023), are extensions that can generate crisis predictability.

As an intriguing analogy to our long-run uncertainty framework, Maxted (2023) considers extrapolative sentiment as the belief in a persistent stochastic growth rate that, in fact, does not exist. We could capture such impacts in our framework by supposing that the state variable Z^1 is "only in the heads of the investors and households" and not in the actual dynamic evolution. We can analyze such a model in same manner as we currently do by including the Z^1 dynamics in the model solution, but omitting it from the simulations, stationary distributions, and elasticity computations. In this way, there is a wedge between beliefs and the actual data generation. We find this alternative perspective on long-term risk to be intriguing; but as we have seen in Section 4.5, an alternative to subjective belief models are ones that acknowledge the measurement challenge of identifying a long-run risk component in data. This challenge seems pertinent not only to econometricians but also economic agents.[16]

The class of models we explored, by design, nests alternative forms of heterogeneity, albeit a rather stark form with two types of investors. For all of the alternatives we investigate, a natural question is "who are the so-called experts?" Should we identify them with insiders at productive firms, or managers of banks, or specialist investors more broadly? The answers to these questions influences the type of market frictions that are reasonable to consider, as well as the calibrations one should adopt.

One related empirical literature explores intermediary asset pricing implications by seeking to identify new pricing factors. Models of the type featured here, when applied to financial intermediaries, highlight forms of state dependence in valuation that could be important. Exposures and market compensations fluctuate as functions of state variables, which suggest a more dynamic approach to empirical investigation. But research on actual market frictions leads to a more nuanced and complex characterization of the actual nature of such frictions and to heterogeneity within the financial sector that is captured by purposefully simplified models we consider here. Nevertheless, we expect some of the forces captured by these models to be pertinent in understanding more generally the impact of heterogeneity in investor types on macroeconomic and financial outcomes. By opening the door to model comparisons, we understand better the similarities and differences among models with alternative forms of heterogeneity.

7 Conclusions

Our essay explores alternative macro-finance models, including many with explicit nonlinearities. The models are highly stylized and perhaps best thought of a devices to engage in "quantitative story telling." The models are not designed to provide fully comprehensive accounting of empirical facts, but rather they offer characterizations of alternative mechanisms for linkages between financial markets and the macroeconomy. We feature model comparisons rather than deep probes into one specific mechanism. While the latter is clearly valuable, we also believe in value of making model comparisons, something that is less common in journal publication. In effect, we are engaged in "quantitative story telling with multiple stories." In this sense, we share a common ambition with Dou et al. (2020), although the class of models we feature is different as are the tools we use. Re-

[16]See Hansen (2014) and Chen et al. (2022) and for related discussions.

lated ambitions are also reflected in the comprehensive Macro Model Data Base (MMB, https://www.macromodelbase.com), although many the models we entertain require special computational challenges because of their nonlinear structure. Moreover, our essay focuses on the substantive comparisons.

Computational methods are required to support this type of analyses. As will be explained in a computational appendix, this is a nontrivial component to our investigation. In each model, we must solve for agents' continuation values, in some cases jointly with asset prices or endogenous risk-sharing constraints. These functions solve systems of highly nonlinear PDEs. Depending on the model, we use either finite-difference based methods or, for larger state spaces, a deep Galerkin method-policy improvement algorithm, incorporating neural net approximations. See Achdou et al. (2022) and d'Avernas et al. (2022) for some additional macro applications of implicit finite-difference schemes for PDEs, based on the seminal work of Barles and Souganidis (1991). See Al-Aradi et al. (2022), Duarte et al. (2023), Gopalakrishna (2022), and Barnett et al. (2023) for recent developments and discussions of deep neural network methods as an alternative designed to accommodate higher dimensional state spaces.

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