Financial Frictions, Amplification, and Sunspots*

Paymon Khorrami
Imperial College

Fernando Mendo
Central Bank of Chile

November 3, 2020

Abstract

Given financial frictions, asset prices and the wealth distribution form a two-way feedback loop, typically referred to as an "amplification" mechanism for real shocks. But this loop also supports self-fulfilling fluctuations: if agents believe in price volatility, sunspot shocks cause trading behavior that shifts allocative efficiency and the wealth distribution, justifying asset-price fluctuations. In a sense, sunspot effects are infinitely amplified. With fundamental uncertainty, amplification is necessarily finite, revealing a tension that constrains the set of sunspot equilibria. We propose a novel method of constructing sunspots in such cases.

JEL Codes: E44, G01.

Keywords: sunspot equilibria, volatility, financial frictions, self-fulfilling beliefs, financial crises.

*Khorrami: `p.khorrami@imperial.ac.uk`; Mendo: `fmendolopez@gmail.com`. The views expressed here are solely those of the authors and do not necessarily reflect the views of the Central Bank of Chile.

It has by now become commonplace, especially after the 2008 global financial crisis, for macroeconomic models to prominently feature banks, limited participation, imperfect risk-sharing, and other such "financial frictions." Incorporating these features allows macroeconomists to speak meaningfully about financial crises and desirable policy responses. The universal feature of these models is *amplification*: small exogenous shocks to fundamentals can create large endogenous movements in economic observables. Despite the dramatic growth in this literature, there remains a sizeable disconnect between the predictions of such models and actual crisis data. Among other things, the literature has had difficulty reproducing the observed severity of economic downturns and asset-price dislocations. One might say that the degree of amplification in the models has been insufficient.[1]

In this paper, we propose *sunspots* as a potential resolution. Throughout our paper, sunspots refer to non-fundamental fluctuations that only occur because agents expect them; that is, we study self-fulfilling fluctuations. Self-fulfilling sunspots obviously give models greater scope to match crisis data. But how and when do they arise? The main contribution of this paper is to demonstrate that standard financial friction models, which have amplification at their core, are already subject to sunspot volatility, with no further assumptions. In this sense, the very same financial frictions activating amplification also provide scope for sunspots. While this has been noted to some extent in the literature (see below), we believe our paper investigates this mechanism in its purest form, helping illuminate the true prerequisites for sunspot volatility.

We begin with the simplest possible dynamic model with financial frictions, similar to Kiyotaki and Moore (1997), Brunnermeier and Sannikov (2014), and many others. In the model, there are two types of agents ("experts" and "households") with different levels of productivity when managing capital. Heterogeneous productivity means the identity of capital holders matters for aggregate output. But incomplete markets prevent agents from sharing risks associated to their capital holdings, so optimal capital holdings

[1] For example, Gertler and Kiyotaki (2015) and Gertler et al. (2020) attempt to integrate bank runs into a conventional financial accelerator model, in order to capture additional amplification and non-linearity (in particular, the suddenness with which financial systems collapse). As a more fundamental critique, financial accelerator models have a difficult time inducing the financial intermediary leverage needed to generate large amounts of amplification. This can be seen in Di Tella (2017), in which the retirement rate of bankers/experts needs to be calibrated to 115% per year, or in Khorrami (2018), in which the implied entry costs needed to match crisis dynamics of asset prices are on the order of 90% of wealth. Krishnamurthy and Li (2020) builds a sentiment process on top of a relatively standard financial accelerator model, in order to help address some of these issues, in particular time-varying financial sector leverage and onset of crises even in absence of fundamental shocks. The similarity to this paper is that their "sentiment" is replaced by our rational "sunspots."

depend to some degree on individual wealth.[2] In this model, fundamental shocks shift the wealth distribution, which transmits to output, then into asset prices, which finally feeds back into the wealth distribution. Typically, this feedback loop is understood as amplification, triggered by a change in fundamentals. The equilibrium mechanics behind amplification are summarized in Figure 1.

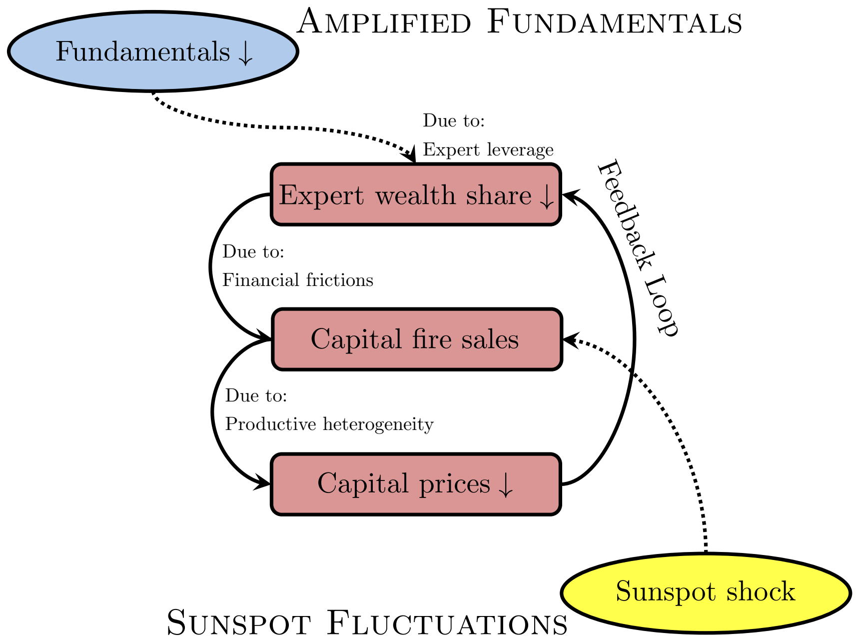

Figure 1: Flow-chart displaying the causal dynamics inside non-sunspot equilibria, which have amplified fundamental shocks, and sunspot equilibria.

But the loop opens the door for self-fulfilling fluctuations, even if fundamentals are deterministic as in our baseline model. Movements in asset prices can be justified solely by agents' coordinated beliefs and active trading in response to those beliefs. For example, a negative sunspot shock that triggers pessimism about asset prices can coordinate fire sales from first-best capital users (experts) to second-best users (households), creating misallocation, lowering capital prices, and thus fulfilling the initial pessimism. Referring back to Figure 1, the only difference is that sunspot shocks directly induce trading even before the wealth distribution shifts.

Importantly, sunspot volatility is intrinsically tied to financial crises-like periods, as our mechanism is only available when experts are insufficiently wealthy. If experts are

[2] In the setting we examine, the market incompleteness can be thought of as an equity-issuance constraint, or skin-in-the-game constraint, for producers. In models like Kiyotaki and Moore (1997), there is additionally a collateral constraint on borrowing, which ensures that optimal capital holdings depend on individual wealth even if capital is fundamentally riskless. Ultimately, sunspots could arise in such a setting as well, as we discuss below with the related literature.

rich, they would prefer to borrow more in response to sunspots, rather than sell their capital, so no sunspot volatility is possible. Given the dramatic increases in volatility in real-world financial crises, this feature of our model offers a promising channel for quantitative macro-finance models.

Our two initial equilibria of focus – a volatile sunspot equilibrium and a deterministic fundamental equilibrium – in fact bookend an entire continuum of equilibria with intermediate volatility. These equilibria differ only in agents' coordinated beliefs about disaster states, i.e., what capital prices would be in the worst-case scenario. As agents coordinate on an inferior disaster outcome, the economy becomes more volatile; conversely, more volatile asset prices justify an inferior disaster belief.

All of these sunspot equilibria are, in a sense, special cases of equilibria with amplified fundamental uncertainty. In the presence of fundamental shocks, amplification is exactly the ratio between overall capital return volatility and fundamental volatility. One can think of sunspot equilibria as an outcome of *infinite amplification*, in order to generate a positive amount of volatility out of zero fundamental shocks. We formalize this by showing that our sunspot equilibria are vanishing-uncertainty limits of non-sunspot equilibria.

Viewing sunspots as extreme amplification suggests an inherent tension. Economically, experts' coordinated buying/selling behavior, in response to net worth shocks, is the driving force behind amplification. Loosely speaking, the infinite amplification of sunspot equilibria requires a lot of this trading; non-sunspot equilibria, which have finite amplification, should see less trading. How can both be true at the same time? How can sunspot volatility co-exist with amplified fundamental volatility?

In the last part of the paper, we resolve this tension by showing how to extend our sunspot equilibrium construction to the case when fundamental shocks are simultaneously present. Sunspots can be added, but their transition dynamics turn out to be endogenous. Our method to construct equilibria here reverses the typical approach: given asset prices as a function (within some admissible class) of the wealth distribution and the sunspot, we solve for the size of sunspot shocks that justifies this price.

For example, when asset prices fall for pure sunspot reasons, capital return volatility must rise going forward. But the fundamental piece of return volatility is already pre-determined (based on experts' leverage and the sensitivity of prices to the wealth distribution). So if the price drop is large enough, the rise in volatility cannot be justified purely by amplification of fundamental shocks. Instead, the difference must be made up by sunspot return volatility, which depends in particular on the size of sunspot shocks. Given a capital price function, the size of sunspot shocks is thus pinned down endogenously.

Two interesting features emerge in the model with both fundamental and sunspot shocks. First, arbitrarily large capital price volatility can be justified; the gap between models and data on financial crises is not because the frictions are not powerful enough, but because we are not choosing the equilibrium closest to the data. Second, relative to low-volatility states, high-volatility states are characterized by a decoupling of the economy from real shocks, as in financial crises.

Related literature. The theoretical focus on financial frictions and sunspots is not new to this paper. Several papers have shown how multiplicity can emerge through the interaction between asset valuations and borrowing constraints. For instance, asset-price bubbles can relax firms' credit constraints, allowing greater investment and thus justifying the existence of the bubble (Farhi and Tirole, 2012; Miao and Wang, 2018). Like us, Liu and Wang (2014) features misallocation among producers, though their self-fulfilling mechanism relies on changes in labor supply that respond to beliefs about future growth, generating a type of endogenous increasing returns to scale. Self-fulfilling credit dynamics can also arise when lending is *unsecured* as opposed to collateralized (Gu et al., 2013; Azariadis et al., 2016).

Relative to these papers, we study a different set of financial frictions (equity-issuance constraints), offer a complete characterization of sunspot fluctuations, and dispense with any features which would be auxiliary to our main point. In a set up close to ours, Mendo (2020) studies self-fulfilled panics that induce collapse of the financial sector. This is an extreme example of the class of self-fulfilled fluctuations we study.

We thus believe our study is a minimalist articulation of the mechanism linking financial frictions and sunspots, which we hope adds clarity. This minimalism also highlights which features are absent. We do not rely on many of the traditional multiplicity-inducing assumptions, such as overlapping generations, non-convexities or externalities in technology,[3] asymmetry of information,[4] or multiple assets.[5]

Our equilibrium construction also differs from the literature in a more technical sense. Building on the seminal studies Azariadis (1981) and Cass and Shell (1983), sunspot equilibria are often constructed by essentially randomizing over a multiplicity of deter-

[3]For example, see Azariadis and Drazen (1990) for multiplicity under threshold investment behavior. See Farmer and Benhabib (1994) for a multiplicity under increasing returns to scale.

[4]In a macro context, Piketty (1997) and Azariadis and Smith (1998) for self-fulfilling dynamics in the presence of screened/rationed credit. In a finance context, Benhabib and Wang (2015) and Benhabib et al. (2016, 2019) generate sunspot fluctuations in dispersed information models.

[5]Hugonnier (2012), Gârleanu and Panageas (2020), Khorrami and Zentefis (2020), and Zentefis (2020) all build "redistributive" sunspots that shift valuations among multiple positive-net-supply assets.

ministic transition paths to a stable steady state. By contrast, our sunspot equilibria are built around an unstable steady state, so we rely critically on the stability induced by the presence of aggregate sunspot volatility.[6]

Non-fundamental fluctuations do not arise solely in fundamentally deterministic economies; sunspot volatility can augment exogenous fundamental volatility. In this part of our analysis, we show how equilibrium provides restrictions on sunspot transition dynamics as a function of experts' wealth share and asset prices, and in our example constructions we find that the sunspot dynamics depend on endogenous objects. In fact, the short-term volatility of the sunspot state variable is almost uniquely pinned down, given experts' wealth and asset prices (the locally-deterministic "drift" component of the sunspot state is almost completely unrestricted, however). Benhabib et al. (2015) have a static counterpart to this finding, whereby the belief distribution is also pinned down by the restrictions of equilibrium.

The paper proceeds as follows. Section 1 presents the baseline model. Section 2 characterizes the sunspot equilibria that emerge in this model. Section 3 illustrates the duality and tension between amplification of fundamental shocks and sunspot volatility. Section 4 resolves this tension, showing how sunspot and fundamental volatility can co-exist. Section 5 concludes. The appendices contain proofs, further details, and model extensions.

1 Baseline model

Technology, Preferences, Markets. Time is continuous and indexed by $t \geq 0$. There are two goods, a non-durable good (the numeraire, "consumption") and a durable good ("capital") that produces the consumption good. The aggregate supply of capital grows exogenously at constant rate g, i.e., $dK_t = gK_t dt$. Individual capital holdings evolve identically. Both goods are freely tradable, with the relative price of capital denoted by q_t.

There are two types of agents, experts and households, who differ in their production technologies. Experts produce a_e units of the consumption good per unit of capital, whereas households' productivity is $a_h \in (0, a_e)$. For simplicity, there is no investment in this model.

[6] Peck and Shell (1991) and Gottardi and Kajii (1999) also obtain sunspot equilibria in models which have a unique non-sunspot equilibrium, though for slightly different reasons than us. In common with these papers, we share the idea that intertemporal wealth effects can provide a self-fulfilling mechanism.

Financial markets consist solely of an instantaneously-maturing, risk-free bond that pays interest rate r_t is in zero net supply. The key financial friction: agents cannot issue equity when managing capital. It is inconsequential that the constraint be this extreme. Partial equity issuance, as long as there is some limit, will generate identical results on sunspot volatility.[7]

Given the stated assumptions, we can write the dynamic budget constraint of an agent of type j (expert or household) as

$$dn_{j,t} = \Big[(n_{j,t} - q_t k_{j,t})r_t - c_{j,t} + a_j k_{j,t}\Big]dt + d(q_t k_{j,t}), \tag{1}$$

where n_j is the agent's net worth, c_j is consumption, and k_j is capital holdings. The term $d(qk)$ represents the capital and price appreciation that accrues while holding capital.

Experts and households have time-separable logarithmic utility, with discount rates ρ_e and $\rho_h \leq \rho_e$, respectively. Thus, they solve the following maximization problem:

$$\sup_{c_j \geq 0, k_j \geq 0, n_j \geq 0} \mathbb{E}\Big[\int_0^\infty e^{-\rho_j t} \log(c_{j,t}) dt\Big] \tag{2}$$

subject to (1). The constraint $n_{j,t} \geq 0$ is the standard solvency constraint. Everything in this optimization problem is homogeneous in (c, k, n), so we can think of the expert and household as representative agents within their class.

Finally, to guarantee a stationary wealth distribution, we also allow an overlapping generation structure: agents perish idiosyncratically at rate δ; perishing agents are replaced by newborns, who inherit an equal share of perishing wealth; a fraction $\nu \in [0, 1]$ of newborns are exogenously designated experts, and $1 - \nu$ are households; there are no annuity markets to trade death risk. As the death rate δ affects an agent's lifetime utility, the subjective discount rates ρ_e, ρ_h are assumed inclusive of δ. Finally, to acknowledge the fact that OLG creates intertemporal transfers across agent types, which do not affect alive agents' individual net worth evolution, let N_e and N_h denote aggregate expert and household net worth. The dynamic evolutions of N_e and N_h will mirror (1), with additional terms capturing OLG-related transfers. We reiterate that OLG is unnecessary for our sunspot results and only serves to obtain stationarity in case we set $\rho_e = \rho_h$.

Equilibrium. The definition of competitive equilibrium is standard: (i) taking prices as

[7] In particular, a partial equity-issuance constraint simply scales the mapping between expert wealth and asset prices. As is well-known, the equilibrium of economies in the class we consider will live in the region where the equity constraint is always-binding. Equity-issuance restrictions, sometimes called "skin-in-the-game" constraints, often arise as the optimal contract in a moral hazard problem, though this micro-foundation is not important for our purposes here.

given, and given an exogenous time-0 allocation of capital and riskless bonds, experts and households solve (2) subject to (1); (ii) the markets for consumption and capital clear at all dates, i.e.,

$$c_{e,t} + c_{h,t} = a_e k_{e,t} + a_h k_{h,t} \tag{3}$$

$$k_{e,t} + k_{h,t} = K_t. \tag{4}$$

The market for riskless bonds clears automatically by Walras' Law.

Despite the fact that all fundamentals of this economy are deterministic, we want to allow the possibility for sunspot volatility in equilibrium. To this end, let Z be a standard Brownian motion, which is extrinsic in the sense that it affects no primitives of the economy. Conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t]. \tag{5}$$

An equilibrium in which $\sigma_q \equiv 0$ is called a *Fundamental Equilibrium* (FE). An equilibrium in which σ_q is not identically zero is called a *Brownian Sunspot Equilibrium* (BSE).

We will show later that any volatility in capital prices is accompanied by output volatility. To benchmark this environment, note that without the equity-issuance friction, the model collapses to a constant-growth path with fully-efficient production and no volatility. With investment added as well, the model becomes a neoclassical growth model. Adding investment is trivial, but we abstract away from it for clarity.[8]

2 Sunspot volatility

2.1 Brownian Sunspot Equilibrium (BSE)

Before determining whether an FE or a BSE exists, we provide a simple characterization of equilibrium that aids much of the future analysis. First, due to all the scalability assumptions embedded in this model, we look for a Markov equilibrium, in which all growing variables scale with aggregate capital K, and experts' wealth share $\eta := N_e/(N_e + N_h) = N_e/qK \in (0, 1)$ serves as the sole non-growing state variable.

[8]One could easily add investment with neoclassical convex adjustment costs. In such a model, capital price q would coincide with Tobin's Q, so sunspot fluctuations in asset prices affect not only in current production (through their impact on allocative efficiency) but also economic growth through investment efficiency.

Conjecture η has dynamics of the following form:

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t}dZ_t, \quad \text{given} \quad \eta_0. \tag{6}$$

Second, given log utility and the scale-invariance of agents' budget sets, optimal consumption satisfies the standard formula $c_j = \rho_j n_j$. Third, let $\kappa := k_e/K$ denote expert's share of capital, which influences total output given heterogeneity in productivity. Substituting optimal consumption, the wealth share η, and the capital share κ, goods market clearing (3) becomes

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h, \tag{7}$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1-\eta)\rho_h$ is the wealth-weighted average discount rate.

Optimal capital holding by experts and households implies

$$\begin{aligned}
&\frac{a_e}{q} + g + \mu_q - r = \sigma_{n_e}\sigma_q \\
&\frac{a_h}{q} + g + \mu_q - r \leq \sigma_{n_h}\sigma_q \quad \text{(with equality if } k_h > 0\text{)},
\end{aligned}$$

where $\sigma_{n_j} := (k_j/n_j)\sigma_q$ denotes the loading of type-j agents on shock dZ. Because all experts (and households) make the same scaled portfolio choices, we have $\sigma_{n_j} = \sigma_{N_j}$, where

$$\sigma_{N_e} = \frac{\kappa}{\eta}\sigma_q \quad \text{and} \quad \sigma_{N_h} = \frac{1-\kappa}{1-\eta}\sigma_q. \tag{8}$$

Using result (8), we can summarize portfolio choices compactly by

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\sigma_q^2\right]. \tag{9}$$

Either experts manage the entire capital stock ($\kappa = 1$) or the excess return experts obtain over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $(\sigma_{N_e} - \sigma_{N_h})\sigma_q = \frac{\kappa-\eta}{\eta(1-\eta)}\sigma_q^2$. Given logarithmic preferences, σ_{N_j} corresponds to the risk price for type-j agents.

This model incorporates a two-way feedback loop between asset prices and the wealth distribution. Applying Itô's formula to the definition of experts' wealth share η, we obtain the diffusion loading $\sigma_\eta = \eta(1-\eta)[\sigma_{N_e} - \sigma_{N_h}]$, which after substituting (8) becomes

$$\sigma_\eta = (\kappa - \eta)\sigma_q. \tag{10}$$

Conversely, in a Markov equilibrium with single state variable η, all stationary variables can be expressed as functions of η, e.g., $q_t = q(\eta_t)$. Then, Itô's formula implies $\sigma_q = \frac{q'}{q}\sigma_\eta$. Plug this back into (10) to solve the two-way feedback loop between σ_q and σ_η:

$$\left[1 - (\kappa - \eta)\frac{q'}{q}\right]\sigma_\eta = 0. \tag{11}$$

Therefore, there are two possibilities: (i) either $\sigma_\eta = 0$, which would correspond to FE; or (ii) there might be an equilibrium with $1 = (\kappa - \eta)\frac{q'}{q}$, in which case σ_η and hence σ_q can be non-zero, corresponding to the BSE.

Finally, one can show that

$$\mu_\eta = \eta(1-\eta)[\rho_h - \rho_e] + \mathbf{1}_{\{\kappa<1\}}(\kappa - 2\eta\kappa + \eta^2)\frac{a_e - a_h}{q} + \delta(\nu - \eta). \tag{12}$$

It turns out, this is enough to determine the equilibrium.

Lemma 1 (Equilibrium Characterization)**.** *An allocation is a Markov equilibrium in η_t if and only if $(\kappa, q, \sigma_q, \sigma_\eta, \mu_\eta)$ are functions of η satisfying (7) and (9)-(12).*

Using this characterization, we can immediately show that an FE exists. As already suggested, this equilibrium corresponds to selecting the solution $\sigma_\eta = 0$ to equation (11).

Lemma 2 (Fundamental Equilibrium)**.** *There exists an equilibrium in which experts manage all capital, $\kappa = 1$, and the price of capital $q_t = a_e/\bar{\rho}(\eta_t)$ evolves deterministically.*

But there is also another class of equilibria, the BSEs, which have volatility. To understand this, consider selecting the solution $1 = (\kappa - \eta)\frac{q'}{q}$ to equation (11), which allows $\sigma_\eta \neq 0$. Substituting $\kappa < 1$ from goods market clearing (7), we obtain an ODE for q:

$$q' = \frac{(a_e - a_h)q}{q\bar{\rho} - \eta a_e - (1-\eta)a_h}, \quad \text{if} \quad \kappa < 1. \tag{13}$$

Suppose we solve this first-order ODE subject to the boundary condition $\kappa(0) = 0$, which translates via (7) to $q(0) = a_h/\rho_h$. For now, we choose this boundary condition – which says that experts fully de-lever as their wealth shrinks – in accordance with the literature. We return to this issue in Section 2.3; note for now only that the boundary condition selects a particular BSE. The ODE (13) is solved on the endogenous region $(0, \eta^*)$ in which households manage some capital, i.e., $\eta^* := \inf\{\eta : \kappa(\eta) = 1\} = \inf\{\eta : q(\eta) = a_e/\bar{\rho}(\eta)\}$.[9] Given the solution for (q, κ), capital price volatility is solved from (9) as

$$\sigma_q^2 = \frac{\eta(1-\eta)}{\kappa - \eta} \frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1. \tag{14}$$

The sign of σ_q is irrelevant, given the symmetry of Brownian shock dZ. Since $\sigma_q \neq 0$, equation (10) confirms the initial conjecture that $\sigma_\eta \neq 0$.

These are the key equations constituting a BSE. Intuitively, if agents believe the sunspot shock dZ can affect asset prices, then the actual arrival of such a shock triggers trading of capital between experts and households. Since experts are more productive than households, capital transfers have real effects and move asset prices. But it does not end there: asset-price fluctuations feed back into the wealth distribution, which initiates another round of capital transfers, and so on. Ultimately, the question is whether there exists an initial belief about asset prices that can be self-justified by this process, and this is tantamount to solving the ODE (13).

It is critical that the aforementioned capital transfers take place. A clear way to see this is to note that sunspot shocks cannot have any effect when $\kappa = 1$. Indeed, equation (7) implies $q = a_e/\bar{\rho}$, whose derivatives are inconsistent with equation (11) under $\kappa = 1$ and $\sigma_\eta \neq 0$. Thus, equilibrium must feature $\sigma_\eta = \sigma_q = 0$, resembling the FE, in states of the world when $\kappa = 1$. Naturally, these states of the world are when experts are sufficiently wealthy, i.e., $\eta_t > \eta^*$.

Proposition 1 (Brownian Sunspot Equilibrium)**.** *There exists a BSE with $\kappa(0) = 0$, in which $\sigma_q(\eta) \neq 0$ on $(0, \eta^*)$ and $\sigma_q(\eta) = 0$ on $(\eta^*, 1)$.*

For exposition purposes, we refer to this equilibrium as *the* BSE. Figure 2 displays a numerical example with the capital price q and volatility σ_q as functions of the expert wealth share.

Does sunspot volatility survive in the long run? This question is tied directly to the stationarity properties of $(\eta_t)_{t \geq 0}$, which boils down to equilibrium behavior around $\eta \approx 0$ and $\eta \approx \eta^*$. As we show in Lemma A.1 in Appendix A.1, this economy possesses

[9] When $\rho_h = \rho_e$, there is a closed form solution for capital price

$$q(\eta) = \frac{1}{\rho}\left[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2}\right], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}.$$

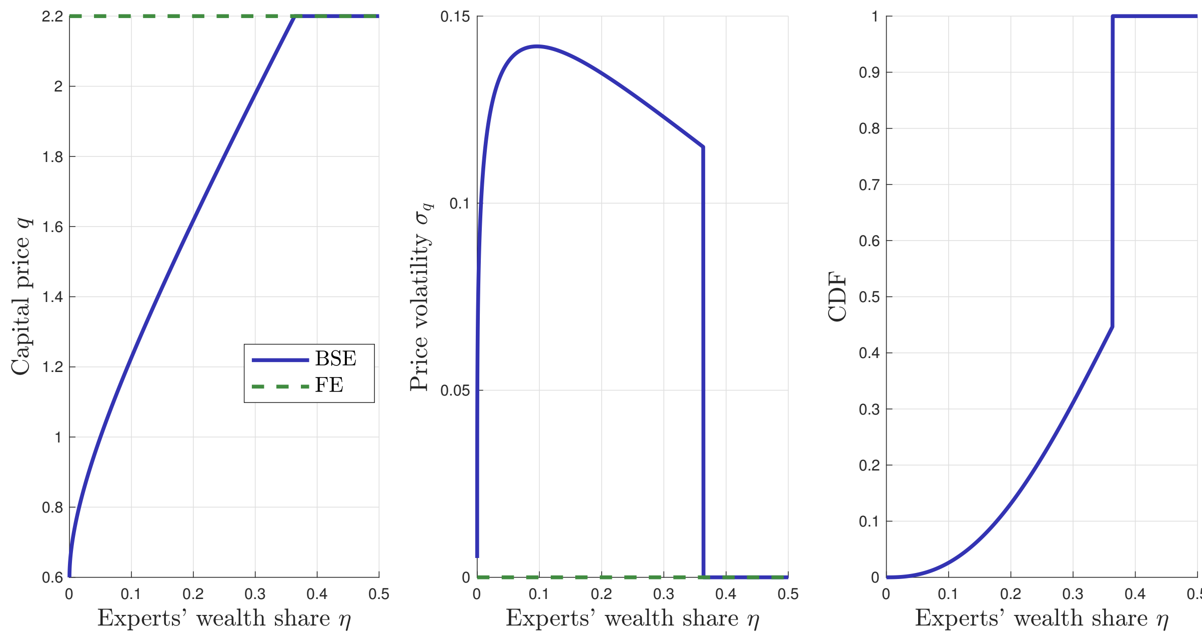

Figure 2: Capital price q, volatility σ_q, and stationary CDF of η. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters (for the CDF): $\nu = 0.1$ and $\delta = 0.04$.

a stationary distribution on $(0, \eta^*]$ under very mild and standard parameter restrictions. For example, if experts are more impatient than households ($\rho_e > \rho_h$), or the economy has an OLG structure with sufficiently few experts ($\delta > 0$ and $\nu < \eta^*$), then sunspot volatility survives. The right panel of Figure 2 shows the stationary CDF of η in the BSE.[10]

These results can be generalized in several directions. While we have chosen to model sunspots as Brownian shocks, an extension in Appendix E.1 shows how to solve an equilibrium with Poisson jump sunspots instead. The intuition and even equations of this extension mirror very closely the Brownian model. We have also chosen to examine the log utility model for simplicity, but similar results can be obtained with more general preferences, as shown in Appendix E.2.

Given our results, a natural conjecture is that sunspots could arise even if the wealth distribution does not have "real consequences" on output. For example, in a pure asset-pricing model with heterogeneous preferences, there is still a link between the wealth distribution and asset prices, modulated by the capital ownership distribution. Nevertheless, we are skeptical of a generic sunspot volatility in such a world: if all agents' unconstrained optimal portfolios are positive, then some additional financial frictions beyond our equity-issuance constraint would be required. Otherwise, markets would be

[10]Note that $\eta_t = \eta^*$ about 55% of the time in this numerical example, i.e., there is a mass point at η^*. This occurs because of a discontinuity in both the drift μ_η and volatility σ_η at that point.

effectively complete, ruling out sunspots.[11]

2.2 Instability of the BSE

In an attempt to differentiate ourselves from the literature, here we examine the stability properties of the BSE. Stability properties are typically studied around the "steady state" of a deterministic equilibrium. The challenge is that our BSE is inherently volatile, and studying a deterministic equilibrium instead puts us in the FE. In the FE, we have $\kappa = 1$ always, which precludes the capital transfers upon which sunspots can be built.

To sidestep this difficulty, we briefly take a detour and study a modified version of our model that has idiosyncratic shocks to capital. Individual capital now evolves as

$$dk_{i,t} = k_{i,t}[gdt + \tilde{\sigma} d\tilde{B}_{i,t}],$$

where $(\tilde{B}_i)_{i\in[0,1]}$ is a continuum of independent Brownian motions. Agents with indexes $i \in [0, I]$ are experts, and those with $i \in [I, 1]$ are households. As before, the aggregate stock of capital $K_t := \int_0^1 k_{i,t} di$ grows deterministically at rate g. With only idiosyncratic shocks, we can study an equilibrium in which capital prices evolve deterministically, even though $\kappa < 1$ in steady state.

The crucial feature of the BSE, preserved in this model, is that capital prices are determined by a function q such that $q_t = q(\eta_t)$. In that case, a steady state is fully characterized by the value $\eta = \eta^{ss}$ such that all non-growing variables are constant over time. This steady state is thus determined by the equation $\dot{\eta} = 0$, where

$$\dot{\eta} = \eta(1-\eta)\Big[\rho_h - \rho_e + \tilde{\sigma}^2\Big((\frac{\kappa}{\eta})^2 - (\frac{1-\kappa}{1-\eta})^2 \Big)\Big] + \delta(\nu - \eta).$$

It is straightforward to show that equilibrium features stable state variable dynamics, in the sense that $\frac{\partial \dot{\eta}}{\partial \eta}|_{\eta=\eta^{ss}} < 0$. However, because the "co-state variable" q is determined explicitly as a function of η, the steady state is not "stable" in the usual sense required by the multiplicity literature. Technically, there is only one stable eigenvalue of the dynamical system (η_t, q_t) near steady state (η^{ss}, q^{ss}).

Lemma 3. *The steady state of the model with idiosyncratic risk is saddle path stable.*

[11] Chan and Kogan (2002) show that their complete-markets heterogeneous preference equilibrium can be solved with a social planner. Bhamra and Uppal (2014) pursue a similar approach with even more dimensions of heterogeneity. Therefore, in these models, there is generically a unique equilibrium. Moreover, a typical finding is that all agents hold positive amounts of the market portfolio, meaning that an equity-issuance constraint would never bind.

As a result of Lemma 3, there is a unique transition path $(\eta_t, q_t)_{t\geq 0}$ to steady state, given an initial condition η_0. Our sunspot equilibrium (the BSE) is not constructed by randomizing over a multiplicity of transition paths that arise due to steady state stability, which is the usual approach (Azariadis, 1981; Cass and Shell, 1983). Instead, the introduction of aggregate uncertainty through sunspots provides additional stability relative to the deterministic equilibrium.

2.3 More sunspot equilibria: beliefs about disaster states

In the BSE, there is a unique viable level of sunspot volatility σ_q at each level of the wealth share η, given by equation (14). This could be seen as restrictive, since agents must somehow coordinate on a particular amount of volatility. In this section, we demonstrate one method for expanding the set of sunspot equilibria, through beliefs about the "tail scenario" in the economy, i.e., what happens when experts are severely undercapitalized.

Recall that we previously have assumed $\kappa(0) = 0$; in other words, experts fully deleverage as their wealth vanishes. This turns out to not be necessary. Consider any $\kappa_0 \in (0,1)$ and put $\kappa(0) = \kappa_0$. We will call κ_0 the *disaster belief* in the economy. The sunspot equilibrium is very similar to Proposition 1, with the exception that the boundary condition to the ODE (13) is now $\kappa(0) = \kappa_0$ rather than $\kappa(0) = 0$.[12]

Proposition 2 (Disaster beliefs)**.** *For a fixed tail belief $\kappa_0 \in (0,1)$, there exists a Markov sunspot equilibrium, with $\sigma_q(\eta) \neq 0$ on a positive measure subset of $(0,1)$. As $\kappa_0 \to 0$, this equilibrium converges to the BSE. As $\kappa_0 \to 1$, the equilibrium converges to the FE.*

Based on Proposition 2, one can view both the BSE and the FE as outcomes of coordination on experts' deleveraging. Intuitively, if experts never sell any capital, there can be no price volatility, with $\sigma_{q,t} = 0$ at all times. If agents expect $\kappa_0 = 0$, which translates to full deleveraging, then the BSE will prevail. But for any $\kappa_0 \in (0,1)$, an intermediate sunspot equilibrium will prevail, with a self-fulfilling amount of expert deleveraging and associated price dynamics. In this way, $\kappa_0 \in [0,1]$ spans an entire range of sunspot equilibria from more to less volatile. An illustration is in Figure 3.

[12] As in footnote 9, there is a closed-form solution when $\rho_h = \rho_e$, which is

$$q(\eta) = \frac{1}{\rho}\Big[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2 + (a_e - a_h)^2\kappa_0^2}\Big], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}(1 - \kappa_0^2).$$

As κ_0 decreases, not only does the worst-case price $q(0)$ decrease, but the slope $q'(\eta)$ increases, consistent with the idea that pessimism about the disaster state raises the sensitivity of equilibrium to sunspot shocks away from disaster. Clearly, this solution converges to the BSE solution in footnote 9 as $\kappa_0 \to 0$, and to the FE solution a_e/ρ as $\kappa_0 \to 1$.

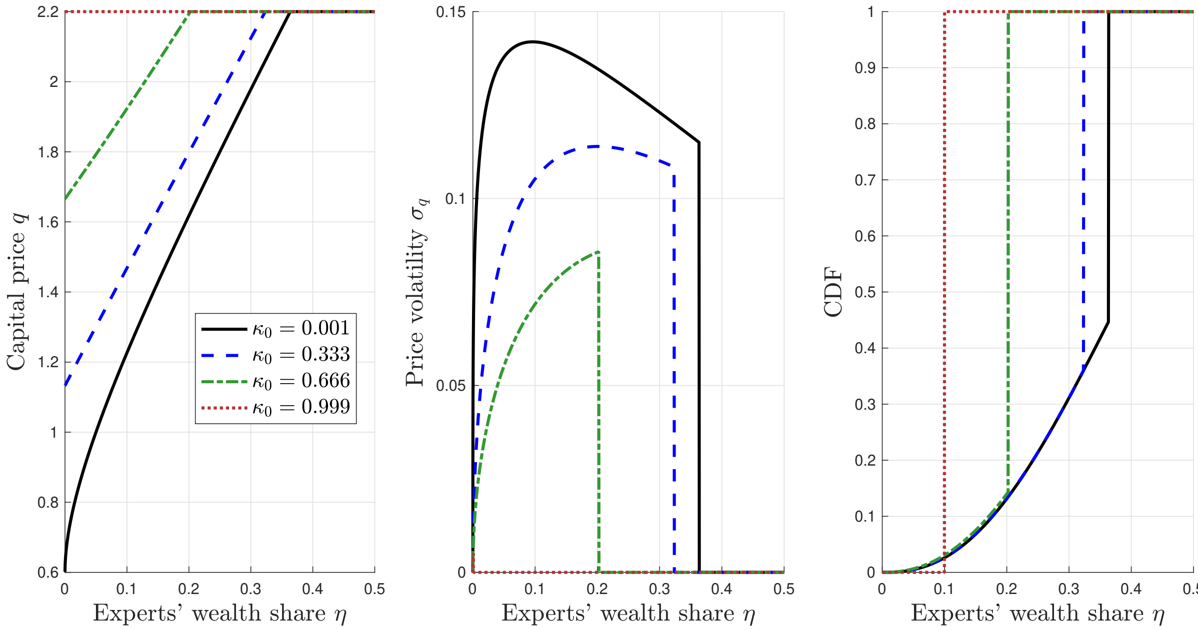

Figure 3: Capital price q, volatility σ_q, and stationary CDFs of η for different levels of the disaster belief κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, and $a_h = 0.03$. OLG parameters (for the CDF): $\nu = 0.1$ and $\delta = 0.04$.

Intuitively, greater optimism about other experts' ability to retain capital in the tail scenario induces smaller capital fire sales in response to sunspot shocks, which keeps volatility low, asset prices high, and justifies the optimism. What if this optimism shifts to pessimism, and vice versa? In Appendix E.3, we allow this possibility, further enlarging the space of self-fulfilling equilibria by modeling time-variation in the disaster belief κ_0.

3 Duality and tension between amplification and sunspots

In the preceding analysis, economic fundamentals are completely deterministic. With some fundamental uncertainty, equilibrium volatility in this class of models is thought to come from amplification of fundamental shocks. Here, we show that sunspots are just a special case of amplification, in the sense that our sunspot equilibria are limiting equilibria as fundamental shocks vanish.

But this connection also reveals a tension between amplification and sunspots. We will show below that sunspot equilibria require infinite amplification, in order to generate a positive amount of volatility out of extrinsic shocks. Infinite amplification is inconsistent with non-zero fundamental shocks, because endogenous volatility cannot be infinite. In this sense, the presence of fundamental shocks rules out certain types of sunspot equilibria.

3.1 Duality: sunspots as a special case amplification

Here, we add fundamental uncertainty to capital, which now follows

$$dK_t = K_t[gdt + \sigma dZ_t],$$

where Z is a one-dimensional Brownian motion, and $\sigma > 0$ is a constant. Conjecture capital prices q and experts' wealth share η follow

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t]$$
$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t}dZ_t.$$

We focus on a Markov equilibrium, in which the sole state variable will still be η.

A crucial equilibrium condition is

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma. \tag{15}$$

Equation (15) is often interpreted as *amplification*, because $\frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}$ takes the form of a convergent geometric series. In words, a negative fundamental shock reduces experts' wealth share η directly through $(\kappa - \eta)\sigma$, which reduces asset prices through q'/q. This explains the numerator of (15). But the reduction in asset prices has an indirect effect: a one percent drop in capital prices reduces experts' wealth share by $(\kappa - \eta)$, which feeds back into a $(\kappa - \eta)q'/q$ percent further reduction capital prices, which then triggers the loop again. The second-round impact is $[(\kappa - \eta)q'/q]^2$, and so on. This infinite series is convergent if $(\kappa - \eta)q'/q < 1$, such that incremental amplification is reduced in each successive round of the feedback loop.

This contrasts with any BSE, in which equation (11) implies $(\kappa - \eta)q'/q = 1$. Intuitively, a BSE has no dampening in successive rounds of the feedback loop, leading to infinite amplification! Infinite amplification is needed to generate price volatility, in fact, because there is no fundamental shock to start with: one can think of zero fundamental uncertainty multiplied by infinite amplification begetting a finite positive amount of price volatility. This logic is formalized by the following limit. As before we let $\kappa_0 = \lim_{\eta\to 0}\kappa(\eta)$ denote the "disaster belief" about experts' capital share when they are impoverished.[13]

[13]We could alternatively model the fundamental uncertainty as jumps, i.e.,

$$dK_t = K_{t-}[gdt + \zeta dJ_t],$$

Proposition 3. *Suppose a Markov equilibrium in η exists for each $\sigma > 0$ and disaster belief $\kappa_0 \in [0, 1)$. As $\sigma \to 0$, the equilibrium converges to the BSE with disaster belief κ_0.*

This limiting result tells us that sunspots can be seen as a special case of amplification. There need not be any discontinuity in the nature of equilibria at $\sigma = 0$. Brunnermeier and Sannikov (2014) provide a related limiting result, arguing that asset-price volatility does not vanish as $\sigma \to 0$, also known as the "volatility paradox."

Proposition 3 goes further by identifying the limiting equilibrium as the BSE. Moreover, the limit applies to any level of disaster belief κ_0. Although our paper is primarily about sunspot equilibria, we note in passing that the literature has not entertained this rich set of non-sunspot equilibria. The coordinated disaster belief κ_0 modulates the strength of the core feedback loop of the model in a self-fulfilling way, exactly as in Section 2.3: greater optimism about other experts' ability to retain capital in crisis begets less capital fire sales at the outset, keeping asset prices high and justifying the optimism. Investigating this story carefully, one sees that the presence or absence of fundamental shocks is irrelevant.[14]

Thus, our entire continuum of sunspot equilibria in Section 2 are just limits of equilibria with fundamental shocks. A related result can be found in the OLG model of Manuelli and Peck (1992), in which sunspot equilibria could be seen as limits of non-sunspot equilibria when fundamental uncertainty vanishes.

3.2 Tension: non-traded fundamental uncertainty kills sunspots

Given that the BSE is consistent with the standard model, in the sense of Proposition 3, one naturally wonders whether it is possible to simply augment the standard model with sunspot shocks. We will address this by trying to add sunspot volatility on top of aggregate fundamental volatility.

where J is a Poisson process with intensity λ, and $\zeta > -1$. Under this specification, we can generalize Proposition 3 to demonstrate the following. As $\zeta \to 0$, the Markov equilibrium (if it exists) converges to another type of sunspot equilibrium, with jumps, which is what we refer to as the Poisson Sunspot Equilibrium (PSE) in Appendix E.1.

[14] Brunnermeier and Sannikov (2014) do not consider these types of equilibria, always assuming $\kappa_0 = 0$. They justify this, in their online appendix, "because in the event that η_t drops to 0, experts are pushed to the solvency constraint and must liquidate any capital holdings to households." This is technically not needed; as shown in Lemma B.2 of Appendix B.1, the dynamics of η_t will not allow it to ever reach 0. In other words, there is no contradiction to equilibrium with both $\kappa_0 > 0$ and $\sigma > 0$. Although we do not prove an existence result, we show several examples of non-sunspot equilibria with $\kappa_0 > 0$ in Appendix B.1. The possible continuum of fundamental equilibria, indexed by κ_0, may be of some independent theoretical interest.

In some sense, the literature has picked the worst possible fundamental equilibrium (minimal-price, maximal-volatility) by imposing $\kappa_0 = 0$. Despite this choice, the quantitative failures outlined in the introduction persist, yet another indicator that sunspot equilibria carry significant promise.

Let $Z := (Z^{(1)}, Z^{(2)})$ be a two-dimensional Brownian motion, where $Z^{(1)}$ represents the fundamental shock, and $Z^{(2)}$ represents the sunspot shock. Aggregate capital follows

$$dK = K_t\Big[gdt + \sigma\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) \cdot dZ_t\Big]. \tag{16}$$

Conjecture capital prices follow

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t]. \tag{17}$$

Let us continue to study a Markov equilibrium with state variable η.

In this environment, we obtain the stark result that capital prices must be completely insensitive to the sunspot shock $Z^{(2)}$, in contrast to our previous existence findings for sunspot equilibria. To see this, note that the wealth share diffusion is now $\sigma_\eta = (\kappa - \eta)[\sigma\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) + \sigma_q]$, the generalization of (10) since return-on-capital volatility is now $\sigma\big(\begin{smallmatrix}1\\0\end{smallmatrix}\big) + \sigma_q$. In a Markov equilibrium in η, capital prices respond to wealth share changes through $\sigma_q = \frac{q'}{q}\sigma_\eta$, as before. Solving this two-way feedback, we obtain

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\sigma_q = \big(\begin{smallmatrix}1\\0\end{smallmatrix}\big)(\kappa - \eta)\sigma\frac{q'}{q}. \tag{18}$$

Given $\sigma > 0$, any equilibrium requires $(\kappa - \eta)\frac{q'}{q} \neq 1$. This immediately proves that $\sigma_q \cdot \big(\begin{smallmatrix}0\\1\end{smallmatrix}\big) = 0$, implying the following proposition.[15]

Proposition 4. *If $\sigma > 0$, any Markov equilibrium in η must be insensitive to sunspot shocks.*

The intuition for this result extends the discussion in the previous subsection: whereas sunspot equilibria require infinite amplification, fundamental shocks cannot be amplified infinitely, because that would lead to unbounded total volatility. This section thus highlights a tension between fundamental shocks and sunspot shocks, casting doubt on whether they can co-exist in any equilibrium.[16]

Before moving on, we provide an alternative intuition for the tension, based on the amount of capital transfers that occur in equilibria with different levels of fundamental volatility. For given $\sigma \geq 0$ and disaster belief $\kappa_0 \in [0, 1)$, let $\tau(\eta; \sigma, \kappa_0)$ denote the loading

[15] If all agents can frictionlessly access a market to trade claims on the fundamental shock $dZ^{(1)}$, but no such market for the sunspot shock $dZ^{(2)}$ exists, then the possibility of the sunspot equilibrium re-emerges. This sunspot equilibrium is exactly the as the BSE, in the sense that $\big(\begin{smallmatrix}0\\1\end{smallmatrix}\big) \cdot \sigma_q$ coincides with the capital price volatility of the BSE. Thus, the precise summary of this section is that *non-traded* fundamental uncertainty can eliminate the sunspot equilibrium.

[16] In Appendix E.4, we also show that the presence of idiosyncratic shocks, if they are large enough, kills sunspot equilibria. The intuition is similar to Proposition 4, though less stark.

of experts' capital share κ on shocks, in a Markov equilibrium with state variable η. We will think of τ as describing the amount of trade.

Lemma 4. *For each $\sigma > 0$ and each $\kappa_0 \in [0,1)$, we have $\tau(\eta; 0, \kappa_0) > \tau(\eta; \sigma, \kappa_0)$ for each η such that $\tau(\eta; 0, \kappa_0) > 0$.*

Lemma 4 says simply that there is more trade in a BSE (which corresponds to $\sigma = 0$) than there is in a non-sunspot equilibrium with positive fundamental volatility. This occurs intuitively because return volatility all must stem from capital prices in BSEs, whereas return volatility has an exogenous fundamental component $\sigma > 0$ in non-sunspot equilibria. But prices can only fluctuate more if capital changes hands more often between experts and households, i.e., if τ is higher. Since capital trading is core to the feedback loop of these models, more trading indicates larger amplification, which provides another explanation for the infinite-amplification nature of BSEs.

Lemma 4 delivers an economic rationale for the tension between sunspot shocks and fundamental shocks. Equilibria with fundamental shocks dictate a particular amount of trading $\tau(\eta; \sigma, \kappa_0)$, which is lower than the amount $\tau(\eta; 0, \kappa_0)$ needed to sustain sunspot volatility. Two different frequencies of trade cannot co-exist in equilibrium.

4 Restoring sunspots with fundamental uncertainty

The last section provides an important caveat to our results; namely, fundamental volatility can eliminate the prospect of sunspot volatility (in a Markovian equilibrium with η as its sole state variable). In this section, we resolve this tension by constructing a different type of sunspot equilibria that co-exists with fundamental uncertainty. In particular, a simple extension that adds a non-fundamental sunspot state variable restores sunspot volatility.

As in the previous section, we continue to assume that there are two shocks: $Z^{(1)}$ is a fundamental shock affecting capital as in (16), whereas $Z^{(2)}$ is a sunspot shock which only affects asset prices if agents believe in it, as in (17). In addition, we posit a sunspot variable s that is irrelevant to economic fundamentals and loads on only the second shock:[17]

$$ds_t = \mu_{s,t} dt + \sigma_{s,t} \binom{0}{1} \cdot dZ_t.$$

[17]This is only for clarity. We have solved examples with sunspots correlated to fundamentals, i.e., with $ds_t = \mu_{s,t} dt + \sigma^{(1)}_{s,t} dZ^{(1)}_t + \sigma^{(2)}_{s,t} dZ^{(2)}_t$. An additional feature that emerges relative to what we show here is that $\sigma^{(1)}_s$ can work to reduce asset price volatility at times, unlike $\sigma^{(2)}_s$. See Appendix E.5 for details.

Equilibrium conditions will impose some restrictions over (σ_s, μ_s), so we leave them unspecified for now. We will study an equilibrium that is Markovian not only in the wealth share η, but also in a sunspot variable s, which we assume lives in an interval $\mathcal{S}$.

Definition 1. A *Markov equilibrium* in state variables $(\eta, s) \in \mathcal{D} := (0,1) \times \mathcal{S}$ consists of functions $(q, \kappa, \sigma_\eta, \mu_\eta, \sigma_s, \mu_s)$, each C^2 almost-everywhere on $\mathcal{D}$, such that

(i) Agents make optimal consumption and portfolio choice decisions, subject to their constraints;

(ii) Goods, capital, and bond markets clear.

A *non-sunspot equilibrium* is defined as a Markov equilibrium of Definition 1 such that q is independent of s. Conversely, a *sunspot equilibrium* is defined as a Markov equilibrium of Definition 1 with the additional property that sunspot volatility σ_s and sunspot sensitivity $\partial_s q$ are non-zero on a positive-measure subset of $\mathcal{D}$.

The equilibrium is derived very similarly to previous sections, so we omit the derivation and state the results. Aggregating optimal portfolio choices, we obtain the following equation linking capital prices, the capital distribution, and sunspot dynamics:

$$0 = \min\Big[1-\kappa,\, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}\Big(\frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2}\Big)\Big]. \tag{19}$$

The squared terms in parentheses represents the return-on-capital variance $|\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\sigma + \sigma_q|^2$, where capital price volatility is given by

$$\sigma_q = \frac{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)(\kappa-\eta)\sigma\partial_\eta \log q + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\sigma_s \partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}. \tag{20}$$

The dynamics of the wealth share are given by

$$\mu_\eta = \eta(1-\eta)\Big[\rho_h - \rho_e + \Big((\frac{\kappa}{\eta})^2 - (\frac{1-\kappa}{1-\eta})^2\Big)\big|\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\sigma + \sigma_q\big|^2\Big] - \frac{|\sigma_\eta|^2}{\kappa-\eta} + \delta(\nu - \eta) \tag{21}$$

$$\sigma_\eta = (\kappa - \eta)\Big[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\sigma + \sigma_q\Big]. \tag{22}$$

There are no additional conditions needed to pin down equilibrium, summarized in the following result.

Lemma 5. *Conditions (i) and (ii) of Definition 1 are satisfied if and only if equations (19)-(22) and (7) are satisfied.*

Based on Lemma 5, we can prove that a Markov sunspot equilibrium exists. In fact, examining the constructive proof of the following proposition, one can see that a continuum of such equilibria could be engineered. One reason for this is that the proposition and the numerical examples that follow impose maximal expert deleveraging, i.e., $\kappa(0, s) = 0$ for all s. As in previous sections, this is unnecessary: many choices for the disaster belief $\kappa(0, s)$ would work, including functions that vary with the sunspot. Another reason for vast multiplicity is that no condition pins down μ_s, and even restricting to drifts μ_s that induce stationarity permits vast degrees of freedom. We return to this point in our numerical example below.

Proposition 5. *Let Assumptions 1-2 in Appendix C hold. Then, for all $\sigma > 0$ small enough, there exists a sunspot equilibrium. Furthermore, this sunspot equilibrium is stationary in the sense that the paths $(\eta_t, s_t)_{t\geq 0}$ induced by $(\sigma_\eta, \mu_\eta, \sigma_s, \mu_s)$ remain in $\mathcal{D}$ almost-surely.*

While the proof of Proposition 5 is somewhat involved and a bit technical, we briefly provide an intuition and outline of the result. The first thing to realize is that sunspot volatility will only emerge if the capital price function is lower than in a non-sunspot equilibrium, since agents are risk averse. For example, as volatility increases, it becomes investors' main concern, eventually swamping any productive efficiency considerations. As $|\sigma_q| \to \infty$, all agents would hold the same portfolios irrespective of their productivity, meaning $\kappa \to \eta$ and $q \to q^1 := \frac{\eta a_e + (1-\eta) a_h}{\bar{\rho}}$. Think of this as the "worst-case" capital price.

Conversely, the non-sunspot equilibrium capital price q^0 can be thought of as the "best-case" capital price. This non-sunspot equilibrium, despite featuring amplification, does not gain any additional volatility coming from sunspots, so volatility is minimal in some sense. Our strategy is essentially to treat the sunspot variable s as a device to shift continuously between the best-case q^0 and the worst-case q^1. Mathematically, we conjecture a capital price that is approximately a weighted average of q^0 and q^1, with weights s and $1 - s$.[18]

But if the economy deteriorates – i.e., sunspot shocks move us closer to q^1 – capital price volatility must increase sufficiently to justify this shift. This points to the novel aspect of our construction: given the conjectured capital price function, we use equation (19) to solve for sunspot volatility σ_s, which will be endogenous as a result. Typically, we would be given σ_s exogenously, and we would regard (19) as a PDE for q.

[18]In this particular equilibrium, capital prices can never literally achieve the "worst-case" capital price q^1, for two technical reasons, both of which ensure that sunspot volatility stays σ_s bounded: (i) we need $q(\eta, s)$ to behave like the non-sunspot solution $q^0(\eta)$ for η close enough to zero, and all s; (ii) we need $q(\eta, s) > q^1(\eta)$, so that $\kappa(\eta, s) > \eta$ for all (η, s). Thus, in the proof of Proposition 5, we actually construct q^1 as a close approximation to the worst-case price, such that (i) and (ii) are satisfied.

Although we do not prove it, there is a sense in which, for a given disaster belief $\kappa(0,s)$, our construction can capture all the relevant sunspot equilibria, being a convex combination of best-case and worst-case scenarios. The technical aspects of the proof are only intended to formalize that this can indeed be done, that q is appropriately monotonic, and that all volatilities are bounded and vanish at the appropriate times in order to ensure the stationarity claim.[19]

Numerical example. We construct a numerical example closely following our proof of Proposition 5. Appendix D provides details of the numerics.

In our example, the sunspot variable s lives in $\mathcal{S} = (0,1)$, and spans between the non-sunspot equilibrium (as $s \to 0$) and the worst-case equilibrium (as $s \to 1$). The left panel of Figure 4 shows the capital price function in this construction. Positive sunspot shocks reduce the capital price, independently of experts' wealth share η.

The middle panel of Figure 4 displays capital return volatility, which can be substantially greater than in the non-sunspot equilibrium. As discussed above, such large volatility is necessary as a self-fulfilling mechanism to justify low capital prices. Implied by capital return volatility is an underlying sunspot shock size σ_s, which is displayed in the right panel of Figure 4. Sunspot dynamics become more volatile both as experts become poor (η shrinks) and as the economy approaches the worst-case equilibrium (s rises). Intuitively, in both of these directions, asset prices fall and households manage a greater fraction of the capital stock, which provides greater scope for the type of self-fulfilling mechanism we have highlighted thusfar. The strong dependence of σ_s on η is the notion of endogenous beliefs needed for self-fulfilling equilibrium.

The magnitude of the difference between the sunspot and non-sunspot equilibrium can be quantified in Figure 5. We plot unconditional cumulative distribution functions for asset prices (left panel) and return volatility (right panel) in both equilibria. We also plot the CDF for $\binom{1}{0} \cdot \sigma_q$, which highlights the contribution to total volatility coming from the fundamental shock $Z^{(1)}$. Relative to the non-sunspot equilibrium, sensitivity of to fundamental shocks $Z^{(1)}$ is reduced in the sunspot equilibrium, with the difference more than made up by sunspot shocks $Z^{(2)}$.

Figure 5 uses an ad-hoc sunspot drift μ_s that allows the sunspot s_t to visit most of the states between 0 and 1. In other words, we have chosen μ_s so that observed capital

[19]One of the assumptions imposed to verify these technical issues is that fundamental volatility σ is small enough. This is only to ensure that a non-sunspot equilibrium exists (i.e., q^0 exists), and that capital prices are more sensitive to the wealth share η in this non-sunspot equilibrium than they would be in the worst-case equilibrium (i.e., q^0 is steeper than q^1). These are very mild requirements that numerically hold for any values of $\sigma > 0$ we have investigated.

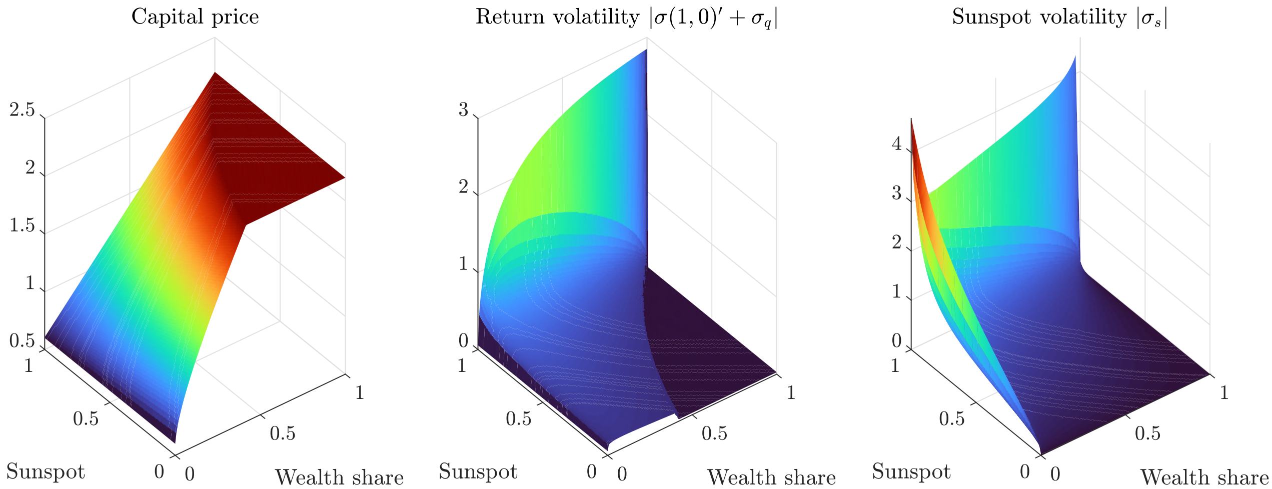

Figure 4: Capital price q, volatility of capital returns $|(1,0)'\sigma + \sigma_q|$, and sunspot shock volatility σ_s. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.025$.

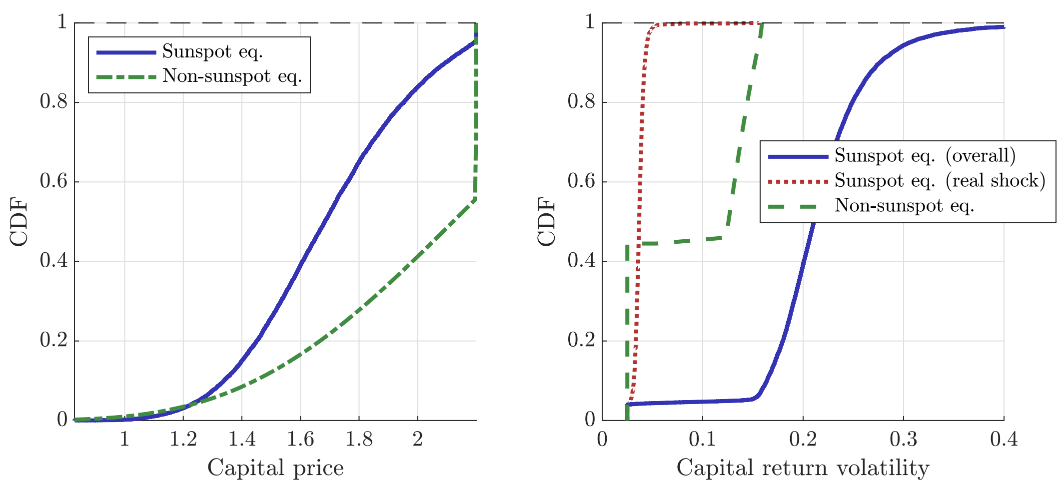

Figure 5: Unconditional CDFs of capital prices and capital return volatility. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = 2 + 2\log(1 - s/s_{\max})$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

prices span most of the range between the best-case capital price q^0 (corresponding to the non-sunspot equilibrium) and the worst-case capital price q^1 (corresponding to an infinite-volatility equilibrium).

But we have the freedom to set μ_s to restrict capital prices to any sub-set of these capital prices. For example, if we set $\mu_s = \epsilon^{-1}[\log(1 - s/s_{\max}) - \log(s/s_{\min} - 1)]$ for $\epsilon > 0$ small enough, the stationary distribution of s_t will live in $(s_{\min}, s_{\max})$. By allowing this interval to be arbitrarily small, we can essentially select a specific sunspot level s and the associated equilibrium capital prices, wealth share dynamics, etc. Due to the flexibility in this choice, there are a continuum of sunspot equilibrium we could observe.

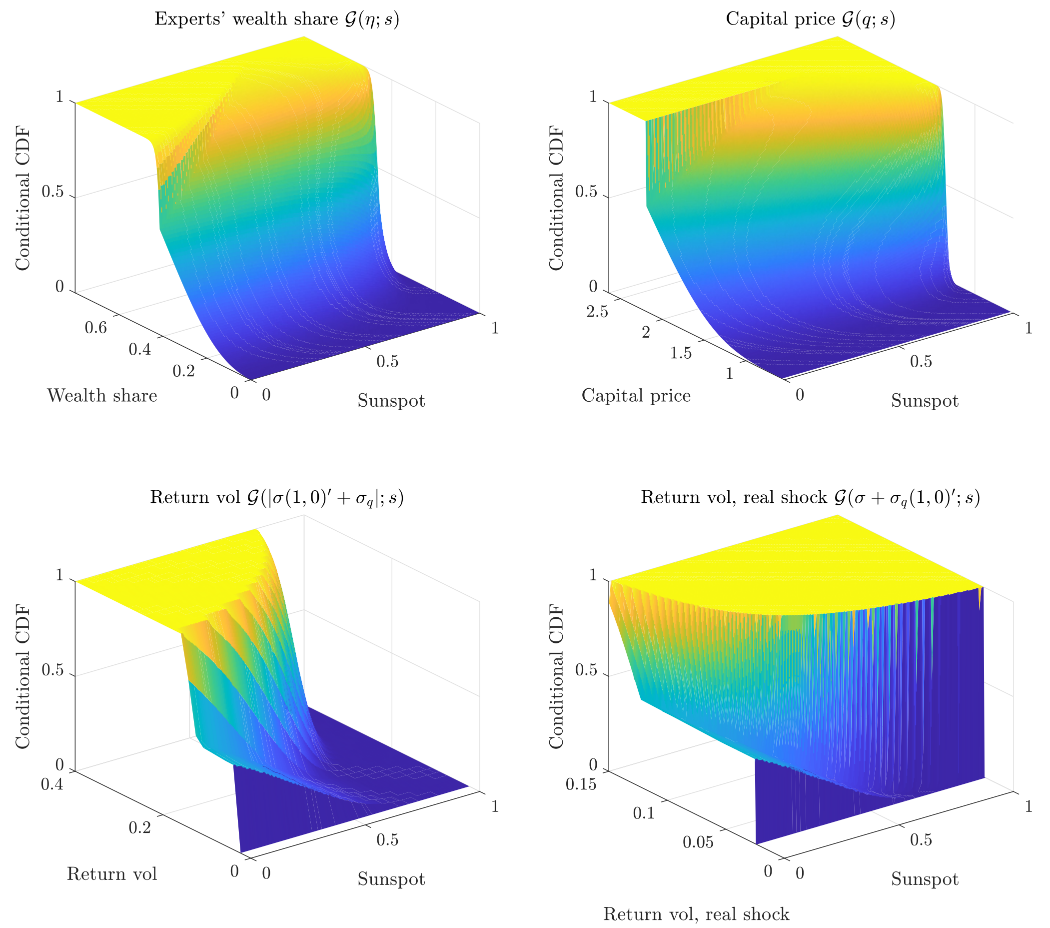

Figure 6: Conditional CDFs. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$.

Motivated by this freedom, Figure 6 shows the conditional CDFs of various equilibrium objects, conditional on levels of the sunspot variable s. In equilibria with higher levels of s, capital prices tend to be much lower (top right panel), with return volatility much higher (bottom left panel). In fact, one can design an equilibrium, based on a close approximation to the worst-case capital price q^1 as $s \to 1$, such that return volatility is arbitrarily high.

Proposition 6. *Given any target variance $\Sigma^* > 0$ and any parameters satisfying the assumptions of Proposition 5, there exists a sunspot equilibrium construction with stationary average return variance exceeding the target, i.e.,* $\mathbb{E}[|\sigma\binom{1}{0} + \sigma_{q,t}|^2] > \Sigma^*$.

As return volatility increases, its nature also changes. Comparing the bottom two panels in Figure 6, higher return volatility equilibria are those in which capital prices depend less on the real shock $Z^{(1)}$. Indeed, at higher levels of s, almost all volatility is non-fundamental, while amplified fundamental volatility vanishes. The trade-off can be formalized.

Proposition 7. *A sunspot equilibrium of Proposition 5 can be constructed such that the following hold for each η on $\{\kappa < 1\}$:*

(i) amplified fundamental volatility $\binom{1}{0} \cdot \sigma_q$ is decreasing in s;

(ii) sunspot price volatility $\binom{0}{1} \cdot \sigma_q$ is increasing in s;

(iii) total return volatility $|\binom{1}{0}\sigma + \sigma_q|$ is increasing in s.

Intuitively, sunspot volatility substitutes for amplified fundamental volatility because experts take less leverage ($\kappa - \eta$ decreases) as s increases. On the one hand, it may seem paradoxical to achieve greater volatility with lower expert leverage, suggesting a minor role for financial frictions. On the other hand, the presence of financial frictions is precisely what allows this sunspot volatility to emerge. Furthermore, the trade-off between sunspot volatility and amplified fundamental volatility, which allows a decoupling of asset prices from fundamental shocks at some times but not others, may be a helpful insight, given the difficulty identifying a "smoking gun" (e.g., TFP shocks, capital efficiency shocks) for financial crises and dramatic price swings.

5 Conclusion

We have shown that macroeconomic models with financial frictions may inherently permit sunspot volatility. Due to the financial frictions, capital may be misallocated, depending on the relative wealth of first-best and second-best users of capital. In such cases, sunspot shocks to asset prices induce capital trading between first- and second-best users, which can self-justify the shift in asset prices. The types of models we study are pervasive in macroeconomics, so this phenomenon cannot be ignored. Future research could explore the implications of endogenous sunspot dynamics, which is a distinctive feature of our analysis.

References

Azariadis, Costas, "Self-fulfilling prophecies," *Journal of Economic Theory*, 1981, *25* (3), 380–396.

— **and Allan Drazen**, "Threshold externalities in economic development," *The Quarterly Journal of Economics*, 1990, *105* (2), 501–526.

— **and Bruce Smith**, "Financial intermediation and regime switching in business cycles," *American economic review*, 1998, pp. 516–536.

— **, Leo Kaas, and Yi Wen**, "Self-fulfilling credit cycles," *The Review of Economic Studies*, 2016, *83* (4), 1364–1405.

Benhabib, Jess and Pengfei Wang, "Private information and sunspots in sequential asset markets," *Journal of Economic Theory*, 2015, *158*, 558–584.

— **,** — **, and Yi Wen**, "Sentiments and aggregate demand fluctuations," *Econometrica*, 2015, *83* (2), 549–585.

— **, Xuewen Liu, and Pengfei Wang**, "Sentiments, financial markets, and macroeconomic fluctuations," *Journal of Financial Economics*, 2016, *120* (2), 420–443.

— **,** — **, and** — , "Financial markets, the real economy, and self-fulfilling uncertainties," *The Journal of Finance*, 2019, *74* (3), 1503–1557.

Bhamra, Harjoat S and Raman Uppal, "Asset prices with heterogeneity in preferences and beliefs," *The Review of Financial Studies*, 2014, *27* (2), 519–580.

Brunnermeier, Markus K and Yuliy Sannikov, "A Macroeconomic Model with a Financial Sector," *The American Economic Review*, 2014, *104* (2), 379–421.

— **and** — , "Macro, money, and finance: A continuous-time approach," in "Handbook of Macroeconomics," Vol. 2, Elsevier, 2016, pp. 1497–1545.

Cass, David and Karl Shell, "Do sunspots matter?," *Journal of Political Economy*, 1983, *91* (2), 193–227.

Chan, Yeung Lewis and Leonid Kogan, "Catching up with the Joneses: Heterogeneous preferences and the dynamics of asset prices," *Journal of Political Economy*, 2002, *110* (6), 1255–1285.

Farhi, Emmanuel and Jean Tirole, "Bubbly liquidity," *The Review of Economic Studies*, 2012, *79* (2), 678–706.

Farmer, Roger EA and Jess Benhabib, "Indeterminacy and increasing returns," *Journal of Economic Theory*, 1994, *63*, 19–41.

Gârleanu, Nicolae and Stavros Panageas, "What to Expect when Everyone is Expecting: Self-Fulfilling Expectations and Asset-Pricing Puzzles," *Journal of Financial Economics*, 2020.

Gertler, Mark and Nobuhiro Kiyotaki, "Banking, liquidity, and bank runs in an infinite horizon economy," *The American Economic Review*, 2015, *105* (7), 2011–43.

—, —, and Andrea Prestipino, "A macroeconomic model with financial panics," *The Review of Economic Studies*, 2020, *87* (1), 240–288.

Gottardi, Piero and Atsushi Kajii, "The structure of sunspot equilibria: the role of multiplicity," *The Review of Economic Studies*, 1999, *66* (3), 713–732.

Gu, Chao, Fabrizio Mattesini, Cyril Monnet, and Randall Wright, "Endogenous credit cycles," *Journal of Political Economy*, 2013, *121* (5), 940–965.

Hugonnier, Julien, "Rational Asset Pricing Bubbles and Portfolio Constraints," *Journal of Economic Theory*, 2012, *147* (6), 2260–2302.

Khasminskii, Rafail, *Stochastic stability of differential equations*, Vol. 66, Springer Science & Business Media, 2011.

Khorrami, Paymon, "Entry and slow-moving capital: using asset markets to infer the costs of risk concentration," 2018. Unpublished working paper. Imperial College London.

— and Alexander Zentefis, "Arbitrage and Beliefs," 2020. Unpublished working paper, CESifo.

Kiyotaki, Nobuhiro and John Moore, "Credit cycles," *Journal of Political Economy*, 1997, *105* (2), 211–248.

Krishnamurthy, Arvind and Wenhao Li, "Dissecting mechanisms of financial crises: Intermediation and sentiment," 2020. Unpublished working paper.

Liu, Zheng and Pengfei Wang, "Credit constraints and self-fulfilling business cycles," *American Economic Journal: Macroeconomics*, 2014, *6* (1), 32–69.

Manuelli, Rodolfo and James Peck, "Sunspot-like effects of random endowments," *Journal of Economic Dynamics and Control*, 1992, *16* (2), 193–206.

Mendo, Fernando, "Risky low-volatility environments and the stability paradox," 2020. Unpublished working paper.

Miao, Jianjun and Pengfei Wang, "Asset bubbles and credit constraints," *The American Economic Review*, 2018, *108* (9), 2590–2628.

Peck, James and Karl Shell, "Market uncertainty: correlated and sunspot equilibria in imperfectly competitive economies," *The Review of Economic Studies*, 1991, *58* (5), 1011–1029.

Piketty, Thomas, "The dynamics of the wealth distribution and the interest rate with credit rationing," *The Review of Economic Studies*, 1997, *64* (2), 173–189.

Tella, Sebastian Di, "Uncertainty shocks and balance sheet recessions," *Journal of Political Economy*, 2017, *125* (6), 2038–2081.

Zentefis, Alexander, "Self-fulfilling asset prices," 2020. Unpublished working paper. Yale University, New Haven, CT.

Appendix

A Proofs for Section 2

Proof of Lemma 1. We omit the proof, because it follows standard arguments, verification of optimality and market clearing, as in Brunnermeier and Sannikov (2014). □

Proof of Lemma 2. Suppose $\kappa = 1$, $q = a_e/\bar{\rho}$, and $\sigma_q = 0$. Both (7) and (9) are satisfied. Furthermore, $\sigma_\eta = 0$ by (11), which confirms that $\sigma_q = \frac{q'}{q}\sigma_\eta = 0$. Thus, (10) is also satisfied. By Lemma 1, if we set the drift μ_η by (12), this constitutes an equilibrium. □

Proof of Proposition 1. This is a direct consequence of Proposition 2, by taking $\kappa_0 \to 0$. □

Proof of Lemma 3. We need only show that q is a function of η, i.e., $q_t = q(\eta_t)$. In such case, the dynamics of q_t are given by $\dot{q}_t = q'(\eta_t)\dot{\eta}_t$, which only depends on η and not q. Consequently, the linearized system near steady state takes the form

$$\begin{bmatrix} \dot{\eta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ m_2 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ q \end{bmatrix}$$

for $m_1, m_2 \neq 0$. The eigenvalues of this system are m_1 and 0, with $m_1 < 0$.

To show that $q_t = q(\eta_t)$, first note that goods market clearing is still characterized by (7). Next, equation (9) is now modified to read

$$0 = \min\left[1 - \kappa,\ \eta(1-\eta)\frac{a_e - a_h}{q} - (\kappa - \eta)\tilde{\sigma}^2\right].$$

The solution to this system can be computed explicitly. Indeed, define

$$\eta^* := \sup\{\eta : (a_e - a_h)\eta\bar{\rho}(\eta) = a_e\tilde{\sigma}^2\}.$$

Then, $\kappa = 1$ for all $\eta \in (\eta^*, 1)$. For $\eta \in (0, \eta^*)$, $\kappa < 1$ is computed as the positive root $\tilde{\kappa}$ from the quadratic equation

$$0 = (a_e - a_h)\tilde{\kappa}^2 + [a_h - \eta(a_e - a_h)]\tilde{\kappa} - \eta a_h - \frac{\eta(1-\eta)(a_e - a_h)\bar{\rho}(\eta)}{\tilde{\sigma}^2}.$$

After determining κ for all values of η, capital price q can be computed from (7), as an explicit function of η. □

Proof of Proposition 2. In the first step, we prove existence of an equilibrium for fixed $\kappa_0 \in (0,1)$. In the second step, we take the limits as $\kappa_0 \to 0$ and $\kappa_0 \to 1$.

Step 1: Existence. Let $F(x,y) := \frac{a_e - a_h}{y\bar{\rho}(x) - xa_e - (1-x)a_h} y$. Fix $\epsilon > 0$. Consider the initial value problem $y' = F(x,y)$, with $y(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h$. As discussed in the text, $y'(0+)$ is bounded, which is enough to ensure that F is bounded and uniformly Lipschitz on $\mathcal{R} := \{(x,y) : 0 < x < 1,\ xa_e + (1-x)a_h < y\bar{\rho}(x)\}$. Thus, the standard Picard-Lidelöf theorem implies that there exists a unique solution q^* to this initial value problem, for $\eta \in (0,b)$, some b. Standard continuation arguments can be used to show that either (i) $b = 1$, (ii) $q^*(\eta)$ is unbounded as $\eta \to b$, or (iii) b satisfies $ba_e + (1-b)a_h = q^*(b)\bar{\rho}(b)$. If case (ii) is true, since $F > 0$ on $\mathcal{R}$, we will in fact have $q^*(b-) = +\infty$. Case (iii) is ruled out by the fact that $F(b-, q^*(b-)) = +\infty$. We are left with cases (i) or (ii).

In case (i), we will set $\eta^* = \inf\{\eta \in (0,1) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that $\eta^* < 1$ in this case: otherwise $q^*(1-)\bar{\rho}(1-) < a_e$, which implies $F(1-, q^*(1-)) < 0$, which by continuity of q^* and F implies an $\eta^\circ \in (0,1)$ such that $\eta^\circ a_e + (1-\eta^\circ)a_h = q^*(\eta^\circ)$, which was just ruled out (case (iii)). In case (ii), we will set $\eta^* = \inf\{\eta \in (0,b) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that we also clearly have $\eta^* < b < 1$ in this case.

Finally, set $q(\eta) = \mathbf{1}_{\eta<\eta^*} q^*(\eta) + \mathbf{1}_{\eta \geq \eta^*} a_e/\bar{\rho}(\eta)$. This function satisfies $q' = F(\eta, q)$ on $(0,\eta^*)$, $q(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h$, and $q(\eta^*) = a_e/\bar{\rho}(\eta)$. Thus, we have found a solution to the capital price satisfying all the desired relations. As discussed in the text, finding such a capital price is enough to prove that a Markov sunspot equilibrium exists.

Since equation (14) implies $\sigma_q^2 > 0$ on $(0,\eta^*)$, in order to establish $\sigma_q(\eta) \neq 0$ on a positive measure subset, it suffices to show that $\eta^* > 0$. But this is automatically implied by the boundary condition $q(0) = (\kappa_0 a_e + (1-\kappa_0)a_h)/\rho_h < a_e/\rho_h$ for $\kappa_0 < 1$, coupled with the continuity of the solution $q(\eta)$.

Step 2: BSE and FE as limiting equilibria. For each initial condition $\kappa(0) = \kappa_0$, let $(q_{\kappa_0}, \eta^*_{\kappa_0})$ be the associated equilibrium capital price and misallocation threshold (at which point households begin purchasing capital).

Define the candidate solution for the BSE, $(q_0, \eta_0^*) := \lim_{\kappa_0 \to 0}(q_{\kappa_0}, \eta^*_{\kappa_0})$. It suffices to show that q_0 satisfies (i) $q_0' = F(\eta, q_0)$ on $(0, \eta_0^*)$, (ii) $q_0(0) = a_h/\rho_h$, and (iii) $q_0(\eta_0^*) = a_e/\bar{\rho}(\eta_0^*)$. Write the integral version of the ODE:

$$q_{\kappa_0}(\eta) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} + \int_0^\eta F(x, q_{\kappa_0}(x))dx.$$

Next, we claim that $q_{\kappa_0}(x)$ is weakly increasing in κ_0, for each x. Indeed, $q_{\kappa_0}(0)$ is strictly increasing in κ_0. By continuity, we may consider $x^* := \inf\{x : q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)\}$ for some $\tilde{\kappa}_0 > \kappa_0$. In that case, since F does not depend on $\tilde{\kappa}_0$ or κ_0, we have $q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)$ for all $x \geq x^*$. This proves $q_{\tilde{\kappa}_0}(x) \geq q_{\kappa_0}(x)$ for all x. Combine this with the fact that $\partial_q F < 0$ to see that $\{F(x, q_{\kappa_0}(x)) : \kappa_0 \in (0,1)\}$ is a sequence which is monotonically (weakly) decreasing in κ_0, for each x. Thus, by the monotone convergence theorem,

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F(x, q_0(x))dx,$$

which proves (i), by differentiating, and (ii), by substituting $\eta = 0$. Similarly,

$$\begin{aligned} q_{\kappa_0}(\eta^*_{\kappa_0}) &= \frac{a_e}{\bar{\rho}(\eta^*_{\kappa_0})} \\ \xrightarrow{\kappa_0 \to 0} q_0(\eta^*_0) &= \frac{a_e}{\bar{\rho}(\eta^*_0)}, \end{aligned}$$

which proves (iii).

Define the candidate solution for the FE, $(q_1, \eta_1^*) := \lim_{\kappa_0 \to 1}(q_{\kappa_0}, \eta^*_{\kappa_0})$. It suffices to show that $\eta_1^* = 0$, so that $q_1(\eta) = a_e/\bar{\rho}(\eta)$ for all η. Note that $q_{\kappa_0}(0) \to a_e/\rho_h$ as $\kappa_0 \to 1$. By continuity of $(q_{\kappa_0}, \eta^*_{\kappa_0})$ in κ_0, we also have $q_{\kappa_0}(0) \to q_1(0)$ as $\kappa_0 \to 1$. Thus, $q_1(0) = a_e/\rho_h$. By the definition of $\eta_1^* = \inf\{\eta : q(\eta) = a_e/\bar{\rho}(\eta)\}$, we must have $\eta_1^* = 0$. □

A.1 Stationarity of sunspot equilibria

Lemma A.1. *In the model with no fundamental volatility, the dynamics prevent η from reaching zero with probability one. Moreover, if one of the following conditions holds*

(P1) $\delta = 0$ *and* $\rho_e > \rho_h$

(P2) $\delta > 0$ *and* $\nu < \eta^* \equiv \min\{\eta : \kappa(\eta) = 1\}$ *(equivalent to* $\nu < \frac{1}{2}(1-\kappa_0^2)\frac{a_e - a_h}{a_e}$ *when* $\rho_e = \rho_h$*)*

then expert wealth share $(\eta_t)_{t\geq 0}$ *has a non-degenerate stationary distribution on* $(0, \eta^*]$*, and when* $\eta_t \in (\eta^*, 1)$*, it follows a deterministic path towards* η^**.*

Proof of Lemma A.1. We consider the baseline model of Section 2 with disaster belief $\kappa(0+) = \kappa_0 \in [0, 1)$. For reference, we re-state the dynamics of η:

$$\mu_\eta = \delta\nu - (\rho_e - \rho_h + \delta)\,\eta + \frac{a_e - a_h}{q}[\kappa - 2\kappa\eta + \eta^2]\mathbf{1}_{\eta<\eta^*} + (\rho_e - \rho_h)\,\eta^2 \tag{A.1}$$

$$\sigma_\eta^2 = \eta(1-\eta)(\kappa - \eta)\frac{a_e - a_h}{q}\mathbf{1}_{\eta<\eta^*}, \tag{A.2}$$

where equation (A.2) follows from $\sigma_\eta = (\kappa - \eta)\sigma_q$ in (10) and $\sigma_q^2 = \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e - a_h}{q}\mathbf{1}_{\eta<\eta^*}$ in (14). We proceed in 3 steps, examining dynamics of η above η^*, in a neighborhood just below η^*, and in a neighborhood just above 0.

Step 1: Dynamics for $\eta > \eta^*$. Equation (A.2) shows that $\sigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. Thus, η it follows a deterministic path towards η^* if $\mu_\eta(\eta) < 0$ for all $\eta \in [\eta^*, 1)$. Substituting $\kappa = 1$ into (A.1) and using either of parameter restriction (P1) or (P2) above delivers the result immediately. Given the deterministic transition toward η^*, we can ignore the sub-interval $(\eta^*, 1)$ in our state space and instead consider only $(0, \eta^*)$.

In general, consider a one-dimensional process $(X_t)_{t\geq 0}$ with $dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t$ that is a regular diffusion on interval $(e_1, e_2) \subset \mathbb{R}$ (i.e., the dynamics of X depend only on X itself, and imply that it reaches every point in (e_1, e_2) with positive probability). Our process $(\eta_t)_{t\geq 0}$ satisfies these conditions for $e_1 = 0$ and $e_2 = \eta^*$.

In such case, we may apply Feller's boundary classification to decide whether boundaries e_1 and e_2 are inaccessible (avoided forever with probability 1) or accessible. To do so, first define

$s(y) := \exp(-\int_{x_0}^{y} \frac{2\mu(u)}{\sigma^2(u)} du)$, $m(x) := \frac{2}{s(x)\sigma^2(x)}$, and let ϵ and x_0 be arbitrary numbers within interval (e_1, e_2). Boundary e_1 is inaccessible if and only if

$$I_1 := \int_{e_1}^{\epsilon} m(x)\Big(\int_{e_1}^{x} s(y)dy\Big)dx = +\infty.$$

Boundary e_2 is accessible if and only if

$$I_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{x}^{e_2} s(y)dy\Big)dx < +\infty.$$

We will prove these results in the next two steps.

Step 2: Dynamics near $e_2 = \eta^*$. Compute

$$\mu_\eta(\eta^*-) = \delta(\nu - \eta^*) - \eta^*(1-\eta^*)(\rho_e - \rho_h) + (1-\eta^*)\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}$$
$$\sigma_\eta^2(\eta^*-) = \eta^*(1-\eta^*)^2\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}.$$

Since $\sigma_\eta^2(\eta^*-)$ is bounded away from zero and $\mu_\eta(\eta^*-)$ is finite, it is easy to check that $I_2 < +\infty$, meaning $e_2 = \eta^*$ is an accessible boundary that is hit in finite time with positive probability. Furthermore, we may also show

$$J_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{\epsilon}^{x} s(y)dy\Big)dx < +\infty,$$

which implies $e_2 = \eta^*$ is a regular boundary that must be included in the state space.

We must establish what occurs when η_t hits boundary point $e_2 = \eta^*$. Recall from step 1 that $\mu_\eta(\eta) < 0$ and $\sigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. This implies that η_t can never enter the region $(\eta^*, 1)$ from η^* and that η_t will not stay at point η^* for an infinite amount of time. Consequently, the region $(0, \eta^*]$ is the ergodic set.

Step 3a: General analysis of dynamics near $e_1 = 0$. First, suppose our diffusion satisfied the following near $e_1 = 0$ (the notation $f(x) \sim g(x)$ means $\lim_{x\to 0} f(x)/g(x) = 1$):

$$\sigma^2(x) \sim \phi x^\beta \quad \phi > 0, \quad \beta \geq 0$$
$$\frac{\mu(x)}{\sigma^2(x)} \sim \theta x^{-\alpha}, \quad \alpha \geq 1, \quad \theta > 0.$$

As we will show below in step 3b, this asymptotic description is flexible enough to cover all cases within our model.

If $\alpha = 1$, we have, for x sufficiently small,

$$S_1(x,\theta) := \int_0^x \frac{s(y)}{s(x)}dy = \int_0^x \exp\Big[2\theta(\log(x) - \log(y))\Big]dy = x^{2\theta}\lim_{z\downarrow 0}\frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta}, \tag{A.3}$$

so letting ϵ be sufficiently small, we obtain

$$I_1 = \int_0^{\epsilon} \frac{2x^{2\theta-\beta}}{\phi}\lim_{z\downarrow 0}\frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta}dx.$$

If $2\theta \geq 1$ (note that $2\theta = 1$ corresponds to $\frac{z^{1-2\theta}}{1-2\theta}$ being replaced by $\log(z)$ in the expression above), then the interior limit is $+\infty$ for all $x > 0$ and therefore $I_1 = +\infty$. This holds independently of the value of β. If $2\theta < 1$, then

$$I_1 = \int_0^{\epsilon} \frac{2}{(1-2\theta)\phi} x^{1-\beta} dx = \frac{2}{(1-2\theta)\phi} \Big(\frac{\epsilon^{2-\beta}}{2-\beta} - \lim_{x \downarrow 0} \frac{x^{2-\beta}}{2-\beta} \Big).$$

So, in this case, $I_1 = +\infty$ only if $\beta \geq 2$ (for $\beta = 2$, $\frac{x^{2-\beta}}{2-\beta}$ is replaced by $\log(x)$).

If $\alpha > 1$ instead, we will show that $I_1 = +\infty$ independent of any other parameters. We have

$$S_\alpha(x,\theta) := \int_0^x \frac{s(y)}{s(x)} dy = \int_0^x \exp\left[\frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha}) \right] dy \tag{A.4}$$

The corresponding expression for the case with $\alpha = 1$ is $S_1(x,\theta)$ in (A.3). We showed above that for $\tau < 1/2$, we have $S_1(x,\tau) = +\infty$. Fix such a τ. We now show that $S_\alpha(x,\theta) \geq S_1(x,\tau)$ for all x sufficiently small and all θ.

Fix any $x > 0$, and define $f(y) := 2\tau(\log(x) - \log(y))$ and $g(y) := \frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})$. Since both functions are strictly positive for $y < x$, and since $\lim_{y \to 0} g(y)/f(y) = \lim_{y \to 0}(\theta/\tau) y^{1-\alpha} = +\infty$, there exists $\bar{y} \in (0,x)$ such that $g(y) > f(y)$ for all $y \in (0,\bar{y})$. From this comparison, we conclude $S_\alpha(\bar{y},\theta) = \int_0^{\bar{y}} \exp(g(y))dy \geq \int_0^{\bar{y}} \exp(f(y))dy = S_1(\bar{y},\tau) = +\infty$. Since this argument is independent of (β,θ,ϕ), this proves that $I_1 = +\infty$ if $\alpha > 1$.

Step 3b: Model-specific analysis of dynamics near $e_1 = 0$. Now, we map our model dynamics into the setup of step 3a. If $\kappa(0+) = \kappa_0 > 0$, then in the limit as $\eta \to 0$, equations (A.1)-(A.2) become

$$\begin{aligned}
\mu_\eta &= \delta\nu + \frac{a_e - a_h}{q(0+)}\kappa_0 - \left(\rho_e - \rho_h + \delta + 2\frac{a_e - a_h}{q(0+)}\kappa_0 \right) \eta + o(\eta) \\
\sigma_\eta^2 &= \frac{a_e - a_h}{q(0+)}\kappa_0 \eta + o(\eta).
\end{aligned}$$

Hence, in terms of the notation in step 3a, we have $\alpha = 1$, $\beta = 1$ and $\theta = \frac{\delta\nu q(0+)}{\kappa_0(a_e - a_h)} + 1 > \frac{1}{2}$. Thus, η avoids zero with probability one.

If $\kappa(0+) = 0$, we need to know the rate at which $\kappa \to 0$ as $\eta \to 0$. Guess, and verify after, that $\kappa = \varphi\eta^\omega + o(\eta^\omega)$ in the limit as $\eta \to 0$. Differentiating goods market clearing (7), we have

$$q' = \frac{1}{\bar{\rho}} \left[(a_e - a_h)\kappa' - (\rho_e - \rho_h)q \right]$$

Combining this with the sunspot differential equation for q, equation (11), we obtain

$$\left[(a_e - a_h)\kappa' - (\rho_e - \rho_h)q \right] (\kappa - \eta) = \bar{\rho} q.$$

Taking the limit as $\eta \to 0$, we have

$$(a_e - a_h) \lim_{\eta \to 0} (\kappa')(\kappa - \eta) = a_h$$

Hence, the guess is verified if $\omega = 1/2$ and $\varphi^2 = 2a_h/(a_e - a_h) > 0$. Substituting this asymptotic behavior into equations (A.1)-(A.2), we have

$$\begin{aligned}
\mu_\eta &= \delta\nu + \sqrt{\frac{2(a_e - a_h)}{a_h}} \rho_h \eta^{1/2} + o(\eta^{1/2}) \\
\sigma_\eta^2 &= \sqrt{\frac{2(a_e - a_h)}{a_h}} \rho_h \eta^{3/2} + o(\eta^{3/2}).
\end{aligned}$$

If $\delta\nu > 0$, then these dynamics match those of step 3a with $\alpha = 3/2$, $\beta = 3/2$ and $\theta > 0$. If $\delta\nu = 0$, then these dynamics match step 3a with $\alpha = 1$, $\beta = 3/2$, and $\theta = 1$. In either case, we have already shown that η cannot reach zero with probability one.

In summary, $(\eta_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution with support $(0, \eta^*]$, the boundary $\{0\}$ is inaccessible, and the boundary η^* is accessible but non-absorbing. $\square$

B Proofs for Section 3

Proof of Proposition 3. Note that the other equations characterizing equilibrium, beyond (15) are (7) and

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\right]. \tag{B.1}$$

Denote the equilibrium solution for $\sigma > 0$ by $(q^{(\sigma)}, \kappa^{(\sigma)})$. Define $q^{(0)} := \lim_{\sigma\to 0} q^{(\sigma)}$ and $\kappa^{(0)} := \lim_{\sigma\to 0} \kappa^{(\sigma)}$. Combine equations (15) and (B.1) and rearrange terms to get

$$\left(1 - (\kappa^{(\sigma)} - \eta)\frac{(q^{(\sigma)})'}{q^{(\sigma)}}\right)^2 = \frac{(\kappa^{(\sigma)} - \eta)q^{(\sigma)}}{\eta(1-\eta)(a_e - a_h)}\sigma, \quad \text{if} \quad \kappa^{(\sigma)} < 1. \tag{B.2}$$

Note that this implies $\kappa^{(\sigma)} > \eta$. Furthermore, continuity of $\kappa^{(\sigma)}(\eta)$ and $\kappa_0 = \kappa^{(\sigma)}(0+) < 1$ imply $\kappa^{(\sigma)}(\eta) < 1$ for all η close enough to 0. Using these facts, and writing (B.2) instead as an integral equation, we obtain

$$\frac{q^{(\sigma)}(\eta_2)}{q^{(\sigma)}(\eta_1)} = \exp\left\{\int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(\sigma)}(x) - x}\left[1 \pm \sqrt{\frac{(\kappa^{(\sigma)}(x) - x)q^{(\sigma)}(x)}{x(1-x)(a_e - a_h)}\sigma}\right]dx\right\}, \quad 0 < \eta_1 < \eta_2,$$

where η_2 is chosen small enough. Because the right-hand-side is continuous in both $q^{(\sigma)}$ and $\kappa^{(\sigma)}$, and both are bounded, taking the limit as $\sigma \to 0$ implies

$$\frac{q^{(0)}(\eta_2)}{q^{(0)}(\eta_1)} = \exp\left\{\int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(0)}(x) - x} dx\right\}.$$

Differentiate this equation with respect to η_2 to obtain

$$\frac{d}{d\eta}\log q^{(0)} = \frac{1}{\kappa^{(0)} - \eta},$$

for all η small enough. Rearranging this equation delivers the ODE characterizing the BSE, i.e., selecting the solution $(\kappa - \eta)q'/q = 1$ in equation (11). Since $\kappa^{(\sigma)}(0+) = \kappa_0$ is fixed for all $\sigma > 0$, we also have the desired boundary condition $\kappa^{(0)}(0+) = \kappa_0$, for any $\kappa_0 \in [0, 1)$. Finally, all the other equations of the BSE can be verified by simply taking limits as $\sigma \to 0$. $\square$

Proof of Proposition 4. In the text leading up to the statement of the proposition. $\square$

Proof of Lemma 4. Fix any $\kappa_0 \in [0,1)$. Let $(q, \kappa, \sigma_q, \sigma_\eta)$ denote the relevant quantities for the non-sunspot Markov equilibrium with $\sigma > 0$ and disaster belief κ_0 (assuming such equilibrium exists). Let $(q^{\text{BSE}}, \kappa^{\text{BSE}}, \sigma_q^{\text{BSE}}, \sigma_\eta^{\text{BSE}})$ denote the same quantities for the Markov sunspot equilibrium with $\sigma = 0$ (this is a BSE with disaster belief κ_0). Let $\eta^* := \inf\{\eta : \kappa^{\text{BSE}} = 1\}$.

Define

$$
\begin{aligned}
F_{\text{BSE}}(y,x) &:= \frac{(a_e - a_h)y}{y\bar{\rho}(x) - xa_e - (1-x)a_h} \\
F(y,x) &:= F_{\text{BSE}}(y,x) - \sigma\Big(\frac{x(1-x)}{y}[\bar{\rho}(x)y - xa_e - (1-x)a_h]\Big)^{-1/2}.
\end{aligned}
$$

We re-state the ODEs that q^{BSE} and q must satisfy, which are (13) and (B.5), respectively:

$$
\frac{d}{d\eta}q^{\text{BSE}} = F_{\text{BSE}}(q^{\text{BSE}}, \eta) \quad \text{and} \quad \frac{d}{d\eta}q = F(q, \eta).
$$

The initial condition for both is the same:

$$
q^{\text{BSE}}(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} = q(0).
$$

First, we prove that $\frac{d}{d\eta}q^{\text{BSE}}(0+) > \frac{d}{d\eta}q(0+)$. This is done separately for the case $\kappa_0 = 0$ and $\kappa_0 > 0$. If $\kappa_0 = 0$, then Lemma B.1 part (iii) shows $\frac{d}{d\eta}q(0+)$ is finite, whereas $\frac{d}{d\eta}q^{\text{BSE}}(0+) = +\infty$ can be obtained by the limit $F_{\text{BSE}}(q^{\text{BSE}}(0+), 0+) = +\infty$. If $\kappa_0 > 0$, then we can take the limit $F(q(0+), 0+) = -\infty$. Conversely, taking the limit $F_{\text{BSE}}(q^{\text{BSE}}(0+), 0+)$ shows that $\frac{d}{d\eta}q^{\text{BSE}}(0+)$ is finite.

Next, we prove that $\frac{d}{d\eta}q^{\text{BSE}} > \frac{d}{d\eta}q$ for each $\eta \in (0, \eta^*)$. Since $F_{\text{BSE}}(y,x) > F(y,x)$ for fixed (y,x), and since $q^{\text{BSE}} > q$ in a neighborhood of zero (by the previous result), it suffices to show that F_{BSE} is increasing in y, which is true. Thus, $\frac{d}{d\eta}q^{\text{BSE}} > \frac{d}{d\eta}q$ on the set of points η such that both ODEs (13) and (B.5) are satisfied. But this also shows that q^{BSE} hits $a_e/\bar{\rho}$ at a lower value of η than does q, so the aforementioned set of points is exactly $(0, \eta^*)$. Note also that this implies $q^{\text{BSE}} > q$ on $(0, \eta^*)$.

Finally, consider the trading functions corresponding to the BSE and non-sunspot equilibrium: $\tau(\eta; 0, \kappa_0)$ and $\tau(\eta; \sigma, \kappa_0)$, respectively. These are mathematically defined as the loadings of κ^{BSE} and κ on shocks. By Itô's formula, we thus have

$$
\tau(\eta; 0, \kappa_0)^2 = (\sigma_\eta^{\text{BSE}}(\eta))^2 (\frac{d}{d\eta}\kappa^{\text{BSE}}(\eta))^2 \tag{B.3}
$$

$$
\tau(\eta; \sigma, \kappa_0)^2 = \sigma_\eta(\eta)^2 (\frac{d}{d\eta}\kappa(\eta))^2. \tag{B.4}
$$

Note that goods market clearing (7) implies $\frac{d}{d\eta}\kappa^{\text{BSE}}(\eta) > \frac{d}{d\eta}\kappa(\eta)$.

Using equation (B.1), which holds in both models, we see that return variances are given by the following:

$$
(\sigma_q^{\text{BSE}})^2 = \frac{a_e - a_h}{q^{\text{BSE}}} \frac{\eta(1-\eta)}{\kappa^{\text{BSE}} - \eta} \quad \text{and} \quad (\sigma + \sigma_q)^2 = \frac{a_e - a_h}{q} \frac{\eta(1-\eta)}{\kappa - \eta}, \quad \text{for } \eta \in (0, \eta^*)
$$

Computing $(\sigma_\eta^{\text{BSE}})^2 = (\kappa^{\text{BSE}} - \eta)^2 (\sigma_q^{\text{BSE}})^2$ and $\sigma_\eta^2 = (\kappa - \eta)^2(\sigma + \sigma_q)^2$, and then using previous results as well as goods market clearing (7), we obtain

$$(\sigma_\eta^{\text{BSE}})^2 = \eta(1-\eta)\frac{\bar{\rho} q^{\text{BSE}} - \eta a_e - (1-\eta)a_h}{q^{\text{BSE}}}$$
$$> \eta(1-\eta)\frac{\bar{\rho} q - \eta a_e - (1-\eta)a_h}{q} = \sigma_\eta^2, \quad \text{for } \eta \in (0, \eta^*).$$

Consequently, using this result in (B.3) and (B.4), we obtain

$$\tau(\eta; 0, \kappa_0)^2 > \tau(\eta; \sigma, \kappa_0)^2.$$

Taking square root delivers the result since $\tau(\eta; 0, \kappa_0) > 0$. □

B.1 Properties of the non-sunspot solution with fundamental volatility

We describe here some properties of non-sunspot equilibria with fundamental volatility $\sigma > 0$. These are used extensively in the construction provided in the proof of Proposition 5.

Lemma B.1. *Assuming it exists, suppose (q, κ) are a non-sunspot equilibrium in η in the sense described just after Definition 1. Assume $\kappa(0+) = 0$. Define $\eta^* := \inf\{\eta : \kappa = 1\}$. Then, the following hold:*

(i) $(\bar{\rho} q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho} q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}$*, for all* $\eta \in (0, \eta^*)$.

(ii) $\eta a_e + (1-\eta)a_h < \bar{\rho} q < a_e$*, for all* $\eta \in (0, \eta^*)$.

(iii) $\frac{q'(0+)}{q(0+)} = \frac{a_e}{a_h} - \frac{\rho_e}{\rho_h} + \rho_h\big(\frac{a_e - a_h}{\sigma a_h}\big)^2$.

(iv) If σ is sufficiently small, then $q' > \frac{a_e - a_h}{\bar{\rho}}$*, for* $\eta \in (0, \eta^*)$.

(v) If σ is sufficiently small, then $\frac{\rho_h}{\rho_e}\big(\frac{1 - a_h/a_e}{\sigma^2} - 1 + \frac{\rho_h}{\rho_e}\big)^{-1} < \eta^* < 1$.

(vi) On $\eta \in (0, \eta^*)$, the solution q is infinitely-differentiable.

Proof of Lemma B.1. Since a non-sunspot equilibrium is assumed to exist, we make use of equations (7) and (B.1). Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1-\eta)a_h$.

(i) Start from equation (B.1), and rearrange to obtain the result, where we have implicitly selected the solution with $1 > (\kappa - \eta)\frac{q'}{q}$.

(ii) The first inequality, which is equivalent to $\kappa > \eta$, is a direct implication of equation (B.1). The second inequality, equivalent to $\kappa < 1$, is a restatement of the definition of η^*.

(iii) Start from equation (B.1). Taking the limit $\eta \to 0$, and using $\kappa(0+) = 0$, delivers an equation for $\kappa'(0+)$. Differentiating (7), we may then substitute $\kappa'(0+) = \frac{\rho_h q'(0+) + (\rho_e - \rho_h)q(0+)}{a_e - a_h}$. Rearranging, we obtain the desired result.

(iv) By part (iii), there exists $\eta^\circ > 0$ and $\bar{\sigma} > 0$ such that uniformly for all $\sigma < \bar{\sigma}$, we have $q' > \frac{a_e - a_h}{\bar{\rho}}$ on the set $\{\eta < \eta^\circ\}$. On the set $\{\eta^\circ \leq \eta < \eta^*\}$, we know that $\kappa - \eta$ is bounded away from zero, uniformly for all $\sigma < \bar{\sigma}$. Using the expression in part (i), the fact that q is bounded by $a_e/\bar{\rho}$ uniformly for all σ, and the previous fact about $\kappa - \eta = \bar{\rho}q - \bar{a}$, we can write

$$q' = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}} q - o(\sigma), \quad \eta \in (\eta^\circ, \eta^*).$$

Therefore,

$$q' + o(\sigma) = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}} q = \frac{a_e - a_h}{\bar{\rho}} \frac{q}{q - \bar{a}/\bar{\rho}} > \frac{a_e - a_h}{\bar{\rho}}, \quad \eta \in (\eta^\circ, \eta^*),$$

where the last inequality is due to $\bar{\rho}q > \bar{a}$ [part (ii)]. Taking σ is small enough implies the result on (η°, η^*), which we combine with the result on $(0, \eta^\circ)$ to conclude.

(v) Consider the function $\tilde{q} := \bar{a}/\bar{\rho}$, whose derivative is $\tilde{q}' = \frac{a_e - a_h}{\bar{\rho}} - \frac{\bar{a}}{\bar{\rho}} \frac{\rho_e - \rho_h}{\bar{\rho}} < \frac{a_e - a_h}{\bar{\rho}}$. Combining this result with part (iv), we obtain $q' > \tilde{q}'$. If $\tilde{q}$ was the capital price, then equation (7) implies the associated capital share $\tilde{\kappa} = \eta$. On the other hand, the fact that $q' > \tilde{q}'$ implies $\kappa' > \tilde{\kappa}' = 1$, which implies $\eta^* < 1$.

Next, consider $\eta \in (\eta^*, 1)$ so that $\kappa = 1$. By equation (B.1), with $q = a_e/\bar{\rho}$, we must have

$$\sigma^2 \leq \eta\bar{\rho}\frac{a_e - a_h}{a_e}\Big(1 + (1-\eta)\frac{\rho_e - \rho_h}{\bar{\rho}}\Big)^2, \quad \eta \geq \eta^*.$$

This is equivalent to

$$1 \leq \eta\frac{\rho_e}{\rho_h}\Big(\frac{a_e - a_h}{a_e\sigma^2}\rho_e - 1 + \frac{\rho_h}{\rho_e}\Big), \quad \eta \geq \eta^*.$$

Substituting $\eta = \eta^*$, and rearranging, we obtain the second inequality. There is no contradiction with $\eta^* < 1$, due to the assumption that σ can be made small enough.

(vi) Note that $F(\eta, q) := q[\frac{a_e - a_h}{\bar{\rho}(\eta)q - \bar{a}(\eta)} - \sigma(\frac{\eta(1-\eta)(\bar{\rho}(\eta)q - \bar{a}(\eta))}{q})]$ is infinitely differentiable in both arguments on $\{(\eta, q) : \eta \in (0,1), \bar{\rho}(\eta)q > \bar{a}(\eta)\}$. Thus, the result is a simple consequence of differentiating part (i), noting that by part (ii) we have $\bar{\rho}(\eta)q(\eta) > \bar{a}(\eta)$, and then using induction. □

It is not necessary to impose the condition $\kappa(0+) = 0$ to have a non-sunspot equilibrium. If we let $\kappa_0 \in (0,1)$ be a given "disaster belief" about experts' deleveraging, and we suppose $\kappa(0+) = \kappa_0$, there is no inherent contradiction to equilibrium. Existence of such an equilibrium boils down simply to existence of a solution to a first-order ODE. Thus, a variety of non-sunspot equilibria could exist, and indeed we provide a numerical example after the following lemma and proof.

Lemma B.2. *A non-sunspot equilibrium with disaster belief $\kappa_0 \in (0,1)$ exists if the free boundary problem*

$$(\bar{\rho}q - \eta a_e - (1-\eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1-\eta)a_h}{\eta(1-\eta)}}, \quad \textit{on} \quad \eta \in (0, \eta^*), \tag{B.5}$$

$$\textit{subject to} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h} \quad \textit{and} \quad q(\eta^*) = \frac{a_e}{\bar{\rho}(\eta^*)}, \tag{B.6}$$

has a solution.

Proof of Lemma B.2. A non-sunspot equilibrium in state variable η exists if and only if equations (7), (15), and (B.1) hold, and if the time-paths $(\eta_t)_{t\geq 0}$ induced by dynamics (σ_η, μ_η) avoid $\eta = 0$ almost-surely. We will demonstrate these conditions.

Suppose (B.5)-(B.6) has a solution (q, η^*) corresponding to $\kappa_0 \in (0,1)$. If there are multiple solutions, we pick the one such that $q(\eta) < a_e/\bar{\rho}(\eta)$ for all $\eta \in (0, \eta^*)$, which is always possible because the boundary conditions (B.6) imply $\bar{\rho}(0)q(0) < \bar{\rho}(\eta^*)q(\eta^*)$. Set $q(\eta) = a_e/\bar{\rho}(\eta)$ for all $\eta \geq \eta^*$. Define $\kappa = \frac{\bar{\rho}q - a_h}{a_e - a_h}$. Note that (7) is automatically satisfied. Note that (15) is also satisfied automatically, by applying Itô's formula to the solution $q(\eta)$ and using $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$.

We show (B.1) holds separately on $(0, \eta^*)$ and $[\eta^*, 1)$. Using (7) and (15) in the ODE (B.5) and rearranging, we show that (B.1) holds when $\kappa < 1$. The boundary condition $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$ is equivalent to $\kappa(\eta^*) = 1$, which shows that $\kappa(\eta) < 1$ for all $\eta < \eta^*$. Therefore, we have proven that (B.1) holds on $(0, \eta^*)$.

If $\eta^* = 1$, then we are done verifying (B.1); otherwise, we need to verify (B.1) on $[\eta^*, 1)$. On this set, $\kappa = 1$, so we need to verify

$$\eta \frac{a_e - a_h}{q} \geq (\sigma + \sigma_q)^2 \quad \text{for all} \quad \eta \geq \eta^*. \tag{B.7}$$

First, we show that it suffices to verify this condition exactly at η^*. Indeed, on $(\eta^*, 1)$, we have $\kappa = 1$ and $q = a_e/\bar{\rho}$. Substituting these and (15) into (B.7), we obtain

$$\text{(B.7) holds} \;\Leftrightarrow\; \Big(\frac{a_e - a_h}{a_e \sigma^2}\rho_e - \frac{\rho_e - \rho_h}{\rho_e}\Big)\eta \geq \frac{\rho_h}{\rho_e} \quad \text{for all} \quad \eta \geq \eta^*.$$

But since the left-hand-side is increasing in η, if it holds at $\eta = \eta^*$, it holds for all $\eta > \eta^*$.

Now, writing (B.7) at η^*, using (15) to replace σ_q, and using ODE (B.5) to replace $\eta^* \frac{a_e - a_h}{q(\eta^*)} = \sigma[1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)]^{-1}$, we need to verify

$$\text{(B.7) holds} \;\Leftrightarrow\; \frac{\sigma}{1 - (1-\eta^*)q'(\eta^*-)/q(\eta^*)} \geq \frac{\sigma}{1 - (1-\eta)q'(\eta^*+)/q(\eta^*)} \;\Leftrightarrow\; q'(\eta^*-) \geq q'(\eta^*+).$$

We clearly have $q'(\eta^*-) \geq q'(\eta^*+)$ by the simple fact that $q < a_e/\bar{\rho}$ for $\eta < \eta^*$ and $q = a_e/\bar{\rho}$ for $\eta \geq \eta^*$.

Finally, it remains to very that η_t almost-surely never reaches the boundary 0. The dynamics are given by

$$\begin{aligned}
\mu_\eta &= -\eta(1-\eta)(\rho_e - \rho_h) + \mathbf{1}_{\{\kappa < 1\}}(\kappa - 2\kappa\eta + \eta^2)\frac{a_e - a_h}{q} + \delta(\nu - \eta) \\
\sigma_\eta^2 &= (\kappa - \eta)^2(\sigma + \sigma_q)^2 = \eta(1-\eta)(\kappa - \eta)\frac{a_e - a_h}{q}.
\end{aligned}$$

Near $\eta = 0$, these are

$$\begin{aligned}
\mu_\eta(\eta) &= \kappa_0 \frac{a_e - a_h}{q(0+)} + \delta\nu + o(\eta) \\
\sigma_\eta^2(\eta) &= \kappa_0 \frac{a_e - a_h}{q(0+)}\eta + o(\eta).
\end{aligned}$$

By the same analysis as in Lemma A.1, the boundary 0 is unattainable [this system corresponds to setting $\alpha = 1$ and $\theta = 1$ in part (3a) of the proof of Lemma A.1]. □

What happens in an equilibrium of Lemma B.2 in which $\kappa_0 > 0$? Behavior at the boundary $\eta = 0$ is substantially different than the $\kappa_0 = 0$ case, because equation (B.1) can only hold there if $\sigma_q \to -\sigma$ as $\eta \to 0$. Capital prices "hedge" fundamental shocks to capital, in a brief region of the state space $(0, \eta^{\text{hedge}})$. Said differently, given the formula (15), the fact that $\sigma_q(0+) = -\sigma$ implies $q'(0+) = -\infty$, so that prices rise as experts lose wealth in a region of the state space. The hedging region is exactly what incentivizes experts to take so much leverage (indeed, expert leverage κ/η blows up near 0). For $\eta > \eta^{\text{hedge}}$, this behavior reverses, and the equilibrium behaves very much like the equilibrium with $\kappa_0 = 0$. Overall, there is no inconsistency with equilibrium even though $q' < 0$ in the region $(0, \eta^{\text{hedge}})$.[20]

Figure B.1 displays several examples of equilibria with different choices of $\kappa_0 > 0$. The solid black lines, which are equilibrium outcomes with $\kappa_0 = 0.001$, corresponds approximately to the equilibrium choice made by Brunnermeier and Sannikov (2014). The other curves, with higher disaster beliefs κ_0, are new to the literature. Similar to the the sunspot results of Section 2.3, more optimistic disaster beliefs raise capital prices and reduce capital price volatility.

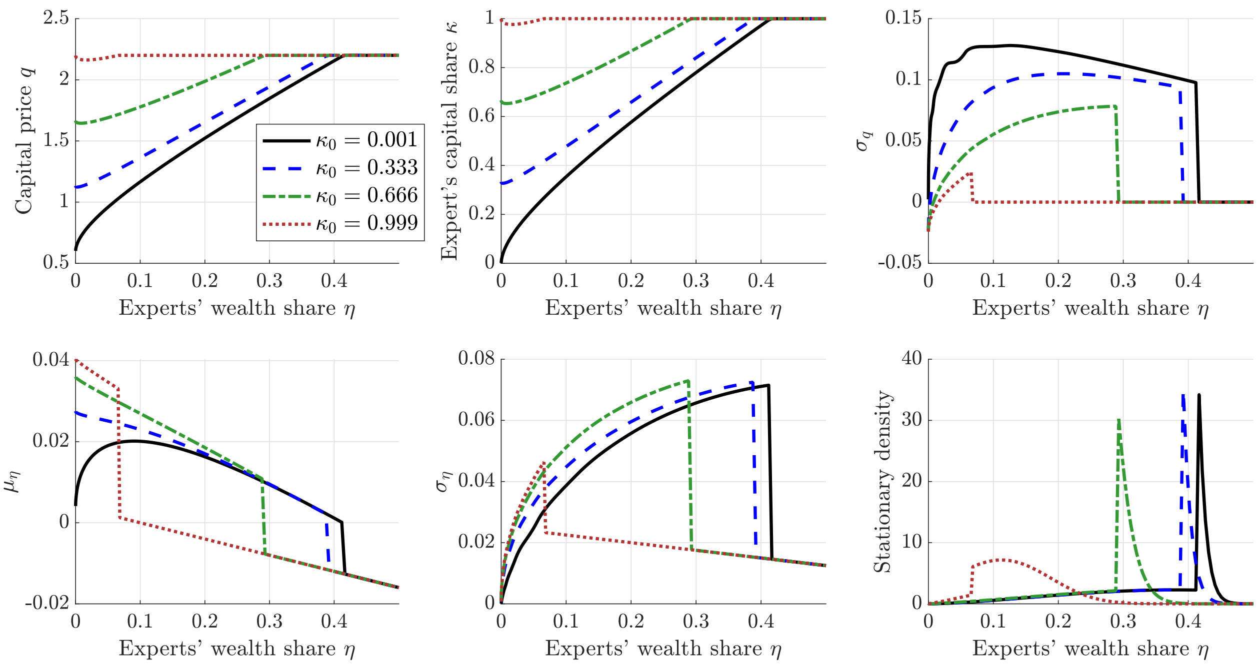

Figure B.1: Non-sunspot equilibria with different disaster beliefs κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.025$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$.

[20] One may think that $q'(0+) = -\infty$, and more generally that $q' < 0$ in some region of the state space, could imply that κ hits η at some point. However, this cannot happen. Indeed, since $\kappa_0 > 0$, we have that $q(0+) > \tilde{q}(0+)$, where $\tilde{q}(\eta) := ((a_e - a_h)\eta + a_h)/\bar{\rho}$ is the price function consistent with $\kappa = \eta$.

To see this, assume there is an $\hat{\eta} \in (0,1)$ such that $\kappa(\hat{\eta}) = \hat{\eta}$ (or equivalently, $q(\hat{\eta}) = \tilde{q}(\hat{\eta})$). If there is more than one, consider the minimum among them, so $q(\eta) > \tilde{q}(\eta)$ for all $\eta \in (0, \hat{\eta})$. From the $\tilde{q}(\eta)$ definition, we have $\tilde{q}'(\eta) = (a_e - a_h)/\bar{\rho} - ((a_e - a_h)\hat{\eta} + a_h)(\rho_e - \rho_h)/\bar{\rho}^2 < \infty$, while from (B.5) it must be that $q'(\hat{\eta}-) \to \infty$. This is a contradiction.

C Proofs for Section 4

Proof of Lemma 5. We omit the proof, because it follows standard arguments, verification of optimality and market clearing, as in Brunnermeier and Sannikov (2014). □

Before proving Proposition 5, we need the following assumptions.

Assumption 1. *There exists $\bar{\sigma} > 0$ such that, for each $\sigma \in (0, \bar{\sigma})$, a non-sunspot equilibrium exists.*

Assumption 2. *Let parameters satisfy $0 < \frac{a_h}{\rho_h} < \frac{a_e}{\rho_e} < +\infty$.*

Assumption 1 is relatively benign, since we know a BSE exists (Proposition 1), and since the BSE corresponds to the limiting economy as $\sigma \to 0$ (Proposition 3). In practice, we can numerically find a non-sunspot equilibrium for any value of $\sigma > 0$. Assumption 2 makes the very mild assumption that the capital price is higher if experts control 100% of wealth than if households control 100% of wealth, which is needed so capital prices are increasing in experts' wealth share η.

Proof of Proposition 5. We proceed by construction. Without loss of generality, let $\mathcal{S} = (0,1)$ so that the domain of the state variables is $\mathcal{D} = (0,1) \times (0,1)$. Recall that $\bar{\rho} := \eta\rho_e + (1-\eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1-\eta)a_h$.

Step 1: Non-sunspot equilibrium. Let $(\hat{q}^0, \hat{\kappa}^0)$ be the solution to the non-sunspot equilibrium (which exists by Assumption 1), and let $\eta^0 := \inf\{\eta : \hat{q}^0 \geq a_e/\bar{\rho}\} = \inf\{\eta : \hat{\kappa}^0 \geq 1\}$. By part (v) of Lemma B.1, there exists $\bar{\sigma}_A > 0$ such that, if $\sigma < \bar{\sigma}_A$, then $\eta^0 < 1$. By part (iv) of Lemma B.1, there exists $\bar{\sigma}_B > 0$ such that, if $\sigma < \bar{\sigma}_B$, then $(\hat{q}^0)' > \frac{a_e - a_h}{\bar{\rho}}$ for $\eta \in (0, \eta^0)$. Assume $\sigma < \min(\bar{\sigma}_A, \bar{\sigma}_B)$. In particular, this implies $\frac{d}{d\eta}[\hat{q}^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$.

Step 2: Two basis functions. We design two "extremal" functions that will assist our construction. First, let φ be a C^2 function with the properties $\varphi(\eta^0) = 0$ and $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})' = \frac{a_e - a_h}{\bar{\rho}}[1 - (1-\eta)\frac{\rho_e - \rho_h}{\bar{\rho}}]$ for all η. Define

$$q^0(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta < \eta^0; \\ \hat{q}^0(\eta) + \varphi(\eta), & \text{if} \quad \eta \geq \eta^0. \end{cases} \tag{C.1}$$

Note that q^0 is C^∞ except at $\eta = \eta^0$, due to part (vi) of Lemma B.1.

To construct the other basis function, fix some $\epsilon \in (0, \eta^0)$, let $\tilde{\epsilon} \in (\epsilon, \eta^0)$, and define a C^∞ (but necessarily non-analytic) function $\beta : (0,1) \mapsto \mathbb{R}_+$ with the following properties

$$\begin{aligned} \beta(\epsilon) &= q^0(\epsilon) - \bar{a}(\epsilon)/\bar{\rho}(\epsilon) \\ \beta^{(k)}(\epsilon) &= \frac{d^k}{d\eta^k}[q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)]|_{\eta=\epsilon} \quad \text{for each derivative of order } k \geq 1 \\ \beta'(\eta) &< \frac{d}{d\eta}[q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)] \quad \text{for all} \quad \eta > \epsilon \\ \beta(\eta) &= 0 \quad \text{for all} \quad \eta > \tilde{\epsilon}. \end{aligned}$$

A particular consequence of $\sigma < \bar{\sigma}_B$ in step 1 is $\frac{d}{d\eta}[q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$. A consequence of $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})'$ is $\frac{d}{d\eta}[q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (\eta^0, 1)$. Together, these properties imply such a function β exists. Then, we put

$$q^1(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if} \quad \eta \leq \epsilon; \\ \bar{a}(\eta)/\bar{\rho}(\eta) + \beta(\eta), & \text{if} \quad \eta > \epsilon. \end{cases} \tag{C.2}$$

Note that $\eta^1 := \inf\{\eta : q^1 \geq a_e/\bar{\rho}\} = 1$. By the properties of β and φ, note the following slope results:

$$(q^0)' > (q^1)' \quad \text{on} \quad \eta \in (\epsilon, 1) \tag{C.3}$$

$$(q^0)^{(k)}(\epsilon) = (q^1)^{(k)}(\epsilon) \quad \text{for all derivatives of order } k \geq 0. \tag{C.4}$$

Step 3: Useful monotonicity results. Before continuing, we make the following claims:

$$\frac{\bar{a}}{\bar{\rho}} < q^1 = q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for} \quad \eta \in (0, \epsilon); \tag{C.5}$$

$$\frac{\bar{a}}{\bar{\rho}} \leq q^1 < q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for} \quad \eta \in (\epsilon, \eta^0); \tag{C.6}$$

$$\frac{\bar{a}}{\bar{\rho}} = q^1 < \frac{a_e}{\bar{\rho}} < q^0, \quad \text{for} \quad \eta \in (\eta^0, 1). \tag{C.7}$$

All inequalities in relationship (C.5), as well as the third inequality in relationship (C.6), hold by part (ii) of Lemma B.1. The first inequality in relationship (C.6) holds because $\beta \geq 0$, whereas the first equality in relationship (C.7) holds because $\beta = 0$ on that set. The second inequality in relationship (C.6) holds due to (C.3). The second inequality in relationship (C.7) holds by the definition of $\eta^1 = 1$. The second inequality in relationship (C.7) holds since $q^0(\eta^0) = a_e/\bar{\rho}(\eta^0)$ combined with $(q^0 - a_e/\bar{\rho})' > (q^1 - a_e/\bar{\rho})' > 0$, for $\eta > \eta^0$.

Step 4: Construct candidate (q, κ). We proceed to combine our basis functions according to the convex combination

$$\tilde{q}(\eta, s) := (1 - \alpha s) q^0(\eta) + \alpha s q^1(\eta), \quad (\eta, s) \in \mathcal{D} = (0,1) \times \mathcal{S}, \tag{C.8}$$

where $\alpha \in (0,1)$ is fixed. For each $s \in \mathcal{S}$, define

$$\eta^*(s) := \inf\{\eta : \tilde{q}(\eta, s) \geq a_e/\bar{\rho}\}.$$

Note that $\tilde{q}$ is C^2 on $(\eta^0, \eta^1) \times \mathcal{S}$, which implies η^* is C^1. Note that $\eta^*(s)$ is strictly increasing by a combination of (C.6)-(C.7).[21] Put

$$q(\eta, s) := \begin{cases} \tilde{q}(\eta, s), & \text{if} \quad \eta < \eta^*(s); \\ a_e/\bar{\rho}(\eta), & \text{if} \quad \eta \geq \eta^*(s). \end{cases} \quad \text{and} \quad \kappa := \frac{\bar{\rho} q - a_h}{a_e - a_h}.$$

[21] Indeed, (C.6) shows that $\tilde{q}(\eta, s) < a_e/\bar{\rho}(\eta)$ on $(\epsilon, \eta^0) \times \mathcal{S}$, which implies $\eta^*(s) \geq \eta^0$ for all $s \in \mathcal{S}$. Then, use the fact that η^* is C^1 to differentiate $\tilde{q}(\eta^*(s), s) = a_e/\bar{\rho}(\eta^*(s))$ with respect to s, and use the fact that $\partial_s \tilde{q} = q^1 - q^0$, and finally rearrange to obtain

$$(\eta^*)'(s) \Big[\partial_\eta \tilde{q}(\eta^*(s), s) + \frac{a_e}{\bar{\rho}(\eta^*(s))} \frac{\rho_e - \rho_h}{\bar{\rho}(\eta^*(s))} \Big] = q^0(\eta^*(s)) - q^1(\eta^*(s)).$$

If at any point s, we had $(\eta^*)'(s) = 0$, we would necessarily have $q^0(\eta^*(s)) = q^1(\eta^*(s))$. But this contradicts the fact from (C.6)-(C.7) that $q^0 > q^1$ for all $\eta > \epsilon$, since $\eta^*(s) \geq \eta^0 > \epsilon$. Thus, $(\eta^*)'(s) \neq 0$ for all s. We can also rule out $(\eta^*)'(s) < 0$ by the fact that $\eta^*(0+) = \eta^0$ and $\eta^*(s) \geq \eta^0$ for all s. Thus, $(\eta^*)'(s) > 0$ for all s.

By construction, the pair (q, κ) satisfy equation (7).

Step 5: Properties of (q, κ). Let $\mathcal{X} := \{(\eta, s) : \eta \in (\epsilon, \eta^*(s)),\ s \in \mathcal{S}\}$. On this set, we have $\kappa > \eta$, or equivalently $\bar{\rho}q > \bar{a}$, by (C.6)-(C.7). In fact, κ is bounded away from η on $\mathcal{X}$, since $\alpha < 1$ in (C.8). We also have the following derivative conditions on $\mathcal{X}$:

$$\partial_s q = \alpha(q^1 - q^0) < 0 \tag{C.9}$$
$$\partial_\eta q = (1 - \alpha s)(q^0)' + \alpha s (q^1)' > 0 \tag{C.10}$$
$$\partial_\eta q < q/(\kappa - \eta). \tag{C.11}$$

Inequality (C.9) holds by (C.6)-(C.7). Inequality (C.10) holds by (C.3) and Assumption 2(ii), which implies $(q^1)' > 0$. Inequality (C.11) is proven as follows. First, note that the function $f(\eta, x) = \frac{(a_e - a_h)x}{\bar{\rho}(\eta)x - \bar{a}(\eta)}$ is strictly decreasing in x on $x > \bar{a}(\eta)/\bar{\rho}(\eta)$. Second, part (i) of Lemma B.1 implies

$$(q^0)' < \frac{(a_e - a_h)q^0}{\bar{\rho}q^0 - \bar{a}} = f(\cdot, q^0).$$

Given (C.3), we thus have $\partial_\eta q < f(\cdot, q^0)$ for any value of s. Finally, since f is decreasing in its second argument, and $q < q^0$ on $\mathcal{X}$, we have $\partial_\eta q < f(\cdot, q)$, which proves the claim.

We remark on one additional smoothness property that holds at $\eta = \epsilon$, due to condition (C.4):

$$\partial_\eta^{(k)} q(\epsilon, s) = (q^0)^{(k)}(\epsilon) \quad \forall s, \quad \text{for all derivatives of order } k \geq 0. \tag{C.12}$$

Step 6: Construct candidate σ_s. Consider solving the following problem.

$$\begin{aligned} &\textit{Problem:} \quad \text{for each } (\eta, s) \in \mathcal{X}, \text{ solve for } y \text{ in the equation} \\ &\qquad\qquad\quad y(\partial_s \log q)^2 = G, \end{aligned} \tag{C.13}$$

where

$$G := \frac{\eta(1-\eta)}{\kappa - \eta} \frac{a_e - a_h}{q} (1 - (\kappa - \eta)\partial_\eta \log q)^2 - \sigma^2.$$

Note that G is bounded, as κ is bounded away from η (step 5). Checking boundedness of the solution y thus boils down to checking $\partial_s q$ at the boundaries of $\mathcal{X}$. By (C.9), as $s \to 0$ or $s \to 1$, $\partial_s q \not\to 0$, so y remains bounded. To check the result as $\eta \to \epsilon$, we first claim that $\lim_{\eta \searrow \epsilon} \partial_\eta^{(k)} G = 0$ for all derivatives of order $k \geq 0$. This is a consequence of parts (i) and (vi) of Lemma B.1, whereby $\partial_\eta^{(k)} G = 0$ for all $k \geq 0$ on $\eta < \epsilon$, combined with result (C.12). Since we also have $\partial_s q \to 0$, we apply L'Hôpital's rule twice to compute $\lim_{\eta \searrow \epsilon} G/(\partial_s \log q)^2 = 0$, noting both times that $\partial_{s\eta} \log q = \frac{\alpha}{q}[(q^1)' - (q^0)'] < 0$ is non-zero. Therefore, the solution $y = G/(\partial_s \log q)^2$ is bounded on $\mathcal{X}$.

Clearly, $\sqrt{y}$ will be a real number if and only if $G \geq 0$. To prove $G \geq 0$, note that $\lim_{s \searrow 0} G = 0$, meaning it suffices to prove $\partial_s G \geq 0$. Differentiating G, we get

$$\begin{aligned} \frac{\partial_s G}{\eta(1-\eta)} = -\frac{a_e - a_h}{(\kappa - \eta)q}(1 - (\kappa - \eta)\partial_\eta \log q)\Big[&(1 - (\kappa - \eta)\partial_\eta \log q)\Big(\frac{\partial_s \kappa}{\kappa - \eta} + \frac{\partial_s q}{q}\Big) \\ &+ 2\frac{(\kappa - \eta)\bar{a}}{\bar{\rho}q - \bar{a}}(\partial_s \log q)(\partial_\eta \log q) + 2(\kappa - \eta)\alpha \frac{(q^1)' - (q^0)'}{q}\Big]. \end{aligned}$$

By properties (C.9)-(C.11), and the fact that $\text{sgn}(\partial_s \kappa) = \text{sgn}(\partial_s q)$, we prove $\partial_s G > 0$ on $\mathcal{X}$. So not only is $\sqrt{y}$ real, it is non-zero.

We set σ_s as follows:

$$\sigma_s(\eta, s) := \begin{cases} \sqrt{y(\eta, s)}, & \text{if} \quad (\eta, s) \in \mathcal{X}; \\ \sqrt{y(\epsilon+, s)} = 0, & \text{if} \quad (\eta, s) \in \{(\eta, s) : \eta \in (0, \epsilon), \, s \in \mathcal{S}\}; \\ \sqrt{y(\eta^*(s)-, s)}, & \text{if} \quad (\eta, s) \in \{(\eta, s) : \eta > \eta^*(s), \, s \in \mathcal{S}\}. \end{cases} \tag{C.14}$$

In passing, we note that we have also shown that $\sigma_s > 0$ on a positive-measure set, as required in a sunspot equilibrium.

Step 7: Verify equation (19) is satisfied. By the construction of σ_s, equation (19) is satisfied on $\mathcal{X}$. On $\{(\eta, s) : \eta \in (0, \epsilon), \, s \in \mathcal{S}\}$, recall $\partial_s q = 0$, so (19) holds by property (i) of Lemma B.1. On $\{(\eta, s) : \eta > \eta^*(s), \, s \in \mathcal{S}\}$, recall $\kappa = 1$, so (19) is satisfied if and only if the second term inside the minimum is non-negative. Substituting $\kappa = 1$ and $q = a_e / \bar{\rho}$, hence $\partial_s q = 0$, into this term shows the non-negativity requirement is

$$\sigma^2 \le \eta \bar{\rho} \frac{a_e - a_h}{a_e} (1 + (1 - \eta) \partial_\eta \log \bar{\rho})^2 \quad \text{for} \quad \eta > \eta^*(s), \, s \in \mathcal{S}. \tag{C.15}$$

On the other hand, property (v) of Lemma B.1, combined with the fact that $\eta^*(s)$ is increasing, imply

$$\eta^*(s) \ge \frac{\rho_h}{\rho_e} \Big(\frac{1 - a_h / a_e}{\sigma^2} \rho_e - 1 + \frac{\rho_h}{\rho_e} \Big)^{-1}, \quad \forall s \in \mathcal{S}. \tag{C.16}$$

Straightforward algebra demonstrates that (C.15) and (C.16) are equivalent, proving (19) holds.

Step 8: Finish equilibrium construction. Having determined q, κ, and σ_s, we define μ_η and σ_η by (21)-(22). It remains to determine μ_s. We will pick μ_s to be a C^2 function with the following properties: $\partial_s \mu_s < 0$, $\lim_{s \searrow s^0} \mu_s(\eta, s) = +\infty$ for some $s^0 \ge 0$ and all $\eta \in (0, 1)$, and $\lim_{s \nearrow s^1} \mu_s(\eta, s) = -\infty$ for some $s^1 < 1$ and all $\eta \in (0, 1)$. We pick μ_s such that it asymptotes sufficiently rapidly.

Step 9: Verify stationarity. Finally, we demonstrate the time-paths $(\eta_t, s_t)_{t \ge 0}$ remain in $\mathcal{D}$ almost-surely. This is a consequence of the following properties:

- σ_s is bounded;
- $\mu_s(\eta, s_0+) = +\infty$ and $\mu_s(\eta, s_1-) = -\infty$, sufficiently rapidly, for all $\eta \in (0, 1)$;
- σ_η is bounded and vanishes, at a super-linear rate, as $\eta \to 0$ and $\eta \to 1$, for all s;
- $\mu_\eta(\eta, s) > 0$ for all η close enough to 0, and all s;
- $\mu_\eta(\eta, s) < 0$ for all η close enough to 1, and all s.

See Chapters 3.5, 3.7, and 4.4 of Khasminskii (2011) for theorems that apply to multi-variate diffusions (i.e., recurrence and stationarity theorems). □

PROOF OF PROPOSITION 6. Fix any $\Sigma^* > 0$. The proof is a simple consequence of the fact that σ_q must be unbounded as κ approaches η, which is as q approaches the worst-case price q^1. We fill in the technical details below.

We construct a sequence of equilibria – indexed by $(\alpha, \epsilon, \zeta)$ – as follows. Recall the capital price construction in Proposition 5:

$$q = (1 - \alpha s)q^0 + \alpha s q^1, \quad \text{when} \quad \kappa < 1,$$

where $\alpha < 1$ is a parameter, q^0 is the non-sunspot equilibrium price, and

$$q^1 = \begin{cases} q^0, & \text{if } \eta < \epsilon; \\ \bar{a}/\bar{\rho} + \beta, & \text{if } \eta \in (\epsilon, \tilde{\epsilon}); \\ \bar{a}/\bar{\rho}, & \text{if } \eta > \tilde{\epsilon}. \end{cases}$$

The function β is a positive mollifier that vanishes uniformly as $\epsilon, \tilde{\epsilon} \to 0$. We set $\tilde{\epsilon} = \epsilon(1+\epsilon)$. Based on the discussion in the text, we may choose μ_s such that equilibrium concentrates on any particular value of s. Thus, pick μ_s such that $s_t \geq \zeta$ almost-surely. Clearly, the choice of μ_s depends on α and ϵ, but such a choice can always be made for any parameters.

Let $p_{\text{low}} > 0$, $p_{\text{high}} > 0$ be given with $p_{\text{low}} + p_{\text{high}} < 1$. First, note that there exist α^*, ζ^*, and ϵ^* such that $\mathbb{P}[\eta_t \leq \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{low}}$ and $\mathbb{P}[\eta_t \geq 1 - \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{high}}$ for all $\alpha > \alpha^*$, $\zeta > \zeta^*$, and $\epsilon < \epsilon^*$. This is a consequence of the fact that in any stationary distribution, we have $\lim_{x \to 0} \mathbb{P}[\eta_t < x] = \lim_{x \to 1} \mathbb{P}[\eta_t > x] = 0$ and the fact that $\lim_{\alpha \to 1} \lim_{s \to 1} \kappa(\eta, s) < 1$ for all η.

At this point, fix such an $\epsilon < \epsilon^*$. Let a constant $M > 0$ be given satisfying

$$M \leq (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e/\rho_h} \frac{\tilde{\epsilon}(1 - \tilde{\epsilon})}{\Sigma^*}. \tag{C.17}$$

Note that

$$\lim_{\alpha \to 1} \lim_{s \to 1} \sup_{\eta \in (\tilde{\epsilon}, 1-\tilde{\epsilon})} \Big| q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta) \Big| = 0.$$

Consequently, we may pick $\alpha > \alpha^*$ close enough to 1 and $\zeta > \zeta^*$ close enough to 1 such that

$$\sup_{s \in (\zeta, 1)} \sup_{\eta \in (\tilde{\epsilon}, 1-\tilde{\epsilon})} \Big| q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta) \Big| \leq M$$

almost-surely.

Finally, using equation (19) and substituting $\kappa < 1$ from (7), we have $|\sigma\binom{1}{0} + \sigma_q|^2 = \frac{(a_e - a_h)^2}{q} \frac{\eta(1-\eta)}{\bar{\rho}q - \bar{a}}$. Note also that $q \leq a_e/\rho_h$ is an upper bound. Then,

$$\mathbb{E}[|\sigma\tbinom{1}{0} + \sigma_{q,t}|^2] > (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e/\rho_h} \frac{\tilde{\epsilon}(1 - \tilde{\epsilon})}{M}.$$

Using (C.17), we obtain $\mathbb{E}[|\sigma\binom{1}{0} + \sigma_{q,t}|^2] > \Sigma^*$. □

Proof of Proposition 7. Revisiting the proof of Proposition 5, we compute on $\{\kappa < 1\}$ and for each $\eta > \epsilon$,

$$\partial_s[(\kappa - \eta)\partial_\eta \log q] = \alpha\Big[(\kappa - \eta)\frac{(q^1)' - (q^0)'}{q} + \frac{\bar{a}(q^1 - q^0)}{(a_e - a_h)q^2}\partial_\eta q\Big] < 0.$$

The inequality uses (C.3) to say $(q^1)' - (q^0)' < 0$, and (C.6)-(C.7) to say $q^1 - q^0 < 0$, and (C.10) to say $\partial_\eta q > 0$. Therefore, $(1 - (\kappa - \eta)\partial_\eta \log q)^{-1}$ is decreasing in s on $\{\kappa < 1\}$ for each $\eta > \epsilon$. By constructing the equilibrium with ϵ arbitrarily small, and noting that $\binom{1}{0} \cdot \sigma_q = \frac{\sigma}{1-(\kappa-\eta)\partial_\eta \log q}$, we obtain part (i).

Next, we prove part (iii). From (19), we can solve for return volatility $|\sigma_R|^2 := |\binom{1}{0} \cdot \sigma + \sigma_q|^2$ on $\{\kappa < 1\}$ to get $|\sigma_R|^2 = \frac{(a_e-a_h)^2}{q} \frac{\eta(1-\eta)}{\bar{\rho}q-\bar{a}}$. Differentiating with respect to s, we obtain

$$\partial_s |\sigma_R|^2 = -\eta(1-\eta) \frac{(a_e-a_h)^2}{q(\bar{\rho}q-\bar{a})} \Big[\frac{1}{q} + \frac{\bar{\rho}}{\bar{\rho}q-\bar{a}}\Big] \partial_s q > 0,$$

since $\partial_s q = \alpha(q^1 - q^0) < 0$ by (C.9).

Finally, part (ii) is a consequence of parts (i) and (iii), because of the equality $|\sigma_R|^2 - [\binom{0}{1} \cdot \sigma_q]^2 = [\sigma + \binom{1}{0} \cdot \sigma_q]^2$. □

D Numerical method for sunspot equilibrium of Section 4

Step 0: Let (i,j) index the state space $\mathcal{N} \times \mathcal{S} = (0,1) \times (0,1)$. Let $i \in \{1,\dots,I\}$ and $j \in \{1,\dots,J\}$.

Step 1: Construct the candidate capital price function $q(\eta_i, s_j)$.

(a) Find the fundamental solution $q^0(\eta_i)$ to the economy from the ODE in part (i) of Lemma B.1. Our example uses this fundamental solution as an upper bound for the sunspot q function. Calculate also the unbounded fundamental solution $\tilde{q}^0(\eta_i)$, i.e., the solution to the ODE that ignores the restriction $\kappa < 1$.

(b) Construct a lower bound $q^1(\eta_i)$. Let $q^1(\eta_i) = q^0(\eta_i)$ for $i = 1,2$. This ensures that $(q^1)'(0+) = (q^0)'(0+)$. Set $q^1(\eta_I) = a_e/\rho_e$ and interpolate all other points, i.e., $q^1(\eta_i)$ for $i = 3,\dots,I-1$. Any interpolation method that delivers a monotonic function should work; we use a linear interpolation.

(c) Set the boundary values by $q(\eta_i, s_1) = q^0(\eta_i)$ and $q(\eta_i, s_J) = q^1(\eta_i)$ for all i. For $j \in \{2,\dots,J-1\}$, put $q(\eta_i, s_j) = \min\{a_e/\bar{\rho}(\eta_i), (1-s_j)\tilde{q}^0(\eta_i) + s_j q^1(\eta_i)\}$ for all i.

Step 2: Solve for κ by plugging q into (7).

Step 3: Compute σ_s as follows.

(a) For points where $\kappa < 1$, the variance of capital returns $|\sigma_R|^2$ is recovered exactly from (19) – it is the term in parentheses. If $\partial_s \log q \neq 0$, then we solve for $\sigma_s^2 = |\sigma_R|^2(1-(\kappa-\eta)(\partial_\eta \log q)^2 - \sigma^2)/(\partial_s \log q)^2$. If $\partial_s \log q = 0$ and $\sigma_R \neq 0$, we must have $\sigma_s \to \infty$ and this is not an equilibrium. This is only the case at $\eta \in \{\eta_1, \eta_2\}$, but as the proof of Proposition 5 shows, this is a non-issue. First, for η_1, equation (19) will hold independently of the value for σ_s. Second, for η_2, the proof shows a mollification ("smoothing out") is possible in a small neighborhood around the point to eliminate this issue. Since we study a finite-difference approximation, we may therefore take the neighborhood smaller than the grid spacing and ignore the issue. Therefore, we may set $\sigma_s(\eta_1, s_j)$ and $\sigma_s(\eta_2, s_j)$ by any extrapolation of $\sigma_s(\eta_i, s_j)$, for $i \geq 3$ and each j.

(b) For points where $\kappa = 1$, we have $\partial_s \log q = 0$, and σ_s can take any value (in our numerical example, we set $\sigma_s(\eta_I, s_j) = 0$ and interpolate the rest of the values).

E Model extensions

E.1 Poisson Sunspot Equilibrium (PSE)

Rather than model sunspots as Brownian shocks, here we conjecture capital prices can "jump" for non-fundamental reasons. Mathematically, write

$$dq_t = q_{t-}[\mu_{q,t-}dt + \zeta_{q,t-}dJ_t],$$

where J is a Poisson process with intensity λ that does not affect physical capital at all. An equilibrium in which ζ_q is not identically zero will be called the *Poisson Sunspot Equilibrium* (PSE).

In a Markov equilibrium, the sole state variable will still be experts' wealth share η, which follows a jump process

$$d\eta_t = \mu_{\eta,t-}dt + \zeta_{\eta,t-}dJ_t.$$

Note that $\zeta_{\eta,t-} := \eta_t - \eta_{t-}$ by definition. Because agents' portfolios (capital and bonds) are predetermined, we can determine the wealth share jump from the jump in q, with the result being[22]

$$\zeta_\eta = (\kappa - \eta)\frac{\zeta_q}{1+\zeta_q}. \tag{E.1}$$

On the other hand, once the post-jump wealth share is known, the capital price is also known, since η is the sole state variable, i.e., we have $q_t = q(\eta_t)$ for some function q. Thus, if we denote the post-jump wealth share by $\hat{\eta}$,

$$\zeta_q = \frac{q(\hat{\eta}) - q}{q}. \tag{E.2}$$

This is the way to solve the two-way feedback between the wealth distribution and capital prices, similar to the Brownian model. Combining (E.1)-(E.2) yields $\hat{\eta} - \eta = (\kappa - \eta)\frac{q(\hat{\eta})-q}{q(\hat{\eta})}$, which is analogous to the sunspot differential equation of the BSE. Indeed, as $\hat{\eta} \to \eta$, this system converges exactly to $q'/q = (\kappa - \eta)^{-1}$.

Because we do not model bankruptcy procedures, we must also make sure the jump renders experts solvent, meaning $\zeta_{\eta,t-} > -\eta_{t-}$, to preserve the risk-free status of the bond. If solvency cannot be ensured, then no self-fulfilling jump can take place.

Portfolio choices are still relatively simple, because the jump size is locally predictable, i.e., ζ_q is known just before the jump actually occurs. Thus, equations characterizing an equilibrium of this model are given by the following simple lemma.

[22] The derivation is as follows. Let variables with hats, e.g., "$\hat{x}$", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K}\kappa - B$ and $\hat{N}_h = \hat{q}\hat{K}(1-\kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\zeta_\eta = \hat{\eta} - \eta = B[1/(qK) - 1/(\hat{q}\hat{K})] = qK(\kappa - \eta)[1/(qK) - 1/(\hat{q}\hat{K})]$. Using the fact that $\hat{K} = K$ and the definition $\zeta_q := \hat{q}/q - 1$, we arrive at $\zeta_\eta = (\kappa - \eta)[1 - (1+\zeta_q)^{-1}]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump. Conceptually, there is no reason why this needs to be true, but it preserves its risk-free conjecture. If bond prices are allowed to jump at the same time, we would find different expressions.

Lemma E.1 (Equilibrium with Jumps). *An allocation is a Markov equilibrium with jumps only if $(q, \kappa, \hat{\eta}, \zeta_q, \zeta_\eta)$ are functions of $\eta \in (0,1)$ satisfying (7), (E.1), (E.2), and the following:*

$$\hat{\eta} = \eta + (\kappa - \eta)\frac{\zeta_q}{1+\zeta_q} > 0$$

$$0 = \min\left[1-\kappa,\ \eta(1-\eta)\frac{a_e - a_h}{q} - (\kappa-\eta)\frac{\lambda\zeta_q^2}{(1+\frac{\kappa}{\eta}\zeta_q)(1+\frac{1-\kappa}{1-\eta}\zeta_q)}\right], \quad \kappa(0) = 0.$$

To show what the PSE looks like, we provide a numerical solution and plot some aspects below. The first panel of figure E.1 displays one simulation of the PSE, also comparing it with a simulation of the BSE. The second panel plots the stationary capital price densities (although note that the BSE "density" in fact has a point mass at $\eta = \eta^*$). Capital prices in the PSE tend to remain at lower levels than in the BSE for our example.

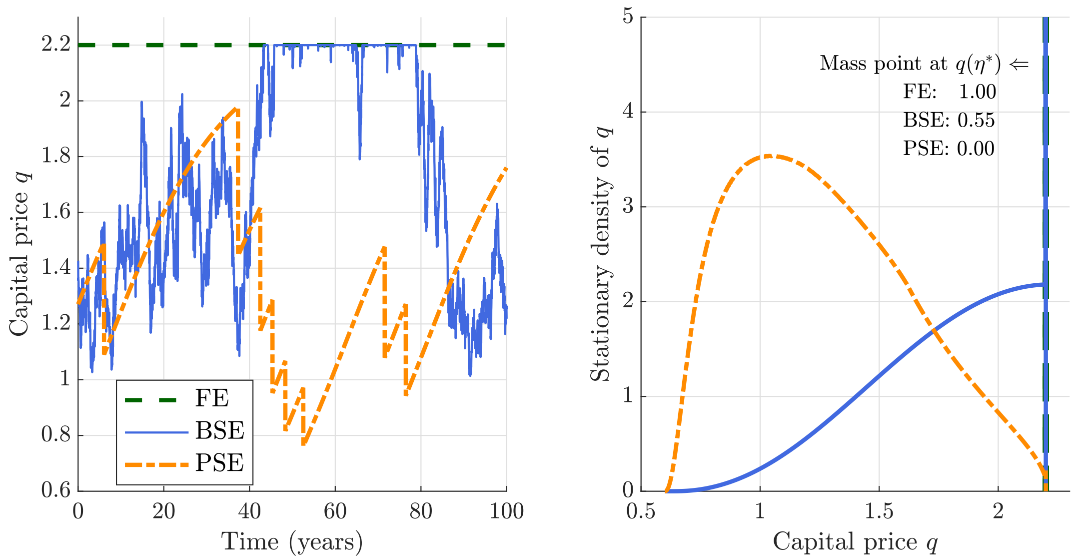

Figure E.1: Time series and stationary density of capital price q in a PSE, BSE, and FE. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. OLG parameters: $\nu = 0.1$ and $\delta = 0.1$.

E.2 Non-myopic preferences

We modify the model of Section 1 by generalizing preferences to the CRRA type. In particular, we replace the $\log(c)$ term in (2) with the flow consumption utility $c^{1-\gamma}/(1-\gamma)$. For simplicity, we consider no OLG structure ($\delta = 0$), but we continue to allow experts' discount rate to exceed households' ($\rho_e \geq \rho_h$). We continue to look for a Markov equilibrium, with experts' wealth share η as the state variable.

Equilibrium. The key equation (11) still holds, repeated here for convenience:

$$\left[1 - (\kappa - \eta)\frac{q'}{q}\right]\sigma_\eta = 0. \tag{E.3}$$

The sunspot equilibrium is associated with the term in brackets being equal to zero. Unlike with logarithmic preferences, this condition does not pin down $q(\eta)$ function, because we can no longer write $\kappa(q,\eta)$ from the goods market clearing condition: the consumption to wealth ratio is not constant anymore, and depends on agents' value functions.

The value function can be written as $V_i = v_i(\eta)K^{1-\gamma}/(1-\gamma)$ where $v_i(\eta)$ is determined in equilibrium. Then, consumption is $c_i/n_i = (\eta_i q)^{1/\gamma-1}/v_i^{1/\gamma}$ where η_i corresponds to the wealth share of sector i. Then, goods market clearing becomes

$$q^{1/\gamma}\Big[\Big(\frac{\eta}{v_e}\Big)^{1/\gamma}+\Big(\frac{1-\eta}{v_h}\Big)^{1/\gamma}\Big]=(a_e-a_h)\kappa+a_h. \tag{E.4}$$

Optimal portfolio decisions imply that

$$0=\min\Big[1-\kappa,\ \frac{a_e-a_h}{q}-\Big(\frac{v_h'}{v_h}-\frac{v_e'}{v_e}+\frac{1}{\eta(1-\eta)}\Big)(\kappa-\eta)\sigma_q^2\Big]. \tag{E.5}$$

The HJB equation for $i\in\{e,h\}$ has the familiar form $\rho_i V_i = u(c)+\mathbb{E}[\frac{dV_i}{dt}]$, which becomes

$$\rho_i=\frac{(\eta_i q)^{1/\gamma-1}}{v_i^{1/\gamma}}+\underbrace{\frac{v_i'}{v_i}\mu_\eta+\frac{1}{2}\frac{v_i''}{v_i}\sigma_\eta^2}_{:=\mu^{v,i}}+(1-\gamma)g. \tag{E.6}$$

The dynamics of η satisfy

$$\sigma_\eta=(\kappa-\eta)\sigma_q \tag{E.7}$$

$$\mu_\eta=\eta(1-\eta)\Big(\varsigma_e\frac{\kappa}{\eta}\sigma_q-\varsigma_h\frac{1-\kappa}{1-\eta}\sigma_q+\frac{c_h}{n_h}-\frac{c_e}{n_e}\Big)-\sigma_\eta\sigma_q \tag{E.8}$$

and agent-specific risk prices satisfy

$$\varsigma_e=-\frac{v_e'}{v_e}\sigma_\eta+\frac{\sigma_\eta}{\eta}+\sigma_q \tag{E.9}$$

$$\varsigma_h=-\frac{v_h'}{v_h}\sigma_\eta-\frac{\sigma_\eta}{1-\eta}+\sigma_q. \tag{E.10}$$

A Markov equilibrium is a set of functions: prices $\{q,\sigma_q,\varsigma_e,\varsigma_h\}$, allocation $\{\kappa\}$, value functions $\{v_h,v_e\}$ and aggregate state dynamics $\{\sigma_\eta,\mu_\eta\}$ that solve the system (E.3)-(E.10).

The fundamental equilibrium corresponds to the solution for (E.3) where $\sigma_\eta=0$, which implies deterministic economic dynamics. Then, the capital price has no volatility ($\sigma_q=0$), risk prices are zero ($\varsigma_e=\varsigma_h=0$), and experts hold the entire capital stock ($\kappa=1$). The capital price is then solved from (E.4), and the value functions satisfy

$$\rho_i=\frac{(\eta_i q)^{1/\gamma-1}}{v_i^{1/\gamma}}+\frac{v_i'}{v_i}\underbrace{\eta(1-\eta)\left(\frac{c_h}{n_h}-\frac{c_e}{n_e}\right)}_{=\mu_\eta}+(1-\gamma)g.$$

Conversely, the sunspot equilibrium corresponds to the solution for (E.3) with $\frac{q'}{q}=(\kappa-\eta)^{-1}$ (and potentially $\sigma_\eta\neq 0$).

Disaster belief. With logarithmic preferences, we proved that any sunspot equilibrium must satisfy $\sigma_q(0+)=0$. This allowed us, in Section 2.3, to construct sunspot equilibria with $\kappa(0+)=\kappa_0$ for any $\kappa_0 \in [0,1)$. With CRRA preferences, we attempt to construct the same class of equilibria, with $\sigma_q(0+) = 0$ and $\kappa_0 \in (0,1)$.

In order to have a non-degenerate stationary distribution, we have the following requirements. Since $\sigma_\eta(0+) = \kappa_0 \sigma_q(0+) = 0$, the state variable avoids the boundary $\{0\}$ if $\mu_\eta(0+) > 0$. Using (E.5) for $\kappa < 1$, we have[23]

$$\frac{a_e - a_h}{q(0+)} = (\varsigma_e(0+) - \varsigma_h(0+))\sigma_q(0+)$$

which allows us to show that[24]

$$\mu_\eta(0+) = \kappa_0 \frac{a_e - a_h}{q(0+)} > 0.$$

In addition, we need $\mu_\eta(\eta^*+) < 0$ where $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. This requirement should be satisfied for $\rho_e - \rho_h$ sufficiently large.[25]

Numerical solution. We do not provide an existence proof – which involves the existence of a solution to the ODE system – but construct numerical examples. For tractability, the numerical examples are constructed for $\kappa_0 > 0$, which keeps $q'(0+) = q(0+)/\kappa_0$ bounded.[26]

The numerical strategy is the following. Construct a grid $\{\eta_1, \ldots, \eta_N\}$ with limit points arbitrarily close to but bounded away from zero and one. Conjecture value functions $v_h(\eta)$ and $v_e(\eta)$. Impose $\kappa(\eta_1) = \kappa_0$ and use (E.4) to solve for $q(\eta_1)$. At each interior grid point, use $q' = q/(\kappa - \eta)$ and (E.4) to solve for $\kappa(\eta)$ and $q(\eta)$ until $\kappa(\eta^*) = 1$. In this region, recover σ_q from (E.5). For $\eta \in (\eta^*, 1]$ impose $\kappa(\eta) = 1$ and $\sigma_q = 0$, and solve capital price from (E.4). The rest of equilibrium objects are calculated directly from the system above. The guesses of the value functions are updated by augmenting the HJBs (E.6) with a time derivative and moving a small time-step backward, as in Brunnermeier and Sannikov (2016). The procedure terminates when the value functions converge to time-independent functions.

In Figure E.2, we plot the equilibrium objects as functions of η, for different levels of risk aversion γ. In Figure E.3, we make the same plots, for different levels of the disaster belief κ_0. Higher risk aversion (higher γ) or more pessimism about disasters (lower κ_0) generates sunspot equilibria featuring lower capital prices and higher volatility.

E.3 Time-varying disaster beliefs

We can further enlarge the space of self-fulfilling equilibria of Section 2.3 by allowing for time-variation in beliefs about experts' deleveraging in disaster states. This can be thought of as stochastic shifts between equilibria described in Section 2.3, each pertaining to a fixed disaster belief κ_0.

Suppose s_t is a time-varying sunspot that follows the process

$$ds_t = \mu_{s,t} dt + \sigma_{s,t} dZ_t,$$

[23] Note that this implies $\varsigma_e(0+) - \varsigma_h(0+)$ diverges.

[24] This expression also assumes that $\varsigma_h(0+)$ remains bounded. This is a mild assumption since households own all capital.

[25] There is an important distinction between the restriction not to reach $\eta = 0$ and $\mu_\eta(\eta^*+) < 0$. Without the first one, the equilibrium for any $\kappa_0 > 0$ unravels, while without the second one, the equilibrium is still valid, but it has a degenerate stationary distribution at some value $\eta^{ss} > \eta^*$.

[26] With logarithmic utility, we obtain a limiting result in Proposition 2, that as $\kappa_0 \to 0$, the equilibrium converges to the sunspot equilibrium. With CRRA, we do not prove such a result analytically, but we do observe numerically what looks like convergence as κ_0 becomes small.

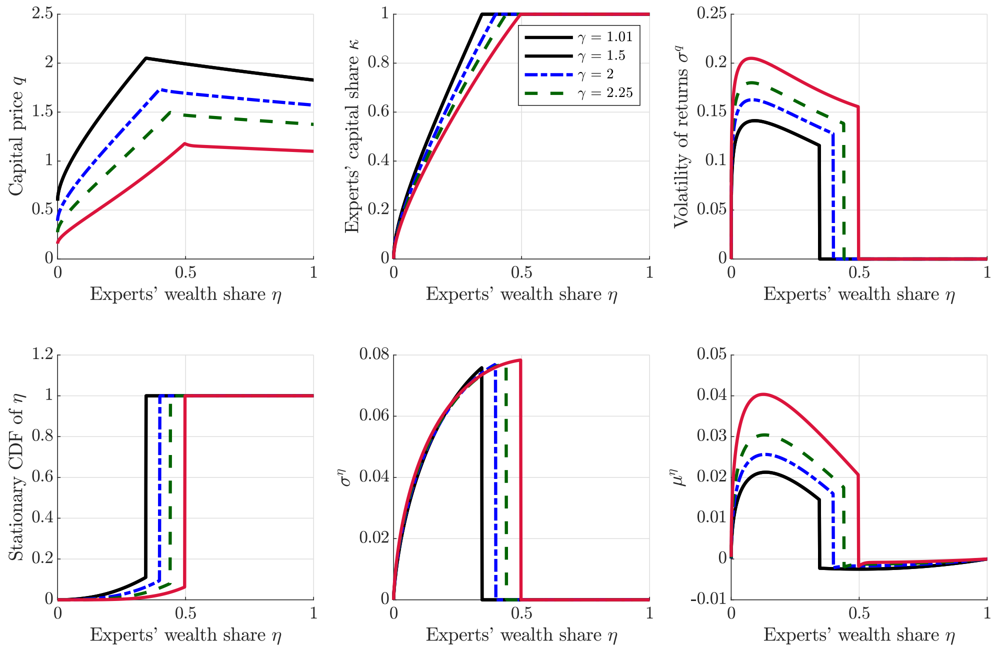

Figure E.2: Sunspot equilibrium for different risk aversion γ. The disaster belief is set to $\kappa_0 = 0.001$. Other parameters are: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, and $g = 0.02$.

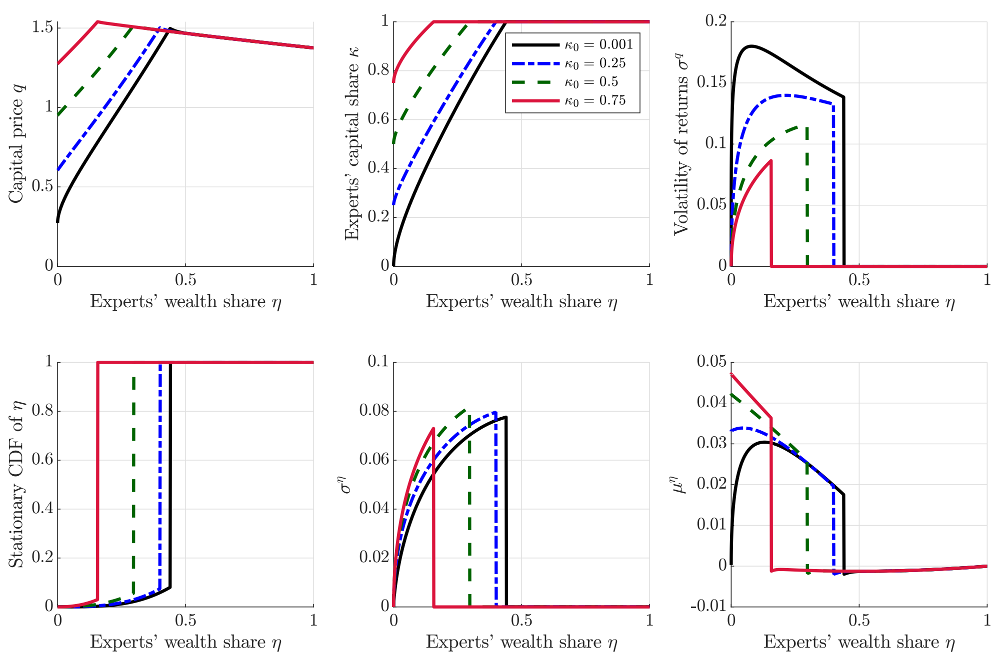

Figure E.3: Sunspot equilibrium for different disaster beliefs κ_0. Risk aversion is set to $\gamma = 2$. Other parameters are: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, and $g = 0.02$.

where Z is a one-dimensional Brownian motion. Assume μ_s and σ_s are such that $s_t \in (0,1)$ for all $t \geq 0$ almost-surely. For convenience, assume $\kappa(0,s) = s$, so that the sunspot corresponds to the belief about experts' deleveraging. To solve for a Markov equilibrium in (η, s), we must solve the following equations, derived similarly to Lemma 1:

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h \tag{E.11}$$

$$\sigma_q = \frac{\sigma_s \partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q} \tag{E.12}$$

$$0 = \min\left[1-\kappa,\, \eta(1-\eta)\frac{a_e - a_h}{q} - (\kappa - \eta)\sigma_q^2\right], \quad \kappa(0,s) = s. \tag{E.13}$$

A requirement for a solution to exist is that $\sigma_s \to 0$ as $\eta \to 0$, for any value of s, so that $\sigma_q \to 0$ as $\eta \to 0$.

Note that the central equation (11) can no longer hold in this model. This is because sunspot shocks move the boundary condition $\kappa(0,s)$, which has a direct effect on the capital price, through the term $\sigma_s \partial_s \log q$. If we had $(\kappa - \eta)\partial_\eta \log q = 1$, this direct effect would be amplified infinitely, and we would have unbounded σ_q in (E.12), which cannot be an equilibrium. Instead, $(\kappa - \eta)\partial_\eta \log q < 1$ is required by equilibrium.

Given a prescribed σ_s, the system above becomes a partial differential equation in q. However, similar to Section 4, we can instead find a solution in reverse: pre-specifying a candidate equilibrium capital price q, we can obtain the sunspot dynamics (σ_s, μ_s) that justify such a price. The resulting sunspot dynamics will be endogenous.

Here, we implement one example with such endogenous sunspot dynamics which can be solved analytically. We suppose experts and households have the same discount rate, $\rho_e = \rho_h$ (stationarity can still be achieved with the OLG features). Let $\beta \in (0,1)$ be a constant. For each $s \in (0,1)$, solve the ODE

$$\frac{y'}{y} = \frac{\beta(a_e - a_h)}{\bar{\rho}y - \beta(xa_e + (1-x)a_h)}, \quad \eta \in (0, x^*), \quad y(0) = \frac{sa_e + (1-s)a_h}{\rho}, \quad y(x^*) = \frac{a_e}{\rho}. \tag{E.14}$$

The idea of equation (E.14) is to mimic the key BSE differential equation (13), but with a milder slope. Similar to footnote 9, this ODE has a closed-form solution, which is

$$y(x;s) = \frac{1}{\rho}\left[\beta\bar{a}(x) + \sqrt{\beta^2\bar{a}(x)^2 + \bar{a}(s)^2 - 2\beta a_h \bar{a}(s)}\right], \quad \bar{a}(x) := xa_e + (1-x)a_h \tag{E.15}$$

$$x^*(s) = \inf\{x : y(x;s) \geq a_e/\rho\}. \tag{E.16}$$

The solution y is well-defined (i.e., the discriminant is strictly positive), strictly increasing in x, and strictly increasing in s, for $x \in (0,1)$ and $s \in (0,1)$.

Using these results, define our solution as follows. Put $\eta^*(s) = x^*(s)$ and

$$q(\eta, s) = \begin{cases} y(\eta; s), & \text{if} \quad \eta < \eta^*(s); \\ a_e/\rho, & \text{if} \quad \eta \geq \eta^*(s). \end{cases}$$

Set $\kappa(\eta, s)$ from (E.11). Finally, substitute (E.12) into (E.13), and use the result to solve for σ_s, i.e.,

$$\sigma_s = \left(\frac{1 - (\kappa - \eta)\partial_\eta \log q}{\partial_s \log q}\right)\sqrt{\frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}}, \quad \text{on} \quad \{(\eta, s) : \eta < \eta^*(s),\, s \in (0,1)\}. \tag{E.17}$$

Since $\partial_s \log q > 0$ on the set $\{(\eta, s) : \eta < \eta^*(s),\, s \in (0,1)\}$, the solution (E.17) is well-defined. On the complement of this set, we can assume any value for σ_s. Note that μ_s is indeterminate, and we pick it to ensure that the bivariate diffusion $(\eta_t, s_t)_{t \geq 0}$ is stationary. By construction, the equilibrium conditions (E.11)-(E.13) are all satisfied.

E.4 Idiosyncratic uncertainty kills sunspots

To investigate whether it is possible to simply layer on sunspot volatility on top of a fundamental equilibrium, l second by trying to add sunspot volatility in the presence of idiosyncratic fundamental shocks We now pursue a different angle by adding idiosyncratic risk. Individual capital evolves in an individual agent's hands as

$$dk_{i,t} = k_{i,t}[gdt + \tilde{\sigma} d\tilde{B}_{i,t}],$$

where $(\tilde{B}_i)_{i \in [0,1]}$ is a continuum of independent Brownian motions. As before, suppose Z is a one-dimensional Brownian motion (a sunspot shock), independent of all $\tilde{B}_i$. Conjecture

$$dq_t = q_t[\mu_{q,t} dt + \sigma_{q,t} dZ_t].$$

We will attempt to solve for a Markov equilibrium in η, in which σ_q is not identically zero.

Following the same analysis as Section 2, equation (13) still determines q if $\sigma_q \neq 0$. In other words, the candidate sunspot equilibrium of this model has a solution (q, κ), both as functions of η, which are independent of the amount of idiosyncratic risk $\tilde{\sigma}$. Denote $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$ the boundary point where households begin managing capital. This is also independent of $\tilde{\sigma}$.

Next, equation (9), the combination of expert and household capital FOCs, is now modified to read

$$0 = \min\left[1 - \kappa, \, \eta(1-\eta)\frac{a_e - a_h}{q} - (\kappa - \eta)(\tilde{\sigma}^2 + \sigma_q^2)\right].$$

We can use this equation to solve for σ_q, given the solutions (q, κ). We get

$$\sigma_q^2 = -\tilde{\sigma}^2 + \frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1.$$

Since $\sigma_q^2 \geq 0$ is required, an immediate consequence is that $\tilde{\sigma}$ high enough eliminates the existence of any sunspot volatility. We collect these results in the following proposition.

Proposition E.1. *Let (q, κ, η^*) be given by the BSE of Proposition 1. If capital has idiosyncratic shocks with volatility $\tilde{\sigma}$, and $\tilde{\sigma}^2 \geq \sup_{\eta < \eta^*} \frac{\eta(1-\eta)}{\kappa(\eta) - \eta}\frac{a_e - a_h}{q(\eta)}$, any Markov equilibrium in η requires $\sigma_q = 0$.*

Intuitively, there is a trade-off between endogenous volatility σ_q and exogenous volatility $\tilde{\sigma}$. With higher idiosyncratic volatility $\tilde{\sigma}$, amplification of the aggregate sunspot shock is necessarily reduced. To understand this, consider Merton's optimal capital portfolio when there is only idiosyncratic volatility

$$\frac{k_j}{n_j} = \frac{a_j/q + g - r}{\tilde{\sigma}^2}, \quad j \in \{e, h\}.$$

As $\tilde{\sigma}$ increases the optimal capital demand becomes much more inelastic to changes in the capital price q. Thus, for a given shift in the wealth distribution η and change in capital price q, the amount of capital that changes hands between experts and households will be dampened as $\tilde{\sigma}$ increases. But it is exactly such capital purchases/sales which are the key ingredient to our sunspot volatility, allowing price fluctuations to be self-fulfilled. As $\tilde{\sigma}$ increases, this mechanism is weakened, leading to a decrease in σ_q. Eventually, the mechanism is severed altogether because $\sigma_q^2 < 0$ is not possible.

E.5 Correlation between sunspots and fundamentals

What happens if sunspot shocks are correlated with fundamental shocks? To model this, we allow

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}^{(1)} dZ_t^{(1)} + \sigma_{s,t}^{(2)} dZ_t^{(2)}.$$

In section 4, we restricted attention to $\sigma_{s,t}^{(1)} = 0$. Without this assumption, equations (19) and (20) are modified to read:

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}\Big(\frac{(\sigma + \sigma_s^{(1)}\partial_s \log q)^2 + (\sigma_s^{(2)}\partial_s \log q)^2}{(1-(\kappa-\eta)\partial_\eta \log q)^2}\Big)\right]$$

$$\sigma_q = \frac{\binom{1}{0}(\kappa-\eta)\sigma\partial_\eta \log q + \sigma_s \partial_s \log q}{1-(\kappa-\eta)\partial_\eta \log q}.$$

The rest of the equilibrium restrictions are identical.

For the present illustration, we additionally assume that $\sigma_{s,t}^{(2)} = 0$, i.e., sunspot shocks *only* load on fundamental shocks. What emerges is the possibility that sunspot shocks "hedge" fundamental shocks: we can have $\sigma_s^{(1)}\partial_s \log q < 0$, which lowers return volatility and raises asset prices. In the extreme, if $\sigma_s^{(1)}\partial_s \log q \to -\sigma$, the economy will converge to the Brownian Sunspot Equilibrium (BSE) of Section 2. At the other end, if $\sigma_s^{(1)}\partial_s \log q \to 0$, the economy resembles the non-sunspot equilibrium (NSE) but with positive fundamental shocks and amplification of those shocks (this NSE was q^0 in our baseline construction in Section 4). Thus, by constructing our conjectured capital price function as a convex combination of the BSE and the NSE, with weight $1-s$ on the BSE and s on the NSE, we can ensure that $\sigma_s^{(1)}\partial_s \log q$ endogenously emerges negative. Figure E.4 displays the equilibrium constructed this way.

Figure E.5 displays the distribution of capital prices and return volatility in this sunspot equilibrium, relative to the distributions in the non-sunspot equilibrium (in which $\sigma_s^{(1)} \equiv 0$). As promised, the presence of $\sigma_s^{(1)}$ allows sunspots to raise asset prices and reduce volatilities.

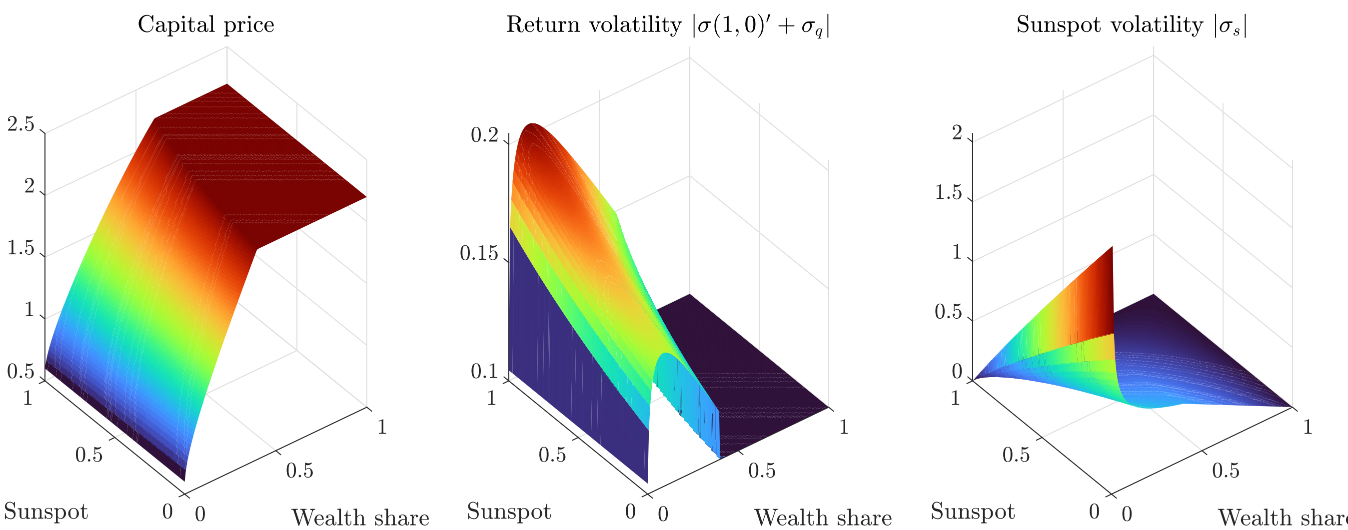

Figure E.4: Capital price q, volatility of capital returns $|(1,0)'\sigma + \sigma_q|$, and sunspot shock volatility $|\sigma_s|$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.10$.

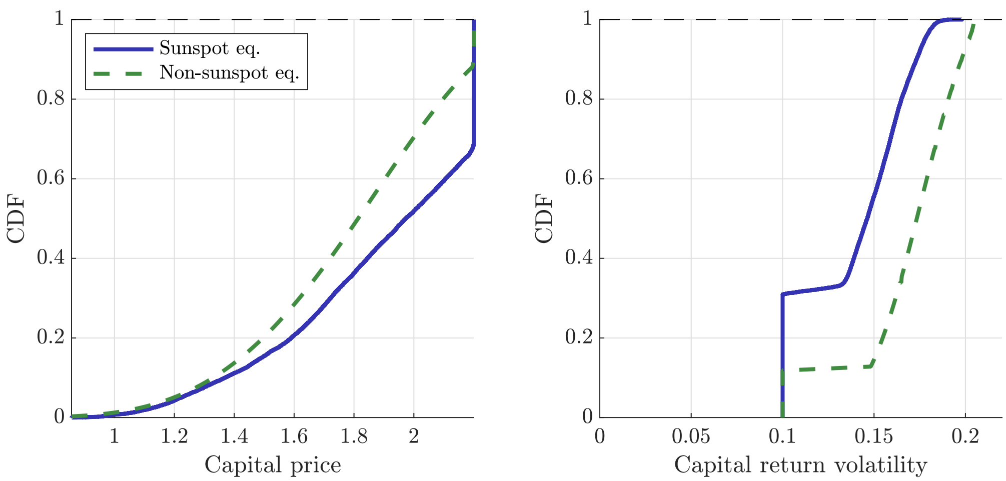

Figure E.5: Unconditional CDFs of capital prices and capital return volatility. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.10$. OLG parameters: $\nu = 0.1$ and $\delta = 0.04$. In this example, we set the sunspot drift $\mu_s = 2 + 2\log(1 - s/s_{\max})$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.