Fear, Indeterminacy, and Policy Responses*

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Abstract

We study the *global* dynamics of the fully *stochastic nonlinear* version of the New Keynesian model and unveil a new class of equilibria characterized by self-fulfilled beliefs about volatility. Self-fulfilling volatility is a real not nominal phenomenon, so these new equilibria survive even in the rigid-price limit. A monetary policy rule that only responds to inflation and output is unable to trim these new equilibria, regardless of its aggressiveness. An enriched monetary rule specifically targeting risk premia can restore determinacy but becomes infeasible in the presence of a lower bound to interest rates. In a variety of specifications, active fiscal policy kills self-fulfilling volatility, showing that the FTPL approach contributes to equilibrium selection beyond disciplining price level dynamics. Overall, this paper supports a theory of macroeconomic dynamics based on equilibrium selection criteria linked to fiscal rather than monetary policy.

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To address topics of inflation, aggregate demand stimulus, and monetary policy, macroeconomists often look to the New Keynesian model for advice. Despite its role as the dominant policy paradigm, this model is plagued by well-known equilibrium multiplicities that influence its answers to those standard macro questions. Currently, there is no consensus on how equilibria are selected and which of the many survive. Among the many alternatives, two popular selection mechanisms are an aggressive monetary policy that responds sufficiently to output and inflation (e.g., the "Taylor principle") versus an active tax and spending policy (e.g., the "Fiscal Theory of the Price Level").

This paper sheds new light on this old controversy by studying the textbook New Keynesian model but with a simple twist: unlike standard practice, we refrain from linearizing the equilibrium around its steady state. Instead, we study the model in its true nonlinear, stochastic form. The model's global dynamics reveal several insights about the nature of the multiplicities and which policies can credibly eliminate them.

As a warm up, and to distinguish our main results, we begin by reviewing the standard deterministic multiplicities in New Keynesian models (Section 2). In that context, we simply generalize the conventional wisdom: a very aggressive Taylor rule can eliminate all equilibria except the steady state. The generalizations involve allowing a nonlinear Phillips curve, which does not alter the conclusions at all, and contrasting linear versus nonlinear Taylor rules.

Our main innovations appear when we study stochastic multiplicities (Section 3). We prove the existence, by construction, of a new class of volatile equilibria that no Taylor rule can completely eliminate. Due to this immunity, our volatile equilibria contrast sharply with the conventional deterministic multiplicities. The key feature that arises in a non-linearized stochastic equilibrium is the presence of a risk premium in agents' Euler equation ("IS curve"). This risk premium, not volatility *per se*, is in fact the source of stochastic equilibrium multiplicities.

Stemming from our volatile equilibria are two additional implications. First, volatility-based indeterminacy is a recessionary phenomenon: it arises only if output is below potential. In a boom, a higher risk premium induces savings that raises agents' consumption growth rate in a way that is unsustainable; in a recession, risk premia sustainably push consumption back toward steady state. Second, because the risk premium is a real object, this indeterminacy is a real rather than nominal phenomenon, in contrast to several theories of self-fulfilling inflations or deflations. In particular, all of these results hold in the rigid-price limit.

We turn next to the Fiscal Theory of the Price Level (Section 4). To be clear about how fiscal policies work to select equilibria, we exclusively consider non-distortionary lump-sum taxes and transfers. In a variety of different settings—including arbitrary exogenous surplus-to-output ratios, different forms of fiscal "rules" that respond to inflation or the output gap, long-term or short-term debt, and different utility functions—FTPL generically kills the volatile equilibria we discovered. Why? The overarching intuition is that, when surpluses are not designed in the knife-edge way to fully absorb output volatility, such short-run volatility must be absorbed either by nominal debt valuation or the price level; sticky prices say that the price level cannot, and the flow budget constraint says nominal debt value cannot either. Consequently, sunspot demand volatility can never be self-justified.

Interestingly, FTPL succeeds as a selection device with or without inflation (e.g., all these results hold as well in the rigid-price limit). The rigid-price limit case effectively corresponds to inflation-indexed government debt, which the FTPL literature typically regards as ineffectual for equilibrium selection. Nevertheless, demand volatility is pinned down by fiscal considerations, even if the price level isn't. For these reasons, we advance an interpretation of FTPL as broadly a theory of aggregate demand management, rather than just a theory of the price level. We conclude that active fiscal policies, in contrast to monetary policies, sharply trim the real indeterminacies endemic to New Keynesian models.

Related literature. This paper relates to two vast literatures: (i) on New Keynesian indeterminacies and (ii) on FTPL as equilibrium selection. We discuss these and various other connections in sequence.

Going back to Sargent and Wallace (1975), it has been widely recognized that exogenous interest rate paths, including pegs, do not pin down the equilibrium. In related work, Benhabib, Schmitt-Grohé, and Uribe (2001) showed that the zero lower bound (ZLB) can lead to "deflationary trap" equilibria, in which low inflation expectations are self-fulfilled by recessionary deflation. More recently, Benigno and Fornaro (2018) showed there can also be "stagnation trap" equilibria, in which low growth expectations are self-fulfilled by low R&D investment at the ZLB. Set against this background, the present paper describes "volatility trap" equilibria. The main distinctive property of volatility trap equilibria is that risk premia are crucial to the self-fulfilment mechanism. (While we primarily study our indeterminacies in the context of monetary policy rules, the appendix shows how the same insights emerge with optimal discretionary monetary policy that is constrained by the ZLB, closer to the papers cited above.)

According to our analysis, active fiscal policies are credible equilibrium pruning devices in New Keynesian models. Seminal contributions to the FTPL literature include

Leeper (1991), Sims (1994), Woodford (1994), and Woodford (1995).[1] Our paper differs in two respects. First, we more often emphasize real indeterminacies and consequently frequently study the rigid-price limit of the model. Second, we analyze the fully nonlinear, stochastic, global dynamics of the model—see also Bassetto and Cui (2018), Mehrotra and Sergeyev (2021), Brunnermeier, Merkel, and Sannikov (2020), and Li and Merkel (2020) for fiscal theory applied to a stochastic nonlinear world.[2]

Volatility trap equilibria are fundamentally nonlinear phenomena. In an important contribution, Caballero and Simsek (2020) study a nonlinear version of the New Keynesian model and illustrate how risk premia are critical to aggregate demand dynamics, but restricting attention to the "fundamental equilibrium." Even more closely related, contemporaneous work by Lee and Dordal i Carreras (2023) also studies a nonlinear IS curve with risk premia driving the multiplicity. Like us, they also argue that "active" Taylor rules do not prune this type of volatility (although they only entertain equilibria local to steady state). Our paper differently performs a more general construction of volatility that can survive in the ergodic distribution. But the most important difference is our exploration of active fiscal policy in tandem with volatility.

At times, our paper is critical of Taylor rules as equilibrium selection devices, following some insights from Cochrane (2011). Recently, Neumeyer and Nicolini (2022) have shown, in a precise sense, that destabilizing Taylor rules are not credible. Overall, one core message conveyed by our results is that fiscal policies are better suited than monetary policies to trim indeterminacies in New Keynesian models. As a plausible alternative, future research could investigate the common knowledge perturbation of Angeletos and Lian (2023), which they applied to the linearized New Keynesian model, in a nonlinear setting like ours.

1 Model

We present a canonical New Keynesian economy with complete markets and nominal rigidities. The setup is a continuous-time version of the model exposited in Galí (2015), which the reader can consult for additional details.

[1]See also Kocherlakota and Phelan (1999) highlighting FTPL as an equilibrium selection criterion; Cochrane (2001) for the important extension to long-term debt; Bassetto (2002) for some microfoundations of the off-equilibrium behavior; and Cochrane (2005) for a discussion of how FTPL is really about the "debt valuation equation" and not the "government budget constraint." Bassetto (2008) and Sims (2013) write excellent reviews. The recent textbook Cochrane (2023) synthesizes all of these results and presents new ones that we will refer to repeatedly throughout the text.

[2]Other recent papers studying the FTPL in nonlinear, but deterministic, environments with "liquidity premia" include Berentsen and Waller (2018), Williamson (2018), and Andolfatto and Martin (2018).

Sunspot shocks. Our baseline model features no fundamental uncertainty in preferences or technologies. Nevertheless, we want to allow the possibility that economic objects evolve stochastically due to coordinated behavior. To do this, we introduce a standard Brownian motion Z that is extrinsic to all economic primitives. All random processes will be adapted to Z.[3]

Preferences. The representative agent has rational expectations and time-separable utility with discount rate ρ, unitary EIS, and labor disutility parameter φ:

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(\log(C_t) - \frac{L_t^{1+\varphi}}{1+\varphi}\Big)dt\Big]. \tag{1}$$

Later, we will generalize some of our arguments to the CRRA utility $\frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\varphi}}{1+\varphi}$. Consumption C_t has the nominal price P_t and labor L_t earns the nominal wage W_t.

Technology. The consumption good is produced by a linear technology $Y_t = L_t$. We abstract from fundamental uncertainty (e.g., productivity shocks) for maximal clarity.

Behind the aggregate production function is a structure common to most of the New Keynesian literature. In particular, there are a continuum of firms who produce intermediate goods using labor in a linear technology. These intermediate goods are aggregated by a competitive final goods sector. The elasticity of substitution across intermediate goods is a constant ε. The intermediate-goods firms behave monopolistically competitively and set prices strategically, described next.

Price setting. Intermediate-goods firms set prices strategically, taking into consideration the impact prices have on their demand. Price setting is not frictionless: firms changing their prices are subject to quadratic adjustment costs, a la Rotemberg (1982). (For simplicity, we assume these adjustment costs are non-pecuniary, so that resource constraints are not directly affected by price adjustments.) In the interest of exposition, we relegate the statement of and solution to this standard problem to Appendix B.

Definition: inflation and output gap. Let P_t denote the aggregate price level and $\pi_t := \dot{P}_t/P_t$ its inflation rate. Note also that the flexible-price level of output is given by $Y^* = (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{1+\varphi}}$. Following the literature, define the output gap $x_t := \log(Y_t/Y^*)$. Conjecture

[3] In the background, the Brownian motion Z exists on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, assumed to be equipped with all the "usual conditions." All equalities and inequalities involving random variables are understood to hold almost-everywhere and/or almost-surely.

that x_t and π_t have dynamics of the form

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t \tag{2}$$

$$d\pi_t = \mu_{\pi,t}dt + \sigma_{\pi,t}dZ_t \tag{3}$$

for some $\mu_x, \sigma_x, \mu_\pi, \sigma_\pi$ to be determined in equilibrium.

Monetary policy. Let ι_t denote the nominal short-term interest rate, which is controlled by the central bank. Monetary policy follows a Taylor rule that targets the output gap and inflation with

$$\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) \tag{MP}$$

for some target rate $\bar{\iota}$ and some response function that satisfies $\Phi(0,0) = 0$. A common linear example that we will use sometimes is $\iota_t = \bar{\iota} + \phi_x x_t + \phi_\pi \pi_t$. When prices are fully rigid, the rule only will only respond to the output gap. In the main paper, we abstract from the zero lower bound (ZLB), which induces well-known indeterminacy issues, and analyze it in Appendix E. For now, think of negative interest rates as a proxy for unconventional monetary policy that can work even when the short rate is zero.

Financial markets. Financial markets are complete. Let M_t be the real stochastic discount factor induced by the real interest rate $r_t := \iota_t - \pi_t$ and the equilibrium price of risk h_t associated to the sunspot shock Z_t. The risk-free bond market is in zero net supply—this will be generalized in Section 4 when we introduce fiscal policies. The equity market is a claim on the profits of the intermediate-goods producers. Alternatively, we can think of these profits as being rebated to the consumers lump-sum.

Definition 1. An *equilibrium* is processes $(C_t, Y_t, L_t, W_t, P_t, M_t, B_t, \iota_t, r_t, \pi_t)_{t\geq 0}$, such that

(i) Taking (M_t, W_t, P_t) as given, consumers choose $(C_t, L_t)_{t\geq 0}$ to maximize (1) subject to their lifetime budget and No-Ponzi constraints[4]

$$\Pi_0 + \mathbb{E}\Big[\int_0^\infty M_t \frac{W_t L_t}{P_t} dt\Big] \geq \mathbb{E}\Big[\int_0^\infty M_t C_t dt\Big] \tag{4}$$

$$\lim_{T\to\infty} M_T \frac{B_T}{P_T} \geq 0, \tag{5}$$

[4] In addition, to prevent arbitrages like "doubling strategies" that can arise in continuous time, we must impose a uniform lower bound on borrowing, e.g., $B_t/P_t \geq -\underline{b}$, although $\underline{b}$ can be arbitrarily large.

where Π represents the real present-value of producer profits and B represents the bond-holdings of the consumer.

(ii) Firms set prices optimally, subject to their quadratic adjustment costs.

(iii) Markets clear, namely $C_t = Y_t = L_t$ and $B_t = 0$.

(iv) The central bank follows the interest rate rule (MP) for some target rate $\bar{\iota}$ and some response function $\Phi(\cdot)$.

In what follows, we refer to a *deterministic equilibrium* as an equilibrium with no real volatility, $\sigma_x \equiv 0$. A *sunspot equilibrium* is an equilibrium with real volatility, $\sigma_x \neq 0$.

Equilibrium characterization. We first provide a summary characterization of all equilibria. Labor supply and consumption decisions satisfy the following optimality conditions:

$$e^{-\rho t} L_t^{\varphi} = \lambda M_t \frac{W_t}{P_t} \tag{6}$$

$$e^{-\rho t} C_t^{-1} = \lambda M_t, \tag{7}$$

where λ is the Lagrange multiplier on the lifetime budget constraint (4).

On the firm side, Appendix B shows that optimal firm price setting gives rise to aggregate inflation dynamics that satisfy

$$\mu_{\pi,t} = \rho \pi_t - \eta \varepsilon \frac{W_t}{P_t} + \eta(\varepsilon - 1), \tag{8}$$

where η is each firm's degree of price flexibility. Notice that as $\eta \to 0$ (prices changes become infinitely costly), one possible equilibrium is to have $\pi_t \to 0$ for all times. We will assume this "rigid-price limit" is the equilibrium that obtains as $\eta \to 0$.

We use these conditions to obtain an "IS curve" and a "Phillips curve." Applying Itô's formula to (7), we obtain the consumption Euler equation, which may be rewritten in terms of the output gap as

$$\mu_{x,t} = \iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2. \tag{IS}$$

Equation (IS) is the IS curve. Next, divide the FOCs (6)-(7), and use goods and labor market clearing $C_t = Y_t = L_t$ to get $Y_t^{1+\varphi} = \frac{W_t}{P_t}$. Substitute this expression into (8) to obtain

$$\mu_{\pi,t} = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big), \tag{PC}$$

where $\kappa := \eta(\varepsilon - 1)(1+\varphi)$. Equation (PC) is the Phillips curve.

Together with the monetary policy rule (MP), equations (IS) and (PC) form the non-linear "three equation model" in standard New Keynesian models. An equilibrium is completely characterized by these three equations, along with some conditions that discipline explosions. We summarize this characterization in the following lemma.

Lemma 1. *Suppose processes $(x_t, \pi_t, \iota_t)_{t\geq 0}$ satisfy the IS equation (IS), the Phillips curve (PC), and the policy rule (MP) Suppose $|x_t| < \infty$ for almost all t, almost-surely. Then, $(x_t, \pi_t, \iota_t)_{t\geq 0}$ corresponds to an equilibrium of Definition 1. Otherwise if $|x_t| = \infty$ with positive probability, the proposed allocation is not an equilibrium.*

The only nuance to Lemma 1, which we are careful to include, is the condition that x_t not explode in finite time. Since $C_t = e^{x_t}Y^* = L_t$, the representative agent would obtain minus infinite utility if $x_t = -\infty$ or $x_t = +\infty$ for any positive measure of times. Such proposed allocations cannot be equilibria, because they involve coordination to make decisions that are obviously not utility maximizing. Consumers would be individually better off ignoring signals to coordinate, unravelling such a proposed allocation.

On the other hand, asymptotic explosions are not ruled out. Indeed, the transversality condition of the representative consumer automatically holds based on the other equations in Lemma 1.[5] So it is perfectly consistent with equilibrium if x_t or π_t diverge asymptotically, an issue that will arise repeatedly in the following sections. This point is also made by Cochrane (2011) when discussing the validity of various equilibria in these models.

Linearized Phillips curve approximation. To simplify the analysis, we will sometimes use a linearized Phillips curve in place of (PC). In those cases, $e^{(1+\varphi)x_t} - 1 \approx (1+\varphi)x_t$ to first order, and the Phillips curve becomes approximately

$$\mu_{\pi,t} = \rho\pi_t - \kappa x_t. \tag{linear PC}$$

[5] To see this, note that, since price adjustment costs are non-pecuniary, real present value of aggregate profits are $\Pi_t = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t}(Y_s - \frac{W_s L_s}{P_s})ds]$. Put this together with the representative consumer labor income to obtain $\Pi_t + \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t}\frac{W_s L_s}{P_s}ds] = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t}Y_s ds]$. Using the resource constraint $C_t = Y_t$, we therefore have that the consumer lifetime budget constraint (4) holds with equality, meaning transversality holds.

We will occasionally work with (linear PC) instead of (PC), because as will become clear the nonlinearity in the IS curve (IS) is the critically novel element, and not so much the nonlinearity in (PC). (We explore the additional insights gained from the nonlinear Phillips curve in Appendix C.) In this approximation, we will sometimes refer to equilibrium as $(x_t, \pi_t, \iota_t)_{t\geq 0}$ that satisfy (IS), (linear PC), and (MP) such that $|x_t| < \infty$ almost surely.

2 Deterministic Equilibria

We start by describing equilibria without volatility, $\sigma_x \equiv 0$. First, we illustrate the basic indeterminacy that arises in New Keynesian models. Second, we show how super-aggressive monetary policy rules can eliminate this indeterminacy. In the process, we generalize some existing results to nonlinear Phillips curves and use nonlinear Taylor rules as "escape clauses."

2.1 Review: conventional indeterminacy in NK models

There is always an equilibrium with $x = \pi = 0$ forever. Using (IS), this equilibrium is supported by a monetary policy rule with $\bar{\iota} = \rho$.

Can there exist other equilibria? As is well known, the answer to this question hinges on the stability/instability properties of the equilibrium dynamical system for (x_t, π_t). We will review this analysis here. First, we specialize the policy rule (MP) to

$$\iota_t = \rho + \phi_x x_t + \phi_\pi \pi_t. \tag{linear MP}$$

Next, combining (linear MP) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{IS'}$$

Thus, the IS curve is automatically linear in a deterministic equilibrium with a linear Taylor rule. Together, equations (IS') and (linear PC) characterize deterministic equilibria.

The typical determinacy analysis picks an aggressive Taylor rule that renders the above system unstable. The system can be written in matrix form as

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \quad \text{where} \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}. \tag{9}$$

The eigenvalues of $\mathcal{A}$ are both strictly positive, and the system unstable, if $\phi_x > 0$ and $\phi_\pi > 1$. This is the continuous-time version of the eigenvalue conditions in Blanchard and Kahn (1980).

However, even when the Taylor rule is aggressive and destabilizing, the result is still a valid equilibrium, as stressed by Cochrane (2011). Nothing about the model rules out asymptotic explosions. No matter how large the central bank picks ϕ_x and ϕ_π, the explosion will always only be at the infinite horizon. For any policy (ϕ_x, ϕ_π), we have a continuum of valid equilibria, indexed by the initial condition (x_0, π_0).[6]

Proposition 1. *Any initial pair (x_0, π_0) is consistent with a deterministic equilibrium with linearized Phillips curve (linear PC). If $\phi_x > 0$ and $\phi_\pi > 1$, then all deterministic equilibria explode asymptotically, except for the one with $(x_0, \pi_0) = (0, 0)$.*

Remark 1 (Real indeterminacy)**.** *The indeterminacy in this model is conceptually about self-fulfilling demand, rather than inflation per se. To see this, consider the rigid price limit ($\kappa \to 0$) so that $\pi_t \equiv 0$. The equilibrium is then summarized by (IS′), or $\dot{x} = \phi_x x$, which has the solution $x_t = e^{\phi_x t} x_0$. For any ϕ_x, there are a continuum of equilibria indexed by x_0. In what follows, we will often consider the rigid price limit to capture the core indeterminacy and facilitate tractability. Mathematically, this reduces the two-dimensional system for (x_t, π_t) to a one-dimensional dynamical system.*

Remark 2 (Nonlinear Phillips curve)**.** *We have used the linearized Phillips curve here for simplicity and exposition. We analyze the nonlinear Phillips curve in Appendix C, and the conclusion is identical to Proposition 1 but the proof is more complicated.*

2.2 Trimming equilibria with a very active Taylor rule

As we have seen, a typical linear Taylor rule does not ensure unique equilibrium. Instead, as noted by Cochrane (2011), uniqueness often requires an additional policy that pledges to "blow up the world" in case the proposed equilibrium is not followed. In our context, this nuclear option is a very aggressive Taylor rule.

In particular, let us dispense with the linear rule (linear MP) and return to (MP). Suppose the response function takes the nonlinear form

$$\Phi(x, \pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \pi \tag{10}$$

[6] The solutions to (9) take the form $\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \exp(\mathcal{A}t) \begin{bmatrix} x_0 \\ \pi_0 \end{bmatrix}$.

with $\phi_x > 0$ and suppose the target rate is again the natural rate $\bar{\iota} = \rho$. Note that the log-linearized version of (10) renders the linear Taylor rule (linear MP) with $\phi_\pi = 1$.

Combining (10) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) \tag{11}$$

This ODE has solution

$$x_t = \log\left(\frac{1 - Ke^{\phi_x t}}{1 + Ke^{\phi_x t}}\right)$$

where $K = \frac{1-e^{x_0}}{1+e^{x_0}}$. This process diverges in *finite time* for any $x_0 \neq 0$: it explodes at time $T = -\phi_x^{-1}\log(|K|)$. Hence, we have proved by construction the following result.

Proposition 2. *Taylor rules exist such that any deterministic equilibrium has $x_t = 0$ forever.*

The analysis above abstracts from any feedback effects from inflation to output gap by setting a monetary policy rule with $\phi_\pi = 1$. This serves two purposes. First, it emphasizes the focus on self-fulfilling demand and not inflation per se. Equilibrium characterization requires output gap to remain bounded for any finite horizon. There is no such requirement for inflation (e.g., hyperinflation might be an equilibrium outcome). Second, it simplifies the analysis and illustrates the point with examples that permit closed form solutions. As an additional benefit, Proposition 2 holds for either the linearized or non-linear Phillips curves.

Determinacy extends beyond the particular response function (10) that has exactly a one-for-one inflation response. In particular, consider inflation sensitivities of more than one-for-one, such as

$$\Phi(x, \pi) = \phi_x(e^x - e^{-x}) + \phi_\pi \pi, \quad \phi_\pi > 1. \tag{12}$$

While more challenging technically to analyze, this rule also selects the zero output gap equilibrium $x_t = 0$. We demonstrate this result formally in Appendix D.

3 A New Class of Sunspot Equilibria

Now, we demonstrate several new results pertaining to volatility in New Keynesian models. In one sense, the deterministic multiplicity in Proposition 1 already suggests the existence of stochastic sunspot equilibria. On the other hand, Proposition 2 shows that the central bank, by adopting a super-aggressive Taylor rule, should be able to eliminate equilibrium multiplicity. In this section, we will show that this logic is wrong, in particular because of the presence of risk premia. As we will then show, a different type of policy rule, which targets the risk premium, is required to eliminate stochastic multiplicities. Finally, we argue that risk premium targeting is fragile because it requires arbitrarily negative interest rates.

3.1 Constructing volatile equilibria

For concreteness, assume that prices are permanently rigid, i.e., $\kappa \to 0$. This clarifies that we are focusing on real indeterminacy rather than inflation indeterminacy. An additional advantage is that we only need to study the dynamics of the output gap, rather than a two-dimensional stochastic system. Let the policy rule have target rate $\bar{\iota} = \rho$ and nonlinear response function (10).

Combining (MP) with (IS), the drift of x_t is given by

$$\mu_x = \phi_x(e^x - e^{-x}) + \frac{1}{2}\sigma_x^2.$$

Building off of the previous analysis, the question is whether the dynamical system characterized by (μ_x, σ_x) keeps x_t finite forever. But here, the volatility σ_x is determined purely via coordination, and some choices will lead to stability. To see why this is possible, examine instead the dynamics of $y_t := e^{x_t}$ and verify that $y_t > 0$ forever. The drift of y_t is

$$\mu_y = \phi_x(y^2 - 1) + y\sigma_x^2$$

and its diffusion $\sigma_y = y\sigma_x$.

Right away, we see that stability is possible, if agents coordinate on sufficiently high volatility. For example, suppose for some $\nu > 0$,

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{13}$$

Putting these equations together, the dynamics for y_t would be

$$dy_t = \begin{cases} \frac{\nu^2}{y_t}dt + \sqrt{\nu^2 + \phi_x y_t(1 - y_t^2)}dZ_t, & \text{if } y_t < 1; \\ \phi_x(y_t^2 - 1)dt & \text{if } y_t \geq 1. \end{cases} \tag{14}$$

It is relatively straightforward to see that $y_t > 0$ for all t in this example: the process above behaves asymptotically (as $y \to 0$) like a Bessel(3) process, which never hits 0. And consequently, $x_t = \log(y_t)$ does not explode negatively in finite time. Provided $y_0 \leq 1$, the process also does not explode positively in finite time. In fact, because there is no volatility for $y_t \geq 1$, the process will eventually converge to and stay stuck at the efficient level $y_t = 1$ (i.e., the sunspot volatility is only temporary in this example). This entire construction works for any $\nu > 0$. In summary, we have just shown that, given the response function (10), many equilibria exist with different σ_x.

As mentioned, the particular construction above only features transitory volatility. That was only to develop an initial understanding and is easily generalized. For example, suppose agents coordinate on the following volatility process for some $\delta \in (0,1)$:

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1-\delta; \\ 0, & \text{if } y \geq 1-\delta. \end{cases} \tag{15}$$

The induced dynamics of $y_t = e^{x_t}$ are

$$dy_t = \begin{cases} \frac{\nu^2}{y_t} dt + \sqrt{\nu^2 + \phi_x y_t (1-y_t^2)}\, dZ_t, & \text{if } y_t < 1-\delta \\ \phi_x (y_t^2 - 1) dt & \text{if } y_t \geq 1-\delta. \end{cases} \tag{16}$$

Provided $y_0 < 1$, this process will eventually exit the deterministic region, enter the volatile region, and remain inefficiently volatile for an infinite amount of time.[7] Figure 1 presents a numerical example in which the economy is permanently inefficient ($y < 1$), and volatility is not transitory.

The key reason for equilibrium multiplicity is the presence of a risk premium, not the presence of volatility per se. To see this most obviously, contrast to the *linearized* version of the Euler equation, which says

$$\mu_x = \iota - \pi - \rho.$$

There is no risk premium term σ_x^2. Repeating the above analysis in this linearized world,

[7]To see all these points, note that the drift is negative when $y \in (1-\delta, 1)$; without volatility, the process exits the region $(1-\delta, 1)$ in finite time almost-surely. Upon entering the volatile region $(0, 1-\delta)$, the process can move around but will never reach $y = 0$, by the same argument established in the text. Finally, the stationary distribution will additionally have a point mass at $y = 1-\delta$, because the dynamics induce y_t to visit the point $1-\delta$ infinitely often.

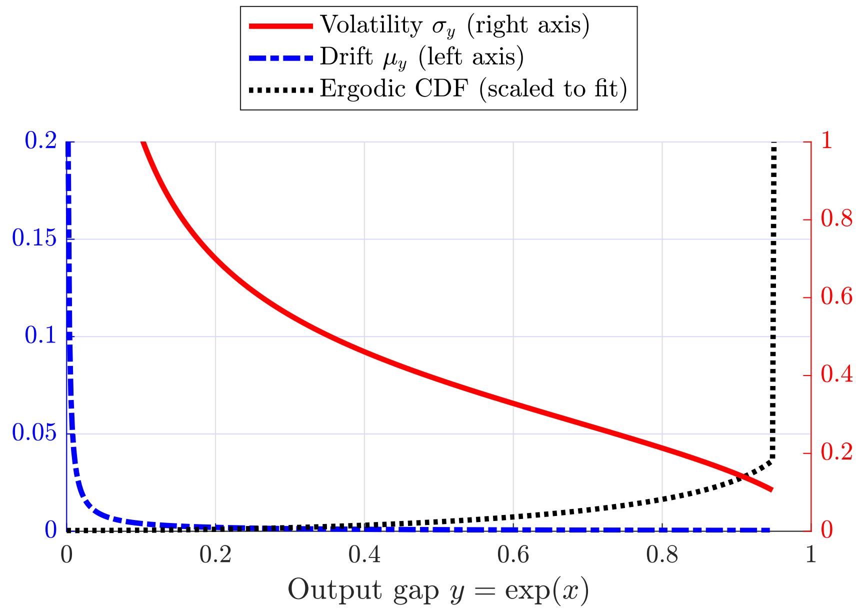

Figure 1: Equilibrium with rigid prices ($\kappa \to 0$) and dynamics given by equations (15)-(16). The stationary CDF is computed via a discretized Kolmogorov Forward equation. The resulting stationary CDF features a mass point at $y = 1 - \delta$. Parameters: $\rho = 0.02$, $\nu = 0.02$, $\delta = 0.05$, $\phi_x = 0.1$.

(14) would be replaced by

$$dy_t = \begin{cases} \phi_x(y_t^2 - 1)dt + \sqrt{\nu^2 + \phi_x y_t(1 - y_t^2)}dZ_t, & \text{if } y_t < 1; \\ \phi_x(y_t^2 - 1)dt, & \text{if } y_t \geq 1. \end{cases} \tag{17}$$

The process in (17) behaves like an arithmetic Brownian motion with negative drift for $y_t \approx 0$. Consequently, one would conclude from the linearized model that $y_t \to 0$ in finite time with positive probability—a nuclear scenario. Thus, the only possible linearized equilibrium can be $y_t = 0$ at all times. A very aggressive Taylor rule trims equilibria in this linearized stochastic world, exactly as in the deterministic equilibria. It is not hard to verify that a similar analysis applies for any arbitrary choice of σ_x.

The presence of a risk premium provides additional stability to the model. Indeed, the risk premium σ_x^2 augments the drift μ_x, pushing the economy back towards 0. We just showed this in a particular example with a specific monetary policy. But perhaps monetary policy could act even more aggressively to destabilize the economy further and eliminate the risk premium effect. Is there some Taylor rule that can kill these equilibria? No. Agents can always coordinate on a level of volatility (hence risk premium) that keeps $y_t > 0$ for *any* level of aggression in the Taylor rule. The following result is proved in Appendix A.

Proposition 3. *Suppose prices are rigid ($\kappa \to 0$). For any Taylor rule (MP) that is increasing in x, there exist a continuum of sunspot equilibria indexed by x_0 and the volatility function $\sigma_x(x)$.*

Intuitively, the idea behind Proposition 3 is contained in the example construction above. For any Taylor rule, agents can coordinate on a level of volatility that "undoes" the effect of interest rates on output gap dynamics. The central bank tries to destabilize the economy, and agents coordinate on a risk premium that stabilizes it.

Comparing Propositions 2-3, our analysis sharply distinguishes stochastic sunspot equilibria from their deterministic equilibria. Typically, they are tightly linked: one often constructs the sunspot equilibria by "randomizing" over deterministic equilibria as in classic studies (Azariadis, 1981). Here instead, the presence of risk premia means that stochastic equilibria can have a markedly different character than their deterministic counterparts.

This result that stability properties can flip in the nonlinear stochastic model is in contrast to the conventional wisdom regarding such models. For example, Cochrane (2023) writes

> this book is really about the broad determinacy and stability properties of monetary models. In one sense, the conclusions of these simple models are likely to be robust, because stability and determinacy depend on which eigenvalues are greater or less than 1. As long as a model modification does not move an eigenvalue across that boundary, the stability and determinacy conclusions are not changed. (Chapter 5.8)

While stability properties may be of some theoretical interest, a practical question to which we now turn is which policies can help trim or eliminate such sunspot equilibria.

3.2 Risk premium targeting

There is one type of rule that can restore determinacy. Suppose we replace the plain-vanilla Taylor rule (MP) with

$$\iota_t = \rho + \Phi(x_t, \pi_t) - (\alpha_- \mathbf{1}_{\{x_t<0\}} + \alpha_+ \mathbf{1}_{\{x_t>0\}})\sigma_{x,t}^2. \tag{MP-vol}$$

The central bank is now targeting not only the output gap but also the risk premium. Although conventional wisdom would suggest that targeting an asset price—which maps one-to-one into the output gap—suffices to target the risk premium, that is not true here, intuitively because coordination on a fearful equilibrium can raise uncertainty $\sigma_{x,t}$ independently, i.e., without affecting x_t in the short run. Rule (MP-vol) directly targets the uncertainty that generates risk premia.

To see how risk premium targeting restores determinacy, substitute rule (MP-vol) into (IS) and rewrite the resulting dynamics in terms of $y_t = e^{x_t}$:

$$dy_t = y_t\Big[\Phi(x_t, \pi_t) - \pi_t + (1 - \alpha(x_t))\sigma_{x,t}^2\Big]dt + y_t\sigma_{x,t}dZ_t, \tag{18}$$

and where $\alpha(x) := \alpha_- \mathbf{1}_{\{x<0\}} + \alpha_+ \mathbf{1}_{\{x>0\}}$ is the state-dependent responsiveness to the risk premium. If $\alpha_- = \alpha_+ = 1$, then the risk premium vanishes from the drift, and we are back in a situation where an aggressive response function Φ can trim equilibria by destabilizing the economy. If $\alpha_+ > 1 > \alpha_-$, then the risk premium itself becomes destabilizing: higher levels of $\sigma_{x,t}^2$ make the drift push x_t further away from zero. Therefore, a modified Taylor rule like (MP-vol), with more aggressive risk premium targeting in bad times, can always eliminate equilibrium multiplicity. Again, for analytical purposes, we state this result in the rigid price limit, with the proof in Appendix A.

Proposition 4. *Suppose prices are rigid ($\kappa \to 0$). Suppose $e^x\sigma_x(x)$ remains bounded as $x \to -\infty$. With sufficiently strong risk premium targeting and sufficiently aggressive responsiveness to the output gap, the modified Taylor rules (MP-vol) ensure that the unique equilibrium is $x_t = 0$.*

The deep difference between the multiplicity of sunspot equilibria and the multiplicity of deterministic equilibria was the presence of a stabilizing risk premium. And this manifests in a qualitatively distinct policy response to restore determinacy: by targeting the risk premium, with the interest rate moving more than one-for-one in bad times, the central bank can use it as a destabilizing nuclear threat.

Remark 3 (Partially-flexible prices)**.** *For analytical convenience, Propositions 3-4 are proved in the rigid price limit. A similar type of analysis could be done with partially-flexible prices, but it is much more tedious and technical. Here is a sketch of the idea. Aggressive risk premium targeting ($\alpha \geq 1$) reduces the drift of x_t, so we have that $x_t \leq \tilde{x}_t$, where $\tilde{x}_t$ follows a related process with $\alpha = 1$:*

$$d\tilde{x}_t = \Big[\Phi(\tilde{x}_t, \pi_t) - \pi_t - \frac{1}{2}\sigma_x(\tilde{x}_t)^2\Big]dt + \sigma_x(\tilde{x}_t)dZ_t, \quad \tilde{x}_0 = x_0.$$

As long as $\tilde{x}_t \to -\infty$ in finite time, so does x_t. But the analysis of $\tilde{x}_t$ is already mostly covered by our previous results. For example, imagine the central bank chooses $\Phi(x, \pi) = \phi_x(e^x - e^{-x}) + \pi$

as in equation (10). Then, the dynamics of $\tilde{y}_t = e^{\tilde{x}_t}$ are

$$d\tilde{y}_t = \phi_x(\tilde{y}_t^2 - 1)dt + \tilde{y}_t \sigma_x(\log(\tilde{y}_t))dZ_t.$$

Assuming the function $\tilde{y} \mapsto \tilde{y}\sigma_x(\log(\tilde{y}))$ is well-behaved, one can prove that $\tilde{y}_t$ hits zero in finite time with positive probability (because it behaves like an arithmetic Brownian motion with negative drift as $\tilde{y}_t \approx 0$). This shows that $\tilde{x}_t \to -\infty$ hence $x_t \to -\infty$ in finite time with positive probability. Using the technique in Appendix D, this argument can then be extended to a response function Φ featuring greater than one-to-one response to inflation.

3.3 Feasibility of aggressive Taylor rules

The policy rules suggested by our analysis, while theoretically interesting, are very aggressive. Are such extreme rules credible?

An immediate thought, following Cochrane (2011), is that "blow up the world" nuclear threats are generically not credible. If the economy ever followed a path away from $x = 0$, could policymakers really commit to sending $x \to -\infty$ in finite time?

Another peculiarity is that all the rules advocated above share the property that $\iota_t \to -\infty$ as $x_t \to -\infty$. This was necessary, in fact: any policy where ι_t remains bounded from below by $\iota_t \geq \underline{\iota}$ cannot trim equilibria. To see this, consider the rigid-price equilibria and inspect output gap dynamics when ι_t is at its lower bound:

$$dx_t = \left[\underline{\iota} - \rho + \frac{1}{2}\sigma_{x,t}^2\right]dt + \sigma_{x,t}dZ_t, \quad \text{when} \quad x_t < 0. \tag{19}$$

In the deterministic case ($\sigma_x = 0$), x_t decays at most linearly, so although the non-zero equilibria may lead to asymptotic explosion, they will not explode in finite time. In the stochastic case, a sufficiently high level of uncertainty can raise the drift and create stable stochastic dynamics. (For instance, a constant variance $\sigma_x^2 > 2(\rho - \underline{\iota})$ induces x_t to behave like an arithmetic Brownian motion with positive drift. More generally, any constant variance is permitted because arithmetic Brownian motions will not diverge in finite time.) We have just proved the following.

Proposition 5. *Suppose prices are rigid ($\kappa \to 0$). If interest rates are lower bounded, $\iota_t \geq \underline{\iota}$, then any $x_0 \leq 0$ corresponds to a valid equilibrium.*

With a zero lower bound (ZLB), or any lower bound, certain monetary threats are not credible. We analyze the ZLB case extensively in Appendix E. There, we even generalize policy by allowing for optimal discretionary monetary policy, yet a tremendous amount of equilibrium multiplicity arises once again, precisely because policy is constrained at the ZLB. In this ZLB case as above, the distinguishing feature of sunspot relative to deterministic equilibria is the presence of risk premia.

In a world with rate constraints such as the ZLB, what restores determinacy? One possibility we pursue is the Fiscal Theory of the Price Level (FTPL). In a world where active monetary policy does not trim equilibria, fiscal policy can even without nuclear threats. We turn to this issue next.

4 Fiscal Theory

Let us now explore a version of "Fiscal Theory of the Price Level" (FTPL). The idea here is to propose some fiscal policies that can prune equilibria. Our contribution to the literature is analysis of FTPL in a nonlinear stochastic model and characterizing its efficacy in a rigid-price limit.

4.1 Equilibrium with lump-sum taxes and transfers

Let us start with a particularly transparent case: lump-sum taxation with government transfers to the representative household. Denote the lump-sum taxes levied by τ_t and the transfers by ξ_t, both in real terms. The real primary surplus of the government is then

$$S_t := \tau_t - \xi_t.$$

Since the government can pick both taxes and transfers, it can effectively choose S_t.

Taxes and transfers do not necessarily offset, so the government borrows by issuing short-term nominally riskless bonds B_t. Later we will generalize to long-term debt. The flow budget constraint of the government is

$$\dot{B}_t = \iota_t B_t - P_t S_t. \tag{20}$$

The nominal interest rate ι_t will be controlled by monetary policy.

Because of the lump-sum nature of the taxes and transfers, there is no impact on the household optimality conditions. Essentially, Ricardian equivalence holds. Indeed, the present-value formula for government debt is

$$\frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big], \tag{GD}$$

where M denotes the real stochastic discount factor process (this is because the transversality condition $\lim_{T\to\infty} \mathbb{E}_t[M_T B_T / P_T] = 0$ holds in our representative agent setup). While the representative household holds the government bonds B_t, it also owes the government future taxes and is owed future transfers. Therefore, the lifetime budget constraint of the representative household is

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}\frac{W_u L_u}{P_u}du\Big] + \frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}S_u du\Big] + \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}C_u du\Big].$$

By (GD), the lifetime budget constraint is equivalent to the budget constraint without any debt at all. And so the household Euler equation is still (IS).

For reference, let us restate the IS curve (IS) and Phillips curve (PC) as the following dynamical system in terms of (x_t, π_t):

$$dx_t = \Big[\iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2\Big]dt + \sigma_{x,t}dZ_t \tag{21}$$

$$d\pi_t = \Big[\rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big)\Big]dt + \sigma_{\pi,t}dZ_t. \tag{22}$$

Together with some nominal interest rate rule for ι_t and some surplus rule for S_t, equilibrium is fully characterized by the government debt valuation (GD) and the dynamical system (21)-(22). Analogously to Lemma 1, we also require x_t to not explode in finite time, because otherwise this would deliver minus infinite utility to the household.

Our previous results did not have government debt or taxes/transfers. However, everything we have said until now still holds with fiscal policies, so long as those policies are "passive" in the language of Leeper (1991). In particular, suppose fiscal policies are chosen so that equation (GD) always holds. Then, government debt valuation can play no role in the analysis, and by the Ricardian equivalence property shown above, the equilibria must be identical to those in Sections 2-3. What we will now characterize is exactly when fiscal policies are "active," as opposed to passive, and thus do provide some equilibrium selection.

4.2 FTPL as equilibrium selection: a first example

Let us now address the question of how fiscal policies trim equilibria.

Consider a fiscal policy with real primary surpluses given by

$$S_t = \bar{s}Y_t, \quad \text{with} \quad \bar{s} > 0. \tag{23}$$

This policy is "active" because its real levels are chosen in a way that does not automatically ensure the government budget constraint holds (e.g., S_t is independent of the price level). Such proportional surpluses are also quite natural, in that they arise in the real world the case of proportional taxes and transfers—although we abstract from the distortionary effects of such policies.

With this policy, and using (GD) along with the FOC (7), we have

$$\frac{B_t}{P_t} = \bar{s}\mathbb{E}_t \int_t^\infty e^{-\rho(u-t)} \frac{Y_t}{Y_u} Y_u du = \rho^{-1} \bar{s} e^{x_t} Y^*.$$

Since B_t/P_t evolves locally deterministically, this proves that x_t also evolves locally deterministically. In particular, $\sigma_x = 0$ is required in such an equilibrium with active fiscal policy. While it is still possible to have inflation volatility $\sigma_\pi \neq 0$, such fluctuations have no real effects. Most strikingly, these results hold for any monetary policy rule and they would hold even in the rigid-price limit $\kappa \to 0$!

How come fiscal policies trim equilibria? The typical analysis suggests *nominal debt* is the key aspect of FTPL. In a world with inflation-indexed government debt, FTPL is inoperative and the price level remains indeterminate (see Chapter 8.1 of Cochrane, 2023). In our model with a permanently rigid price level, we effectively have inflation-indexed debt, and so indeterminacies should remain. As mentioned above, the exact same math goes through even if $\kappa \to 0$. So how are indeterminacies trimmed?

The answer is that the government debt valuation equation (GD) is a "no-default" condition in our rigid price world. Rather than determine the price level, or future inflation, it says that surpluses must eventually be positive enough to justify the current debt value. But given the government's exogenous taxation and spending regime, and without the flexibility that inflation provides, the only way a government can fulfill its no-default commitment is if demand takes a particular path. The government debt valuation equation (GD) thus pins down demand. FTPL is not really a theory of the price level, but a theory of aggregate demand management.

In fact, "demand management" corresponds to the typical stories told about FTPL. Cochrane (2023), Chapter 2.3, writes

> What force pushes the price level to its equilibrium value? ...If the price level is too low, money may be left overnight. Consumers try to spend this money, raising aggregate demand. Alternatively, a too-low price level may come because the government soaks up too much money from bond sales. Consumers either consume too little today relative to the future or too little

overall, violating intertemporal optimization or the transversality condition. Fixing these, consumers again raise aggregate demand, raising the price level.

The key margin of adjustment in these stories is aggregate demand. The equilibrium price that reflects this adjustment would be the price level in a frictionless model. But in models with sticky prices, the price level cannot jump, so the equilibrium adjusts via either future inflation or current and future output, or both. In our specification for surpluses, taxes and spending are proportional to aggregate demand, a property which is then inherited by the debt valuation. And so a necessary feature of a non-volatile debt valuation is a non-volatile demand.

Before generalizing the results just shown, we note that these effects of fiscal policy represent a non-trivial pruning of equilibria. The reader may already anticipate this from our earlier indeterminacy results and from the New Keynesian literature, but to make absolutely sure, we give an example here.

Example 1. Suppose volatilities are constant, i.e., $\sigma_{x,t} = \sigma_x$ and $\sigma_{\pi,t} = \sigma_\pi$. Let us also use the linearized Phillips curve, which replaces $(1+\varphi)^{-1}[e^{(1+\varphi)x} - 1] \approx x$ in equation (22). Finally, suppose also that the interest rate rule takes the linear Taylor form (linear MP), but with an arbitrary target rate $\bar{\iota}$. In this case, the dynamical system (21)-(22) becomes linear. Defining $F_t := (x_t, \pi_t)'$, we have

$$dF_t = [\mu_0 + \mathcal{A}F_t]dt + \sigma dZ_t,$$

where

$$\mu_0 := \begin{bmatrix} \bar{\iota} - \rho + \frac{1}{2}\sigma_x^2 \\ 0 \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}, \quad \text{and} \quad \sigma := \begin{bmatrix} \sigma_x \\ \sigma_\pi \end{bmatrix}.$$

The solution for these linear vector dynamics is

$$F_t = \exp(\mathcal{A}t)\Big[F_0 + \int_0^t \exp(-\mathcal{A}u)\mu_0 du + \int_0^t \exp(-\mathcal{A}u)\sigma dZ_u\Big],$$

Given a monetary policy rule, the entire family of solutions is indexed by the initial value F_0 and the volatilities σ. Hence, there are a continuum of valid stochastic equilibria without active fiscal policy. Among these, FTPL picks the ones that have $\sigma_x = 0$.

It is best to pair passive monetary policy with active fiscal policy (Leeper, 1991). To see this in our linear example, recall that the eigenvalues of $\mathcal{A}$ are both positive when $\phi_x > 0$ and $\phi_\pi > 1$. In such case, the solution for F_t is necessarily explosive asymptotically. An interest rate peg works better. For instance, putting $\bar{\iota} = \rho$ and $\phi_x = \phi_\pi = 0$ implies

$$dF_t = \begin{bmatrix} 0 & -1 \\ -\kappa & \rho \end{bmatrix} F_t dt + \begin{bmatrix} 0 \\ \sigma_\pi \end{bmatrix} dZ_t,$$

which has one stable eigenvalue. Even better is to use $\phi_x < -\rho$ and $\phi_\pi < 1$ so that $\mathcal{A}$ has two stable eigenvalues. Under this monetary policy, the equilibria picked by FTPL will be stable and non-explosive. □

In what follows, we generalize these results. We will explore (i) other exogenous surplus processes; (ii) non-proportionality to output; (iii) fiscal "rules"; (iv) long-term debt; and (v) non-logarithmic utility. In every case, the same equilibrium selection results obtain for FTPL.

4.3 FTPL with more general exogenous surpluses

Our first generalization uses $S_t = s_t Y_t$, where

$$ds_t = \lambda(\bar{s} - s_t)dt + \sigma_{s,t} dZ^s_t, \tag{24}$$

and where Z^s is independent of the sunspot shock Z, and $\sigma_{s,t}$ is an arbitrary potentially stochastic volatility. Surpluses are thus still exogenous but no longer a constant fraction of output.

The present-value equation (GD) still holds, in which case

$$\begin{aligned}
\frac{B_t}{P_t} &= \rho^{-1} e^{x_t} Y^* \mathbb{E}_t \Big[\int_t^\infty \rho e^{-\rho(u-t)} s_u du \Big] \\
&= \rho^{-1} e^{x_t} Y^* \mathbb{E}_t \Big[\int_t^\infty \rho e^{-\rho(u-t)} \Big(e^{-\lambda(u-t)} s_t + (1 - e^{-\lambda(u-t)})\bar{s} + \int_t^u \sigma_{s,t'} e^{-\lambda(u-t')} dZ^s_{t'} \Big) \Big] \\
&= \rho^{-1} e^{x_t} Y^* \Big[\frac{\rho}{\rho+\lambda} s_t + \frac{\lambda}{\rho+\lambda} \bar{s} \Big].
\end{aligned} \tag{25}$$

Again, since B_t/P_t has no loading on the sunspot shock Z_t, and neither does s_t, we necessarily have that x_t is independent of Z_t. Again, this holds for any monetary policy rule and it is unrelated to inflation dynamics.

In this world, the real economy is volatile, however. From equation (25), we have

$$\mathrm{Cov}_t[dx_t, dZ^s_t] = -\frac{\rho \sigma_{s,t}}{\rho s_t + \lambda \bar{s}}.$$

With its stochastic spending needs, the government introduces demand volatility. Absent other considerations, it is optimal for the government to minimize this volatility and set $\sigma_{s,t} = 0$. Of course, risk-management considerations must be balanced against the political realities of taxation and spending, so the true advice is to minimize σ_s to the extent possible.

4.4 FTPL without proportionality

Our second generalization examines a more general dependence of surpluses on output to make sure that the strict proportionality of $S_t = \bar{s}Y_t$ to output is not driving equilibrium selection. Suppose surpluses are now given by

$$S_t = \zeta(x_t)Y_t, \quad \text{with} \quad \zeta(\cdot) > 0. \tag{26}$$

Repeating the analysis from before, the present-value equation (GD) now says

$$\frac{B_t}{P_t} = \rho^{-1}e^{x_t}Y^*\mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(u-t)}\zeta(x_u)du\Big].$$

The key object is $\mathbb{E}_t[\int_t^\infty \rho e^{-\rho(u-t)}\zeta(x_u)du]$. To analyze this object most transparently, let us inspect the rigid price limit ($\kappa \to 0$) so that the dynamics of x are autonomous, hence

$$f(x_t) = \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(u-t)}\zeta(x_u)du\Big]$$

for some $f(\cdot)$. From the pre-determined nature of B_t/P_t, we will have $\sigma_x = 0$ except potentially in the case that $f(x_t)$ is proportional to e^{-x_t}. But such proportionality only arises in the knife-edge case that the interest rate rule is $\iota_t \propto e^{x_t}\zeta(x_t)$.[8]

For example, if the total surplus is independent of output, corresponding to $\zeta(x) = \bar{s}e^{-x}$, then anything other than an interest rate peg will require $\sigma_x = 0$ in equilibrium. If total surplus is increasing in output but less than one-for-one, corresponding to $\zeta(x) = \bar{s}e^{-(1-\epsilon)x}$, then the interest rate rule must be $\iota(x) \propto e^{\epsilon x}$, implying not only a particular functional form but a particular sensitivity to the output gap.

[8]To see this, note that f solves the differential equation

$$0 = \rho\zeta - \rho f + [\iota - \rho + \frac{1}{2}\sigma_x^2]f' + \frac{1}{2}\sigma_x^2 f''$$

Guess that $f(x) = Ke^{-x}$, as required for indeterminacy of σ_x. Plugging this back into the ODE yields $\iota(x) = K^{-1}\rho e^x\zeta(x)$.

4.5 FTPL with fiscal rules

The previous section allowed some notion of a "fiscal rule" where surplus levels responded to the output gap. Our next generalization allows surpluses to respond to endogenous variables in changes, similarly to the interest rate rule. Suppose again that $S_t = s_t Y_t$, where

$$ds_t = \Big(\theta_x x_t + \theta_\pi \pi_t\Big)dt + \sigma_{s,t} dZ_t^s, \tag{27}$$

where again Z^s is independent of the sunspot shock Z. Suppose the interest rate rule is given by the linear Taylor rule (linear MP).

Repeating the debt valuation computation from (GD), we obtain

$$\frac{B_t}{P_t} = \rho^{-1} e^{x_t} Y^* \Big[s_t + f(x_t, \pi_t)\Big], \tag{28}$$
$$\text{where} \quad f(x_t, \pi_t) := \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(T-t)}(s_T - s_t)dT\Big].$$

The reason this expectation can be written as a function solely of x_t and π_t is that

$$\mathbb{E}_t[s_T - s_t] = \int_t^T \mathbb{E}_t\big(\theta_x x_u + \theta_\pi \pi_u\big)du,$$

and because the drifts of (x_t, π_t) are autonomous (this is because both the Taylor rule and surplus rule solely depend on x_t and π_t).

Even without computing the function f, by applying Itô's formula to (28) and examining the loading on the sunspot shock dZ_t, we can say that

$$\sigma_{x,t} = -\frac{\partial_\pi f(x_t, \pi_t)}{s_t + f(x_t, \pi_t) + \partial_x f(x_t, \pi_t)} \sigma_{\pi,t}. \tag{29}$$

Therefore, output volatility inherits inflation volatility, multiplied by a factor capturing the sensitivity of future surpluses, through their endogeneity, to current sunspot shocks. For instance, in the case of fully-rigid prices ($\kappa \to 0$), we necessarily have $\sigma_\pi = 0$ and hence $\sigma_x = 0$ generically.

To make further progress, let us return to our linear Example 1. To keep the algebra simple, specify $\bar{\iota} = \rho + \frac{1}{2}\sigma_x^2$ so that $\mu_0 = 0$. Then, in that linear environment, we may compute

$$f(x, \pi) = \begin{bmatrix} \theta_x \\ \theta_\pi \end{bmatrix} \cdot \hat{\mathcal{A}} \begin{bmatrix} x \\ \pi \end{bmatrix},$$

where $\hat{\mathcal{A}} := (\rho I - \mathcal{A})^{-1}$ is the resolvent matrix of $\mathcal{A}$. Substituting into equation (29), we see that no equilibrium of this linear form can generically exist unless $\sigma_x = \sigma_\pi = 0$.[9]

4.6 FTPL with long-term debt

Our next generalization replaces short-term debt with long-term debt. Such an extension is naturally of interest because short-term debt prices can never respond to shocks (i.e., their interest rate is pre-determined). This may lead one to think that short-term debt mechanically, in a knife-edge sense, rules out self-fulfilling demand volatility.

To fix ideas and keep things tractable, let us assume that debt has a constant exponential maturity structure. Per unit of time dt, a constant fraction ωdt of outstanding debts mature. The per-unit price of this debt is thus

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty \frac{M_T}{M_t}\frac{P_t}{P_T}\omega e^{-\omega(T-t)}dT\Big]. \tag{30}$$

Note that this debt is nominal and thus priced using the nominal SDF M/P. The total outstanding real value of debt is then $Q_t B_t / P_t$.

For concreteness, we will combine this long-term debt setup with proportional process for surpluses $S_t = \bar{s} Y_t$. (Although the following discussion will apply for any surplus process that previously led to the result $\sigma_x = 0$.) Recall result (25) from that section that the total real value of the government debt portfolio equals $\rho^{-1} e^{x_t} Y^* [\frac{\rho}{\rho+\lambda} s_t + \frac{\lambda}{\rho+\lambda}\bar{s}]$. Consequently, we must have

$$\sigma_{Q,t} = \sigma_{x,t}, \tag{31}$$

where σ_Q is the loading of $\log(Q_t)$ on the sunspot shock dZ_t. In other words, the self-fulfilling demand shocks must be absorbed by long-term debt prices. The key question is whether the pricing of long-term debt is consistent with this absorption.

To see that debt pricing is generically inconsistent with (31), consider first an interest rate peg, $\iota_t = \bar{\iota}$. In that case, the nominal SDF takes the form

$$\frac{M_t}{P_t} = \exp\Big[-\bar{\iota}t - \frac{1}{2}\int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u\Big].$$

[9] Indeed, define the row vector $\hat{\Theta} := (\theta_x, \theta_\pi)\hat{\mathcal{A}}$. The term "generically" refers to the fact that $\hat{\Theta}$ has two non-zero entries except in knife-edge cases of the parameters. Then, from equation (29), which says that $(s + \hat{\Theta}(x,\pi)' + \hat{\Theta}_1)\sigma_x + \hat{\Theta}_2\sigma_\pi = 0$, we have that $\sigma_x = \sigma_\pi = 0$.

Using the notation $\tilde{\mathbb{E}}$ for the risk-neutral expectation, the debt price is then

$$Q_t = \tilde{\mathbb{E}}_t\Big[\int_t^\infty \omega e^{-(\bar{\iota}+\omega)(T-t)} dT\Big] = \frac{\omega}{\bar{\iota}+\omega}.$$

Debt prices are constant, $\sigma_Q = 0$, and cannot absorb any demand volatility!

We can allow nominal interest rates to vary, potentially endogenously in response to x and π. To see this, let us generalize this example to have the linear Taylor rule (linear MP). In such case, equation (31) for the sunspot volatility requires the debt price to take the form $Q(x,\pi) = K(\pi)e^x$ for some function K that is independent of x (and furthermore, if K depends on π, then we must have $\sigma_\pi = 0$). But this debt price function is inconsistent with equation (30).[10] More broadly, there is an inconsistency between (31)—which arises due to the valuation of the overall government debt portfolio—and equation (30) that prices each debt contract, unless sunspot volatility is nil.

4.7 FTPL with general CRRA utility

Our final generalization replaces log utility with general CRRA $u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\varphi}}{1+\varphi}$. This extension is of interest because the pricing formulas with log utility almost mechanically imply a non-stochastic demand. In particular, the SDF with log utility is related to the inverse of output, and so the present-value of future surpluses can have no contribution from "discount rate fluctuations."

In this CRRA world, the IS curve (IS) and hence dynamics (21) is replaced by

$$dx_t = \Big[\frac{r_t-\rho}{\gamma} + \frac{1}{2}\gamma\sigma_{x,t}^2\Big]dt + \sigma_{x,t}dZ_t, \tag{32}$$

where $r_t = \iota_t - \pi_t$ is the real rate. The real SDF is thus given by $M_t = e^{-\rho t}Y_t^{-\gamma}$. Besides this CRRA extension, let us return to an environment with short-term debt and the origi-

[10] Pricing equation (30) in differential form reads:

$$-\omega + \iota + \mu_Q + \frac{\omega}{Q} - \sigma_Q\sigma_x = 0,$$

where (μ_Q, σ_Q) are the geometric drift and diffusion of Q. If $Q(x,\pi) = K(\pi)e^x$ (and $\sigma_\pi = 0$ whenever $K'(\pi) \neq 0$), then we have $\mu_Q = \mu_x + \frac{1}{2}\sigma_x^2 + \frac{K'(\pi)}{K(\pi)}\mu_\pi$ and $\sigma_Q = \sigma_x$. Using the expressions in (21)-(22) for (μ_x, μ_π), the differential pricing equation then becomes

$$-\omega - \rho - \pi + \frac{K'(\pi)}{K(\pi)}\Big[\rho\pi - \kappa\big(\frac{e^{(1+\varphi)x}-1}{1+\varphi}\big)\Big] + \frac{\omega}{K(\pi)}e^{-x} = 0$$

This latter equation cannot possibly hold for all x, which contradicts the proposed sunspot equilibrium.

nal surplus specification $S_t = \bar{s}Y_t$, although the following argument is easily generalized to the AR(1) example (24).

There are two formulas for the value of government debt. The first is (GD), which we have been using extensively. The second discounts surpluses with the ex-post realized returns on the debt, i.e., in the case of short-term debt,

$$\frac{B_t}{P_t} = \int_t^\infty e^{-\int_t^T r_u du} S_T dT. \tag{33}$$

If (33) holds path-by-path it also holds in expectation. Specializing these two formulas to the proportional surplus process $S_t = \bar{s}Y_t$, we have, respectively

$$\begin{aligned} &(\text{from (GD)}) \quad \frac{B_t}{P_t} = \bar{s}Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(T-t)} e^{(1-\gamma)(x_T - x_t)} dT\Big] \\ &(\text{from (33)}) \quad \frac{B_t}{P_t} = \bar{s}Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^T r_u du} e^{x_T - x_t} dT\Big] \end{aligned}$$

Using the equilibrium output gap dynamics from (32), these two formulas become

$$\begin{aligned} &(\text{from (GD)}) \quad \frac{B_t}{P_t} = \bar{s}Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\frac{\rho}{\gamma}(T-t)} e^{-\frac{\gamma-1}{\gamma}\int_t^T r_u du} e^{\frac{\gamma}{2}\int_t^T \sigma_{x,u}^2 du + \int_t^T \sigma_{x,u} dZ_u} e^{-\frac{\gamma^2}{2}\int_t^T \sigma_{x,u}^2 du - \gamma \int_t^T \sigma_{x,u} dZ_u} dT\Big] \\ &(\text{from (33)}) \quad \frac{B_t}{P_t} = \bar{s}Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\frac{\rho}{\gamma}(T-t)} e^{-\frac{\gamma-1}{\gamma}\int_t^T r_u du} e^{\frac{\gamma}{2}\int_t^T \sigma_{x,u}^2 du + \int_t^T \sigma_{x,u} dZ_u} dT\Big] \end{aligned}$$

These two formulas are identical except for the martingale $e^{-\frac{\gamma^2}{2}\int_t^T \sigma_{x,u}^2 du - \gamma \int_t^T \sigma_{x,u} dZ_u}$ that is present in the first expression—this martingale is used to convert from the objective probability to the risk-neutral probability. Generically, then, the two formulas can only coincide if this change-of-measure is degenerate (in which case $\sigma_x \equiv 0$) or if the integrand of the second formula is deterministic (in which case $\sigma_x \equiv 0$ is also required). Either way, we cannot have sunspot demand volatility.

As before, this argument does not hinge on the monetary policy rule, nor does it depend on the inflation process (and in particular holds in the rigid-price limit $\kappa \to 0$). The proof above, although different than the previous methods, was more transparent and simple in this case. For completeness, we note that we can also arrive at the same conclusion by analyzing (GD) alone and proving directly that its solution is generically inconsistent with $\sigma_x \neq 0$.[11]

[11] For $\sigma_x \neq 0$ to prevail, it must be that the present value $\mathbb{E}_t[\int_t^\infty e^{-\rho(T-t)} e^{(1-\gamma)(x_T - x_t)} dT] = e^{-x_t} K_t$, where K_t does not have sunspot fluctuations. In the case of a Taylor rule that only depends on (x, π), the joint dynamics of (x_t, π_t) are autonomous, and so $K_t = K(\pi_t)$ for some function K (in addition, we have $\sigma_\pi = 0$ if $K' \neq 0$). Then, rewriting the present value in differential form, and then using the dynamics (21)-(22),

4.8 Summary of FTPL

Using various examples, we have just shown that FTPL generically kills real sunspot volatility in New Keynesian models, corresponding to $\sigma_x = 0$. (The question of whether the equilibrium is unique overall is a question we do not address here.) This type of finding is consistent with conventional wisdom about the power of FTPL as a selection mechanism, and so one can think of our results as generalizing this conventional wisdom to a fully nonlinear, stochastic version of the New Keynesian model.

5 Conclusion

We have shown that macroeconomies with nominal rigidities—New Keynesian models—may inherently permit a novel type of sunspot volatility that appears only in the nonlinear version of the model. The distinguishing features of our volatility are that it is self-fulfilled by the presence of risk premia and can arise only in recessionary times. While monetary policy has almost no power to trim these volatile equilibria, active fiscal policies generically do so. Our results broadly support fiscal policies as effective tools for aggregate demand and risk management, perhaps moreso than monetary policies.

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we have the requirement that

$$\iota = \pi + (1-\gamma)\sigma_x^2 + \frac{K'(\pi)}{K(\pi)}\Big[\rho\pi - \kappa(\frac{e^{(1+\varphi)x}-1}{1+\varphi})\Big]$$

Thus, only one particular feedback rule can be consistent with $\sigma_x \neq 0$.

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Appendix:
Fear, Indeterminacy, and Policy Responses

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A Proofs

Proof of Proposition 3. Since inflation is rigid, consider any Taylor rule with target rate $\bar{\imath}$ and response function $\Phi(x)$ that is increasing in x. The dynamics of x_t are given by

$$\mu_x = \bar{\imath} - \rho + \Phi(x) + \frac{1}{2}\sigma_x^2.$$

The dynamics of $y_t = e^{x_t}$ are given by

$$\mu_y = y[\bar{\imath} - \rho] + y\Phi(\log(y)) + y\sigma_x^2.$$

Specify volatility by, for any $\nu > 0$ and $\delta \in (0,1)$,

$$\sigma_x^2 = \begin{cases} \max\left[0, \left(\frac{\nu}{y}\right)^2 - \Phi(\log(y))\right], & \text{if } y < 1-\delta; \\ 0, & \text{if } y \geq 1-\delta. \end{cases} \tag{A.1}$$

Since $\Phi(\cdot)$ is increasing and continuous, we may re-write (A.2), for some $\hat{\delta} > 0$ determined by the unique solution to $(\frac{\nu}{1-\hat{\delta}})^2 = \Phi(\log(1-\hat{\delta}))$, as

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 - \Phi(\log(y)), & \text{if } y < 1-\max(\delta,\hat{\delta}); \\ 0, & \text{if } y \geq 1-\max(\delta,\hat{\delta}). \end{cases} \tag{A.2}$$

Consequently, an identical argument applies as in the text. In particular, the only difference is the term $y[\bar{\imath} - \rho]$ in the drift, but this term vanishes, hence y_t still behaves like a Bessel(3) process, asymptotically as $y \to 0$. □

Proof of Proposition 4. It suffices to prove the proposition in the case $\alpha_- = \alpha_+ = 1$, because when $\alpha_- > 1 > \alpha_+$, the drift of dx_t is increased (decreased) when x_t is positive (negative). And hence the dynamics push x_t further away from zero than they would in the case $\alpha_- = \alpha_+ = 1$. (Formally, standard diffusion comparison theorems imply that $|x_t|$ will be forever further from zero, almost surely, than it would in the case $\alpha_- = \alpha_+ = 1$.)

If $\alpha_- = \alpha_+ = 1$, then the dynamics of $y_t = e^{x_t}$ in the rigid-price limit are given by

$$dy_t = y_t \Phi(\log(y_t))dt + y_t \sigma_x(\log(y_t))dZ_t$$

Recall the assumption that the volatility $y_t \sigma_x(\log(y_t))$ remains bounded as $y_t \to 0$. Write $\bar{\sigma} := \lim_{y \to 0} y\sigma_x(\log(y))$. Choose $\Phi(x) = \frac{\phi_x}{2}(e^x - e^{-x})$. Then, asymptotically as $y \to 0$, the drift of y_t is equal to -1 and the volatility equal to $\bar{\sigma}$. This asymptotic behavior is exactly identical to an arithmetic Brownian motion. Hence, $y_t \to 0$ in finite time with positive probability. This cannot be an equilibrium, and so we must have $\sigma_x = 0$ when $y < 1$. By examining the dynamics of $\tilde{y}_t := 1/y_t$,

$$d\tilde{y}_t = -\tilde{y}_t[\Phi(-\log(\tilde{y}_t)) + \sigma_x(\log(y_t))^2]dt - \tilde{y}_t \sigma_x(-\log(\tilde{y}_t))dZ_t,$$

we can argue analogously that $\tilde{y}_t \to 0$ in finite time with positive probability. Together, these arguments imply the unique equilibrium is $y_t = 1$, hence $x_t = 0$ forever. $\square$

B Inflation Dynamics under Rotemberg

Here, we generalize the sticky-price model of Rotemberg (1982) to our environment. Since firms in our economy are ex-ante identical, they will have identical utilization and price-setting incentives, allowing us to study a representative firm's problem and a symmetric equilibrium.

To set up the representative intermediate-goods-producer problem, let l_t denote the firm's hired labor, at some equilibrium wage W_t. The firm produces $y_t = l_t$. The firm makes its price choice p_t, internalizing its demand $y_t = (p_t/P_t)^{-\varepsilon} Y_t$, where P_t and Y_t are the aggregate price and output. This demand curve comes from an underlying Dixit-Stiglitz structure with CES preferences (with substitution elasticity $\varepsilon > 1$) and monopolistic competition in the intermediate goods sector.

Letting M_t denote the real SDF process, the representative firm solves

$$\sup_{p,l} \mathbb{E}\Big[\int_0^\infty M_t\Big(\frac{p_t}{P_t}y_t - \frac{W_t l_t}{P_t} - \frac{1}{2\eta}\big(\frac{dp_t}{p_t}\big)^2 Y_t\Big)dt\Big] \tag{B.1}$$

$$\text{subject to} \quad y_t = (p_t/P_t)^{-\varepsilon} Y_t \tag{B.2}$$

$$y_t = l_t \tag{B.3}$$

The quadratic price adjustment cost in (B.1) has a penalty parameter η. As $\eta \to 0$ ($\eta \to \infty$), prices become permanently rigid (flexible). We assume that this price adjustment cost is purely non-pecuniary for simplicity (this means that adjustment costs do not affect the resource constraint). Alternatively, we could redistribute these adjustment costs lump-sum to the representative household.

The firm's optimal price sequence solves a dynamic optimization problem. Substituting the demand curve from (B.2) and the production function from (B.3), we may rewrite problem (B.1) as

$$\sup_{\dot{p}} \mathbb{E}_t\Big[\int_t^\infty \frac{M_s Y_s}{M_t Y_t}\Big(\big(\frac{p_s}{P_s}\big)^{1-\varepsilon} - \frac{W_s}{P_s}\big(\frac{p_s}{P_s}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_s}{p_s}\big)^2\Big)ds\Big].$$

Furthermore, note that in the log utility model used in the text, we have $M_t Y_t = e^{-\rho t}$. Letting J denote this firm's value function, the HJB equation is

$$0 = \sup_{\dot{p}_t}\left\{\big(\frac{p_t}{P_t}\big)^{1-\varepsilon} - \frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_t}{p_t}\big)^2 - \rho J_t + \frac{1}{dt}\mathbb{E}_t\big[dJ_t\big]\right\}$$

The firm value function follows a process of the form

$$dJ_t = [\mu_{J,t} + \dot{p}_t \frac{\partial}{\partial p} J_t]dt + \sigma_{J,t} dZ_t,$$

where $\mu_{J,t}$ and $\sigma_{J,t}$ are only functions of aggregate states (not the individual price). The only part that the firm can affect is $\dot{p}_t \frac{\partial}{\partial p} J_t$. Plugging these results back into the HJB equation and taking the FOC, we have

$$0 = -\frac{1}{\eta}\big(\frac{\dot{p}_t}{p_t}\big)\frac{1}{p_t} + \frac{\partial}{\partial p} J_t \tag{B.4}$$

Differentiating the HJB equation with respect to the state variable p_t, we have the envelope condition

$$(\varepsilon - 1)\big(\frac{p_t}{P_t}\big)^{-\varepsilon}\frac{1}{P_t} - \varepsilon\frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon-1}\frac{1}{P_t} = \frac{1}{\eta}\big(\frac{\dot{p}_t}{p_t}\big)^2\frac{1}{p_t} - \rho\frac{\partial}{\partial p} J_t + \frac{1}{dt}\mathbb{E}_t\big[d\big(\frac{\partial}{\partial p} J_t\big)\big], \tag{B.5}$$

where the last term uses the stochastic Fubini theorem. Combining equations (B.4) and (B.5), we have

$$\eta(\varepsilon - 1)\big(\frac{p_t}{P_t}\big)^{-\varepsilon}\frac{1}{P_t} - \eta\varepsilon\frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon-1}\frac{1}{P_t} = \big(\frac{\dot{p}_t}{p_t}\big)^2\frac{1}{p_t} - \rho\big(\frac{\dot{p}_t}{p_t}\big)\frac{1}{p_t} + \frac{1}{dt}\mathbb{E}_t\big[d\big(\big(\frac{\dot{p}_t}{p_t}\big)\frac{1}{p_t}\big)\big] \tag{B.6}$$

At this point, define the firm-level inflation rate $\pi_t := \dot{p}_t / p_t$, note that $\mathbb{E}_t\big[d\big(\pi_t \frac{1}{p_t}\big)\big] =$

$\frac{1}{p_t}\mathbb{E}_t[d\pi_t] - \frac{1}{p_t}\pi_t^2 dt$, and use the symmetry assumption $p_t = P_t$ in (B.6) to get

$$\eta(\varepsilon - 1) - \eta\varepsilon\frac{W_t}{P_t} = -\rho\pi_t + \frac{1}{dt}\mathbb{E}_t[d\pi_t]. \tag{B.7}$$

Equation (B.7) is the continuous-time stochastic Phillips curve, with π_t interpreted also as the aggregate inflation rate (given a symmetric equilibrium).

Finally, note that the firm's optimization problem also requires the following transversality condition (see Theorem 9.1 of Fleming and Soner (2006)):

$$\lim_{T\to\infty} \mathbb{E}_t[M_T Y_T J_T] = 0.$$

However, since the optimality condition (B.6) is independent of J, the transversality condition becomes irrelevant (i.e., it is only a boundary condition on the HJB equation to verify optimality, but does not discipline aggregate inflation).

C Nonlinear Phillips Curve

This section briefly explores the nonlinear Phillips curve, in contrast the linearized version used throughout the paper. We will do this only in the context of deterministic equilibria, for simplicity. For convenience, we repeat this nonlinear equation here:

$$\dot{\pi}_t = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big). \tag{C.1}$$

We also repeat the IS curve after substituting the linear Taylor rule with target rate $\bar{\imath} = \rho$:

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{C.2}$$

A deterministic equilibrium in this environment is (x_t, π_t) that satisfy (C.1)-(C.2).

The following result shows that the nonlinearity of the Phillips curve does not trim deterministic equilibria nor changes the steady state solution. In other words, the result of Proposition 1 is unchanged.

Proposition C.1. *Consider the system (C.1)-(C.2) with $\phi_x > 0$ and $\phi_\pi > 1$. Then, any initial pair (x_0, π_0) is consistent with a deterministic equilibrium. Nevertheless, $(x_t, \pi_t) = (0, 0)$ remains the unique steady state and it is locally unstable.*

Proof of Proposition C.1. Define $f(x) := (1+\varphi)^{-1}[e^{(1+\varphi)x} - 1]$. From (C.1)-(C.2), we have

$$e^{-\rho t}\pi_t - \pi_0 = -\kappa \int_0^t e^{-\rho s} f(x_s) ds \tag{C.3}$$

$$e^{-\phi_x t} x_t - x_0 = (\phi_\pi - 1) \int_0^t e^{-\phi_x s} \pi_s ds. \tag{C.4}$$

Substituting (C.3) into (C.4) and integrating, we have

$$\begin{aligned} e^{-\phi_x T} x_T - x_0 &= (\phi_\pi - 1) \int_0^T e^{-\phi_x t} \Big[e^{\rho t}\pi_0 - \kappa e^{\rho t} \int_0^t e^{-\rho s} f(x_s) ds \Big] dt \\ &= (\phi_\pi - 1) \Big[\Big(\frac{e^{(\rho-\phi_x)T} - 1}{\rho - \phi_x} \Big) \pi_0 - \kappa \int_0^T \int_s^T e^{(\rho-\phi_x)t} e^{-\rho s} f(x_s) dt ds \Big] \\ &= (\phi_\pi - 1) \Big[\Big(\frac{e^{(\rho-\phi_x)T} - 1}{\rho - \phi_x} \Big) \pi_0 - \kappa \int_0^T \Big(\frac{e^{(\rho-\phi_x)(T-s)} - 1}{\rho - \phi_x} \Big) e^{-\phi_x s} f(x_s) ds \Big]. \end{aligned} \tag{C.5}$$

This expression evidently holds for $\rho \neq \phi_x$, but is also continuous in the $\rho = \phi_x$ limit, since $\frac{e^{at}-1}{a} \to t$ as $a \to 0$. Using the fact that $f(x) \geq -(1+\varphi)^{-1}$, and the fact that $\frac{e^{at}-1}{a} > 0$ for any a and all $t > 0$, we have the inequality

$$\int_0^T \Big(\frac{e^{(\rho-\phi_x)(T-s)} - 1}{\rho - \phi_x} \Big) e^{\phi_x (T-s)} f(x_s) ds \geq -\frac{1}{1+\varphi} \frac{1}{\rho - \phi_x} \Big[\frac{e^{\rho T} - 1}{\rho} - \frac{e^{\phi_x T} - 1}{\phi_x} \Big]$$

Using this in (C.5), we obtain

$$x_T \leq e^{\phi_x T} \Big[x_0 + (\phi_\pi - 1) \Big(\frac{e^{(\rho-\phi_x)T} - 1}{\rho - \phi_x} \Big) \pi_0 \Big] + \frac{(\phi_\pi - 1)\kappa}{1+\varphi} \frac{1}{\rho - \phi_x} \Big[\frac{e^{\rho T} - 1}{\rho} - \frac{e^{\phi_x T} - 1}{\phi_x} \Big]$$

so that x_T can only increase upward sub-exponentially in time. This proves that x_t does not diverge upward in finite time.

On the other hand, assume leading to contradiction that x_t diverged to $-\infty$ by some finite time T. By path-continuity of x, we would then have $\sup_{t \in [0,T]} x_t \leq \bar{x}$ for some finite $\bar{x}$, and so $f(x_t) \leq \bar{f} := (1+\varphi)^{-1}[e^{(1+\varphi)\bar{x}} - 1]$. Returning to equation (C.5), we then have

$$x_T \geq e^{\phi_x T} \Big[x_0 + (\phi_\pi - 1) \Big(\frac{e^{(\rho-\phi_x)T} - 1}{\rho - \phi_x} \Big) \pi_0 \Big] - (\phi_\pi - 1)\kappa \bar{f} \frac{1}{\rho - \phi_x} \Big[\frac{e^{\rho T} - 1}{\rho} - \frac{e^{\phi_x T} - 1}{\phi_x} \Big]$$

The right-hand-side does not diverge in finite time, a contradiction.

This proves that x_t does not explode in finite time. From (C.3), we also have that π_t does not explode in finite time.

Finally, from (C.1)-(C.2), the steady state solves

$$-\phi_x x = (\phi_\pi - 1)\kappa\rho^{-1}\Big(\frac{e^{(1+\varphi)x} - 1}{1+\varphi}\Big)$$

The two sides of this equation have opposite slopes in x, so the unique solution is $x = 0$. This then proves that the unique steady state is $(x, \pi) = (0, 0)$. □

D Nuclear Taylor Rule with Inflation

Here, we extend the analysis of the "nuclear Taylor rule" (10) to allow for a more general responsiveness to inflation. Suppose the monetary rule is a target rate $\bar{\imath} = \rho$ and a response function

$$\Phi(x,\pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \phi_\pi \pi, \quad \phi_x > 0,\, \phi_\pi > 1. \tag{D.1}$$

We will show that this entire class of Taylor rules leads to the unique deterministic equilibrium $x_t = 0$, thus generalizing Proposition 2.

Under this rule, the dynamical system for (x_t, π_t) is

$$\dot{\pi}_t = \rho\pi_t - \kappa f(x_t) \tag{D.2}$$

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) + (\phi_\pi - 1)\pi_t \tag{D.3}$$

where $f(x) := (1+\varphi)^{-1}[e^{(1+\varphi)x} - 1]$.

Proposition D.1. *Consider the system (D.2)-(D.3) with $\phi_x > 0$ and $\phi_\pi > 1$. Then, $(x_t, \pi_t) = (0, 0)$ is the unique equilibrium.*

Proof of Proposition D.1. Suppose the solution $(x_t(\phi_\pi), \pi_t(\phi_\pi))_{t\geq 0}$ associated to some $\phi_\pi > 1$ (which is unique prior to an explosion by the standard ODE uniqueness theorem) did not explode in finite time. In that case, because the solution is continuous in ϕ_π (again, there is a standard ODE theorem for this), it would follow that the solution $(x_t(\tilde{\phi}_\pi), \pi_t(\tilde{\phi}_\pi))_{t\geq 0}$ associated with $\tilde{\phi}_\pi < \phi_\pi$ also does not explode in finite time. (Continuity requires this: otherwise, the two solutions would be infinitely far apart at some finite time T when one of the solutions does explode.) But Proposition 2 has already shown that $(x_t(1), \pi_t(1))_{t\geq 0}$ is explosive in finite time, a contradiction. □

E Zero Lower Bound

Let us address the fact that a zero lower bound (ZLB) constrains monetary policy. To simplify the exposition, we work exclusively in the rigid-price limit $\kappa \to 0$, and so inflation is zero ($\pi_t = 0$) and the nominal rate is equal to the real rate ($\iota_t = r_t$). To make matters interesting, we will assume that monetary policy aims to achieve the flexible-price allocation whenever possible, but they are subject to the ZLB $r_t \geq 0$.

In particular, monetary authorities set the nominal rate (hence the real rate) to implement $x_t = 0$ whenever possible, subject to the ZLB. This is the same idea behind the policy in Caballero and Simsek (2020), who consider a version of the New Keynesian model with risky capital. Under this policy rule, zero output gap prevails whenever the real rate is positive, and a negative output gap must arise at the ZLB (because recall raising the interest rate will lower output):

$$0 = \min[-x_t, r_t]. \tag{E.1}$$

In Lemma E.1 below, we show that within the class of equilibria we study, (E.1) is the outcome of optimal discretionary monetary policy (i.e., monetary policy without commitment to future policies). More deeply, the implementation of $x_t = 0$ "whenever possible" itself requires some kind of commitment to off-equilibrium threats, for instance to reduce interest rates if x_t ever fell below 0—this is the standard notion of "active" monetary policy that pervades the New Keynesian literature, but it becomes somewhat hidden by the outcome (E.1). In that sense, the rule (E.1) actually embeds some amount of commitment power.

Lemma E.1. *Optimal discretionary monetary policy—which maximizes (1) subject to $r_t \geq 0$, optimal household and firm decisions, and its own future decisions—implements (E.1).*

Proof of Lemma E.1. Since there is no upper bound on interest rates, the central bank can always threaten r_t high enough to ensure that $x_t \leq 0$. Since positive output gaps are undesirable, they will implement this. Then, we can restate the problem as: optimal discretionary monetary policy seeks to pick a r_t to maximize (1), subject to (IS), $x_t \leq 0$, the ZLB $r_t \geq 0$, and subject to its own future decisions.

We will discretize the problem to time intervals of length Δ and later take $\Delta \to 0$.

Noting that $C_t = e^{x_t} Y^*$, the time-t household utility is proportional to

$$\begin{aligned} \mathbb{E}_t\Big[\int_0^\infty \rho e^{-\rho s} x_{t+s} ds\Big] &\approx \rho x_t \Delta + \mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big] \\ &\approx -\rho\Delta \mathbb{E}_t[x_{t+\Delta} - x_t] + \underbrace{\mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big] + \rho\Delta\mathbb{E}_t[x_{t+\Delta}]}_{\text{taken as given by discretionary central bank}}. \end{aligned}$$

The term with brackets underneath is taken as given by the time-t discretionary central bank, because it involves expectations of future variables that the future central bank can influence.

Thus, taking $\Delta \to 0$, the time-t central bank solves

$$\min_{r_t \geq 0} \mathbb{E}_t[dx_t]$$

subject to the constraints

$$\begin{aligned} &r_t = \rho + \mu_{x,t} - \frac{1}{2}\sigma_{x,t}^2 \\ &x_t \leq 0 \quad \text{and if} \quad x_t = 0 \quad \text{then} \quad \mu_{x,t} = \sigma_{x,t} = 0. \end{aligned}$$

Note that $\sigma_{x,t}$ is independent of policy when $x_t < 0$. There are two cases. If $x_t = 0$, then the constraints imply that $r_t = \rho$. If $x_t < 0$, we may substitute the dynamics of x_t (replacing μ_x from the first constraint) to re-write the problem as

$$\min_{r_t \geq 0} [r_t - \rho + \frac{1}{2}\sigma_{x,t}^2].$$

Since σ_x is taken as given, the optimal solution is $r_t = 0$. Thus, the discretionary central bank optimally sets

$$r_t = \rho \mathbf{1}_{\{x_t = 0\}}.$$

In other words, the complementary slackness condition $x_t r_t = 0$ holds, which together with $r_t \geq 0$ and $x_t \leq 0$ implies (E.1). □

The entire model dynamics are characterized by the IS curve (IS) with volatility when $r_t = 0$ and $x_t < 0$ and not otherwise, i.e.,

$$\mu_{x,t} = (-\rho + \frac{1}{2}\sigma_{x,t}^2)\mathbf{1}_{\{x_t < 0\}}. \tag{E.2}$$

The entire previous analysis from Section 3 goes through with $\phi_x = 0$ and $\bar{\iota} = 0$.

However, just to see a different construction, let $y = e^x$ and suppose

$$\sigma_x = \begin{cases} \nu(1-y), & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{E.3}$$

(If we had set $\sigma_x = \nu/y$ when $y < 1$, then the argument would be identical to that in Section 3.) In this case, the dynamics of y_t are

$$dy_t = y_t\Big[-\rho + \nu^2(1-y_t)\Big]\mathbf{1}_{\{y_t<1\}}dt + y_t(1-y_t)\nu\mathbf{1}_{\{y_t<1\}}dZ_t. \tag{E.4}$$

This process never reaches $y = 0$, since it behaves asymptotically as a geometric Brownian motion as $y_t \to 0$. Thus, we have constructed a valid equilibrium with volatility at the ZLB.

If agents expect volatility to be sufficiently countercyclical, then the volatility is forever recurrent. To see this, suppose $\nu^2 > 2\rho$ so that $\log(y_t)$ has a positive drift as $y_t \to 0$. By standard arguments, y_t will not concentrate mass near $y = 0$ in the long run. On the other hand, the drift of $\log(y_t)$ is negative as $y_t \to 1$, and its volatility vanishes, so y_t will not ever reach $y = 1$ either. There will be a non-degenerate ergodic distribution of y_t, hence volatility $\sigma_{x,t}$. This economy is persistently demand-driven and stuck at the ZLB.

By adding coordinated jumps in σ_x, we believe we can make the equilibria even more realistic. Initially, volatility can be non-existent and the economy sitting at $x_t = 0$. All of a sudden, fear can rise sufficiently that x_t must jump to negative territory. Because of the ZLB, it is not possible for monetary policy to correct this fear-driven recession. The rise in volatility essentially forces r to the ZLB, similar to Caballero and Simsek (2020). Once $x_t < 0$, volatility can vary continuously, and sunspot shocks will be moving demand. Imagine at some later time T, demand reverts back to the flexible-price outcome $x_T = 0$. At some still later date, volatility can re-emerge. In this way, we can construct equilibria that alternate between efficiency and inefficient, self-fulfilling, volatile recessions.