Differential Geometry Notes of 03/25/2013

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1 Isometries

- In this note, S and $\bar{S}$ will always denote regular surfaces.
- **Definition 1.1.** *A differomorphism $\varphi : S \to \bar{S}$ is an* **isometry** *if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$, we have*

  $$\langle w_1, w_2 \rangle_p = \langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle_{\varphi(p)}.$$

  The surfaces S and $\bar{S}$ are then said to be isometric.
- In other words, φ is an isometry if the differential $\mathrm{d}\varphi$ preserves the inner product.
- **Proposition 1.2.** *φ is an isometry if and only if it preserves the first fundamental form.*

  Proof. ($\rightarrow$) Suppose φ is an isometry. Then,

  $$I_p(w) = \langle w, w \rangle_p = \langle \mathrm{d}\varphi_p(w), \mathrm{d}\varphi_p(w) \rangle_{\varphi(p)} = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

  for all $w \in T_p(S)$.

  ($\leftarrow$) Suppose φ preserves the first fundamental form; that is,

  $$I_p(w) = I_{\varphi(p)}(\mathrm{d}\varphi_p(w))$$

  for all $w \in T_p(S)$. Then,

  $$\begin{aligned} 2\langle w_1, w_2 \rangle &= I_p(w_1 + w_2) - I_p(w_1) - I_p(w_2) \\ &= I_{\varphi(p)}(\mathrm{d}\varphi_p(w_1 + w_2)) - I_\varphi(p)(\mathrm{d}\varphi_p(w_1)) - I_\varphi(p)(\mathrm{d}\varphi_p(w_2)) \\ &= 2\langle \mathrm{d}\varphi_p(w_1), \mathrm{d}\varphi_p(w_2) \rangle_{\varphi(p)}, \end{aligned}$$

  and φ is an isometry. □
- **Definition 1.3.** *A map $\varphi : V \to \bar{S}$ of a neighborhood V of $p \in S$ is a* **local isometry** *at p if there exists a neighborhood $\bar{V}$ of $\varphi(p) \in \bar{S}$ such that $\varphi : V \to \bar{V}$ is an isometry.*

  If there exists a local isometry ino $\bar{S}$ at every $p \in S$ the surface S is said to be **locally isometric** *to $\bar{S}$*

  We say that S and $\bar{S}$ are locally isometric to each other if S is locally isometric to $\bar{S}$ and $\bar{S}$ is locally isometric to S.
- It is clear that if $\varphi : S \to \bar{S}$ is a diffeomorphism and a local isoemetry for every $p \in S$, then ϕ is a local isometry globally.
- It may happen that two surfaces are locally isometric without being globally isometric.

- Let $U = \{(u,v) : 0 < u < 2\pi, -\infty < v < \infty\}$
Let $\bar{\mathbf{x}} : U \to \mathbb{R}^3$ be given $\bar{\mathbf{x}}(u,v) = (\cos u, \sin u, v)$, which is a parameterization of a cylinder.
Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $\mathbf{x}(u,v) = p_0 + uw_1 + vw_2$ where $p_0, w_1, w_2 \in \mathbb{R}^3$ and w_1 and w_2 are unit orthogonal vectors. (That is, $\mathbf{x}$ is a parametermization of a plane.)
Define $\varphi = \mathbf{x} \circ \bar{\mathbf{x}}^{-1}$, which is a map from a coordinate neighborhood of a cylinder to a plane.

  We have that φ is a local isometry.

  In particular, each vector w tangent to the cylinder at a point $p \in \bar{x}(U)$ is tangent to a curve $\bar{\mathbf{x}}(u(t), v(t))$ where $(u(t), v(t))$ is a curve in U. Thus, $w = \bar{\mathbf{x}}_u u' + \bar{\mathbf{x}}_v v'$.

  On the other hand, $\mathrm{d}\phi(w)$ is tangent ot the curve $\varphi(\bar{\mathbf{x}}(u(t), v(t))) = \mathbf{x}(u(t), v(t))$. As a result, we have that $\mathrm{d}\varphi_p(w) = \mathbf{x}_u u' + \mathbf{x}_v v'$.

  We have that

$$
\begin{aligned}
\bar{\mathbf{x}}_u &= (-\sin u, \cos u, 0) \\
\bar{\mathbf{x}}_v &= (0,0,1) \\
\bar{E} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_u \rangle = 1 \\
\bar{F} &= \langle \bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v \rangle = 0 \\
G &= \langle \bar{\mathbf{x}}_v, \bar{\mathbf{x}}_v \rangle = 1 \\
E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle w_1, w_1 \rangle = 1 \\
F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = \langle w_1, w_2 \rangle = 0 \\
G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \langle w_2, w_2 \rangle = 1.
\end{aligned}
$$

  Therefore,

$$I_p(w) = \bar{E}(u')^2 + \bar{F}u'v' + \bar{G}(v')^2 = E(u')^2 + Fu'v' + G(v')^2 = I_{\varphi(p)}(\mathrm{d}\varphi_p(w)).$$

  Notee that this isometry cannot be extended to the entire cylinder because the cylinder is not even homeomorphic to a plane. The idea is that any simple closed curve in a plane can be shrunk continuously to a point without leaving the plane. This property is preserved under a homeomorphism. However, a parallel to the cylinder cannot be shrunk continuously to a point. So, there does not exist a homeomorphism between a plane an a point

- **Proposition 1.4.** *Assume the existence of a paramerization $\mathbf{x} : U \to S$ and $\bar{\mathbf{x}} : U \to \bar{S}$ such that $E = \bar{E}$, $F = \bar{F}$, and $G = \bar{G}$. Then the map $\varphi = \bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is a local isometry.*

  Proof. Let $p \in \mathbf{x}(U)$ and $w \in T_p(S)$. Then, w is tangent ot a curve $\mathbf{x}(\alpha(t))$ at $t = 0$, where $\alpha(t) = (u(t), v(t))$ is a curve in U. Thus, w may be written as:

$$w = \mathbf{x}_u u' + \mathbf{x}_v v'.$$

  By definition, the vector $\mathrm{d}\varphi_p(w)$ is tangent to the curve $\varphi(\mathbf{x}(\alpha(t))) = \bar{\mathbf{x}} \circ \mathbf{x}^{-1} \circ \mathbf{x}(\alpha(t)) = \bar{\mathbf{x}}(\alpha(t))$. Hence,

$$\mathrm{d}\varphi_p(w) = \bar{\mathbf{x}}_u u' + \bar{\mathbf{x}}_v v'.$$

  Since,

$$
\begin{aligned}
I_p(w) &= E(u')^2 + Fu'v' + G(v')^2 \\
I_{\varphi(p)}(\mathrm{d}\varphi_p(w)) &= \bar{E}(u')^2 + \bar{F}u'v' + \bar{G}(v')^2,
\end{aligned}
$$

  we can conclude that $I_p(w) = I_{\phi(p)}(\mathrm{d}\varphi_p(w))$ for all $p \in \mathbf{x}(U)$. So, φ is a local isoemetry. $\square$

- Let S be surface of revolution and let

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

where $a \leq v \leq b$, $0 < u < 2\pi$, and $f(v) > 0$, be a paramieterization of S.
The coefficients of the first fundamental form of S with respect to $\mathbf{x}$ is given by:

$$E = (f(v))^2, \qquad F = 0, \qquad G = (f'(v))^2 + (g'(v))^2.$$

- The **caternary** is a curve given by:

$$\begin{aligned} x &= a\cosh v \\ z &= av \end{aligned}$$

where $-\infty < v < \infty$.

- The surface of revolution of the caternary has the following parameterizaton:

$$\mathbf{x}(u, v) = (a\cosh v\cos u, a\cosh v\sin u, av)$$

where $0 < u < 2\pi$, and $-\infty < v < \infty$. The coefficients of the fundamental forms are:

$$E = a^2\cosh^2 v, \qquad F = 0, \qquad G = a^2(1 + \sinh^2 v) = a^2\cosh^2 v.$$

This surface of revolution is called the **cartenoid.**

- The **helicoid** is a regular surface of revolution given by the parametermization:

$$\bar{\mathbf{x}}(\bar{u}, \bar{v}) = (\bar{v}\cos\bar{u}, \bar{v}\sin\bar{v}, a\bar{u})$$

where $0 < \bar{u} < 2\pi$ and $-\infty < \bar{v} < \infty$.
Let us make the following change of parameter:

$$\begin{aligned} \bar{u} &= u \\ \bar{v} &= a\sinh v \end{aligned}$$

where $0 < u < 2\pi$ and $-\infty < v < \infty$.
This is possible since the map is one-to-one. (Hyperbolic sine is a bijection.) Moreover, the Jacobian

$$\begin{vmatrix} \partial\bar{u}/\partial u & \partial\bar{u}/\partial v \\ \partial\bar{v}/\partial u & \partial\bar{v}/\partial v \end{vmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a\cosh v \end{bmatrix} = a\cosh v$$

is non-zero everywhere. (Therefore, this change of variable is a diffeomorphism.)
Therefore, we have another parametermization of the helicoid:

$$\bar{x}(u, v) = (a\sinh v\cos u, a\sinh v\sin u, au)$$

relative to which the first fundamental form is given by:

$$E = a^2\cosh^2 v, \qquad F = 0, \qquad G = a^2\cosh^2 v.$$

It follows that the cartenoid and the helicoid are locally isometric.

- The one-sheeted cone (minus the vertex) is given by:

$$z = +k\sqrt{x^2 + y^2}$$

  where $(x, y) \neq (0, 0)$.

  We shall show that the one-sheeted cone is locally isometric to a plane. The idea is to show that a cone minus a generator can be "rolled" onto a piece of a plane.

  Let $U \subseteq \mathbb{R}^2$ be the open set given in polar coordinates (ρ, θ) where $0 < \rho < \infty$ and $0 < \theta < 2\pi \sin\alpha$ with 2α $(0 < 2\alpha < \pi)$ is the angle at the vertex of the cone. (That is, $\cot\alpha = k$.) Let $F : U \to \mathbb{R}^2$ be the map

$$F(\rho, \theta) = \left(\rho \sin\alpha \cos\left(\frac{\theta}{\sin\alpha}\right), \rho \sin\alpha \sin\left(\frac{\theta}{\sin\alpha}\right), \rho\cos\alpha\right).$$

  We have that $F(U)$ is contained in the cone. This is because

$$k\sqrt{x^2 + y^2} = \cot\alpha\sqrt{\rho^2 \sin^2\alpha} = \rho\cos\alpha = z.$$

  Moreover, when θ takes all the values from the interval $(0, 2\pi \sin\alpha)$, we have that $\theta / \sin\alpha$ takes the all values from the interval $(0, 2\pi)$. Hence, all points except those with $\theta = 0$ (the generator) are covered by $F(U)$.

  We can check easily that F and $\mathrm{d}F$ are one-to-one in U. Therefore, F is a differeomorphism of U onto the cone minus a generator.

  We shall now show that F is an isometry. First, realize that U may be thought of as a regular surface, parameterized by:

$$\bar{\mathbf{x}}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$$

  with $0 < \rho < \infty$ and $0 < \theta < 2\pi \sin\alpha$.

  The coefficients of the first fundamental form is given by:

$$\bar{E} = 1, \qquad \bar{F} = 0, \qquad \bar{G} = \rho^2.$$

  On the other hand, the coefficients of the first fundamental form of the cone relative to F is given by:

$$E = 1, \qquad \bar{F} = 0, \qquad \bar{G} = \rho^2.$$

  So, the cone is locally isometric to the plane.

- The fact that we can compute lengths of curves on a surface S by using only its first fundamental form allows us to introduce a notion of "intrinsic" distance for points in S.

- We may define the **intrinsic distance** $d(p, q)$ between two points of S as the infimum of the length of curves on S joining p and q.

  This distance is clearly greater than or equal to the distance $\|p - q\|$ between p and q as points in $\mathbb{R}^3$. It may be shown that the distance d is invariant uder isometries.

- The notion of isometry is the natural concept of equivalence for the metric properties of regular surfaces.

  A diffeomorphism captures the equivalence from the point of view of differentiability.

2 Conformal Maps

- **Definition 2.1.** *A diffeomorphism $\varphi : S \to \bar{S}$ is called a* **conformal map** *if for all $p \in S$ and all $v_1, v_2 \in T_p(S)$, we have*

  $$\langle \mathrm{d}\phi_p(v_1), \mathrm{d}\phi_p(v_2) \rangle_{\varphi(p)} = \lambda^2(p) \langle v_1, v_2 \rangle_p$$

  where $\lambda^2(p)$ is a nowhere-zero differentiable function on S.

  The surfaces S and $\bar{S}$ are then said to be conformal.

  A map $\varphi : V \to \bar{S}$ of a neighborhood V of $p \in S$ into $\bar{S}$ is a local conformal map at p if there exists a neighborhood $\bar{V}$ of $\varphi(p)$ such that $\varphi : V \to \bar{V}$ is a conformal map.

  If for each $p \in S$, there exists a local conformal map at p, the surface S is said to be locally conformal to $\bar{S}$.

- The geomeetric meaning of the above definition is that the angles (but not necessarily the lengths) are preserved by conformal maps.

  In fact, let $\alpha : I \to S$ and $\beta : I \to S$ be two curves in S which intersects at $t = 0$. Their angle θ at $t = 0$ is given by:

  $$\cos\theta = \frac{\langle \alpha', \beta' \rangle}{|\alpha'||\beta'|}.$$

  A conformal map $\varphi : S \to \bar{S}$ maps these curves into curves $\varphi \circ \alpha : I \to \bar{S}$ and $\varphi \circ \beta : I \to \bar{S}$, which intersect at $t = 0$ and make an angle $\bar{\theta}$ given by:

  $$\cos\bar{\theta} = \frac{\langle \mathrm{d}\varphi(\alpha').\mathrm{d}\varphi(\beta') \rangle}{|\mathrm{d}\varphi(\alpha')||\mathrm{d}\varphi(\beta')|} = \frac{\lambda^2 \langle \alpha', \beta' \rangle}{\lambda^2 |\alpha'||\beta'|} = \cos\theta.$$

- **Proposition 2.2.** *Let $\mathbf{x} : U \to S$ and $\bar{x} : U \to \bar{S}$ be parametermizations such that $E = \lambda^2 \bar{E}$, $F = \lambda^2 \bar{F}$, $G = \lambda^2 \bar{G}$ in U, where λ^2 is a nowhere-zero differentiable function in U. Then, the map $\varphi = \bar{\bar{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \bar{S}$ is a local conformal map.*

- Local conformality is eaisly seen to be an equivalence relation; that is, if S_1 is locally conformal to S_2, and S_2 is loally conformal to S_3, then S_1 is locally conformal to S_3.

- **Theorem 2.3.** *Any two regular surfaces are conformal.*

  The proof is based on the possibility of parametrizing a neighborhood of any point of a regbular surface in such a way tha the coefficienst of the first fundamental form are $E = \lambda^2(u, v) > 0$, $F = 0$, and $G = \lambda^2(u, v)$. Such a coordinate system is called **isothermal**. Once the existence of an isothermal coordinate sysmte of a regular surface S is assume, then S is clearly conformal to a plane. So, by composition, it is locally conformal to any other surface.