

# Quaternion Exponentiation and Logarithm

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This note is about quaternion exponentiation. I'm basing this note on the note by Glenn Rowe [Row].

## 1 Quaternions

- A quaternion is a mathematical object of the form

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where  $a, b, c, d$  are real numbers, and  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are basis vectors that conform to the multiplication rules below:

$$\begin{array}{lll} \mathbf{i}^2 = -1, & \mathbf{ij} = \mathbf{k}, & \mathbf{jk} = -\mathbf{i}, \\ \mathbf{ji} = -\mathbf{k}, & \mathbf{j}^2 = -1, & \mathbf{jk} = \mathbf{i}, \\ \mathbf{ki} = \mathbf{j}, & \mathbf{kj} = -\mathbf{i}, & \mathbf{k}^2 = -1. \end{array}$$

- Let us make note of an interesting property. Let  $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ . In other words,  $s$  is a quaternion with no real part. Then,

$$\begin{aligned} s^2 &= (b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^2 \\ &= -b^2\mathbf{i}^2 - c^2\mathbf{j}^2 - d^2\mathbf{k}^2 + bc\mathbf{ij} + bc\mathbf{ji} + cd\mathbf{jk} + cd\mathbf{kj} + bd\mathbf{ki} + bd\mathbf{ik} \\ &= -b^2 - c^2 - d^2 + bck - bck + cd\mathbf{i} - cd\mathbf{i} + bd\mathbf{j} - bd\mathbf{j} \\ &= -(b^2 + c^2 + d^2) \end{aligned}$$

- The norm of the quaternion  $q$  is defined as

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

- So, if  $s$  is a quaternion with no real part, then

$$s^2 = -\|s\|^2.$$

In particular, for  $k \in \mathbb{N} \cup \{0\}$ ,

$$s^k = \begin{cases} (-1)^{k/2} \|s\|^k, & k \text{ is even} \\ (-1)^{(k-1)/2} \|s\|^{k-1} s, & k \text{ is odd} \end{cases}.$$

- The conjugate of the quaternion  $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$  is defined as

$$q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

Again, if we write  $q = a + s$  where  $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ . Then,  $q = a - s$ . Moreover,

$$qq^* = q^*q = (a + s)(a - s) = a^2 - s^2 = a^2 + \|s\|^2 = a^2 + b^2 + c^2 + d^2 = \|q\|^2.$$

## 2 Quaternion Exponentiation

- Let  $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$  be a quaternion with no real part. Let us assume that  $\|s\| \neq 0$ . We have that

$$\begin{aligned}
 e^s &= \sum_{k=0}^{\infty} \frac{s^k}{k!} \\
 &= \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{s^{2k+1}}{(2k+1)!} \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k}}{(2k+1)!} s \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k}}{(2k)!} + \frac{s}{\|s\|} \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k+1}}{(2k+1)!} \\
 &= \cos \|s\| + \frac{s}{\|s\|} \sin \|s\|.
 \end{aligned}$$

- As a result, for  $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + s$ , we have that

$$e^q = e^{a+s} = e^a e^s = e^a \left( \cos \|s\| + \frac{s}{\|s\|} \sin \|s\| \right).$$

## 3 Quaternion Logarithm

- Let  $q$  be a unit quaternion. Then, we can find  $\theta \in \mathbb{R}$  and  $u = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  such that  $\|u\| = 1$  such that

$$q = \cos \theta + u \sin \theta.$$

- Notice that  $\|u\theta\| = \theta$ , and  $u = (u\theta)/\|u\theta\|$ . It follows that

$$e^{u\theta} = \cos \theta + u \sin \theta$$

As a result, we may say that

$$\log(\cos \theta + u \sin \theta) = u\theta.$$

- For a general quaternion  $q$ , we may write  $q = \|q\|(\cos \theta + u \sin \theta)$ . Hence,

$$\log q = \log (\|q\|(\cos \theta + u \sin \theta)) = \log \|q\| + \log(\cos \theta + u \sin \theta) = \log \|q\| + u\theta.$$

## 4 Rotation and Logarithm

- Let  $q = \cos(\theta/2) + \mathbf{u} \sin(\theta/2)$ . For any vector  $\mathbf{v} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ , it is well-known that  $q\mathbf{v}q^*$  is the rotation of  $\mathbf{v}$  around the axis  $\mathbf{u}$  by an angle of  $\theta$ . As a result, a rotation in  $\mathbb{R}^3$  can be represented by a unit quaternion.
- We can go even further. When we represent a rotation by a unit quaternion  $q$ , we can take the logarithm of  $q$  to get a vector  $u\theta \in \mathbb{R}^3$ . So, a rotation in  $\mathbb{R}^3$  can also be represented by a vector in  $\mathbb{R}^3$ .

## References

- [Row] G. Rowe, *Exponentiation of a quaternion*, <https://physicspages.com/pdf/Group%20theory/Exponential%20of%20a%20quaternion.pdf>, Accessed: 2025-07-14.