Quaternion Exponentiation and Logarithm

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This note is about quaternion exponentiation. I'm basing this note on the note by Glenn Rowe [Row].

1 Quaternions

- A quaternion is a mathematical object of the form

  $$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

  where a, b, c, d are real numbers, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis vectors that conform to the multiplication rules below:

  $$\begin{aligned}
  \mathbf{i}^2 &= -1, & \mathbf{ij} &= \mathbf{k}, & \mathbf{jk} &= -\mathbf{i}, \\
  \mathbf{ji} &= -\mathbf{k}, & \mathbf{j}^2 &= -1, & \mathbf{jk} &= \mathbf{i}, \\
  \mathbf{ki} &= \mathbf{j}, & \mathbf{kj} &= -\mathbf{i}, & \mathbf{k}^2 &= -1.
  \end{aligned}$$

- Let us make not of an interesting property. Let $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. In other words, s is a quaternion with no real part. Then,

  $$\begin{aligned}
  s^2 &= (b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^2 \\
  &= -b^2\mathbf{i}^2 - c^2\mathbf{j}^2 - d^2\mathbf{k^2} + bc\mathbf{ij} + bc\mathbf{ji} + cd\mathbf{jk} + cd\mathbf{kj} + bd\mathbf{ki} + bd\mathbf{ik} \\
  &= -b^2 - c^2 - d^2 + bc\mathbf{k} - bc\mathbf{k} + cd\mathbf{i} - cd\mathbf{i} + bd\mathbf{j} - bd\mathbf{j} \\
  &= -(b^2 + c^2 + d^2)
  \end{aligned}$$

- The norm of the quaternion q is defined as

  $$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

- So, if s is a quaternion with no real part, then

  $$s^2 = -\|s\|^2.$$

  In particular, for $k \in \mathbb{N} \cup \{0\}$,

  $$s^k = \begin{cases} (-1)^{k/2}\|s\|^k, & k \text{ is even} \\ (-1)^{(k-1)/2}\|s\|^{k-1}s, & k \text{ is odd} \end{cases}.$$

- The conjugate of the quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is defined as

  $$q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

  Again, if we write $q = a + s$ where $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Then, $q = a - s$. Moreover,

  $$qq^* = q^*q = (a+s)(a-s) = a^2 - s^2 = a^2 + \|s\|^2 = a^2 + b^2 + c^2 + d^2 = \|q\|^2.$$

2 Quaternion Exponentiation

- Let $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ be a quaternion with no real part. Let us assume that $\|s\| \neq 0$. We have that

$$\begin{aligned}
e^s &= \sum_{k=0}^{\infty} \frac{s^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{s^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k}}{(2k+1)!} s \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k}}{(2k)!} + \frac{s}{\|s\|} \sum_{k=0}^{\infty} \frac{(-1)^k \|s\|^{2k+1}}{(2k+1)!} \\
&= \cos \|s\| + \frac{s}{\|s\|} \sin \|s\|.
\end{aligned}$$

- As a result, for $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + s$, we have that

$$e^q = e^{a+s} = e^a e^s = e^a \left(\cos \|s\| + \frac{s}{\|s\|} \sin \|s\| \right).$$

3 Quaternion Logarithm

- Let q be a unit quaternion. Then, we can find $\theta \in \mathbb{R}$ and $u = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $\|u\| = 1$ such that

$$q = \cos\theta + u \sin\theta.$$

- Notice that $\|u\theta\| = \theta$, and $u = (u\theta)/\|u\theta\|$. It follows that

$$e^{u\theta} = \cos\theta + u\sin\theta$$

  As a result, we may say that

$$\log(\cos\theta + u\sin\theta) = u\theta.$$

- For a general quaternion q, we may write $q = \|q\|(\cos\theta + u\sin\theta)$. Hence,

$$\log q = \log\big(\|q\|(\cos\theta + u\sin\theta)\big) = \log\|q\| + \log(\cos\theta + u\sin\theta) = \log\|q\| + u\theta.$$

4 Rotation and Logarithm

- Let $q = \cos(\theta/2) + \mathbf{u}\sin(\theta/2)$. For any vector $\mathbf{v} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, it is well-known that $q\mathbf{v}q^*$ is the rotation of $\mathbf{v}$ around the axis $\mathbf{u}$ by an angle of θ. As a result, a rotation in $\mathbb{R}^3$ can be represented by a unit quaternion.

- We can go even further. When we represent a rotation by a unit quaternion q, we can take the logarithm of q to get a vector $u\theta \in \mathbb{R}^3$. So, a rotation in $\mathbb{R}^3$ can also be represented by a vector in $\mathbb{R}^3$.

References

[Row] G. Rowe, *Exponentiation of a quaternion*, `https://physicspages.com/pdf/Group%20theory/Exponential%20of%20a%20quaternion.pdf`, Accessed: 2025-07-14.