# Complex Analysis Review

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## 1 Polar Form and Complex Exponentials

- A complex number $z = x + iy$ can be written in polar coordinate $(r, \theta)$ where $r = |z|$,

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta.$$

- The angle $\theta$ is not unique.
We know that $\theta = \tan^{-1}(y/x)$ works.
However, if $\theta$ works, then $\theta \pm 2\pi k$, where $k \in \mathbb{Z}$, also works.

- We call the value of any of these $\theta$ angles the **phase** or the **argument** of $z$. It is denoted by

$$\arg z.$$

For example, we write

$$\arg i = \frac{\pi}{2} + 2\pi k \quad (k \in \mathbb{Z}).$$

Note that $\arg z$ is a multi-valued function.

- To make $\arg z$ a single-valued function, we specify an interval that $\arg z$ can take.
We commonly use $\arg_\tau z$ to denote the argument that takes value from $(\tau, \tau + 2\pi]$.
Note that the function has a $2\pi$ value jump on the line containing complex number whose $\arg z = \tau$.

- The **principal branch** of $\arg z$ is denoted $\operatorname{Arg} z$, and it is defined to be $\operatorname{Arg} z = \arg_{-\pi} z$.

## 2 Analytic Functions

- Let $f$ be a complex-value function defined in a neighborhood of $z_0$. Then the **derivative** of $f$ at $z_0$ is given by

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z_0) \equiv f'(z_0) := \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

provided that the limit exists.

- $f(z)$ is **complex differentiable** at $z_0$ if $f'(z_0)$ exists and independent of how $\Delta z$ approaches 0.

- If $f$ is complex differentiable in an open region, we say that $f$ is **analytic** in that region.

- All polynomials and quotients of polynomials are anlaytic where they are defined.
That is, except where the denominator is 0.

- The function $f(z) = \bar{z}$ is nowhere differentiable.

# 3 Cauchy–Riemann Equatnions

- **Theorem 3.1.** *If $f(z) = u(x,y) + iv(x,y)$ is analytic in an open region, then the following* **Cauchy–Riemann equations***:*

  $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

  *must hold at every point in the region.*

- The converse is almost true:

  **Theorem 3.2.** *If the partial derivatives exist in an open region around point $z_0$, are continuous at $z_0$, and satisfy the Cauchy–Riemann equations at $z_0$, then $f$ is analytic at $z_0$.*

  It follows that, if the partial derivatives exist, are continuous, and satisfy Cauchy–Riemann equations in an open region, then $f$ is analytic in that region.

- If $u(x,y)$ and $v(x,y)$ satisfies the Cauchy–Riemann equations, then they are **harmonic**. That is,

  $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0, \text{ and } \nabla^2 v = \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} = 0.$$

  This is because

  $$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}, \text{ and} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}. \end{aligned}$$

  The same derivation can be shown for $\nabla^2 v$.

- We also have that the contours of constant $u$ and $v$ almost always intersect at right angles.
  The exceptions are the plances where $f'(z) = 0$.
  This is the consequence of $f$ being analytic, so conformal.

- **Example 3.3.** *Find all functions $f(z)$ which are analytic in whole complex plane and have their real part $u(x,y) = xy - \cos x \cosh y$.*

  **Solution.** Since $f$ is analytic everywhere, it must satisfy the Cauchy–Riemann equations everywhere. We have that

  $$\frac{\partial u}{\partial x} = y + \sin x \cosh y = \frac{\partial v}{\partial y}.$$

  Therefore,

  $$v = \frac{y^2}{2} + \sin x \sinh y + g(x)$$

  for some function $g(x)$. Now,

  $$\frac{\partial v}{\partial x} = \cos x \sinh y + g'(x) = -\frac{\partial u}{\partial y} = -x + \cos x \sinh y.$$

  As such,

  $$g'(x) = -x,$$

so $g(x) = -x^2/2 + C$ for some constant $C$. As such, we have that

$$v(x, y) = \frac{y^2}{2} + \sin x \sinh y - \frac{x^2}{2} + C.$$

Now,

$$\begin{aligned}
f(z) &= u(x, y) + iv(x, y) \\
&= xy - \cos x \cosh y + i\left(\frac{y^2}{2} + \sin x \sinh y - \frac{x^2}{2} + C\right) \\
&= -\frac{i}{2}(x^2 - y^2 + 2ixy) - (\cos x \cosh y - i \sin x \sinh y) + iC \\
&= -\frac{i}{2}z - \cos z + iC.
\end{aligned}$$

# 4 Application to Fluid Dynamics

- We let $\mathbf{v}(x, y)$ denote a 2D velocity field.
- We are interested in a velocity field $\mathbf{v}$ such that it is:
  - **Incompressible:** $\nabla \cdot \mathbf{v} = 0$. That is, there exists a scalar function $\psi(x, y)$ such that

    $$\mathbf{v}_x = \frac{\partial \psi}{\partial y}, \text{ and } \mathbf{v}_y = -\frac{\partial \psi}{\partial x}.$$

    Here, $\psi$ is called the **stream function**.
  - **Irrotational:** There exists a scalar function $\phi(x, y)$ such that

    $$\mathbf{v}_x = \frac{\partial \phi}{\partial x}, \text{ and } \mathbf{v}_y = \frac{\partial \phi}{\partial y}.$$

- Note that $\phi$ and $\psi$ satisfies Cauchy–Riemann equations. So, $\phi(x, y) + i\psi(x, y)$ is an analytic function.
- The contours of $\psi$ are all the **streamlines**.
- Fluid flows along stream lines.

# 5 Complex Exponentials, Sines, and Cosines

- We know that the exponential function

  $$e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$$

  is analytic on the whole complex plane.
- Define

  $$\begin{aligned}
  \cos z &= \frac{e^{iz} + e^{-iz}}{2}, \\
  \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, \\
  \cosh z &= \frac{e^z + e^{-z}}{2}, \text{ and} \\
  \sinh z &= \frac{e^z - e^{-z}}{2}.
  \end{aligned}$$

- Familiar trigonometric identities hold with $\cos z$ and $\sin z$.
- Here are some useful identities about hyperbolic functions:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\sinh z &= \cosh z\\ \frac{\mathrm{d}}{\mathrm{d}z}\cosh z &= \sinh z\\ \cos(iz) &= \cosh z\\ \sin(iz) &= i\sinh(z)\\ \cos z &= \cosh(iz)\\ \sin z &= i\sinh(iz)\\ \cosh^2 z - \sinh^2 z &= 1\end{aligned}$$

- We also have that

$$\cos z = \cos(x+iy) = \cos x\cos(iy) - \sin x\sin(iy) = \cos x\cosh y - i\sin x\sinh y,$$

  and

$$\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y.$$

# 6 Complex Logarithms

- If $z \neq 0$, define $\log z$ to be the set of infinitely many values

$$\log z := \ln|z| + i\arg z = \ln|z| + i\mathrm{Arg}\, z + i2\pi k\ (k \in \mathbb{Z}).$$

- Let $f(z)$ be a multi-valued complex function.
  We say that $g(z)$, defined on some open region $D$, to be a **branch** of $f(z)$ if:
  - $g$ is continuous in $D$, and
  - $g(z)$ agrees with one of the possible values of $f(z)$.
- The **principal branch** of the logarithm function $\log z$, denoted by $\mathrm{Log}\, z$, is defined as:

$$\mathrm{Log}\, z := \ln|z| + i\mathrm{Arg}\, z.$$

  It is defined on the set $\mathbb{C} - \{z : \mathrm{Re}\,(z) \leq 0, \mathrm{Im}\,(z) = 0\}$.
- Once a branch of $\log z$ is chosen, we have that

$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z}$$

  at any $z$ such that the branch is defined.
- The point $z_0$ is called a **branch point** for $f(z)$ if $f(z)$ changes its value as one starts out at a point, traces a closed path enclosing $z_0$ and returns to the starting point.
- The point $z = 0$ is a branch point of $\log z$.
- The poin $z = \infty$ is also a branch point of $\log z$.
  This is because, if we let $\zeta = 1/z$. We have that $\log z = -\log\zeta$.
  Hence, there's a branch point at $\zeta = 0$, which means $z = \infty$.

- To see if a finite $z_0$ is a branch point, encircle it with a small loop. If $f(z)$ changes after one circuit, it is a branch point.
- To see if $\infty$ is a branch point, circle *all* finite branch points. If $f(z)$ changes after one circuit, then $\infty$ is a branch point.
- The function $f(z) = z^\alpha = e^{\alpha \log z}$, where $\alpha \notin \mathbb{Z} \cup \{0\}$, is a multi-valued function.

# 7 Complex Integration

- A point set $\gamma$ in the complex plane is said to be a **smooth arc** if it is the range of some continuous complex-value function $z = z(t)$, $a \leq t \leq b$ that satisfies the following conditions:
  - $z(t)$ has a continuous derivative on $[a, b]$,
  - $z'(t)$ does not vanishes on $[a, b]$.
  - $z(t)$ is one-to-one on $[a, b]$.
- A point set $\gamma$ is said to be a **smooth closed curve** if it is the range of some continuous function $z = z(t), a \leq t \leq b$, satisfying the conditions:
  - $z(t)$ has a continuous derivative on $[a, b]$,
  - $z'(t)$ does not vanishes on $[a, b]$.
  - $z(t)$ is one-to-one on $[a, b)$ and $z(a) = z(b)$ and $z'(a) = z'(b)$.
- A **contour** $\Gamma$ is either a single point $z_0$ or a finite sequence of directed smooth curve $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of $\gamma_k$ coincides with the initial point of $\gamma_{k+1}$.

  We write $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.
- If $f(z)$ is a continuous function on $\gamma$, the complex integral on a smooth curve $\gamma$ is given as:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,dt.$$

- If $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\,\mathrm{d}z.$$

- **ML bound.** If $f$ is continuous on the contour $\Gamma$ and if $|f(z)| \leq M$ on every point $z \in \Gamma$, then

$$\left| \int_\Gamma f(z)\,\mathrm{d}z \right| \leq ML$$

  where $L$ is the length of the contour $\Gamma$.
- If $f(z) = F'(z)$ for some function $F$, then

$$\int_\gamma f(z)\,\mathrm{d}z = F(z_T) - F(z_I),$$

  where $z_T$ is the terminal point and $z_I$ is the initial point of the smooth curve $\gamma$.

  As a result, if $\gamma$ is a closed contour, the integral is identically zero.

- Any domain $D$ possessing the property that every loop in $D$ can be continuously deformed in $D$ to a point is called a **simply connected domain**.

- **Theorem 7.1 (Cauchy's Integral Theorem).** *Let $f(z)$ be analytic in a simply connected domain $D$ and $\Gamma$ any loop in $D$. Then,*

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

- **Corollary 7.2.** *If $f$ is analytic in a simply connected region $D$, then $f$ has path independence.*

- **Corollary 7.3 (Deforming the Contour).** *Let $f$ be a function analytic in a domain $D$ containing loops $\Gamma_0$ and $\Gamma_1$. If these loops can be continuously deformed into one another in $D$, then*

$$\int_{\Gamma_0} f(z)\,\mathrm{d}z = \int_{\Gamma_1} f(z)\,\mathrm{d}z.$$

- **Corollary 7.4.** *In a simply connected domain, an analytic function has an anti-derivative, its contour integrals are independent of path, and its loop integrals vanish.*

- Consider the following integral:

$$\int_C (z-z_0)^n\,\mathrm{d}z$$

where $C$ is any circle centered at $z_0$ traversed in the counterclockwise direction (i.e., positively oriented). If $n \neq -1$, then $(z-z_0)^n$ has an anti-derivative, which is $(z-z_0)^{n+1}/(n+1)$. So, the integral vanishes. If $n = -1$, then we deform the circle to the circle of radius one. That is, we can now assume $C = \{z : |z-z_0| = 1\}$. Now, take the parameterization $z(t) = z_0 + e^{it}$ where $0 \leq t \leq 2\pi$. We have that $z'(t) = ie^{it}$. So,

$$\int_C \frac{1}{z-z_0}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{e^{it}}(ie^{it})\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

Thus,

$$\int_C (z-z_0)^n\,\mathrm{d}z = \begin{cases} 0, & \text{if } n \neq -1, \\ 2\pi i, & \text{if } n = -1. \end{cases}$$

It follows that, if $\Gamma$ is any simple positively oriented contour containing $z_0$ in the inside, then

$$\int_\Gamma (z-z_0)^n\,\mathrm{d}z = \begin{cases} 0, & \text{if } n \neq -1, \\ 2\pi i, & \text{if } n = -1. \end{cases}$$

- **Theorem 7.5 (Cauchy's Integral Formula).** *Let $\Gamma$ be any simple positive oriented contour. If $f$ is analytic in some simply connected domain $D$ containing $\Gamma$ and $z_0$ is any point inside $\Gamma$, then*

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

*In general,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z$$

*for any integer $n = 0, 1, 2, \ldots$.*

- Using the Cauchy's integral formula, we can establish the following facts:
  - If $f$ is analytic in a domain $D$, then all of its derivatives exist and are analytic in $D$.
  - If $f = u + iv$ is analytic in a domain $D$, then all partial derivatives of $u$ and $v$ exist and are continuous in $D$.
  - If $f$ is continuous in a domain $D$ and if

$$\int_\Gamma f(z)\,\mathrm{d}z = 0$$

    for every closed contour $\Gamma$ in $D$, then $f$ is analytic in $D$.

- Using the Cauchy's integral formula and the ML bound, we have the following lemma:

  **Lemma 7.6.** *Let $f$ be analytic inside and on a circle $C_R$ of radius $R$ centered about $z_0$. If $|f(z)| \le M$ for all $z$ on the circle $C_R$, then the derivatives of $f$ at $z_0$ satisfies:*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n}.$$

- **Theorem 7.7 (Liouville's).** *The only bounded entire functions are constant functions.*

  *Proof.* Let us say that $|f(z)| \le M$ for all $z$. Take a circle $C_R$ around any point $z_0$ and $n = 1$. We have that $|f'(z_0)| \le M/R$. As $R \to \infty$, we have that the bound goes to zero. Hence, $f'(z_0) = 0$ everywhere. So, $f$ must be constant. □

- **Theorem 7.8 (Fundamental Theorem of Algebra).** *Every non-constant polynomial with complex coefficients has at least one root.*

- **Lemma 7.9 (Mean-value property).** *If $f$ is analytic inside and on the circle $C_R$ of radius $R$ around $z_0$, then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,\mathrm{d}t.$$

  *In other words, $f(z_0)$ is the average of its values around the circle $C_R$.*

- **Lemma 7.10.** *Suppose that $f$ is analytic in a disk centered at $z_0$ and that the maximum value of $|f(z)|$ over this disk is $f(z_0)$. Then $|f(z)|$ is constant in the disk.*

- **Theorem 7.11.** *If $f$ is analytic in a domain $D$ and $|f(z)|$ achieves its maximum value at a point $z_0$ in $D$, then $f$ is constant in $D$.*

- **Theorem 7.12.** *A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.*

- **Theorem 7.13.** *Let $\phi$ be a function harmonic on a simply connected domain $D$. Then there is an analytic function $f$ such that $\phi = \operatorname{Re} f$ on $D$.*

- **Theorem 7.14.** *If $\phi$ is harmonic in a simply connected domain $D$ and $\phi$ achieves its maximum or minimum value at some point $z_0$ in $D$, then $\phi$ is constant in $D$.*

- **Theorem 7.15.** *If $\phi$ is harmonic in a simply connected domain $D$ and continuous up to and including the boundary attains its maximum and minimum on the boundary.*

# 8 Series Representation of Analytic Functions

- The series $\sum_{j=0}^{\infty} c^j$ converse to $1/(1-c)$ if $|c|<1$.
- Suppose that the terms $c_j$ satisfies the inequality $|c_j| \le M_j$ for all $j$ larger than some integer $J$. Then, if the series $\sum_{j=0}^{\infty} M_j$ converges, so does $\sum_{j=0}^{\infty} c_j$
- **Lemma 8.1 (Ratio Test).** *Suppose the terms of the series $\sum_{j=0}^{\infty} c_j$ have the property that the ratio $|c_{j+1}/c_j|$ approaches a limit $L$ as $j \to \infty$. Then the series converges if $L < 1$ and diverges if $L > 1$.*
- The sequene $\{F_n(z)\}_{n=1}^{\infty}$ is said to **converge uniformly to** $F(z)$ **on the set** $T$ if for any $\varepsilon > 0$ there exists an integer $N$ such that when $n > N$, we have

  $$|F(z) - F_n(z)| < \varepsilon$$

  for all $z \in T$.
- The series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly to $f(z)$ on $T$ if its sequence of partial sums converges uniformly to $f(z)$ on $T$.
- If $f$ is analytic at $z_0$, then the series

  $$f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$$

  is called the **Taylor's series** for $f$ around $z_0$. If $z_0 = 0$, it is also known as the **Maclaurin series** for $f$.
- **Theorem 8.2.** *If $f$ is analytic in the disc $|z-z_0| < R$, then the Taylor's series of $f$ around $z_0$ converges to $f(z)$ for all $z$ in this disk. Furthermore, the convergence of the series is uniform in any closed subdisk $|z-z_0| \le R' < R$.*
- The series of the form $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ is called a **power series**. The constant $a_j$ are the **coefficients** of the power series.
- For any power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ there is a real number $R$ between 0 and $\infty$ inclusive, which depends only on the coefficients $\{a_j\}$ such that
  - the series converges for $|z-z_0| < R$,
  - the series converges uniformly in any closed subdisk $|z-z_0| \le R' < R$, and
  - the series diverges for $|z-z_0| < R$.

  The number $R$ is called the **radius of convergence** of the power series.
- If a power series $\sum_{j=0}^{\infty} a_j z^j$ converges at a point having modulus $r$, then it converges at every poin in the disk $|z| < r$.
- Let $f_n$ be a sequence of functions continuous on a set $T \subset \mathbb{C}$ and converging uniformly to $f$ on $T$. Then $f$ is also continous on $T$.
- Let $f_n$ be a sequence of functions continuous on a set $T \subseteq \mathbb{C}$ containing the contour $\Gamma$, and suppose that $f_n$ converges uniformly to $f$ on $T$. Then, the sequence $\int_\Gamma f_n(z)\,\mathrm{d}z$ converges to $\int_\Gamma f(z)\,\mathrm{d}z$.
- Let $f_n$ be a sequence of functions analytic in a simply connected domain $D$ and converging uniformly to $f$ in $D$. Then $f$ is analytic in $D$.

- **Theorem 8.3.** *A power series sums to a function that is analytic at every point in its circle of convergence.*
- If $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ converges to $f(z)$ in some circular neighborhood of $z_0$, then

$$a_j = \frac{f^{(j)}(z_0)}{j!}.$$

  That is, the Taylor's series of $f$ around $z_0$ is unique.

- **Theorem 8.4.** *Let $f$ be analytic in the annulus $r < |z-z_0| < R$. Then $f$ can be expressed there as the sum of two series:*

$$f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$$

  *both series converging in the annulus, converging uniformly in any closed subannulus $r < \rho_1 \le |z-z_0| \le \rho_2 < R$. The coefficients of $a_j$ is given by:*

$$a_j = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, \mathrm{d}\zeta$$

  *where $C$ is a positively oriented simple closed contour lying in the annulus and containing $z_0$ in its interior. The sum of the two series is called the* **Laurent series** *of $f$ around $z_0$ in the annulus $r < |z-z_0| < R$.*

- The Laurent series is unique.
- A point $z_0$ is called a **zero of order** $m$ for the function $f$ if $f$ is analytic at $z_0$ and its first $m-1$ derivatives vanishes at $z_0$, but $f^{(m)}(z_0) \neq 0$.
- Let $f$ by analytic at $z_0$. Then $f$ has a zero of order $m$ at $z_0$ if and only if $f$ can be written as $f(z) = (z-z_0)^m g(z)$ where $g$ is an analytic function at $z_0$ and $g(z_0) \neq 0$.
- If $f$ is an analytic function such that $f(z_0) = 0$, then either $f$ is identically zero in a neighborhood of $z_0$ or there is a punctured disk about $z_0$ in which $f$ has no zeros.
- An **isolated singularity** of $f$ is a point $z_0$ suc hthat $f$ is analytic in some punctured disk $0 < |z-z_0| < R$, but not analytic at $z_0$ itself.
- **Definition 8.5.** *Let $f$ have an isolated singularity at $z_0$, and let $\sum_{-\infty}^{\infty} a_j(z-z_0)^j$ be the Laurent series expansion of $f$ in $0 < |z-z_0| < R$. Then,*
  - *If $a_j = 0$ for all $j < 0$, we say that $z_0$ is a* **removable singularity** *of $f$;*
  - *If $a_{-m} \neq 0$ for some positive integer $m$ but $a_j = 0$ for all $j < -m$, we say that $z_0$ is a* **pole of order** *$m$ of $f$;*
  - *If $a_j \neq 0$ for infinitely many negative $j$s, we say that $z_0$ is an* **essential singularity** *of $f$.*
- If $f$ has a removable singularity at $z_0$, then
  - $f(z)$ is bounded in some punctured disk around $z_0$,
  - $f(z)$ has a (finite) limit at $z$ appraoches $z_0$, and
  - $f(z)$ can be redefined at $z_0$ so that the new function is analytic at $z_0$.

- If the function $f$ has a pole of order $m$ at $z_0$, then $|(z-z_0)^\ell f(z)| \to \infty$ as $z \to z_0$ for all integer $\ell < m$, while $(z-z_0)^m f(z)$ has a removable singularity at $z_0$. In particular $|f(z)| \to \infty$ as $z$ approaches a pole.
- A function $f$ as a pole of order $m$ at $z_0$ if and only if in some punctured neighborhood of $z_0$,

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where $g$ is analytic at $z_0$ and $g(z_0) \neq 0$.
- If $f$ has a zero of order $m$ at $z_0$, then $1/f$ has a pole of order $m$ at $z_0$. Conversely, if $f$ has a pole of order $m$ at $z_0$, then $1/f$ has a removable singularity at $z_0$, and if we define $(1/f)(z_0) = 0$, then $1/f$ has a zero of order $m$ at $z_0$.
- **Theorem 8.6 (Picard's).** *A function with an essential singularity assumes every complex number, with possibly one exception, as a vlue in any neighborhood of this singularity.*
- We can distinguishes singularity by looking at the limit of $f(z)$ as $z \to z_0$. If the limit is bounded, that indicates a removable singularity. If the limit approaches infinity, that indicates a pole. Anything else indicates an essential singularity.

# 9 Residue Theory

- **Definition 9.1.** *If $f$ has an isolated singularity at the point $z_0$, then the coefficient of $a_{-1}$ of $1/(z-z_0)$ in the Laurent expansion of $f$ around $z_0$ is called the* **residue** *of $f$ at $z_0$ and is denoted by*

$$\mathrm{Res}(f; z_0) \text{ or } \mathrm{Res}(z_0).$$

- Consider evaluating the integral

$$\int_\Gamma f(z)\,\mathrm{d}z$$

where $\Gamma$ is a simple closed positively oriented contour and $f(z)$ is analytic on and inside $\Gamma$ *except* for a single isolated singularity $z_0$ inside $\Gamma$.

We know that $f(z)$ has Laurent series expansion

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j.$$

Thus,

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{j=-\infty}^{\infty} a_j \int_\Gamma (z-z_0)^j\,\mathrm{d}z = 2\pi i a_{-1} = 2\pi i \mathrm{Res}(f; z_0)$$

- Now, if $\Gamma$ has my isolated singularities inside it, say $z_1, z_2, \ldots, z_m$, then

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f; z_k).$$

- If $f$ has a pole of order $m$ at $z_0$, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)].$$

  In particular, if $f$ has a pole of order 1 at $z_0$, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

- To find integrals of the form $\int_0^{2\pi} U(\cos\theta, \sin\theta)\,\mathrm{d}\theta$, convert it to a contour integral around the circle $C = \{z : |z| = 1\}$ with:

$$\begin{aligned}
\cos\theta &= \frac{1}{2}\left(z + \frac{1}{z}\right) \\
\sin\theta &= \frac{1}{2i}\left(z - \frac{1}{z}\right) \\
\mathrm{d}\theta &= \frac{dz}{iz}.
\end{aligned}$$

  Then, use Residue theory to find the integral.

- Given any function $f$ continuous on $(-\infty, \infty)$, the limit

$$\lim_{\rho \to \infty} \int_{-\rho}^{\rho} f(x)\,\mathrm{d}x$$

  is called the **Cauchy's principal value** of the integral of $f$ over $(-\infty, \infty)$. We denote it by the symbol:

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

- If
  - $f$ is analytic on and above the real axis except for a finite number of isolated singularities in the open upper half-plane, and
  - $\lim_{\rho \to \infty} \int_{C_\rho^+} f(z)\,\mathrm{d}z = 0$ where $C_\rho^+$ is the half circular arc from $(\rho, 0)$ to $(-\rho, 0)$ traversed positively,

  then p.v. $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ can be found by integrating the half circle contour.

- **Lemma 9.2.** *If $f(z) = P(z)/Q(z)$ is a quotient of two polynomials such that $\deg Q \geq 2 + \deg P$, then*

$$\lim_{\rho \to \infty} \int_{C_\rho^+} f(z)\,\mathrm{d}z = 0.$$

- One can also find the Cauchy's principal value of the following integrals using Residue theory:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx\,\mathrm{d}x$$
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx\,\mathrm{d}x$$

where $m$ is a positive real number. Again, the trick is to transform the sine and cosine to complex exponetials with:

$$\cos mx = \frac{e^{imx} + e^{-imx}}{2}, \text{ and}$$
$$\sin mx = \frac{e^{imx} - e^{-imx}}{2i}.$$

Consider the case of evaluating the integral with cosine. We have that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \, \mathrm{d}x &= \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \frac{e^{imx} + e^{-imx}}{2} \, \mathrm{d}x \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{imx} \frac{P(x)}{Q(x)} \, \mathrm{d}x + \frac{1}{2} \int_{-\infty}^{\infty} e^{-imx} \frac{P(x)}{Q(x)} \, \mathrm{d}x.\end{aligned}$$

We shall convert the two integrals on the RHS to contour integrals. Which contour should we use? Consider $e^{imz}$. We have that $e^{imz} = e^{im(x+iy)} = e^{imx}e^{-my}$. So, we have that $e^{imz}$ is bounded in the upper half plane. On the other hand, $e^{-imz}$ is bounded in the lower half plane. As a result, we should use the upper half circle contour with the integral involving $e^{imx}$ and the lower half circle contour with integral involving $e^{-imx}$.

- Evaluating the contour in the last item involves integrating over the half circular arc. The process can be simplified if the integral is zero.

- **Lemma 9.3 (Jordan's).** *If $m > 0$ and $P/Q$ is the quotient of two polynomials such that $degQ \geq 1 + \deg P$, then*

$$\lim_{\rho \to \infty} \int_{C_\rho^+} e^{imz} \frac{P(z)}{Q(z)} \, \mathrm{d}z = 0.$$

- **Lemma 9.4 (Hung Cheng's).** *Consider $\int_{\Gamma_R} f(z)e^{iz}dz$ where $\Gamma_R$ is the contour*

$$(-R, 0) \to (R, 0) \to (R, 2R) \to (-R, 2R).$$

*If $f(z)$ is bounded on $\Gamma_R$ with $\max_{\Gamma_R} |f(z)| \to 0$ as $R \to \infty$. Then, the integral is 0.*

- The following lemma is useful for evaluating indented contour:

**Lemma 9.5.** *If $f$ has a simple pole at $z = c$ and $T_r$ is the circular arc define by:*

$$T_r = \{z : z = c + re^{i\theta}, \theta_1 \leq \theta \leq \theta_2\}.$$

*Then,*

$$\lim_{r \to 0^+} \int_{T_r} f(z) \, \mathrm{d}z = i(\theta_2 - \theta_1)\mathrm{Res}(f; c)$$

- Given a closed curve $\gamma$, the **winding number** of $\gamma$ is defined as

$$W(\gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}w}{w}.$$

It is equal to the number of times $\gamma$ winds around 0 in the positive orientation.

- Now, if $w = f(z)$, then

$$W(\gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}w}{w} = \frac{1}{2\pi i} \int_\beta \frac{f'(z)}{f(z)} \, \mathrm{d}z$$

where $\beta$ is the closed curve such that $f(\beta) = \gamma$.

- **Theorem 9.6 (Argument Principle).** *If $f$ is analytic and non-zero at each point of a simple closed positively oriented contour $C$ and is meromorphic inside $C$, then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, \mathrm{d}z = N_0(f) - N_p(f)$$

*where $N_0(f)$ and $N_p(f)$ are, respectively, the number of zeros and poles of $f$ inside $C$ (multiplicity included).*

- If $f$ is analytic inside and on a simple closed positiveliy oriented contour $C$ and if $f$ is non-zero on $C$, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, \mathrm{d}z = N_0(f).$$

- **Theorem 9.7 (Rouche's).** *If $f$ and $h$ are each functions that are analytic inside and on a simple closed contour $C$ and if the strict inequality $|h(z)| < |f(z)|$ holds at each point on $C$, then $f$ and $f + h$ must have the same total number of zeros (counting multiplicities) inside $C$.*

- If $f$ is a non-constant and analytic in an open domain $D$, then its image $f(D)$ is an open set.

# 10 Conformal Mapping

- If $\phi(x, y)$ is harmonic, then it satisfies **Laplace's equation:**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

- We know that $\phi$ is the real part of some analytic function.

- Suppose $f$ sends $z = x + yi$ to $w = u + vi$ such that $f$ is one-to-one. Suppose that the function $\psi(w) = \psi(u, v)$ satisfies Laplace's equation:

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0.$$

Then, the function

$$\phi(z) = \psi(f(z))$$

satisfies Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

- The inversion mapping $w = 1/z$ maps lines and circles to lines and circles.

- Lines and circles passing through 0 will be mapped to lines.

- Lines and circles not passing through 0 will mapped to circles.
- It maps the line $x = 1/2$ to a circle centered at $z = 1$ with radius 1.
- A **Mobius transformation** is any function of the form

$$w = \frac{az + b}{cz + d}$$

  with the restriction that $ab \neq bc$.

- **Theorem 10.1.** *If $f$ is any Mobius transformation, then*
  - *$f$ can be expressed as the compotion of a finite number of translations, magnificatios, rotations, and inversions.*
  - *$f$ maps the extended complex plane one-to-one onto itself.*
  - *$f$ maps the class of circles and lines to itself.*
  - *$f$ is conformal at every point except its pole.*