Distance Between Skew Lines

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December 11, 2019

- Suppose there are two lines.
  The first one is given by the function $\ell_0(\lambda_0) = \mathbf{p}_0 + \lambda_0 \mathbf{q}_0$.
  The second one is given by the function $\ell_0(\lambda_1) = \mathbf{p}_1 + \lambda_1 \mathbf{q}_1$.
  Here, $\mathbf{p}_0 = (p_0^x, p_0^y, p_0^z) \in \mathbb{R}^3$. The same notion is used for the other points and vectors.

- We would like to find a pair of λ_0 and λ_1 such that the distance squared

$$F(\lambda_1, \lambda_2) = \|\ell_0(\lambda_0) - \ell_1(\lambda_1)\|^2$$

  between the point $\ell_0(\lambda_0)$ and $\ell_1(\lambda_1)$ is minimal.

- Now,

$$\begin{aligned}
&F(\lambda_0, \lambda_1) \\
&= \|\mathbf{p}_0 - \mathbf{p}_1 + \lambda_0 \mathbf{q}_0 - \lambda_1 \mathbf{q}_1\|^2 \\
&= (\mathbf{p}_0 - \mathbf{p}_1 + \lambda_0 \mathbf{q}_0 - \lambda_1 \mathbf{q}_1)^T (\mathbf{p}_0 - \mathbf{p}_1 + \lambda_0 \mathbf{q}_0 - \lambda_1 \mathbf{q}_1) \\
&= (\mathbf{p}_0 - \mathbf{p}_1)^T (\mathbf{p}_0 - \mathbf{p}_1) + 2\lambda_0 (\mathbf{p_0} - \mathbf{p}_1)^T \mathbf{q}_0 - 2\lambda_1 (\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_1 - 2\lambda_0 \lambda_1 \mathbf{q}_0^T \mathbf{q}_1 + \lambda_0^2 \mathbf{q}_0^T \mathbf{q}_0 + \lambda_1^2 \mathbf{q}_1^T \mathbf{q}_1
\end{aligned}$$

  So,

$$\begin{aligned}
\frac{\partial F}{\partial \lambda_0} &= 2(\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_0 - 2\lambda_1 \mathbf{q}_0^T \mathbf{q}_1 + 2\lambda_0 \mathbf{q}_0^T \mathbf{q}_0 \\
\frac{\partial F}{\partial \lambda_1} &= 2(\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_1 - 2\lambda_0 \mathbf{q}_0^T \mathbf{q}_1 + 2\lambda_1 \mathbf{q}_1^T \mathbf{q}_1.
\end{aligned}$$

  Setting both partial derivatives to zero, we have the system of linear equations:

$$\begin{bmatrix} \mathbf{q}_0^T \mathbf{q}_0 & -\mathbf{q}_0^T \mathbf{q_1} \\ -\mathbf{q}_0^T \mathbf{q}_1 & \mathbf{q}_1^T \mathbf{q_1} \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} -(\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_0 \\ -(\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_1 \end{bmatrix}$$

  So, by Cramer's rule:

$$\begin{aligned}
\lambda_0 &= \frac{-\|\mathbf{q}_1\|^2 (\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_0 - (\mathbf{q_0}^T \mathbf{q}_1)[(\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_1]}{\|\mathbf{q}_0\|^2 \|\mathbf{q}_1\|^2 - (\mathbf{q}_0^T \mathbf{q}_1)^2} \\
\lambda_1 &= \frac{-\|\mathbf{q}_0\|^2 (\mathbf{p}_0 - \mathbf{p}_1)^T \mathbf{q}_1 - (\mathbf{q_0}^T \mathbf{q}_1)[(\mathbf{p}_0 - \mathbf{p}_0)^T \mathbf{q}_0]}{\|\mathbf{q}_0\|^2 \|\mathbf{q}_1\|^2 - (\mathbf{q}_0^T \mathbf{q}_1)^2}
\end{aligned}$$

- The minimum distance is given by:

$$\left| \frac{\mathbf{q}_0 \times \mathbf{q}_1}{\|\mathbf{q}_0 \times \mathbf{q}_1\|} \cdot (\mathbf{p}_0 - \mathbf{p}_1) \right|$$