Volume Rendering

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1 The Equation of Transfer

- When light travels in a participating medium, there are three processes which can change the radiance.
  - *Absorption* – light collides with particles and is converted to other types of energy.
  - *Emission* – the material itself adds light to the environment.
  - *Scattering* – light collides with particles and changes direction.
- The *equation of transfer* is an integro-differential equation that describes the change of radiance as light travels through participating media. From 10 miles above, the equation looks like:

  $$\text{change of radiance} = \text{absorption term} \ + \ \text{emission term} \ + \ \text{scattering term}. \tag{1}$$

  We will discuss each of the terms in turn.
- The "change of radiance" term is modeled as the directional derivative of the radiance function $L(x, \omega)$.

  $$\text{change of radiance} = \omega \cdot \nabla L(x, \omega).$$

  Here, x is a position in 3D, and ω is a direction, i.e. a unit vector. This directional derivative can be thought of as differentiation with respect to distance along direction ω. That is,

  $$\omega \cdot \nabla L(x, \omega) = \frac{\mathrm{d}L(x + s\omega, \omega)}{\mathrm{d}s},$$

  where s denotes distance along the ω direction.
- The "absorption term" seeks to model the absorption process as a Poisson process.

  The probability of an absorption event occurring while the light travels an infinitesimal distance ds around x in direction ω is given by $\sigma_a(x, \omega) ds$, where $\sigma_a(x, \omega)$ is the *absorption cross section.*

  When an absorption event occur, all the radiance is taken away. Thus, we have that

  $$\begin{aligned} \mathrm{d}L(x + s\omega, \omega) &= -(\sigma_a(x, \omega)\mathrm{d}s)L(x, \omega) \\ \frac{\mathrm{d}L(x + s\omega, \omega)}{\mathrm{d}s} &= -\sigma_a(x, \omega)L(x, \omega) \end{aligned}$$

  Thus, we have that

  $$\text{absorption term} = -\sigma_a(x, \omega)L(x, \omega).$$

  Note that as the absorption cross section gives the probability that absorption occurs per unit distance. Hence, its unit is m^{-1}.

- The "emission term" is modeled by a function $Q(x, \omega)$:

$$\text{emission term} = Q(x, \omega).$$

The function Q gives additional radiance per unit distance. Therefore, its unit is W m^{-3} sr^{-1}, which is basically the unit of radiance divided by distance.

- The "scattering term" is the most complicated. There are two components: the *out-scattering* and *in-scattering*.

$$\text{scattering term} = \text{out-scattering term} \; + \; \text{in-scattering term}$$

  - Scattering is, again, modeled as a Poisson process. When a scattering event occurs, both out-scattering and in-scattering occur at the same time. The *scattering coefficient* $\sigma_s(x, \omega)$ gives the probability that a scattering event occurs per unit length. (So, the unit is m^{-1}.)
  - The out-scattering term accounts for the event that light traveling along direction ω change its direction to another direction. The event removes all the radiance and thus looks pretty much like the absorption term.

$$\text{out-scattering term} = -\sigma_s(x, \omega)L(x, \omega).$$

  - The in-scattering term accumulates light that changes direction from other direction to ω. The probability that light from direction ω' change its direction to ω is accounted by the *phase function* $p(x, \omega' \to \omega)$. So,

$$\text{in-scattering term} = \sigma_s(x, \omega) \int_{S^2} p(x, \omega' \to \omega)L(x, \omega') \; d\omega'$$

    Here, S^2 is the unit sphere in 3D. The unit of the phase function is sr^{-1}.

In conclusion,

$$\text{scattering term} = -\sigma_s(x, \omega)L(x, \omega) + \int_{S^2} p(x, \omega' \to \omega)L(x, \omega') \; d\omega'$$

- Writing the transfer equation in full, we have

$$\omega \cdot \nabla L(x, \omega) = -(\sigma_a(x, \omega) + \sigma_s(x, \omega))L(x, \omega) + Q(x, \omega) + \int_{S^2} p(x, \omega' \to \omega)L(x, \omega) \; d\omega'.$$

We usually combine the absorption cross section and the scattering coefficient to one *extinction coefficient*:

$$\sigma_t(x, \omega) = \sigma_a(x, \omega) + \sigma_s(x, \omega).$$

So, the widely used form of the transfer equation is

$$\omega \cdot \nabla L(x, \omega) = -\sigma_t(x, \omega)L(x, \omega) + Q(x, \omega) + \int_{S^2} p(x, \omega' \to \omega)L(x, \omega) \; d\omega'.$$

2 Solutions to Some Special Cases

2.1 Extinction Only

- **Extinction only:** In this case, only the extinction (absorption and out-scattering) term has effect.

- Let x_0 and x_1 be points in space, surrounded by a medium. We shall find the radiance traveling along direction ω, starting from x_0.

  Let s denote the distance along ω. We can think of L and σ_a as functions of s. That is, we can write $L(x, \omega)$ as $L(x_0 + s\omega, \omega)$ or simply $L(s)$ or L. In the same way, $\sigma_t(x, \omega)$ becomes $\sigma(x_0 + s\omega, \omega)$ or $\sigma_t(s)$. So, the transfer equation becomes

$$\begin{aligned} \frac{dL}{ds} &= -\sigma_t(s) L \\ \frac{1}{L}\ dL &= -\sigma_t(s)\ ds \\ \int \frac{1}{L}\ dL &= -\int \sigma_t(s)\ ds \\ \log L &= -\int \sigma_t(s) ds + C \\ L &= Ae^{-\int \sigma_t(s) ds}. \end{aligned}$$

  Bringing back x_0 and ω, we have

$$L(x_0 + r\omega, \omega) = Ae^{-\int_0^r \sigma_t(x_0 + s\omega, \omega) ds}.$$

  Substituting $d = 0$, we have that $A = L(x_0, \omega)$. Hence,

$$L(x_0 + r\omega, \omega) = L(x_0, \omega) e^{-\int_0^r \sigma_t(x_0 + s\omega, \omega) ds}.$$

- Let $x_1 = x_0 + r\omega$. The integral $\int_0^r \sigma_t(x_0 + s\omega, \omega)\ ds$ is called the *optical thickness* and is denoted $\tau(x_0 \to x_1)$.

- The quantity $e^{\int_0^r \sigma_t(x_0 + s\omega, \omega) ds}$ is called the *beam transmittance* and is denoted by $T_r(x_0 \to x_1)$.

- Below are some properties of the optical thickness and the beam transmittance.

$$\begin{aligned} L(x_1, \omega) &= L(x_0, \omega) T_r(x_0 \to x_1) = L(x_0, \omega) e^{-\tau(x_0 \to x_1)} \\ T_r(x_0 \to x_2) &= T_r(x_0 \to x_1) T_r(x_1 \to x_2) \\ \tau(x_0 \to x_2) &= \tau(x_0 \to x_1) + \tau(x_1 \to x_2) \end{aligned}$$

  given that x_0, x_1, x_2 lie in this order along the line whose direction is ω.

- If $\sigma_t(x, \omega)$ is constant, we say that the material is *homogeneous*. In this case, we have that

$$\tau(x_0 \to x_1) = \sigma_t \|x_1 - x_0\|.$$

  Hence,

$$T_r(x_0 \to x_1) = e^{-\sigma_t \|x_1 - x_0\|}.$$

  The above equation is called *Beer's law.*

2.2 Extinction and Emission Only

- In this case, the transfer equation becomes

$$\begin{aligned} \frac{dL}{ds} &= -\sigma_t(s) L + Q(s) \\ \frac{dL}{ds} + \sigma_t(s) L &= Q(s), \end{aligned}$$

which is a first order linear ODE. To solve it, we multiply both sides by $I(s) = e^{\int_0^s \sigma(v) dv}$.

$$\begin{aligned}
I(s)\frac{dL}{ds} + I(s)\sigma_t(s)L &= I(s)Q(s) \\
\frac{d}{ds}\Big\{I(s)L\Big\} &= I(s)Q(s) \\
I(s)L &= \int_0^s I(u)Q(u)\ du + C \\
L &= \frac{\int_0^s I(u)Q(u)\ du + C}{I(s)} \\
L &= e^{-\int_0^s \sigma(v)dv}\Big[\int_0^s e^{\int_0^u \sigma(v)dv} Q(u)\ du + C\Big] \\
L &= \int_0^s e^{\int_0^u \sigma(v)dv - \int_0^s \sigma(v)dv} Q(u)\ du + Ce^{-\int_0^s \sigma(v)dv} \\
L &= \int_0^s e^{-\int_u^0 \sigma(v)dv - \int_0^s \sigma(v)dv} Q(u)\ du + Ce^{-\int_0^s \sigma(v)dv} \\
L &= \int_0^s e^{-\int_u^s \sigma(v)dv} Q(u)\ du + Ce^{-\int_0^s \sigma(v)dv} \\
L(x_0 + s\omega, \omega) &= \int_0^s T_r(x_0 + u\omega \to x_0 + s\omega) Q(x + u\omega, \omega)\ du + CT_r(x_0 \to x_0 + s\omega)
\end{aligned}$$

Substituting $s = 0$ yields $C = L(x_0, \omega)$. The solution, in full, is then

$$L(x_0 + s\omega, \omega) = \int_0^s T_r(x_0 + u\omega \to x_0 + s\omega) Q(x + u\omega, \omega)\ du + L(x_0, \omega) T_r(x_0 \to x_0 + s\omega).$$

Simplifying the notation by letting $x_1 = x_0 + s\omega$ and integrating points on the line between x_0 and x_1, we have

$$L(x_1, \omega) = T_r(x_0 \to x_1) L(x_0, \omega) + \int_{x_0}^{x_1} T_r(x \to x_1) Q(x, \omega)\ dx.$$

The above equation has a very nice interpretation. To compute radiance at x_1, we need to sum all contribution from emission from every point x along the line from x_0 to x_1. The emission $Q(x, \omega)$ at x gets attenuated by a factor of $T_r(x \to x_1)$, so its contribution is $T_r(x \to x_1)Q(x, \omega)dx$. Lastly, the outgoing radiance from x_0 gets attenuated by a factor of $T_r(x_0 \to x_1)$ before it reaches x_1.

2.3 Single Scattering

- In this case, we assume there is a point light source at position x_L whose intensity is given by $I_L(x_L, \omega)$.
- Again, we are interested in finding the radiance $L(x_1, \omega)$ in terms of radiance $L(x_0, \omega)$ where ω is the unit vector pointing from x_0 to x_1.
- However, we assume that light from the light source scatters once into the direction ω. That is, for any point x along the segment from x_0 to x_1, light travels from x_L to x, being attenuated along the way, and then scatters into direction ω.

  Let ω_L be the direction from x_L to x, we have that incoming radiance due to the light source is $L(x, \omega') = V(x_L, x) T_r(x_L \to x) I_L(x_L, \omega_L) \delta(x_L, \omega)$ where $V(x_L, x)$ is the visibility between x_L and x, and δ is the Dirac delta function.

As such, the scattering integral simplifies to

$$\int_{S^2} p(\omega' \to \omega) L(x, \omega') d\omega' = p(\omega_L \to \omega) V(x_L, x) T_r(x_L \to x) I_L(x_L, \omega_L),$$

and the transfer equation becomes

$$\omega \cdot \nabla L(x, \omega) = -\sigma_t(x, \omega) L(x, \omega) + Q(x, \omega) + \sigma_s(x, \omega) p(\omega_L \to \omega) V(x_L, x) T_r(x_L \to x) I_L(x_L, \omega_L).$$

Notice that the scattering term can be written as a function of s. So, the equation is a first-order ODE, which can be solved in the same way as the last case. Hence, the solution is

$$L(x_1, \omega) = T_r(x_0 \to x_1) L(x_0, \omega) + \int_{x_0}^{x_1} T_r(x \to x_1) \Big(Q(x, \omega) + p(\omega_L \to \omega) V(x_L, x) T_r(\omega_L \to \omega) I_L(x_L, \omega_L) \Big)\, dx.$$

3 Diffusion Approximation

- The diffusion approximation gives a low-frequency approximation of the radiance field. The approximation works in practice because, in highly scattering media, light distribution becomes blurred very quickly.
- The radiance field is approximated as follows:

$$L(x, \omega) = \frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega \cdot E(x)$$

  where

  - $\phi(x) = \mu_0[L] = \int_{S^2} L(x, \omega)\ d\omega$ is the *fluence*, and
  - $E(x) = \mu_1[L] = \int_{S^2} L(x, \omega) \omega\ d\omega$ is the *vector irradiance*.

  See the "Angular Moments" note for more details.

- Substituting the approximation into the transfer equation we have:

$$\begin{aligned} &\omega \cdot \nabla \left[\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega \cdot E(x) \right] + \sigma_t(x, \omega) \left[\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega \cdot E(x) \right] \\ &\quad = Q(x, \omega) + \sigma_s(x, \omega) \int_{S^2} p(\omega' \to \omega) \left[\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega' \cdot E(x) \right] d\omega' \end{aligned} \tag{2}$$

3.1 Isotropic Homogeneous Material

- In this section, we assume that the material is homogeneous. That is, $\sigma_t(x, \omega)$ and $\sigma_s(x, \omega)$ are constant for all x and ω.
- We also assume that the material is isotropic. That is, $p(\omega' \to \omega)$ only depends on the angle between ω' and ω. In other words, $p(\omega' \to \omega) = p(\omega' \cdot \omega)$.
- With these assumptions, equation 2 becomes

$$\begin{aligned} &\omega \cdot \nabla \left[\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega \cdot E(x) \right] + \sigma_t \left[\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega \cdot E(x) \right] \\ &\quad = Q(x, \omega) + \sigma_s \int_{S^2} p(\omega' \to \omega) \left[\frac{1}{4\pi} \phi(x) + \frac{3}{4\pi} \omega' \cdot E(x) \right] d\omega' \\ &\frac{1}{4\pi} \omega \cdot \nabla \phi(x) + \frac{3}{4\pi} \omega \cdot \nabla (\omega \cdot E(x)) + \frac{\sigma_t}{4\pi} \phi(x) + \frac{3\sigma_t}{4\pi} \omega \cdot E(x) \\ &\quad = Q(x, \omega) + \frac{\sigma_s}{4\pi} \phi(x) \int_{S^2} p(\omega' \cdot \omega)\ d\omega' + \frac{3\sigma_s}{4\pi} \int_{S^2} p(\omega' \cdot \omega) \omega' \cdot E(x)\ d\omega' \end{aligned} \tag{3}$$

- Equation 3 can be simplified in a number of ways. First, notice that since p, the phase function, is a probability distribution on both ω and ω'. We have that $\int_{S^2} p(\omega' \cdot \omega)\ d\omega' = 1$ for all ω.
- Second, consider the term $\omega \cdot \nabla(\omega \cdot E(x))$. We have that

$$\omega \cdot \nabla(\omega \cdot E(x)) = \omega^T \nabla(\omega \cdot E(x)) = \omega^T \nabla(E(x))^T \omega = \omega^T (E(x)\nabla^T)^T \omega.$$

Now, $E(x)\nabla^T$ is just the Jacobian $J_E(x)$. Hence, $\omega \cdot \nabla(\omega \cdot E) = \omega^T (J_E(x))^T \omega$.

- With the above simplifications, Equation 3 becomes

$$\begin{aligned} &\frac{1}{4\pi}\omega \cdot \nabla\phi(x) + \frac{3}{4\pi}\omega^T(J_E(x))^T\omega + \frac{\sigma_t}{4\pi}\phi(x) + \frac{3\sigma_t}{4\pi}\omega \cdot E(x) \\ &\quad = Q(x,\omega) + \frac{\sigma_s}{4\pi}\phi(x) + \frac{3\sigma_s}{4\pi}\int_{S^2} p(\omega' \cdot \omega)\omega' \cdot E(x)\ d\omega' \end{aligned} \tag{4}$$

- In order to get a solvable equation, we will take the 0th moment of both sides of the above equation. Let us do it term by term.
  - First term of LHS:

$$\mu_0\left[\frac{1}{4\pi}\omega \cdot \nabla\phi(x)\right] = \frac{1}{4\pi}\mu_0[\omega \cdot \nabla\phi(x)] = 0 \qquad \text{(Lemma 3.4 of Angular Moments note)}$$

  - Second term of LHS:

$$\begin{aligned} \mu_0\left[\frac{3}{4\pi}\omega^T(J_E(x))^T\omega\right] &= \frac{3}{4\pi}\mu_0\left[\omega^T(J_E(x))^T\omega\right] \\ &= \frac{3}{4\pi}\cdot\frac{4\pi}{3}\mathrm{tr}(J_E(x)^T) \qquad \text{(Lemma 3.6 of Angular Moments note)} \\ &= \frac{\mathrm{d}E_1(x)}{\mathrm{d}x_1} + \frac{\mathrm{d}E_2(x)}{\mathrm{d}x_2} + \frac{\mathrm{d}E_3(x)}{\mathrm{d}x_3} \\ &= \nabla \cdot E(x). \end{aligned}$$

  - Third term of LHS:

$$\mu_0\left[\frac{\sigma_t}{4\pi}\phi(x)\right] = \frac{\sigma_t}{4\pi}\phi(x)\mu_0[1] = \sigma_t\phi(x).$$

  - Fourth term of LHS:

$$\mu_0\left[\frac{3\sigma_t}{4\pi}\omega \cdot E(x)\right] = \frac{3\sigma_t}{4\pi}\mu_0[\omega \cdot E(x)] = 0 \qquad \text{(Lemma 3.4 of Angular Moments note).}$$

  So, the LHS becomes $\nabla \cdot E(x) + \sigma_t\phi(x)$. However, we still have the RHS to work on.

  - First term of RHS: We have $\mu_0[Q(x,\omega)]$, which we shall abbreviate as $Q_0(x)$.
  - Second term of RHS:

$$\mu_0\left[\frac{\sigma_s}{4\pi}\phi(x)\right] = \frac{\sigma_s}{4\pi}\phi(x)\mu_0[1] = \sigma_s\phi(x).$$

– Third term of RHS:

$$\begin{aligned}\mu_0\left[\frac{3\sigma_s}{4\pi}\int_{S^2} p(\omega'\cdot\omega)(\omega'\cdot E(x))\ \mathrm{d}\omega'\right] &= \frac{3\sigma_s}{4\pi}\int_{S^2}\int_{S^2} p(\omega'\cdot\omega)(\omega'\cdot E(x))\ \mathrm{d}\omega'\mathrm{d}\omega \\ &= \frac{3\sigma_s}{4\pi}\int_{S^2}(\omega'\cdot E(x))\left(\int_{S^2} p(\omega'\cdot\omega)\ \mathrm{d}\omega\right)\ \mathrm{d}\omega' \\ &= \frac{3\sigma_s}{4\pi}\int_{S^2}(\omega'\cdot E(x))\ \mathrm{d}\omega' \\ &= 0.\end{aligned}$$

The reason we could remove $\int_{S^2} p(\omega'\cdot\omega)\ \mathrm{d}\omega$ was because p is a probability distribution over ω. Also, notice that we applied Lemma 3.4 of the Angular Moments note.

Hence, the RHS becomes $Q_0(x)+\sigma_s\phi(x)$. Hence, the equation becomes

$$\begin{gathered}\nabla\cdot E(x)+\sigma_t\phi(x) = Q_0(x)+\sigma_s\phi(x),\ \text{ or} \\ \nabla\cdot E(x) = Q_0(x)-\sigma_a\phi(x).\end{gathered}$$

- We will also take the 1st moment of both sides. We'll do it term by term again.
  - First term of LHS:

$$\begin{aligned}\mu_1\left[\frac{1}{4\pi}\omega\cdot\nabla\phi(x)\right] &= \frac{1}{4\pi}\mu_1[\omega\cdot\nabla\phi(x)] \\ &= \frac{1}{4\pi}\cdot\frac{4\pi}{3}\nabla\phi(x) && \text{(Lemma 3.8 of Angular Moments note)} \\ &= \frac{1}{3}\nabla\phi(x)\end{aligned}$$

  - Second term of LHS:

$$\begin{aligned}\mu_1\left[\frac{3}{4\pi}\omega^T(J_E(x))^T\omega\right] &= \frac{3}{4\pi}\mu_1\left[\omega^T(J_E(x))^T\omega\right] \\ &= 0 && \text{(Lemma 3.10 of Angular Moments note)}\end{aligned}$$

  - Third term of LHS:

$$\mu_0\left[\frac{\sigma_t}{4\pi}\phi(x)\right] = \frac{\sigma_t}{4\pi}\phi(x)\mu_1[1] = 0. \qquad \text{(Lemma 3.7 of Angular Moments note)}$$

  - Fourth term of LHS:

$$\begin{aligned}\mu_1\left[\frac{3\sigma_t}{4\pi}\omega\cdot E(x)\right] &= \frac{3\sigma_t}{4\pi}\mu_1[\omega\cdot E(x)] = \frac{3\sigma_t}{4\pi}\cdot\frac{4\pi}{3}E(x) && \text{(Lemma 3.8 of Angular Moments note)} \\ &= \sigma_t E(x).\end{aligned}$$

So, the LHS becomes $\frac{1}{3}\nabla\phi(x)+\sigma_t E(x)$. We work on RHS next.

  - First term of RHS: We have $\mu_1[Q(x,\omega)]$, which we shall abbreviate as $Q_1(x)$.
  - Second term of RHS:

$$\mu_1\left[\frac{\sigma_s}{4\pi}\phi(x)\right] = \frac{\sigma_s}{4\pi}\phi(x)\mu_1[1] = 0. \qquad \text{(Lemma 3.7 of Angular Moments note)}$$

– Third term of RHS:

$$\begin{aligned}
\mu_1\left[\frac{3\sigma_s}{4\pi}\int_{S^2} p(\omega'\cdot\omega)(\omega'\cdot E(x))\ \mathrm{d}\omega'\right] &= \frac{3\sigma_s}{4\pi}\int_{S^2}\omega\int_{S^2} p(\omega'\cdot\omega)(\omega'\cdot E(x))\ \mathrm{d}\omega'\mathrm{d}\omega \\
&= \frac{3\sigma_s}{4\pi}\int_{S^2}(\omega'\cdot E(x))\left(\int_{S^2} p(\omega'\cdot\omega)\ \mathrm{d}\omega\right)\ \mathrm{d}\omega' \\
&= \frac{3\sigma_s}{4\pi}\int_{S^2}(\omega'\cdot E(x))\ \mathrm{d}\omega' \\
&= 0.
\end{aligned}$$

The reason we could remove $\int_{S^2} p(\omega'\cdot\omega)\ \mathrm{d}\omega$ was because p is a probability distribution over ω. Also, notice that we applied Lemma 3.4 of the Angular Moments note.