Dual Scattering Implementation

Pramook Khungurn

November 17, 2015

This document is written as I try to implement the dual scattering algorithm [Zinke et al., 2008] and coupling it with the fiber scattering function defined in [Khungurn et al., 2015]. To do so, I will consult the implementation note from Walt Disney Animation Studios [Sadeghi and Tamstorf, 2010].

1 Dual Scattering

Dual scattering converts the incoming direct radiance field L_d into an approximate muliply-scattered radiance field L_i. This is done using the *multiple scattering function* Ψ:

$$L_i(x, \omega_i) = \int_{S^2} \Psi(x, \omega_d, \omega_i) L_d(x, \omega_d)\ \mathrm{d}\omega_d.$$

When shading a point, we compute:

$$\begin{aligned} L_o(x, \omega_o) &= \int_{S^2} L_i(x, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i \\ &= \int_{S^2} \left(\int_{S^2} \Psi(x, \omega_d, \omega_i) L_d(x, \omega_d)\ \mathrm{d}\omega_d \right) f_s(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i \\ &= \int_{S^2} \left(\int_{S^2} \Psi(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i \right) L_d(x, \omega_d)\ \mathrm{d}\omega_d. \end{aligned}$$

When estimating the above integral (such as when implementing the `Li` method of an `Integrator`), we sample ω_d by emitter sampling and compute:

$$L_o(x, \omega_o) \approx \left(\int_{S^2} \Psi(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i \right) \frac{L_d(x, \omega_d)}{p_{\mathrm{emitter}}(\omega_d)}. \tag{1}$$

The main task of the algorithm then is to evaluate the integral in the RHS.

The multiple scattering function consists of two terms, the *global multiple scattering function* Ψ^G and the *local multiple scattering function* Ψ^L.

$$\Psi(x, \omega_d, \omega_i) = \Psi^G(x, \omega_d, \omega_i)(1 + \Psi^L(x, \omega_d, \omega_i)).$$

The local multiple scattering function is not defined directly. It is defined in combination with the fiber scattering function $f_s(\omega_i, \omega_o)$:

$$\Psi^L(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) = d_b f_{\mathrm{back}}(\omega_i, \omega_o).$$

where d_b is a constant, which is set to 0.7 in the paper. With this, we can rewrite the integral in (1) as:

$$\begin{aligned}
&\int_{S^2} \Psi(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i \\
&= \int_{S^2} \Psi^G(x, \omega_d, \omega_i)(1 + \Phi^L(x, \omega_d, \omega_i)) f_s(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i \\
&= \int_{S^2} \Psi^G(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i + \int_{S^2} \Psi^G(x, \omega_d, \omega_i)\Phi^L(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i \\
&= \int_{S^2} \Psi^G(x, \omega_d, \omega_i) f_s(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i + d_b \int_{S^2} \Psi^G(x, \omega_d, \omega_i) f_{\mathrm{back}}(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i \\
&= F^G(x, \omega_d, \omega_o) + d_b F^L(x, \omega_d, \omega_o).
\end{aligned}$$

We shall call F^G the *global term* and F^L the *local term*. The main focus of this document deals with the computation of these terms.

2 The Fiber Scattering Model

In [Khungurn et al., 2015], we proposed a fiber scattering with two modes:

$$f_s(\omega_i, \omega_o) = f_R(\omega_i, \omega_o) + f_{TT}(\omega_i, \omega_o)$$

where each mode is seperable into the *longitudinal scattering function (LSF)* M and the *azimuthal scattering function (ASF)* N:

$$\begin{aligned}
f_R(\omega_i, \omega_o) &= M_R(\theta_i, \theta_o) N_R(\phi_i, \phi_o) \\
f_{TT}(\omega_i, \omega_o) &= M_{TT}(\theta_i, \theta_o) N_{TT}(\phi_i, \phi_o).
\end{aligned}$$

We note that both f_R and f_{TT} are colored. We, however, will think of N_R and N_{TT} as probability distribution on $[0, 2\pi)$, which is the domain of the azimuthal angle. This implies that M_R and M_{TT} must be colored.

The LSFs are defined as a normalized gaussian times a color, which varies according to the incoming longitudinal angle θ_i:

$$\begin{aligned}
M_R(\theta_i, \theta_o) &= \mathscr{F}_R(\theta_i) \ \bar{g}(\theta_o; -\theta_i, \beta_R) \\
M_{TT}(\theta_i, \theta_o) &= (1 - \mathscr{F}_R(\theta_i)) \ C_{TT} \ \bar{g}(\theta_o; -\theta_i, \beta_{TT})
\end{aligned}$$

where $\mathscr{F}_R$ is the Schlick's approximation of the Fresnel factor

$$\mathscr{F}_R(\theta_i) = C_R + (1 - C_R)(1 - \cos\theta_i)^5,$$

and C_R, β_R, C_{TT}, and β_{TT} are all model parameters. Note that C_R and C_{TT} are colors while β_R and β_{TT} are scalars. The normalized Gaussian function $\bar{g}$ is defined on the domain $(-\pi/2, \pi/2)$ as:

$$\bar{g}(x; \mu, \sigma) = \frac{g(x; \mu, \sigma)}{G(\mu, \sigma)}$$

with

$$G(\mu, \sigma) = \int_{-\pi/2}^{\pi/2} g(x; \mu, \sigma) Q(x) \ \mathrm{d}x$$

where $Q(x)$ is a polynomial that approximates $\cos^2\theta x$ from above, and $g(x; \mu, \sigma)$ is the Gaussian function with mean μ and standard deviation σ.

For reference purpose, we shall expand the LSF a little bit:

$$M_R(\theta_i, \theta_o) = \mathscr{F}_R(\theta_i)\ \frac{g(\theta_o; -\theta_i, \beta_R)}{G(-\theta_i, \beta_R)}$$
$$M_{TT}(\theta_i, \theta_o) = (1 - \mathscr{F}_R(\theta_i))\ C_{TT}\ \frac{g(\theta_o; -\theta_i, \beta_{TT})}{G(-\theta_i, \beta_{TT})}.$$

On to the ASFs, we define:

$$N_R(\phi_i, \phi_o) = \frac{1}{2\pi}$$
$$N_{TT}(\phi_i, \phi_o) = v(\phi_o; \phi_i + \pi, \gamma_{TT})$$

where v is the Von Mises distribution, and γ_{TT} is a model parameters.

3 The Global Multiple Scattering Function

There are two cases for the global scattering function. If the ray from x in direction ω_d is not occluded by other fibers, then $\Psi^G(x, \omega_d, \omega_i)$ is the delta function $\delta(\omega_d - \omega_i)$.

Otherwise, the global scattering function $\Psi^G(x, \omega_d, \omega_i)$ is defined as:

$$\Psi^G(x, \omega_d, \omega_i) = T_f(x, \omega_d) S_f(x, \omega_d, \omega_i)$$

where T_f is the *forward scattering transmittance function*, and S_f is the *forward scattering spread function*.

3.1 Forward Scattering Transmittance Function

The forward scattering transmittance function is defined as:

$$T_f(x, \omega_d) = d_f \prod_{k=1}^{n} \bar{a}_f(\theta_d^k)$$

where d_f is a constant which is set to 0.7 in the paper, and $\bar{a}_f$ is the *average attenuation function*, and θ_d^k is the inclination angle at the kth fiber along the ray from x in the direction of ω_d. The $\bar{a}_f$ is defined as:

$$\begin{aligned}\bar{a}_f(\theta_d) &= \frac{1}{\pi} \int_0^\pi \int_{\Omega_f} f_s((\theta_d, \phi_d), \omega) \cos\theta\ \mathrm{d}\omega \mathrm{d}\phi_d \\ &= \frac{1}{\pi} \int_0^\pi \left(\int_{\Omega_f} f_s((\theta_d, \phi_d), \omega) \cos\theta\ \mathrm{d}\omega \right) \mathrm{d}\phi_d.\end{aligned}$$

Here, ω_f is the forward scattering hemisphere (i.e., the bottom hemisphere, which is the directions where $\phi \in (\pi, 2\pi)$). The rationale for the above function is that it averages outgoing irradiance over the forward scattering hemisphere due to uniform coming radiance from backward directions with longitudinal θ_d.

We note that there are a number of differences between our formulation of $\bar{a}_f$ and the one in the original paper. (These differences are highlighted in red.) First, the integral on ϕ has range $(0, \pi)$ instead of $(-\pi/2, \pi/2)$. This might be because we think of the light as coming from the top instead of the right as might be meant in the paper. Second, we think, without the $\cos\theta$ factor, the calculation in the paper does not make sense. This is because the integral value can exceed 1 even the scattering function f_s is energy conserving.

With this corrected definition in mind, we can compute the average attentuation of our scattering function. We start with the inner integral:

$$\begin{aligned}
&\int_{\Omega_f} f_s((\theta_d, \phi_d), \omega) \cos\theta \ \mathrm{d}\omega \\
&= \int_{\Omega_f} f_R((\theta_d, \phi_d), \omega) \cos\theta \ \mathrm{d}\omega + \int_{\Omega_f} f_{TT}((\theta_d, \phi_d), \omega) \cos\theta \ \mathrm{d}\omega \\
&= \int_{\pi}^{2\pi} \int_{-\pi/2}^{\pi/2} M_R(\theta_d, \theta) N_R(\phi_d, \phi) \cos^2\theta \ \mathrm{d}\theta \mathrm{d}\phi + \int_{\pi}^{2\pi} \int_{-\pi/2}^{\pi/2} M_{TT}(\theta_d, \theta) N_{TT}(\phi_d, \phi) \cos^2\theta \ \mathrm{d}\theta \mathrm{d}\phi \\
&= \bigg(\int_{-\pi/2}^{\pi/2} M_R(\theta_d, \theta) \cos^2\theta \ \mathrm{d}\theta \bigg) \bigg(\int_{\pi}^{2\pi} N_R(\phi_d, \phi) \ \mathrm{d}\phi \bigg) \\
&\quad + \bigg(\int_{-\pi/2}^{\pi/2} M_{TT}(\theta_d, \theta) \cos^2\theta \ \mathrm{d}\theta \bigg) \bigg(\int_{\pi}^{2\pi} N_{TT}(\phi_d, \phi) \ \mathrm{d}\phi \bigg).
\end{aligned}$$

Now, by the definition of the normalized Gaussian function, we have that

$$\int_{-\pi/2}^{\pi/2} \frac{\bar{g}(x; \mu, \sigma)}{G(\mu, \sigma)} \cos^2 x \ \mathrm{d}x \approx 1.$$

So,

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} M_R(\theta_d, \theta) \cos^2\theta \ \mathrm{d}\theta &= \mathscr{F}_R(\theta_d) \int_{-\pi/2}^{\pi/2} \frac{\bar{g}(\theta; -\theta_d, \beta_R)}{G(-\theta_d, \beta_R)} \cos^2\theta \ \mathrm{d}\theta \approx \mathscr{F}_R(\theta_d) \\
\int_{-\pi/2}^{\pi/2} M_{TT}(\theta_d, \theta) \cos^2\theta \ \mathrm{d}\theta &= (1 - \mathscr{F}_R(\theta_d)) C_{TT} \int_{-\pi/2}^{\pi/2} \frac{\bar{g}(\theta; -\theta_d, \beta_{TT})}{G(-\theta_d, \beta_{TT})} \cos^2\theta \ \mathrm{d}\theta \approx (1 - \mathscr{F}_R(\theta_d)) C_{TT}.
\end{aligned}$$

Thus, we have that:

$$\begin{aligned}
&\int_{\Omega_f} f_s((\theta_d, \phi_d), \omega) \cos\theta \ \mathrm{d}\omega \\
&\approx \mathscr{F}_R(\theta_d) \bigg(\int_{\pi}^{2\pi} N_R(\phi_d, \phi) \ \mathrm{d}\phi \bigg) + (1 - \mathscr{F}_R(\theta_d)) C_{TT} \bigg(\int_{\pi}^{2\pi} N_{TT}(\phi_d, \phi) \ \mathrm{d}\phi \bigg) \\
&= \frac{\mathscr{F}_R(\theta_d)}{2} + (1 - \mathscr{F}_R(\theta_d)) C_{TT} \bigg(\int_{\pi}^{2\pi} N_{TT}(\phi_d, \phi) \ \mathrm{d}\phi \bigg)
\end{aligned}$$

The average attenuation then is given by:

$$\begin{aligned}
\bar{a}_f(\theta_d) &= \frac{1}{\pi} \int_0^{\pi} \bigg(\int_{\Omega_f} f_s((\theta_d, \phi_d), \omega) \cos\theta \ \mathrm{d}\omega \bigg) \ \mathrm{d}\phi_d \\
&= \frac{1}{\pi} \int_0^{\pi} \bigg(\frac{\mathscr{F}_R(\theta_d)}{2} + (1 - \mathscr{F}_R(\theta_d)) C_{TT} \bigg(\int_{\pi}^{2\pi} N_{TT}(\phi_d, \phi) \ \mathrm{d}\phi \bigg) \bigg) \ \mathrm{d}\phi_d \\
&= \frac{1}{\pi} \int_0^{\pi} \bigg(\frac{\mathscr{F}_R(\theta_d)}{2} + (1 - \mathscr{F}_R(\theta_d)) C_{TT} \bigg(\int_{\pi}^{2\pi} N_{TT}(\phi_d, \phi) \ \mathrm{d}\phi \bigg) \bigg) \ \mathrm{d}\phi_d \\
&= \frac{\mathscr{F}_R(\theta_d)}{2} + \frac{(1 - \mathscr{F}_R(\theta_d)) C_{TT}}{\pi} \int_0^{\pi} \int_{\pi}^{2\pi} N_{TT}(\phi_d, \phi) \ \mathrm{d}\phi \mathrm{d}\phi_d.
\end{aligned}$$

The last double integral can be precomputed. I will use Gaussian quadrature for that.

3.2 Forward Scattering Spread Function

The spread function approximates the angular distribution of the front scatttered light. I found that there are many things wrong with the formulation in the original paper:

1. Since it accounts for angular distribution, the function should be a *probability distribution on the sphere.* This means that

$$\int_{S^2} S_f(x,\omega_d,\omega_i)\ \mathrm{d}\omega_i = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} S_f(x,\omega_d,\omega_i)\cos\theta_i\ \mathrm{d}\theta_i \mathrm{d}\phi_i = 1.$$

   However, both the original paper and [Sadeghi and Tamstorf, 2010] define S_f as a fraction with $\cos\theta_d$ in the denominator. This makes the function loses energy, which might be okay, but I will not follow that approach.

2. The distribution is on the *incoming directions*, yet the original paper insists the function is non-zero on the "front" hemisphere. However, it should be the back hemisphere that gets the energy.

3. The original paper says the longitudinal lobe should be be centered around $\theta_i = -\theta_d$. However, with the same logic as in the last item, it should be centered around $\theta_i = \theta_d$ instead.

As such, we define the spread function as:

$$S_f(x,\omega_d,\omega_i) = \frac{\tilde{s}_f(\phi_d,\phi_i)}{\cos\theta_i}\tilde{g}(\theta_i;\theta_d,\sigma_f(x,\omega_d))$$

where

$$\tilde{s}_f(\phi_d,\phi_i) = \begin{cases} 1/\pi, & \phi_d - \pi/2 \le \phi_i \le \phi_d + \pi/2 \\ 0, & \text{otherwise} \end{cases},$$

and $\tilde{g}$ is the following normalized Gaussian function:

$$\tilde{g}(x;\mu,\sigma) = \frac{g(x;\mu,\sigma)}{\int_{-\pi/2}^{\pi/2} g(x;\mu,\sigma)\ \mathrm{d}x} = \frac{g(x;\mu,\sigma)}{\tilde{G}(\mu,\sigma)}.$$

The standard deviation $\sigma_f(x,\omega_d)$ is computed by summing up the variances of the TT modes of the fibers encountered along the ray from x in the direction ω_d. Supposing that all fibers have the same BCSDF, we have that:

$$\sigma_f(x,\omega_d) = \sqrt{n\beta_{TT}^2}$$

where n is the number of fibers on the ray.

4 Global Term

We now turn our attention of the global term $F^G(x,\omega_d,\omega_o)$. We have that

$$\begin{aligned}
F^G(x,\omega_d,\omega_o) &= \int_{S^2} \Psi^G(x,\omega_d,\omega_i) f_s(\omega_i,\omega_o)\cos\theta_i\ \mathrm{d}\omega_i \\
&= T_f(x,\omega_d)\int_{S^2} \frac{\tilde{s}_f(\phi_d,\phi_i)}{\cos\theta_i}\tilde{g}(\theta_i;\theta_d,\sigma_f) f_s(\omega_i,\omega_o)\cos\theta_i\ \mathrm{d}\omega_i \\
&= T_f(x,\omega_d)\int_{S^2} \tilde{s}_f(\phi_d,\phi_i)\tilde{g}(\theta_i;\theta_d,\sigma_f)[f_R(\omega_i,\omega_o) + f_{TT}(\omega_i,\omega_o)]\ \mathrm{d}\omega_i \\
&= T_f(x,\omega_d)\left[\int_{S^2} \tilde{s}_f(\phi_d,\phi_i)\tilde{g}(\theta_i;\theta_d,\sigma_f) f_R(\omega_i,\omega_o)\ \mathrm{d}\omega_i + \int_{S^2} \tilde{s}_f(\phi_d,\phi_i)\tilde{g}(\theta_i;\theta_d,\sigma_f) f_{TT}(\omega_i,\omega_o)\ \mathrm{d}\omega_i\right]
\end{aligned}$$

We deal with the integral involving f_R first.

$$
\begin{aligned}
&\int_{S^2} \tilde{s}_f(\phi_d,\phi_i)\tilde{g}(\theta_i;\theta_d,\sigma_f) f_R(\omega_i,\omega_o)\mathrm{d}\omega_i \\
&= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \tilde{s}_f(\phi_d,\phi_i)\frac{g(\theta_i;\theta_d,\sigma_f)}{\tilde{G}(\theta_d,\sigma_f)}\frac{M_R(\theta_i,\theta_o)}{2\pi}\cos\theta_i\ \mathrm{d}\theta_i\mathrm{d}\phi_i \\
&= \frac{1}{2\pi}\left(\int_0^{2\pi}\tilde{s}_f(\phi_d,\phi_i)\ \mathrm{d}\phi_i\right)\frac{1}{\tilde{G}(\theta_d,\sigma_f)}\left(\int_{-\pi/2}^{\pi/2} g(\theta_i;\theta_d,\sigma_f)M_R(\theta_i,\theta_o)\cos\theta_i\ \mathrm{d}\theta_i\right) \\
&= \frac{1}{2\pi\tilde{G}(\theta_d,\sigma_f)}\int_{-\pi/2}^{\pi/2} g(\theta_i;\theta_d,\sigma_f)M_R(\theta_i,\theta_o)\cos\theta_i\ \mathrm{d}\theta_i.
\end{aligned}
$$

I will compute the last integral by Gaussian quadrature.

For the integral involing f_{TT}, we have that:

$$
\begin{aligned}
&\int_{S^2} \tilde{s}_f(\phi_d,\phi_i)\tilde{g}(\theta_i;\theta_d,\sigma_f) f_{TT}(\omega_i,\omega_o)\ \mathrm{d}\omega_i \\
&= \frac{1}{\tilde{G}(\theta_d,\sigma_f)}\left(\int_0^{2\pi}\tilde{s}(\phi_d,\phi_i)N_{TT}(\phi_i,\phi_o)\ \mathrm{d}\phi_i\right)\left(\int_{-\pi/2}^{\pi/2} g(\theta_i;\theta_d,\sigma_f)M_{TT}(\theta_i,\theta_o)\cos\theta_i\ \mathrm{d}\theta_i\right) \\
&= \frac{1}{\tilde{G}(\theta_d,\sigma_f)}\left(\frac{1}{\pi}\int_{\phi_d-\pi/2}^{\phi_d+\pi/2} N_{TT}(\phi_i,\phi_o)\ \mathrm{d}\phi_i\right)\left(\int_{-\pi/2}^{\pi/2} g(\theta_i;\theta_d,\sigma_f)M_{TT}(\theta_i,\theta_o)\cos\theta_i\ \mathrm{d}\theta_i\right).
\end{aligned}
$$

Again, the two integrals can be evaluated with Gaussian quadrature.

5 f_{back}

The function f_{back} has the same type as the BCSDF. So, it should be energy conserving, meaning that:

$$
\int_{S^2} f_{\text{back}}(\omega_i,\omega_o)\cos\theta_o\ \mathrm{d}\omega_o = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} f_{\text{back}}(\omega_i,\omega_o)\cos^2\theta_o\ \mathrm{d}\theta_o\mathrm{d}\phi_o \le 1.
$$

The way the paper defines the function eventually has $\cos^2\frac{\theta_o-\theta_i}{2}$ in the denominator, but I'm not so sure whether all of that is sensible. Instead, I'm redefining it as:

$$
f_{\text{back}}(\omega_i,\omega_o) = \bar{A}_b(\theta_i)\bar{S}_b(\omega_i,\omega_o)
$$

where $\bar{A}_b(\theta_i)$ is the *average backscattering attenuation*, and $\bar{S}_b(\omega_i,\omega_o)$ is the average backscattering spread, which has the same type as a BCSDF.

5.1 Average Backscattering Attentuation

The average backscattering attenuation is a function on the incoming longitudinal angle θ_i. It arises from the hand calculation of scattering in a simplified world where there is an infinite array of parallel fibers arranged in a plane. The function is the sum of two terms:

$$
\bar{A}_b(\theta_i) = \bar{A}_1(\theta_i) + \bar{A}_3(\theta_i)
$$

where

- $\bar{A}_1(\theta_i)$ describes the fraction of light that survived the interaction with the parallel fibers in light paths that involves one backscattering, and

- $\bar{A}_3(\theta_i)$ describes the fraction of light that survived the interaction with the parallel fibers in light paths that involves three backscattering.

By calculation, we have that

$$\bar{A}_1(\theta_i) = \frac{\bar{a}_b(\theta_i)\bar{a}_f^2(\theta_i)}{1-\bar{a}_f^2(\theta_i)}$$
$$\bar{A}_3(\theta_i) = \frac{\bar{a}_b^3(\theta_i)\bar{a}_f^2(\theta_i)}{(1-\bar{a}_f^2(\theta_i))^3}$$

The forward average attenuation $\bar{a}_f$ was defined and calculated in Section 3.1. The *backward average attenuation* $\bar{a}_b$ is defined as:

$$\begin{aligned}\bar{a}_b(\theta_i) &= \frac{1}{\pi}\int_0^\pi \int_{\Omega_b} f_s((\theta_i,\phi_i),\omega)\cos\theta_i\cos\theta \ \mathrm{d}\omega\mathrm{d}\phi_i \\ &= \frac{1}{\pi}\int_0^\pi \left(\int_{\Omega_b} f_s((\theta_i,\phi_i),\omega)\cos\theta \ \mathrm{d}\omega\right)\mathrm{d}\phi_i.\end{aligned}$$

Here, Ω_b is the backward scattering hemisphere, i.e., those directions with $\phi \in (0,\pi)$.

With the calculation we did in Section 3.1, we can say that:

$$\bar{a}_b(\theta_i) = \frac{\mathscr{F}_R(\theta_i)}{2} + \frac{(1-\mathscr{F}_R(\theta_i))C_{TT}}{\pi}\int_0^\pi\int_0^\pi N_{TT}(\phi_i,\phi)\ \mathrm{d}\phi\mathrm{d}\phi_i.$$

5.2 Average Backscattering Spread

Since the average backscattering spread is supposed to be a BCSDF, I define it to be:

$$\bar{S}_b(\omega_i,\omega_o) = \tilde{s}_b(\phi_i,\phi_o)\bar{g}(\theta_o;-\theta_i,\bar{\sigma}_b(\theta_i)).$$

Here,

$$\tilde{s}_b(\phi_i,\phi_o) = \begin{cases}1/\pi, & \phi_i-\pi/2 \le \phi_o \le \phi_i+\pi/2 \\ 0, & \text{otherwise}\end{cases}.$$

The $\bar{g}$ function is the normalized Gaussian function used to define our BCSDF. The standard deviation $\bar{\sigma}_b(\theta_i)$ is given by:

$$\bar{\sigma}_b(\theta_i) \approx (1+0.7\bar{a}_f^2)\frac{\bar{a}_b\sqrt{2\beta_{TT}^2+\beta_R^2}+\bar{a}_b^3\sqrt{2\beta_{TT}^2+3\beta_R^2}}{\bar{a}_b+\bar{a}_b^3(2\beta_{TT}+3\beta_R)}$$

The above is taken directly from the original paper. I'm not so sure how currect it is, but let us not discuss this now.

6 Local Term

Now, the local term is given by:

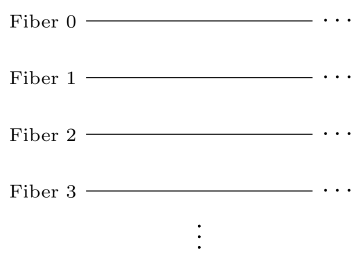
$$\begin{aligned}F^L(x,\omega_d,\omega_o) &= \int_{S^2}\Psi^G(x,\omega_d,\omega_i)f_{\text{back}}(\omega_i,\omega_o)\cos\theta_i\ \mathrm{d}\omega_i \\ &= T_f(x,\omega_d)\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\frac{\tilde{s}_f(\phi_d,\phi_i)}{\cos\theta_i}\bar{g}(\theta_i;\theta_d,\sigma_f)\bar{A}_b(\theta_i)\tilde{s}_f(\phi_i,\phi_o)\bar{g}(\theta_o;-\theta_i,\sigma_f(\theta_i))\cos^2\theta_i\ \mathrm{d}\theta_i\mathrm{d}\phi_i \\ &= \frac{T_f(x,\omega_d)}{G(\theta_d,\sigma_f)}\left(\int_0^{2\pi}\tilde{s}_f(\phi_d,\phi_i)\tilde{s}_f(\phi_i,\phi_o)\ \mathrm{d}\phi_i\right)\left(\int_{-\pi/2}^{\pi/2}\bar{A}_b(\theta_i)g(\theta_i;\theta_d,\sigma_f)\bar{g}(\theta_o;-\theta_i,\sigma_f(\theta_i))\cos\theta_i\ \mathrm{d}\theta_i\right).\end{aligned}$$

The first integral can be evaluated symbolically by finding the length of the interval which is the intersection of $(\phi_d - \frac{\pi}{2}, \phi_d + \frac{\pi}{2})$ and $(\phi_o - \frac{\pi}{2}, \phi_o + \frac{\pi}{2})$ and then dividing the length by π^2. The second integral needs to be evaluated with Gaussian quadrature.

7 The Complete Attenuation Term

The paper only proposes approximates the attenuation $\bar{A}_b$ as the sum of two terms $\bar{A}_1 + \bar{A}_3$. However, I think it is possible to do the infinite sum with some math.

First, let us define the situation that we are in. We have an infinite arrangement of fibers in a plane, with Fiber 0 above Fiber 1, Fiber 1 above Fiber 2, and so on.



Let a_i^+ denote the attenuation that the light that strikes Fiber i from above experiences after it travels through all the fibers and finally scatters upward from Fiber 0. Let a_i^- denote the same attenuation, but now with the light striking Fiber i from below.

For convience, we may think that there is actually Fiber -1 above Fiber 0. The goal of the process is to reach Fiber -1, so $a_{-1}^- = 1$ because that's our goal. With this definition, we have that

$$\begin{aligned} a_i^+ &= \bar{a}_f a_{i+1}^+ + \bar{a}_b a_{i-1}^- \\ a_i^- &= \bar{a}_b a_{i+1}^+ + \bar{a}_f a_{i-1}^- \end{aligned}$$

for all $i \geq 0$. Consider Fiber 0, we have that:

$$a_0^+ = \bar{a}_f a_1^+ + \bar{a}_b \tag{2}$$

$$a_0^- = \bar{a}_b a_1^+ + \bar{a}_f. \tag{3}$$

Now, consider a_1^+. We have that, to reach Fiber -1, the light must first travels back to Fiber 0 from below. This situation is the same as starting at Fiber 0 and arriving at Fiber -1 from below. In this first step, the light is attenuated by a factor of a_0^+. Then, to reach Fiber -1, it is attenuated by another factor of a_0^-. It follows that:

$$a_1^+ = a_0^+ a_0^-.$$

Multiplying (2) and (3) together, we have that:

$$\begin{aligned} a_1^+ &= (\bar{a}_f a_1^+ + \bar{a}_b)(\bar{a}_b a_1^+ + \bar{a}_f) \\ a_1^+ &= \bar{a}_f \bar{a}_b (a_1^+)^2 + (\bar{a}_f^2 + \bar{a}_b^2) a_1^+ + \bar{a}_f \bar{a}_b \\ 0 &= \bar{a}_f \bar{a}_b (a_1^+)^2 + (\bar{a}_f^2 + \bar{a}_b^2 - 1) a_1^+ + \bar{a}_f \bar{a}_b \\ a_1^+ &= \frac{1 - \bar{a}_f^2 - \bar{a}_b^2 \pm \sqrt{(\bar{a}_f^2 + \bar{a}_b^2 - 1)^2 - 4\bar{a}_f^2 \bar{a}_b^2}}{2\bar{a}_f \bar{a}_b} \end{aligned}$$

Only one of the solution will be less than 1, so we will take that one.

The attenuation to output should be $\bar{a}_f a_0^+$.

References

[Khungurn et al., 2015] Khungurn, P., Schroeder, D., Zhao, S., Bala, K., and Marschner, S. (2015). Matching real fabrics with micro-appearance models. To appear in ACM Transactions on Graphics.

[Sadeghi and Tamstorf, 2010] Sadeghi, I. and Tamstorf, R. (2010). Efficient implementation of the dual scattering model in renderman. Technical report, Walt Disney Animation Studios.

[Zinke et al., 2008] Zinke, A., Yuksel, C., Weber, A., and Keyser, J. (2008). Dual scattering approximation for fast multiple scattering in hair. *ACM Trans. Graph.*, 27(3):32:1–32:10.