Differential Geometry Notes of 03/29/2013

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1 Christoffel Symbols

- S will denote a regular, orientable, and oriented surface. Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ be a parameterization in the orientation of S.
- It is possible to assign each point of $\mathbf{x}(U)$ a basis given by the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and N.
- We can now express the derivatives of the vectors $\mathbf{x}$, $\mathbf{y}$, and N in this basis.

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + L_1 N \\
\mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + L_2 N \\
\mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + L_3 N \\
N_u &= a_{11} \mathbf{x}_u + a_{21} \mathbf{x}_v \\
N_v &= a_{12} \mathbf{x}_u + a_{22} \mathbf{x}_v
\end{aligned}$$

  The coefficients Γ_{ij}^k are called the **Christoffel symbols** of S in the parameterization $\mathbf{x}$.

- Now, we have that

$$\begin{aligned}
\langle \mathbf{x}_{uu}, N \rangle &= \Gamma_{11}^1 \langle \mathbf{x}_u, N \rangle + \Gamma_{11}^2 \langle \mathbf{x}_v, N \rangle + L_1 \langle N, N \rangle \\
e &= L_1.
\end{aligned}$$

  Similar, by computing $\langle \mathbf{x}_{uv}, N \rangle$ and $\langle \mathbf{x}_{vv}, N \rangle$, we have that $L_2 = f$ and $L_3 = g$. Hence, we can rewrite the equations as:

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + eN \\
\mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + fN \\
\mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + gN \\
N_u &= a_{11} \mathbf{x}_u + a_{21} \mathbf{x}_v \\
N_v &= a_{12} \mathbf{x}_u + a_{22} \mathbf{x}_v
\end{aligned}$$

- Now,

$$E_u = \frac{\partial}{\partial u} \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 2 \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle.$$

  So,

$$\frac{1}{2} E_u = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle$$

Now,

$$\frac{1}{2}E_u = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \Gamma_{11}^1 \langle \mathbf{x}_u, \mathbf{x}_u \rangle + \Gamma_{11}^2 \langle \mathbf{x}_v, \mathbf{x}_u \rangle + e\langle N, \mathbf{x}_u \rangle = \Gamma_{11}^1 E + \Gamma_{11}^2 F$$

Moreover,

$$\begin{aligned} F_u &= \frac{\partial}{\partial u}\langle \mathbf{x}_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle + \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle \\ E_v &= \frac{\partial}{\partial v}\langle \mathbf{x}_u, \mathbf{x}_u \rangle = 2\langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle \end{aligned}$$

So,

$$\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \frac{1}{2}E_v.$$

Now,

$$F_u - \frac{1}{2}E_v = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = \Gamma_{11}^1 \langle \mathbf{x}_u, \mathbf{x}_v \rangle + \Gamma_{11}^2 \langle \mathbf{x}_v, \mathbf{x}_v \rangle + e\langle N, \mathbf{x}_v \rangle = \Gamma_{11}^1 F + \Gamma_{11}^2 G$$

As a result, we have the following equations:

$$\begin{aligned} \frac{1}{2}E_u &= \Gamma_{11}^1 E + \Gamma_{11}^2 F \\ F_u - \frac{1}{2}E_v &= \Gamma_{11}^1 F + \Gamma_{11}^2 G. \end{aligned}$$

In other words,

$$\begin{aligned} \begin{bmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{bmatrix} &= \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} \\ \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}E_u \\ F_u - \frac{1}{2}E_v \end{bmatrix} \end{aligned}$$

Now that the determinat of the matrix being inverted is $EG - F^2$, which is always non-zero because this is the differential of the area of the regular surface.

Hence, **we can write Γ_{11}^1 and Γ_{11}^2 in terms of E, F, G, and their derivatives.**

- With similar derivation as in the last item, we have that

$$\begin{aligned} \Gamma_{12}^1 E + \Gamma_{12}^2 F &= \frac{1}{2}E_v \\ \Gamma_{12}^1 F + \Gamma_{12}^2 G &= \frac{1}{2}G_u \end{aligned}$$

  and

$$\begin{aligned} \Gamma_{22}^1 E + \Gamma_{22}^2 F &= F_v - \frac{1}{2}G_u \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G &= \frac{1}{2}G_v \end{aligned}$$

  These equations tell us that **we can write all the Christoffel symbols in terms of E, F, G and their derivatives.**

- The consequence is that **all geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.**

2 Surface of Revolution

- In this section, we shall compute the Christoffel symbols for the surface of revolution given by the parameterization:

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

  where $f(v) \neq 0$.

- It can be easily shown that

$$\begin{aligned} E &= (f(v))^2 \\ F &= 0 \\ G &= (f'(v))^2 + (g'(v))^2 \end{aligned}$$

  So,

$$\begin{aligned} E_u &= 0 \\ E_v &= 2ff' \\ F_u &= F_v = 0 \\ G_u &= 0 \\ G_v &= 2(f'f'' + g'g'') \end{aligned}$$

  Now, using the approach discussed in the last section, we have that

$$\begin{aligned} \Gamma^1_{11} &= 0 & \Gamma^2_{11} &= -\frac{ff'}{(f')^2 + (g')^2} \\ \Gamma^1_{12} &= \frac{ff'}{f^2} & \Gamma^2_{12} &= 0 \\ \Gamma^2_{22} &= 0 & \Gamma^2_{22} &= \frac{f'f'' + g'g''}{(f')^2 + (g')^2} \end{aligned}$$

3 Gauss Formula

- We shall rewrite the following identity using the expansion of $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$ in the basis $\{\mathbf{x}_u, \mathbf{x}_v, N\}$:

$$(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u = 0.$$

- Starting with $(\mathbf{x}_{uu})_v$, we have

$$\begin{aligned} (\mathbf{x}_{uu})_v &= (\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN)_v \\ &= (\Gamma^1_{11})_v\mathbf{x}_u + \Gamma^1_{11}\mathbf{x}_{uv} + (\Gamma^2_{11})_v\mathbf{x}_v + \Gamma^2_{11}\mathbf{x}_{vv} + e_vN + eN_v \\ &= (\Gamma^1_{11})_v\mathbf{x}_u + \Gamma^1_{11}(\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN) \\ &\quad + (\Gamma^2_{11})_v\mathbf{x}_v + \Gamma^2_{11}(\Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + gN) \\ &\quad + e_vN + e(a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v) \\ &= [(\Gamma^1_{11})_v + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + a_{12}e]\mathbf{x}_u \\ &\quad + [\Gamma^1_{11}\Gamma^2_{12} + (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} + a_{22}e]\mathbf{x}_v \\ &\quad + [f\Gamma^1_{11} + g\Gamma^2_{11} + e_v]N \end{aligned}$$

Now, for $(\mathbf{x}_{uv})_u$, we have

$$\begin{aligned}
(\mathbf{x}_{uv})_u &= (\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN)_u \\
&= (\Gamma^1_{12})_u\mathbf{x}_u + \Gamma^1_{12}\mathbf{x}_{uu} + (\Gamma^2_{12})_u\mathbf{x}_v + \Gamma^2_{12}\mathbf{x}_{uv} + f_uN + fN_u \\
&= (\Gamma^1_{12})_u\mathbf{x}_u + \Gamma^1_{12}(\Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN) \\
&\quad + (\Gamma^2_{12})_u\mathbf{x}_v + \Gamma^2_{12}(\Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN) \\
&\quad + f_uN + f(a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v) \\
&= [(\Gamma^1_{12})_u + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^1_{12}\Gamma^2_{12} + a_{11}f]\mathbf{x}_u \\
&\quad + [\Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})_u + (\Gamma^2_{12})^2 + a_{21}f]\mathbf{x}_v \\
&\quad + [e\Gamma^1_{12} + f\Gamma^2_{12} + f_u]N.
\end{aligned}$$

Because $(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u = 0$ and because $\mathbf{x}_u$, $\mathbf{x}_v$, and N are linearly independent, it must be the case that the coefficient of $\mathbf{x}_v$ in the expression of $(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u$ must be 0. In other words,

$$\Gamma^1_{11}\Gamma^2_{12} + (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} + ea_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})_u - (\Gamma^2_{12})^2 - fa_{21} = 0$$

Taking,

$$\begin{aligned}
a_{21} &= \frac{eF - fE}{EG - F^2} \\
a_{22} &= \frac{fF - gE}{EG - F^2},
\end{aligned}$$

we have

$$\Gamma^1_{11}\Gamma^2_{12} + (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} + e\frac{fF - gE}{EG - F^2} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})_u - (\Gamma^2_{12})^2 - f\frac{eF - fE}{EG - F^2} = 0.$$

In other words,

$$\begin{aligned}
\Gamma^1_{11}\Gamma^2_{12} + (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - (\Gamma^2_{12})_u - (\Gamma^2_{12})^2 &= f\frac{eF - fE}{EG - F^2} - e\frac{fF - gE}{EG - F^2} \\
(\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} &= \frac{efF - f^2E - efF + egE}{EG - F^2} \\
(\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} &= E\frac{eg - f^2}{EG - F^2} \\
(\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} &= EK.
\end{aligned}$$

This last equation is called the **Gauss formula**.

- The last equation tells us that the Gaussian curvature K can be written as an expression of E, F, G, and their derivatives.

- **Theorem 3.1 (Gauss).** *The Gaussian curvature K of a surface is invariant by local isometries.*

  Proof. Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ is a parameterization of $p \in S$. Let $\varphi : V \subseteq S \to S$, where $V \subseteq \mathbf{x}(U)$ is a neighborhood of p, be a local isometry. Define $\mathbf{y} = \varphi \circ \mathbf{x}$. We have that $\mathbf{y}$ is a parameterization around $\varphi(p)$

  Since φ is an isometry, the coefficients of the first fundamental forms in the parameterization $\mathbf{x}$ and $\mathbf{y}$ agree at corresponding points q and $\varphi(q)$ for all point $q \in V$. It follows that $K(q) = K(\varphi(q))$ for all $q \in V$ because the Gaussian curvature can be written as an expression of the coefficients of the first fundamental forms and their derivatives. $\square$

4 Mainardi–Cordazzi Equations

- By setting the coefficient of N in $(\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u = 0$ to 0, we have that

$$e_v - f_u = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}. \tag{1}$$

- Also, by setting the coefficient of N in $(\mathbf{x}_{vv})_u - (\mathbf{x}_{uv})_v = 0$ to 0, we have that

$$f_v - g_u = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}. \tag{2}$$

- Equation (1) and (2) are known collectively as the **Mainardi–Cordazzi** equations.
- The Gauss formula and the Mainardi–Cordazzi equations are known under the name of the **compatibility equations of the theory of surfaces**.
- The compatibility equations assert relations among the coefficients of the first and second fundamental forms of a regular surfaces.
- The converse is also true. If a collection of six functions satisfy the compatibility equations, then there exists a surface having them as the coefficients of the first and second fundamental forms.

  Theorem 4.1 (Bonnet). *Let E, F, G, e, f, g be differentiable functions defined in an open set $V \subseteq \mathbb{R}^2$, with $E > 0$ and $G > 0$. Assume that the given functions satisfy the compatibility equations and that $EG - F^2 > 0$. Then, for every $q \in V$, there exists a neighborhood $U \subseteq V$ of q and a diffeomorphism $\mathbf{x} : U \to \mathbb{R}^3$ such that $\mathbf{x}(U)$ is a regular surface that has E, F, G, e, f, g as coefficients of the first and second fundamental forms, respectively.*

  Furthermore, if U is connected and if $\bar{\mathbf{x}} : U \to \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exists a translation T and a proper linear orthogonal transformation ρ in $\mathbb{R}^3$ such that $\mathbf{x} = T \circ \rho \circ \mathbf{x}$. (In other words, the coefficients of the first and second fundamental forms determine the surface up to a rigid motion.)

- The Mainardi–Cordazzi equations simplify when the coordinate neighborhoods contains no umbilical points and the coordinate curves are lines of curvature ($F = 0 = f$). The equations becomes:

$$\begin{aligned} e_v &= e\Gamma^1_{12} - g\Gamma^2 11 \\ g_u &= g\Gamma^2_{12} - e\Gamma^1 22. \end{aligned}$$

  Also,

$$\begin{aligned} \Gamma^2_{11} &= -\frac{1}{2}\frac{E_v}{G} \\ \Gamma^1_{12} &= \frac{1}{2}\frac{E_v}{E} \\ \Gamma^2_{12} &= \frac{1}{2}\frac{G_u}{G} \\ \Gamma^1_{22} &= -\frac{1}{2}\frac{G_u}{E}. \end{aligned}$$

  Hence,

$$\begin{aligned} e_v &= \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right) \\ g_u &= \frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right) \end{aligned}$$