Differential Geometry Notes of 01/28/2013

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1 Continuity

- A **ball** in $\mathbb{R}^n$ with center $p_0 = (x_1^0, x_2^0, \dots, x_n^0)$ and radius $\epsilon > 0$ is the set
$$B_\epsilon(p_0) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2 < \epsilon^2\}.$$
- The set $U \subseteq \mathbb{R}^n$ is an **open set** if for each $p \in U$ there exists a ball $B_\epsilon(p) \subseteq U$.
- An open set $\mathbb{R}^n$ containing a point $p \in \mathbb{R}^n$ is a **neighborhood** of p.
- Let U be an open set. A map $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $p \in U$ if given $\epsilon > 0$, there exists $\delta > 0$ such that
$$F(B_\delta(p)) \subseteq B_\epsilon(F(p)).$$
We say that F is continuous in U is it is continuous at any point p in U.
- **Proposition 1.1.** *Let U be an open set. $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if each component function $f_i : U \subseteq \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \dots, m$ is continuous.*
- **Proposition 1.2.** *A map $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $p \in U$ if and only if, given a neighborhood V of $F(p)$ in $\mathbb{R}^m$, there exists a neighborhood W of p in $\mathbb{R}^n$ such that $F(W) \subseteq V$.*
- **Proposition 1.3.** *Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $G : V \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous maps, where U and V are open sets such that $F(U) \subseteq V$. Then $G \circ F : U \subseteq \mathbb{R}^n \to \mathbb{R}^k$ is a continuous map.*
- Let $F : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where A is an arbitrary set in $\mathbb{R}^n$ (not an open set like U). We say that F is **continuous** in A if there exists an open set $U \subseteq \mathbb{R}^n$ and $A \subseteq U$ and a continuous $\bar{F} : U \to \mathbb{R}^n$ such that the restriction of $\bar{F}$ to A is F.
- We say that a continuous map $F : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a **homeomorphism** onto $F(A)$ if F is one-to-one and the inverse $F^{-1} : F(A) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuous.
We say that A and $F(A)$ are **homeomorphic sets**.
- **Proposition 1.4.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on the closed interval $[a, b]$. Assume that $f(a)$ and $f(b)$ have opposite signs. Then, there exists a point $c \in (a, b)$ such that $f(c) = 0$.*
- **Proposition 1.5.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, f reaches its maximum and minimum in $[a, b]$; that is, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in [a, b]$.*
- **Proposition 1.6 (Heine–Borel Theorem).** *Let $[a, b]$ be a closed interval and let $I_\alpha, \alpha \in A$, be a collection of open interval in $[a, b]$ such that $\bigcup_\alpha I_\alpha = [a, b]$. Then, it is possible to choose a finite number $I_{k_1}, I_{k_2}, \dots, I_{k_n}$ of I_α such that $\bigcup I_{k_i} = I$ for $i = 1, 2, \dots, n$.*

2 Differentiability in $\mathbb{R}^n$

- Let $f : U \subseteq \mathbb{R} \to \mathbb{R}$. The **derivative** $f'(x_0)$ of f at $x_0 \in U$ is the limit (when it exists)

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- If f has derivatives at all points of neighborhood V of x_0, we can consider the derivative of $f' : V \to \mathbb{R}$ at x_0, which is called the **second derivative** $f''(x_0)$ of f at x_0. We can define the derivative of higher order in a similar manner.

- We say that f is **differentiable** at x_0 if it has continous derivatives of all orders at x_0. We say that f is **differentiable** in U if it is differentiable at all points in U.

- Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}$. The **partial derivative** of f with respect to x at $(x_0, y_0) \in U$, denoted by $(\partial f / \partial x)(x_0, y_0)$ is the derivative at x_0 for the function of one variable $x \mapsto f(x, y_0)$.

  The partial derivative with respect y is defined similarly.

- The **second partial derivatives** at (x_0, y_0) are:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \, \partial y}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \, \partial x} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

  Partial derivatives of higher order are defined similarly.

- We say that f is **differentiable** at (x_0, y_0) if it has continuous partial derivatives of all orders at (x_0, y_0). We say that f is differentiable in U if it is differentiable at all points in U.

- When f is differentiable, the partial derivatives of f are independent of the order in which they are performed; that is,

$$\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial^2 f}{\partial y \, \partial x}, \qquad \frac{\partial^3 f}{\partial x^2 \, \partial y} = \frac{\partial^3 f}{\partial x \, \partial y \, \partial x}, \qquad \text{etc.}$$

- We sometimes denote the partial derivatives with the following notations:

$$f_x = \frac{\partial f}{\partial x} \qquad f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \, \partial y} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

- The definitions of partial derivatives for $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ can be similarly defined as the 2D case.

- Partial derivatives obey the **chain rule**.

  For example, if $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ are real differentiable functions in $U \subseteq \mathbb{R}^2$, and $f(x, y, z)$ is a real differential function in $\mathbb{R}^3$, then the partial derivative of f with respect to u is given by:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

- We say that the function $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

  is **differentiable** at $(x_1, \ldots, x_n)$ if all of its component functions are differentiable at $(x_1, \ldots, x_n)$.

  We say that F is differentiable in U if it is differentiable at all points in U.

- When $n = 1$, we have that F define a **differentiable curve** in $\mathbb{R}^n$.
- A **tangent** vector to a differentiable curve $\alpha : U \subseteq \mathbb{R} \to \mathbb{R}^m$ at $t_0 \in U$ is the vector
$$\alpha'(t_0) = (\alpha_1'(t_0), \ldots, \alpha_m'(t_0)).$$
- Given a vector $w \in \mathbb{R}^m$ and a point $p \in U \subseteq \mathbb{R}^n$, we can always find a differentiable curve $alpha : (-\epsilon, \epsilon) \to U$ with $\alpha(0) = p$ and $\alpha'(0) = w$. This is done simply by taking $\alpha(t) = p + wt$.
- **Definition 2.1.** *Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. To each $p \in U$, we associate a linear map $\mathrm{d}F_p : \mathbb{R}^n \to \mathbb{R}^m$ (that is, an $m \times n$ matrix) which is called the* **differential** *of F at p, which is defined as follows.*

  Let $w \in \mathbb{R}^n$ and let $\alpha : (-\epsilon, \epsilon) : \mathbb{R} \to \mathbb{R}^n$ be a differentiable curve such that $\alpha(0) = p$ and $\alpha'(0) = w$. By the chain rule, the function $\beta = F \circ \alpha : \mathbb{R} \to \mathbb{R}^m$ is also differentiable. We define
$$\mathrm{d}F_p(w) = \beta'(0).$$
- **Proposition 2.2.** *$\mathrm{d}F_p$ is well-defined. That is, the value of $\mathrm{d}F_p(w)$ does not depend on the particular choice of the curve α.*

  Proof. Let
$$F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)),$$
  and
$$\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)).$$
  As a result,
$$\beta(t) = (F \circ \alpha)(t) = \begin{bmatrix} f_1(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)) \\ f_2(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)) \\ \vdots \\ f_M(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)) \end{bmatrix}.$$
  So,
$$\begin{aligned} \beta'(0) = \left[\frac{\mathrm{d}}{\mathrm{d}t}(F \circ \alpha)\right]_{t=0} &= \begin{bmatrix} \left[\frac{\mathrm{d}}{\mathrm{d}t} f_1(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))\right]_{t=0} \\ \left[\frac{\mathrm{d}}{\mathrm{d}t} f_2(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))\right]_{t=0} \\ \vdots \\ \left[\frac{\mathrm{d}}{\mathrm{d}t} f_M(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))\right]_{t=0} \end{bmatrix} \\ &= \begin{bmatrix} \left[\frac{\partial f_1}{\partial x_1}\right]_{x_1=\alpha_1(0)} + \cdots + \left[\frac{\partial f_1}{\partial x_n}\right]_{x_n=\alpha_n(0)} \alpha_n'(0) \\ \left[\frac{\partial f_2}{\partial x_1}\right]_{x_1=\alpha_1(0)} + \cdots + \left[\frac{\partial f_2}{\partial x_n}\right]_{x_n=\alpha_n(0)} \alpha_n'(0) \\ \vdots \\ \left[\frac{\partial f_m}{\partial x_1}\right]_{x_1=\alpha_1(0)} + \cdots + \left[\frac{\partial f_m}{\partial x_n}\right]_{x_n=\alpha_n(0)} \alpha_n'(0) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{x=p} \begin{bmatrix} \alpha_1'(0) \\ \alpha_2'(0) \\ \vdots \\ \alpha_n'(0) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{x=p} w, \end{aligned}$$

and the matrix on the RHS is $\mathrm{d}F_p$. $\square$

- The matrix $[\partial f_i/\partial x_j]$ where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ is called the **Jacobian matrix** of F at p. When $n = m$, this is a square matrix and its determinatn is called the **Jacobian determinant**, which is usually denoted by:

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}.$$

- **Proposition 2.3 (Chain rule for maps).** *Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $G : V \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be differentiable maps, where U and V are open sets and $F(U) \subseteq V$. Then, $G \circ F = U \to \mathbb{R}^k$ is a differentiable map, and*

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p.$$

- We say that an open set $U \subseteq \mathbb{R}^n$ is **connected** if given two points $p, q \in U$, there exists a continuous map $\alpha : [a, b] \to U$ such that $\alpha(a) = p$ and $\alpha(b) = q$.

- **Proposition 2.4.** *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable map defined on a connected open subset U of $\mathbb{R}^n$. Assume that $\mathrm{d}f_p : \mathbb{R}^n \to \mathbb{R}$ is zero at every point $p \in U$. Then f is constant on U.*

  Proof. Let $p \in U$, and let $B_\delta(p) \subseteq U$ be an open ball around p inside U. Any point $q \in B_\delta(p)$ can be joined to p by the straight line $\beta : [0, 1] \to B_\delta(p)$ where $\beta(t) = (1 - t)p + tq$. Since U is open, we expand β's domain to $(0 - \epsilon, 1 + \epsilon)$. Now, we consider $f \circ \beta : (0 - \epsilon, 1 + \epsilon) \to \mathbb{R}$ is a function defined on open interval. We have that

$$\mathrm{d}(f \circ \beta)_t = (\mathrm{d}f \circ \mathrm{d}\beta)_t = 0$$

  because $\mathrm{d}f_p$ is zero at every point p. Because $\mathrm{d}(f \circ \beta)_t$ is a 1×1 matrix whose entry is simply $\mathrm{d}(f \circ \beta)/\mathrm{d}t$, we have that the derivative is 0. This means that $f \circ \beta$ is constant. So, $f(p) = f(\beta(0)) = f(\beta(1)) = f(q)$. Thus, because p and q are arbitrary, f is constant in $B_\delta(U)$.

  Next, we need to show that $f(p) = f(q)$ for any two points p, q in U. Since U is connected, there is a connected curve $\alpha : [a, b] \to U$ that joints p and q. By the first part of the proof, for reach $t \in [a, b]$, there exists an open interval I_t such that $f \circ \alpha$ is constant. Note also that $\bigcap_t I_t = [a, b]$. By the Heine–Borel theorem, we can choose a finite number of open intervals $I_1, I_2, \ldots, I_k$ that covers $[a, b]$. By renumbering the intervals, we can assume that consecutive intervals overlap. This implies that f is constant over the union of all intervals. Therefore, $f(p) = f(q)$, and we are done. $\square$

- **Proposition 2.5 (Inverse Function Theorem).** *Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapp and suppose that at $p \in U$ the differential $\mathrm{d}F_p : \mathbb{R}^a \to \mathbb{R}^n$ is invertible. Then, there exists a neighborhood V of p in U and a neighborhood W of $F(p)$ in $\mathbb{R}^n$ such that $F : V\mathbb{R}W$ has a differentiable inverse $F^{-1} : W \to V$.*

- A differentiable map $F : V \subseteq \mathbb{R}^n \to W \subseteq \mathbb{R}^n$, where V and W are open sets, is called a **diffeomorphism** of V with W if F has a differentiable inverse.