Quaternion Exponentiation and Logarithm

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This note is about quaternion exponentiation. I'm basing this note on the note by Glenn Rowe [Row].

1 Quaternions

- A quaternion is a mathematical object of the form

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

  where a, b, c, d are real numbers, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis vectors that conform to the multiplication rules below:

$$\begin{aligned} \mathbf{i}^2 &= -1, & \mathbf{ij} &= \mathbf{k}, & \mathbf{jk} &= -\mathbf{i}, \\ \mathbf{ji} &= -\mathbf{k}, & \mathbf{j}^2 &= -1, & \mathbf{jk} &= \mathbf{i}, \\ \mathbf{ki} &= \mathbf{j}, & \mathbf{kj} &= -\mathbf{i}, & \mathbf{k}^2 &= -1. \end{aligned}$$

- Let us make note of an interesting property. Let $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. That is, s is a quaternion without a real part, which means that it is *purely imaginary*. Then,

$$\begin{aligned} s^2 &= (b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^2 \\ &= b^2\mathbf{i}^2 + c^2\mathbf{j}^2 + d^2\mathbf{k}^2 + bc\mathbf{ij} + bc\mathbf{ji} + cd\mathbf{jk} + cd\mathbf{kj} + bd\mathbf{ki} + bd\mathbf{ik} \\ &= -b^2 - c^2 - d^2 + bc\mathbf{k} - bc\mathbf{k} + cd\mathbf{i} - cd\mathbf{i} + bd\mathbf{j} - bd\mathbf{j} \\ &= -(b^2 + c^2 + d^2) \end{aligned}$$

- The norm of the quaternion q is defined as

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

- If s is a purely imaginary quaternion, then

$$s^2 = -\|s\|^2.$$

  In particular, for $k \in \mathbb{N} \cup \{0\}$,

$$s^k = \begin{cases} (-1)^{k/2}\|s\|^k, & k \text{ is even} \\ (-1)^{(k-1)/2}\|s\|^{k-1}s, & k \text{ is odd} \end{cases}.$$

- Another way to denote the above fact is to write $s = u\theta$ where $\theta = \|s\|$ and u is a unit vector in $\mathbb{R}^3$ that makes the equation true. (In other words, u is uniquely determined if $\|s\| \neq 0$, but we can pick any unit vector if $\|s\| = 0$.) We have that

$$u^k = \begin{cases} (-1)^{k/2}, & k \text{ is even} \\ (-1)^{(k-1)/2}u, & k \text{ is odd} \end{cases}.$$

So,

$$s^k = (u\theta)^k = \begin{cases} (-1)^{k/2}\theta^k, & k \text{ is even} \\ (-1)^{(k-1)/2}u\theta^k, & k \text{ is odd} \end{cases}.$$

2 Quaternion Exponentiation

- Let $s = u\theta$ be a purely imaginary quaternion. We have that

$$\begin{aligned} e^s = e^{u\theta} &= \sum_{k=0}^{\infty} \frac{(u\theta)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(u\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(u\theta)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k u\theta^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k}}{(2k)!} + u\sum_{k=0}^{\infty} \frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \\ &= \cos\theta + u\sin\theta. \end{aligned}$$

- As a result, for $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + s = a + u\theta$, we have that

$$e^q = e^{a+u\theta} = e^a e^{u\theta} = e^a\left(\cos\theta + u\sin\theta\right).$$

3 Quaternion Logarithm

- Let q be a unit quaternion. We can always find $\theta \in \mathbb{R}$ and $u = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ such that $\|u\| = 1$ such that

$$q = \cos\theta + u\sin\theta.$$

- From the last section, we know that $e^{u\theta} = \cos\theta + u\sin\theta$. As a result, we may say that

$$\log q = \log(\cos\theta + u\sin\theta) = u\theta.$$

- For a general quaternion q, we may write $q = \|q\|(\cos\theta + u\sin\theta)$. Hence,

$$\log q = \log\big(\|q\|(\cos\theta + u\sin\theta)\big) = \log\|q\| + \log(\cos\theta + u\sin\theta) = \log\|q\| + u\theta.$$

4 Rotation and Logarithm

- The *conjugate* of the quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is defined as

$$q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

- If we write $q = a + s$ where $s = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Then, $q^* = a - s$. Moreover,

$$qq^* = (a+s)(a-s) = a^2 - s^2 = a^2 + \|s\|^2 = a^2 + b^2 + c^2 + d^2 = \|q\|^2.$$

  We can show that $q^*q = \|q\|^2 = qq^*$ as well.

- Let $q = \cos(\theta/2) + u\sin(\theta/2)$ be a unit quaternion. For any purely imaginary quaternion $v = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, it is well-known that qvq^* is the rotation of $\mathbf{v}$ around the axis u by an angle of θ. As a result, a rotation in $\mathbb{R}^3$ can be represented by a unit quaternion.
- We can go even further. When we represent a rotation by a unit quaternion q, we can take the logarithm of q to get a vector $u\theta/2 \in \mathbb{R}^3$. So, a rotation in $\mathbb{R}^3$ can also be represented by a vector in $\mathbb{R}^3$.
- Note, however, that the logarithm representation is not unique. This is because $e^{u\theta} = e^{u(\theta+2\pi k)}$ for any integer k.

References

[Row] G. Rowe, *Exponentiation of a quaternion*, `https://physicspages.com/pdf/Group%20theory/Exponential%20of%20a%20quaternion.pdf`, Accessed: 2025-07-14.