Sampling Surface Points

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Given a point x on a surface, we would like to sample a point y such that the the probability density that $\|y - x\| = r$ is proportional to $f(r)$ for some fixed function f.

1 Motivation

- This sampling problem arises in the computation of outgoing radiance from surface with subsurface scattering.
- Let x be a point on such a material. The outgoing radiance at x in direction ω is given by

$$L_{\mathrm{out}}(x, \omega) = \int_A \int_{H^2} S(y, \omega', x, \omega) L_{\mathrm{in}}(y, \omega') \cos\theta' \ \mathrm{d}\omega' \mathrm{d}y$$

  where

  - A is the surface where x is on,
  - H^2 is the hemisphere of direction around the surface normal at x,
  - S is the BFFRDF,
  - L_{in} is the incoming radiance, and
  - θ' is the angle between ω' and the surface normal.

- The BSSRDF used is typically of the following form:

$$S(y, \omega_i, x, \omega_o) = T(\eta, \omega_i) f(\|y - x\|) T(\eta, \omega_o)$$

  where

  - η is the index of refraction,
  - $T(\eta, \omega)$ is the Fresnel transmittance, and
  - f is an arbitrary weight function.

- We can evaluate the above integral by Monte Carlo integration with importance sampling:

$$L_{\mathrm{out}}(x, \omega) = \frac{1}{N} T(\eta, \omega) \sum_{j=1}^{N} \frac{f(r_j) L_i(y_j, \omega'_j) T(\eta, \omega') \cos\theta'_j}{p(y_j, \omega'_j)}$$

  where $p(y, \omega')$ is the probability density of sampling y and ω'.

- To simplify the above sum, we want $p(y, \omega')$ to be proportional to $f(r)\cos\theta'$. That is, we want the probability density of sampling y be proportional to $f(r)$.

2 The Sampling Algorithm

- Our sampling algorithm relies on the following fact.

  Theorem 2.1. *Let A be a flat surface that lies entirely in a sphere of radius r. Pick two points x_0 and x_1 unifomly at random from the surface of the sphere and draw a segment between them. Let X be the number of points the segment intersects A. Then,*

  $$E[X] = \frac{A}{2\pi r^2}.$$

  In other words, the probability density that point x on the surface is on the segment as well is $1/2\pi r^2$.

- The above theorem suggest the following algorithm for uniformly sampling a point on a flat surface.

  1. Form a sphere that contains the surface.
  2. Pick two points uniformly at random from the surface of the sphere.
  3. Draw a segment between them.
  4. Report the intersection between the segment and the surface, if there is one.

- Nevertheless, we would like to sample point y around x such that the probability density of picking y is proportional to $f(r)$. We do so by a modified version of the previous algorithm.

  1. Sample a number $s \in [0, \infty)$ according to another probability density function $g(s)$.
  2. Form a sphere of radius s around x.
  3. Pick two points uniformly at random from the surface of the sphere.
  4. Draw a segment between them.
  5. Report the intersection between the segment and the surface, if there is one.

- Let y be a point on the surface at distance r from x, and let dA be an infinitesimal surface area containing y. We have that

  $$\begin{aligned}
  \Pr(\text{pick } y) &= f(r)\ \mathrm{d}A \\
  &= \int_0^\infty \Pr(\text{pick } s) \Pr(\text{pick } y \mid \text{pick } s)\ \mathrm{d}s \\
  &= \int_0^\infty g(s) \Pr(\text{pick } y \mid \text{pick } s)\ \mathrm{d}s \\
  &= \int_0^r g(s) \Pr(\text{pick } y \mid \text{pick } s)\ \mathrm{d}s + \int_r^\infty g(s) \Pr(\text{pick } y \mid \text{pick } s)\ \mathrm{d}s
  \end{aligned}$$

  When $s < r$, the probability of picking y is 0. Otherwise, the probability is $\mathrm{d}A/(2\pi s^2)$. Hence,

  $$\begin{aligned}
  f(r)\ \mathrm{d}A &= \int_r^\infty g(s) \frac{\mathrm{d}A}{2\pi s^2}\ \mathrm{d}s \\
  f(r) &= \int_r^\infty \frac{g(s)}{2\pi s^2}\ \mathrm{d}s \\
  -f(r) &= \int_\infty^r \frac{g(s)}{2\pi s^2}\ \mathrm{d}s
  \end{aligned}$$

  Differentiating both sides with r, we have

  $$\begin{aligned}
  -f'(r) &= \frac{g(r)}{2\pi r^2} \\
  g(r) &= -2\pi r^2 f'(r).
  \end{aligned}$$

- In order for $g(r)$ to be a probability density, $f'(r)$ must be non-positive, which means that f must be non-increasing. Moreover, $\int_0^\infty g(r)\ \mathrm{d}r$ must be 1. Now,

$$\int_0^\infty g(r)\ \mathrm{d}r = \int_0^\infty -2\pi r^2 f'(r)\ \mathrm{d}r$$
$$1 = -2\pi \Big[r^2 f(r) \Big]_0^\infty + 4\pi \int_0^\infty r f(r)\ \mathrm{d}r.$$

  If we require that $\lim_{r\to\infty} r^2 f(r) = 0$, we have that

$$1 = 4\pi \int_0^\infty r f(r)\ \mathrm{d}r = 2\int_0^{2\pi}\int_0^\infty f(r) r\ \mathrm{d}r\mathrm{d}\phi = 2\int_A f(r)\ \mathrm{d}A.$$

  In other words,

$$\int_A f(r)\ \mathrm{d}A = \frac{1}{2},$$

  or the weight function must integrate to one half on the whole plane.

- **Theorem 2.2.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(y)$ depends only on the distance between y and the origin. (That is, we can write $f(y) = f(r)$ where r is the distance.) Moreover, let f satisfies the following properties:*

  - *f is not increasing in r,*
  - *$\lim_{r\to\infty} r^2 f(r) = 0$, and*
  - *$\int_A f(r)\ \mathrm{d}y = 1/2$.*

  If we sample the radius s of the sphere according to probability density funcion

$$g(r) = -2\pi r^2 f'(r),$$

  then the probability density point y at distance r from x gets picked is $f(r)$.

- When sampling the radius s, we need to evaluate the cumulative distribution function of g. This is given by

$$\int_0^r g(r)\ \mathrm{d}r = -2\pi r^2 f(r) + 4\pi \int_0^r s f(s)\ \mathrm{d}s.$$

3 Sampling Distribution for Some Weight Function

3.1 Uniform Weight

- If we want every point inside a circle of radius R centered at x to have equal weight, we choose

$$f(r) = \begin{cases} 1/(2\pi R^2), & 0 \le r \le R \\ 0, & r > R \end{cases}.$$

- Hence,

$$g(r) = 2\pi r^2 f(r) = \begin{cases} 0, & 0 \le r < R \\ \delta(r-R)/(2\pi R^2), & r = R \\ 0, & r > R \end{cases}.$$

  This g suggests that we set $s = R$, which makes perfect sense.

3.2 Polynomial Weight 1

- In this section, we want f be proportional to

$$f^*(r) = \begin{cases} (1-r/R)^d, & 0 \le r \le R \\ 0, & r > R \end{cases}$$

We want to find a constant c such that $\int_A c(1-r/R)^d \, \mathrm{d}A = \frac{1}{2}$. We have that

$$\begin{aligned}
\int_A (1-r/R)^d \, \mathrm{d}A &= \int_0^\infty \int_0^{2\pi} r(1-r/R)^d \, \mathrm{d}\phi \mathrm{d}r \\
&= 2\pi \int_0^R r(1-r/R)^d \, \mathrm{d}r \\
&= 2\pi \left(\left[-\frac{Rr(1-r/R)^{d+1}}{d+1} \right]_0^R - \int_0^R \frac{-R(1-r/R)^{d+1}}{d+1} \, \mathrm{d}r \right) \\
&= -2\pi \left[\frac{R^2(1-r/R)^{d+2}}{(d+1)(d+2)} \right]_0^R \\
&= \frac{2\pi R^2}{(d+1)(d+2)}.
\end{aligned}$$

So, $c = (d+1)(d+2)/(4\pi R^2)$ and

$$f(r) = \frac{(d+1)(d+2)(1-r/R)^d}{4\pi R^2}, \; 0 \le r \le R$$

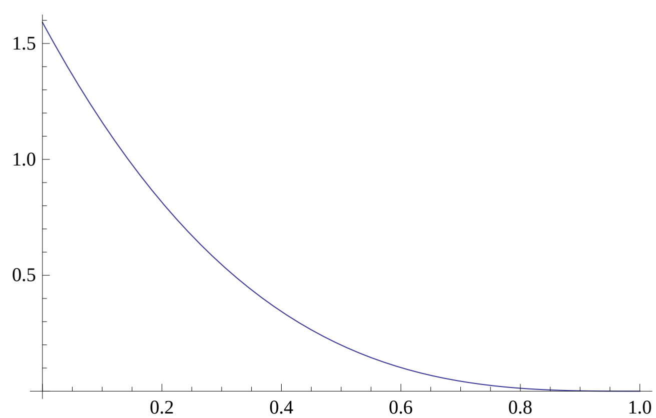

Figure 1: The weight function propotional to $(1-r/R)^3$ with $R = 1$.

- The probability distribution g is then

$$\begin{aligned}
g(r) &= -2\pi r^2 f'(r) \\
&= -2\pi r^2 \left[\frac{(d+1)(d+2)(1-r/R)^d}{4\pi R^2} \right]' \\
&= \frac{d(d+1)(d+2)}{2R^3} r^2 (1-r/R)^{d-1}, \; 0 \le r \le R.
\end{aligned}$$

The cumulative distribution function is

$$\begin{aligned}
\int_0^r g(r)\ \mathrm{d}r &= \frac{d(d+1)(d+2)}{2R^3}\int_0^r s^2(1-s/R)^{d-1}\ \mathrm{d}s \\
&= \frac{d(d+1)(d+2)}{2R^3}\left[-\frac{R(1-s/R)^d(2R^2+2Rds+d(d+1)s^2)}{d(d+1)(d+2)}\right]_0^r \\
&= \frac{1}{2R^2}\left[-(1-s/R)^d(2R^2+2Rds+d(d+1)s^2)\right]_0^r \\
&= 1-\frac{(1-r/R)^d(2R^2-2Rdr+d(d+1)r^2)}{2R^2} \\
&= 1-(1-r/R)^d\left(1-\frac{d}{R}r+\frac{d(d+1)}{2R^2}r^2\right),\ 0\le r\le R.
\end{aligned}$$

3.3 Polynomial Weight 2

- In this section, we want f be proportional to

$$f^*(r)=\begin{cases}1-(r/R)^d, & 0\le r\le R\\ 0, & r>R\end{cases}$$

We want to find a constant c such that $\int_A c(1-(r/R)^d)\ \mathrm{d}A=\frac{1}{2}$. We have that

$$\begin{aligned}
\int_A (1-(r/R)^d)\ \mathrm{d}A &= \int_0^\infty\int_0^{2\pi} r(1-(r/R))^d\ \mathrm{d}\phi\mathrm{d}r \\
&= 2\pi\int_0^R r(1-(r/R)^d)\ \mathrm{d}r = 2\pi\left(\int_0^R r\ \mathrm{d}r-\int_0^R r^{d+1}/R^d\ \mathrm{d}r\right) \\
&= 2\pi\left(\left[\frac{r^2}{2}\right]_0^R-\left[\frac{r^{d+2}}{(d+2)R^d}\right]_0^R\right) \\
&= 2\pi\left(\frac{R^2}{2}-\frac{R^2}{d+2}\right) \\
&= \frac{\pi dR^2}{d+2}.
\end{aligned}$$

So, $c=(d+2)/(2\pi dR^2)$ and

$$f(r)=\frac{d+2}{2\pi dR^2}\left(1-\frac{r^d}{R^d}\right),\ 0\le r\le R$$

- The probability density function is then

$$g(r)=-2\pi r^2 f'(r)=-2\pi r^2\times\frac{d+2}{2\pi dR^2}\times(-d)\frac{r^{d-1}}{R^d}=\frac{d+2}{R^{d+2}}r^{d+1}.$$

- The cdf is

$$\int_0^r g(s)\ \mathrm{d}s=\frac{d+2}{R^{d+2}}\int_0^r r^{d+1}\ \mathrm{d}r=\frac{d+2}{R^{d+2}}\left[\frac{s^{d+2}}{d+2}\right]_0^r=\frac{r^{d+2}}{R^{d+2}}.$$

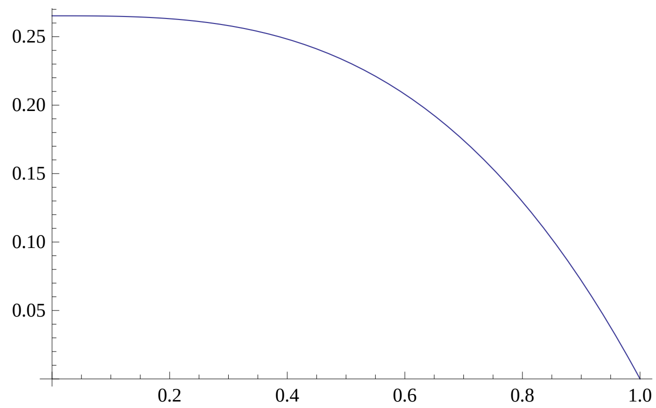

Figure 2: The weight function proportional to $1 - (r/R)^3$ with $R = 1$.

3.4 Polynomial Weight 3

- In this section, we want f to be proportional to

$$f^*(r) = \begin{cases} (1 - r^2/R^2)^2, & 0 \le r \le R, \\ 0, & r > 1 \end{cases}$$

- Integrating $f^*(r)$ over the plane, we have

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} r f^*(r)\ \mathrm{d}r\mathrm{d}\theta &= 2\pi\int_0^R r(1 - r^2/R^2)^2\ \mathrm{d}r = 2\pi\int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)\ \mathrm{d}r \\ &= 2\pi\left(\left[\frac{r^2}{2}\right]_0^R - \left[\frac{r^4}{2R^2}\right]_0^R + \left[\frac{r^6}{6R^4}\right]_0^R\right) \\ &= 2\pi\left(\frac{R^2}{2} - \frac{R^2}{2} + \frac{R^2}{6}\right) = \frac{\pi R^2}{3}.\end{aligned}$$

Hence,

$$f(r) = \frac{3(1 - r^2/R^2)^2}{2\pi R^2}.$$

- We have that

$$\begin{aligned}g(r) &= -2\pi r^2 f'(r) = -\frac{3}{R^2}r^2\{(1 - r^2/R^2)^2\}' = -\frac{3}{R^2}r^2\left\{1 - \frac{2r^2}{R^2} + \frac{r^4}{R^4}\right\}' = -\frac{3}{R^2}r^2\left(-\frac{4r}{R^2} + \frac{4r^3}{R^4}\right) \\ &= 12\left(\frac{r^3}{R^4} - \frac{r^5}{R^6}\right)\end{aligned}$$

- The cdf is then:

$$\begin{aligned}\int_0^r g(s)\ \mathrm{d}s &= \frac{12}{R^4}\int_0^r s^3\ \mathrm{d}s - \frac{12}{R^4}\int_0^r s^5\ \mathrm{d}s \\ &= \frac{12}{R^6}\frac{r^4}{4} - \frac{12}{R^6}\frac{r^6}{6} = \frac{3r^4}{R^4} - \frac{2r^6}{R^6}.\end{aligned}$$

One nice thing about this cdf is that it is a cubic polynomial in r^2/R^2, which means we can solve $\int_0^r g(s)\ \mathrm{d}s = \xi$ for any given ξ exactly.

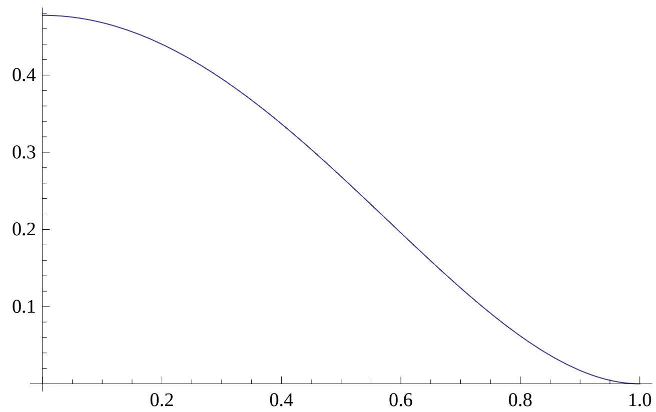

Figure 3: The weight function proportional to $(1-(r/R)^2)^2$ with $R=1$.

3.5 Exponential Weight

- In this section, we want f be proportional to

$$f^*(r) = e^{-\sigma r}.$$

for some positive constant σ. We want to find a constant c such that $\int_A ce^{-\sigma r}\ \mathrm{d}A = \frac{1}{2}$. We have that

$$\begin{aligned}\int_A e^{-\alpha r}\ \mathrm{d}A &= \int_0^\infty \int_0^{2\pi} e^{-\sigma r}\ \mathrm{d}\phi \mathrm{d}r \\ &= 2\pi \int_0^\infty e^{-\sigma r}\ \mathrm{d}r \\ &= 2\pi \left[-\frac{e^{-\sigma r}(\sigma r + 1)}{\sigma^2} \right]_0^\infty \\ &= 2\pi/\sigma^2.\end{aligned}$$

So, $c = \sigma^2/(4\pi)$ and

$$f(r) = \frac{\sigma^2 e^{-\sigma r}}{4\pi}$$

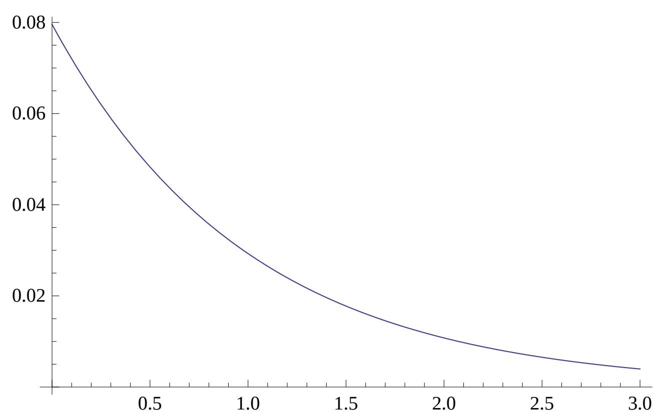

Figure 4: The weight function proportional to $e^{-\sigma r}$ with $\sigma = 1$.

- The probability distribution g is then

$$\begin{aligned} g(r) &= -2\pi r^2 f'(r) \\ &= \frac{\sigma^3 r^2}{2} e^{-\sigma r}. \end{aligned}$$

The cumulative distribution function is

$$\begin{aligned} \int_0^r g(r)\ \mathrm{d}r &= \frac{\sigma^3}{2} \int_0^r s^2 e^{-\sigma s}\ \mathrm{d}s \\ &= \frac{\sigma^3}{2} \bigg[-\frac{1}{\sigma^3} e^{-\sigma s} (\sigma^2 s^2 + 2\sigma s + 2) \bigg]_0^r \\ &= \frac{1}{2} \bigg[-e^{-\sigma s} (\sigma^2 s^2 + 2\sigma s + 2) \bigg]_0^r \\ &= 1 - \frac{1}{2} e^{-\sigma r} (\sigma^2 r^2 + 2\sigma r + 2). \end{aligned}$$

3.6 Jensen's Dipole Weight

- A popular weight to use is the dipole diffuse scattering weight. The dipole weight function has four parameters:
  - the coefficient of absorption σ_a,
  - the coefficient of scattering σ_s,
  - the mean cosine g, and
  - the index of refraction η.

  From these parameters, we define the following quantities:
  - $\sigma'_s = \sigma_s(1 - g)$,
  - $\sigma'_t = \sigma'_s + \sigma_a$,
  - $D = \frac{1}{3\sigma'_t}$,
  - $F_{dr} = \frac{-1.44}{\eta^2} + \frac{0.71}{\eta} + 0.668 + 0.0636\eta$,
  - $A = \frac{1+F_{dr}}{1-F_{dr}}$,
  - $\sigma = \sqrt{3\sigma_a \sigma'_t}$,
  - $z_r = 1/\sigma'_t$,
  - $z_v = z_r + 4AD$, and
  - $\alpha' = \sigma'_s / \sigma'_t$.

  Then, the dipole weight is given by:

$$f^*(r) = \frac{\alpha'}{4\pi} \left[z_r (\sigma(r^2 + z_r^2)^{1/2} + 1) \frac{e^{-\sigma(r^2+z_r^2)^{1/2}}}{(r^2 + z_r^2)^{3/2}} + z_v (\sigma(r^2 + z_v^2)^{1/2} + 1) \frac{e^{-\sigma(r^2+z_v^2)^{1/2}}}{(r^2 + z_v^2)^{3/2}} \right]$$

- Let

$$h(r, z) = (\sigma(r^2 + z^2)^{1/2} + 1) \frac{e^{-\sigma(r^2+z^2)^{1/2}}}{(r^2 + z^2)^{3/2}}.$$

We have that

$$\int r h(r, z)\ \mathrm{d}r = -\frac{e^{-\sigma(r^2+z^2)^{1/2}}}{(r^2 + z^2)^{1/2}} + C$$

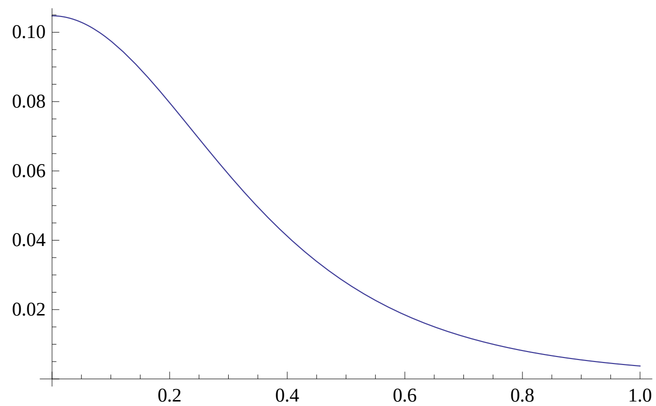

Figure 5: The (unnormalized) dipole weight function with $\sigma_a = \sigma_s = 1$, $g = 0$, and $\eta = 1.2$.

- Rewriting $f^*(r)$, we have

$$f^*(r) = \frac{\alpha'}{4\pi}[z_r h(r, z_r) + z_v h(r, z_v)].$$

Let us integrate $f^*(r)$ over the plane.

$$\begin{aligned}
\int_0^\infty \int_0^{2\pi} r f^*(r)\ \mathrm{d}\phi \mathrm{d}r &= 2\pi \int_0^\infty r f^*(r)\ \mathrm{d}r \\
&= \frac{\alpha'}{2}\left(z_r \int_0^\infty r h(r, z_r)\ \mathrm{d}r + z_v \int_0^\infty r h(r, z_v)\ \mathrm{d}v \right) \\
&= \frac{\alpha'}{2}\left(z_r \left[-\frac{e^{-\sigma(r^2+z_r^2)^{1/2}}}{(r^2+z_r^2)^{1/2}} \right]_0^\infty + z_v \left[-\frac{e^{-\sigma(r^2+z_v^2)^{1/2}}}{(r^2+z_v^2)^{1/2}} \right]_0^\infty \right) \\
&= \frac{\alpha'}{2}(e^{-\sigma z_r} + e^{-\sigma z_v}).
\end{aligned}$$

- So, the weight function f is

$$f(r) = \frac{1}{2} \times \frac{2}{\alpha'(e^{-\sigma z_r} + e^{-\sigma z_v})} \times f^*(r) = \frac{z_r h(r, z_r) + z_v h(r, z_v)}{4\pi(e^{-\sigma z_r} + e^{-\sigma z_v})}.$$

- Since the calculation of g is a pain in the neck, we compute the cdf of g instead. Recall from last section that

$$\int_0^r g(r)\ \mathrm{d}r = -2\pi r^2 f(r) + 4\pi \int_0^r s f(s)\ \mathrm{d}s.$$

Let us work on the above expression term by term. For the first time, we have

$$-2\pi r^2 f(r) = -2\pi r^2 \left(\frac{z_r h(r, z_r) + z_v h(r, z_v)}{4\pi(e^{-\sigma z_r} + e^{-\sigma z_v})} \right) = -r^2 \left(\frac{z_r h(r, z_r) + z_v h(r, z_v)}{2(e^{-\sigma z_r} + e^{-\sigma z_v})} \right).$$

For the second term, we have

$$
\begin{aligned}
4\pi \int_0^r sf(s)\ \mathrm{d}s &= 4\pi \int_0^r s\frac{z_r h(s,z_r) + z_v h(s,z_v)}{4\pi(e^{-\sigma z_r} + e^{-\sigma z_v})}\ \mathrm{d}s \\
&= \frac{1}{e^{-\sigma z_r} + e^{-\sigma z_v}} \int_0^r z_r s h(s,z_r) + z_v s h(s,z_v)\ \mathrm{d}s \\
&= \frac{1}{e^{-\sigma z_r} + e^{-\sigma z_v}} \left[- z_r \frac{e^{-\sigma(s^2+z_r^2)^{1/2}}}{(s^2+z_r^2)^{1/2}} - z_v \frac{e^{-\sigma(s^2+z_v^2)^{1/2}}}{(s^2+z_v^2)^{1/2}} \right]_0^r \\
&= 1 - \frac{1}{e^{-\sigma z_r} + e^{-\sigma z_v}} \left(z_r \frac{e^{-\sigma(r^2+z_r^2)^{1/2}}}{(r^2+z_r^2)^{1/2}} + z_v \frac{e^{-\sigma(r^2+z_v^2)^{1/2}}}{(r^2+z_v^2)^{1/2}} \right).
\end{aligned}
$$

Then,

$$
\int_0^r g(r)\ \mathrm{d}r = 1 - \frac{1}{e^{-\sigma z_r} + e^{-\sigma z_v}} \left(z_r \frac{e^{-\sigma(r^2+z_r^2)^{1/2}}}{(r^2+z_r^2)^{1/2}} + z_v \frac{e^{-\sigma(r^2+z_v^2)^{1/2}}}{(r^2+z_v^2)^{1/2}} \right) - r^2 \left(\frac{z_r h(r,z_r) + z_v h(r,z_v)}{2(e^{-\sigma z_r} + e^{-\sigma z_v})} \right).
$$

4 Images

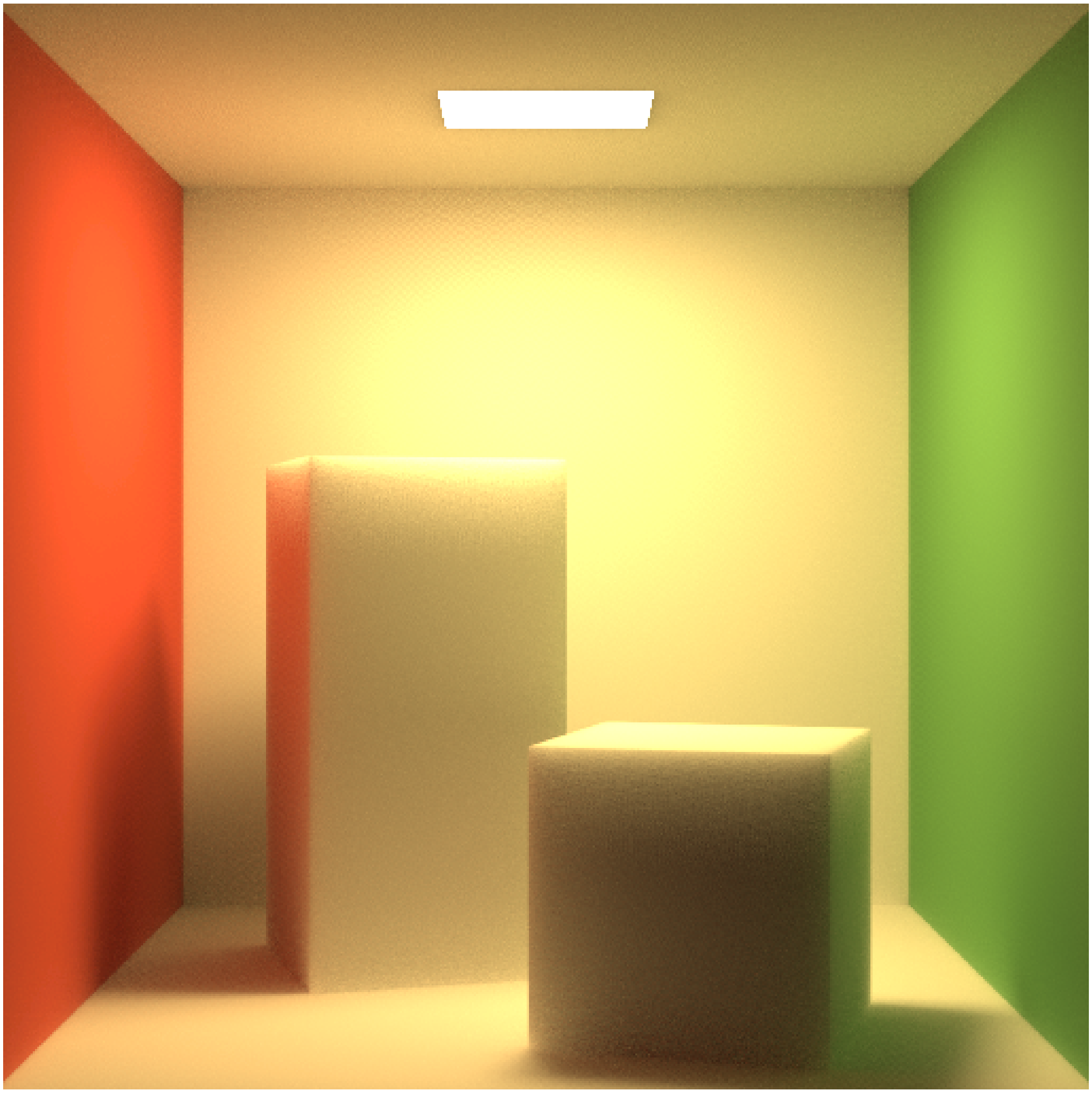

Figure 6: $f(r) \propto (1 - r/80)^3$

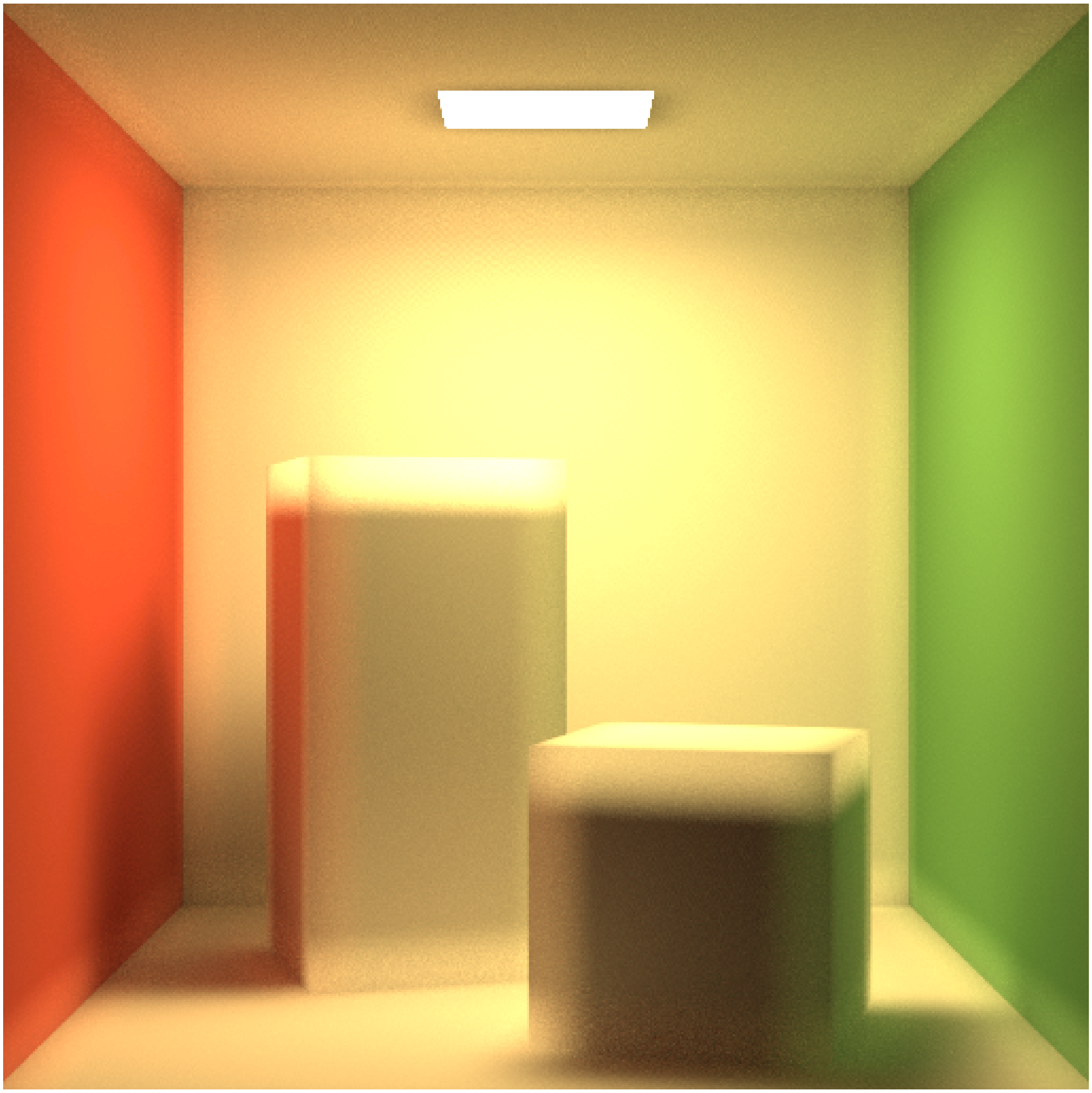

Figure 7: $f(r) \propto 1 - r^3/40^3$

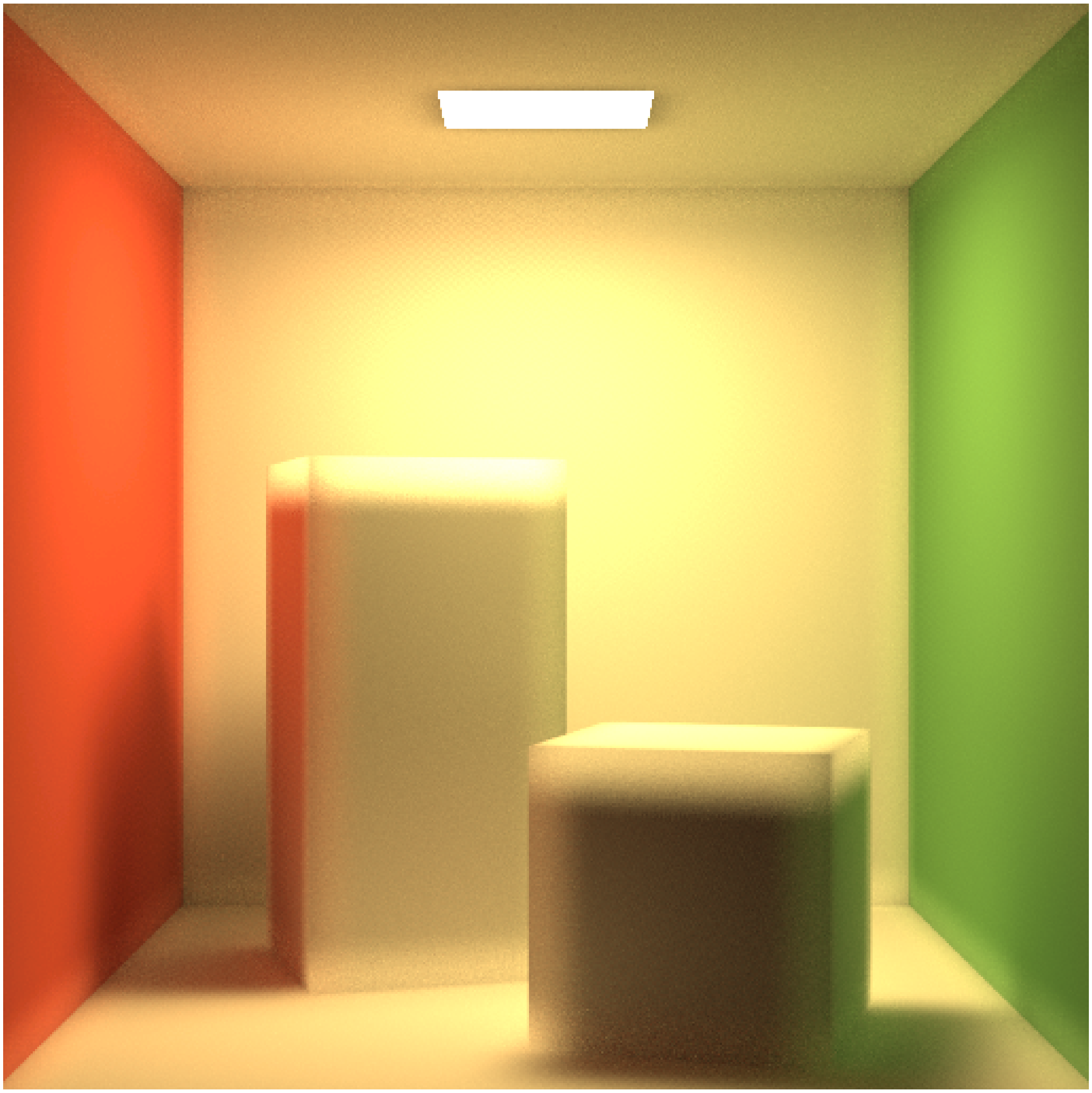

Figure 8: $f(r) \propto (1 - r^2/40^2)^2$

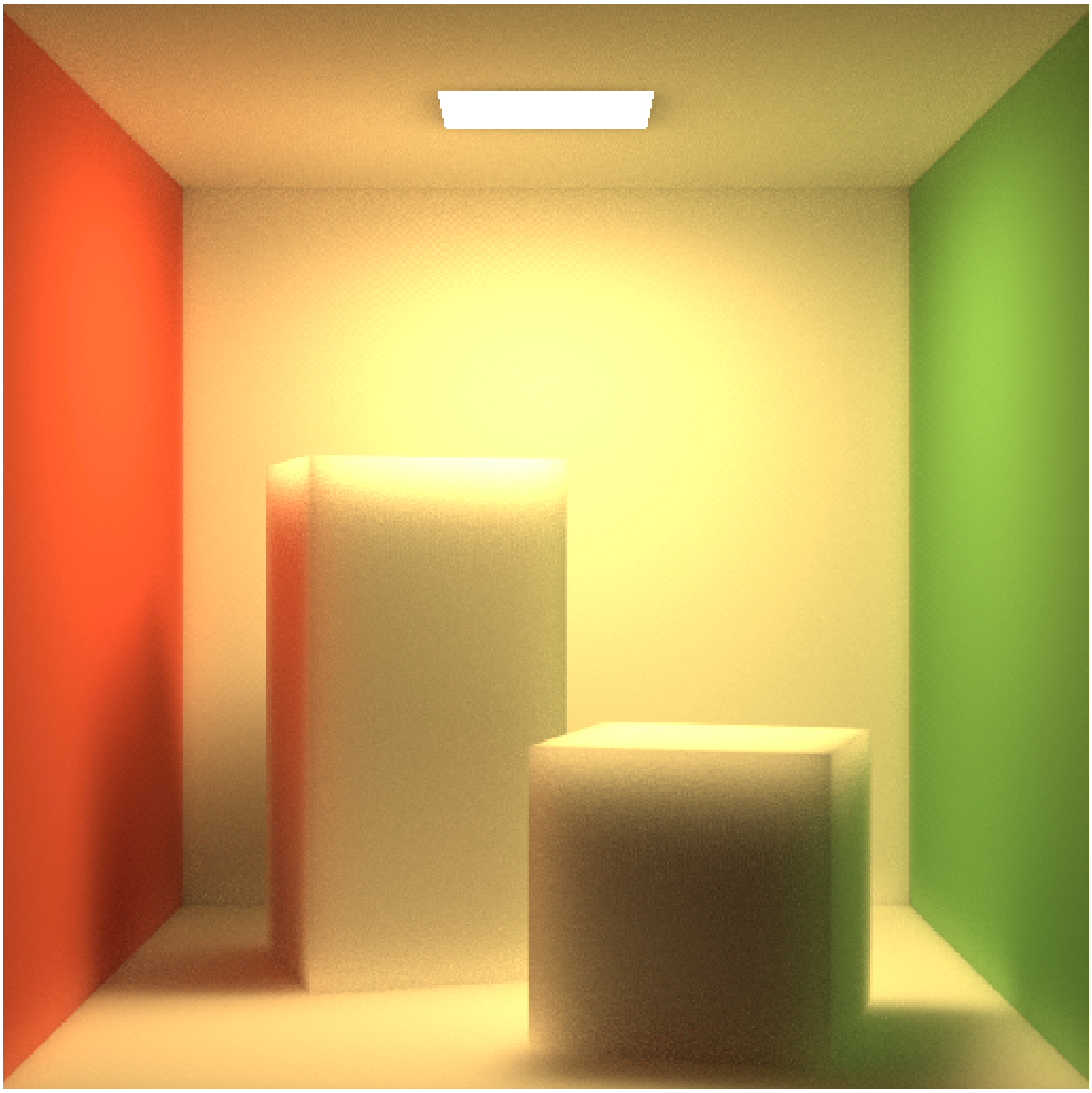

Figure 9: $f(r) \propto e^{-0.1r}$

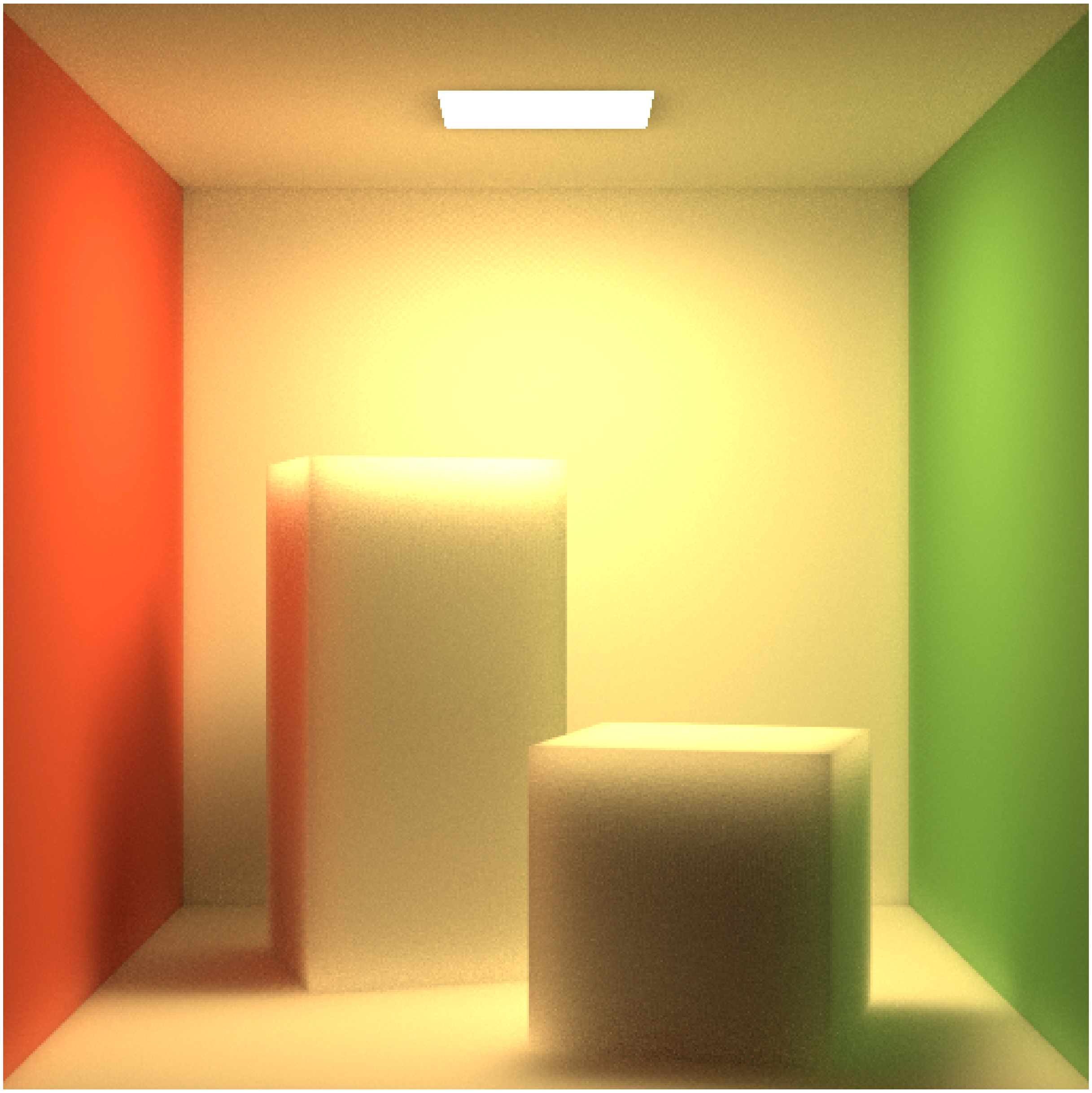

Figure 10: Dipole weight function with $\sigma_s = \sigma_a = 0.025$, $g = 0$, $\eta = 1.2$. I'm shocked it looks almost the same as the exponential weight function.