Spherical Gaussian Technology

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The spherical Gaussian function has become a power tool in real-time rendering research. This article is written to aid my study of its use. We'll start with a look at the paper that "All-Frequency Rendering of Dynamic, Spatially-Varying Reflectance" [Wang et al., 2009], which uses the function to great effect. We will then look at a generalization into anisotropic spherical Gaussians [Xu et al., 2013] and an optimization for integrating the function [Iwasaki et al., 2012]. Lastly, we will look at how it is applied to render cloth models [Iwasaki et al., 2014].

1 Spherical Gaussians

- A spherical Gaussian (SG) has the form:

$$G(\mathbf{v}; \mathbf{p}, \lambda, \mu) = \mu e^{\lambda(\mathbf{v} \cdot \mathbf{p} - 1)}$$

  where

    - $\mathbf{p} \in S^2$ is the *lobe axis*,
    - $\lambda \in (0, \infty)$ is the *lobe sharpness*,
    - $\mu \in \mathbb{R}$ is the *lobe amplitude*. and
    - $\mathbf{v} \in S^2$ is the spherical direction which is the variable of the function.

- A spherical Gaussian is *compactly ε-supported*. This means that the closure of the region in which $G(\mathbf{v}) \geq \varepsilon$ covers less than the entire sphere. More specifically, if $\lambda > -\ln \varepsilon$, we have that

$$\int_{G(\mathbf{v}, \mathbf{p}, \lambda, 1) \geq \varepsilon} \mathrm{d}\mathbf{v} = -2\pi \frac{\ln \varepsilon}{\lambda}.$$

- Define $f_a(\lambda)$ to be the above integral value when $\varepsilon = 0.1$.

- The rotation of a spherical Gaussian is given by the rotation of its axis $\mathbf{p}$.

- There are two types of products that can be defined for SGs:

  1. the inner product, and
  2. the vector product.

- The *inner product* of two SGs G_1 and G_2 is denoted by $G_1 \cdot G_2$ and is given by:

$$G_1 \cdot G_2 = \int_{S^2} G_1(\mathbf{v}) G_2(\mathbf{v}) \, \mathrm{d}\mathbf{v} = \frac{4\pi \mu_1 \mu_2}{e^{\lambda_1 + \lambda_2}} \frac{\sinh(d_m)}{d_m}$$

  where $d_m = \|\lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2\|$.

- The *vector product* is denoted by $G_1 \otimes G_2$ and is given by:

$$(G_1 \otimes G_2)(\mathbf{v}) = G_1(\mathbf{v})G_2(\mathbf{v}) = G\left(\mathbf{v}; \frac{\mathbf{p}_m}{\|\mathbf{p}_m\|}, \lambda_m\|\mathbf{p}_m\|, \mu_1\mu_2 e^{\lambda_m(\|\mathbf{p}_m\|-1)}\right)$$

  where

  - $\mathbf{p}_m = (\lambda_1\mathbf{p}_1 + \lambda_2\mathbf{p}_2)/(\lambda_1 + \lambda_2)$, and
  - $\lambda_m = \lambda_1 + \lambda_2$.

- It is because of the ease of computing the inner and outer products that make SGs great for representing rendering related functions.

- Given a spherical function $F : S^2 \to \mathbb{R}$, we can approximate it with a sum of SGs:

$$F(\mathbf{v}) \approx F^*(\mathbf{v}) = \sum_{i=1}^{n} G(\mathbf{v}; \mathbf{p}_i, \lambda_i, \mu_i).$$

  The sum is also known as a *mixture of Gaussians*.

- Given a rotation matrix $\mathfrak{R}$, the rotation of a mixture of Gaussians $\mathfrak{R}F^*(\mathbf{v})$ as the sum of the rotated version of the constituents Gaussians:

$$\mathfrak{R}F^*(\mathbf{v}) = \sum_{i=1}^{n} G(\mathbf{v}; \mathfrak{R}\mathbf{p}_i, \lambda_i, \mu_i).$$

2 Shading with Spherical Gaussians

This section is about the methods employed by the [Wang et al., 2009] paper.

2.1 BRDF

- Let $\rho(\mathbf{o}, \mathbf{i})$ denote a BRDF. We decompose it to a diffuse and a specular component:

$$\rho(\mathbf{o}, \mathbf{i}) = k_d + k_s\rho_s(\mathbf{o}, \mathbf{i}).$$

- Let us say that the specular component is modeled by a microfacet BRDF. Then, ρ_s is given by:

$$\rho_s(\mathbf{o}, \mathbf{i}) = M_{\mathbf{o}}(\mathbf{i})D(\mathbf{h})$$

  where

  - $\mathbf{h} = (\mathbf{o} + \mathbf{i})/\|\mathbf{o} + \mathbf{i}\|$ is the half vector,
  - $D(\mathbf{h})$ is the *normal distribution function* (NDF), and
  - $M_{\mathbf{o}}(\mathbf{i})$ is a function that combines the shadow masking term and the Fresnel reflectance.

- The NDF is modeled using a single SG lobe for isotropic models and multiple lobes for anisotropic models.

- For example, the Cook–Torrance model is given by:

$$\rho_s(\mathbf{i}, \mathbf{i}) = \frac{F_{CT}(\mathbf{o}, \mathbf{i})S_{CT}(\mathbf{o}, \mathbf{i})}{\pi(\mathbf{n} \cdot \mathbf{i})(\mathbf{n} \cdot \mathbf{o})} e^{-(\arccos(\mathbf{h}\cdot\mathbf{n})/m)^2}.$$

In other words,

$$M_{\mathbf{o}}(\mathbf{i}) = \frac{F_{CT}(\mathbf{o}, \mathbf{i})S_{CT}(\mathbf{o}, \mathbf{i})}{\pi(\mathbf{n} \cdot \mathbf{i})(\mathbf{n} \cdot \mathbf{o})}$$

$$D(\mathbf{h}) = e^{-(\arccos(\mathbf{h} \cdot \mathbf{n})/m)^2}.$$

The Wang paper approximates the NDF with a single Gaussian lobe:

$$D(\mathbf{h}) \approx G(\mathbf{h}; \mathbf{n}, 2/m^2, 1).$$

It said the $M_{\mathbf{o}}(\mathbf{i})$ term is "analytic." This is okay as we will see later.

- As another example, the Wang paper also has a model where:

$$\rho_s(\mathbf{o}, \mathbf{i}) = \frac{F(\mathbf{o}, \mathbf{i})S(\mathbf{o})S(\mathbf{i})}{\pi(\mathbf{n} \cdot \mathbf{i})(\mathbf{n} \cdot \mathbf{o})}D(\mathbf{h})$$

where F denotes the Fresnel factor:

$$F(\mathbf{o}, \mathbf{i}) = \frac{(g - c)^2}{2(g + c)^2}\left(1 + \frac{(c(g + c) - 1)^2}{(c(g - c) + 1)^2}\right)$$

with $g = \sqrt{\eta^2 - c^2 - 1}$, $c = \cos\theta = |\mathbf{i} \cdot \mathbf{h}|$, and η is the index of refraction. S denotes the shadow masking function, which the paper assumes to be isotropic. D is tabulated, and the paper approximates it with a mixture of Gaussians. Fitting is done by Levenberg–Marquardt optimization.

- When performing reflectance integration, the vector $\mathbf{o}$ is fixed, and we need ρ_s as a function of $\mathbf{i}$. We call this view of ρ_s a *slice* of ρ_s and denote it by $\rho_s(\mathbf{i}; \mathbf{o})$. Assuming that D is approximate by a mixture of Gaussian D^*, we would like to compute:

$$\rho_s(\mathbf{i}; \mathbf{o}) = M_{\mathbf{o}}(\mathbf{i})D^*(\mathbf{h}).$$

However, the problem here is that D^* is a function of $\mathbf{h}$ instead of $\mathbf{i}$. As a result, we must write D^* in terms of $\mathbf{i}$. Let us call this new mixture W^*.

- The Wang paper proposes approximating each SG lobe in D^* with an SG lobe in $\mathbf{i}$. The idea is that $\mathbf{i}$ and $\mathbf{h}$ are related by the following transformation:

$$\mathbf{i} = \psi(\mathbf{h}) = 2(\mathbf{o} \cdot \mathbf{h})\mathbf{h} - \mathbf{o}.$$

So, if you have a Gaussian $G(\mathbf{h}; \mathbf{p}_i^D, \lambda_i^D, \mu_i^D)$ in $\mathbf{h}$, then it should be approximated well by a Gaussian in $\mathbf{i}$ such that

    – the new Gaussian's axis is the old Gaussian's axis transformed by ψ,

    – the new Gaussian's "mass" should be the same as the old Gaussian's "mass," and

    – the amplitudes of the two Gaussians are the same.

More formally, if $G(\mathbf{i}; \mathbf{p}_i^W, \lambda_i^W, \mu_i^W)$ is the approximation of $G(\mathbf{h}; \mathbf{p}_i^D, \lambda_i^D, \mu_i^D)$, then

$$\mathbf{p}_i^W = \psi(\mathbf{p}_i^D), \text{ and}$$

$$\mu_i^W = \mu_i^D.$$

The equal mass idea is more complicated. The Wang paper says that the mass of SG with lobe sharpness λ is just $f_a(\lambda)$. So, we have to find λ_i^W such that

$$f_a(\lambda_i^W) = \int_{G(\mathbf{i}; \mathbf{p}_i^W, \lambda_i^W, \mu_i^W) \geq 0.1} d\mathbf{i} = \int_{G(\mathbf{h}; \mathbf{p}_i^D, \lambda_i^D, \mu_i^D) \geq 0.1} d\mathbf{h} = f_a(\lambda_i^D)$$

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Now, we have that

$$\mathrm{d}\mathbf{i} = |D\psi(\mathbf{h})| \, \mathrm{d}\mathbf{h} = 4|\mathbf{h} \cdot \mathbf{o}| \, \mathrm{d}\mathbf{h}.$$

Let $A = \{\mathbf{h} : G(\mathbf{h}; \mathbf{p}_i^D, \lambda_i^D, \mu_i^D) \geq 0.1\}$. We have that:

$$f_a(\lambda_i^D) = \int_A \mathrm{d}\mathbf{h} = \int_{\psi(A)} \frac{\mathrm{d}\mathbf{i}}{4|\mathbf{h} \cdot \mathbf{o}|}.$$

Assuming $\psi(A)$ is a small set, we may say that the change-of-variable factor is constant. So,

$$f_a(\lambda_i^D) \approx \frac{1}{4|\mathbf{h} \cdot \mathbf{o}|} \int_{\psi(A)} \mathrm{d}\mathbf{i}.$$

Because we want the mass of $G(\mathbf{i}; \mathbf{p}_i^W, \lambda_i^W, \mu_i^W)$ to approximate the mass of $G(\mathbf{h}; \mathbf{p}_i^D, \lambda_i^D, \mu_i^D)$, we can say that we require

$$\int_{\psi(A)} \mathrm{d}\mathbf{i} = \int_{G(\mathbf{i}; \mathbf{p}_i^W, \lambda_i^W, \mu_i^W) \geq 0.1} \mathrm{d}\mathbf{i} = f_a(\lambda_i^W).$$

Therefore,

$$f_a(\lambda_i^D) = \frac{1}{4|\mathbf{h} \cdot \mathbf{o}|} f_a(\lambda_i^W)$$

$$-2\pi \frac{\ln 0.1}{\lambda_i^D} = \frac{1}{4|\mathbf{h} \cdot \mathbf{o}|}\left(-2\pi \frac{\ln 0.1}{\lambda_i^W}\right)$$

$$\lambda_i^W = \frac{\lambda_i^D}{4|\mathbf{h} \cdot \mathbf{o}|}.$$

- Now, we have that:

$$\rho_s(\mathbf{i}; \mathbf{o}) = M_\mathbf{o}(i)D(\mathbf{h}) \approx M_\mathbf{o}(i)D^*(\mathbf{h}) \approx M_\mathbf{o}(i)W^*(\mathbf{i}).$$

So, we need to approximate $M_\mathbf{o}(i)W^*(\mathbf{i})$. The Wang paper assumes that $M_\mathbf{o}(i)$ changes slowly over the ε-support of each SG, so it uses the following approximation:

$$\rho_s(\mathbf{i}; \mathbf{o}) \approx M_\mathbf{o}(i)W^*(\mathbf{i}) \approx \sum_{i=1}^{n} G(\mathbf{i}; \mathbf{p}_i^W, \lambda_i^W, M_\mathbf{o}(\mathbf{p}_i^W)\mu_i^W).$$

2.2 Visibility Representation

- The visibility function at point $\mathbf{x}$ is denoted by $V_\mathbf{x}(\mathbf{i})$ where $V_\mathbf{x} = 1$ if the light is visible from $\mathbf{x}$ in the direction $\mathbf{i}$, and $V_\mathbf{x}(\mathbf{i}) = 0$ otherwise.

- The Wang paper represents visibility with something called the *spherical signed distance function* (SSDF). One of this is stored at each mesh vertex and is interpolated to get fragment values.

- The SSDF supports *ghost-free* interpolation: the interpolated value does not feature double images of vertex values. (See the image from the paper for what this actually is.)

- Let $V_\mathbf{x}^d(\mathbf{i})$ denote the SSDF encoding of the real visiblity function $V_\mathbf{x}(\mathbf{i})$. The value of the SSDF is defined according to the following rules:

  - $V_\mathbf{x}^d(\mathbf{i})$ is negative if direction $\mathbf{i}$ is occluded and positive otherwise.

– The absolute value of $V_{\mathbf{x}}^d(\mathbf{i})$ is the angular distancde to the nearest direction $\mathbf{t}$ that is on the shadow boundary.

Mathematically,

$$V_{\mathbf{x}}^d(\mathbf{i}) = \begin{cases} + \min\limits_{V_{\mathbf{x}}(\mathbf{t})=0} \arccos(\mathbf{t} \cdot \mathbf{i}), & \text{if } V_{\mathbf{x}}(\mathbf{i}) = 1 \\ - \min\limits_{V_{\mathbf{x}}(\mathbf{t})=1} \arccos(\mathbf{t} \cdot \mathbf{i}), & \text{if } V_{\mathbf{x}}(\mathbf{i}) = 0 \end{cases}$$

- The Wang paper computes the SSDFs as 512×512 images and then downsample them to 128×128 resolution. Then, it performs PCA of all vertex SSDF and keeps the weight vectors of 48 principal components.

- The central computation in lighting with SGs is multiplication of an SG lobe with the SSDF. Let $\mathbf{p}$ denote the lobe axis. Then, the value $V_{\mathbf{x}}^d(\mathbf{p})$, if it is positive, gives you the size of the unoccluded spherical cap centered at $\mathbf{p}$. The paper argues that modulating the lobe with this spherical cap is a good approximation to visibility because:

  – If the lobe is narrow, then it is a good approximation already.

  – If the lobe is broad, detailed vibility knowledge is not necessary. (I'm not sure I'm buying this, but their results say it's good.)

- Let us get formal with the above process. Let $\theta^d = V_{\mathbf{x}}^d(\mathbf{p})$. The spherical cap visiblity function $V'(\mathbf{i})$ centered at $\mathbf{p}$ is given by:

$$V'(\mathbf{i}) = \begin{cases} 1, & \mathbf{p} \cdot \mathbf{i} \geq \cos(\theta_d) \\ 0, & \text{otherwise} \end{cases}.$$

We are interested in the computing the dot product and the vector product of the Gaussian lobe $G(\mathbf{i}; \mathbf{p}, \lambda, \mu)$ with $V'(\mathbf{i})$.

- Let us work with the dot product first. We have that

$$G(\mathbf{i}; \mathbf{p}, \lambda, \mu) \cdot V'(\mathbf{i}) = \int_{S^2} G(\mathbf{i}; \mathbf{p}, \lambda, \mu) V'(\mathbf{i}) \, d\mathbf{i}.$$

Paramerizing the direction $\mathbf{i}$ with spherical coordinates (θ, φ) in the coordinate system where $\mathbf{p}$ is the positive z-axis, the above integral becomes:

$$\int_{S^2} G(\mathbf{i}; \mathbf{p}, \lambda, \mu) V'(\mathbf{i}) \, d\mathbf{i} = \mu \int_0^{\theta_d} \int_0^{2\pi} G(\mathbf{i}; \mathbf{z}, \lambda, 1) \sin \theta \, d\varphi d\theta.$$

The paper precomputes the 2D function

$$f_h(\theta_d, \lambda) = \int_0^{\theta_d} \int_0^{2\pi} G(\mathbf{i}; \mathbf{z}, \lambda, 1) \sin \theta \, d\varphi d\theta$$

and approximates it with a sigmoid multiplies with a polynomial.

- For the vector product, the paper approximates it with a Gaussian lobe with the same center and sharpness, but with amplitude scaled to give the same integral value as the product:

$$G(\mathbf{i}; \mathbf{p}, \lambda, \mu) \otimes V'(\mathbf{i}) \approx G\left(\mathbf{i}; \mathbf{p}, \lambda, \frac{f_h(\theta_d, \lambda)}{f_h(\pi/2, \lambda)} \mu\right).$$

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2.3 Lighting Representation

- The paper approximates the radiance field incident on point x from a spherical light source with radius r and intensity s yields, located at $\mathbf{l}$, with the following sphercial Gaussian:

$$L_{\mathbf{x}}^*(\mathbf{i}) = G\left(\mathbf{i}; \frac{\mathbf{l} - \mathbf{x}}{\|\mathbf{l} - \mathbf{x}\|}, f^{-1}\left(\frac{2\pi r^2}{\|\mathbf{l} - \mathbf{x}\|^2}, \frac{s}{\|\mathbf{l} - \mathbf{x}\|^2}\right)\right)$$

If the light source is infinitely far away, the approximation is:

$$L_{\mathbf{x}}^*(\mathbf{i}) = G(\mathbf{i}; \mathbf{l}, f_a^{-1}(2\pi r^2), s).$$

Here, note that $\mathbf{l}$ becomes a direction, not a 3D position like in the previous case.

- The paper represents environment lighting differently based on whether it is convolved with diffuse or specular BRDFs.

  - For diffuse BRDFs, the paper simply fit < 10 SG lobes to the environment map.
  - For specular BRDFs, the paper preconvolves the environmental radiance with SGs of varying lobe sharpness. Formally, the paper precompues a 3D function $\Gamma_L(\mathbf{p}, \lambda)$ where

$$\Gamma_L(\mathbf{p}, \lambda) = \int_{S^2} G(\mathbf{i}; \mathbf{p}, \lambda, 1) L(\mathbf{i}) \, d\mathbf{i}$$

where L is the environment lighting. In this way, the convolution between environment lighting and an SG lobe that represents the BRDF is given by:

$$G(\mathbf{i}; \mathbf{p}, \lambda, \mu) = \mu \int_{S^2} G(\mathbf{i}; \mathbf{p}, \lambda, 1) L(\mathbf{i}) \, d\mathbf{i} = \mu \Gamma_L(\mathbf{p}, \lambda).$$

2.4 Shading

- First, let us review the data that the paper uses. At each vertex $\mathbf{x}$, we have

  - the position, the texture coordinates, the local coordinate frame, and
  - the 48 PC coefficients $w_{\mathbf{x},j}^V$ of the SSDF.

  The textures contain:

  - the diffuse color,
  - parameters for the specular lobe:
    * In case of the Cook–Torrance model, this is just a single number.
    * In case of measured specular BRDF, a list of Gaussian lobes that makes up the approximate normal distribution fuction D^*.

- We are ready to work out the rendering integral:

$$R(\mathbf{o}) = \int_{S^2} (k_d + k_s \rho_s(\mathbf{o}, \mathbf{i})) L(\mathbf{i}) V(\mathbf{i}) \max\{0, \mathbf{i} \cdot \mathbf{n}\} \, d\mathbf{i}$$

$$= k_d \int_{S^2} L(\mathbf{i}) V(\mathbf{i}) \max\{0, \mathbf{i} \cdot \mathbf{n}\} \, d\mathbf{i} + k_s \int_{S^2} \rho_s(\mathbf{o}, \mathbf{i}) L(\mathbf{i}) V(\mathbf{i}) \max\{0, \mathbf{i} \cdot \mathbf{n}\} \, d\mathbf{i}.$$

- For the diffuse integral, we have that:

$$\int_{S^2} L(\mathbf{i})V(\mathbf{i})\max 0, \mathbf{i} \cdot \mathbf{n} \, d\mathbf{i} = (L(\mathbf{i})\max\{0, \mathbf{i} \cdot \mathbf{n}\}) \cdot V(\mathbf{i})$$

$$\approx (L(\mathbf{i})\max\{0 \cdot \mathbf{i}, \mathbf{n}\}) \cdot V_{\mathbf{x}}^d(\mathbf{i})$$

where $V_{\mathbf{x}}^d(\mathbf{i})$ is the SSDF.

Since we have discussed how to compute the dot product of SGs with SSDF in Section 2.2, it remains to figure out how represent $L(\mathbf{i})\max\{0, \mathbf{i}, \mathbf{n}\}$ with SGs. However, this is quite simple because we already represent $L(\mathbf{i})$ with a mixture of SGs $L^*(\mathbf{i})$. The function $\max\{0, \mathbf{i} \cdot \mathbf{n}\}$ is approximated with the following SG:

$$C^*(\mathbf{i}, \mathbf{n}) = G(\mathbf{i}; \mathbf{n}, 2.133, 1.170).$$

So,

$$L(\mathbf{i})\max\{0, \mathbf{i} \cdot \mathbf{n}\} \approx L^*(\mathbf{i}) \otimes C^*(\mathbf{i}, \mathbf{n})$$

and we already know how to compute the scalar product of two SGs.

- For the specular integral, we approximate it as:

$$\int_{S^2} \rho_s(\mathbf{o}, \mathbf{i})L(\mathbf{i})V(\mathbf{i})\max\{0, \mathbf{i} \cdot \mathbf{n}\} \, d\mathbf{i} \approx (C^*(\mathbf{i}, \mathbf{n}) \otimes \rho_{s,\mathbf{x}}^*(i; \mathbf{o}) \otimes V_{\mathbf{x}}^d(\mathbf{i})) \cdot L(\mathbf{i})$$

This means that, we compute the SG lobes in the parenthesis first and look up Γ_L for the dot product.

3 Anisotropic Gaussian

This section is based on the [Xu et al., 2013] paper.

3.1 Basic Definition

- An *anisotropic spherical Gaussian* (ASG) is the function:

$$G(\mathbf{v}; [\mathbf{x}, \mathbf{y}, \mathbf{z}], [\lambda, \mu], c) = c \cdot S(\mathbf{v}; \mathbf{z}) \cdot e^{-\lambda(\mathbf{v} \cdot \mathbf{x})^2 - \mu(\mathbf{v} \cdot \mathbf{y})^2}$$

where

- $S(\mathbf{v}; \mathbf{z}) = \max\{0, \mathbf{v} \cdot \mathbf{z}\}$ is called the *smooth term*.
- The vectors $\mathbf{z}$, $\mathbf{x}$, and $\mathbf{y}$ are called the *lobe*, *tangent*, and *bitangent* directions, respectively, and they should form an orthonormal frame.
- The λ and μ constants are called the *bandwidths*.

The definition above is called the *geometric form* of ASG.

- The above definition is based on the *Bingham distribution* in directional statistics. Basically, if you remove the smooth term, you get the Bingham distribution.

- There is another equivalent definition of ASG called the *algebric form*:

$$G(\mathbf{v}; A) = S(\mathbf{v}; \mathbf{z}) \cdot e^{-\mathbf{v}^T A \mathbf{v}}$$

where $\mathbf{A}$ is a 3×3 symmetric matrix, and $\mathbf{z}$ is its eigenvector with the smallest eigenvalue.

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- Spherical Gaussians are defined in terms of the Von Mises distribution:

$$G_{\text{iso}}(\mathbf{v}; \mathbf{p}, \nu, c) = c \cdot e^{2\nu(\mathbf{v} \cdot \mathbf{p} - 1)}.$$

As a result, it is not the same as ASG with two equal bandwidths because it lacks the smooth term. However, it is still roughly the same:

$$G_{\text{iso}}(\mathbf{v}; \mathbf{p}, \nu, c) \approx G(\mathbf{v}; [\mathbf{x}, \mathbf{y}, \mathbf{p}], [\nu, \nu], c).$$

3.2 Operations

- In order to create a rendering system from ASG, we need to know how to compute the following operations on ASGs:

  - **Integral**: $\int_{S^2} G(\mathbf{v}) \, d\mathbf{v}$
  - **Product**: $G_1(\mathbf{v})G_2(\mathbf{v})$
  - **Convolution**: $\int_{S^2} G_1(\mathbf{v})G_2(\mathbf{v}) \, d\mathbf{v}$

3.2.1 Integral

- The integral of an ASG over the sphere does not have a closed form. However, it can be approximated as follows:

$$\int_{S^2} G(\mathbf{v}) \, d\mathbf{v} = \frac{\pi}{\sqrt{\lambda\mu}} - \frac{e^{-\mu}}{2\lambda}\left(F(\nu) + \frac{\nu}{\mu}F\left(\nu + \frac{\nu}{\mu}\right)\right)$$

where $\nu = \lambda - \mu$, and

$$F(a) = \int_0^{2\pi} e^{-a \cos^2 \phi} \, d\phi.$$

Here, we assume that $\lambda \geq \mu$ WLOG.

- The function F can be precomputed into a lookup table, or it can be approximated by a rational function.

- If $\lambda, \mu > 5$, the terms involving F becomes small, so the integral is well approximated by:

$$\int_{S^2} G(\mathbf{v}) \, d\mathbf{v} \approx \frac{\pi}{\sqrt{\lambda\mu}}.$$

3.2.2 Product

- Given two ASGs in algebraic form:

$$G_1(\mathbf{v}) = G(\mathbf{v}; \mathbf{A}_1) = \max(\mathbf{v} \cdot \mathbf{z}_1, 0) \cdot e^{-\mathbf{v}^T \mathbf{A}_1 \mathbf{v}}$$
$$G_2(\mathbf{v}) = G(\mathbf{v}; \mathbf{A}_2) = \max(\mathbf{v} \cdot \mathbf{z}_2, 0) \cdot e^{-\mathbf{v}^T \mathbf{A}_2 \mathbf{v}}.$$

The product of the two ASGs is given by:

$$G_1(\mathbf{v})G_2(\mathbf{v}) = S(\mathbf{v}; \mathbf{z}_1, \mathbf{z}_2) \cdot e^{-\mathbf{v}^T (\mathbf{A}_1 + \mathbf{A}_2)\mathbf{v}}$$

where

$$S(\mathbf{v}; \mathbf{z}_1, \mathbf{z}_2) = S(\mathbf{v}; \mathbf{z}_1)S(\mathbf{v}; \mathbf{z}_2).$$

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- Let $\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2$ and $\mathbf{z}_3$ be the smallest eigenvector of $\mathbf{A}_3$. We have that:

$$G_1(\mathbf{v})G_2(\mathbf{v}) = \frac{S(\mathbf{v}; \mathbf{z}_1, \mathbf{z}_2)}{\max(\mathbf{v} \cdot \mathbf{z}_3, 0)} \left(\max(\mathbf{v} \cdot \mathbf{z}_3) \cdot e^{-\mathbf{v}^T \mathbf{A}_3 \mathbf{v}} \right)$$
$$= \frac{S(\mathbf{v}; \mathbf{z}_1, \mathbf{z}_2)}{\max(\mathbf{v} \cdot \mathbf{z}_3, 0)} G(\mathbf{v}; \mathbf{A}_3).$$

The paper makes the following approximation:

$$\frac{S(\mathbf{v}; \mathbf{z}_1, \mathbf{z}_2)}{\max(\mathbf{v} \cdot \mathbf{z}_3, 0)} \approx \frac{S(\mathbf{z}_3; \mathbf{z}_1, \mathbf{z}_2)}{\max(\mathbf{z}_3 \cdot \mathbf{z}_3, 0)} = S(\mathbf{z}_3; \mathbf{z}_1, \mathbf{z}_2).$$

Basically, they approximate the ratio with its value at $\mathbf{z}_3$, which is the peak of the Gaussian term. Xu et al. argues that this approximation is acceptable because the Gaussian term changes much faster than the smooth term.

3.2.3   Convolution of ASG with SG

- Surprisingly, the Xu et al. paper only gives a convolution of an ASG with an SG. Maybe they don't have good approximations for the convolution of two ASGs?

- Let us denote the (isotropic) SG with:

$$G_{\text{iso}}(\mathbf{v}; \mathbf{p}, \nu) = e^{2\nu(\mathbf{v} \cdot \mathbf{p} - 1)} = e^{-\nu \|\mathbf{v} - \mathbf{p}\|^2}.$$

- Let $C(\mathbf{p})$ denotes the convolution between ASG $G(\mathbf{v})$ and SP $G_{\text{iso}}(\mathbf{v}; \mathbf{p}, \nu)$:

$$C(\mathbf{p}) = \int_{S^2} G(\mathbf{v}) G_{\text{iso}}(\mathbf{v}; \mathbf{p}, \nu) \, d\mathbf{v} = \int_{S^2} S(\mathbf{v}; \mathbf{z}) e^{-\lambda(\mathbf{v} \cdot \mathbf{x})^2 - \mu(\mathbf{v} \cdot \mathbf{y})^2 - \nu \|\mathbf{v} - \mathbf{p}\|^2} \, d\mathbf{v}.$$

- The first approximate the Xu et al. paper made is to approximate $S(\mathbf{v}; \mathbf{z})$ with its value at $\mathbf{p}$:

$$\int_{S^2} S(\mathbf{v}; \mathbf{z}) e^{-\lambda(\mathbf{v} \cdot \mathbf{x})^2 - \mu(\mathbf{v} \cdot \mathbf{y})^2 - \nu \|\mathbf{v} - \mathbf{p}\|^2} \, d\mathbf{v} \approx S(\mathbf{p}; \mathbf{z}) \int_{S^2} e^{-\lambda(\mathbf{v} \cdot \mathbf{x})^2 - \mu(\mathbf{v} \cdot \mathbf{y})^2 - \nu \|\mathbf{v} - \mathbf{p}\|^2} \, d\mathbf{v}.$$

- Next, the paper approximates the $\|\mathbf{v} - \mathbf{p}\|^2$ term by a Taylor series expansion around $\mathbf{z}$, which is the peak of $G(\mathbf{v})$. To do so, observe that:

$$\mathbf{p} = \mathbf{z} + (\mathbf{p} - \mathbf{z})$$
$$= \mathbf{z} + [(\mathbf{p} - \mathbf{z}) \cdot \mathbf{x}]\mathbf{x} + [(\mathbf{p} - \mathbf{z}) \cdot \mathbf{y}]\mathbf{y} + [(\mathbf{p} - \mathbf{z}) \cdot \mathbf{z}]\mathbf{z}$$
$$= \mathbf{z} + (\mathbf{p} \cdot \mathbf{x})\mathbf{x} + (\mathbf{p} \cdot \mathbf{y})\mathbf{y} + [(\mathbf{p} - \mathbf{z}) \cdot \mathbf{z}]\mathbf{z}$$
$$\approx \mathbf{z} + (\mathbf{p} \cdot \mathbf{x})\mathbf{x} + (\mathbf{p} \cdot \mathbf{y})\mathbf{y}.$$

So,

$$\mathbf{v} \approx \mathbf{z} + (\mathbf{v} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y})\mathbf{y}.$$

Thus,

$$\mathbf{v} - \mathbf{p} \approx (\mathbf{v} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y} - \mathbf{p} \cdot \mathbf{y})\mathbf{y}.$$

As a result,

$$\|\mathbf{v} - \mathbf{p}\|^2 \approx (\mathbf{v} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{x})^2 + (\mathbf{v} \cdot \mathbf{y} - \mathbf{p} \cdot \mathbf{y})^2.$$

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- Substituing the above approximation to into the convolution integral, we have that:

$$C(\mathbf{p}) \approx S(\mathbf{p}; \mathbf{z}) \int_{S^2} e^{-\lambda(\mathbf{v}\cdot\mathbf{x})^2 - \mu(\mathbf{v}\cdot\mathbf{y})^2 - \nu(\mathbf{v}\cdot\mathbf{x} - \mathbf{p}\cdot\mathbf{x})^2 - \nu(\mathbf{v}\cdot\mathbf{y} - \mathbf{p}\cdot\mathbf{y})^2} \, d\mathbf{v}.$$

We now simplify the exponent of e of the integrand:

$$-\lambda(\mathbf{v}\cdot\mathbf{x})^2 - \mu(\mathbf{v}\cdot\mathbf{y})^2 - \nu(\mathbf{v}\cdot\mathbf{x} - \mathbf{p}\cdot\mathbf{x})^2 - \nu(\mathbf{v}\cdot\mathbf{y} - \mathbf{p}\cdot\mathbf{y})^2$$
$$= -\lambda(\mathbf{v}\cdot\mathbf{x})^2 - \mu(\mathbf{v}\cdot\mathbf{y})^2 - \nu[(\mathbf{v}\cdot\mathbf{x})^2 + (\mathbf{p}\cdot\mathbf{x})^2 - 2(\mathbf{v}\cdot\mathbf{x})(\mathbf{p}\cdot\mathbf{x})] - \nu[(\mathbf{v}\cdot\mathbf{y})^2 + (\mathbf{p}\cdot\mathbf{y})^2 - 2(\mathbf{v}\cdot\mathbf{y})(\mathbf{p}\cdot\mathbf{y})]$$
$$= -\big[(\lambda+\nu)(\mathbf{v}\cdot\mathbf{x})^2 - 2\nu(\mathbf{v}\cdot\mathbf{x})(\mathbf{p}\cdot\mathbf{x}) + \nu(\mathbf{p}\cdot\mathbf{x})^2\big] - \big[(\mu+\nu)(\mathbf{v}\cdot\mathbf{y})^2 - 2\nu(\mathbf{v}\cdot\mathbf{y})(\mathbf{p}\cdot\mathbf{y}) + \nu(\mathbf{p}\cdot\mathbf{y})^2\big].$$

Let us simplify the left term:

$$(\lambda+\nu)(\mathbf{v}\cdot\mathbf{x})^2 - 2\nu(\mathbf{v}\cdot\mathbf{x})(\mathbf{p}\cdot\mathbf{x}) + \nu(\mathbf{p}\cdot\mathbf{x})^2$$
$$= \left[(\lambda+\nu)(\mathbf{v}\cdot\mathbf{x})^2 - 2\nu(\mathbf{v}\cdot\mathbf{x})(\mathbf{p}\cdot\mathbf{x}) + \frac{\nu^2}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2\right] - \frac{\nu^2}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2 + \nu(\mathbf{p}\cdot\mathbf{x})^2$$
$$= \left[\sqrt{\lambda+\nu}(\mathbf{v}\cdot\mathbf{x}) - \frac{\nu}{\sqrt{\lambda+\nu}}(\mathbf{p}\cdot\mathbf{x})\right]^2 + \frac{-\nu^2 + \nu(\lambda+\nu)}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2$$
$$= (\lambda+\nu)\left[\mathbf{v}\cdot\mathbf{x} - \frac{\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})\right]^2 + \frac{\lambda\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2.$$

As a result,

$$-\lambda(\mathbf{v}\cdot\mathbf{x})^2 - \mu(\mathbf{v}\cdot\mathbf{y})^2 - \nu(\mathbf{v}\cdot\mathbf{x} - \mathbf{p}\cdot\mathbf{x})^2 - \nu(\mathbf{v}\cdot\mathbf{y} - \mathbf{p}\cdot\mathbf{y})^2$$
$$= -(\lambda+\nu)\left(\mathbf{v}\cdot\mathbf{x} - \frac{\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})\right)^2 - \frac{\lambda\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2 - (\mu+\nu)\left(\mathbf{v}\cdot\mathbf{y} - \frac{\nu}{\mu+\nu}(\mathbf{p}\cdot\mathbf{y})\right)^2 - \frac{\mu\nu}{\mu+\nu}(\mathbf{p}\cdot\mathbf{y})^2$$
$$= -(\lambda+\nu)\left(\mathbf{v}\cdot\mathbf{x} - \frac{\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})\right)^2 - (\mu+\nu)\left(\mathbf{v}\cdot\mathbf{y} - \frac{\nu}{\nu+\nu}(\mathbf{p}\cdot\mathbf{y})\right)^2 - \frac{\lambda\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2 - \frac{\mu\nu}{\mu+\nu}(\mathbf{p}\cdot\mathbf{y})^2.$$

Hence,

$$C(\mathbf{p}) \approx S(\mathbf{p}; \mathbf{z}) \cdot e^{-\frac{\lambda\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})^2 - \frac{\mu\nu}{\mu+\nu}(\mathbf{p}\cdot\mathbf{y})^2} \int_{S^2} e^{-(\lambda+\nu)\left(\mathbf{v}\cdot\mathbf{x} - \frac{\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})\right)^2 - (\mu+\nu)\left(\mathbf{v}\cdot\mathbf{y} - \frac{\nu}{\mu+\nu}(\mathbf{p}\cdot\mathbf{y})\right)^2} \, d\mathbf{v}.$$

The integral on the right evaluates to a constant. The Xu et al. paper says that it can be approximated as an integral of an ASG on the sphere:

$$\int_{S^2} e^{-(\lambda+\nu)\left(\mathbf{v}\cdot\mathbf{x} - \frac{\nu}{\lambda+\nu}(\mathbf{p}\cdot\mathbf{x})\right)^2 - (\lambda+\mu)\left(\mathbf{v}\cdot\mathbf{y} - \frac{\nu}{\mu+\nu}(\mathbf{p}\cdot\mathbf{y})\right)^2} \, d\mathbf{v} \approx \frac{\pi}{\sqrt{(\lambda+\nu)(\mu+\nu)}}.$$

The logic on this part is VERY sketchy, and I shall ask Run-Dong for clarification.

If you believe in the simplication above, we have that:

$$C(\mathbf{p}) \approx G\Big(\mathbf{p}, [\mathbf{x}, \mathbf{y}, \mathbf{z}], \Big[\frac{\lambda\nu}{\lambda+\nu}, \frac{\mu\nu}{\mu+\nu}\Big], \frac{\pi}{\sqrt{(\lambda+\nu)(\mu+\nu)}}\Big).$$

- The approximation above is accurate when $(\mathbf{p} - \mathbf{v}) \cdot \mathbf{z}$ is small. However, when it is large, the SG is small, so the error will be small too.

- The approximation becomes more accurate when λ and μ becomes larger. When $\lambda, \mu > 3$, the L2-error is less than 0.2%. When $\lambda, \mu > 1$, the error is bounded by 2.8%.

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3.3 ASG-Based Rendering Framework

- The framework is based on the SG-based framework in [Wang et al., 2009] paper, but BRDFs and lighting are represented with ASGs instead of SGs.

- Assuming distant lighting and static geometry, the outgoing radiance $L(\mathbf{o})$ is given by:

$$L(\mathbf{o}) = \int_{S^2} L(\mathbf{i})V(\mathbf{i})\rho(\mathbf{i}, \mathbf{o}) \max(\mathbf{i} \cdot \mathbf{n}, 0) \, d\mathbf{i}$$

  where the BRDF $\rho(\mathbf{i}, \mathbf{o})$ is the sum of diffuse and specular component:

$$\rho(\mathbf{i}, \mathbf{o}) = k_d + k_s \rho_s(\mathbf{i}, \mathbf{o}).$$

As a result, $L(\mathbf{o}) = k_d L_d + k_s L_s(\mathbf{o})$ where

$$L_d = \int_{S^2} L(\mathbf{i})V(\mathbf{i}) \max(\mathbf{i} \cdot \mathbf{n}, 0) \, d\mathbf{i}$$

$$L_s(\mathbf{o}) = \int_{S^2} L(\mathbf{i})V(\mathbf{i})\rho_s(\mathbf{i}, \mathbf{o}) \max(\mathbf{i} \cdot \mathbf{n}, 0) \, d\mathbf{i}$$

3.3.1 Light Approximation

- The environment light is represented by a mixture of ASGs.

- Given an environment map, the Xu et al. paper first determine some initial ASGs to cover the mininum value of the environment map. Then, the yuse the L-BFGS-B solver (aka scipy) to fit the rest of the ASGs.

3.3.2 BRDF Approximation

- According the Cook–Torrance model, the specular term is given by:

$$\rho_s(\mathbf{i}, \mathbf{o}) = M(\mathbf{i}, \mathbf{o})D(\mathbf{h})$$

  where $\mathbf{h} = (\mathbf{i} + \mathbf{o})/\|\mathbf{i} + \mathbf{o}\|$ is the half vector, D is the normal distribution function (NDF), and M is the product of the shadowing and Fresnel terms.

- For the Ward model, M and D are approximated as:

$$M(\mathbf{i}, \mathbf{o}) = \frac{1}{4\pi\alpha_x\alpha_y\sqrt{(\mathbf{i} \cdot \mathbf{n})(\mathbf{o} \cdot \mathbf{n})}}$$

$$D(\mathbf{h}) = \exp\left(-\frac{2}{1 + \mathbf{h} \cdot \mathbf{n}}\left(\frac{(\mathbf{h} \cdot \mathbf{x})^2}{\alpha_x^2} + \frac{(\mathbf{h} \cdot \mathbf{y})^2}{\alpha_y^2}\right)\right) \approx G\left(\mathbf{h}; [\mathbf{x}, \mathbf{y}, \mathbf{n}], \left[\frac{1}{\alpha_x^2}, \frac{1}{\alpha_y^2}\right]\right)$$

- For the Ashikhmin model, M and D are approximate as:

$$M(\mathbf{i}, \mathbf{o}) = \frac{\sqrt{(n_u + 1)(n_v + 1)}F(\mathbf{i} \cdot \mathbf{h})}{8\pi(\mathbf{i} \cdot \mathbf{h})\max(\mathbf{i} \cdot \mathbf{n}, \mathbf{o} \cdot \mathbf{n})}$$

$$D(\mathbf{h}) = (\mathbf{h} \cdot \mathbf{n})^{\frac{n_u(\mathbf{h} \cdot \mathbf{u})^2 + n_v(\mathbf{h} \cdot \mathbf{v})^2}{1 - (\mathbf{h} \cdot \mathbf{n})^2}} \approx G\left(\mathbf{h}; [\mathbf{u}, \mathbf{v}, \mathbf{n}], \left[\frac{n_u}{2}, \frac{n_v}{2}\right]\right)$$

- The Xu et al. paper reports that measured BRDFs' NDFs can be fitted with 3-4 ASGs.

- In the same way as the last section on isotropic SGs, $D(h)$ is written in terms of the half vector, but the shading integral is written in terns of $\mathbf{i}$. There is a change of variable that needs to be performed, and it is best if the resulting function can be approximated again by an ASG.

- Suppose that $D(h)$ is approximated by $G(\mathbf{h}; [\mathbf{x}_h, \mathbf{y}_h, \mathbf{z}_h], [\lambda_h, \mu_h])$. We want to write this as a function of $\mathbf{i}$ and approximates it as an ASG:

$$G_h(\mathbf{h}) = G(\mathbf{h}; [\mathbf{x}_h, \mathbf{y}_h, \mathbf{z}_h], [\lambda_h, \mu_h]) = G\left(\frac{\mathbf{i}+\mathbf{o}}{\|\mathbf{i}+\mathbf{o}\|}; [\mathbf{x}_h, \mathbf{y}_h, \mathbf{z}_h], [\lambda_h, \mu_h]\right) \approx G(\mathbf{i}; [\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i], [\lambda_i, \mu_i]) = G_i(\mathbf{i})$$

The main task to write $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$, λ_i, and μ_i in terms of $\mathbf{x}_h$, $\mathbf{y}_h$, $\mathbf{z}_h$, λ_h, and μ_h.

- First, we preserve the peak location by writing $\mathbf{z}_i$ in terms of $\mathbf{z}_h$:

$$\mathbf{z}_i = 2(\mathbf{o} \cdot \mathbf{z}_h)\mathbf{z}_h = \mathbf{o}.$$

- Other parameters are determined by preserving the second-order derivative of the exponential term in $G_h(\mathbf{h})$ at the lobe position $\mathbf{i} = \mathbf{z}_i$.

- Let $G_h(\mathbf{h}(\mathbf{i})) = S(\mathbf{h}(\mathbf{i})) \cdot e^{-g(\mathbf{i})}$. We have that:

$$g(\mathbf{i}) = \lambda_h(\mathbf{h}(\mathbf{i}) \cdot \mathbf{x}_h)^2 + \mu_h(\mathbf{h}(\mathbf{i}) \cdot \mathbf{y}_h)^2$$

Define the coordinate frame $[\mathbf{x}_i', \mathbf{y}_i', \mathbf{z}_i]$ where $\mathbf{x}_i'$ is chosen to be perpendicular to $\mathbf{o}$. We have that $g(\mathbf{i})$ can be approximate by a second order Tyalor expansion at $\mathbf{i} = \mathbf{z}_i$ as:

$$g(\mathbf{i}) \approx \begin{bmatrix} \mathbf{i} \cdot \mathbf{x}_i' & \mathbf{i} \cdot \mathbf{y}_i' \end{bmatrix} \begin{bmatrix} \frac{\partial^2 g}{\partial x_i'^2} & \frac{\partial^2 g}{\partial x_i' \partial y_i'} \\ \frac{\partial^2 g}{\partial x_i' \partial y_i'} & \frac{\partial^2 g}{\partial y_i'^2} \end{bmatrix} \begin{bmatrix} \mathbf{i} \cdot \mathbf{x}_i' \\ \mathbf{i} \cdot \mathbf{y}_i' \end{bmatrix}$$

where

$$\frac{\partial^2 g}{\partial x_i'^2} = \frac{\lambda_h(\mathbf{x}_h \cdot \mathbf{x}_i') + \mu_h(\mathbf{y}_h \cdot \mathbf{x}_h')^2}{4(\mathbf{o} \cdot \mathbf{z}_h)^2}$$

$$\frac{\partial^2 g}{\partial x_i' \partial y_i'} = \frac{(\mu_h - \lambda_h)(\mathbf{x}_h \cdot \mathbf{x}_i')(\mathbf{y}_h \cdot \mathbf{x}_i')}{4(\mathbf{o} \cdot \mathbf{z}_h)}$$

$$\frac{\partial^2 g}{\partial y_i'^2} = \frac{\mu_h(\mathbf{x}_h \cdot \mathbf{x}_i')^2 + \lambda_h(\mathbf{y}_h \cdot \mathbf{x}_i')}{4}.$$

The middle matrix in the approximation for $g(\mathbf{i})$ is the Hessian matrix $\mathbf{H}(g)$, which can then be factored with the eigen-decomposition:

$$\mathbf{H}(g) = \mathbf{U} \begin{bmatrix} \lambda_i & 0 \\ 0 & \mu_i \end{bmatrix} \mathbf{U}^T.$$

Now, we have that:

$$g(\mathbf{i}) \approx \lambda_i(\mathbf{i} \cdot \mathbf{x}_i)^2 + \mu_i(\mathbf{i} \cdot \mathbf{y}_i)^2$$

with

$$\begin{bmatrix} \mathbf{x}_i & \mathbf{y}_i \end{bmatrix} = \mathbf{U} \begin{bmatrix} \mathbf{x}_i' & \mathbf{y}_i' \end{bmatrix}.$$

- I have a bad feeling about all these formulas because of several reasons:

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– Is it valid to say

$$g(\mathbf{i}) \approx \begin{bmatrix} \mathbf{i} \cdot \mathbf{x}'_i & \mathbf{i} \cdot \mathbf{y}'_i \end{bmatrix} \mathbf{H}(g) \begin{bmatrix} \mathbf{i} \cdot \mathbf{x}'_i \\ \mathbf{i} \cdot \mathbf{y}'_i \end{bmatrix}?$$

Where is $\Delta\mathbf{i}$? Shouldn't the LHS be $g(\mathbf{i} + \Delta\mathbf{i})$? Should't the RHS contain the $g(\mathbf{i})$ term? Where is the first order term? How does this approximation come about?

– Where is the change-of-variable factor? Shouldn't it become a part of the constant of $G_i(\mathbf{i})$?

– It's hard to verify the correctness of the second-order derivatves given in the paper.

Really, I need to ask Run-Dong or Zhao about this.

3.3.3 Visibility Approximation

- The Xu et al. paper still uses the spherical signed distance function (SSDF) to represent visibility. To recap, given a binary visibility function $V(\mathbf{i})$, its corresponding SSDF $\theta_d(\mathbf{i})$ is the signed angular distance to the nearest point to $\mathbf{i}$ in which $V(\mathbf{i})$ changes value. The sign is positive when $V(\mathbf{i}) = 1$ and negative when $V(\mathbf{i}) = 0$.

- The convolution between an SG and visibility function $V(\mathbf{i})$ is approximated as:

$$\int_{S^2} G_{\mathrm{iso}}(\mathbf{i}; \mathbf{p}, \nu) V(\mathbf{i}) \, \mathrm{d}\mathbf{i} \approx f_h(\theta_d(\mathbf{p}), \nu)$$

where f_h is a sigmoid multipled with a polynomial.

- The above calculation cannot be applied to ASGs. The Xu et al. paper proposes turning an ASG into an isotropic SG when doing the above convolution. Given an ASG with bandwidths λ and μ, we say that the ASG's *effective bandwidth* is $\sqrt{\lambda\mu}$. When computing convolution with visibility, we use the formula above, substituting ν with the effective bandwidth:

$$\int_{S^2} G(\mathbf{i}; [\mathbf{x}, \mathbf{y}, \mathbf{p}], [\lambda, \mu]) \approx f_h(\theta_d(\mathbf{i}), \sqrt{\lambda\mu}).$$

- We also need to compute the vector product of an ASG with visibility function. The paper proposes using the effective bandwidth to compute the scaling term as follows:

$$G(\mathbf{i}; [\mathbf{x}, \mathbf{y}, \mathbf{p}], [\lambda, \mu]) = \frac{f_h(\theta_d(\mathbf{p}), \sqrt{\lambda\mu})}{f_h(\pi/2, \sqrt{\lambda\mu})} G(\mathbf{i}; [\mathbf{x}, \mathbf{y}, \mathbf{p}], [\lambda, \mu]).$$

- The use of effective bandwidth causes much error when the aspect ratio is large. The error manifests in the form of softness of shadow boundary, which can be visually subtle.

- The SSDF is compressed with PCA. The paper says 48 PCA coefficients yield good restuls.

3.3.4 Rendering

- When computing the diffuse term, the environment lighting $L(\mathbf{i})$ is approximated with a mixture of ASGs:

$$L(\mathbf{i}) \approx \sum_j l_j G_j(\mathbf{i})$$

where $G_j(\mathbf{i})$ stands for $G(\mathbf{i}; [\mathbf{x}_j, \mathbf{y}_j, \mathbf{z}_j], [\lambda_j, \mu_j])$.

The diffuse term is then approximated as follows:

$$L_d \approx \int_{S^2} \left(\sum_j l_j G_j(\mathbf{i}) \right) V(\mathbf{i}) \max(\mathbf{i} \cdot \mathbf{n}, 0) \, d\mathbf{i}$$

$$\approx \sum_j l_j \max(\mathbf{z}_j \cdot \mathbf{n}, 0) \int_{S^2} G_j(\mathbf{i}) V(\mathbf{i}) \, d\mathbf{i}.$$

- For the specular term, the environment lighting is represented with an environment map. To enable faster computation, the environment map must be prefiltered with Gaussian kernels. I believe this is the Γ_L table that the Wang et al. paper uses.

  The Xu et al. paper approximates $\rho_s(\mathbf{i}, \mathbf{o})$ as:

  $$\rho_s(\mathbf{i}, \mathbf{o}) \approx M(\mathbf{i}, \mathbf{o}) \sum_k G_k(\mathbf{i})$$

  where k is usually no more than 4. So,

  $$L_s(\mathbf{o}) \approx \int_{S^2} L(\mathbf{i}) V(\mathbf{i}) \left(\sum_k M(\mathbf{i}, \mathbf{o}) G_k(\mathbf{i}) \right) \max(\mathbf{i} \cdot \mathbf{n}, 0) \, d\mathbf{i}$$

  $$\approx \sum_k M(\mathbf{z}_k, \mathbf{o}) \max(\mathbf{z}_k, \mathbf{n}, 0) \int_{S^2} L(\mathbf{i}) V(\mathbf{i}) G_k(\mathbf{i}) \, d\mathbf{i}$$

  The paper then approximate $V(\mathbf{i}) G_k(\mathbf{i})$ with $g_k G_k(\mathbf{i})$ where

  $$g_k = \frac{f_h(\theta_d(\mathbf{z}_k), \sqrt{\lambda_k \mu_k})}{f_h(\pi/2, \sqrt{\lambda_k \mu_k})}.$$

  So, the specular term becomes:

  $$L_s(\mathbf{o}) \approx \sum_k M(\mathbf{z}_k, \mathbf{o}) \max(\mathbf{z}_k, \mathbf{n}, 0) g_k \int_{S^2} L(\mathbf{i}) G_k(\mathbf{i}) \, d\mathbf{i}$$

  The calculation is then reduced to convolving environment map $L(\mathbf{i})$ with ASGs. The paper says they perform anisotropic lookup into the texture with the `textureGrad` function in GLSL. However, `textureGrad` can only handle aspect ratio of at most 16, so they must place more samples by themselves. How this is actually done is not documented in the paper. (Is this even reproducible?)

4 Rendering Microcylinder Cloth with Spherical Gaussians

This section is about [Iwasaki et al., 2014], which deals with rendering Sadeghi et al.'s microcylinder model under environment lighting [Sadeghi et al., 2013].

4.1 Gaussians

- A *spherical Gaussian* is defined by:

  $$G(\omega; \xi, \sigma) = \exp \left(\frac{2}{\sigma^2} (\omega \cdot \xi - 1) \right)$$

  where $\omega, \xi \in S^2$. Here, ξ is the lobe direction, σ is the bandwidth.

- A *circular Gaussian* is defined as:

$$g^c(x; \mu, \sigma) = \exp\left(\frac{2}{\sigma^2}(\cos(x - \mu) - 1)\right).$$

Here g^c is a function on the circular angles, σ is the bandwidth, and μ is the mean.

- A (canonical) Gaussian is defined as:

$$g(x; \mu, \sigma) = \exp\left(-\frac{(x - \mu)^2}{\sigma^2}\right)$$

The normalized version is written as:

$$g^u(x; \mu, \sigma) = \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{\sigma^2}\right).$$

Note that this is a little bit different on the way the Gaussian function is defined in most literature. Basically, the bandwidth σ here is $\sqrt{2}$ times larger than the bandwidth in other literator.

- The paper works with fiber-based coordinate system discussed in [Marschner et al., 2003]. Basically, we have the longitudinal angle θ and the azimuthal angle ϕ. The paper let

$$\theta_d = (\theta_i - \theta_o)/2$$
$$\theta_h = (\theta_i + \theta_o)/2, \text{ and}$$
$$\phi_d = \phi_i - \phi_o.$$

- One interesting thing to note is that a spherical Gaussian can be decomposed into a product of two circular Gaussians. First, let us write:

$$\omega = \begin{bmatrix} \sin\theta \\ \cos\theta\cos\phi \\ \cos\theta\sin\phi \end{bmatrix} \qquad \text{and} \qquad \xi = \begin{bmatrix} \sin\theta' \\ \cos\theta'\cos\phi' \\ \cos\theta'\sin\phi' \end{bmatrix}.$$

We have that

$$
\begin{aligned}
\omega \cdot \xi - 1 &= \sin\theta\sin\theta' + \cos\theta\cos\phi\cos\theta'\cos\phi' + \cos\theta\sin\phi\cos\theta'\sin\phi' - 1 \\
&= \sin\theta\sin\theta' + \cos\theta\cos\theta'\cos\phi\cos\phi' + \cos\theta\cos\theta'\sin\phi\sin\phi' - 1 \\
&= \sin\theta\sin\theta' + \cos\theta\cos\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi') \\
&= \sin\theta\sin\theta' + \cos\theta\cos\theta'\cos(\phi - \phi') - 1 \\
&= \sin\theta\sin\theta' + (\cos\theta\cos\theta' - \cos\theta\cos\theta') + \cos\theta\cos\theta'\cos(\phi - \phi') - 1 \\
&= (\sin\theta\sin\theta' + \cos\theta\cos\theta' - 1) + [\cos\theta\cos\theta'\cos(\phi - \phi') - \cos\theta\cos\theta'] \\
&= (\sin\theta\sin\theta' + \cos\theta\cos\theta' - 1) + \cos\theta\cos\theta'(\cos(\phi - \phi') - 1) \\
&= (\cos(\theta - \theta') - 1) + \cos\theta\cos\theta'(\cos(\phi - \phi') - 1).
\end{aligned}
$$

As a result:

$$\frac{2}{\sigma^2}(\omega \cdot \xi - 1) = \frac{2}{\sigma^2}(\cos(\theta - \theta') - 1) + \frac{2}{(\sigma/\sqrt{\cos\theta\cos\theta'})^2}(\cos(\phi - \phi') - 1)$$

$$\exp\left(\frac{2}{\sigma^2}(\omega \cdot \xi - 1)\right) = \exp\left(\frac{2}{\sigma^2}(\cos(\theta - \theta') - 1)\right)\exp\left(\frac{2}{(\sigma/\sqrt{\cos\theta\cos\theta'})^2}(\cos(\phi - \phi') - 1)\right)$$

$$G(\omega; \xi, \sigma) = g^c(\theta; \theta', \sigma)g^c\left(\phi; \phi', \frac{\sigma}{\sqrt{\cos\theta\cos\theta'}}\right).$$

The paper abbreviates $\sigma/\sqrt{\cos\theta\cos\theta'}$ as σ'.

4.2　Microcylinder Model

- According to the model, the outgoing radiance $L(\omega_o)$ is given by $\alpha_1 L_1(\omega_o) + \alpha_2 L_2(\omega_o)$ where

  - $L_1(\omega_o)$ is the outgoing radiance from the warp yarn,
  - $L_2(\omega_o)$ is the outgoing radiance from the weft yarn, and
  - α_1 and α_2 are percentage of surface area of the two yarns.

- Each yarn has a tangent curve, which the Sadeghi et al. samples at discretes points. Let C_j denote the set of sampled tangent vectors of the tangent curve of Yarn j where $j \in \{1, 2\}$. Then, the outgoing radiance from the yarn is given by:

$$L_j(\omega_o) = \sum_{\mathbf{t} \in C_j} \int_{S^2} L(\omega_i) f_s(\omega_i, \omega_o) W(\mathbf{t}, \omega_i, \omega_o) \cos \theta_i \; \mathrm{d}\omega_i$$

  where f_s is the *yarn scattering function*, and W is the *weighting function*. Note that each integral is evaluated in a coordinate frame with $\mathbf{t}$ acting as the fiber tangent, so $\theta_i = \sin^{-1}(\omega_i \cdot \mathbf{t})$.

- The yarn scattering function is given by:

$$f_s(\omega_i, \omega_o) = \frac{f_{r,s}(\omega_i, \omega_o) + f_{r,v}(\omega_i, \omega_o)}{\cos^2 \theta_d}$$

$$f_{r,s}(\omega_i, \omega_o) = F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) g^u(\theta_h; 0, \gamma_s)$$

$$f_{r,v}(\omega_i, \omega_o) = F \frac{(1 - k_d) g^u(\theta_h; 0, \gamma_v) + k_d}{\cos \theta_i + \cos \theta_d} A$$

  Here, η is the index of refraction, F_r is the Fresnel reflectance term, and F is the product of two Fresnel transmittance terms (which I'm still fucking unclear what it actually is, even after looking at [Sadeghi et al., 2013]).

- The weighting function W is given by:

$$W(\mathbf{t}, \omega_i, \omega_o) = M(\mathbf{t}, \omega_i, \omega_o) \frac{P(\mathbf{t}, \omega_i, \omega_o)}{\sum_{t' \in C_1 \cup C_2} P(\mathbf{t}', \omega_i, \omega_o)}$$

  where M is called the *shadowing and masking term*, and P is called the *projection term*. They are defined as follows:

$$M(\mathbf{t}, \omega_i, \omega_o) = D(g(\phi_d; 0, \sigma), \max(\cos \phi_i, 0), \max(\cos \phi_o, 0))$$

$$P(\mathbf{t}, \omega_i, \omega_o) = D(g(\psi_d; 0, \sigma), \max(\cos \psi_i, 0), \max(\cos \psi_o, 0))$$

  where $D(x, y, z) = (1 - x) \cdot y \cdot z + x \min(y, z)$, and σ is a bandwidth parameter between $15°$ and $20°$. Also, given a direction ω, the angle ψ is the angle between the cloth surface's normal $\mathbf{n}$ and the projection of ω onto the plane spanned by $\mathbf{n}$ and $\mathbf{t}$.

4.3　Shading

- The paper represents environment lighting as a mixture of Gaussian:

$$L_{\mathrm{env}}(\omega_i) \approx \sum_{k=1}^{K} L_k G(\omega_i; \xi_k, \sigma_k).$$

  We shall abbreviate $G(\omega_i; \xi_k, \sigma_k)$ with $G_k(\omega_i)$.

- Assuming environment lighting, the outgoing radiance for Yarn j is given by:

$$L_j(\omega_o) = \sum_{k=1}^{K} L_k \sum_{\mathbf{t} \in C_j} B_{j,k}(\mathbf{t}, \omega_o)$$

where

$$B_{j,k}(\mathbf{t}, \omega_o) = \int_{S^2} G_k(\omega_i) V(\omega_i) f_s(\omega_i, \omega_o) W(\mathbf{t}, \omega_i, \omega_o) \cos \theta_i \ \mathrm{d}\omega_i,$$

which is the central integral that we have to evaluate.

- The weighting function is the bane of the above integral, the Iwasaki et al. paper factors it out first:

$$B_{j,k}(\mathbf{t}, \omega_o) = \frac{\int_{S^2} G_k(\omega_i) W(\mathbf{t}, \omega_i, \omega_o) \ \mathrm{d}\omega_i}{\int_{S^2} G_k(\omega_i) \ \mathrm{d}\omega_i} \times \int_{S^2} G_k(\omega_i) V(\omega_i) f_s(\omega_i, \omega_o) \cos \theta_i \ \mathrm{d}\omega_i.$$

The factored out term is denoted $T(\mathbf{t}, \xi_k, \sigma_k, \omega_o)$.

The paper precomputed T for all the sampled $\mathbf{t}$ and a discrete set of σ_k. For a fixed pair of $\mathbf{t}$ and σ_k, the paper computes the 4D function $T_{\mathbf{t}, \sigma_k}(\xi_k, \omega_o)$, performs SVD on it, and keeps a few largest singular values and vectors. The resolution of $T_{\mathbf{t}, \sigma_k}(\xi_k, \omega_o)$ is 48×48, and average number singular values used is in the range of 7.2 to 14.4

- Now, we turn to the integral on the right. Let's consider integral involving $f_{r,s}$ term because the paper says the $f_{r,v}$ term can be thought about in the same way. That is, we want to evaluate:

$$\begin{aligned} I_s &= \int_{S^2} G_k(\omega_i) V(\omega_i) f_{r,s}(\omega_i, \omega_o) \cos \theta_i \ \mathrm{d}\omega_i \\ &= \int_{S^2} G_k(\omega_i) V(\omega_i) F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) g^u(\theta_h; 0, \gamma_s) \frac{\cos \theta_i}{\cos^2 \theta_d} \ \mathrm{d}\omega_i \\ &= \int_{S^2} g^c(\theta_i; \theta_k, \sigma_k) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos \theta_i \cos \theta_k}}\right) V(\omega_i) F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) g^u(\theta_h; 0, \gamma_s) \frac{\cos \theta_i}{\cos^2 \theta_d} \ \mathrm{d}\omega_i. \end{aligned}$$

The circular Gaussian $g^u(\theta_h; 0, \gamma_s)$ may be approximated by $g^c(\theta_i; -\theta_o, 2\gamma_s)/(\sqrt{\pi}\gamma_s)$. Because the product of two circular Gaussian is another circular Gaussian, we have that

$$g^c(\theta_i; \theta_k, \sigma_k) g^c(\theta_i; -\theta_o, 2\gamma_s)/(\sqrt{pi}\gamma_s) = \alpha \cdot g^c(\theta_i; \theta_l, \sigma_l)$$

for some θ_l, σ_l, and α. So,

$$\begin{aligned} I_s &\approx \int_{S^2} \alpha g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos \theta_i \cos \theta_k}}\right) V(\omega_i) F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) \frac{\cos \theta_i}{\cos^2 \theta_d} \ \mathrm{d}\omega_i \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \alpha g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos \theta_i \cos \theta_k}}\right) V(\omega_i) F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) \frac{\cos^2 \theta_i}{\cos^2 \theta_d} \ \mathrm{d}\phi_i \mathrm{d}\theta_i \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left(\alpha F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) \frac{\cos^2 \theta_i}{\cos^2 \theta_d}\right) g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos \theta_i \cos \theta_k}}\right) V(\omega_i) \ \mathrm{d}\phi_i \mathrm{d}\theta_i. \end{aligned}$$

Let us define $C(\theta_i, \phi_i) = \alpha F_r(\eta, \theta_d, \phi_d) \cos(\theta_d/2) \frac{\cos^2 \theta_i}{\cos^2 \theta_d}$. The above expression simplifies to:

$$I_s \approx \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} C(\theta_i, \phi_i) g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos \theta_i \cos \theta_k}}\right) V(\omega_i) \ \mathrm{d}\phi_i \mathrm{d}\theta_i.$$

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Because the functions that make up $C(\theta_i, \phi_i)$ changes slowly relative to the Gaussians, the paper approximates it as a constant $C(\theta_l, \phi_k)$, which is at the peak of the two Gaussians in the integrand. We now have:

$$I_s \approx C(\theta_l, \phi_k) \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) V(\omega_i) \, \mathrm{d}\phi_i \mathrm{d}\theta_i.$$

It remains to approximate the remaining integral.

- To make approximation of the integral easier, the Iwasaki et al. paper represents the visibility function $V(\omega_i)$ with the following signed distance function:

$$d(\theta_i, \phi_i) = \begin{cases} + \min_{V(\theta,\phi)=0} \sqrt{(\theta_i - \theta)^2 + (\phi_i - \phi)^2} & \text{if } V(\theta_i, \phi_i) = 1 \\ - \min_{V(\theta,\phi)=1} \sqrt{(\theta_i - \theta)^2 + (\phi_i - \phi)^2} & \text{if } V(\theta_i, \phi_i) = 0 \end{cases}.$$

I think this definition is problematic because it relies on a coordinate frame defined at each tangent vector. Do you have to store a separate d for each tangent vector $\mathbf{t}$? Can you define this in a way that is not specific to a particular coordinate system?

- With the distance function d defined, let $d'_{\theta_l, \phi_k}(\theta_i, \phi_i)$ denote the first-order Taylor expansion of d around (θ_l, ϕ_k):

$$d'_{\theta_l, \phi_k}(\theta_i, \phi_i) = d(\theta_l, \phi_k) + (\theta_i - \theta_l) \frac{\partial d}{\partial \theta_i}\bigg|_{\theta_l, \phi_k} + (\phi_i - \phi_k) \frac{\partial d}{\partial \phi_i}\bigg|_{\theta_l, \phi_k}$$

- The paper then approximates V with the following function:

$$V'_{\theta_l, \phi_k}(\theta_i, \phi_i) = \begin{cases} 1, & \text{if } d'_{\theta_l, \phi_k}(\theta_i, \phi_i) > 0 \\ 0, & \text{if } d'_{\theta_l, \phi_k}(\theta_i, \phi_i) < 0 \end{cases}.$$

- The integral involving visibility function is then approximated as:

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) V(\omega_i) \, \mathrm{d}\phi_i \mathrm{d}\theta_i$$

$$\approx \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) V'_{\theta_l, \phi_k}(\theta_i, \phi_i) \, \mathrm{d}\phi_i \mathrm{d}\theta_i.$$

- The above is still hard to approximate, so V' is factored out:

$$I \approx \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) \mathrm{d}\phi_i \mathrm{d}\theta_i$$

$$\times \frac{\int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) V'_{\theta_l, \phi_k}(\theta_i + \theta_l, \phi_i + \phi_k) \, \mathrm{d}\phi_i \mathrm{d}\theta_i}{\int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) \mathrm{d}\phi_i \mathrm{d}\theta_i}$$

- The first term is approximate as:

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\pi} g^c(\theta_i; \theta_l, \sigma_l) g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_i \cos\theta_k}}\right) \mathrm{d}\phi_i \mathrm{d}\theta_i$$

$$\approx \left(\int_{-\pi/2}^{\pi/2} g^c(\theta_i; \theta_l, \sigma_l) \, \mathrm{d}\theta_i\right) \left[\int_0^{2\pi} g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_l \cos\theta_k}}\right) \mathrm{d}\phi_i\right].$$

The left term in the product is approximated as:

$$\int_{-\pi/2}^{\pi/2} g^c(\theta_i; \theta_l, \sigma_l) \, \mathrm{d}\theta_i = \int_{-\pi/2}^{\pi/2} g(\theta_i; \theta_l, \sigma_l) \, \mathrm{d}\theta_i$$

$$= \frac{\sqrt{\pi}\sigma_l}{2} \left[\mathrm{erf}\left(\frac{\pi/2 - \theta_l}{\sigma_l} \right) - \mathrm{erf}\left(\frac{-\pi/2 - \theta_l}{\sigma_l} \right) \right].$$

The right term can be computed exactly. From the von Mises distribution, we know that

$$\int_0^{2\pi} \exp(\kappa \cos(x - \mu)) = 2\pi I_0(\kappa)$$

where I_0 is the modified Bessel function of order 0. So,

$$\int_0^{2\pi} g^c\left(\phi_i; \phi_k, \frac{\sigma_k}{\sqrt{\cos\theta_l \cos\theta_k}} \right) \, \mathrm{d}\phi_i$$

$$= \int_0^{2\pi} \exp\left(\frac{2\cos\theta_l \cos\theta_k}{\sigma_k^2} (\cos(\phi_i - \phi_k) - 1) \right) \, \mathrm{d}\phi_i$$

$$= \exp\left(-\frac{2\cos\theta_l \cos\theta_k}{\sigma_k^2} \right) \int_0^{2\pi} \exp\left(\frac{2\cos\theta_l \cos\theta_k}{\sigma_k^2} \cos(\phi_i - \phi_k) \right) \, \mathrm{d}\phi_i$$

$$= \exp\left(-\frac{2\cos\theta_l \cos\theta_k}{\sigma_k^2} \right) 2\pi I_0\left(\frac{2\cos\theta_l \cos\theta_k}{\sigma_k^2} \right).$$

- The second term is problematic. I just felt that the paper was written in a way that prevents comprehension.

  - First, why is V' even introduced in the first place? You see that $V'_{\theta_l, \phi_k}(\theta_i, \phi_i)$ depends on $d'_{\theta_l, \phi_k}(\theta_i, \phi_i)$. So, we have that $d'_{\theta_l, \phi_k}(\theta_l + \theta_i, \phi_k + \phi_i) = d(\theta_l, \phi_k)$, and $V'_{\theta_l, \phi_k}(\theta_i, \phi_i) = V(\theta_l, \phi_k)$, which is a constant! The value of the integral would be reduced to 1.
  - Second, they try to approximate the integral ratio, which is a function of σ_l, $\sigma'_k = \frac{\sigma_k}{\sqrt{\cos\theta_l \cos\theta_k}}$, d, and ∇d. They say this is a 4D function, but d is a 2D function! The type doesn't check!

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