Differential Geometry Notes of 03/03/2013

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April 1, 2013

1 Self-Adjoint Linear Maps and Quadratic Forms

- Let V denote a real vector space of dimension 2 endowed with an inner product $\langle \ , \ \rangle$.
- We say that a linear map $A : V \to V$ is **self-adjoint** if $\langle Av, w \rangle = \langle v, Aw \rangle$ for all $v, w \in V$.
- If $\{e_1, e_2\}$ is an orthonormal basis for V and a_{ij} for $i, j = 1, 2$ is the matrix for A in this basis, then

  $$a_{ji} = \langle Ae_i, e_j \rangle = \langle e_i, Ae_j \rangle = a_{ij}.$$

  So, if A is self-adjoint, then it is represented by a symmetric matrix in any orthonormal basis.
- To each self-adjoint linear map, we associate a map $B : V \times V \to \mathbb{R}$ defined by

  $$B(v, w) = \langle Av, \rangle.$$

  We have that B is bilinear; that is, it is linear in both v and w.
  Moreover, the fact that A is self-adjoint means that $B(v, w) = B(w, v)$.
  So, B is a symmetric bilinear form.
- If B is a symmetric bilinear form in V, we can define a linear map $A : V \to V$ by $\langle Av, w \langle = B(v, w)$.
  (That is, you can get the coefficients of the matrix of A by computing $B(e_i, e_j)$)
  Because B is symmetric, it implies that A is self-adjoint.
- A quadratic form is a polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ such that

  $$Q(x, y) = ax^2 + 2bxy + cy^2$$

  for some $a, b, c \in Real^2$.
- For each symmetric bilinear form, B in V, there corresponds a quadratic form Q in V given by:

  $$Q(v) = B(v, v).$$

- Q determines B completely because

  $$\begin{aligned} B(v + w, v + w) &= B(v + w, v) + B(v + w, w) \\ &= B(v, v) + B(w, v) + B(v, w) + B(w, w) \\ &= B(v, v) + 2B(v, w) + B(w, w). \end{aligned}$$

  So,

  $$B(v, w) = \frac{1}{2}\big(Q(v + w) - Q(v) - Q(w)\big).$$

- As a result, there's a one-to-one correponsdence between quadratic forms and self-adjoint linear maps in V.

- Given a self-adjoint linear map $A : V \to V$, there exists an orthonormal basis for V such that, relative to the basis, the matrix of A is a diagonal matrix.

  Furthermore, the elements of the diagonal are the maximum and the minimum of the corresponding quadratic form restricted to the unit circle of V.

- **Lemma 1.1.** *If the function $Q(x, y) = ax^2 + 2bxy + cy^2$, restricted to the unit circle $x^2 + y^2 = 1$, as a maximum at the point $(1, 0)$, then $b = 0$.*

  Proof. Parametermize the cirlce $x^2 + y^2 = 1$ by $x = \cos t$ and $y = \sin t$. Write Q as a function of t. Differentiate and set equal to 0. Substitute $t = 0$, and you'll get $b = 0$. □

- **Proposition 1.2.** *Given a quadratic form Q in V, there exists an orthonormal basis $\{e_1, e_2\}$ of V such that, if $v \in V$ is given by $v = xe_1 + ye_2$, then*

  $$Q(v) = \lambda_1 x^2 + \lambda_2 y^2$$

  where λ_1 and λ_2 are the maximum and minimum, respectively, of Q on the unit circle $|v| = 1$.

  Proof. Let λ_1 be the maximum of Q on the unit circle $|v| = 1$, and let e_1 be the unit vector with $Q(e_1) = v$. Such e_1 exists by continuity of Q on the compact set $|v| = 1$. Let e_2 be a unit vector orthogonal to e_1 and set $\lambda_2 = Q(e_2)$. We will show that the basis $\{e_1, e_2\}$ satisfies the conditions of the proposition.

  Let B be the symmetric bilinear form that is associated to Q. Let $v = xe_1 + ye_2$.

  $$\begin{aligned} Q(v) = B(v, v) &= B(xe_1 + ye_2, xe_1 + ye_2) \\ &= B(xe_1, xe_1) + 2B(xe_1, ye_2) + B(ye_2, y_{e2}) \\ &= x^2 B(e_1, e_1) + 2xyB(e_1, e_2) + y^2 B(e_2, e_2) \\ &= \lambda_1 x^2 + 2bxy + \lambda_2 y^2 \end{aligned}$$

  where $b = B(e_1, e_2)$. By the lemma, $b = 0$, and it only remains to prove that λ_2 is the minimum of Q in the circle $|v| = 1$. However,

  $$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 \geq \lambda_2(x^2 + y^2) = \lambda_2$$

  since $\lambda_2 \leq 1$. So, λ_2 is the minimum. □

- **Theorem 1.3.** *Let $A : V \to V$ be a self-adjoint linear map. Then, there exists an orthonormal basis $\{e_1, e_2\}$ of V such that $A(e_1) = \lambda_1 e_1, A(e_2) = \lambda_2 e_2$. In the basis $\{e_1, e_2\}$, the matrix of A is clearly diagonal and the elements λ_1, λ_2 with $\lambda_1 \geq \lambda_2$ on the diagonals are the maximum and the minimum, respectively, of the quadratic form $Q(v) = \langle Av, v \rangle$.*

  Proof. Consider the quadratic form $Q(v) = \langle Av, v \rangle$. By the proposition above, there exists an orthonormal basis $\{e_1, e_2\}$ of V with $Q(e_1) = \lambda_1$, and $Q(e_2) = \lambda_2 \leq \lambda_1$, where λ_1 and λ_2 are the maximum and the minimum, respectively, of Q in the unit circle. It remains to show that $A(e_1) = \lambda_1 e_1$ and $A(e_2) = \lambda_2 e_2$.

  Since $B(e_1, e_2) = \langle Ae_1, e_2 \langle = 0$ (by the lemma) and $e_2 \neq \mathbf{0}$, we have that either Ae_1 is parallel to e_1 or $Ae_1 = 0$. If Ae_1 is parallel to e_1, then $Ae_1 = \alpha e_1$, and since $\langle Ae_1, e_1 \rangle = \lambda_1 = \lambda \alpha e_1, e_1 = \alpha$, we conclude thet $Ae_1 = \lambda_1 e_1$. If $Ae_1 = \mathbf{0}$, then ${}_1 = \langle Ae_1, e_1 \rangle = 0$. So, $Ae_1 = \mathbf{0} = \lambda_1 e_1$. In any case, we have that $Ae_1 = \lambda_1 e_1$.

  Using the same argument, we can show that $Ae_2 = \lambda_2 e_2$. □