Differential Geometry Notes of 05/05/2013

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1 Complete Surfaces

- We will concern ourselves with regular and connected surfaces, except when otherwise stated.
- **Definition 1.1.** *A regular (connected) surface S is said to be* **extendable** *if there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$ as a proper subset. If there exists no such $\bar{S}$, we say S is* **nonextendable**.
- **Definition 1.2.** *A regular surface S is said to be* **complete** *when, for every point $p \in S$, any parametermized geodesic $\gamma : [0, \epsilon) \rightarrow S$, starting from $p = \gamma(0)$, may be extended into a parametermized geodesic $\bar{\gamma} : \mathbb{R} \rightarrow S$, defined on the entire line $\mathbb{R}$.*
- Every complete surface is nonextendable.
  However, there exist nonextendable surfaces which are not complete.
  Every closed surface in $\mathbb{R}^3$ is complete.

  So, nonextendable is weaker than complete, and complete is weaker than compact.
- Examples
  - The plane is a complete surface.
  - The cone minus the vertex is not a complete surface. Extending the generator (which is a geodesic) sufficiently will reach the vertex, which does not belong to the surface.
  - A sphere is a complete surface, since its parametermized geodesic may be defined for every real value.
  - The cylinder is a complete surface. Its geodesics are circles, lines, and helices, which are defined for all real values.
  - The surface $S - \{p\}$ obtained by removing a point $\{p\}$ for a complete surface is not complete. By taking a point q near p, there exists a parametermized geodesic of $S - \{p\}$ that starts from q that cannot be extended through p.
- **Proposition 1.3.** *A complete surface S is nonextendable.*

  Proof. By way of contradiction, let us assume that S is extendable. Since S is extendable, there is a regular (connected) surface $\bar{S}$ with $S \subset \bar{S}$. Since S is a regular surface, S is open in $\bar{S}$. (For every point $p \in S$, there exists an open coordinate neighborhood in $S \subset \bar{S}$.) The boundary $\mathrm{Bd}\, S$ of S in $\bar{S}$ is nonempty; otherwise, $\bar{S} = S \cup (\bar{S} - S)$ would be the union of two disjoint sets S and $\bar{S} - S$, which contradicts the connectedness of $\bar{S}$. Therefore, there exists a point $p \in \mathrm{Bd}\, S$, and since S is open in $\bar{S}$, $p \notin S$.

  Let $\bar{V} \subseteq \bar{S}$ be a neighborhood of p in $\bar{S}$ such that every $q \in \bar{V}$ may be joined to p by a unique geodesic of $\bar{S}$. (The normal neighborhood is such a neighborhood.) Since $p \in \mathrm{Bd}\, S$, some $q_0 \in V$ belongs to S.

Let $\bar{\gamma} : [0, 1] \to \bar{S}$ be a geodesic of $\bar{S}$, with $\bar{\gamma}(0) = p$ and $\bar{\gamma}(1) = q_0$. It is clear that $\alpha : [0, \epsilon] \to \bar{S}$, given by $\alpha(t) = \bar{\gamma}(1 - t)$, is a geodesic of S, with $\alpha(0) = q_0$, the extension of which to the line $\mathbb{R}$ would pass through p for $t = 1$. Since $p \notin S$, the geodesic cannot be extended. Contradiction. □

- The one-sheeted cone $S = \{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}\}$ is not a complete surface.

  We will argue that it is also nonextendable. By way of contradiction, assume that there exists a regular (connected) surface $\bar{S}$ with $S \subset \bar{S}$. We will show that the boundary of S in $\bar{S}$ reduces to the vertex p_0, and there exists a neighborhood $\bar{W}$ of p_0 in $\bar{S}$ such that $\bar{W} - \{p_0\} \subseteq S$. However, this contradicts the fact that cone (vertex p_0) included is not a regular surface.

  First, we observe that the only geodesic of S, starting from a point $p \in S$ that cannot be extended for every value of the parameter is the meridian that passes through p. This may be seen by using Clairau's relation.

  Let $p \in \operatorname{Bd} S$. We know that $p \neq S$ because S is open in $\bar{S}$. Let $\bar{V}$ be a neighborhood of p in $\bar{S}$ such that every point $\bar{V}$ may be joined to p by a unique geodesic of $\bar{S}$ in $\bar{V}$. Since $p \in \operatorname{Bd} S$, there exist $q \in \bar{V} \cap S$. Let $\bar{\gamma}$ be a geodesic of $\bar{S}$ joining p and q. Because S is open in $\bar{S}$, $\bar{\gamma}$ agrees with a geodesic γ of S in a neighborhood of q. Let p_0 be the first point of $\bar{\gamma}$ that does not belong to S. By the initial observation, $\bar{\gamma}$ is a meridian and p_0 is a vertex of S. Furthermore, $p_0 = p$; otherwise, there would exist a neighborhood of p that does not contain p_0. By repeating the argument for the neighborhood, we obtain a vertex diffferent from p_0, which is a contradiction. It follows that $\operatorname{Bd} S$ reduces to the vertex p_0.

  Let $\bar{W}$ be a neighborhood of p_0 in $\bar{S}$ such that any two points of $\bar{W}$ may be joined by a geodesic of $\bar{S}$. We shall prove that $\bar{W} - \{p_0\} \subseteq S$. In fact, the points of γ belong to S. On the other hand, a point $r \in \bar{W}$ such does not belong to γ or to its extension may be joined to a point t of γ, $t \neq p_0$, $t \subseteq \bar{W}$, by a geodesic α, differentent from γ. By the initial observation, eveyr point of α, in partcular r, belongs to S. Finally, the points of the extension of γ, except p_0, also belong to S; otherwise, they would belong to the boundary of S, which we have proved to be made up of only p_0.

2 Intrinsic Distance

- A continuous mapping $\alpha : [a, b] \to S$ of a closed interval $[a, b] \subseteq \mathbb{R}$ onto the surface S is said to be a **parametermized, piecewise differentiable curve** joining $\alpha(a)$ to $\alpha(b)$ if there exists a partition of $[a, b]$ by points $a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$ such that α is differentiable in $[t_i, t_{i+1}]$ for all i.

  The length $l(\alpha)$ of α is defined as

$$l(\alpha) = \sum_{t=0}^{k} \int_{t_i}^{t_{i+1}} |\alpha'(t)| \, \mathrm{d}t$$

- **Proposition 2.1.** *Given two points $p, q \in S$ of a regular (connected) surface S, there exists a parametermized piecewise differentiable curve jointing p to q.*

  Proof. Because S is connected, there exists a curve $\alpha : [a, b] \to S$ such that $\alpha(a) = p$ and $\alpha(b) = b$. It is possible to divide the intervals $[a, b]$ into a finite number of intervals so that the image of each interval is contained in a coordinate neighborhood. For each interval, it is possible to connect the first point to the last point with a differentiable curve. □

- Let $p, q \in S$ be two points of a regular surface S. We denote by $\alpha_{p,q}$ a parametermized piecewise differentiable curve joining p to q. Let $l(\alpha_{p,q}$ denote its length.

- **Definition 2.2.** *The* **(intrinsic) distance** $d(p,q)$ *from the point* $p \in S$ *to the point* $q \in S$ *is the number:*

$$d(p,q) = \inf_{\alpha_{p,q}} l(\alpha_{p,q}).$$

Here, the infemum is taken over all the parametermized piecewise differentiable curve connecting p *to* q*.*

- **Proposition 2.3.** d *is a metric. That is,*

  1. $d(p,q) = d(q,p)$

  2. $d(p,q) + d(q,r) \geq d(p,r)$

  3. $d(p,q) \geq 0$

  4. $d(p,q) = 0$ *if and only if* $p = q$

  where p, q, r *are arbitrary points on* S.

  Proof. For Property 1, for every curve connecting p to q, there exists a curve connecting q to p with the same length.

  For Property 2, $d(p,q) + d(q,r)$ is the infemum of the set A of curves that goes from p to r passing through q. We have that $A \subseteq B$ where B is the set of curves connecting p to r. Because, if $A \subseteq B$, $\inf A \geq \inf B$, we have that $d(p,q) + d(q,r) \geq d(p,r)$.

  Property 3 follows from the fact that the length of a curve is never less than 0.

  Let us now prove Property 4. If $p = q$, then we can take the constant curve, which has length 0.

  For the converse, suppose for contradiction that $\inf l(\alpha_{p,q}) = 0$, but $p \neq q$. Let V be a neighborhood of p in S with $q \neq V$, such that every point of V may be joint to p by a unique geodesic in V. Let $B_r(p) \subseteq V$ be the region bounded by a geodesic circle of radius r centered at p and contained in V. By the definition of infimum, given $\epsilon > 0$ and $0 < \epsilon < r$, there exist a parametermized, piecewise differentiable curve $\alpha : [a,b] \to S$ joining p to q with $l(\alpha) < \epsilon$. Since $\alpha([a,b])$ is connected and $q \neq B_r(p)$, there exists a point $t_0 \in [a,b]$ such that $\alpha(t_0)$ belongs to the boundary of $B_r(p)$. It follows that $l(\alpha) \geq r > e$, which is a contradiction. ☐

- **Corollary 2.4.** $|d(p,r) - d(r,q)| \leq d(p,q)$

  Proof. We have that

$$\begin{aligned} d(p,r) &\leq d(p,q) + d(q,r) \\ d(r,q) &\leq d(r,p) + d(p,q) \end{aligned}$$

  Therefore,

$$\begin{aligned} d(p,r) - d(q,r) &\leq d(p,q) \\ -d(p,q) &\leq d(r,p) - d(r,q). \end{aligned}$$

  In other words,

$$-d(p,q) \leq d(p,r) - d(r,q) \leq d(p,q).$$

- **Proposition 2.5.** *If we let $p_0 \in S$ be a fixed point, then the function $f : S \to \mathbb{R}$ given by $f(p) = d(p_0, p)$ is continuous on S.*

  Proof. We have to show that for each p in S, given $\epsilon > 0$, there exists $\delta > 0$ such that if $q \in B_\delta(p) \cup S$, then $|f(p) - f(q)| = |d(p_0, p) - d(p_0, q)| < \epsilon$.

  Let $\epsilon' < \epsilon$ be such that the exponential map $\exp_p$ is a diffeomorphism in the disc $B_{\epsilon'}(\mathbf{0}) \subseteq T_p(S)$. Set $V = \exp_p(B_{\epsilon'}(\mathbf{0}))$. Clearly, V is an open set in S; hence, there exists an open ball $B_\delta(p) \in \mathbb{R}^3$ such that $B_\delta(p) \cap S \subseteq V$. Thus, if $q \in B_\delta(p) \cap S$, we have that

  $$|d(p_0, p) - d(p_0, q)| \leq d(p, q) \leq \epsilon' < \epsilon$$

  as required. □

- **Proposition 2.6.** *A closed surface $S \subseteq \mathbb{R}^3$ is complete.*

  Proof. Let $\gamma : [0, \epsilon) \to S$ be a parametermized geodesic of S with $\gamma(0) = p$. WLOG, let us assume that γ is parameterized with arc length. We need to show that it is possible to extend γ to a geodesic $\bar{\gamma} : \mathbb{R} \to S$, defined on the entire real line.

  Observe first that when $\bar{\gamma}(s_0)$, $s_0 \in \mathbb{R}$, is defined, then, by the theorem of existence and uniqueness of geodesics, it is possible to extend γ to a neighborhood of s_0 in $\mathbb{R}$. As a result, the set of all $s \in \mathbb{R}$ where $\bar{\gamma}$ is defined is open in $\mathbb{R}$. If we can prove that this set is closed in $\mathbb{R}$ (which is connected), it will be possible to define $\bar{\gamma}$ for all of $\mathbb{R}$, and the proof will be completed.

  Let us assume that $\bar{\gamma}$ is defined for $s < s_0$ and let us show that $\bar{\gamma}$ is defined for $s = s_0$. Consider a sequence $\{s_n\} \to s_0$ with $s_n < s_0$ for all n.

  We shall first prove that the sequence $\{\bar{\gamma}(s_n)\}$ converges in S. Because $\{s_n\}$ is convergent, it is Cauchy. So, given $\epsilon = 0$, there exists n_0 such that if $n, m > n_0$, then $|s_n - s_m| < \epsilon$. Denote by $\bar{d}$ the Euclidean distance in $\mathbb{R}^3$. Observe that, if $p, q \in S$, then $\bar{d}(p, q) < d(p, q)$. Thus,

  $$\bar{d}(\bar{\gamma}(s_n), \bar{\gamma}(s_m)) \leq d(\gamma(s_n), \gamma(s_m)) = |s_n - s_m| < \epsilon.$$

  If follows that $\{\bar{\gamma}(s_n)\}$ is a Cauchy sequence in $\mathbb{R}^3$. Therefore, it converges to a point $q \in \mathbb{R}^3$. Since q is a limit point of $\{\bar{\gamma}(s_n)\}$ and S is closed, $q \in S$.

  Let W and δ be the neighborhood of q and the number such that, for every point $r \in W$, $\exp_r$ is a diffeomorphism in $B_\delta(\mathbf{0}) \in T_r(S)$ and $W \subseteq \exp_r(B_\delta(\mathbf{0}))$. Let $\bar{\gamma}(s_n), \bar{\gamma}(s_m) \in W$ be points such that $|s_n - s_m| < \delta$, and let γ be the unique geodesic with $l(\gamma) < \delta$ joining $\bar{\gamma}(s_n)$ and $\bar{\gamma}(s_m)$. Clearly, $\bar{\gamma}$ agrees with γ. Since $\exp_{\bar{\gamma}(s_n)}$ is a diffeomorphism in $B_\delta(\mathbf{0})$ and $W \subseteq \exp_{\bar{\gamma}(s_n)}(B_\delta(\mathbf{0}))$, it follows that γ extends $\bar{\gamma}$ beyond q. Thus, $\bar{\gamma}$ is defined at $s = s_0$, which completes the proof. □

- **Corollary 2.7.** *A compact surface is complete.*

- A complete surface need not be closed.

3 Theorem of Hopf–Rinow

- A geodesic γ joining two points $p, q \in S$ is **minimal** if its length $l(\gamma)$ is smaller than or equal to the length of any piecewise regular curve joining p to q.

  This is equivalent to saying that $l(\gamma) = d(p, q)$ because, for any given piecewise differentiable curve α joining p to q, we can find a piecewise regular curve joining p to q that is not longer than α.

- **Theorem 3.1 (Hopf–Rinow).** *Let S be a complete surface. Given two points $p, q \in S$, there exists a minimal geodesic joining p to q.*

Proof. Let $r = d(p, q)$ be the distance between the points p and q. Let $B_\delta(\mathbf{0}) \subseteq T_p(S)$ be a disk of radius δ, centered in the origin $\mathbf{0}$ of the tangent plane $T_p(S)$ and contained in a neighborhood $U \subseteq T_p(S)$ of $\mathbf{0}$, where $\exp_p$ is a diffeomorphism. Let $B_\delta(p) = \exp_p(B_\delta(\mathbf{0}))$. Observe that the boundary $\mathrm{Bd}\, B_\delta(p) = \Sigma$ is compact since it is a continuous image of the compact set $\mathrm{Bd}\, B_\delta(\mathbf{0}) \subseteq T_p(S)$.

If $x \in \Sigma$, the continous function $d(x, q)$ reaches a minimum at a point x_0 of the compact set Σ. The point x_0 may be written as $x_0 = \exp_p(\delta v)$ where $|v| = 1$ and $v \in T_p(S)$.

Let γ be a geodesic paraetermized by arc length, given by $\gamma(s) = \exp_p(sv)$. Since S is complete, γ is defined for eveyr $s \in \mathbb{R}$. In particular, γ is defined in the interval $[0, r]$. If we show that $\gamma(r) = q$, then γ is the geodesic joining p to q, which is minimal, since $l(\gamma) = r = d(p, q)$.

To prove this, we shall show that, if $s \in [\delta, r]$, then $d(\gamma(s), q) = r - s$. The equation implies that $d(\gamma(r), q) = 0$, which means $\gamma(r) = q$.

We shall first show that the equation holds for $s = \delta$. If this is true, then the set

$$A = \{s \in [\delta, r] : d(\gamma(s), q) = r - s\}$$

is not empty. Moreover, it is closed. Next, we show that if $s_0 \in A$ and $s_0 < r$, then the equation holds for $s_0 + \delta' > 0$ and δ' is sufficiently small. It follows that $A = [\delta, r]$.

We now show that the equation holds for $s = \delta$. Because every curve joining p to q intersects Σ, we have, denoting by x an arbitrary point of Σ,

$$\begin{aligned} d(p, q) &= \inf_{\alpha_{p,q}} l(\alpha_{p,q}) = \inf_{x \in \Sigma} \{inf_{\alpha_{p,x}} l(\alpha_{p,x}) + \inf_{\alpha_{x,q}} l(\alpha_{x,q})\} \\ &= \inf_{x \in \Sigma} (d(p, x) + d(q, x)) = \inf_{x \in \Sigma} (\delta + d(x, q)) \\ &= \delta + d(x_0, q). \end{aligned}$$

In other words,

$$d(\gamma(\delta), q) = r - \delta.$$

Now, we shall show that if the equation holds for $s_0 \in [\delta, r]$, then, for sufficiently small $\delta' > 0$, it holds for $s_0 + \delta'$.

Let $B_{\delta'}(\mathbf{0})$ be a disk in the tangent plane $T_{\gamma(s_0)}(S)$, centered in the origin $\mathbf{0}$ of this tangent plane and contained in a neighborhood U' where $\exp_{\gamma(s_0)}$ is a diffeomorphism. Let $B_{\delta'}(\gamma(s_0)) = \exp_{\gamma(s_0)}(B_{\delta'}(\mathbf{0}))$ and $\Sigma' = \mathrm{Bd}\, B_{\delta'}(\gamma(s_0))$. If $x' \in \Sigma'$, the continuous function $d(x', q)$ reachees a minimum at $x'_0 \in \Sigma'$. Then, as argued previously,

$$d(\gamma(s_0), q) = \inf_{x' \in \Sigma'} \{d(\gamma(s_0), x') + d(x', q)\} = \delta' + d(x'_0, q).$$

Since the equation holds at s_0, we have that $d(\gamma(s_0), q) = r - s_0$. Therefore, $d(x'_0, q) = r - s_0 - \delta'$. Furthermore, since

$$d(p, x'_0) \geq d(p, q) - d(q, x'_0),$$

we have that

$$d(p, x'_0) \geq r - (r - s_0 - \delta') = s_0 + \delta'.$$

We see that the curve that goes from p to $\gamma(s_0)$ through γ, and then from $\gamma(s_0)$ to x'_0 through a geodesic radius of $B_{\delta'}(\gamma(s_0))$ has length exactly equal to $s_0 + \gamma'$. Since $d(p, x'_0) \geq s_0 + \delta'$, this curve, which joins p to x'_0 has minimal length. It follows that it is a geodesic, and hence regular in all its point. Therefore, it should coincide with γ; hence, $x'_0 = \gamma(s + \gamma')$. Thus, we can write:

$$d(\gamma(s_0 + \delta'), q) = r - (s_0 + \delta')$$

which means that the equations holds for $s_0 + \delta'$. $\square$

- **Corollary 3.2.** *Let S be complete. Then, for every point $p \in S$, the map $\exp_p : T_p(S) \to S$ is onto S.*

  Proof. If $q \in S$ and $d(p, q) = r$, then $q = \exp_p(rv)$ where $v = \gamma'(0)$ is the tangent vector of a minimal geodesic parametermized by arc length joining p to q. $\square$

- **Corollary 3.3.** *Let S be complete and bounded in the matrix d (that is, there exists $r > 0$ such that $d(p, q) < r$ for every $p, q \in S$). Then S is compact.*

  Proof. By fixing $p \in S$, the fact that S is bounded implies the existence of a closed ball $B \subseteq T_p(S)$ of radius r, centered at the origin $\mathbf{0}$ of the tangent plane $T_p(S)$ such that $\exp_p(B) = \exp_p(T_p(S))$. By the fact that $\exp_p$ is onto, we have that $S = \exp_p(B)$. Since B is compact and $\exp_p$ is continuous, we conclude that S is compact. $\square$

- The diameter of the surface S, denoted by $\rho(S)$, is defined as:

$$\rho(S) = \sup_{p,q \in S} d(p, q).$$

- The diameter of the unit sphere S^2 is $\rho(S^2) = \pi$.