Polarization

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This document is written as I read a small booklet called "Field Guide to Polarization" by Edward Collett. This is done so that I take the time to remember the symbols and terms.

1 Foundations of Polarized Light

- In 1670, *Bartholinus* discovered that wne a single ray of natural incident light propagated through a rhombohedral calcite crystal, to rays emerged. This demonstrated that a single ray of light actually consists of two rays.
- Because the two rays refract at different angles, calcite crystal is said to be double refractive or **birefringent**. That is, the two rays experience different refractive indices.
- *Huygens* put another crystal to receive the two rays.
  - He rotated the second crytal and found that he could get the intensity of one ray to maximize and the other to vanish.
  - Rotating 90° from that angle, the second ray's intensity is maximized while the first ray's intensity vanishes.
  - At 45°, the intensities of the two rays are equal.
- Because the opposite behavior with respect to rotation, the two rays are said to be **polarized**. The two polarization states are called the **s-** and **p-polarization** states. The p and s notation from the German words for parallel (paralelle) and perpendicular (senkrecht).
- In 1808, *Malus* made the following discoveries and proposals:
  - Natural incident light became polarized when it was reflected by a glass surface.
  - The light reflected close to the incident angle of 57° is extinguished when viewed through a calcite crystal.
  - Malus proposed that the s- and p-polarization are *perpendicular* to each other.
  - The intensity of the reflected light varied from a maximum to a mininum as the crystal was rotated.
  - Malus poposed that the amplitude of the reflected beam must be

    $$A = A_0 \cos\theta.$$

  - However, intensity is given by the square of the amplitude. Hence, the intensity is given by:

    $$I(\theta) = I_0 \cos^2\theta$$

    where $I_0 = A_0^2$. This is called **Malus's law**.

- In 1812, *Brester* made the following discoveries and proposals
  - For different glasses, after passing the reflected ray through an analyzing calcite crystal, the p-polarized ray vanishes completely when the incident light's angle is at a particular angle i.
  - By rotating the analyzing calcite crystal through $90°$, the s-polarized ray also vanishes.
  - The refracted ra angle r is simply related to the angle i by:

    $$i + r = 90°.$$

  - By Snell's law:

    $$\begin{aligned} n_1 \sin i &= n_2 \sin r \\ n_1 \sin i &= n_2 \sin(90° - i) \\ n_1 \sin i &= n_2 \cos i \\ \tan i &= \frac{n_2}{n_1} = n. \end{aligned}$$

    This equation is known as **Brewster's law**, and the angle i is called the **Brewster's angle**.
  - Brewster's law allows the index of refraction of a glass to be determined by reflection rather than refraction.

2 The Wave Theory of Light

- In 1820, *Fresnel* came up with **Fresnel's wave theory**.
  - The opticla field consisted of only two orthogonal components in the plane transverse to the direction of propagation.
  - The field components are described by the following two **wave equations**:

    $$\begin{aligned} \nabla^2 E_x(\mathbf{r}, t) &= \frac{1}{v^2}\frac{\partial^2 E_x(\mathbf{r}, t)}{\partial t^2} \\ \nabla^2 E_y(\mathbf{r}, t) &= \frac{1}{v^2}\frac{\partial^2 E_y(\mathbf{r}, t)}{\partial t^2} \end{aligned}$$

    where
    * $E_x(\mathbf{r}, t)$ and $E_y(\mathbf{r}, t)$ are the optical-field components,
    * $\mathbf{r}$ is a position in space,
    * t is the time, and
    * v is the velocity of the weave.
  - The solutions of the wave equations are:

    $$\begin{aligned} E_x(\mathbf{r}, t) &= E_{0x} \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_x) \\ E_y(\mathbf{r}, t) &= E_{0y} \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_y) \end{aligned}$$

    where
    * $\mathbf{k}$ is the (vector) wave number and describes the direction of propagation,
    * $\omega = 2\pi f$ is the angular frequency, and
    * δ_x and δ_y are arbitrary phase offsets.

    The vector $\mathbf{k}$ is perpendicular to the plane containing E_x and E_y.

– In practice, we often assume that the wave propagates along the z-direction, i.e., $\mathbf{k} = (0, 0, k)$. Here, $k = 2\pi/\lambda$ is the wave number magnitude. So, the solutions become:

$$E_x(z,t) = E_{0x}\cos(\omega t - kz + \delta_x)$$
$$E_y(z,t) = E_{0y}\cos(\omega t - kz + \delta_y).$$

The term $\omega t - kz$ is called the **propagator**.

– We may also express the components with the following complex functions:

$$\mathcal{E}_x(z,t) = E_{0x}\exp(i(\omega t - kz + \delta_x)),$$
$$\mathcal{E}_y(z,t) = E_{0y}\exp(i(\omega t - kz + \delta_y)).$$

Using the functions, we have that

$$E_x(z,t) = \Re\{\mathcal{E}_x(z,t)\},$$
$$E_y(z,t) = \Re\{\mathcal{E}_y(z,t)\}.$$

– The E_y component is called the p-polarization component, and E_x is called the s-polarization component.

- The two components of the light field satisfies the following equation of an ellipse, called the **polarization ellipse**:

$$\frac{E_x(z,t)^2}{E_{0x}^2} + \frac{E_y(z,t)^2}{E_{0y}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta$$

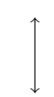
where $\delta = \delta_y - \delta_x$. The polarization ellipse remains constant as the polarized beam propagate.

- There are several combinations of amplitude and phase that are especially important, and they are called **degenerate polarization states**.

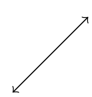
– **Linearly horizontal polarized light (LHP)**. $E_{0y} = 0$.

$$\longleftrightarrow$$

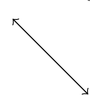
– **Linearly vertical polarized light (LVP)**. $E_{0x} = 0$.



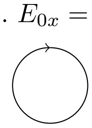
– **Linear $+45°$ polarized light (L+45P)**. $E_{0x} = E_{0y} = E_0$ and $\delta = 0$.



– **Linear $-45°$ polarized light (L-45P)**. $E_{0x} = E_{0y} = E_0$ and $\delta = \pi$.

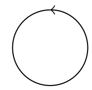


– **Right circularly polarized light (RCP)**. $E_{0x} = E_{0y} = E_0$ and $\delta = \pi/2$.



This one rotates clockwise.

- **Left circularly polarized light (RCP)**. $E_{0x} = E_{0y} = E_0$ and $\delta = -\pi/2$.



This one rotates counterclockwise.

These polarization states are important because

- they are relatively easy to create in a laboratory using linear and circular polarizers, and
- polarization measurements as well as polarization calcluations are greatly simplified using these states.

- The polarization ellipse can be expressed in terms of two angular parameters: the **orientation angle** ψ $(0 \leq \psi \leq \pi)$ and the **ellipticity angle** χ $(-\pi/2 \leq \chi \leq \pi/4)$. The angles are given by:

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2} \cos\delta,$$
$$\sin 2\chi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 + E_{0y}^2} \sin\delta.$$

The above two equations can be rewritten in trigonometric terms by introducing the **auxiliary angle** α $(0 \leq \alpha \leq \pi/2)$:

$$\tan\alpha = \frac{E_{0y}}{E_{0x}}.$$

With α, we can write ψ and χ as:

$$\tan 2\psi = (\tan 2\alpha)\cos\delta$$
$$\sin 2\chi = (\sin 2\alpha)\sin\delta.$$

- The polarization ellipse can also be represented as a point on the **Poincaré sphere**.

Given a polarization ellipse with parameters ψ and χ, the coordinate of the corresponding point on the Poincaré sphere is:

$$\begin{bmatrix} \cos(2\chi)\cos(2\psi) \\ \cos(2\chi)\sin(2\psi) \\ \sin(2\chi) \end{bmatrix}.$$

That is, 2χ is the longitudinal angle, and 2ψ is the azimuthal angle. (Be careful though. This is not the standard parameterization used in computer graphics, where the longitudinal angle runs from 0 to π instead of $-\pi/2$ to $\pi/2$.)

- Interestingly, the degenerate polarization states corresponds to important points on the Poincaré sphere:

  - LHP corresponds to $(1, 0, 0)$.
  - L+45P corresponds to $(0, 1, 0)$.
  - LVP corresponds to $(-1, 0, 0)$.
  - L-45P corresponds to $(0, -1, 0)$.
  - RCP corresponds to $(0, 0, 1)$.
  - LCP corresponds to $(0, 0, -1)$.

Moreover, all linear polarization states lie on the equator.

3 The Observables of Polarized Light

- The **Stokes polarization parameters** are 4 measurable quantitites of the polarized field. They are defined in terms of the time average of the product of two polarization components:

$$\langle \mathcal{E}_i(z,t), \mathcal{E}_j(z,t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{E}_i(z,t)\mathcal{E}_j^*(z,t)\ \mathrm{d}t$$

where $i, j \in \{x, y\}$, and * denotes complex conjugation.

The Stokes parameters are:

$$\begin{aligned} S_0 &= \langle \mathcal{E}_x, \mathcal{E}_x \rangle + \langle \mathcal{E}_y, \mathcal{E}_y \rangle = E_{0x}^2 + E_{0y}^2 \\ S_1 &= \langle \mathcal{E}_x, \mathcal{E}_x \rangle - \langle \mathcal{E}_y, \mathcal{E}_y \rangle = E_{0x}^2 - E_{0y}^2 \\ S_2 &= \langle \mathcal{E}_x, \mathcal{E}_y \rangle + \langle \mathcal{E}_y, \mathcal{E}_x \rangle = 2E_{0x}E_{0y}\cos\delta \\ S_3 &= i(\langle \mathcal{E}_x, \mathcal{E}_y \rangle - \langle \mathcal{E}_y, \mathcal{E}_x \rangle) = 2E_{0x}E_{0y}\sin\delta. \end{aligned}$$

- It is convenient to arrange the Stokes parameters as a column matrix, which is referred to as the **Stokes vector for elliptically polarized light**:

$$S = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta. \end{bmatrix}$$

- The Stokes vectors for degenerate polarization states are:

$$S_{\mathrm{LHP}} = I_0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad S_{\mathrm{LVP}} = I_0 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad S_{\mathrm{L+45P}} = I_0 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S_{\mathrm{L-45P}} = I_0 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad S_{\mathrm{RCP}} = I_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad S_{\mathrm{LCP}} = I_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

where I_0 is the intensity, which is often normalized to unity.

Hence, they can be interpreted as follows:

  - S_0 is the total intensity of the light beam.
  - S_1 is the preponderance on LHP over LVP.
  - S_2 is the preponderance on L+45P over L-45P.
  - S_3 is the preponderance on RCP over LCP.

- The Stokes parameters can be shown to be related to the Pointcaré sphere and the orientation and ellipticity angles as follows:

$$\begin{aligned} S_1 &= S_0 \cos(2\chi)\cos(2\psi) \\ S_2 &= S_0 \cos(2\chi)\sin(2\psi) \\ S_3 &= S_0 \sin(2\chi) \\ \psi &= \frac{1}{2}\tan^{-1}\frac{S_2}{S_1} \\ \chi &= \frac{1}{2}\sin^{-1}\frac{S_3}{S_0}. \end{aligned}$$

- The Stokes parameters describe not only completely polarized light but also *unpolarized* and *partially polarized light* as well.
- The Stokes vector for unpolarized light is:

$$S_{\text{unp}} = S_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

  where S_0 is the first Stokes parameter (total intensity).
- Partially polarized light is a mizxture of completely polarized light and unpolarized light:

$$S = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = (1 - \mathcal{P}) \begin{bmatrix} S_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathcal{P} \begin{bmatrix} S_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

  where $\mathcal{P}$ $(0 \leq \mathcal{P} \leq 1)$ is called the **degree of polarization** (DOP).
- The DOP is defined by the equation:

$$\mathcal{P} = \frac{I_{\text{pol}}}{I_{\text{total}}} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^3}}{S_0}$$

- Hence, for partially polarized light, we have that the following relationship holds:

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2.$$

- The Stokes parameters of a polarized beam can be measured by passing a beam sequentially through two polarizing elements know as a **wave plate** and a **polarizer**. The emerging beam then goes to an optical detector.

  The wave plate introduces a phase shift ϕ between the orthogonal components of the incident beam. The polarizer then transmits the resultant field along its transmission axis at an angle θ relative to the x-axis.

  The intensity $I(\theta, \phi)$ on the detect is given by:

$$I(\theta, \phi) = \frac{1}{2}(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta \cos \phi - S_3 \sin 2\theta \sin \phi).$$

  So, we have that

$$\begin{aligned} S_0 &= I(0,0) + I(\pi/2, 0) \\ S_1 &= I(0,0) - I(\pi/2, 0) \\ S_2 &= 2I(\pi/4, 0) - S_0, \\ S_3 &= S_0 - 2I(\pi/4, \pi/2). \end{aligned}$$

- A **polarizing material** changes a light beam's polarization state as the beam propagates through it.

  Let the input beam be characterized by a Stokes vector S, and the output beam by S'. An assumption is often made that S and S' are linearly related by a transformation matrix known as the **Mueller matrix**:

$$S' = MS.$$

- Polarizing elements include:
  - **polarizers**, which change the amplitude,
  - **wave plates**, which change the phase, and
  - **rotator**, which rotates the polarizing ellipse.

  Using thse three elements, nay elliptical polarization state can be obtained.

- A **linear polarizer** changes the amplitude. It is characterized by two absorption coefficients that differ along the x- and y-axes. The absorption coefficients in the amplitude domain are denoted by p_x and p_y, both of which are members of $[0, 1]$.

  The Mueller matrix for a linear polarizer is given by:

$$M_{\mathrm{POL}}(p_x, p_y) = \frac{1}{2}\begin{bmatrix} p_x^2 + p_y^2 & p_x^2 - p_y^2 & 0 & 0 \\ p_x^2 - p_y^2 & p_x^2 + p_y^2 & 0 & 0 \\ 0 & 0 & 2p_x p_y & 0 \\ 0 & 0 & 0 & 2p_x p_y \end{bmatrix}$$

- For an **ideal linear polarizer**, there is complete transmission along one axis and no transmission along the other axis.

  The Mueller matrix for an ideal linear polarizer with its transmission axis along the x-axis is given by:

$$M_{\mathrm{POL}}(1, 0) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

  On the other hand, the Mueller matrix for an ideal linear polarizer with its transmission axis along the y-axis is given by:

$$M_{\mathrm{POL}}(0, 1) = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

  Another interesting case is the **neutral density (ND) filter**, which has the same absorption coefficients for both axes ($p_x = p_y = p$). The Mueller matrix for the ND filter is:

$$M_{\mathrm{POL}}(p, p) = p^2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

  One can see that, after the transformation, the polarization state remains the same, except for the fact that intensity is scaled down by a factor of p^2.

- Passing light through a **wave plate**, the light field experiences a phase shift of $+\phi/2$ along the x-axis (called the **fast axis**) and a phase shift of $-\phi/2$ along the y-axis (called the **slow axis**).

  The Mueller matrix for a wave plate is given by:

$$M_{\mathrm{WP}}(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix}.$$

- Important wave plates are the **quarter wave plate (QWP)** ($\phi = \pi/4$) and the **half wave plate (HWP)** ($\phi = \pi/2$). The Mueller matrices for the two wave plates are:

$$M_{QWP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{HWP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- The QWP transforms L+45P light into RCP light and RCP light into L-45P light.
- The HWP changes the orientation and ellipticity angles of the incoming beam as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ -\cos 2\chi \sin 2\psi \\ -\sin 2\chi \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2(\chi - \pi/2) \cos 2(\pi/2 - \psi) \\ \cos 2(\chi - \pi/2) \sin 2(\pi/2 - \psi) \\ \sin 2(\chi - \pi/2). \end{bmatrix}$$

- The Mueller matrix of a **rotator** is given by:

$$M_{\text{ROM}}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where θ is the angle of rotation. A rotator only rotates the polarization ellipse. It does not affect the ellipticity:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2\chi \cos(2\psi + 2\theta) \\ \cos 2\chi \sin(2\psi + 2\theta) \\ \sin 2\chi \end{bmatrix}$$

- All matrices discussed so far are defined with the canonical coordiante system with the standard x- and y-axis. That is, it is assumed that the polarizing element is aligned with the canonical coordinate system. If the element is rotated by an angle of θ wrt the canonical coordinate system, then the Mueller matrix of the rotated polarizing element is given by:

$$M(\theta) = M_{\text{ROT}}(-\theta)\, M\, M_{\text{ROT}}(\theta).$$

- The Mueller matrix for a rotated ideal linear polarizer is

$$M_{\text{POL}}(\theta) = \frac{1}{2} \begin{bmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Stokes vector of the output beam is given by:

$$\frac{1}{2} \begin{bmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \frac{1}{2}(S_0 + S_1 \cos 2\theta + S_2 \sin 2\theta) \begin{bmatrix} 1 \\ \cos 2\theta \\ \sin 2\theta \\ 0 \end{bmatrix}.$$

So, regardless of the polarization state of the input beam, the output beam does not have circular polarization component.

- For $\theta = 0°$, $45°$, $90°$, and $135°$, the Mueller matrix reduces to the following special forms:

$$M_{\text{LHP}} = M_{\text{POL}}(0°) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_{\text{L+45P}} = M_{\text{POL}}(45°) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{\text{LVP}} = M_{\text{POL}}(90°) = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M_{\text{L}-45\text{P}} = M_{\text{POL}}(135°) = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- A circular plarizer is constructed from an L+45P polarizer and a QWP. The Mueller matrix of a circular polarizer is given by:

$$M_{\text{CP}} = M_{\text{QWP}} M_{\text{L+45P}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

We have that

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \frac{1}{2}(S_0 + S_2) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the output beam is always circularly polarized regardless of the polarization state of the input beam.

4 Reflection and Transmission

- Consider the plane in which light reflects off and refracts into a dielectric material. Let us set the coordinate system so that the two components are the one that is parallel to the plane (p-polarization) and the one that is perpendicular to the plane (s-polarization).

- Let i and r be the incident angle and the refraction angle, respectively. The amplitudes of the reflected and refracted rays are governed by **Fresnel's equations**:

$$R_p = \frac{\tan(i-r)}{\tan(i+r)} E_p, \qquad R_s = \frac{\sin(i-r)}{\sin(i+r)} E_s$$

$$T_p = \frac{2\sin r \cos i}{\sin(i+r)\cos(i-r)} E_p, \qquad T_s = \frac{2\sin r \cos i}{\sin(i+r)} E_s.$$

- The Stokes parameters for the reflected light in terms of the Stokes parameters of the incoming light are given by:

$$\begin{aligned} S_{0R} &= f_R[(\cos^2(i-r) + \cos^2(i+r))S_0 + (\cos^2(i-r) - \cos^2(i+r))S_1] \\ S_{1R} &= f_R[(\cos^2(i-r) - \cos^2(i+r))S_0 + (\cos^2(i-r) + \cos^2(i+r))S_1] \\ S_{2R} &= -f_R(2\cos(i-r)\cos(i+r))S_2 \\ S_{3R} &= -f_R(2\cos(i-r)\cos(i+r))S_3 \end{aligned}$$

where

$$f_R = \frac{1}{2}\left(\frac{\tan(i-r)}{\sin(i+r)}\right)^2.$$

So, the Mueller matrix for reflection is:

$$M_R = f_R \begin{bmatrix} \cos^2(i-r)+\cos^2(i+1) & \cos^2(i-r)-\cos^2(i+1) & 0 & 0 \\ \cos^2(i-r)-\cos^2(i+1) & \cos^2(i-r)+\cos^2(i+1) & 0 & 0 \\ 0 & 0 & -2\cos(i-r)\cos(i+r) & 0 \\ 0 & 0 & 0 & -2\cos(i-r)\cos(i+r) \end{bmatrix}$$

- The Stokes parameters for the refracted light in terms of the Stokes parameters of the incoming light are given by:

$$\begin{aligned} S_{0T} &= f_T[(\cos^2(i-r)+1)S_0 + (\cos^2(i-r)-1)S_1] \\ S_{1T} &= f_T[(\cos^2(i-r)-1)S_0 + (\cos^2(i-r)+1)S_1] \\ S_{2T} &= -f_T(2\cos(i-r))S_2 \\ S_{3T} &= -f_T(2\cos(i-r))S_3 \end{aligned}$$

where

$$f_T = \frac{1}{2}\frac{\sin 2i \sin 2r}{(\sin(i+r)\cos(i-r))^2}.$$

So, the Mueller matrix for refraction is:

$$M_T = f_T \begin{bmatrix} \cos^2(i-r)+1 & \cos^2(i-r)-1 & 0 & \\ \cos^2(i-r)-1 & \cos^2(i-r)+1 & 0 & \\ 0 & 0 & -2\cos(i-r) & 0 \\ 0 & 0 & 0 & -2\cos(i-r) \end{bmatrix}.$$

- We have that $S_0 = S_{0R} + S_{0T}$, which means that the total energy is conserved (as it should be).
- Consider the behavior of unpolarized light after it reflects from the surface.

$$S_R = M_R \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\left(\frac{\tan(i-r)}{\tan(i+r)}\right)^2 \begin{bmatrix} \cos^2(i-r)+\cos^2(i+r) \\ \cos^2(i-r)-\cos^2(i+r) \\ 0 \\ 0 \end{bmatrix}.$$

The degree of polarization $\mathcal{P}$ is given by:

$$\mathcal{P} = \left|\frac{S_{1R}}{S_{0R}}\right| = \left|\frac{\cos^2(i-r)-\cos^2(i+r)}{\cos^2(i-r)+\cos^2(i+r)}\right|.$$

In general, $\mathcal{P}$ is less than 1. If, however, $\cos(i+r) = 0$, then $\mathcal{P} = 1$ and $i + r = \pi/2$. This is the Brewster's angle, and S_R is reduced to:

$$S_R = \frac{1}{2}\cos^2 2i \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

That is, at Brewster's angle, the reflected light is LHP and the LVP component vanishes. So, if the beam passes through an LVP filter, the intensity of the beam that emerges is 0.

- Now, consider the behavior of unpolarized light after it transmits into the material. We have that:

$$S_T = M_T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \frac{\sin 2i \sin 2r}{2(\sin(i+r)\cos(i-r))^2} \begin{bmatrix} \cos^2 2(i-r) + 1 \\ \cos^2 2(i-r) - 1 \\ 0 \\ 0 \end{bmatrix}$$

The DOP of the transmitted light is:

$$\mathcal{P} = \left| \frac{S_{1T}}{S_{0T}} \right| = \left| \frac{\cos^2(i-r) - 1}{\cos^2(i-r) + 1} \right|.$$

The transmitted light is always partially polarized.

- When light propagates from a material with higher index of refraction (say n) to one with lower index of refraction (say 1), total internal reflection occurs when $n \sin i > 1$. The reflected light experiences the following phase shift. The Mueller matrix is given by:

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\delta & -\sin\delta \\ 0 & 0 & \sin\delta & \cos\delta \end{bmatrix}$$

where

$$\tan\frac{\delta}{2} = \frac{\cos i \sqrt{n^2 \sin^2 i - 1}}{n \sin^2 i}.$$

- A simpler notation for the Mueller matrices for reflection and transmission uses the **Fresnel reflection and transmission coefficients**.

The Fresnel reflection coefficients are:

$$\rho_s = \left(\frac{R_s}{E_s}\right)^2 = \left(\frac{\sin(i-r)}{\sin(i+r)}\right)^2,$$
$$\rho_p = \left(\frac{R_p}{E_p}\right)^2 = \left(\frac{\tan(i-r)}{\tan(i+r)}\right)^2.$$

The Fresnel transmission coefficients are:

$$\tau_s = \frac{n\cos r}{\cos i}\left(\frac{T_s}{E_s}\right)^2 = \frac{\sin 2i \sin 2r}{\sin^2(i+r)}$$
$$\tau_p = \frac{n\cos r}{\cos i}\left(\frac{T_p}{E_p}\right)^2 = \frac{\sin 2i \sin 2r}{\sin^2(i+r)\cos^2(i-r)}.$$

Note that $\rho_s + \tau_s = 1$ and $\rho_p + \tau_p = 1$.

- At the Brewster angle i_B, the Fresnel coefficients are:

$$\rho_{s,B} = \cos^2 2i_B, \qquad \rho_{p,B} = 0, \qquad \tau_{s,B} = \sin^2 2i_B, \qquad \tau_{p,B} = 1$$

- The Mueller matrices for reflection and transmission are given by:

$$M_r = \frac{1}{2}\begin{bmatrix} \rho_s + \rho_p & \rho_s - \rho_p & 0 & 0 \\ \rho_s - \rho_p & \rho_s + \rho_p & 0 & 0 \\ 0 & 0 & 2\sqrt{\rho_s \rho_p} & 0 \\ 0 & 0 & 0 & 2\sqrt{\rho_s \rho_p} \end{bmatrix}$$

$$M_t = \frac{1}{2}\begin{bmatrix} \tau_s + \tau_p & \tau_s - \tau_p & 0 & 0 \\ \tau_s - \tau_p & \tau_s + \tau_p & 0 & 0 \\ 0 & 0 & 2\sqrt{\tau_s \tau_p} & 0 \\ 0 & 0 & 0 & 2\sqrt{\tau_s \tau_p} \end{bmatrix}$$

5 Other Polarization Matrix Calculi

5.1 The Jones Matrix Calculus

- The **Jones matrix calculus** is a matrix formulation of polarized light that consists of 2×1 **Jones vectors** to describe the field components and 2×2 **Jones matrices** to describe polarizing components.
- It is simpler than the Mueller calculus and is limited to treating only completely polarized light. It is used when treating interference phenomena or in problems where field amplitudes must be superposed.
- The 2×1 Jones vector for polarized light field is:

$$\mathbf{E} = \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{bmatrix} = \begin{bmatrix} E_{0x} e^{i\delta_x} \\ E_{0y} e^{i\delta_y} \end{bmatrix}$$

  where E_{0x} and E_{0y} are the amplitudes, and δ_x and δ_y are the phases.
- Intensity I of the field can be calculated by dotting the Jones vector:

$$I = \begin{bmatrix} \mathcal{E}_x^* & \mathcal{E}_y^* \end{bmatrix} \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{bmatrix} = \mathcal{E}_x^* \mathcal{E}_x + \mathcal{E}_y^* \mathcal{E}_y = \mathbf{E}^H \mathbf{E}.$$

- The Jones vectors for the degenerate polarization states are:

$$\mathbf{E}_{\text{LHP}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{E}_{\text{LVP}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{E}_{\text{L+45P}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{E}_{\text{L}-45\text{P}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{E}_{\text{RCP}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \mathbf{E}_{\text{LCP}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

- Given an arbitrary polarized light, we can decompose it into $\mathbf{E}_{\text{LHP}}$ and $\mathbf{E}_{\text{LVP}}$:

$$\mathbf{E} = \begin{bmatrix} E_{0x} e^{i\delta_x} \\ E_{0y} e^{i\delta_y} \end{bmatrix} = E_{0x} e^{i\delta_x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + E_{0y} e^{i\delta_y} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = E_{0x} e^{i\delta_x} \mathbf{E}_{\text{LHP}} + E_{0y} e^{i\delta_y} \mathbf{E}_{\text{LVP}}.$$

- A polarizing element is represented by a 2×2 Jones matrix:

$$J = \begin{bmatrix} j_{xx} & j_{xy} \\ j_{yx} & j_{yy} \end{bmatrix}$$

- For a linear polarizer, the Jones matrix is:

$$J_{\text{POL}} = \begin{bmatrix} p_x & 0 \\ 0 & p_y \end{bmatrix}$$

where $0 \leq p_x, p_y \leq 1$.

For an ideal linear horizontal and linear vertical polarizer, the Jones matrices tkae the form, respectively:

$$J_{\text{LHP}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad J_{\text{LVP}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

- The Jones matrices for a wave plate with a phase shift of $\phi/2$ along the x-axis and $-\phi/2$ along the y-axis is:

$$J_{\text{WP}} = \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix},$$

and the above is equivalent (modulo the absolute phase) to:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{bmatrix}$$

The Jones matrices for QWP and HWP are:

$$J_{QWP} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \qquad J_{HWP} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- The Jones matrix for a rotator is:

$$J_{\text{ROT}}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

5.2 Wolf's Coherency Matrix Calculus

- The **Wolf's coherency matrix calculus** serves as a usefu bridge between the Mueller and Jones matrix calculi.

- The coherency matrix C is deinfed in terms of the complex product of the optical field:

$$\begin{aligned} C &= \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix} = \begin{bmatrix} \langle \mathcal{E}_x, \mathcal{E}_x \rangle & \langle \mathcal{E}_x, \mathcal{E}_y \rangle \\ \langle \mathcal{E}_y, \mathcal{E}_x \rangle & \langle \mathcal{E}_y, \mathcal{E}_y \rangle \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} = \frac{1}{2} \left(S_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + S_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + S_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + S_4 \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right). \end{aligned}$$

Here, the 4 basis matrices are called the **Pauli spin matrices**.

- The matrices of degenerate polarization states are:

$$\begin{aligned} C_{\text{LHP}} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & C_{\text{LVP}} &= \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ C_{\text{L+45P}} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & C_{\text{L}-45\text{P}} &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ C_{\text{RCP}} &= \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} & C_{\text{LCP}} &= \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}. \end{aligned}$$

- The matrix for unpolarized light is given by:

$$C_{\text{UNP}} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the general matrix for elliptically polarized light is given by:

$$C_{\text{ELP}} = \frac{1}{2}\begin{bmatrix} 1+\cos 2\alpha & e^{i\delta}\sin 2\alpha \\ e^{-i\delta}\sin 2\alpha & 1-\cos 2\alpha \end{bmatrix}$$

where $\alpha = \tan^{-1}(E_{0y}/E_{0x})$ and δ is the phase.