Smoothed Particle Hydrodynamics Note

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1 2D Kernel Functions

- Müller et al. [1] uses three kernel functions to do SPH. For anything but pressure and viscocity, they use the following kernel:

$$W_{\mathrm{poly6}}(r,h) = \frac{315}{64\pi h^6}\begin{cases}(h^2-r^2)^3 & 0 \le r \le h \\ 0 & \text{otherwise}\end{cases}.$$

For pressure, the following kernel is used:

$$W_{\mathrm{spiky}} = \frac{15}{\pi h^6}\begin{cases}(h-r)^3 & 0 \le r \le h \\ 0 & \text{otherwise}\end{cases}$$

Lastly, for viscosity

$$W_{\mathrm{viscosity}}(r,h) = \frac{15}{2\pi h^3}\begin{cases}-\frac{r^3}{2h^3}+\frac{r^2}{h^2}+\frac{h}{2r}-1 & 0 \le r \le h \\ 0 & \text{otherwise}\end{cases}$$

- The above kernels were constructed for the 3D case. We shall derive kernels with the same properties for the 2D cases.

- For the first kernel, let us say that we want to use $W(r,h) = A(h^2-r^2)^2$. Since the kernel has to integrate to one on the whole plane, we have that

$$\begin{aligned}1 &= \int_0^h \int_0^{2\pi} A(h^2-r^2)^2 r\,\mathrm{d}\theta\mathrm{d}r \\ A &= \bigg(\int_0^h \int_0^{2\pi} (h^2-r^2)^2 r\,\mathrm{d}\theta\mathrm{d}r\bigg)^{-1}.\end{aligned}$$

Now,

$$\begin{aligned}\int_0^h \int_0^{2\pi} (h^2-r^2)^2 r\,\mathrm{d}\theta\mathrm{d}r &= 2\pi\int_0^h (h^4-2h^2r^2+r^4)r\,\mathrm{d}r = 2\pi\int_0^h h^4r-2h^2r^3+r^5\,\mathrm{d}r \\ &= 2\pi\bigg[h^4\frac{r^2}{2}-2h^2\frac{r^4}{4}+\frac{r^6}{6}\bigg]_0^h = 2\pi\bigg(\frac{h^6}{2}-\frac{h^6}{2}+\frac{h^6}{6}\bigg) \\ &= \frac{\pi h^6}{3}.\end{aligned}$$

Therefore, $A = 3/(\pi h^6)$. Therefore, the normal kernel should be:

$$W(r,h) = \frac{3}{\pi h^6}\begin{cases}(h^2-r^2)^2 & 0 \le r \le h \\ 0 & \text{otherwise}\end{cases}.$$

- For the pressure kernel, we shall use the kernel of the form $W_{\text{pressure}}(h,r) = A(h-r)^2$. We have that:

$$
\begin{aligned}
A &= 2\pi \int_0^h (h-r)^2 r \,\mathrm{d}r = 2\pi \int_0^h (h^2 - 2hr + r^2) r \,\mathrm{d}r = 2\pi \int_0^h h^2 r - 2hr^2 + r^3 \,\mathrm{d}r \\
&= 2\pi \left[h^2 \frac{r^2}{2} - 2h \frac{r^3}{3} + \frac{r^4}{4} \right]_0^h = 2\pi \left(\frac{h^4}{2} - \frac{2h^4}{3} + \frac{h^4}{4} \right) \\
&= 2\pi \frac{h^4}{12} = \frac{\pi h^4}{6}.
\end{aligned}
$$

Therefore,

$$
W_{\text{pressure}}(r,h) = \frac{6}{\pi h^4} \begin{cases} (h-r)^2 & 0 \le r \le h \\ 0 & \text{otherwise} \end{cases}.
$$

- Before we derive the last kernel, we derive some useful facts:

$$
\frac{\partial r}{\partial x} = \frac{\partial \sqrt{x^2+y^2}}{\partial x} = \frac{\mathrm{d}\sqrt{x^2+y^2}}{\mathrm{d}(x^2+y^2)} \frac{\partial (x^2+y^2)}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r}.
$$

As a result, we also have that $\partial r / \partial y = y/r$.

- Now,

$$
\begin{aligned}
\frac{\partial^2 f(r)}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f(r)}{\partial x} \right) \\
&= \frac{\partial}{\partial x} \left(f'(r) \frac{x}{r} \right) \\
&= \frac{\partial f'(r)}{\partial x} \frac{x}{r} + \frac{f'(r)}{r} - f'(r) x \cdot \frac{1}{r^2} \cdot \frac{x}{r} \\
&= f''(r) \frac{x^2}{r^2} + \frac{f'(r)}{r} \left(1 - \frac{x^2}{r^2} \right)
\end{aligned}
$$

So,

$$
\begin{aligned}
\nabla^2 f(r) &= \frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} \\
&= \frac{f''(r)}{r^2}(x^2+y^2) + \frac{f'(r)}{r} \left(2 - \frac{x^2+y^2}{r^2} \right) \\
&= \frac{f''(r)}{r^2} r^2 + \frac{f'(r)}{r} \left(2 - \frac{r^2}{r^2} \right) \\
&= f''(r) + \frac{f'(r)}{r}.
\end{aligned}
$$

- Unfortunately, the viscosity kernel proposed in the paper only works in 3D. There's no function of the same form that works for the 2D case. So, instead, we use Monaghan's kernel:

$$
W(r,h) = \frac{\sigma}{h^v} \begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & 0 \le q \le 1 \\ \frac{1}{4}(2-q)^3 & 1 \le q \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

where

- $q = 2r/h$
- σ is a constant with value $4/3$, $40/(7\pi)$, and $8/\pi$ for 1-, 2-, and 3-dimensional system, respectively, and
- v is the number of dimensions of the system.

2 The Monaghan Kernel

- Monaghan proposes the following kernel:
$$W(r,h) = \frac{\sigma}{h^v}\begin{cases} 1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 & 0 \le q \le 1 \\ \frac{1}{4}(2-q)^3 & 1 \le q \le 2 \\ 0 & \text{otherwise} \end{cases}$$
where
  - $q = 2r/h$
  - σ is a constant with value $4/3$, $40/(7\pi)$, and $8/\pi$ for 1-, 2-, and 3-dimensional system, respectively, and
  - v is the number of dimensions of the system.
- We shall write it in terms of r.
$$W(r,h) = \frac{\sigma}{h^v}\begin{cases} 1 - 6r^2/h^2 + 6r^3/h^3 & 0 \le r \le h/2 \\ 2 - 6r/h + 6r^2/h^2 - 2r^3/h^3 & h/2 \le r \le h \\ 0 & \text{otherwise} \end{cases}$$
.
- We shall check the normalization constant for the 2D case:
$$\begin{aligned} 2\pi \int_0^h W(r,h)\,\mathrm{d}r &= 2\pi\biggl(\int_0^{h/2} (1 - 6r^2/h^2 - 6r^3/h^3) r\,\mathrm{d}r + \int_{h/2}^h 2 - 6r/h + 6r^2/h^2 - 2r^3/h^3\,\mathrm{d}r \biggr) \\ &= \frac{\sigma}{h^v}\frac{7\pi h^2}{40}. \end{aligned}$$
So, $v = 2$ and $\sigma = 40/(7\pi)$.
- The gradient is given by:
$$\begin{aligned} \nabla W(\mathbf{r},h) &= \frac{\mathbf{r}}{r}W'(r) \\ &= \frac{\mathbf{r}}{r}\frac{\sigma}{h^v}\begin{cases} -12r/h^2 + 18r^2/h^3 & 0 \le r \le h/2 \\ -6/h + 12r/h^2 - 6r^2/h^3 & h/2 \le r \le h \\ 0 & \text{otherwise} \end{cases} \\ &= \mathbf{r}\frac{\sigma}{h^v}\begin{cases} -12/h^2 + 18r/h^3 & 0 \le r \le h/2 \\ -6/(hr) + 12/h^2 - 6r/h^3 & h/2 \le r \le h \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$
- We have that
$$W''(r) = \frac{\sigma}{h^v}\begin{cases} -12/h^2 + 36r/h^3 & 0 \le r \le h/2 \\ 12/h^2 - 12r/h^3 & h/2 \le r \le h \\ 0 & \text{otherwise} \end{cases}$$

So, the Laplacian is given by:

$$\nabla^2 W(r) = W''(r) + W'(r)/r = \frac{\sigma}{h^v} \begin{cases} -24/h^2 + 54r/h^3 & 0 \le r \le h/2 \\ -6/(hr) + 24/h^2 - 18r/h^3 & h/2 \le r \le h \\ 0 & \text{otherwise} \end{cases} .$$

References

[1] Matthias Müller, David Charypar, and Markus Gross. Particle-based fluid simulation for interactive applications. In *Proceedings of the 2003 ACM SIGGRAPH/Eurographics symposium on Computer animation*, SCA '03, pages 154–159, Aire-la-Ville, Switzerland, Switzerland, 2003. Eurographics Association.