Differential Geometry Notes of 02/08/2013

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1 Differentiable Functions on Surfaces

- We are interested in defining differential functions on a surface.
- A natural approach to defining differentiability on a surface is as follows. We say that a function $f : S \to \mathbb{R}$ is differentiable at point $p \in S$ if there is a coordinate neighborhood parameterized by u and v such that f's expression in terms of u and v admits partial derivatives of all orders.
- However, the problem with this approach is that there can be many coordinate neighborhoods around p. Some may satisfy the conditions. Some may not.
- For the above definition to make sense, the differentiability should not depend on the chosen coordinate neighborhood. As such, we must show that if p belongs to two coordinate neighborhoods—one with parameters (u, v) and other with (ξ, η)—it is possible to pass from one to the other by means of a differentiable transformation.
- **Proposition 1.1 (Change of Parameters).** *Let p be a point of a regular surface S. Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ and $\mathbf{y} : V \subseteq \mathbb{R}^2 \to S$ be two parameterizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{x}(V) = W$. Then, the "change of coordinates"*

  $$h = \mathbf{x}^{-1} \circ \mathbf{y},$$

  which maps $\mathbf{y}^{-1}(W)$ to $\mathbf{x}^{-1}(W)$, is a diffeomorphism. That is, it is differentiable and has a differentiable inverse h^{-1}.

  Proof. First, we note that $\mathbf{x}$ and $\mathbf{y}$ are homeomorphisms. They are both one-to-one and have continuous inverses. Thus, $h = \mathbf{x}^{-1} \circ \mathbf{y}$ is a homeomorphism.

  However, we cannot conclude that h is differentiable yet. The problem is that, while $\mathbf{y}$ is differentiable by definition, we do not know how to differentiate $\mathbf{x}^{-1} : S \to U$ because its domain S is not an open set.

  The trick is to extend $\mathbf{x}$ so that we know that its inverse is differentiable. This is done by applying the inverse function theorem. So, let $\mathbf{x} = (x(u, v), y(u, v), z(u, v))$. We define $\bar{\mathbf{x}} : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

  $$\bar{\mathbf{x}}(u, v, t) = (x(u, v), y(u, v), z(u, v) + t).$$

  Let $r = \mathbf{y}^{-1}(p)$ and let $q = h(r)$ so that $\mathbf{x}(q) = p$. Because $\mathrm{d}\mathbf{x}_q$ is injective, one of its Jacobian determinant is not zero. WLOG, let us assume that

  $$\left| \frac{\partial(x, v)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0.$$

Hence, we have that the Jacobian $\mathrm{d}\bar{\mathbf{x}}_q$ is given by:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial v}{\partial t} \\ 0 & 0 & 1 \end{vmatrix} = \left| \frac{\partial(x,v)}{\partial(u,v)} \right| \neq 0.$$

As such, there's a neighborhood M around p such that $\bar{\mathbf{x}}^{-1}$ is defined and is differentiable.

Let $N = W \cap M$. Consider the function $F = \bar{\mathbf{x}}^{-1} \circ \mathbf{y}$ from $\mathbf{y}^{-1}(N) \to \mathbb{R}^3$. We have that F is differentiable, and $\mathbf{y}^{-1}(N)$ contains r. Now, for all point $n \in N$, we have that $\bar{\mathbf{x}}^{-1}$ is of the form $(*, *, 0)$ where the first two coordinates must agree with $\mathbf{x}^{-1}(n)$. Hence, in a neighborhood of q, we can say that $h = \pi \circ \bar{\mathbf{x}}^{-1} \circ \mathbf{y}$ where π is the projection that drops the last component. Since π, $\bar{\mathbf{x}}$, and $\mathbf{y}$ are all differentiable, we have that h is differentiable. □

- **Definition 1.2.** *Let $f : V \subseteq S \to \mathbb{R}$ be a function defined on an open subset V of a regular surface S. Then f is said to be differentiable at point $p \in V$ if, for some parameterization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subseteq V$, the composition $f \circ \mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$.*

  We say that f is differentiable in V if it is differentiable at all points of V.

- From now on, when f is a function from a surface to the reals, we will sometimes write $f(u, v)$ instead of $f(\mathbf{x}(u, v))$ for some coordinate function $\mathbf{x}$.