Differential Geometry Notes of 01/25/2013

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1 Change of Orientation

- Given a curve $\alpha : (a, b) \to \mathbb{R}^3$ parameterized by arc length, we can define another curve $\beta : (-b, -a) \to \mathbb{R}^3$ such that $\beta(s) = \alpha(-s)$ for any $s\ in (-b, -a)$

  β has the same trace as α, but the points are traced in the opposite direction.

  We say that α and β differ by a change of orientation.

- If α changes the direction, then the tangent vector also changes direction. This is because, for any $s_0 \in (-b, -a)$

$$t_\beta(s_0) = \frac{\mathrm{d}\beta}{\mathrm{d}s}\bigg|_{s=s_0} = \left(\frac{\mathrm{d}}{\mathrm{d}s}\alpha(-s)\right)\bigg|_{s=s_0}.$$

  Let $u = -s$, we have that

$$t_\beta(s_0) = \left(\frac{\mathrm{d}}{\mathrm{d}s}\alpha(-s)\right)\bigg|_{s=s_0} = \left(\frac{\mathrm{d}\alpha(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}s}\right)\bigg|_{s=s_0} = -t_\alpha(u)\bigg|_{s=s_0} = -t_\alpha(-s)\bigg|_{s=s_0} = -t_\alpha(-s_0).$$

  So, the tangent at the "same" point on α and β are anti-parallel.

- Now, consider the derivative of the tangent:

$$\begin{aligned}\frac{\mathrm{d}t_\beta}{\mathrm{d}s}\bigg|_{s=s_0} &= \left(\frac{\mathrm{d}}{\mathrm{d}s}(-t_\alpha(-s))\right)\bigg|_{s=s_0} = \left(-\frac{\mathrm{d}}{\mathrm{d}s}(t_\alpha(-s))\right)\bigg|_{s=s_0} = \left(-\left(\frac{\mathrm{d}t_\alpha(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}s}\right)\right)\bigg|_{s=s_0}\\ &= t'_\alpha(u)\bigg|_{s=s_0} = t'_\alpha(-s_0).\end{aligned}$$

  This means that the normal of the curve remains the same as well as the curvature.

- Because $b(s) = t(s) \wedge n(s)$, we have that the binormal changes direction after a change of direction because the tangent changes direction, but the normal does not. That is,

$$b_\beta(s) = -b_\alpha(-s).$$

- Now, consider the derivative of the binormal:

$$\frac{\mathrm{d}b_\beta}{\mathrm{d}s}\bigg|_{s=s_0} = \left(\frac{\mathrm{d}}{\mathrm{d}s}(-b_\alpha(-s))\right)\bigg|_{s=s_0} = b'_\alpha(-s)\bigg|_{s=s_0} = b'_\alpha(-s_0).$$

  As such, the derivative of the binormal remains the same, and so does the torsion.

2 Frenet Formulas

- Let $\alpha : I \to \mathbb{R}^3$ be a curve parameterized by arc length with no singular points of order 1.
- At each point s, we can derive three vectors:
  - the tangent $t(s)$,
  - the normal $n(s)$, and
  - the binormal $b(s)$.
- We have that
  - $t'(s) = k(s)n(s)$, and
  - $b'(s) = \tau(s)n(s)$.

  What can we say about $n'(s)$?
- Because $n(s) = b(s) \wedge t(s)$, we have

$$n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) = \tau(s)(n(s) \wedge t(s)) + k(s)(b(s) \wedge n(s)) \; = -\tau(s)b(s) - k(s)t(s).$$

- The following three equations:

$$\begin{aligned} t'(s) &= k(s)n(s), \\ n'(s) &= -\tau(s)b(s) - k(s)t(s), \text{ and} \\ b'(s) &= \tau(s)n(s) \end{aligned}$$

  are called the **Frenet formulas**.
- The tb plane is called the **rectifying plane**.
  The nb plane is called the **normal plane**.
  The tn plane is called the **osculating plane**.
- The line which contains $n(s)$ and passes through $\alpha(s)$ is called the **principal normal**.
  The line which contains $b(s)$ and passes through $\alpha(s)$ is called the **binormal**.
- The inverse of the curvature $R(s) = 1/k(s)$ is called the **radius of curvature**.

3 Fundamental Theorem of the Local Theory of Curves

- We can think of a curve being formed from a line segment by bending (curvature) and twisting (torsion).
- It turns out that k and τ completely describe the local properties of curves.
- **Theorem 3.1.** *Given differentiable function $k(s) > 0$ and $\tau(s)$ where $s \in I = (a, b)$. There exists a regular parameterized curve $\alpha : I \to \mathbb{R}^3$ such that*
  - *s is the arc length,*
  - *$k(s)$ is the curvature, and*
  - *$\tau(s)$ is the torsion*

  of α. Morever, any other curve $\bar{\alpha}$ satisfying the same conditions differs from α by a rigid motion. That is, there exists an orthogonal linear map $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ with positive determinant and a vector c such that $\bar{\alpha}(s) = \rho \circ \alpha(s) + c$ for all s.

Proof. We will only prove uniqueness upto rigid motion.

First, we state that arc length, curvature, and torsion are invariant under rigid motion.

Assume that two curves α and $\bar{\alpha}$ satisfy the property that $k(s) = \bar{k}(s)$ and $\tau(s) = \bar{\tau}(s)$. Let t_0, n_0, b_0 and $\bar{t}_0, \bar{n}_0, \bar{n}_0$ be the Frenet frame at point $s = s_0 \in I$ of α and $\bar{\alpha}$, respectively. There's a rigid motion that takes $\bar{\alpha}(s_0), \bar{t}_0, \bar{n}_0, \bar{b}_0$ to $\alpha(s_0), t_0, n_0, b_0$. After performing this rigid motion, we have that $\alpha(s_0) = \bar{\alpha}(s_0)$. Moreover, the following Frenet equations hold for all s:

$$\begin{aligned} t' &= kn & \bar{t}' &= k\bar{n} \\ n' &= -\tau b - kt & \bar{n}' &= -\tau\bar{b} - k\bar{t} \\ b' &= \tau n & \bar{b}' &= \tau\bar{n} \end{aligned}$$

with $t(s_0) = \bar{t}(s_0), n(s_0) = \bar{n}(s_0)$, and $b(s_0) = \bar{b}(s_0)$.

Consider the function

$$\begin{aligned} E(s) &= |t(s) - \bar{t}(s)|^2 + |n(s) - \bar{n}(s)|^2 + |b(s) - \bar{b}(s)|^2 \\ &= (t - \bar{t}) \cdot (t - \bar{t}) + (n - \bar{n}) \cdot (n - \bar{n}) + (b - \bar{b}) \cdot (b - \bar{b}). \end{aligned}$$

We have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} E(s) &= 2(t' - \bar{t}') \cdot (t - \bar{t}) + 2(n' - \bar{n}') \cdot (n - \bar{n}) + 2(b' - \bar{b}') \cdot (b - \bar{b}) \\ &= 2k(n - \bar{n}) \cdot (t - \bar{t}) + 2(-kt - \tau b + k\bar{t} + \tau\bar{b}) \cdot (n - \bar{n}) + 2\tau(n - \bar{n}) \cdot (b - \bar{b}) \\ &= 2(n - \bar{n})\big[k(t - \bar{t}) - k(t - \bar{t}) - \tau(b - \bar{b}) + \tau(b - \bar{b})\big] \\ &= 0. \end{aligned}$$

Hence, $E(s)$ is constant. Because $E(s_0) = 0$, we have that $E(s) = 0$ for all s. It follows that $t(s) = \bar{t}(s)$ for all s.

Now, since $\alpha'(s) = t(s) = \bar{t}(s) = \bar{\alpha}'(s)$, we have that $\frac{\mathrm{d}}{\mathrm{d}s}(\alpha - \bar{\alpha}) = \mathbf{0}$ for all s. As such $\alpha(s) = \bar{\alpha}(s) + a$ for some constant vector a. Since $\alpha(s_0) = \bar{\alpha}(s_0)$, we have that $a = 0$. Thus, $\alpha(s) = \bar{\alpha}(s)$ for all s. It follows that α and $\bar{\alpha}$ differs by a rigid motion. $\square$

4 Arc Length Parameterization

- Given a regular parameterized curve $\alpha : I \to \mathbb{R}^3$, it is possible to obtain a curve $\beta : J \to \mathbb{R}^3$ parameterized by arc length which has the same trace as α.
- First, let us define the arc length:

  $$s = s(t) = \int_{t_0}^{t} |\alpha'(t)| \,\mathrm{d}t$$

  where $t, t_0 \in I$.
- Because $\mathrm{d}s/\mathrm{d}t = |\alpha'(t)| \neq 0$, the function $s = s(t)$ has a differentiable inverse $t = t(s)$ where $s \in s(I) = J$.
- Now, we can set $\beta = \alpha \circ t$, which maps J to $\mathbb{R}^3$.

  We have that $\beta(J) = \alpha(I)$ so the curves have the same trace.

  Also, $|\beta'(s)| = |\alpha'(t) \cdot \mathrm{d}t/\mathrm{d}s| = 1 = |\alpha'(t)|/|\alpha'(t)| = 1$. So, β is parameterized by arc length.