Approximating Hair Scattering Function for Interactive Rendering

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This document is written as a companion to reading "Interactive Hair Rendering under Environment Lighting" by Ren et al. [2] and "Interative Hair Rendering and Appearance Editing under Environtmnet Lighting" by Xu et al. [3].

We focus on these two papers approximate the hair scattering function $S(\omega_i, \omega_o)$ for interactive rendering under environment lighting. We begin with a tutorial on the sphecial radial basis function, which is central to both papers.

1 Spherical Radial Basis Functions

- Let $\mathbb{S}^m$ denote the unit sphere in $\mathbb{R}^{m+1}$.
- Let ω and ξ be two points on $\mathbb{S}^m$. We set θ be the geodesic distance between the two points. In other words, $\theta = \arccos(\omega \cdot \xi)$.
- A **spherical radial basis function** (SBRF) $G(\omega)$ is a function from $\mathbb{S}^m \to \mathbb{R}$ that depends only on $\omega \cdot \xi$ where ξ is a fixed point on the sphere. That is:

$$G(\omega) = G(\omega \cdot \xi) = G(\cos\theta).$$

  We call ξ the **center** of the SRBF. When writing an SBRF, we will write $G(\omega \cdot \xi)$ instead of $G(\omega)$.
- Most of the time, SRBFs are represented in terms of expansions in Legendre polynomials:

$$G(\omega \cdot \xi) = \sum_{l=0}^{\infty} G_l P_l(\omega \cdot \xi)$$

  where $P_l(x)$ is the normalized Legendre polynomial of degree l. The **Legendre coefficients** G_l satisfy the property that $G_l \geq 0$ and $\sum_{l=0}^{\infty} G_l < \infty$.
- Let G be an SRBF with center ξ_g, and H be another SBRF with center ξ_h. The convolution of G and H are given by the following equation:

$$G *_m H = \int_{\mathbb{S}^m} G(\omega \cdot \xi_g) H(\omega \cdot \xi_h)\ \mathrm{d}\omega(\omega) = \sum_{l=0}^{\infty} G_l H_l \frac{\Omega_m}{d_{m,l}} P_l(\xi_g \cdot \xi_h).$$

  Here, Ω_m is the volume of $\mathbb{S}^m$:

$$\Omega_m = \frac{(2\pi)^{(m+1)/2}}{\Gamma((m+1)/2)},$$

  and $d_{m,l}$ i the imension of the space of the order-l spherical harmonics on $\mathbb{S}^m$:

$$d_{m,l} = \begin{cases} \frac{(2l+m-1)!(l+m-2)!)}{l!(m-l)!}, & \text{if } l \geq 1 \\ 1, & \text{if } l = 0 \end{cases}$$

- There are two examples of SBRFs.
  - The **Abel–Poisson SRBF kernel** is defined as:

$$G^{\text{Abel}}(\omega \cdot \xi; \lambda) = \frac{1 - \lambda^2}{[1 - 2\lambda(\omega \cdot \xi) + \lambda^2]^{3/2}}$$

    for $0 < \lambda < 1$.
  - The **Gaussian SRBF kernel** is defined as:

$$G^{\text{Gau}}(\omega \cdot \xi; \lambda) = e^{-\lambda} e^{\lambda(\omega \cdot \xi)}$$

    for $\lambda > 0$.

  In both of these functions, λ denotes the "bandwidth" parameter which controls the width of the lobes.

- Let G^{Abel} be an Abel–Poisson SBRF kernel with center ξ_g and bandwidth λ_g. Let H^{Abel} be the same with center ξ_h and λ_h. We have that

$$G^{\text{Abel}} *_m H^{\text{Abel}} = \frac{1 - (\lambda_g \lambda_h)^2}{[1 - 2(\lambda_g \lambda_h)(\xi_g \cdot \xi_h) + (\lambda_g \lambda_h)^2]^{3/2}}.$$

  It's interesting to note that, if we view $G^{\text{Abel}} *_m H^{\text{Abel}}$ as a function of $\xi_g \cdot \xi_h$, then it is an Abel–Poisson SBRF kernel.

- Define G^{Gau} and H^{Gau} similarly as the previous item but with the Gaussian SRBF kernel. We have that:

$$G^{\text{Gau}} *_m H^{\text{Gau}} = e^{-(\lambda + \lambda_h)} \Omega_m \Gamma\left(\frac{m+1}{2}\right) I_{\frac{m-1}{2}}(\|r\|) \left(\frac{2}{\|r\|}\right)^{\frac{m-1}{2}}.$$

  where $r = \lambda_g \xi_g + \lambda_h \xi_h$ and I_ν is the *modified Bessel function of the first kind of order* ν:

$$I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^{1} e^{-xz}(1 - z^2)^{\nu - 1/2}\, \mathrm{d}z.$$

  For the special case of $m = 2$ (sphere in 3D), we can simplify the convolution to:

$$G^{\text{Gau}} *_2 H^{\text{Gau}} = 4\pi \frac{\sinh(\|r\|)}{\|r\|}.$$

- Given a spherical function $F(\omega)$, its **SRBF kernel expansion** is basically an approximation of F as a linear combination of a finite number of SRBF kernels:

$$F(\omega) \approx \sum_{k=1}^{n} F_k G(\omega \cdot \xi_k; \lambda_k).$$

  If the centers $\xi_1, \xi_2, \ldots, \xi_n$ and bandwidths $\lambda_1, \lambda_2, \ldots, \lambda_n$ are given, the coefficient $F_1, F_2, \ldots, F_n$ can be obtained by least squares.

2 Ren et al. [2]

- Following [1], the outgoing *curve intensity* can be written as:

$$\overline{L}(\omega_o) = D \int_{\mathbb{S}^2} L(\omega_i) T(\omega_i) S(\omega_i, \omega_o) \cos\theta_i \, \mathrm{d}\omega_i$$

  where

- D is the diameter of the hair fiber,
- $L(\omega_i)$ is the incoming radiance,
- $T(\omega_i)$ is the transmittance in the incident direction generated by the hair volume:

$$T(\omega_i) = T(x, \omega_i) = \exp\bigg(- \sigma_a \int_x^\infty \rho(x')\ \mathrm{d}x' \bigg)$$

  where $\rho(x)$ is the density function at x, and
- $S(\omega_i, \omega_o)$ is the bidirectional curve radiance distribution function.

Note that the positional dependence of each term is omitted for brevity.

- It's important to distinguish between *curved intensity* $\overline{L}(\omega)$ and *radiance* $L(\omega)$. Curved intensity is energy per unit length of the fiber along a direction, but radiance is energy per unit area of the fiber's surface along a direction.

- In a physically based renderer, we are concerned with radiance, not curved intensity. The formula that relates outgoing radiance to incoming radiance is:

$$L(\omega_o) = \int_{\mathbb{S}^2} L(\omega_i) T(\omega_i) S(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i.$$

  Notice that there is no D here.

- The environment light is approximate as a linear combination of a finite number of Gaussian SRBF kernels:

$$L(\omega_i) \approx \sum_{j=1}^N L_j G(\omega_i \cdot \xi_j; \lambda_j).$$

  Using $G_j(\omega_i)$ to denote $G(\omega_i \cdot \xi_j; \lambda_j)$, this means that the outgoing curve intensity is given by:

$$\overline{L}(\omega_i) \approx D \sum_{j=1}^N L_j \int_{\mathbb{S}^2} G_j(\omega_i) T(\omega_i) S(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i$$

- The transmittance term is also factored out as the *effective transmittance* $\widetilde{T}$:

$$\overline{L}(\omega_o) \approx D \sum_{j=1}^N L_j \widetilde{T}(\xi_j, \lambda_j) \int_{\mathbb{S}^2} G_j(\omega_i) S(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i.$$

  where

$$\widetilde{T}(\xi_j, \lambda_j) = \frac{\int_{\mathbb{S}^2} G_j(\omega_i) T(\omega_i) S(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i}{\int_{\mathbb{S}^2} G_j(\omega_i) S(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i}.$$

- The effective transmittance depends on the position x and is approximated exponentiating the optical depth at x averaged over the directions covered by the SRBF light. We will not cover the details in this document.

- Note that after factoring out the effective transmittance, the term

$$I(\omega_o; \xi, \lambda) = \int_{\mathbb{S}^2} G(\omega_i \cdot \xi; \lambda) S(\omega_i, \omega_o) \cos\theta_i\ \mathrm{d}\omega_i,$$

  which the paper calls the *convolution term*, is indenpendent of the hair geometry. The paper precomputes this term for a set of ω_o, ξ, and λ parameters and stores the results in a table for run-time lookup.

- If we use the Marschner model (or any other scattering model with circular cross section), then the above table is a 4D table $\mathbf{I}_M(\cos\theta_\xi, \cos\theta_o, \cos(\phi_\xi - \phi_o), 1/\lambda)$.

  Ren et al. chooses to use $1/\lambda$ instead of λ because it is proportional to the width of the SRBF. The cosines are used to avoid the computation of inverse trigonometric functions.

3 Xu et al. [3]

- The paper approximates the Marschner model with SRBF kernels.
- We will not give a complete formulation of the Marschner model here. Please refer to the Marschner model note written earlier.
- As a matter of notational different, the original Marschner paper uses p to denote a scattering mode. However, the Xu paper uses the variable t. Therefore, the scattering function for Mode t is denoted by $S_t(\omega_i, \omega_o)$.
- The paper also changes the definition of the Gaussian SRBF a little. It defines:

  $$G(\omega_i \cdot \xi_j; \lambda_j) = e^{-2(\omega_i \cdot \xi_j - 1)/\lambda_j^2}.$$

  It does this to be consistent with the 1D Gaussian:

  $$g^u(x; \mu, \lambda) = \frac{1}{\sqrt{\pi}\lambda} e^{-(x-\mu)^2/\lambda^2}.$$

- Define $\mathcal{M}_t(\xi_j, \lambda_j)$ to be the result of convolving $S_t(\omega_i, \omega_o)$ with $G_j(\omega_i) = G(\omega_i \cdot \xi_j; \lambda_j)$:

  $$\begin{aligned} \mathcal{M}_t(\xi_j, \lambda_j) &= \int_{\mathbb{S}^2} G_j(\omega_i) S_t(\omega_i, \omega_o) \cos\theta_i \ \mathrm{d}\omega_i \\ &= \int_{\mathbb{S}^2} G_j(\omega_i) M_t(\theta_h) N_t(\theta_d, \phi) \cos\theta_i / \cos^2\theta_d \ \mathrm{d}\omega_i \end{aligned}$$

  where $\theta_h = (\theta_i + \theta_o)/2$, and $\theta_d = \|\theta_i - \theta_o\|/2$, and $\phi = \phi_o - \phi_i$.

- We write ω_i and ξ_j in Cartesian coordinates:

  $$\begin{aligned} \omega_i &= (\sin\theta_i, \cos\theta_i \cos\phi_i, \cos\theta_i \sin\phi_i) \\ \xi_j &= (\sin\theta_j, \cos\theta_j \cos\phi_j, \cos\theta_j \sin\phi_j). \end{aligned}$$

  Their dot product minus one is given by:

  $$\omega_i \cdot \xi_j - 1 = [\cos(\theta_i - \theta_j) - 1] + \cos\theta_i \cos\theta_j [\cos(\phi_i - \phi_j) - 1].$$

  Hence,

  $$\begin{aligned} G(\omega_i \cdot \xi_j; \lambda_j) &= e^{-2([\cos(\theta_i - \theta_j) - 1] + \cos\theta_i \cos\theta_j [\cos(\phi_i - \phi_j) - 1])/\lambda_j^2} \\ &= e^{-2[\cos(\theta_i - \theta_j) - 1]/\lambda_j^2} e^{-2\cos\theta_i \cos\theta_j [\cos(\phi_i - \phi_j) - 1]/\lambda_j^2} \\ &= g^c(\theta_i; \theta_j, \lambda_j) g^c(\phi_i; \phi_j, \lambda_i / \sqrt{\cos\theta_i \cos\theta_j}) \end{aligned}$$

  where $g^c(x; \mu, \lambda) = e^{2[\cos(x-\mu) - 1]/\lambda^2}$. To make the notation even shorter, let

  $$\begin{aligned} &g_j^c(\theta_i) = g^c(\theta_i; \theta_j, \lambda_j), \text{ and} \\ &g_j^c(\phi_i) = g^c(\phi_i; \phi_j, \lambda_j / \sqrt{\cos\theta_i \cos\theta_j}), \end{aligned}$$

  so that we can write $G_j(\omega_i) = g_j^c(\theta_i) g_j^c(\phi_i)$.

- Now that we have factored the SRBF, we can write:

$$\begin{aligned}\mathcal{M}_t(\xi_j,\lambda_j) &= \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} g_j^c(\theta_i)g_j^c(\phi_i)M_t(\theta_h)N_t(\theta_d,\phi)\frac{\cos^2\theta_i}{\cos^2\theta_d}\ \mathrm{d}\phi_i\mathrm{d}\theta_i \\ &= \int_{-\pi/2}^{\pi/2} g_j^c(\theta_i)M_t(\theta_h)\frac{\cos^2\theta_i}{\cos^2\theta_d}\left(\int_0^{2\pi} g_j^c(\phi_i)N_t(\theta_d,\phi)\ \mathrm{d}\phi_i\right)\mathrm{d}\theta_i.\end{aligned}$$

- Now consider the azimuthal integral:

$$\int_0^{2\pi} g_j^c(\phi_i)N_t(\theta_d,\phi)\ \mathrm{d}\phi_i = \int_0^{2\pi} g^c(\phi_i;\phi_j,\lambda_j/\sqrt{\cos\theta_i\cos\theta_j})N_t(\theta_d,\phi_o-\phi_i)\ \mathrm{d}\phi_i.$$

  Let $\phi_i' = \phi_i - \phi_j$ and $\lambda_j' = \lambda_j/\sqrt{\cos\theta_i\cos\theta_j}$. We can write the above integral in terms of ϕ_i':

$$\int_0^{2\pi} g^c(\phi_i;\phi_j,\lambda_j')N_t(\theta_d,\phi_o-\phi_i)\ \mathrm{d}\phi_i = \int_0^{2\pi} g^c(\phi_i';0,\lambda_j')N_t(\theta_d,(\phi_o-\phi_j)-\phi_i')\ \mathrm{d}\phi_i'.$$

  We shall abbreviate the RHS as $\mathcal{N}_t(\lambda_j',\phi_o-\phi_j,\theta_d)$.

- Now make the following approximations:

$$g_j^c(\theta_i) \approx e^{-(\theta_i-\theta_j)^2/\lambda_j} = g(\theta_i;\theta_j,\lambda_j) = g_j(\theta_i).$$

  Therefore,

$$\mathcal{M}_t(\xi_j,\lambda_j) = \int_{-\pi/2}^{\pi/2} g_j(\theta_i)M_t(\theta_h)\frac{\cos^2\theta_i}{\cos^2\theta_d}\mathcal{N}_t(\lambda_j',\phi_o-\phi_j,\theta_d)\ \mathrm{d}\theta_i$$

  Thet task now is to evaluate the inner integral $\mathcal{N}_t$ and the outer integral $\mathcal{M}_t$ for each of the three modes.

3.1 Approximating the R Mode

- For the R mode, we have

$$N_R(\eta,\theta_d,\phi) = \frac{1}{4}|\cos(\phi/2)|F(\eta,\theta_d,-\sin(\phi/2))$$

  were $F(\eta,\theta_d,h)$ is the Fresnel reflectance.

- We approximate the Fresnel term with Schlick's approximation:

$$F(\eta,\theta_d,h) \approx F_0 + (1-F_0)(1-\cos\theta_d\sqrt{1-h^2})^5$$

  where $F_0 = (1-\eta)^2/(1+\eta)^2$.

- Substituting $h = -\sin(\phi/2)$, we have

$$\begin{aligned}F(\eta,\theta_d,-\sin(\phi/2)) &\approx F_0 + (1-F_0)(1-\cos\theta_d\sqrt{1-\sin^2(\phi/2)})^5 \\ &= F_0 + (1-F_0)(1-\cos\theta_d|\cos(\phi/2)|)^5.\end{aligned}$$

  Now, N_R becomes a polynomial of $|\cos(\phi/2)|$ of degree 6. Let us write:

$$N_R(\eta,\theta_d,\phi) \approx \sum_{k=0}^{6} C_k(\theta_d,\eta)|\cos^k(\phi/2)|.$$

- As such, the inner integral can be approximate as:

$$\mathcal{N}_R(\lambda_j', \phi_o - \phi_j, \theta_d) = \sum_{k=0}^{6} C_k(\theta_d, \eta) \int_0^{2\pi} \left| \cos^k \left(\frac{\phi_o - \phi_j - \phi_i'}{2} \right) \right| g^c(\phi_i'; 0, \lambda_j')\ \mathrm{d}\phi_i'$$
$$= \sum_{k=0}^{6} C_k(\theta_d, \eta) \mathcal{C}_k(\lambda_j', \phi_o - \phi_j)$$

  where $\mathcal{C}_k(\lambda_j', \phi_o - \phi_j)$ stands for the integral in the RHS. The paper precomputes this for various values of λ_j' and $\phi_o - \phi_j$ and look up its value in run time.

- Now, we need to figure out the outer integral $\mathcal{M}_R(\xi_j, \lambda_j)$. Consider the term $M_R(\theta_h)$. We have that

$$M_R(\theta_h) = g^u(\theta_h; \alpha_R, \beta_R) = 2g^u(\theta_i; 2\alpha_R - \theta_o, 2\beta_R).$$

  Therefore, the term $g_j(\theta_i)M_R(\theta_h)$ is a product of two 1D Gaussians, which means that it is also a Gaussian. Hence, $\mathcal{M}_R(\xi_j, \lambda_j)$ is an integral of a product of the function $\frac{\cos^2 \theta_i}{\cos^2 \theta_d} \mathcal{N}_t(\lambda_j', \phi_o - \phi_j, \theta_d)$ with a Gaussian. The paper estimates this product integral with a numerical quadrature procedure that we will detail in the next section.

3.2 Approximate Product Integral with 1D Gaussian

- Given a Guassian $g(x; \mu, \lambda)$ and an arbitrary function $f(x)$, we would like to approximate the integral

$$\int_{r_0}^{r_1} g(x; \mu, \lambda) f(x)\ \mathrm{d}x,$$

  which generally has no analytic solution.

- We assume that $f(x)$ is smooth and approximate it as being piecewise linear. To do so:
  - We take $m + 1$ samples $r_0 = x_0 < x_1 < x_2 < \ldots < x_m = r_1$ from the interval $[r_0, r_1]$.
  - We approximate $f(x)$ on the interval $[x_s, x_{s_1}]$ as $f(x) \approx b_s x + c_s$.
    Here, b_s and c_s are linear coefficients computed from $f(x_s)$ and $f(x_{s+1})$.

- The product integral becomes:

$$\sum_{s=0}^{m-1} \int_{x_s}^{x_{s+1}} g(x; \mu, \lambda)(b_s x + c_s)\ \mathrm{d}x$$

  which can be computed analytically because the product of a Gaussian with a linear function has an analytic solution involving the error function. (See the supplement material of the paper for details.)

- To compute $\mathcal{M}_t(\xi_j, \lambda_j)$ is to compute the product integral over the interval $[-\pi/2, \pi/2]$. The paper picks $m + 1$ samples by:
  - Pick the two end points $-\pi/2$ and $\pi/2$.
  - Pick $m - 1$ samples uniformly from the interval $[u - (m-2)\lambda/2, \mu + (m-2)\lambda/2]$.

  The paper found that using $m = 4$ is enough.

3.3 Approximating the TT Mode

- For the TT mode, we have that

$$N_{TT}(\eta, \theta_d, \sigma_a, \phi) = \frac{1}{2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}h}\right|^{-1}(1 - F(\eta, \theta_d, h))^2 T(\sigma_a, \theta_d, h).$$

- The paper approximates the above function by a single circular Gaussian centered at $\phi = \pi$ (apparently because the TT mode is forward scattering):

$$N_{TT}(\eta, \theta_d, \sigma_a, \phi) \approx b_{TT}(\eta, \sigma_d, \sigma_a) g^c(\phi; \pi, \lambda_{TT}(\eta, \theta_d, \sigma_a))$$

  where $b_T T$ and $\lambda_T T$ denote the scaling factor and the width of the Gaussian. Both depends on η, θ_d, and σ_a.

- b_{TT} is set to be the value of N_{TT} at $\phi = \pi$.

- $\lambda_T T$ is defined as:

$$\lambda_{TT}(\eta, \theta_d, \sigma_a) = \frac{1}{\sqrt{\pi} b_{TT}(\eta, \theta_d, \sigma_a)} \int_{-\pi}^{\pi} N_{TT}(\eta, \theta_d, \sigma_a, \phi)\ \mathrm{d}\phi.$$

  Thus, our task comes down to approximating the integral of N_{TT} over $[-\pi, \pi]$.

- To approximate the integral, we first approximate the transmittance function $T(\sigma_a, \theta_d, h)$ with a fourth order Taylor expansion:

$$T(\sigma_a, \theta_d, h) \approx \sum_{k \in \{0,2,4\}} a_k(\theta_d, \sigma_a) h^k$$

  where $a_k(\theta_d, \sigma_a)$ is a Taylor expansion coefficient which has an analytic expression.

- Now, we have that

$$\begin{aligned}
\int_{-\pi}^{\pi} N_{TT}(\eta, \theta_d, \sigma_a, \phi)\ \mathrm{d}\phi &= \int_{-\pi}^{\pi} \frac{1}{2}\left|\frac{\mathrm{d}\phi}{\mathrm{d}h}\right|^{-1}(1 - F(\eta, \theta_d, h))^2 T(\sigma_a, \theta_d, h)\ \mathrm{d}\phi \\
&= \frac{1}{2}\int_{-1}^{1} (1 - F(\eta, \theta_d, h))^2 T(\sigma_a, \theta_d, h)\ \mathrm{d}h \\
&\approx \frac{1}{2}\int_{-1}^{1} (1 - F(\eta, \theta_d, h))^2 \bigg(\sum_{k \in \{0,2,4\}} a_k(\theta_d, \sigma_a) h^k \bigg)\ \mathrm{d}h \\
&= \frac{1}{2} \sum_{k \in \{0,2,4\}} a_k(\theta_d, \sigma_a) \bigg(\int_{-1}^{1} (1 - F(\eta, \theta_d, h))^2 h^k\ \mathrm{d}h \bigg).
\end{aligned}$$

  Let us the denote the integral $\int_{-1}^{1}(1 - F(\eta, \theta_d, h))^2 h^k\ \mathrm{d}h$ as $\mathcal{K}_k^{TT}(\eta, \theta_d)$. The paper precomputes these integrals for various values of η and θ_d and stores the results in a 2D table for run-time lookups.

- Now that we have approximated N_{TT}, our next task is to evaluate the integral

$$\mathcal{N}_{TT}(\lambda'_j, \phi_o - \phi_j, \theta_d) = \int_0^{2\pi} g^c(\phi'_i; 0, \lambda'_j) N_{TT}(\theta_d, (\phi_o - \phi_j) - \phi'_i)\ \mathrm{d}\phi'_i.$$

  Note that we have approximated N_{TT} as a circular Gaussian, we have that the integrand is a product of two circular Gaussians, which is still a circular Gaussian. The integral has an analytic solution which can be easily computed.

- The outer integral $\mathcal{M}_{TT}$ is a product integral between a Gaussian and the function $\mathcal{N}_{TT} \cos^2\theta_i / \cos^2\theta_d$, which is smooth. We evaluate this integral the same way we evaluated the integral for the R mode.

3.4 Approximating the TRT Mode

- In the Marschner model, the TRT has two types of behavior. When $\eta' < 2$, the azimuthal scattering function has two peaks. When $\eta' \geq 2$, it has no peaks.
- When $\eta' < 2$, Xu et al. approximates the azimuthal scattering function as the sum of three Gaussians:

$$N_{TRT} \approx b_1(g^c(\phi; \phi^*, w_c) + g^c(\phi; -\phi^*, w_c)) + b_2 g^c(\phi; 0, \phi^*).$$

  where ϕ^* is the peak location which is given by:

$$\phi^* = \phi(h^*) = \phi\bigg(\sqrt{\frac{4 - \eta'^2}{3}} \bigg),$$

  and w_c is the width of the peak, which the user sets.

- The coefficient b_2 is set to:

$$b_2 = (1 - g^c(0; \phi^*, w_c))^2 N_{TRT}(0).$$

- The coefficient b_1 is set to:

$$b_1 = \frac{\int_{-\pi}^{\pi} N_{TRT}(\phi)\ \mathrm{d}\phi - \int_{-\pi}^{\pi} b_2 g^c(\phi)\ \mathrm{d}\phi}{2\int_{-\pi}^{\pi} g^c(\phi; \phi^2, w_c)\ \mathrm{d}\phi} = \frac{\int_{-\pi}^{\pi} N_{TRT}(\phi)\ \mathrm{d}\phi - b_2\sqrt{\pi}\phi^*}{2\sqrt{\pi}w_c}.$$

- To approximate the $\int_{-\pi}^{\pi} N_{TRT}\mathrm{d}\phi$ term, we use the same technique used to apprixmate $\int_{-\pi}^{\pi} N_{TT}\mathrm{d}\phi$. First, we write:

$$\int_{-\pi}^{\pi} N_{TRT}(\phi)\ \mathrm{d}\phi = \frac{1}{2}\int_{-1}^{1} (1 - F(\eta, \theta_d, h))^2 F(\eta, \theta_d, h) T^2(\sigma_a, \theta_d, h)\ \mathrm{d}h$$

  Then, we approximate

$$T^2(\sigma_a, \theta_d, h) \approx \sum_{k \in \{0,2,4\}} c_k(\theta_d, \sigma_a) h^k$$

  where each $c_k(\theta_d, \sigma_a)$ is a Taylor series expansion coefficient that has analytic expressions.

  Thus, we have that

$$\int_{-\pi}^{\pi} N_{TRT}(\phi)\ \mathrm{d}\phi \approx \frac{1}{2} \sum_{k \in \{0,2,4\}} c_k(\theta_d, \sigma_a) \bigg(\int_{-1}^{1} (1 - F(\eta, \theta_d, h))^2 F(\eta, \theta_d, h) h^k\ \mathrm{d}h \bigg)$$

  Let $\mathcal{K}^{TRT}(\eta, \theta_d)$ denote the integral $\int_{-1}^{1} (1 - F(\eta, \theta_d, h))^2 F(\eta, \theta_d, h) h^k\ \mathrm{d}h$. We precompute a table of $\mathcal{K}^{TRT}(\eta, \theta_d)$ for real-time lookups.

- When $\eta' \geq 2$, N_{TRT} is approximated using a single Guassian:

$$N_{TRT} \approx b_3 g^c(\phi; 0, \lambda_3).$$

  The coefficient b_3 is set to the value of N_{TRT} at $\phi = 0$, and λ_3 is determined by trying to preserve the energy just like in the TT mode case:

$$\lambda_3 = \frac{1}{\sqrt{\pi} b_3} \int_{-\pi}^{\pi} N_{TRT}(\phi)\ \mathrm{d}\phi.$$

- The approximation of N_{TRT} when $\eta' \geq 2$ might lead to discontinuity when η' changes across 2.

  This is a problem because, when $\eta' < 2$, the center Gaussian bandwidth is w_c. However, when $\eta' \geq 2$, when Gaussian bandwidth is λ_3.

  Xu et al. interpolates the bandwidth of the center Gaussian between w_c and λ_3 over a small range of $\Delta\eta'$:

$$\lambda_3' = \mathrm{interp}(w_c, \lambda_3, \mathrm{smoothstep}(2, 2 + \Delta\eta', \eta')).$$

  The coefficient b' is modified to $b_3\lambda_3/\lambda_3'$ to preserve the energy of the lobe.

- The outer integral $\mathcal{M}_{TRT}$ is approximate using linear quadrature as was done in the TT mode.

3.5 Handling Eccentricity in the TRT Mode

- Marschner proposed using varying refraction index η^* that depends on the azimuthal half angle ϕ_h to simulate hair eccentricity:

$$\begin{aligned} \eta_1^* &= 2(\eta - 1)a^2 - \eta + 2 \\ \eta_2^* &= 2(\eta - 1)a^{-2} - \eta + 2 \\ \eta^*(\phi_h) &= ((\eta_1^* + \eta_2^*) + \cos(2\phi_h)(\eta_1^* - \eta_2^*))/2 \end{aligned}$$

  where a is the eccentricity parameter.

- To incorporate eccentricity, the inner integral $\mathcal{N}_{TRT}$ is changed to:

$$\begin{aligned} \mathcal{N}_{TRT} &= \int g^c(\phi_i'; 0, \lambda_j') N_{TRT}(\eta^*(\phi_h), \theta_d, \phi_o - \phi_j - \phi_i')\ \mathrm{d}\phi_i' \\ &\approx \int g^c(\phi_i'; 0, \lambda_j') N_{TRT}(\overline{\eta^*}, \theta_d, \phi_o - \phi_j - \phi_i')\ \mathrm{d}\phi_i' \end{aligned}$$

  where

$$\overline{\eta^*} = \frac{\int g^c(\phi_i'; 0, \lambda_j)\eta^*(\phi_h)\ \mathrm{d}\phi_i'}{\int g^c(\phi_i'; 0, \lambda_j)\ \mathrm{d}\phi_i'}.$$

  The average index of refraction $\overline{\eta^*}$ can be efficiently computed on the fly by reusing the precomputed $\mathcal{C}_k(\lambda_j', \phi_o - \phi_j)$. This because we have that $\cos(2\phi_h) = 2\cos^2(\phi_h) - 1 = 2\cos^2((\phi_i' - \phi_j + \phi_o)/2) - 1$, which means we only need $\mathcal{C}_0$ and $\mathcal{C}_2$.

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