# Differential Geometry Notes of 02/17/2013

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## 1 The Tangent Plane

- By a **tangent vector** to a regular surface $S$, at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parameterized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

- **Proposition 1.1.** *Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ be a parameterization of regular surface $S$. Let $q \in U$. The vector subspace of dimension 2,*

$$\mathrm{d}\mathbf{x}_q(\mathbb{R}^2) \subseteq \mathbb{R}^3$$

*coincides with the set of tangent vectors to $S$ at $\mathbf{x}(q)$.*

*Proof.* Let $w$ be a tangent vector at $\mathbf{x}$. That is, let $w = \alpha'(0)$, where $\alpha : (\epsilon, -\epsilon) \to \mathbf{x}(U) \subseteq S$ is differentiable and $\alpha(0) = \mathbf{x}(q)$. Because $\mathbf{x}^{-1}$ is a differentiable function (See Example 2 of Section 2-3 of Do Carmo.), we have that $\beta = \mathbf{x}^{-1} \circ \alpha : (-\epsilon, \epsilon) \to U$ is a differentiable function. By the definition of differentials, we have that $\mathrm{d}\mathbf{x}_q(\beta'(0)) = w$. Hence, $w \in \mathrm{d}\mathbf{x}_q(\mathbb{R}^2)$.

On the other hand, let $w = \mathrm{d}\mathbf{x}_q(v)$, where $v \in \mathbb{R}^2$. It is lcear that $v$ is the velocity vector of the curve $\gamma : (-\epsilon, \epsilon) \to U$ given by:

$$\gamma(t) = tv + q.$$

By the definition of the differential, $w = \alpha'(0)$ where $\alpha = \mathbf{x} \circ \gamma$. □

- By the above proposition, the plane $\mathrm{d}\mathbf{x}_q(\mathbb{R}^2)$ does not depend on the parametermization $\mathbf{x}$.
We call this plane the **tangent plane** to $S$ at $p$.
We denote the plane by the symbol $T_p(S)$.

- The choice of parametermization $\mathbf{x}$ around $p$ determine the basis vectors $(\partial \mathbf{x}/\partial u)(q)$ and $(\partial \mathbf{y}/\partial v)(q)$ of $T_p(S)$.
We call them the **basis associated to** x.

- We sometimes write $\partial \mathbf{x}/\partial u$ as $\mathbf{x}_u$ and $\partial \mathbf{x}/\partial v$ as $\mathbf{x}_v$.

- If $\alpha = \mathbf{x} \circ \beta$ where $\beta : (-\epsilon, \epsilon) \to U$ is given by $\beta(t) = (u(t), v(t))$, then the tangent vector $\alpha'(0)$ can be written in terms of the above basis vectors as follows:

$$\alpha'(0) = \frac{\mathrm{d}(\mathbf{x} \circ \beta)}{\mathrm{d}t}(0) = \frac{\mathrm{d}\mathbf{x}(u(t), v(t))}{\mathrm{d}t}(0) = \left.\frac{\partial \mathbf{x}}{\partial u}\right|_q u'(0) + \left.\frac{\partial \mathbf{x}}{\partial v}\right|_q v'(0) = \mathbf{x}_u(q)u'(0) + \mathbf{x}_v(q)v'(0).$$

So, the vector $\alpha'(0)$ has coordinate $(u'(0), v'(0))$ in the basis $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$.

## 2 Differentials of Maps between Surfaces

- Let $S_1$ and $S_2$ be two regular surfaces.
  Let $\varphi : V \subseteq S_1 \to S_2$ be a differentiable mapping of an open set $V$ of $S_1$ to $S_2$.

- If $p \in V$, we know that every tangent vector $w \in T_p(S)$ is the velocity vector $\alpha'(0)$ for a differentiable curve $\alpha : (-\epsilon, \epsilon) \to V$ where $\alpha(0) = p$.

- Let $\beta = \varphi \circ \alpha$. We have that $\beta(0) = \varphi(p)$.
  As a result $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$

- We now define $\mathrm{d}\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ as follows:

$$\mathrm{d}\varphi_p(w) = \beta'(0)$$

  for any differentiable curve $\alpha : (-\epsilon, \epsilon) \to S_1$ such that $\alpha(0) = p$ and $\alpha'(0) = w$ and $\beta = \varphi \circ \alpha$.

- **Proposition 2.1.** *Given $w$, the definition above does not depend on the choice of $\alpha$. Moreover, the map $\mathrm{d}\varphi_p$ is linear.*