Vector Fields in Cylindrical and Spherical Coordinates

Pramook Khungurn

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1 Vector Fields in Cartesian Coordinate

- A vector field $\mathbf{f}$ in 3 dimentions is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$.
- To specify a vector field in Cartesian coordinate, you have to specify three scalar functions f_x, f_y, and f_z so that

$$\mathbf{f}(x, y, z) = f_x(x, y, z)\hat{\mathbf{i}} + f_y(x, y, z)\hat{\mathbf{j}} + f_z(x, y, z)\hat{\mathbf{k}}$$

2 Vector Fields in Cylindrical Coordinate

- The cylindrical coordinate system specifies a point in $\mathbb{R}^3$ by three numbers: r, θ, and z.
- The point (r, θ, z) in cylindrical coordinate is equivalent to the point (x, y, z) in Cartesian coordinate where

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

- A vector field can also be specified in cylindrical coordinate system. What we mean by this is that, for any given point (r, θ, z) in $\mathbf{R}^3$, we construct a coordinate system specific to that point such that:
  - the point (r, θ, z) being the origina,
  - there are three orthonomal basis vectors, corresponding to the r, θ, and z corodinate, and
  - each basis vector points in the direction where the corresponding coordinate is increasing.
- Let

$$\mathbf{r}(r, \theta, z) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}.$$

- If u is one of r, θ, and z, then the vector $\partial\mathbf{r}/\partial u$ points in the direction where u is increasing. We define the basis vector as:

$$\hat{u} = \frac{\partial\mathbf{r}/\partial u}{\|\partial\mathbf{r}/\partial u\|}.$$

- So, in case of spherical coordinate, we have

$$\frac{\partial \mathbf{r}}{\partial r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{r} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \frac{\partial \mathbf{r}}{\partial r}, \qquad \hat{\theta} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta}, \qquad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\partial \mathbf{r}}{\partial z}$$

- A vector field $\mathbf{f}$, then can be specified by specifying three scalar functoins f_r, f_θ, and f_z so that

$$\mathbf{f}(r,\theta,z) = f_r(r,\theta,z)\hat{\mathbf{r}}(r,\theta,z) + f_\theta(r,\theta,z)\hat{\theta}(r,\theta,z) + f_z(r,\theta,z)\hat{\mathbf{z}}.$$

- When calculating the div, the grad, or the curl, it helps to have the derivative of the basis vectors:

$$\begin{aligned}
\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \hat{\theta}\dot{\theta} \\
\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\hat{r}\dot{\theta} \\
\frac{\mathrm{d}\hat{z}}{\mathrm{d}t} &= \mathbf{0}.
\end{aligned}$$

3 Vector Field in Spherical Coordinate

- With spherical coordinate, we specify a position by three parameters—r, θ, φ—with the function:

$$\mathbf{r}(r,\theta,\varphi) = \begin{bmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{bmatrix}.$$

- As a result,

$$\frac{\partial \mathbf{r}}{\partial r} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{bmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{bmatrix}.$$

So,

$$\begin{aligned}
\hat{r} &= \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} = \frac{\partial \mathbf{r}}{\partial r}, \\
\hat{\theta} &= \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{r}\begin{bmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix}, \\
\hat{\varphi} &= \frac{1}{r\sin\theta}\frac{\partial \mathbf{r}}{\partial \varphi} = \frac{1}{r\sin\theta}\begin{bmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix}.
\end{aligned}$$

- For derivatives, we have

$$
\begin{aligned}
\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} &= \frac{\partial \hat{r}}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial \hat{r}}{\partial \varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \begin{bmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{bmatrix}\dot{\theta} + \begin{bmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{bmatrix}\dot{\varphi} = \hat{\theta}\dot{\theta} + \hat{\varphi}\dot{\varphi}\sin\theta \\
\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} &= \frac{\partial \hat{\theta}}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\partial \hat{\theta}}{\partial \varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \begin{bmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{bmatrix}\dot{\theta} + \begin{bmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{bmatrix}\dot{\phi} = -\hat{r}\dot{\theta} + \hat{\varphi}\dot{\varphi}\cos\theta \\
\frac{\mathrm{d}\hat{\varphi}}{\mathrm{d}t} &= \begin{bmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -(\sin^2\theta + \cos^2\theta)\cos\varphi \\ -(\sin^2\theta + \cos^2\theta)\sin\varphi \\ -\sin\theta\cos\theta + \cos\theta\sin\theta \end{bmatrix}\dot{\varphi} = \left(\begin{bmatrix} -\sin^2\theta\cos\phi \\ -\sin^2\theta\sin\phi \\ -\sin\theta\cos\phi \end{bmatrix} + \begin{bmatrix} -\cos^2\theta\cos\phi \\ -\cos^2\theta\sin\phi \\ +\sin\theta\cos\theta \end{bmatrix} \right)\dot{\varphi} \\
&= \left(-\sin\theta \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix} - \cos\theta \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix} \right)\dot{\varphi} = -(\hat{r}\sin\theta + \hat{\theta}\cos\theta)\dot{\varphi}.
\end{aligned}
$$