Differential Geometry Notes of 05/02/2013

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1 The Exponential Map

- Given a point p of a regular surface S,
  and a non-zero vector $v \in T_p(S)$,
  tehre exist a unique parameterized geodesic $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ and $\gamma'(0) = v$.

- We shall denote $\gamma(t, v) = \gamma$ to indicate the depencence of the geodesic on v.

- **Lemma 1.1.** *If the geodesic $\gamma(t, v)$ is defined for $t \in (-\epsilon, \epsilon)$, then the geodesic $\gamma(t, \lambda v)$ with $\lambda \in \mathbb{R}$, $\lambda \neq 0$, is defined for $t \in (-\epsilon/\lambda, \epsilon/\lambda)$, and $\gamma(t, \lambda v) = \gamma(\lambda t, v)$.*

  Proof. Let $\alpha : (-\epsilon/\lambda, \epsilon/\lambda) \to S$ be a parameterized curve defined by $\alpha(t) = \gamma(\lambda t)$. Then, $\alpha(0) = \gamma(0) = p$. Also, $\alpha'(0) = \frac{\mathrm{d}\gamma(\lambda t)}{\mathrm{d}t}\big|_{t=0} = \lambda\gamma'(0) = \lambda v$. By the linearity of

  $$\frac{D\alpha'(t)}{\mathrm{d}t} = \frac{D(\gamma'(\lambda t))}{\mathrm{d}t} = \frac{D(\gamma'(\lambda t))}{\mathrm{d}(\lambda t)}\frac{\mathrm{d}(\lambda t)}{\mathrm{d}t} = \mathbf{0}.$$

  This is because $\gamma(\lambda t)$ is a geodesic. It follows that α is a geodesic whose $\alpha(0) = \lambda(0)$ and $\alpha'(0) = \lambda\gamma'(0)$. By uniqueness of geodesic,

  $$\alpha(t) = \gamma(t, \lambda v) = \gamma(\lambda t, v)$$

  as required. □

- If $v \in T_p(S)$, $v \neq \mathbf{0}$, is that $\gamma(|v|, v/|v|) = \gamma(1, v)$ is defined, we set

  $$\exp_p(v) = \gamma(1, v), \text{ and } \exp_p(\mathbf{0}) = p.$$

- **Proposition 1.2.** *Given $p \in S$, there exists an $\epsilon > 0$ such that $\exp_p$ is defined and differentiable in the interior of B_ϵ of a disk of radius ϵ of $T_p(S)$, with the center in the origin.*

  Proof. For every direction of $T_p(S)$, it is possible by the last lemma to take v sufficiently small so that the definition of $\gamma(t, v)$ contains 1. Thus, $\gamma(1, v) = \exp_p(v)$ is defined.

  The next problem is that, if we let v varies through all the direction, ϵ does not go to zero. However, the following proposition is true:

  > Given $p \in S$, there exists numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and a differentiable map
  >
  > $$\gamma : (-\epsilon_2, \epsilon_2) \times B_{\epsilon_1} \to S$$
  >
  > such that, for $v \in B_{\epsilon_1}$, $v \neq \mathbf{0}$, $t \in (-\epsilon_2, \epsilon_2)$, the curve $\gamma(t, v)$ is a geodesic of S with $\gamma(0, v) = p$ and $\gamma'(0, v) = v$. Moreover, $\gamma(t, \mathbf{0}) = p$.

Since $\gamma(t,v)$ is defined for $|t| < \epsilon_2$ and $|v| < \epsilon_1$, we can set $\lambda = \epsilon_2/2$, so that $\gamma(t,(\epsilon_2/2)v)$ is defined for $|t| < 2$ and $|v| < \epsilon_1$. Hence, $\exp_p(v) = \gamma(1,v)$ is defined for all $|v| < \epsilon_1\epsilon_2/2$. The differentiability of $\exp_p$ follows from the differentiability of $\gamma(t,v)$. □

- **Proposition 1.3.** $\exp_p : B_\epsilon \subseteq T_p(S) \to S$ *is a diffeomorphism in a neighborhood* $U \subseteq B_\epsilon$ *of the origin* $\mathbf{0}$ *of* $T_p(S)$.

  Proof. We shall show that $\mathrm{d}(\exp_p)$ is non-singular at $\mathbf{0} \in T_p(S)$. To do this, we identify the space of the tangent vectors to $T_p(S)$ at $\mathbf{0}$ with $T_p(S)$ itself.

  Consider the curve $\alpha(t) = tv$, $v \in T_p(S)$. We have that $\alpha(0) = \mathbf{0}$ and $\alpha'(0) = v$. The curve $(\exp_p \circ \alpha)(t) = \exp_p(tv)$. Therefore,

  $$\frac{\mathrm{d}}{\mathrm{d}t}(\exp_p(tv))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t,v))\bigg|_{t=0} = v.$$

  If follows that $\mathrm{d}(\exp_p)_{\mathbf{0}}(v) = v$, which means that it is non-singular. The proposition is true by applying the inverse function theorem. □

- We call $V \subseteq S$ a **normal neighborhood of** $p \in S$ if V is the image of $\exp_p(U)$ of the origin of $T_p(S)$, restricted to which $\exp_p$ is a diffeomorphism.

2 Coordinates Defined by Exponential Maps

- The exponential map at $p \in S$ is diffeomorphism on U, it can be used to define coordinates in V. The most usual coordinate systems are:
  - The **normal coordinates** which corresponds to a system of rectangular coordinates in the tangent space $T_p(S)$.
  - The **geodesic polar coordinates** which corresponds to the polar coordinates in the tangent space $T_p(S)$.

- The normal coordinate system can be obtained by choosing two orthogonal vectors e_1 and e_2 in $T_p(S)$. Now, we can define the parameterization $\mathbf{x} : U \subseteq \mathbb{R}^2 \to S$ as:

  $$\mathbf{x}(u,v) = \exp_p(ue_1 + ve_2).$$

  The parameterization, of course, depends on e_1 and e_2.

- In the normal coordinate system, the geodesics that pass through p are the images of $\exp_p$ of the line:

  $$\begin{aligned} u &= at \\ v &= vt, \end{aligned}$$

  which pass through $(0,0)$, which maps to p.

- Let us calculate $\mathbf{x}_u$ and $\mathbf{x}_v$ at p. We have that

  $$\frac{\mathrm{d}(\mathbf{x}(u'te_1 + v'te_2))}{\mathrm{d}t}\bigg|_{t=0} = \mathbf{x}_u u' + \mathbf{x}_v v'.$$

  Therefore,

  $$\mathbf{x}_u = \frac{\mathrm{d}(\mathbf{x}(te_1))}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}\gamma(1,te_1)}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}\gamma(t,e_1)}{\mathrm{d}t}\bigg|_{t=0} = e_1.$$

  Also, we can similarly argue that $\mathbf{x}_v = e_2$.

  Hence, the coefficients of the first fundamental form are $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1$, $G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1$, and $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$.

- We now study the geometric polar coordinates.
  We pick a system of polar coordinate (ρ, θ) around p in $T_p(S)$.
  Here, $\theta \in (0, 2\pi)$, and $\rho \in (0, \infty)$.

  The polar coordinates are not defined in the half line $l = \{(x, 0) : x \in [0, \infty)\}$.
  Let $L = \exp_p(l)$.
  So, the geodesic polar coordinate is a function from $U - l$ to $V - L$.

- The images by $\exp_p : U \to V$ of circles in U centered at $\mathbf{0}$ are called the **geodesic circles**.
  The images of $\exp_p$ of the lines through $\mathbf{0}$ are called the **radial geodesics**.
  These are curves with $\rho = const.$ and $\theta = const.$, respectively.

- **Proposition 2.1.** *Let $\mathbf{x} : U - l \to V - L$ be a system of geodesic polar coordinate (ρ, θ). Then, the coefficients $E = E(\rho, \theta)$. $F = F(\rho, \theta)$, and $G = G(\rho, \theta)$ of the first fundamental form statisfy the coditions.*

  $$E = 1, \qquad F = 0, \qquad \lim_{\rho \to 0} G = 0, \qquad \lim_{\rho \to 0} (\sqrt{G})_\rho = 1.$$

  Proof. We first show that $E = 1$. Fix θ and pick a curve with $\rho = \rho_0 + t$. We have that

  $$\mathbf{x}_\rho(\rho, \theta) = \frac{\partial \mathbf{x}(\rho, \theta)}{\partial \rho} = \frac{\partial \gamma(\rho, (\cos\theta, \sin\theta))}{\partial \rho} = \gamma'(\rho, (\cos\theta, \sin\theta)).$$

  So, $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1$ because the velocity of the geodesic is constant and is equal to $|(\cos\theta, \sin\theta)| = 1$.

  Next, we will show that $F = 0$. To do so, we proceed in two steps.

  1. We will show that F does not depend on ρ; that is $F_\rho = 0$.
  2. Second, we will show that $\lim_{\rho \to 0} F(\rho, \theta) = 0$.

  The two assertions together show that $F = 0$ identically.

  Now, we show that $F_\rho = 0$. Notice that the curve given by setting $\theta = const.$ and $\rho = t$ is a geodesic. The curve satistfies the following differential equations of the geodesics:

  $$\rho'' + \Gamma^1_{11}(\rho')^2 + 2\Gamma^1_{12}\rho'\theta' + \Gamma^1_{22}(\theta')^2 = 0$$
  $$\theta'' + \Gamma^2_{11}(\rho')^2 + 2\Gamma^2_{12}\rho'\theta' + \Gamma^2_{22}(\theta')^2 = 0.$$

  Because $\theta' = 0$ and $\rho' = 1$, we have that the second equation becomes:

  $$\Gamma^2_{11} = 0.$$

  Now, the definition of the Christoffel symbols requires that:

  $$\Gamma^1_{11}E + \Gamma^2_{11}F = \frac{1}{2}E_\rho.$$

  Because $E = 1$ and $\Gamma^2_{11} = 0$, we also have that

  $$\Gamma^1_{11} = 0.$$

  Also, because

  $$\Gamma^1_{11}F + \Gamma^2_{11}G = F_\rho - \frac{1}{2}E_\theta,$$

  we have that

  $$F_\rho = 0.$$

Hence, $F(\rho,\theta)$ does not depend on ρ.

Next, we show that $\lim_{\rho\to 0} F(\rho,\theta) = 0$. For each $q \in V$, denote by $\alpha(\sigma)$ the geodesic circle that passes through q. Here, $\sigma \in (0, 2\pi)$. (Notice that, if $q = p$, then $\alpha(\sigma) = p$ reduces to a point.) Also, denote by $\gamma(s)$, where s is arclength of γ, the radial geodesics that passes through q. With this notation, we may write:

$$F(\rho,\theta) = \left\langle \frac{\mathrm{d}\alpha}{\mathrm{d}\sigma}, \frac{\mathrm{d}\gamma}{\mathrm{d}s} \right\rangle.$$

Notice that $F(\rho,\theta)$ is not defined at p. However, if we fix the radial geodesic $\theta = const.$, the derivative $\mathrm{d}\gamma/\mathrm{d}s$ is defined for every point on the geodesic. Also, since at p, $\alpha(\sigma) = p$ for all σ, it means that $\mathrm{d}\alpha/\mathrm{d}s = \mathbf{0}$. Thus, we have that

$$\lim_{\rho\to 0} F(\rho,\theta) = \lim_{\rho\to 0} \left\langle \frac{\mathrm{d}\alpha}{\mathrm{d}\sigma}, \frac{\mathrm{d}\gamma}{\mathrm{d}s} \right\rangle = 0.$$

It remains to show that $\lim_{\rho\to 0} G = 0$, and $\lim_{\rho\to 0} \sqrt{G}_\rho = 1$. Now, observe that since $E = 1$ and $F = 0$, we have that

$$\sqrt{EG - F^2} = \sqrt{G}.$$

Hence,

$$\lim_{\rho\to 0} \sqrt{G} = \lim_{\rho\to 0} \sqrt{EG - F^2},$$
$$\lim_{\rho\to 0} (\sqrt{G})_\rho = \lim_{\rho\to 0} (\sqrt{EG - F^2})_\rho.$$

Therefore, we can study the behavior of $\sqrt{EG - F^2}$ instead of G.

To study the behavior of $\sqrt{EG - F^2}$, we reparameterize the neighborhood with thenew variables $\bar{u}$ and $\bar{v}$ such that:

$$\bar{u} = \rho\cos\theta, \qquad \bar{v} = \rho\sin\theta$$

which is just the normal coordinate system. Recall that

$$\sqrt{EG - F^2} = \sqrt{\bar{E}\bar{G} - \bar{F}^2}\frac{\partial(\bar{u},\bar{v})}{\partial(\rho,\theta)}.$$

We know that $\sqrt{\bar{E}\bar{G} - \bar{F}^2} = 1$ at p. Also,

$$\frac{\partial\bar{u}}{\partial\rho} = \cos\theta, \qquad \frac{\partial\bar{v}}{\partial\rho} = \sin\theta, \qquad \frac{\partial\bar{u}}{\partial\theta} = -\rho\sin\theta \qquad \frac{\partial\bar{v}}{\partial\theta} = \rho\cos\theta.$$

So,

$$\frac{\partial(\bar{u},\bar{v})}{\partial(\rho,\theta)} = \frac{\partial\bar{u}}{\partial\rho}\frac{\partial\bar{v}}{\partial\theta} - \frac{\partial u}{\partial\theta}\frac{\partial\bar{v}}{\partial\rho} = \rho\cos^2\theta + \rho\sin^2\theta = \rho.$$

Hence, $\sqrt{G} = \sqrt{EG - F^2} = \rho$ at p. Thus,

$$\lim_{\rho\to 0} G = \lim_{\rho\to 0} \rho^2 = 0,$$
$$\lim_{\rho\to 0} \sqrt{G}_\rho = \lim_{\rho\to 0} 1 = 1$$

as required. $\square$

- The fact that $F = 0$ means that, in the normal neighborhood, the family of geodesic circles is orthogonal to the family of radial geodesics.

  This is known as the **Gauss lemma**.

- Since in the polar geodesic coordinate system, we have that $E = 1$ and $F = 0$. Now,

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_\theta}{\sqrt{EG}}\right)_\theta + \left(\frac{G_\rho}{\sqrt{EG}}\right)_\rho\right\} = -\frac{1}{2\sqrt{G}}\left(\frac{G_\rho}{\sqrt{G}}\right)_\rho = -\frac{1}{\sqrt{G}}\left(\frac{G_\rho}{2\sqrt{G}}\right)_\rho$$
$$= -\frac{1}{\sqrt{G}}\big((\sqrt{G})_\rho\big)_\rho = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}.$$

  The expression

$$K = -\frac{\sqrt{G}_{\rho\rho}}{\sqrt{G}}$$

  can be thought of as the differential equation which $\sqrt{G}(\rho, \theta)$ should satisfy if we want to have the surface to have the curvature $K(\rho, \theta)$.

3 Theorem of Minding

- If K is constant, the equation simplifies to

$$(\sqrt{G})_{\rho\rho} + K\sqrt{G} = 0,$$

  which is a linear differential equation of second order with constant coefficient.

- Let us study what E, F, and G have to be when K is constant.
  There are three cases: $K = 0$, $K > 0$, and $K < 0$.

- If $K = 0$, we have htat $(\sqrt{G}_{\rho\rho}) = 0$. Thus $(\sqrt{G})_\rho = g(\theta)$, a function of θ. Since

$$\lim_{\rho\to 0}(\sqrt{G})_\rho = 1,$$

  we conclude that $(\sqrt{G})_\rho = 1$ identically. So, $\sqrt{G} = \rho + f(\theta)$ with $f'(\theta) = g(\theta)$. Now,

$$0 = \lim_{\rho\to 0}\sqrt{G} = \lim_{\rho\to 0}\rho + \lim_{\rho\to 0}f(\theta) = f(\theta).$$

  Hence, we can conclude that $\sqrt{G} = \rho$. So,

$$E = 1, \quad F = 0, \quad G(\rho, \theta) = \rho^2.$$

- If $K > 0$, the general solution of $(\sqrt{G})_{\rho\rho} + K\sqrt{G} = 0$ is given by:

$$\sqrt{G} = A(\theta)\cos(\sqrt{K}\rho) + B(\theta)\sin(\sqrt{K}\rho).$$

  Since $\lim_{\rho\to 0}\sqrt{G} = 0$, we have that $A(\theta) = 0$. Thus,

$$\sqrt{G} = B(\theta)\sin(\sqrt{K}\rho).$$

  Also, we have that

$$1 = \lim_{\rho\to 0}(\sqrt{G})_\rho = \lim_{\rho\to 0}B(\theta)\sqrt{K}\cos(\sqrt{K}\rho) = B(\theta)\sqrt{K}.$$

  If follows that $B(\theta) = 1/\sqrt{K}$. Hence,

$$E = 1, \quad F = 0, \quad G = \frac{1}{K}\sin^2\sqrt{K}\rho.$$

- If $K < 0$, the general solution of $(\sqrt{G})_{\rho\rho} + K\sqrt{G} = 0$ is given by:

$$\sqrt{G} = A(\theta)\cosh(\sqrt{-K}\rho) + B(\theta)\sinh(\sqrt{-K}\rho).$$

  Again, we can find that:

$$E = 1, \quad F = 0, \quad G = \frac{1}{-K}\sinh^2(\sqrt{-K}\rho).$$

- **Theorem 3.1 (Minding).** *Any two regular surfaces with the same constant Gaussian curvature are locally isometric.*

  More precisely, let S_1, S_2 be two regular surfaces with the same constant curvature K.
  Choose point $p_1 \in S_1$ and $p_2 \in S_2$.
  Choose orthonormal basis $\{e_1, e_2\} \in T_{p_1}(S_1)$ and $\{f_1, f_2\} \in T_{p_2}(S_2)$.
  Then, there exists a neighborhood V_1 of p_1 and V_2 of p_2, and
  an isometry $\psi : V_1 \to V_2$ such that $\mathrm{d}\psi_{p_1}(e_1) = f_1$ and $\mathrm{d}\psi_{p_1}(e_2) = f_2$.

  Proof. Let V_1 and V_2 be normal neighborhood of p_1 and p_2, respectively. Let $\varphi : T_{p_1}(S_1) \to T_{p_2}(S_2)$ be the linear map such that $\varphi(e_1) = f_1$ and $\varphi(e_2) = f_2$. We have that φ is an isometry from $T_{p_1}(S_1)$ to $T_{p_2}(S_2)$. Let $\psi : V_1 \to V_2$ be defined by:

$$\psi = \exp_{p_2} \circ \varphi \circ (\exp_{p_1})^{-1}.$$

  We claim that ψ is the required isometry.

  Take a poloar coordinate system (ρ, θ) in $T_{p_1}(S_1)$ with axis l and set $L_1 = \exp_{p_1}(l)$ and $L_2 = \exp_{p_2}(\varphi(l))$. The restriction of $\bar{\psi}$ of ψ to $V_1 - L_1$ maps a polar coordinate neighborhood with coordinates (ρ, θ) centered at p_1 into a polar coordinate neighborhood with coordinates (ρ, θ) centered at p_2. Through the study of the coefficients of the first fundamental forms above, we have that the coefficients of the fundamental forms before and after the isometry are equal. So, $\bar{\psi}$ is an isometry. By continuity, ψ still preserves inner products of points of L_1, and so is an isometry. It is also easy to check that $\mathrm{d}\psi_{p_1}(e_1) = f_1$ and $\mathrm{d}\psi_{p_1}(e_2) = f_2$. □

- When K is not constant but maintains its sign, the expression $\sqrt{G}K = -(\sqrt{G})_{\rho\rho}$ has a nice intuitive meaning.

- Consider the arc length $L(\rho)$ of the curve $\rho = \mathit{const.}$ between two close geodesics $\theta = \theta_0$ and $\theta = \theta_1$:

$$L(\rho) = \int_{\theta_0}^{\theta_1} \sqrt{E(\rho')^2 + F\rho'\theta' + G(\rho,\theta)(\theta')^2}\,\mathrm{d}\theta = \int_{\theta_0}^{\theta_1} \sqrt{G(\rho,\theta)}\,\mathrm{d}\theta$$

  where $\rho' = 0$ because $\rho = \mathit{const.}$ and $\theta' = 1$ because we want θ to vary constantly.

  Assume that $K < 0$. Since,

$$\lim_{\rho\to 0}(\sqrt{G})_\rho = 1, \quad \text{and} \quad (\sqrt{G})_{\rho\rho} = -K\sqrt{G} > 0.$$

  This means that $(\sqrt{G})_\rho$ is increasing. Since $(G)_\rho$ is always positive, it means that $\sqrt{G}$ is increasing with ρ. Hence, $L(\rho)$ is increasing with ρ. That is, as ρ increases, $\theta = \theta_0$ and $\theta = \theta_1$ get farther and farther apart.

  On the other hand, if $K < 0$, $L(p)$ may or may not get closer to gether. It depends on whether $\sqrt{G}_\rho$ becomes negative or not. However, the rate that the two radial geodesic get further from each other will become slower.

4 Geometric Interpretation of Gaussian Curvature

- The expression of K in geodesic polar coordinate with center $p \in S$ is given by:

$$K = -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}}.$$

So,

$$\begin{aligned}(\sqrt{G})_{\rho\rho} &= -K\sqrt{G} \\ \frac{\partial^3(\sqrt{G})}{\partial\rho^3} &= -K(\sqrt{G})_\rho - K_\rho(\sqrt{G}).\end{aligned}$$

Now, because

$$\lim_{\rho\to 0}\sqrt{G} = 0, \quad \text{and} \quad \lim_{\rho\to 0}(\sqrt{G})_\rho = 1,$$

we have

$$-K(p) = \lim_{\rho\to 0}\frac{\partial^3(\sqrt{G})}{\partial\rho^3}$$

- By Taylor's theorem, we have that

$$\sqrt{G}(\rho,\theta) = \sqrt{G}(0,\theta) + \rho(\sqrt{G})_\rho(0,\theta) + \frac{\rho^2}{2!}(\sqrt{G})_{\rho\rho}(0,\theta) + \frac{\rho^3}{3!}(\sqrt{G})_{\rho\rho\rho}(0,\theta) + R(\rho,\theta)$$

where

$$\lim_{\rho\to 0}\frac{R(\rho,\theta)}{\rho^3} = 0$$

uniformly in θ. Substituting the values obtained above, we have that

$$\sqrt{G}(\rho,\theta) = 0 + \rho - \frac{\rho^3}{3!}K(p) + R.$$

The $\rho^2/2!(\sqrt{G})_{\rho\rho}(0,\theta)$ disappear because $\sqrt{G}_{\rho\rho}(0,\theta) = -K(0,\theta)\sqrt{G}(0,\theta) = 0$.

- With the value for $\sqrt{G}$, we compute the arc length L of a geodesic circle of radius $\rho = r$:

$$\begin{aligned}L &= \lim_{\epsilon\to 0}\int_{0+\epsilon}^{2\pi-\epsilon}\sqrt{G}(r,\theta)\,\mathrm{d}\theta \\ &= \lim_{\epsilon\to 0}\int_{0+\epsilon}^{2\pi-\epsilon} r - \frac{r^3}{6}K(p) + R(r,\theta)\,\mathrm{d}\theta \\ &= 2\pi r - \frac{\pi r^3}{3}K(p) + R_1\end{aligned}$$

where

$$\lim_{r\to 0}\frac{R_1}{r^3} = 0.$$

It follows that

$$K(p) = \frac{3}{\pi}\frac{2\pi r - L}{r^3} - \frac{3R_1}{\pi r^3}$$

So,

$$K(p) = \lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r - L}{r^3}.$$

This gives an intrinsic interpretation of $K(p)$ in terms of the length of the geodesic circle L and the length of the circle or radius r in $T_p(S)$ that gives rise to it.

5 Geodesics Minimize Distance

- **Proposition 5.1.** *Let p be a point on a surface S. Then, there exists a neighborhood $W \subseteq S$ of p such that, if $\gamma : I \to W$ is a parameterized beodesci with $\gamma(0) = p$ and $\gamma(t_1) = q$, $t_1 \in I$, and $\alpha : [0, t_1] \to S$ is a paraetermized regular curve joining p to q, we have that*

  $$l_\gamma \leq l_\alpha$$

  where l_α denotes the length of the curve α. Moreover, if $l_\gamma = l_\gamma$, then the trace of α coincides with the trace of γ between p and q.

  Proof. Let V be a normal neighborhood of p. Let $\bar{W}$ be the closed region bounded by a geodesic circle of radius r contained within V. Let (ρ, θ) be geodesic polar coordinates in $\bar{W} - L$ centered in p such that $q \in L$.

  Suppose first that $\alpha((0, t_1)) \subseteq \bar{W} - L$, and set $\alpha(t) = (\rho(t), \theta(t))$. Observe initially that

  $$\sqrt{(\rho')^2 + G(\theta')^2} \geq \sqrt{(\rho')^2},$$

  and equality holds if and only if $\theta' \equiv 0$; that is $\theta = const$. Therefore, the length $l_\alpha(\epsilon)$ of α between ϵ and $t_1 - \epsilon$ satisfies:

  $$l_\alpha(\epsilon) = \int_\epsilon^{t_1 - \epsilon} \sqrt{(\rho')^2 + G(\theta')^2}\, \mathrm{d}t \geq \int_\epsilon^{t_1 - \epsilon} \sqrt{(\rho')^2}\, \mathrm{d}t \geq \int_\epsilon^{t_1 - \epsilon} \rho'\, \mathrm{d}t = l_\gamma - 2\epsilon.$$

  Equation holds if and only if $\theta = const.$ and $\rho' > 0$. By making $\epsilon \to 0$ in the expression above, we bontain that $l_\alpha \geq l_\gamma$, and that equality holds if and only if α is the radius geodesic $\theta = const.$ with a parameterization $\rho = \rho(t)$ where $\rho'(t) > 0$. It follows that, if $l_\alpha = l_\gamma$, then the traces of α and γ between p and q coincide.

  Suppose now that $\alpha((0, t_1))$ intersects L, and assume that this occurs for the first time at, say, $\alpha(t_2)$. Then, by the previous argument, $l_\alpha \geq l_\gamma$ between t_0 and t_2, and $l_\alpha = l_\gamma$ implies that the traces of α and γ conincide. Since $\alpha([0, t_1])$ and L are compact, there exists a $\bar{t} \geq t_2$ such that either $\alpha(\bar{t})$ is the last point where $\alpha((0, t_1))$ intersects L or $\alpha([\bar{t}, t_1]) \subseteq L$. In any case, applying the above case, the conclusions of the proposition follows.

  Suppose finally that $\alpha([0, t_1])$ is not entirely contained in $\bar{W}$. Let $t_0 \in [0, t_1]$ be the first value for which $t_0) = x$ belongs to the boundary of $\bar{W}$. Let $\bar{\gamma}$ be the radial geodesic px and let $\bar{\alpha}$ be the restriction of the curve α to the interval $[0, t_0]$. It is clear that $l_\alpha \geq l_{\bar{\alpha}}$. By the previous argument, $l_{\bar{\alpha}} \geq l_{\bar{\gamma}}$. Since q is a point in the interior of $\bar{W}$, we have that $l_{\bar{\gamma}} > l_\gamma$. We conclude that $l_\alpha > l_\gamma$, which ends the proof. □

- The above prosition is true for piecewise regular curve as well.

- The converse of the proposition is true. However, if we relax the requirement and make α a piecewise regular curve, then the converse is not true.

- The proposition is not true globally.

- **Proposition 5.2.** *Let $\alpha : I \to S$ be a regular parameterized curve with a parameter proportional to arc length. Suppose that the arc length of α between any two points $t, \tau \in I$ is smaller than or equal to the arc length of any regular parameterized curve joining $\alpha(t)$ to $\alpha(\tau)$. Then, α is a geodesic.*

  Proof. Let $t_0 \in I$ be an arbitrary point on I and let W be the neighborhood of $\alpha(t_0) = p$ given by the last proposition. Let $q = \alpha(t_1) \in W$. From the case of equality in the last proposition, it follows that α is a geodesic in (t_0, t_1). Otherwise, α would have, between t_0 and t_1, a length greater than the radial geodesic joining $\alpha(t_0)$ and $\alpha(t_1)$, a contradiction to the hypothesis. Since α is regular, we have, by continuity, that α is still a geodesic in t_0. □