Implementing the Marschner's Hair Scattering Model

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The Marshner's hair scattering model [1] is a complex piece of mathematical modeling that has a lot of details. This document attempts to iron out all the components of the model, so that the reader can easily implement it in a program.

1 Azimuthal Scattering

1.1 Bravais Index of Refraction

- The **Bravais index of refraction** determines the effective index of refraction when refraction in 3D is projected on the normal plane of the hair fiber.
- Let η be the index of refraction of the hair material. Suppose that the light strikes the hair at incoming direction, which makes an angle θ_d with the normal plane. Then, the Bravias index of refraction is given by:

$$\eta' = \frac{\sqrt{\eta^2 - \sin^2\theta_d}}{\cos\theta_d}$$

- When computing Fresnel reflectance, we need another index of refraction, which we shall call the **anti-Bravais index of refraction**. It is given by:

$$\eta'' = \frac{\eta\cos\theta_d}{\sqrt{1 - \eta^{-2}\sin^2\theta_d}}$$

- Using the two indices, the Fresnel reflectance, resulting from light coming in at angle γ_i (made with the normal) and refracted to the angle γ_t (again, made with the normal), is given by

$$F = \frac{1}{2}(F_p^2 + F_s^2)$$

  where

$$F_p = \frac{\eta''\cos\gamma_i - \cos\gamma_t}{\eta''\cos\gamma_i + \cos\gamma_t}, \text{ and}$$
$$F_s = \frac{\cos\gamma_i - \eta'\cos\gamma_t}{\cos\gamma_i + \eta'\cos\gamma_t}.$$

1.2 Height Parameterization, Transmitted Angle, and Exit Angles

- Marschner models light arriving at a hair fiber's cross section as being distributed through the "height" of the cross section. The height is denoted by the variable h, and it has value ranging from -1 to 1.

- The angle γ_i that the light's direction make with the normal is given by:

$$\gamma_i = \sin^{-1}(h).$$

- The transmitted angle γ_t that the light direction make with the normal is given by:

$$\gamma_t = \sin^{-1}(h/\eta').$$

  This is derived from Snell's law:

$$\sin\gamma_i = \eta' \sin\gamma_t.$$

- Given the scattering mode $p = 0, 1, 2, \ldots$ ($p = 0$ is the R mode, $p = 1$ is the TT mode, and $p = 2$ is the TRT mode) and height h, the azimuthal angle that the light particle exits the hair fiber is given by:

$$\begin{aligned}\phi_{\text{exit}}(\theta_d, p, h) &= 2\gamma_i - 2p\gamma_t + p\pi \\ &= 2\sin^{-1}(h) - 2p\sin^{-1}(h/\eta') + p\pi.\end{aligned}$$

  We put θ_d in the function because η' depends on θ_d.

- The model requires us to compute the derivative of the exit angle with respect to h. The derivatives for $p = 0$, 1, and 2 are given below:

$$\begin{aligned}\frac{\mathrm{d}\phi_{\text{exit}}(\theta_d, 0, h)}{\mathrm{d}h} &= \frac{2}{\sqrt{1-h^2}}, \\ \frac{\mathrm{d}\phi_{\text{exit}}(\theta_d, 1, h)}{\mathrm{d}h} &= \frac{2}{\sqrt{1-h^2}} - \frac{2}{\eta'\sqrt{1-(h/\eta')^2}}, \\ \frac{\mathrm{d}\phi_{\text{exit}}(\theta_d, 2, h)}{\mathrm{d}h} &= \frac{2}{\sqrt{1-h^2}} - \frac{4}{\eta'\sqrt{1-(h/\eta')^2}}.\end{aligned}$$

1.3 Azimuthal Scattering Function

- The azimuthal scattering function is given in terms of the longitudinal angle difference $\theta_d = |\theta_i - \theta_r|/2$, the azimuthal difference $\phi_d = \phi_r - \phi_i$:

$$N(\theta_d, \phi_d).$$

  It is the sum of the azimuthal scattering functions of different modes:

$$N(\theta_d, \phi_d) = N_0(\theta_d, \phi_d) + N_1(\theta_d, \phi_d) + N_2(\theta_d, \phi_d).$$

1.4 Approximation for R and TT Mode

- For $p = 0$ and $p = 1$, the azimuthal scattering function is found using the following procedure. Given ϕ_i and ϕ_r, we find h such that $\phi_{\text{exit}}(\theta_d, p, h) = \phi_d$. Let us call this $h_{\text{solve}}(\theta_d, p, \phi_d)$.

- For $p = 0$, we have that

$$h_{\text{solve}}(\theta_d, 0, \phi)d) = \sin\left(\frac{\phi_d}{2}\right).$$

- For other p, we have to solve the equation:

$$\begin{aligned}\phi_d &= 2\sin^{-1}(h) - 2p\sin^{-1}(h/\eta') + p\pi \\ &= 2\gamma_i - 2p\gamma_t + p\pi.\end{aligned}$$

  Marschner makes the following approximation:

$$\gamma_t = \frac{3\sin^{-1}(1/\eta')}{\pi}\gamma_i - \frac{4\sin^{-1}(1/\eta')}{\pi^3}\gamma_i^3.$$

  The equation then becomes:

$$\begin{aligned}\phi_d &= 2\gamma_i - \frac{6p\sin^{-1}(1/\eta')}{\pi}\gamma_i + \frac{8p\sin^{-1}(1/\eta')}{\pi^3}\gamma_i^3 + p\pi \\ &= \frac{8p\sin^{-1}(1/\eta')}{\pi^3}\gamma_i^3 + \left(2 - \frac{6p\sin^{-1}(1/\eta')}{\pi}\right)\gamma_i + p\pi\end{aligned}$$

  which we solve for the root.

- Once we have $h_{\text{solve}}(\theta_d, p, \phi_d)$, we have that the azimuthal scattering function is given by

$$N_p(\theta_d, \phi_d) = A(\theta_d, p, h_{\text{solve}}(\theta_d, p, \phi_d))\left|2\frac{\mathrm{d}}{\mathrm{d}h}\phi_{\text{exit}}(\theta_d, p, h_{\text{solve}}(\theta_d, p, \phi_d))\right|^{-1}.$$

- The $A(\theta_d, p, h)$ function gives the attenuation due to absorption by the material of the hair. It is given by

$$A(\theta_d, p, h) = \begin{cases} F(\eta', \eta'', \gamma_i), & \text{if } p = 0 \\ (1 - F(\eta', \eta'', \gamma_i))^2 F(\frac{1}{\eta'}, \frac{1}{\eta''}, \gamma_t)^{p-1}(\exp(2\sigma_a \cos\gamma_t))^p, & \text{if } p > 0 \end{cases}$$

  where

  - σ_a is the absorption coefficient of the hair material,
  - $F(\eta', \eta'', \gamma_i)$ is the Fresnel reflectance:

$$F(\eta', \eta'', \gamma_i) = \frac{1}{2}\left(\frac{\eta''\cos\gamma_i - \cos\gamma_t}{\eta''\cos\gamma_i + \cos\gamma_t}\right)^2 + \frac{1}{2}\left(\frac{\cos\gamma_i - \eta'\cos\gamma_t}{\cos\gamma_i + \eta'\cos\gamma_t}\right)^2.$$

  - $F(1/\eta', 1/\eta'', \gamma_t)$ is the "anti"-Fresnel reflectance:

$$\begin{aligned}F\left(\frac{1}{\eta'}, \frac{1}{\eta''}, \gamma_t\right) &= \frac{1}{2}\left(\frac{\frac{1}{\eta''}\cos\gamma_t - \cos\gamma_i}{\frac{1}{\eta''}\cos\gamma_t + \cos\gamma_i}\right)^2 + \frac{1}{2}\left(\frac{\cos\gamma_t - \frac{1}{\eta'}\cos\gamma_i}{\cos\gamma_t + \frac{1}{\eta'}\cos\gamma_i}\right)^2 \\ &= \frac{1}{2}\left(\frac{\cos\gamma_t - \eta''\cos\gamma_i}{\cos\gamma_t + \eta''\cos\gamma_i}\right)^2 + \frac{1}{2}\left(\frac{\eta'\cos\gamma_t - \cos\gamma_i}{\eta'\cos\gamma_t + \cos\gamma_i}\right)^2 \\ &= F(\eta', \eta'', \gamma_i).\end{aligned}$$

- Note that, although we don't use this formula for $p = 2$, it is a component of the TRT approximation which is described in the next section.

1.5 Approximation for TRT Mode

- The TRT generates two caustics at

$$h_c = \sqrt{\frac{4 - \eta'^2}{3}}$$

and at $-h_c$. Therefore, if $\eta' \geq 2$, there will be no caustics.

- The Marschner model differentiates between the following two cases:
  - If $\eta' < 2$, it tries to generate blurred caustics at ϕ_c and $-\phi_c$
  - If $\eta' \geq 2$, it tries to generate a caustic at $\phi_c = 0$ and fade it out smoothly.

1.5.1 Approximation for $\eta' < 2$ Case

- First of all, we cannot evaluate $N_2(\theta_d, \phi_d)$ directly. Because, if $\phi_d = \pm\phi_c$, then $N_2(\theta_d, \phi_d)$ is undefined.
- Instead, in case $\phi_d = \pm\phi_c$, the Marschner model replaces $N_p(\theta_d, \phi_d)$ by the sum of two Gaussians:

$$k_G A(\theta_d, 2, h_c)\Delta h(g(\phi_d; \phi_c, w_c) + g(\phi_d; -\phi_c, w_c)). \tag{1}$$

The variables in the above equations are:

  - k_G is the caustic lobes scaling factor. Its value has range 0.5 to 5.
  - w_c is the width of the caustic lobe. It has value typically from 10° to 25°.
  - $g(x; \mu, \sigma)$ is the Gaussian distribution:

$$g(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right).$$

  - h_c is the h value that corresponds to the position of the caustics: $h_c = h_{\mathrm{solve}}(\theta_d, 2, \phi_c)$.
  - Δh is the estimate the width of the interval of h that maps within w_c of ϕ_c:

$$\Delta h = \min\left\{ \Delta h_M, 2\sqrt{2w_c \left| \frac{\mathrm{d}^2}{\mathrm{d}h^2}\phi_{\mathrm{exit}}(\theta_d, 2, h_c) \right|^{-1}} \right\} \tag{2}$$

    where Δh_M is the maximum width of the h-interval, which is a parameter that the user has to set manually.

- To evaluate the above interval width, we need to evaluate the second derivative of the ϕ_{exit} function, which is given below:

$$\begin{aligned}
\frac{\mathrm{d}\phi_{\mathrm{exit}}(\theta_d, 2, h)}{\mathrm{d}h^2} &= \frac{\mathrm{d}}{\mathrm{d}h}\left\{ \frac{2}{\sqrt{1-h^2}} - \frac{4}{\eta'\sqrt{1-(h/\eta')^2}} \right\} \\
&= 2\frac{\mathrm{d}}{\mathrm{d}h}\left\{ \frac{1}{\sqrt{1-h^2}} \right\} - \frac{4}{\eta'}\frac{\mathrm{d}}{\mathrm{d}h}\left\{ \frac{1}{\sqrt{1-(h/\eta')^2}} \right\} \\
&= 2\left(-\frac{1}{2}(1-h^2)^{-3/2}(-2h) \right) - \frac{4}{\eta'}\left(-\frac{1}{2}(1-(h/\eta')^2)^{-3/2}\left(-\frac{2h}{\eta'^2} \right) \right) \\
&= \frac{2h}{(1-h^2)^{3/2}} - \frac{4h}{\eta'^3(1-(h/\eta')^2)^{3/2}}.
\end{aligned}$$

- In case, $\phi_d \neq \pm\phi_c$, the Marschner model attempts to interpolate smoothly between the value in (1) and $N_2(\theta_d, \phi_d)$. This is done by setting the azimuthal scattering function to:

$$N_2(\theta_d, \phi_d)\left(1 - \frac{g(\phi_d; \phi_c, w_c)}{g(\phi_c; \phi_c, w_c)}\right)\left(1 - \frac{g(\phi_d; -\phi_c, w_c)}{g(-\phi_c; \phi_c, w_c)}\right) + k_G A(\theta_d, 2, h_c)\Delta h(g(\phi_d; \phi_c, w_c) + g(\phi_d; -\phi_c, w_c)).$$

  The intuition is that, when ϕ_d is near $\pm\phi_c$, the contribution of (1) should be very strong. Otherwise, the constribution of $N_2(\theta_d, \phi_d)$ should be very strong.

1.5.2 Approximation for $\eta' \geq 2$ Case

- The value should be $N_2(\theta_d, \phi_d)$.
- However, the caustic lobes given in (1) are present and fully used in the values $\eta' = 2 - \varepsilon$. So, we should not use $N_2(\theta_d, \phi_d)$, but we should face out the caustics as η' increases.
- The Marschner model introduces an interpolation parameter t, which is 1 if $\eta' < 2$, and 0 if $\eta' > 2+\Delta\eta'$:

$$t = \text{smoothstep}(2, 2 + \Delta\eta', \eta').$$

  Here, $\Delta\eta'$ is a parameter that the user sets. Its value is from 0.2 to 0.4. The $\text{smoothstep}(a, b, x)$ is 1 if $x < a$, and 0 if $x > b$, and smooth between.

- The value of the azimuthal scattering function is given by:

$$N_2(\theta_d, \phi_d)\left(1 - t\frac{g(\phi_d; \phi_c, w_c)}{g(\phi_c; \phi_c, w_c)}\right)\left(1 - t\frac{g(\phi_d; -\phi_c, w_c)}{g(-\phi_c; \phi_c, w_c)}\right) + tk_G A(\theta_d, 2, h_c)\Delta h(g(\phi_d; \phi_c, w_c) + g(\phi_d; -\phi_c, w_c)).$$

  where ϕ_c should be set to 0 (because the caustic has merged), and Δh should always be set to Δh_M.

1.5.3 Pseudocode for the TRT Mode

We reproduce the pseudocode for the TRT mode here for completeness.

function $N_{TRT}(\eta, \theta_d, \phi_d; w_c, k_G, \Delta\eta', \Delta h_M)$
  $\eta' \leftarrow \sqrt{\eta - \sin^2\theta_d}/\cos\theta_d$
  if $\eta' < 2$ **then**
    Compute h_c, ϕ_c using η'.
    Compute Δh using (2).
    $t \leftarrow 1$
  else
    $\phi_c \leftarrow 0$
    $\Delta h \leftarrow \Delta h_M$
    $t \leftarrow \text{smoothstep}(2, 2 + \Delta\eta', \eta')$
  fi
  $L \leftarrow N_2(\theta_d, \phi_d)$
  $L \leftarrow L \cdot (1 - tg(\phi_d; \phi_c, w_c)/g(\phi_c; \phi_c, w_c))(1 - tg(\phi_d; -\phi_c, w_c)g(-\phi_c; -\phi_c, w_c))$
  $L \leftarrow L + tk_G A(\theta_d, 2, h_c)\Delta h(g(\phi_d; \phi_c, w_c) + g(\phi_d; -\phi_c, w_c))$
  return L
end

1.5.4 Approximation for Eccentricity

- The paper claims that the TRT caustics is affected by the slightest deviation in the eccentricity.
- The Marschner model simulates the effect of eccentricity to the TRT mode by changing the index of refraction.
- Let a be the cross section's eccentricity. The model computes the index of refraction $\eta^*(\phi_h)$ to be fed to the procedure N_{TRT} as follows:

$$\eta^*(\phi_h) = \frac{1}{2}((\eta_1^* + \eta_2^*) + \cos(2\phi_h)(\eta_1^* - \eta_2^*))$$

  where

  - $\phi_h = (\phi_i + \phi_r)/2$ is the *azimuthal half angle.*
  - η_1^* and η_2^* are given by:

$$\begin{aligned} \eta_1^* &= 2(\eta - 1)a^2 - \eta + 2, \text{ and} \\ \eta_2^* &= 2(\eta - 1)a^{-2} - \eta + 2. \end{aligned}$$

2 Longitudinal Scattering Function

- The longitudinal scattering functions of the Marschner models are very simple: they are Gaussians of the half azimuthal angle $\theta_h = (\theta_i + \theta_r)/2$:

$$\begin{aligned} M_R(\theta_h) &= g(\theta_h; -\alpha_R, \beta_R) \\ M_{TT}(\theta_h) &= g(\theta_h; -\alpha_{TT}, \beta_{TT}) \\ M_{TRT}(\theta_h) &= g(\theta_h; -\alpha_{TRT}, \beta_{TRT}) \end{aligned}$$

  where

  - α_R is the shift of the R mode's reflection peak,
  - β_R is the width of the R mode's scattering lobe,
  - $\alpha_{TT} = \alpha_R/2$,
  - $\beta_{TT} = \beta_R/2$,
  - $\alpha_{TRT} = 3\alpha_R/2$, and
  - $\beta_{TRT} = 2\beta_R$.

3 The Whole Scattering Model

The whole scattering model is given by the following equation:

$$\begin{aligned} S(\theta_i, \phi_i, \theta_r, \phi_r) = {} & M_R(\theta_h) N_0(\theta_d, \phi_d)/\cos^2\theta_d \\ & + M_{TT}(\theta_h) N_1(\theta_d, \phi_d)/\cos^2\theta_d \\ & + M_{TRT}(\theta_h) N_{TRT}(\eta^*(\phi_i), \theta_d, \phi_d)/\cos^2\theta_d \end{aligned}$$

4 Solving Cubic Equation

- Computing $h_{\text{solve}}(\theta_d, p, \phi_d)$ requires us to solve a cubic equation of the form $x^3 + Ax + B = 0$. We reproduce the method to solve it here for completeness.

- We would like to find two numbers s and t such that

$$\begin{aligned} 3uv &= A, \text{ and} \\ u^3 - v^3 &= -B. \end{aligned}$$

  If we find such a number, we have that $x = u - v$ is a solution to the equation. This is because

$$(u-v)^3 + 3uv(u-v) + v^3 - u^3 = 0.$$

- To find u and v, we note that $u = A/3v$, and we have that

$$\begin{aligned} \left(\frac{A}{3v}\right)^3 - v^3 &= -B \\ \frac{A^3}{27v^3} - v^3 &= -B \\ A^3 - 27v^6 &= -27Bv^3 \\ -27v^6 + 27Bv^3 + A^3 &= 0 \end{aligned}$$

  which is a quadratic equation of v^3. So, we solve for v^3, and then u, and then x.

References

[1] Stephen R. Marschner, Henrik Wann Jensen, Mike Cammarano, Steve Worley, and Pat Hanrahan. Light scattering from human hair fibers. In *ACM SIGGRAPH 2003 Papers*, SIGGRAPH '03, pages 780–791, New York, NY, USA, 2003. ACM.