Multiplication of Real Spherical Harmonics

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December 14, 2016

1 Preliminaries

- Let $\omega \in S^2$. The direction is parametermized by two angles—the **elevation angle** θ and the **azimuthal angle** ϕ—such that:

$$\omega = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}.$$

- We denote a **real spherical harmonic basis function** by the symbol $Y_{l,m} : S^2 \to \mathbb{R}$ where $l \geq 0$ and $-l \leq m \leq l$. We have that:

$$Y_{l,m}(\omega) = \begin{cases} \sqrt{2}K_l^m \cos(m\phi) P_l^m(\cos\theta), & m > 0 \\ \sqrt{2}K_l^m \sin(|m|\phi) P_l^{|m|}(\cos\theta), & m > 0 \\ K_l^0 P_l^0(\cos\theta), & m = 0 \end{cases}.$$

Here,

$$K_l^m \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}},$$

and P_l^m is the **associated Legendre polynomial**, which may be computed by the following recurrence relations:

$$\begin{aligned} P_0^0(x) &= 1 \\ P_{l+1}^{l+1}(x) &= -(2l+1)\sqrt{1-x^2}P_l^l(x) \\ P_{l+1}^l(x) &= x(2l+1)P_l^l(x) \end{aligned}$$

- The **complex spherical harmonics basis function** is denoted by $\mathcal{Y}_{l,m}$. We have that

$$\mathcal{Y}_{0,0}(\omega) = \frac{1}{\sqrt{4\pi}},$$

and

$$\mathcal{Y}_{l,m}(\omega) = i^{m+|m|}\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{im\phi}$$

for all $(l,m) \neq (0,0)$.

- The complex basis functions are related to the real basis functions as follows:

$$Y_{l,m}(\omega) = \begin{cases} \mathcal{Y}_{l,0}(\omega), & m = 0 \\ \frac{1}{\sqrt{2}}[\mathcal{Y}_{l,m}(\omega) + (-1)^m \mathcal{Y}_{l,-m}(\omega)], & m > 0 \\ \frac{i}{\sqrt{2}}[(-1)^m \mathcal{Y}_{l,m}(\omega) - \mathcal{Y}_{l,-m}(\omega)], & m < 0 \end{cases}$$

- We shall call the basis functions with the same l-index as belonging to the same **band**. We will also refer to the bands by its l-index; for examples, Band 0, Band 1, and so on.

- The **real SH expansion of order** L of a spherical function f is an approximation of f by a linear combination of SH basis functions:

$$f(\omega) \approx \sum_{l=0}^{L} \sum_{m=-l}^{l} \tilde{f}_{l,m} Y_{l,m}(\omega)$$

  where the coefficient $\tilde{f}_l^m$ is given by:

$$\tilde{f}_{l,m} = \int_{S^2} f(\omega) Y_{l,m}(\omega)\ \mathrm{d}\omega.$$

  It follows that an expansion of order L has $(L+1)^2$ coefficients.

2 Spherical Harmonics Multiplication

- In this document, the problem we are interested in this: we are given two functions f and g that are expanded in the real SH basis. We would like to compute the real SH expansion of $h = fg$.

- The tool of the trade is the following identify:

$$\mathcal{Y}_{l_1,m_1}(\omega)\mathcal{Y}_{l_2,m_2}(\omega) = \sum_l \sum_m \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix} (-1)^m \mathcal{Y}_{l,m}(\omega).$$

  For convenience, let

$$\begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{Bmatrix} = \sum_l \sum_m \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m \end{pmatrix} (-1)^m.$$

  So, we may write:

$$\mathcal{Y}_{l_1,m_1}(\omega)\mathcal{Y}_{l_2,m_2}(\omega) = \sum_l \sum_m \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{Bmatrix} \mathcal{Y}_{l,m}(\omega)$$

- The term

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix}$$

  is called the **Wigner** $3j$**-symbol**. The symbol is zero any of the following conditions is not satisifed:

  1. $-l_1 \le m_1 \le l_1$, $-l_2 \le m_2 \le l_2$, and $-l \le m \le l$.
  2. $m = -(m_1 + m_2)$,
  3. $|l_1 - l_2| \le l \le l_1 + l_2$.

The general formula of the symbol is complicated. However, Mathematica has a function called `ThreeJSymbol` that computes it.

As a result of Condition 2, we have that:

$$\mathcal{Y}_{l_1,m_1}(\omega)\mathcal{Y}_{l_2,m_2}(\omega) = \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix} \mathcal{Y}_{l,m_1+m_2}(\omega)$$

- The $3j$-symbol satisfies the following identity:

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{m_1+m_2+m} \begin{pmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m \end{pmatrix}$$

So, when $m = -(m_1+m_2)$, we have that

$$\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -(m_1+m_2) \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & m_1+m_2 \end{pmatrix}$$

. As a result,

$$\begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & -m_1+m_2 \end{Bmatrix} = \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix}$$

- We would like to derive an expression similiar to the big identity above for real SH basis functions. There are six cases, and we will evaluate them all.
- When $m_1 = 0, m_2 = 0$, we have that

$$\begin{aligned} Y_{l_1,0}(\omega)Y_{l_2,0}(\omega) &= \mathcal{Y}_{l_1,0}(\omega)\mathcal{Y}_{l_2,0}(\omega) \\ &= \sum_l \sum_m \begin{Bmatrix} l_1 & l_2 & l \\ 0 & 0 & m \end{Bmatrix} (-1)^m \mathcal{Y}_{l,m}(\omega) \\ &= \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{Bmatrix} \mathcal{Y}_{l,0}(\omega) \\ &= \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{Bmatrix} Y_{l,0}(\omega). \end{aligned}$$

- When $m_1 = 0$ and $m_2 > 0$, we have that

$$\begin{aligned} &Y_{l_1,0}(\omega)Y_{l_2,m_2}(\omega) \\ &= \mathcal{Y}_{l_1,0}(\omega)\frac{1}{\sqrt{2}}\Big(\mathcal{Y}_{l_2,m_2}(\omega) + (-1)^{m_2}\mathcal{Y}_{l_2,-m_2}(\omega)\Big) \\ &= \frac{1}{\sqrt{2}}\mathcal{Y}_{l_1,0}(\omega)\mathcal{Y}_{l_2,m_2}(\omega) + \frac{(-1)^{m_2}}{\sqrt{2}}\mathcal{Y}_{l_1,0}(\omega)\mathcal{Y}_{l_2,-m_2}(\omega) \\ &= \frac{1}{\sqrt{2}}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix} \mathcal{Y}_{l,m_2}(\omega) + \frac{(-1)^{m_2}}{\sqrt{2}}\sum_l \sum_m \begin{Bmatrix} l_1 & l_2 & l \\ 0 & -m_2 & -m_2 \end{Bmatrix} \mathcal{Y}_{l,-m_2}(\omega) \\ &= \frac{1}{\sqrt{2}}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix} \mathcal{Y}_{l,m_2}(\omega) + \frac{(-1)^{m_2}}{\sqrt{2}}\sum_l \sum_m \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix} \mathcal{Y}_{l,-m_2}(\omega) \\ &= \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix} \left(\frac{1}{\sqrt{2}}\mathcal{Y}_{l,m_2}(\omega) + \frac{(-1)^{m_2}}{\sqrt{2}}\mathcal{Y}_{l,-m_2}(\omega)\right) \\ &= \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix} Y_{l,m_2}(\omega). \end{aligned}$$

- When $m_1 = 0$ and $m_2 < 0$, we have that

$$
\begin{aligned}
&Y_{l_1,0}(\omega)Y_{l_2,m_2}(\omega)\\
&= \mathcal{Y}_{l_1,0}(\omega)\frac{i}{\sqrt{2}}\Big((-1)^{m_2}\mathcal{Y}_{l_2,m_2}(\omega) - \mathcal{Y}_{l_2,-m_2}(\omega)\Big)\\
&= (-1)^{m_2}\frac{i}{\sqrt{2}}\mathcal{Y}_{l_1,0}(\omega)\mathcal{Y}_{l_2,m_2}(\omega) - \frac{i}{\sqrt{2}}\mathcal{Y}_{l_1,0}(\omega)\mathcal{Y}_{l_2,-m_2}(\omega)\\
&= (-1)^{m_2}\frac{i}{\sqrt{2}}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix}\mathcal{Y}_{l,m_2}(\omega) - \frac{i}{\sqrt{2}}\sum_l\sum_m \begin{Bmatrix} l_1 & l_2 & l \\ 0 & -m_2 & -m_2 \end{Bmatrix}\mathcal{Y}_{l,-m_2}(\omega)\\
&= (-1)^{m_2}\frac{i}{\sqrt{2}}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix}\mathcal{Y}_{l,m_2}(\omega) - \frac{i}{\sqrt{2}}\sum_l\sum_m \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix}\mathcal{Y}_{l,-m_2}(\omega)\\
&= \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix}\Big((-1)^{m_2}\frac{i}{\sqrt{2}}\mathcal{Y}_{l,m_2}(\omega) - \frac{i}{\sqrt{2}}\mathcal{Y}_{l,-m_2}(\omega)\Big)\\
&= \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ 0 & m_2 & m_2 \end{Bmatrix}Y_{l,m_2}(\omega).
\end{aligned}
$$

- When $m_1 > 0$ and $m_2 > 0$, we have that

$$
\begin{aligned}
&Y_{l_1,m_1}Y_{l_2,m_2}\\
&= \frac{1}{\sqrt{2}}\Big(\mathcal{Y}_{l_1,m_1} + (-1)^{m_1}\mathcal{Y}_{l_1,-m_1}\Big)\frac{1}{\sqrt{2}}\Big(\mathcal{Y}_{l_2,m_2} + (-1)^{m_2}\mathcal{Y}_{l_2,-m_2}\Big)\\
&= \frac{1}{2}\Big(\mathcal{Y}_{l_1,m_1}\mathcal{Y}_{l_2,m_2} + (-1)^{m_2}\mathcal{Y}_{l_1,m_1}\mathcal{Y}_{l_2,-m_2} + (-1)^{m_1}\mathcal{Y}_{l_1,-m_1}\mathcal{Y}_{l_2,m_2} + (-1)^{m_1+m_2}\mathcal{Y}_{l_1,-m_1}\mathcal{Y}_{l_2,-m_2}\Big)\\
&= \frac{1}{2}\Big(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix}\mathcal{Y}_{l,m_1+m_2}\\
&\qquad + (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1-m_2 \end{Bmatrix}\mathcal{Y}_{l,m_1-m_2}\\
&\qquad + (-1)^{m_1}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & -m_1+m_2 \end{Bmatrix}\mathcal{Y}_{l,-m_1+m_2}\\
&\qquad + (-1)^{m_1+m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix}\mathcal{Y}_{l,-m_1-m_2}\Big)\\
&= \frac{1}{\sqrt{2}}\Big(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix}\frac{1}{\sqrt{2}}\Big(\mathcal{Y}_{l,m_1+m_2} + (-1)^{m_1+m_2}\mathcal{Y}_{l,-m_1-m_2}\Big)\\
&\qquad + (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1-m_2 \end{Bmatrix}\frac{1}{\sqrt{2}}\Big(\mathcal{Y}_{l,m_1-m_2} + (-1)^{m_1-m_2}\mathcal{Y}_{l,-m_1+m_2}\Big)\Big)\\
&= \frac{1}{\sqrt{2}}\Big(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix}Y_{l,m_1+m_2}\\
&\qquad + (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1-m_2 \end{Bmatrix}\frac{1}{\sqrt{2}}\Big(\mathcal{Y}_{l,m_1-m_2} + (-1)^{m_1-m_2}\mathcal{Y}_{l,-m_1+m_2}\Big)\Big).
\end{aligned}
$$

Here, there are three cases.
If $m_1 > m_2$, we have that:

$$
Y_{l_1,m_1}Y_{l_2,m_2} = \frac{1}{\sqrt{2}}\Big(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix}Y_{l,m_1+m_2} + (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1-m_2 \end{Bmatrix}Y_{l,m_1-m_2}\Big).
$$

If $m_1 = m_2$, we have that:

$$Y_{l_1,m_1}Y_{l_2,m_2} = \frac{1}{\sqrt{2}}\left(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix} Y_{l,m_1+m_2} + (-1)^{m_2}\sqrt{2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & 0 \end{Bmatrix} Y_{l,0}\right).$$

If $m_1 < m_2$, we can swap them to get back to the case where $m_1 > m_2$.

- When $m_1 < 0$ and $m_2 < 0$, we have that

$$\begin{aligned}
&Y_{l_1,m_1}Y_{l_2,m_2} \\
&= \frac{i}{\sqrt{2}}\left((-1)^{m_1}\mathcal{Y}_{l_1,m_1} - \mathcal{Y}_{l_1,-m_1}\right)\frac{i}{\sqrt{2}}\left((-1)^{m_2}\mathcal{Y}_{l_2,m_2} - \mathcal{Y}_{l_2,-m_2}\right) \\
&= -\frac{1}{2}\left((-1)^{m_1+m_2}\mathcal{Y}_{l_1,m_1}\mathcal{Y}_{l_2,m_2} - (-1)^{m_1}\mathcal{Y}_{l_1,m_1}\mathcal{Y}_{l_2,-m_2} - (-1)^{m_2}\mathcal{Y}_{l_1,-m_1}\mathcal{Y}_{l_2,m_2} + \mathcal{Y}_{l_1,-m_1}\mathcal{Y}_{l_2,-m_2}\right) \\
&= -\frac{1}{2}\left((-1)^{m_1+m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{Bmatrix}\mathcal{Y}_{l,m_1+m_2}\right. \\
&\qquad - (-1)^{m_1}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1-m_2 \end{Bmatrix}\mathcal{Y}_{l,m_1-m_2} \\
&\qquad - (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & -m_1+m_2 \end{Bmatrix}\mathcal{Y}_{l,-m_1+m_2} \\
&\qquad \left. + \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix}\mathcal{Y}_{l,-m_1-m_2}\right) \\
&= -\frac{1}{\sqrt{2}}\left(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix}\frac{1}{\sqrt{2}}\left(\mathcal{Y}_{l,-m_1-m_2} + (-1)^{m_1+m_2}\mathcal{Y}_{l,m_1+m_2}\right)\right. \\
&\qquad \left. - (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & -m_1+m_2 \end{Bmatrix}\frac{1}{\sqrt{2}}\left(\mathcal{Y}_{l,-m_1+m_2} + (-1)^{m_1-m_2}\mathcal{Y}_{l,m_1-m_2}\right)\right) \\
&= -\frac{1}{\sqrt{2}}\left(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix} Y_{l,-m_1-m_2}\right. \\
&\qquad \left. - (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & -m_1+m_2 \end{Bmatrix}\frac{1}{\sqrt{2}}\left(\mathcal{Y}_{l,-m_1+m_2} + (-1)^{m_1-m_2}\mathcal{Y}_{l,m_1-m_2}\right)\right).
\end{aligned}$$

There are three cases again.

If $-m_1 + m_2 > 0$, we have

$$\begin{aligned}
Y_{l_1,m_1}Y_{l_2,m_2} = -\frac{1}{\sqrt{2}}&\left(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix} Y_{l,-m_1-m_2}\right. \\
&\left. - (-1)^{m_2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & -m_1+m_2 \end{Bmatrix} Y_{l,-m_1+m_2}\right).
\end{aligned}$$

If $-m_1 + m_2 = 0$, we have

$$\begin{aligned}
Y_{l_1,m_1}Y_{l_2,m_2} = -\frac{1}{\sqrt{2}}&\left(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1-m_2 \end{Bmatrix} Y_{l,-m_1-m_2}\right. \\
&\left. - (-1)^{m_2}\sqrt{2}\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & 0 \end{Bmatrix} Y_{l,0}\right).
\end{aligned}$$

If $-m_1 + m_2 < 0$, we can flip it so that we end up with the $-m_1 + m_2 > 0$ case.

- If $m_1 > 0$ and $m_2 < 0$, we have that

$$
\begin{aligned}
&Y_{l_1,m_1} Y_{l_2,m_2} \\
&= \frac{1}{\sqrt{2}} \Big(\mathcal{Y}_{l_1,m_1} + (-1)^{m_1} \mathcal{Y}_{l_1,-m_1} \Big) \frac{i}{\sqrt{2}} \Big((-1)^{m_2} \mathcal{Y}_{l_2,m_2} - \mathcal{Y}_{l_2,-m_2} \Big) \\
&= \frac{i}{2} \Big(-\mathcal{Y}_{l_1,m_1} \mathcal{Y}_{l_2,-m_2} + (-1)^{m_1+m_2} \mathcal{Y}_{l_1,-m_1} \mathcal{Y}_{l_2,m_2} + (-1)^{m_2} \mathcal{Y}_{l_1,m_1} \mathcal{Y}_{l_2,m_2} - (-1)^{m_1} \mathcal{Y}_{l_1,-m_1} \mathcal{Y}_{l_2,-m_2} \Big) \\
&= \frac{i}{2} \Bigg(-\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} \mathcal{Y}_{l,m_1-m_2} \\
&\qquad + (-1)^{m_1+m_2} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & m_2 & -m_1 + m_2 \end{Bmatrix} \mathcal{Y}_{l,-m_1+m_2} \\
&\qquad + (-1)^{m_2} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} \mathcal{Y}_{l,m_1+m_2} \\
&\qquad - (-1)^{m_1} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ -m_1 & -m_2 & -m_1 - m_2 \end{Bmatrix} \mathcal{Y}_{l,-m_1-m_2} \Bigg) \\
&= \frac{1}{\sqrt{2}} \Bigg(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} \frac{i}{\sqrt{2}} \Big(-\mathcal{Y}_{l,m_1-m_2} + (-1)^{m_1+m_2} \mathcal{Y}_{l,-m_1+m_2} \Big) \\
&\qquad + \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} \frac{i}{\sqrt{2}} \Big((-1)^{m_2} \mathcal{Y}_{l,m_1+m_2} - (-1)^{m_1} \mathcal{Y}_{l,-m_1-m_2} \Big) \Bigg). \\
&= \frac{1}{\sqrt{2}} \Bigg(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} Y_{l,-m_1+m_2} \\
&\qquad + \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} \frac{i}{\sqrt{2}} \Big((-1)^{m_2} \mathcal{Y}_{l,m_1+m_2} - (-1)^{m_1} \mathcal{Y}_{l,-m_1-m_2} \Big) \Bigg).
\end{aligned}
$$

There are three cases.
If $m_1 + m_2 > 0$, we have that

$$
\begin{aligned}
&Y_{l_1,m_1} Y_{l_2,m_2} \\
&= \frac{1}{\sqrt{2}} \Bigg(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} Y_{l,-m_1+m_2} \\
&\qquad - (-1)^{m_2} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} \frac{i}{\sqrt{2}} \Big(-\mathcal{Y}_{l,m_1+m_2} + (-1)^{m_1+m_2} \mathcal{Y}_{l,-m_1-m_2} \Big) \Bigg) \\
&= \frac{1}{\sqrt{2}} \Bigg(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} Y_{l,-m_1+m_2} - (-1)^{m_2} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} Y_{l,-m_1-m_2} \Bigg).
\end{aligned}
$$

If $m_1 + m_2 = 0$, we have that

$$
Y_{l_1,m_1} Y_{l_2,m_2} = \frac{1}{\sqrt{2}} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} Y_{l,-m_1+m_2}.
$$

If $m_1 + m_2 < 0$, we have that

$$\begin{aligned}
&Y_{l_1,m_1}Y_{l_2,m_2}\\
&= \frac{1}{\sqrt{2}}\Bigg(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} Y_{l,-m_1+m_2}\\
&\qquad + (-1)^{m_1} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} \frac{i}{\sqrt{2}}\Big((-1)^{m_1+m_2}\mathcal{Y}_{l,m_1+m_2} - \mathcal{Y}_{l,-m_1-m_2}\Big)\Bigg)\\
&= \frac{1}{\sqrt{2}}\Bigg(\sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m_1 - m_2 \end{Bmatrix} Y_{l,-m_1+m_2} + (-1)^{m_1} \sum_l \begin{Bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1 + m_2 \end{Bmatrix} Y_{l,m_1+m_2}\Bigg).
\end{aligned}$$