Random Walk and Electrical Networks

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1 Random Walks in One Dimension

- Consider a random walk on integers $0, 1, 2, \ldots, N$.

  Let $p(x)$ be the probability that, starting at x, the walker reach N before reading 0.

  We have that:

  - $p(0) = 0$.
  - $p(N) = 1$.
  - $p(x) = \frac{1}{2}p(x-1) + \frac{1}{2}p(x+1)$ for all x such that $1 \leq x \leq N-1$.

- Consider an electrical network in Figure 1.

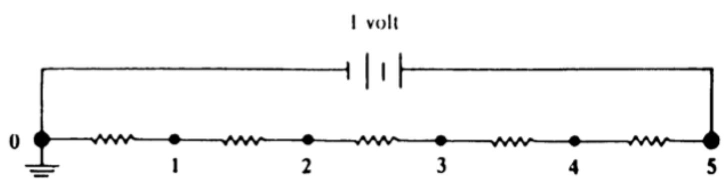

Figure 1: A 1D electrical network. Each resistor is 1 Ohm.

  Let $v(i)$ denote the voltage at point i. We have that

  - $v(0) = 0$.
  - $v(5) = 1$.

- We will derive expressions for voltage at other points by laws of electrical networks. So, it is good to review those laws now. We will be using two laws:

  - **Ohm's Law.** The current flowing through a resister R connecting points x and y in the direction from x to y is by
  $$\frac{v(x) - v(y)}{R}.$$
  - **Kirchhoff's Current Law.** The current flowing out of any point in an electrical circuit is always zero.

- Let us go back to the electrical network in Figure 1. We have that the current flowing out of x is given by
$$\frac{v(x) - v(x+1)}{1} + \frac{v(x) - v(x-1)}{1} = 2v(x) - v(x+1) - v(x-1).$$

By Kirchhoff's Current Law, the above expression is 0. In other words,

$$v(x) = \frac{1}{2}v(x-1) + \frac{1}{2}v(x+1)$$

for any $x = 1, 2, \ldots, N-1$.

- We can see that p and v are very similar. In turns out that we can show that they must be the same.

2 Harmonic Functions in One Dimension

- Let $S = \{0, 1, 2, \ldots, N\}$.

  We call the points in the set $D = \{1, 2, \ldots, N-1\}$ the *interior points*, and the points in $B = \{0, N\}$ the *boundary points*.

- **Definition 2.1.** *A function f defined on S is* harmonic *if*

  $$f(x) = \frac{f(x-1) + f(x+1)}{2}$$

  for all $x \in D$. In other words, $f(x)$ is the average of its neighbors.

- We can see from the last section that p and v are harmonic functions. We also see that they have the same boundary values. That is, $p(0) = v(0) = 0$ and $p(N) = v(N) = 1$.

- We shall determine whether p and v are actually the same. The tool that lets us do that is the following *maximum principle.*

  Lemma 2.2 (Maximum Principle). *A harmonic function $f(x)$ defined on S takes on its maximum value M and its minimum value m on the boundary.*

  Proof. Suppose by way of contradiction that f takes on its maximum value M at an interior point x. Moreover, assume that $f(x) > f(0)$ and $f(x) > f(N)$. Then, because $f(x)$ is the average of $f(x-1)$ and $f(x+1)$, one of $f(x-1)$ and $f(x+1)$ must be at least M. Continuing this way, we have that $f(0)$ or $f(N)$ must be equal to M, a contradiction.

  The same argument also works for m. □

- **Corollary 2.3.** *If f is a harmonic function such that $f(0) = c$ and $f(N) = c$ for some constant c, then $f(x) = c$ for all $x \in S$.*

- Finding harmonic functions that having specified boundary values is called the *Dirichlet problem.* The *uniqueness principle* states that there can be only one solution to the *Dirichlet problem.*

  Theorem 2.4 (Uniqueness Principle). *If f and g are harmonic functions such that $f(0) = g(0)$ and $f(N) = g(N)$, then $f = g$.*

  Proof. Let $h = f - g$. We have that h is a harmonic function, and $h(0) = h(N) = 0$. By Corollary 2.3, $h(x) = 0$ for all x, which means that $f(x) = g(x)$ for all x. □

- Returning to the random walk and the electrical network in the last section, we have that $p = v$.