A Primer on Measure Theory

Pramook Khungurn

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This is a primer on measure theory and Lebesgue integration. The materials are taken from Bartle's "The Elements Of Integration And Lebesgue Measure" [Bartle, 1995] and Hunter's note on measure theory [Hunter, 2011].

1 Introduction

- Why do we care about measure theory and Lebesgue integration?
  - They expand the class of functions for which integrations are defined compared to what can be achieved by Riemann integration.
  - Theorems relating to the intechange of limits and integrals are valid under less stringent conditions (again, compared to Reimann integration).
  - In particular, the dominated convergence theorem [1] is a very powerful tool. For examples, it can be use to easily show that

    $$\lim_{n\to\infty} \int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\,\mathrm{d}x = 0$$

    and

    $$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty x^2 e^{-tx}\,\mathrm{d}x = -\int_0^\infty x^3 e^{-tx}\,\mathrm{d}x.$$

  - Measure theory is the foundation of modern probability theory, and the dominated convergence theorem shows up everywhere in it.
- How is Lebesgue integration different from Reimann integration?
  - Riemann integrals are defined in terms of approximating a function with constant functions over intevals.
  - An **interval** is a subset of the real line which is of one of the following forms:

    $$\begin{aligned}
    [a,b] &= \{x \in \mathbb{R} : a \le x \le b\},\\
    [a,b) &= \{x \in \mathbb{R} : a \le x < b\},\\
    (a,b] &= \{x \in \mathbb{R} : a < x \le b\},\\
    (a,b) &= \{x \in \mathbb{R} : a < x < b\}.
    \end{aligned}$$

    The real number a and b are said to be the **endpoints** of the interval, and $b-a$ is the **length** of the interval.

[1] https://en.wikipedia.org/wiki/Dominated_convergence_theorem

- A **step function** φ is a linear combination of a finite number of characteristic functions of intevals.

$$\varphi(x) = \sum_{i=1}^{n} c_j \chi_{E_j}(x)$$

where $c_j \in \mathbb{R}$ and each E_j is an interval with endpoints a_j and b_j.
- The integral of a step function φ is defined to be

$$\int \varphi = \sum_{i=1}^{n} c_j (b_j - a_j).$$

- If f is a bounded function on $[a, b]$, then the **Reimann integral** is defined to the limit of the integrals of step functions that approximate f.
- The **lower Rieman integral** is defined to be the supremum of integrals of all step functions ϕ such that $\phi(x) \leq f(x)$ for all $x \in [a, b]$ and $\phi(x) = 0$ for all $x \notin [a, b]$.

• The Lebesgue integral is defined similarly, with some differences.
  - Intervals are replaced by a larger collection of sets (called **measurable sets**).
  - The notion of "length" is generalize to the notation of **measure**.
    * Here, the measure is a function μ that maps a set of a non-negative real number.
  - The step function is replaced by the **simple function**, which is a finite linear combination of characteristic functions of measurable sets.

$$\varphi(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x)$$

where each E_j is a measurable set. The integral of ϕ is defined to be

$$\int \varphi = \sum_{j=1}^{n} c_j \mu(E_j).$$

  - If f is a non-negative function defined on $\mathbb{R}$, then the **Lebesgue integral** of f is the supremum of all simple functions ϕ such that $\phi(x) \leq f(x)$ for all $x \in \mathbb{R}$.
    * This notation can later be generalized to functions taking both signs.

• When studying integration, it is convenient to work with the **extended real number system** $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.
  - For any $x \in \mathbb{R}$, we have that $-\infty < x < \infty$.
  - We say that the length of the real line is ∞.
  - We define the supremum of non-empty set of real numbers which does not have an upper bound to be ∞, and the infemum of the a non-empty set of real numbers which does not have a lower bound to be $-\infty$.
    * In this way, all non-empty sets of real numbers (or subsets of $\overline{\mathbb{R}}$) have unique supremums and infemums.

– The arithematic operations between the infiniites and real numbers are as follows:

$$\begin{aligned}(\pm\infty)+(\pm\infty)=x+(\pm\infty)=(\pm\infty)+x&=\pm\infty\\(\pm\infty)(\pm\infty)&=+\infty\\(\pm\infty)(\mp\infty)&=-\infty\\(\pm\infty)x=x(\pm\infty)&=\begin{cases}\pm\infty, & \text{if } x>0,\\0, & \text{if } x=0,\\\mp\infty, & \text{if } x<0\end{cases}\end{aligned}$$

for any (finite) real number x.

– Note that we do not define $(\pm\infty)-(\pm\infty)$. We also do not define quotients when the denominators are $\pm\infty$.

- If (x_n) is a sequence of extended real numbers, define the **limit superior** and **limit inferior** by

$$\begin{aligned}\limsup_{n\to\infty}\ x_n&=\lim_{n\to\infty}\Big(\sup_{m\geq n}x_m\Big)=\inf_n\Big(\sup_{m\geq n}x_m\Big)\\\liminf_{n\to\infty}\ x_n&=\lim_{n\to\infty}\Big(\inf_{m\geq n}x_m\Big)=\sup_n\Big(\inf_{m\geq n}x_m\Big).\end{aligned}$$

If the limit superior and limit inferere of a sequence both exist and are equal, then the **limit** of the sequence exists and is equal to that value.

2 Measures

- Let us denote the power set of set X with $\mathcal{P}(X)$.
- A σ-algebra is the domain upon which we define measures. It is a collection of sets with some nice properties.

Definition 1. *A* **σ-algebra** *(or a* **σ-field***) on a set X is a collection $\mathcal{X}\in\mathcal{P}(X)$ of subsets of X, called* **measurable sets***, such that the following properties hold.*

1. *$\emptyset, X\in\mathcal{X}$.*
2. *If $A\in\mathcal{X}$, then so is its complement. That is, $A^c=X-A\in\mathcal{X}$.*
3. *If (A_n) is a sequence of sets in $\mathcal{X}$, then*

$$\bigcup_{n=1}^{\infty}A_n\in\mathcal{X}.$$

- For a σ-algebra, we can show that

$$\bigcap_{i=1}^{\infty}A_n\in\mathcal{X}$$

as well. To see this, we note that $A_n^c\in\mathcal{X}$ for all $n\in\mathbb{N}$, and so

$$\bigcup_{i=1}^{\infty}A_n^c\in\mathcal{X}$$

As result,

$$\left(\bigcup_{i=1}^{\infty} A_n^c\right)^c \in \mathcal{X}.$$

Applying de Morgan's law, we have that

$$\bigcap_{i=1}^{\infty} A_n = \left(\bigcup_{i=1}^{\infty} A_n^c\right)^c \in \mathcal{X}.$$

- **Definition 2.** *A measurable space $(X, \mathcal{X})$ is a non-empty set X equipped with a σ-algebra $\mathcal{X}$ on X.*

- **Definition 3.** *Let $\mathcal{A}$ be a non-empty collection of subsets of X. The* **σ-algebra generated by** *$\mathcal{A}$, denoted by $\sigma(\mathcal{A})$ is the smallest σ-algebra that contains $\mathcal{A}$. In other words,*

$$\sigma(\mathcal{A}) = \bigcap \left\{ \tilde{\mathcal{A}} \subseteq \mathcal{P}(X) : \mathcal{A} \subseteq \tilde{\mathcal{A}} \text{ and } \tilde{\mathcal{A}} \text{ is a } \sigma\text{-algebra} \right\}.$$

- **Definition 4.** *The* **Borel algebra** *is the σ-algebra $\mathcal{B}$ generated by all the open intervals (a, b) in $\mathbb{R}$. Any set in $\mathcal{B}$ is called a* **Borel set**.

- Observe that we can write any open interval (a, b) as a countable unions of closed intervals:

$$(a, b) = \bigcup_{n \geq N}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n} \right]$$

  where N is an integer such that $b - a - \frac{2}{N} > 0$. As a result, $\mathcal{B}$ is also generated by the collection of close intervals $[a, b]$ in $\mathbb{R}$. The same is also true for half-open intervals of the form $(a, b]$ and $[a, b)$.

- Let X be the set $\overline{\mathbb{R}}$ of extended real numbers. If E is a Borel set, then define

$$\begin{aligned} E_1 &= E \cup \{\infty\} \\ E_2 &= E \cup \{-\infty\} \\ E_3 &= E \cup \{-\infty, \infty\} \end{aligned}$$

  Let $\overline{\mathcal{B}}$ the collection of all sets E, E_1, E_2, and E_3 as E varies over $\mathcal{B}$. We have that $\overline{B}$ is a σ-algebra, and it is called the **extended Borel algebra**.

- A "measure" encapsulates the notion of length, area, volume, mass, etc. of a set.

  Definition 5. *Let $(X, \mathcal{X})$ be a measurable space. A* **measure** *is a function $\mu : \mathcal{X} \to [0, \infty]$ with the following properties.*

  1. $\mu(\emptyset) = 0$.
  2. μ *is* **countably additive**. *That is, for a sequence (E_n) of disjoint sets, it holds that*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

- If $\mu(E) < \infty$ for all $E \in \mathcal{X}$, we say that μ is **finite**.

- A **probability measure** is a finite measure with $\mu(X) = 1$.

- If there exists a sequence (E_n) of sets in $\mathcal{X}$ with $\bigcup_{i=1}^{\infty} E_n = X$ and such that $\mu(E_n) < \infty$ for all n, then we say that μ is **σ-finite**.

- Here is an example of a measure that is σ-finite but not finite. Let $X = \mathbb{N}$, and $\mathcal{X} = \mathcal{P}(\mathbb{N})$. Define $\mu(E)$ to be the number of elements in E with the convention that $\mu(E) = \infty$ when E is infinite. Obviously, $\mu(\mathbb{N}) = \infty$. However, $\mathbb{N} = \{1\} \cup \{2\} \cup \cdots$, and $\mu(\{n\}) = 1$ for all $n \in \mathbb{N}$. The measure μ is called the **counting measure** on $\mathbb{N}$.

- **Proposition 6.** *Let μ be a measure defined on a σ-algebra $\mathcal{X}$. Let $E, F \in \mathcal{X}$ be such that $E \subseteq F$, then $\mu(E) \leq \mu(F)$. If $\mu(E) < \infty$, then $\mu(F - E) = \mu(F) - \mu(E)$.*

  Proof. Since $F = E \cup (F - E)$ and $E \cap (F - E) = \emptyset$, it follows that

  $$\mu(F) = \mu(E) + \mu(F - E).$$

  Because $\mu(F - E) \geq 0$, it follows that $\mu(F) \geq \mu(E)$. If $\mu(E) < \infty$, we can subtract from both sides of the equation. □

- A sequence of sets (E_n) is **increasing** if $E_n \subseteq E_{n+1}$ for all n.

- A sequence of sets (E_n) is **decreasing** if $E_n \supseteq E_{n+1}$ for all n.

- **Proposition 7.** *If (E_n) is an increasing sequence of measurable sets, then*

  $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

  If (E_n) is a decreasing sequence of measurable sets and $\mu(E_1) < \infty$, then

  $$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

  Proof. Let (E_n) be increasing. Set $F_0 = E_1$, and $F_n = E_{n+1} - E_n$ for all $n \geq 1$. We have that (F_n) is a sequence of disjoint sets. So,

  $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=0}^{\infty} F_n\right) = \sum_{n=0}^{\infty} \mu(F_n)$$

  Also because $E_n = \bigcup_{i=0}^{n} F_i$, we have that

  $$\mu(E_n) = \sum_{i=0}^{n} \mu(F_i).$$

  So,

  $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \mu(F_n) = \lim_{n\to\infty} \sum_{i=0}^{n} \mu(F_i) = \lim_{n\to\infty} \mu(E_n).$$

  Next, let (E_n) be decreasing and $\mu(E_1) < \infty$. Let $F_n = E_1 - E_n$. We have that (F_n) is increasing and $\mu(F_n) = \mu(E_1) - \mu(E_n)$. If follows that

  $$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(E_1) - \mu(E_n) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n)$$

Now,

$$\bigcap_{n=1}^{\infty} E_n = E_1 - \bigcup_{n=1}^{\infty} F_n$$

So,

$$\mu\bigg(\bigcap_{n=1}^{\infty} E_n\bigg) = \mu(E_1) - \mu\bigg(\bigcup_{n=1}^{\infty} F_n\bigg) = \mu(E_1) - \Big(\mu(E_1) - \lim_{n\to\infty}\mu(E_n)\Big) = \lim_{n\to\infty}\mu(E_n)$$

are required. □

- **Definition 8.** *A* **measure space** *is a triple* $(X, \mathcal{X}, \mu)$ *where* X *is a non-empty set,* $\mathcal{X}$ *is a* σ*-algebra on* X*, and* μ *is a measure on* $\mathcal{X}$*.*

- **Definition 9.** *In a measure space* $(X, \mathcal{X}, \mu)$*, a set* $N \in \mathcal{X}$ *is set to be of* **measure zero** *or a* **null set** *if* $\mu(N) = 0$*. A property that holds on* N^c *is said to hold* μ**-almost everywhere**. *In the context where* μ *is clear, we says that a property holds just* **almost everywhere**.

- For examples, we say that two functions f and g are equal almost everywhere if $f(x) = g(x)$ for all $x \notin N$ where N is a set of measure zero. We also say that a sequence of functions (f_n) converges almost everywhere in X if $\lim_{n\to\infty} f_n(x)$ exists for all $x \notin N$.

- **Definition 10.** *A measure space* $(X, \mathcal{X}, \mu)$ *is* **complete** *if every subset of a set of measure zero is measureable.*

- **Theorem 11.** *Let* $(X, \mathcal{X}, \mu)$ *be a measure space. Define* $(X, \overline{\mathcal{X}}, \overline{\mu})$ *by*

$$\overline{\mathcal{X}} = \{A \cup M : A \in \mathcal{X}, M \subseteq N \text{ where } N \in \mathcal{X} \text{ and } \mu(N) = 0\}$$

and

$$\overline{\mu}(A \cup M) = \mu(A).$$

Then, $(X, \overline{\mathcal{X}}, \overline{\mu})$ *is a complete measure space such that* $\mathcal{X} \subseteq \overline{X}$ *and* $\overline{\mu}$ *is the unique extension of* μ *to* $\overline{X}$*.*

Proof (sketch). The hardest bit of the proof is to show that $\overline{\mathcal{X}}$ is close under complementation. Let $A \in \mathcal{X}$, $N \in \mathcal{X}$ be a set of measure zero, and $M \subseteq N$. We have that $(A \cup M)^c = A^c \cap M^c$. Because $M^c = N^c \cup (N - M)$, we have that

$$(A \cup M)^c = A^c \cap M^c = A^c \cap (N^c \cup (N - M)) = (A^c \cap N^c) \cup (A^c \cap (N - M)).$$

We note that $A^c \cap N^c \in \mathcal{X}$ and $A^c \cap (N - M) \subseteq N$, so $(A \cup M)^C \in \overline{\mathcal{X}}$. The other parts of the proof seems straightforward, and we refer to [Hunter, 2011] for a longer proof sketch. □

3 Lebesgue Measure on the Real Line

- The Lebesgue measure on $\mathbb{R}$ is a measure that corresponds to the notion of "length" on the real line. We will construct it in this section.

- The natural notion of **length** can be defined as follows.
  - Let ℓ denote the length function.

- The length of the half-open interval $(a, b]$ is defined to be $b - a$.
- The lengths of $(-\infty, b]$, $(a, +\infty)$, and $(-\infty, \infty)$ are defined to be ∞.
- The length of the union of a finite number of disjoint sets of intervals of these forms is defined to be the sum of the corresponding lengths.

$$\ell\bigg(\bigcup_{i=1}^{n}(a_i, b_n]\bigg) = \sum_{i=1}^{n}(b_i - a_i).$$

- By the notion above, we have that ℓ is defined on intervals of the form

$$(a, b], (-\infty, b], (a, \infty), (\infty, \infty) \tag{1}$$

and their finite unions. However, we cannot claim that it is a measure because we have not defined how to deal with countable unions of such intervals yet. So, we are not so sure whether it is a measure on the Borel algebra $\mathcal{B}$.

- **Definition 12.** *A family $\mathcal{A}$ of subsets of a set X is said to be an* **algebra** *or a* **field** *on X if the following properties are satisfied.*

1. *$\emptyset, X \in \mathcal{A}$.*
2. *If $E \in \mathcal{A}$, then $E^c = X - E \in \mathcal{A}$.*
3. *If $E_1, E_2, \dots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} E_i \in \mathcal{A}$.*

- Let $\mathcal{F}$ be the collection of subsets of $\mathbb{R}$ that contains
  - all intervals of the forms in (1), and
  - all the finite unions of such intervals.

- We have that $\mathcal{F}$ is an algebra on $\mathbb{R}$.
  - $\emptyset \in \mathcal{F}$ because $\emptyset = \{x : 1 < x \leq 1\} = (1, 1]$.
  - $\mathbb{R} = (-\infty, \infty) \in \mathcal{F}$ by definition.
  - $\mathcal{F}$ is closed under complementation.
    * $(-\infty, b]^c = (b, \infty) \in \mathcal{F}$.
    * $(a, \infty)^c = (-\infty, a] \in \mathcal{F}$.
    * $(a, b]^c = (\infty, a] \cup (b, \infty) \in \mathcal{F}$.
    * $\emptyset$ and $(-\infty, \infty)$ are in $\mathcal{F}$.
  - $\mathcal{F}$ is closed under finite unions by definition.

- **Definition 13.** *Let $\mathcal{A}$ be an algebra on X. A* **premeasure** *on $\mathcal{A}$ is an extended real valued function μ defined on $\mathcal{A}$ that satisfies the following properties.*

1. *$\mu(\emptyset) = 0$.*
2. *$\mu(E) \geq 0$ for all $E \in \mathcal{A}$.*
3. *If (E_n) is any disjoint sequence of sets in $\mathcal{A}$ such that $\bigcup_{i=n}^{\infty} E_n$ belongs to $\mathcal{A}$, then*

$$\mu\bigg(\bigcup_{n=1}^{\infty} E_n\bigg) = \sum_{n=1}^{\infty} \mu(E_n).$$

- **Lemma 14.** *The length function ℓ is a premeasure on $\mathcal{F}$.*

Proof. First, $\ell(\emptyset) = \ell((1,1]) = 1 - 1 = 0$.

Second, if $E \in \mathcal{F}$, then $\mathcal{E}$ is a finite union of intervals of the forms in (1). We can subdivide these intervals into disjoint pieces, and the length of the union is the sum of the length of the individual pieces, each of which is non-negative. So, we have that $\mu(E) \geq 0$.

Lastly, suppose that (E_n) is a sequence of disjoint sets in $\mathcal{F}$ such that $\bigcup_{i=n}^{\infty} E_n$ is also in $\mathcal{F}$. Note that $\bigcup_{i=n}^{\infty} E_n$ is a finite union of intervals, so we can again partition them into disjoint pieces, each of which is of a form in (1). Each piece is now a union of a countable collection of disjoint elements of $\mathcal{F}$. WLOG, we may treat each of the E_n's as an interval which is disjoint from any other. Our goal now would be to show that the lengths of the constituent intervals add up to the length of the piece.

A piece can be any of the 4 types. We will only deal with the $(a,b]$ type in this proof as the proof of other types are similar. Suppose, then, that

$$(a,b] = \bigcup_{j=1}^{\infty} (a_j, b_j]$$

where the intervals are disjoint. Consider the first n intervals. We may assume that

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b.$$

We have that

$$\sum_{i=1}^{n} \ell((a_i, b_i]) = \sum_{i=1}^{n} (b_i - a_i) \leq b_n - a_1 \leq b - a = \ell((a,b]).$$

Because n is arbitrary, we have that

$$\sum_{i=1}^{\infty} \ell((a_i, b_i]) \leq \ell((a,b]).$$

For the other direction, let $\varepsilon > 0$ be arbitrary. Let (ε_j) be a sequence of positive numbers with $\sum \epsilon_j < \varepsilon/2$. Consider the interval $I_j = (a_j - \varepsilon_j, b_k + \varepsilon_j)$. The collection $\{I_j\}$ of open sets is a cover of the interval $[a,b]$. Since $[a,b]$ is compact, it has a finite subcover, say, $I_1, I_2, \ldots, I_m$. By reordering and discarding some intervals, we may assume that

$$\begin{aligned} a_1 - \varepsilon_1 &< a \\ b &< b_m + \varepsilon_m \\ a_j - \varepsilon_j &< b_{j-1} + \varepsilon_{j-1}. \end{aligned}$$

If follows that

$$b - a \leq (b_m + \varepsilon_m) - (a_1 - \varepsilon_1) \leq \sum_{j=1}^{m} [(b_j + \varepsilon_j) - (a_j - \varepsilon_j)] \leq \varepsilon + \sum_{j=1}^{m} (b_j - a_j) \leq \varepsilon + \sum_{j=1}^{\infty} (b_j - a_j).$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\ell((a,b]) \leq \sum_{i=1}^{\infty} \ell((a_i, b_i]).$$

As a result, $\ell((a,b]) = \sum_{i=1}^{\infty} \ell((a_i, b_i])$.

Combining the results of all cases, we can conclude that ℓ is countably additive in $\mathcal{F}$. $\square$

- Given a premeasure μ on an algebra $\mathcal{A}$, we can show that μ can be extended to a measure on a σ-algebra.
  - In other words, there exist a σ-algebra $\mathcal{A}^*$ containing $\mathcal{A}$ and a measure μ^* defined on $\mathcal{A}^*$ such that $\mu^*(E) = \mu(E)$ for all $E \in \mathcal{A}$.

  As a result, we can extend ℓ so that it becomes a measure in a σ-algebra.

- The way to extend μ is as follows.

  Definition 15. *Given a premeasure μ defined on an algebra $\mathcal{A}$ on set X, define $\mu^* : \mathcal{P}(\mathbf{X}) \to [0,\infty]$ to be*

$$\mu^*(B) = \inf\left\{ \sum_{j=1}^{\infty} \mu(E_j) : (E_j) \in \mathfrak{C}(B) \right\}$$

  where $\mathfrak{C}(B)$ is the set of all sequences (E_j) of sets in $\mathcal{A}$ such that $B \subseteq \bigcup_{j=1}^{\infty} E_j$.

- **Lemma 16.** *The function μ^* has the following properties.*

  (a) $\mu^*(\emptyset) = 0$.
  (b) $\mu^*(B) \geq 0$ *for any* $B \subseteq X$.
  (c) *If* $A \subseteq B$, *then* $\mu^*(A) \leq \mu^*(B)$.
  (d) *If* $A \in \mathcal{A}$, *then* $\mu^*(A) = \mu(A)$.
  (e) *If* (B_n) *is a sequence of subsets of* X, *then*

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n \right) \leq \sum_{i=1}^{\infty} \mu^*(B_n).$$

  Proof. Let us call a sequence $(E_j) \in \mathfrak{C}(B)$ a *cover* of B.

  For (b), recall that μ is a measure, so $\mu(E_j) \geq 0$ for all j. Hence, $\sum \mu(E_j) \geq 0$ for any sequence (E_j). As a result, $\mu^*(B) = \inf\{\sum \mu(E_j) : (E_j) \in \mathfrak{C}(\emptyset)\} \geq 0$.

  For (a), note that the sequence $(\emptyset, \emptyset, \dots)$ is a cover of $\emptyset$, so $\mu^*(\emptyset) = \inf\{\sum \mu(E_j) : (E_j) \in \mathfrak{C}(\emptyset)\} \leq 0$. However, from (b), we have that $\mu^*(\emptyset) \geq 0$, so $\mu^*(\emptyset) = 0$.

  For (c), let $A \subseteq B$. Let $(E_j) \in \mathfrak{C}(B)$. We have that $A \subseteq B \subseteq \bigcup E_j$, so $(E_j) \in \mathfrak{C}(A)$ too. It follows that $\mathfrak{C}(A) \supseteq \mathfrak{C}(B)$, which implies that

$$\left\{ \sum \mu(E_j) : (E_j) \in \mathfrak{C}(A) \right\} \supseteq \left\{ \sum \mu(E_j) : (E_j) \in \mathfrak{C}(B) \right\},$$

  and so

$$\mu^*(A) = \inf\left\{ \sum \mu(E_j) : (E_j) \in \mathfrak{C}(A) \right\} \leq \inf\left\{ \sum \mu(E_j) : (E_j) \in \mathfrak{C}(B) \right\} = \mu^*(B).$$

  For (d), let $A \in \mathcal{A}$. We have that $\{A, \emptyset, \emptyset, \dots)$ is a cover of A. As a result,

$$\mu^*(A) \leq \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots = \mu(A).$$

  Let (E_j) be a cover of A. We have that $A = \bigcup (A \cap E_j)$. Because μ is a measure,

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(A \cap E_j) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

It follows that $\mu(A) \leq \inf\{\sum \mu(E_j) : (E_j) \in \mathfrak{C}(A)\} = \mu^*(A)$. Hence, $\mu(A) = \mu^*(A)$.

For (e), let $\varepsilon > 0$ be arbitrary. For each n, choose a sequence (E_{nk}) of sets in $\mathcal{A}$ such that (E_{nk}) covers B_n and

$$\sum_{k=1}^{\infty} \mu(E_{nk}) \leq \mu^*(B_n) + \frac{\varepsilon}{2^n}.$$

Since $\{E_{nk} : n, k \in \mathbb{N}\}$ is a countable collection from $\mathcal{A}$ whose union contains $\bigcup B_n$, it follows that

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(E_{nk}) \leq \varepsilon + \sum_{k=1}^{\infty} \mu^*(B_n).$$

Since ε can be arbitrarily small, we have that Property (e) holds. $\square$

- **Definition 17.** *An* **outer measure** μ^* *on a set* X *is a function* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *such that the following properties hold.*

  1. $\mu^*(\emptyset) = 0$.
  2. *If* $E \subseteq F \subseteq X$*, then* $\mu^*(E) \leq \mu^*(F)$.
  3. μ^* *is* **countably subadditive**. *In other words, if* $\{E_i \subseteq X : i \in \mathbb{N}\}$ *is a countable collection of subsets of* X*, then*

  $$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

- Lemma 16 shows that μ^* is an outer measure on X if μ is a premeasure on an algebra $\mathcal{A}$ on X. The outer measure μ^* is called the **outer measured generated by** μ.

- μ^* is defined for arbitrary subsets of X, so it is also defined on countable unions of subsets of X as well. However, it is not yet a fully fledged measure because we cannot yet find a σ-algebra on which it is countably additive on.

- The following criterion is used to classify members of such a σ-algebra.

  Definition 18. *A subset* E *of* X *is said to be* μ^***-measurable** *if*

  $$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

  for every subset A *of* X.

- A μ^*-measurable set E splits any set A into pieces whose output measures add up to the outer measure of A. In other words, a set is μ^*-measurable if it splits other sets in a "nice" way.

- **Theorem 19 (Carathéodory's extension).** *Let* μ^* *be an outer measure on* X*. The collection of* μ^**-measurable sets is a* σ*-algebra on* X*. Moreover,* μ^* *is a measure on this collection.*

  Proof. Let $\mathcal{X}^*$ denote the set of μ^*-measurable sets on X. We shall show that $\mathcal{X}^*$ is a σ-algebra, and μ^* is a measure on it.

  ($\infty, X \in \mathcal{X}^*$) Because μ^* is a premeasure, we have that $\mu^*(\emptyset) = 0$. For any set $A \subseteq X$, we have that

  $$\mu^*(A) = \mu^*(\emptyset) + \mu^*(A) = \mu^*(A \cap \emptyset) + \mu^*(A \cap X).$$

  Because $\emptyset$ and X are complements of each other, it follows that they are μ^*-measurable and so belong to $\mathcal{X}^*$.

(Closure under complementation) Let E be a μ^*-measurable set. It follows that

$$\mu^*(A) = \mu^*(\emptyset) + \mu^*(A \cap E) = \mu^*(A \cap E^c)$$

This implies that E^c is also μ^*-measurable.

(Closure under finite unions) This step is required to show closure under countable union. Suppose that E and F are μ^*-measurable. Let $A \subseteq X$. We need to show that

$$\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c).$$

Because $A = (A \cap (E \cup F)) \cup (A \cap (E \cup F)^c)$ and μ^* is subadditive, we have that

$$\mu^*(A) \leq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c).$$

Thus, it remains to show that $\mu^*(A) \geq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c)$. Because A is μ^*-measurable, we have that

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Since both $A \cap E$ and $A \cap E^c$ are subsets of X and B is μ^*-measurable, we have

$$\begin{aligned} \mu^*(A \cap E) &= \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) \\ \mu^*(A \cap E^c) &= \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c). \end{aligned}$$

As a result,

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c) \\ &= \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap (E \cup F)^c). \end{aligned}$$

Because $E \cup F = (E \cap F) \cup (E \cap F^c) \cup (E^c \cap F)$, we have that

$$A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E \cap F^c) \cup (A \cap E^c \cap F).$$

So,

$$\mu^*(A \cap (E \cup F)) \leq \mu^*(A \cap E \cap F) \cup \mu^*(A \cap E \cap F^c) \cup \mu^*(A \cap E^c \cap F).$$

Thus,

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap (E \cup F)^c) \\ &\geq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c). \end{aligned}$$

It follows that $\mu^*(A) = \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c)$, and so $E \cup F$ is also measurable.

(Closure under finite intersections) We briefly mention that closure under complementation and closure under finite union implies closure under complementation. This is a consequence of de Morgan's law: $E \cap F = (E^c \cup F^c)^c$.

(Closure under disjoint countable unions implies closure under countable unions) We need to show closure under countable union. However, it suffices to only show that $\bigcup E_j$ is measurable for all sequences (E_j) where the sets are measurable and disjoint. To see this, let F_j denote the union of the first j sets:

$$F_j = \bigcup_{i=1}^{j} E_j,$$

and let $G_0 = E_1$ and $G_j = F_{j+1} - F_j = F_{j+1} \cap F_j^c$ for $j \geq 1$. It follows that, G_j is measurable for all j because $\mathcal{X}^*$ is closed under finite unions and intersections. Moreover, the G_j's are disjoint, and

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=0}^{\infty} G_j.$$

Hence, the measurability of $\bigcup_{j=0}^{\infty} G_j$ imples the measurability of $\bigcup_{j=1}^{\infty} E_j$.

(Finite additivity) In order to establish closure under countable unions, it is useful to show that μ^* is additive for finite disjoint unions. Let E and F be any disjoint measurable sets. Because E is measurable, we have that

$$\mu^*(E \cup F) = \mu^*((E \cup F) \cap E) + \mu^*((E \cup F) \cap E^c) = \mu^*(E) + \mu^*((E \cup F) - E) = \mu^*(E) + \mu^*(F).$$

(Closure under disjoint countable unions) Let (E_j) be a sequence of disjoint measurable sets. Let

$$F_j = \bigcup_{i=1}^{j} E_j, \qquad F = \bigcup_{i=1}^{\infty} E_j.$$

We need to show show that, for any set $A \subseteq X$, it is true that $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$. Because μ^* is subadditive, we already know that $\mu^*(A) \leq \mu^*(A \cap F) + \mu^*(A \cap F^c)$. So, we only need to show that $\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c)$.

For any j, we have that F_j is measurable because of closure under finite unions. So

$$\begin{aligned}
\mu^*(A) &= \mu^*(A \cap F_j) + \mu^*(A \cap F_j^c) \\
&= \mu^*\left(A \cap \bigcup_{i=1}^{j} E_i\right) + \mu^*(A \cap F_j^c) \\
&= \mu^*\left(\bigcup_{i=1}^{n} (A \cap E_i)\right) + \mu^*(A \cap F_j^c).
\end{aligned}$$

Because $A \cap E_1$, $A \cap E_2$, ..., and $A \cap E_j$ are mutually disjoint, we have that

$$\mu^*(A) = \sum_{i=1}^{j} \mu^*(A \cap E_j) + \mu^*(A \cap F_j^c).$$

Moreover, because $F_j \subseteq F$, it follows that $A \cap F_j^c \supseteq A \cap F^c$. Hence, $\mu^*(A \cap F_j^c) \geq \mu^*(A \cap F^c)$. Thus,

$$\mu^*(A) \geq \sum_{i=1}^{j} \mu^*(A \cap E_i) + \mu^*(A \cap F^c).$$

Taking the limit as $j \to \infty$, we have that

$$\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap F^c) \\
&\geq \mu^*\left(\bigcup_{i=1}^{\infty} (A \cap E_i)\right) + \mu^*(A \cap F^c) \\
&= \mu^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + \mu^*(A \cap F^c) \\
&= \mu^*(A \cap F) + \mu^*(A \cap F^c),
\end{aligned}$$

which implies that F is μ^*-measurable.

(Countable additivity) In the proof of closure under countable union, we established that

$$\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap F^c) \\
&\geq \mu^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) + \mu^*(A \cap F^c) \\
&= \mu^*(A \cap F) + \mu^*(A \cap F^c) \\
&\geq \mu^*(A).
\end{aligned}$$

Therefore, it must be the case that

$$\sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap F^c) = \mu^*\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) + \mu^*(A \cap F^c),$$

and this is true for any set $A \subseteq X$. Taking $A = F$, we have that

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu^*(F \cap E_i) + \mu^*(F \cap F^c) &= \mu^*\left(\bigcup_{i=1}^{\infty}(F \cap E_i)\right) + \mu^*(F \cap F^c) \\
\sum_{i=1}^{\infty} \mu^*(E_i) + \mu^*(\emptyset) &= \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) + \mu^*(\emptyset) \\
\sum_{i=1}^{\infty} \mu^*(E_i) &= \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right),
\end{aligned}$$

which shows that μ^* is countably additive. □

- **Corollary 20.** *Let μ be a premeasrue on an algebra $\mathcal{A}$ on X. Let μ^* be the outer measure generated by μ, and let $\mathcal{A}^*$ be the collection of μ^*-measurable sets.*

4 Product Measures

5 Measurable Functions

6 Integration

- In this section, consider a fixed measurable space $(X, \mathcal{X})$.
- **Definition 21.** *A function $f : X \to \mathbb{R}$ is said to be* X**-measurable** *(or simply* **measurable***) if, for every real number α, the set $\{x \in X : f(x) > \alpha\}$ is measurable (in other words, belongs to $\mathcal{X}$).*
- **Proposition 22.** *For a function $f : X \to \mathbb{R}$, the following statements are equivalent.*
  (a) For every $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ belongs to $\mathcal{X}$.
  (b) For every $\alpha \in \mathbb{R}$, the set $B_\alpha = \{x \in X : f(x) \geq \alpha\}$ belongs to $\mathcal{X}$.
  (c) For every $\alpha \in \mathbb{R}$, the set $C_\alpha = \{x \in X : f(x) \leq alpha\}$ belongs to $\mathcal{X}$.
  (d) For every $\alpha \in \mathbb{R}$, the set $D_\alpha = \{x \in X : f(x) < \alpha\}$ belongs to $\mathcal{X}$.

References

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