Angular Moments

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1 Definitions

- In this section, we talk about spherical functions and their moments.
- A function f that associates a point on the unit sphere S^2 to a real number is called a *spherical function*.
- Here, S^2 is parameterized by two angles: the *azimuthal angle* $\theta \in [0, 2\pi)$ and the *inclination angle* $\varphi \in [0, \pi)$. The point associated with the ordered pair (θ, ϕ) is
$$\begin{bmatrix} \cos\theta \sin\varphi \\ \sin\theta \sin\varphi \\ \cos\varphi \end{bmatrix}.$$
Such a point is often denoted by ω. The three components of ω, from top to bottom, are denoted by ω_1, ω_2, and ω_3.
- Note that $d\omega = \sin\varphi\ d\varphi\ d\theta$. Hence, integrating a spherical function f over the sphere can be rewritten in terms of θ and φ as follows:
$$\int_{S^2} f(\omega)\ d\omega = \int_0^{2\pi} \int_0^{\pi} f(\theta, \varphi) \sin\varphi\ d\varphi\ d\theta.$$
- Let f be a spherical function.

  The 0*th moment* of f is
$$\mu_0[f] = \int_{S^2} f(\omega)\ d\omega.$$
  The 1*st moment* of f is
$$\mu_1[f] = \begin{bmatrix} \mu_1[f]_1 \\ \mu_1[f]_2 \\ \mu_1[f]_3 \end{bmatrix} = \begin{bmatrix} \int_{S^2} f(\omega)\omega_1\ d\omega \\ \int_{S^2} f(\omega)\omega_2\ d\omega \\ \int_{S^2} f(\omega)\omega_3\ d\omega \end{bmatrix} = \int_{S^2} f(\omega)\omega\ d\omega.$$
  The 2*nd moment* of f is
$$\mu_2[f] = \begin{bmatrix} \mu_2[f]_{11} & \mu_2[f]_{12} & \mu_2[f]_{13} \\ \mu_2[f]_{21} & \mu_2[f]_{22} & \mu_2[f]_{23} \\ \mu_2[f]_{31} & \mu_2[f]_{32} & \mu_2[f]_{33} \end{bmatrix}$$
  where
$$\mu_2[f]_{ij} = \int_{S^2} f(\omega)\omega_i\omega_j\ \mathrm{d}\omega.$$
  The 3rd moment, 4th moment, and so on can be defined in a similar way, but we will not go there.

2 Integrals of Powers of Sine and Cosine

- We shall develop some identities for evaluating the moments. These identities involve integrals of sine and cosine.

- **Definition 2.1.** *Let m and n be non-negative integers. Define*

$$\mathcal{I}^{m,n}(a,b) = \int_a^b \sin^m\theta\cos^n\theta\ \mathrm{d}\theta.$$

Moreover, let

$$\begin{aligned}
\mathcal{S}^{m,n} &= \mathcal{I}^{m,n}(0,2\pi), & & \\
\mathcal{H}_1^{m,n} &= \mathcal{I}^{m,n}(0,\pi), & \mathcal{H}_2^{m,n} &= \mathcal{I}^{m,n}(\pi,2\pi), \\
\mathcal{Q}_1^{m,n} &= \mathcal{I}^{m,n}(0,\pi/2), & \mathcal{Q}_2^{m,n} &= \mathcal{I}^{m,n}(\pi/2,\pi), \\
\mathcal{Q}_3^{m,n} &= \mathcal{I}^{m,n}(\pi,3\pi/2), & \mathcal{Q}_4^{m,n} &= \mathcal{I}^{m,n}(3\pi/2,2\pi).
\end{aligned}$$

- **Lemma 2.2.** $\mathcal{Q}_1^{m,n} = (-1)^n\mathcal{Q}_2^{m,n} = (-1)^{m+n}\mathcal{Q}_3^{m,n} = (-1)^m\mathcal{Q}_4^{m,n}$.

Proof. We have that

$$\mathcal{Q}_2^{m,n} = \int_{\pi/2}^{\pi} \sin^m\theta\cos^n\theta\ \mathrm{d}\theta.$$

Let $u = \pi - \theta$. We have that $\mathrm{d}u = -\mathrm{d}\theta$, and

$$\begin{aligned}
\int_{\pi/2}^{\pi} \sin^m\theta\cos^n\theta\ \mathrm{d}\theta &= -\int_{\pi/2}^{0} \sin^m(\pi-u)\cos^n(\pi-u)\ \mathrm{d}u \\
&= \int_0^{\pi/2} \sin^m u\ (-1)^n\cos^n u\ \mathrm{d}u = (-1)^n\mathcal{Q}_2^{m,n}.
\end{aligned}$$

Other equations are similar. □

- **Lemma 2.3.** *If m or n is odd, then $\mathcal{S}^{m,n} = 0$. Otherwise, $\mathcal{S}^{m,n} = 4\mathcal{Q}_1^{m,n}$.*

Proof. If m or n is odd, then exactly two of m, n, and $m+n$ are odd. So,

$$\mathcal{S}^{m,n} = \mathcal{Q}_1^{m,n} + \mathcal{Q}_2^{m,n} + \mathcal{Q}_3^{m,n} + \mathcal{Q}_4^{m,n} = \mathcal{Q}_1^{m,n} + (-1)^n\mathcal{Q}_1^{m,n} + (-1)^{m+n}\mathcal{Q}_1^{m,n} + (-1)^m\mathcal{Q}_1^{m,n} = 0.$$

Otherwise, all of m, n, and $m+n$ are positive.

$$\mathcal{S}^{m,n} = \mathcal{Q}_1^{m,n} + \mathcal{Q}_2^{m,n} + \mathcal{Q}_3^{m,n} + \mathcal{Q}_4^{m,n} = \mathcal{Q}_1^{m,n} + \mathcal{Q}_1^{m,n} + \mathcal{Q}_1^{m,n} + \mathcal{Q}_1^{m,n} = 4\mathcal{Q}_1^{m,n}.$$

- **Lemma 2.4.** $\mathcal{Q}_1^{m,n} = \mathcal{Q}_1^{n,m}$ *and* $\mathcal{S}^{m,n} = \mathcal{S}^{n,m}$.

Proof. Let $u = \pi/2 - \theta$. We have $\mathrm{d}\theta = -\mathrm{d}u$, and

$$\begin{aligned}
\mathcal{Q}_1^{m,n} &= \int_0^{\pi/2} \sin^m\theta\cos^n\theta\ \mathrm{d}\theta \\
&= -\int_{\pi/2}^{0} \sin^m(\pi/2-u)\cos^n(\pi/2-u)\ \mathrm{d}u \\
&= \int_0^{\pi/2} \cos^m u\sin^n u\ \mathrm{d}u = \mathcal{Q}_1^{n,m}.
\end{aligned}$$

The equation involving $\mathcal{S}^{m,n}$ follows from Lemma 2.3. □

- **Lemma 2.5.** *If $m \geq 2$ and $n \geq 0$, then $\mathcal{Q}_1^{m,n} = \frac{m-1}{m+n}\mathcal{Q}_1^{m-2,n}$. Moreover, if $n \geq 2$ and $m \geq 0$, then $\mathcal{Q}_1^{m,n} = \frac{n-1}{m+n}\mathcal{Q}_1^{m,n-2}$.*

  Proof. Let $u = \sin^{m-1}\theta\cos^n\theta$, and $v = \cos\theta$. We have that $\mathrm{d}v = -\sin\theta\ \mathrm{d}\theta$, and

$$\begin{aligned}\mathrm{d}u &= [(m-1)\sin^{m-2}\theta\cos^{n+1}\theta - n\sin^m\theta\cos^{n-1}\theta]\ \mathrm{d}\theta \\ &= [(m-1)\sin^{m-2}\theta(1-\sin^2\theta)\cos^{n-1}\theta - n\sin^m\theta\cos^{n-1}\theta]\ \mathrm{d}\theta \\ &= [(m-1)\sin^{m-2}\theta\cos^{n-1}\theta - (m+n-1)\sin^m\theta\cos^{n-1}\theta]\ \mathrm{d}\theta.\end{aligned}$$

  So,

$$\mathcal{Q}_1^{m,n} = \int_0^{\pi/2}\sin^m\theta\cos^n\theta\ \mathrm{d}\theta = -\int_0^{\pi/2} u\ \mathrm{d}v = -[uv]_0^{\pi/2} + \int_0^{\pi/2} v\ \mathrm{d}u.$$

  Now, $[uv]_0^{\pi/2} = [\sin^{m-1}\theta\cos^{n+1}\theta]_0^{\pi/2}$. Since both $m-1$ and $n+1$ are at least one, we have that $[uv]_0^{\pi/2} = 0$. Therefore,

$$\begin{aligned}\mathcal{Q}_1^{m,n} &= \int_0^{\pi/2} v\ \mathrm{d}u \\ &= (m-1)\int_0^{\pi/2}\sin^{m-2}\theta\cos^n\theta\ \mathrm{d}\theta - (m+n-1)\int_0^{\pi/2}\sin^m\theta\cos^n\theta\ \mathrm{d}\theta \\ &= (m-1)\mathcal{Q}_1^{m-2,n} - (m+n-1)\mathcal{Q}_1^{m,n}.\end{aligned}$$

  Thus, $\mathcal{Q}_1^{m,n} = \frac{m-1}{m+n}\mathcal{Q}_1^{m-2,n}$.
  Moreover, if $n \geq 2$ and $m \geq 0$, then $\mathcal{Q}_1^{n,m} = \mathcal{Q}_1^{m,n} = \frac{n-1}{m+n}\mathcal{Q}_1^{n-2,m} = \frac{n-1}{m+n}\mathcal{Q}_1^{m,n-2}$ as required. □

- **Definition 2.6.** *Let n be a non-negative integer. The* double factorial *of n, denoted by $n!!$, is defined as follows:*

$$n!! = \begin{cases} 1, & n \leq 1 \\ n \times (n-2)!!, & n \geq 2 \end{cases}.$$

- **Theorem 2.7.**

$$\mathcal{Q}_1^{m,n} = \begin{cases} \pi/2, & m = n = 0 \\ 1, & m = 1, n = 0 \\ 1, & m = 0, n = 1 \\ 1/2, & m = 1, n = 1 \\ \frac{(m-1)!!(n-1)!!}{(m+n)!!}\mathcal{Q}_1^{m \bmod 2, n \bmod 2}, & \textit{otherwise} \end{cases}$$

  Proof. By induction on $m+n$ and repeated use of Lemma 2.5. □

3 Moment Integrals

- **Lemma 3.1.** *Let k be a positive integer. Let $i_1, i_2, \ldots, i_k \in \{1,2,3\}$. Let π be any permutation of $\{1,2,3\}$. Then,*

$$\int_{S^2}\omega_{i_1}\omega_{i_2}\cdots\omega_{i_k}\ \mathrm{d}\omega = \int_{S^2}\omega_{\pi(i_1)}\omega_{\pi(i_2)}\cdots\omega_{\pi(i_k)}\ \mathrm{d}\omega.$$

  That is, you can change the indices without changing the value of the integral.

Proof. Symmetry. ☐

- **Lemma 3.2.** $\mu_1[1]_1 = \mu_1[1]_2 = \mu_1[1]_3 = 0$.

  Proof. By Lemma 3.1, we only need to show that $\mu_1[1]_3 = 0$. We have that

$$\begin{aligned}\mu_1[1]_3 &= \int_{S^2} \omega_3 \, \mathrm{d}\omega = \int_0^{2\pi}\int_0^{\pi} \cos\varphi \sin\varphi \, \mathrm{d}\varphi \mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \cos\varphi\sin\varphi \, \mathrm{d}\varphi \\ &= 2\pi\mathcal{H}_1^{1,1} = 2\pi(\mathcal{Q}_1^{1,1} + \mathcal{Q}_2^{1,1}) = 2\pi(\mathcal{Q}_1^{1,1} - \mathcal{Q}_1^{1,1}) = 0.\end{aligned}$$

- **Lemma 3.3.**

$$\mu_2[1]_{ij} = \begin{cases} 0, & i \neq j \\ 4\pi/3, & i = j \end{cases}$$

  Proof. If $i \neq j$, we have

$$\mu_2[1]_{ij} = \mu_2[1]_{12} = \int_0^{2\pi}\int_0^{\pi} \sin\theta\cos\theta\sin^3\varphi \, \mathrm{d}\varphi\mathrm{d}\theta = \int_0^{2\pi} \sin\theta\cos\theta \, \mathrm{d}\theta \int_0^{\pi} \sin^3\varphi \, \mathrm{d}\varphi = \mathcal{S}^{1,1}\mathcal{H}_1^{3,0} = 0.$$

  If $i = j$, we have

$$\begin{aligned}\mu_2[1]_{ii} = \mu_2[1]_{33} &= \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi\sin\varphi \, \mathrm{d}\varphi\mathrm{d}\theta = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \cos^2\varphi\sin\varphi \, \mathrm{d}\varphi \\ &= 2\pi\mathcal{H}_1^{2,1} = 4\pi Q_1^{2,1} = 4\pi \cdot \frac{1}{3} Q_1^{0,1} = \frac{4\pi}{3}.\end{aligned}$$

- **Lemma 3.4.** *Let* $\mathbf{a}$ *be any constant vector. Then,* $\omega_0[\mathbf{a}\cdot\omega] = 0$.

  Proof. Let $\mathbf{a} = (a_1, a_2, a_3)^T$. Then,

$$\omega_0[\mathbf{a}\cdot\omega] = \int_{S^2} \omega\cdot\mathbf{a} \, \mathrm{d}\omega = \int_{S^2} (a_1\omega_1 + a_2\omega_2 + a_3\omega_3) \, \mathrm{d}\omega = a_1\mu_1[1]_1 + a_2\mu_1[1]_2 + a_3\mu_1[1]_3 = 0.$$

- **Lemma 3.5.** *Let* a *be any scalar. Then,* $\omega_0[a] = 4\pi a$.

  Proof. Obvious. ☐

- **Lemma 3.6.** *Let* A *be any* 3×3 *constant matrix. Then,* $\mu_0[\omega^T A\omega] = \frac{4\pi}{3}\mathrm{tr}(A)$.

  Proof. We have that

$$\omega^T A \omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3} \omega_i\omega_j a_{ij}.$$

  Thus,

$$\mu_0[\omega^T A\omega] = \mu_0\left[\sum_{i=1}^{3}\sum_{j=1}^{3} \omega_i\omega_j a_{ij}\right] = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\mu_0[\omega_i\omega_j] = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\mu_2[1]_{ij}.$$

  By Lemma 3.3, the only non-zero terms in the sum are those where $i = j$. Thus,

$$\mu_0[\omega^T A\omega] = a_{11}\mu_2[1]_{11} + a_{22}\mu_2[1]_{22} + a_{33}\mu_2[1]_{33} = \frac{4\pi}{3}(a_{11} + a_{22} + a_{33}) = \frac{4\pi}{3}\mathrm{tr(A)}.$$

- **Lemma 3.7.** *Let a be any constant. We have that $\mu_1[a] = \mathbf{0}$.*

  Proof. We have that

$$\mu_1[a] = \begin{bmatrix} \mu_1[a]_1 \\ \mu_1[a]_2 \\ \mu_1[a]_3 \end{bmatrix} = \begin{bmatrix} a\mu_1[1]_1 \\ a\mu_1[1]_2 \\ a\mu_1[1]_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

- **Lemma 3.8.** *If $\mathbf{a}$ is a constant vector, then $\mu_1[\omega \cdot \mathbf{a}] = \frac{4\pi}{3}\mathbf{a}$.*

  Proof. Let $\mathbf{a} = (a_1, a_2, a_3)^T$. Then,

$$\begin{aligned} \mu_1[\omega \cdot \mathbf{a}] &= \begin{bmatrix} \mu_1[\omega \cdot \mathbf{a}]_1 \\ \mu_1[\omega \cdot \mathbf{a}]_2 \\ \mu_1[\omega \cdot \mathbf{a}]_3 \end{bmatrix} = \begin{bmatrix} \mu_1[a_1\omega_1 + a_2\omega_2 + a_3\omega_3]_1 \\ \mu_1[a_1\omega_1 + a_2\omega_2 + a_3\omega_3]_2 \\ \mu_1[a_1\omega_1 + a_2\omega_2 + a_3\omega_3]_3 \end{bmatrix} = \begin{bmatrix} a_1\mu_1[\omega_1]_1 + a_2\mu_1[\omega_2]_1 + a_3\mu_1[\omega_3]_1 \\ a_1\mu_1[\omega_1]_2 + a_2\mu_1[\omega_2]_2 + a_3\mu_1[\omega_3]_2 \\ a_1\mu_1[\omega_1]_3 + a_2\mu_1[\omega_2]_3 + a_3\mu_1[\omega_3]_3 \end{bmatrix} \\ &= \begin{bmatrix} a_1\mu_2[1]_{11} + a_2\mu_2[1]_{12} + a_3\mu_2[1]_{13} \\ a_1\mu_2[1]_{21} + a_2\mu_2[1]_{22} + a_3\mu_2[1]_{23} \\ a_1\mu_2[1]_{31} + a_2\mu_2[1]_{32} + a_3\mu_2[1]_{33} \end{bmatrix} = \begin{bmatrix} a_1(4\pi/3) \\ a_2(4\pi/3) \\ a_3(4\pi/3) \end{bmatrix} = \frac{4\pi}{3}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{4\pi}{3}\mathbf{a}. \end{aligned}$$

- **Lemma 3.9.** *$\mu_3[1]_{ijk} = 0$ for all i, j, k.*

  Proof. We start with the case where i, j, and k are all different.

$$\begin{aligned} \mu_3[1]_{ijk} = \mu_3[1]_{123} &= \int_0^{2\pi}\int_0^{\pi} (\sin\varphi\sin\theta)(\sin\varphi\cos\theta)\cos\varphi\sin\varphi \ \mathrm{d}\varphi \ \mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\pi} (\sin^3\varphi\cos\varphi)(\sin\theta\cos\theta) \ \mathrm{d}\varphi \ \mathrm{d}\theta \\ &= \left(\int_0^{2\pi} \sin\theta\cos\theta \ \mathrm{d}\theta\right)\left(\int_0^{\pi} \sin^3\varphi\cos\varphi \ \mathrm{d}\varphi\right) \\ &= \mathcal{S}^{1,1}\left(\int_0^{\pi} \sin^3\varphi\cos\varphi \ \mathrm{d}\varphi\right) = 0. \end{aligned}$$

  We then deal with the case where two of i, j, and k are the same.

$$\begin{aligned} \mu_3[1]_{ijk} = \mu_3[1]_{113} &= \int_0^{2\pi}\int_0^{\pi} (\sin\varphi\cos\theta)^2\cos\varphi\sin\varphi \ \mathrm{d}\varphi \ \mathrm{d}\theta \\ &= \left(\int_0^{2\pi} \cos^2\theta \ \mathrm{d}\theta\right)\left(\int_0^{\pi} \cos^3\varphi\sin\varphi \ \mathrm{d}\phi\right) \\ &= \mathcal{S}^{0,2}\mathcal{H}_1^{1,3} = \mathcal{S}^{0,2}(\mathcal{Q}_1^{1,3} + \mathcal{Q}_2^{1,3}) = \mathcal{S}^{0,2}(\mathcal{Q}_1^{1,3} + (-1)^3\mathcal{Q}_1^{1,3}) = 0. \end{aligned}$$

  Lastly, we work on the case where $i = j = k$.

$$\begin{aligned} \mu_3[1]_{ijk} = \mu_3[1]_{111} &= \int_0^{2\pi}\int_0^{\pi} (\sin\varphi\cos\theta)^3\sin\varphi \ \mathrm{d}\varphi \ \mathrm{d}\theta \\ &= \left(\int_0^{2\pi} \cos^3\theta \ \mathrm{d}\theta\right)\left(\int_0^{\pi} \sin^4\varphi \ \mathrm{d}\varphi\right) \\ &= \mathcal{S}^{0,3}\mathcal{H}_1^{4,0} = 0. \end{aligned}$$

- **Lemma 3.10.** *Let A be a constant 3×3 matrix. Then, $\mu_1[\omega^T A\omega] = \mathbf{0}$.*

  Proof. We have that

$$\mu_1[\omega^T A\omega] = \begin{bmatrix} \mu_1[\omega^T A\omega]_1 \\ \mu_1[\omega^T A\omega]_2 \\ \mu_1[\omega^T A\omega]_3 \end{bmatrix} = \begin{bmatrix} \mu_1[\sum\sum a_{ij}\omega_i\omega_j]_1 \\ \mu_1[\sum\sum a_{ij}\omega_i\omega_j]_2 \\ \mu_1[\sum\sum a_{ij}\omega_i\omega_j]_3 \end{bmatrix} = \begin{bmatrix} \sum\sum a_{ij}\mu_1[\omega_i\omega_j]_1 \\ \sum\sum a_{ij}\mu_1[\omega_i\omega_j]_2 \\ \sum\sum a_{ij}\mu_1[\omega_i\omega_j]_3 \end{bmatrix} = \begin{bmatrix} \sum\sum a_{ij}\mu_3[1]_{1ij} \\ \sum\sum a_{ij}\mu_3[1]_{2ij} \\ \sum\sum a_{ij}\mu_3[1]_{3ij} \end{bmatrix} = \mathbf{0}.$$