# Summary: Continuity

## Definition of continuity at a point

We say that a function $f$ is **continuous at a point** $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

In particular, if either $f(a)$ or $\lim_{x \to a} f(x)$ fails to exist, then $f$ is discontinuous at $a$.

We say that a function $f$ is **right-continuous at a point** $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a).$$

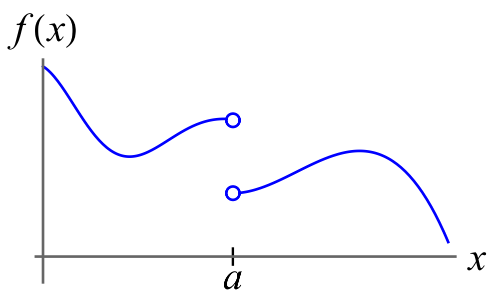
We say that a function $f$ is **left-continuous at a point** $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a).$$

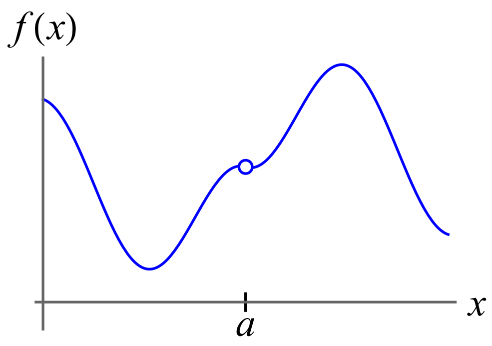
## Types of Discontinuities

It is sometimes useful to classify certain types of discontinuities.

If the left-hand limit $\lim_{x \to a^-} f(x)$ and the right-hand limit $\lim_{x \to a^+} f(x)$ both exist at a point $x = a$, but they are not equal, then we say that $f$ has a **jump discontinuity** at $x = a$.



If the overall limit $\lim_{x \to a} f(x)$ exists (i.e., the left- and right-hand limits agree), but the overall limit does not equal $f(a)$, then we say that $f$ has a **removable discontinuity** at $x = a$.




## Definition of continuous functions

A function $f(x)$ is **continuous** if for every point $c$ in the domain of $f(x)$, the function $f$ is continuous at the point $x = c$.

## Basic Continuous Functions

Note: we have not proven all of the following facts, but you should feel free to use them.

The following functions are continuous at *all real numbers*:

- all polynomials
- $\sqrt[3]{x}$
- $|x|$
- $\cos x$ and $\sin x$
- exponential functions $a^x$ with base $a > 0$

The following functions are continuous *at the specified values* of $x$:

- $\sqrt{x}$, for $x > 0$
- $\tan x$, at all $x$ where it is defined
- logarithmic functions $\log_a x$ with base $a > 0$, for $x > 0$

## Limit laws and continuity

If the functions $f$ and $g$ are continuous everywhere, then:

- $f + g$ is continuous everywhere.
- $f - g$ is continuous everywhere.
- $f \cdot g$ is continuous everywhere.
- $\dfrac{f}{g}$ is continuous where it is defined.

## Intermediate Value Theorem

If $f$ is a function which is continuous on the interval $[a, b]$, and $M$ lies between the values of $f(a)$ and $f(b)$, then there is at least one point $c$ between $a$ and $b$ such that $f(c) = M$.

(A function $f$ is **continuous on a closed interval** $[a, b]$ if it is right-continuous at $a$, left-continuous at $b$, and continuous at all points between $a$ and $b$.)