Wave Project

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1 Mathematical problem

The project addresses the two-dimensional, standard, linear wave equation, with damping,

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(q\left(x,y\right)\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial y}\left(q\left(x,y\right)\frac{\partial u}{\partial x}\right) + f\left(x,y,t\right) \tag{1}$$

Assosiated boundary conditions

$$\frac{\partial u}{\partial n} = 0 \tag{2}$$

In a rectangular spatial domain $\Omega = [0,L_x]X[0,L_y]$. The initial conditions are

$$u\left(x,y,0\right) = I\left(x,y\right) \tag{3}$$

$$u_t\left(x,y,0\right) = V\left(x,y\right) \tag{4}$$

2 Discretization

General scheme for computing $u_{i,j}^{n+1}$

$$u_{i,j}^{n+1} = \frac{2}{(2+b\Delta t)}[2u_{i,j}^{n} + \left(\frac{b\Delta t}{2} - 1\right) u_{i,j}^{n-1} + \left(\frac{\Delta t}{\Delta x}\right)^2 [D_x q D_x u]_{i,j}^{n} + \left(\frac{\Delta t}{\Delta y}\right)^2 [D_y q D_y u]_{i,j}^{n} + f_{i,j}^{n}]$$

Modified scheme for first step

$$u_{i,j}^{1} = \frac{1}{2}[2u_{i,j}^{0} - \left(\frac{b\Delta t}{2} - 1\right) 2\Delta t V_{i,j} + \left(\frac{\Delta t}{\Delta x}\right)^2 [D_x q D_x u]_{i,j}^{0} + \left(\frac{\Delta t}{\Delta y}\right)^2 [D_y q D_y u]_{i,j}^{0} + f_{i,j}^{0}]$$

and used the initial condition

$$u_t\left(x, y, 0\right) \approx \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} = V_{i,j}$$

to approximate the $u_{i,j}^{-1}$ value. The terms with the variable coefficient is approximated by

$$[D_x q D_x u]_{i,j}^{n} = \frac{1}{2}\left(q_{i,j} + q_{i+1,j}\right)\left(u_{i+1,j}^{n} + u_{i,j}^{n}\right) - \frac{1}{2}\left(q_{i,j} + q_{i-1,j}\right)\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right) \approx \frac{\partial u}{\partial x}\left(q\left(x, y\right)\frac{\partial u}{\partial x}\right)$$

and

$$[D_x q D_x u]_{i,j}^{n} = \frac{1}{2}\left(q_{i,j} + q_{i,j+1}\right)\left(u_{i,j+1}^{n} + u_{i,j}^{n}\right) - \frac{1}{2}\left(q_{i,j} + q_{i,j-1}\right)\left(u_{i,j}^{n} - u_{i,j-1}^{n}\right) \approx \frac{\partial u}{\partial y}\left(q\left(x, y\right)\frac{\partial u}{\partial x}\right)$$

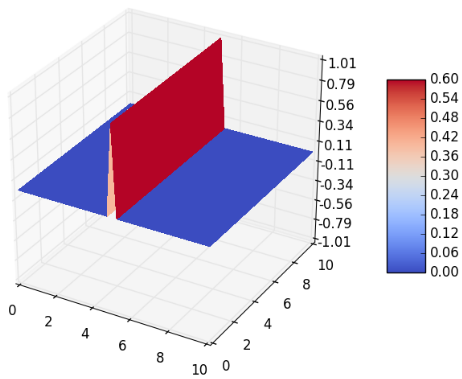
The Neumann boundary conditions is approximated by

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1,j}^{n} - u_{i-1,j}}{2\Delta x} = 0$$

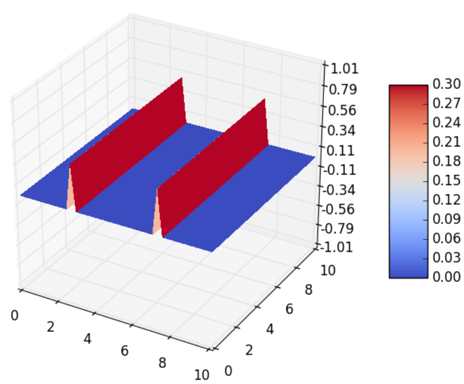
$$\frac{\partial u}{\partial y} \approx \frac{u_{i,j+1}^{n} - u_{i,j+1}}{2\Delta y} = 0$$

3 Exact 1D plug-wave solution in 2D

pulse-function for simulating the propagation of a plug wave, where I(x) is constant in some region of the domain and zero elsewhere.



plug wave at t = 0



plug wave at some point t > 0

The initial condition I(x)

$$m(T) = \begin{cases} 0.1 & for(x - Lx/2.0) > 0.1 \\ 1 & \end{cases} \tag{5}$$

4 Physical problem

Exploring what happens to a wave that enters a medium with different wave velocity. The particular physical interpretation is a tsunami over a subsea hill. Wave velocity in this case is given by $q = gH(x, y)$, where g is the acceleration of gravity and $H(x, y)$ is the stillwater depth.
initial surface is taken as a smooth Gaussian function

$$I(x; I_0, I_a, I_m, I_s) = I_0 + I_a \exp(-(\frac{x - I_m}{I_s})^2)$$

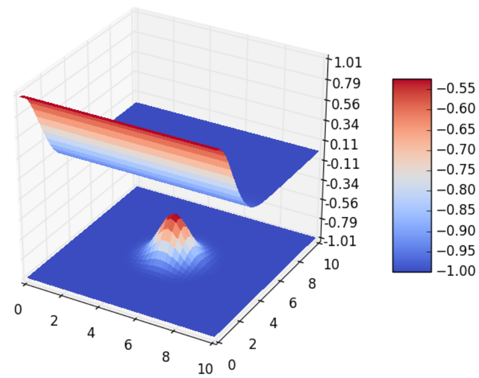
the seabed is modeled by a 2D Gaussian hill

$$B(x, y; B_0, B_a, B_{mx}, B_{my}, B_s, b) = B_0 + B_a \exp(-(\frac{x - B_{mx}}{b_s})^2 - (\frac{y - B_{my}}{bB_s})^2) \quad (6)$$

Further, the wave velocity q is given by

$$q = gH(x, y) = g(H_0 - B) \quad (7)$$

Tsunami over a Gaussian seabed at $t = 0$



Tsunami over a Gaussian seabed at $t > 0$