1 Chapter 4: Number Theory

1.1 Even and Odd

The sum of any two even integers is even.

The product of any two even integers is even.

1.2 Rational

1.3 Divisibility

1.4 Quotient-Remainder Theorem

The square of any integer can be written as $3s$ or $3s+1$ for some integer s.

In symbols: $\forall n \in \mathbb{Z}\ \exists s \in \mathbb{Z}\ (n^2 = 3s) \vee (n^2 = 3s+1)$

Proof:

Suppose n is an arbitrary integer.

By the quotient-remainder theorem, there exists an integer q such that $n = 3q$ or $n = 3q+1$ or $n = 3q+2$.

Case 1: Assume $n = 3q$.

$$\begin{aligned} n^2 &= (3q)^2 && \text{by substitution} \\ &= 3(3q^2) && \text{by algebra} \end{aligned}$$

Let $s = 3q^2$. s is an integer by closure of the integers under multiplication and addition.

Therefore, $n^2 = 3s$.

Case 2: Assume $n = 3q+1$.

$$\begin{aligned} n^2 &= (3q+1)^2 && \text{by substitution} \\ &= 9q^2 + 6q + 1 && \text{by algebra} \\ &= 3(3q^2 + 2q) + 1 && \text{by algebra} \end{aligned}$$

Let $s = 3q^2 + 2q$. s is an integer by closure of the integers under multiplication and addition.

Therefore, $n^2 = 3s+1$.

Case 3: Assume $n = 3q+2$.

$$\begin{aligned} n^2 &= (3q+2)^2 && \text{by substitution} \\ &= 9q^2 + 12q + 4 && \text{by algebra} \\ &= 9q^2 + 12q + 3 + 1 && \text{by algebra} \\ &= 3(3q^2 + 4q + 1) + 1 && \text{by algebra} \end{aligned}$$

Let $s = 3q^2 + 4q + 1$. s is an integer by closure of the integers under multiplication and addition.

Therefore, $n^2 = 3s+1$.

The QRT tells us that one of these three cases must apply, and in each case, we have proved that $n^2 = 3s$ or $n^2 = 3s+1$ for some integer s.

1.5 Proof by Contradiction and Contraposition