Prove that the square of any odd integer has the form $8m + 1$ for some integer m.

Restatement in symbols: $\forall n \in \mathbb{Z}^{\text{odd}}\ \exists m \in \mathbb{Z}\ x^2 = 8m + 1$.

Proof:

Suppose n is an arbitrary odd integer.

By the definition of odd, there exists an integer k such that $n = 2k + 1$.

[Normally, we'd now write something like $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$, but the problem is that we need to write n^2 as $8 \cdot (\text{integer}) + 1$, but we only have 4s in our equation, not 8s. So we use the quotient-remainder theorem.]

By the quotient-remainder theorem, there exists an integer q such that $k = 2q$ or $k = 2q + 1$.

*[We have used the QRT on k with $d = 2$ to learn that there must exist an integer r such that $k = 2q + r$ and $0 \leq r < 2$, which means r can only be 0 or 1. Note how the book does this differently: they use the QRT on **n**, not k, and use $d = 4$, not $d = 2$. Either way works, but I like this way better because the book glosses over showing that $4q$ and $4q + 2$ cannot be odd.]*

Case 1: Assume $k = 2q$.

$$
\begin{aligned}
n^2 = (2k+1)^2 &= 4k^2 + 4k + 1 && \text{by algebra} \\
&= 4(2q)^2 + 4(2q) + 1 && \text{by substitution} \\
&= 16q^2 + 8q + 1 && \text{by algebra} \\
&= 8(2q^2 + q) + 1 && \text{by algebra}
\end{aligned}
$$

Let $m = 2q^2 + q$. m is an integer by closure of the integers under multiplication and addition.

Therefore, $n^2 = 8m + 1$ for some integer m.

Case 2: Assume $k = 2q + 1$.

$$
\begin{aligned}
n^2 = (2k+1)^2 &= 4k^2 + 4k + 1 && \text{by algebra} \\
&= 4(2q+1)^2 + 4(2q+1) + 1 && \text{by substitution} \\
&= 4(4q^2 + 4q + 1) + (8q + 4) + 1 && \text{by algebra} \\
&= (16q^2 + 16q + 4) + (8q + 4) + 1 && \text{by algebra} \\
&= 16q^2 + 24q + 8 + 1 && \text{by algebra} \\
&= 8(2q^2 + 3q + 1) + 1 && \text{by algebra}
\end{aligned}
$$

Let $m = 2q^2 + 3q + 1$. m is an integer by closure of the integers under multiplication and addition.

Therefore, $n^2 = 8m + 1$ for some integer m.

Because one of Cases 1 and 2 must apply by the QRT, we can conclude that $n^2 = 8m + 1$ for some integer m.