Discrete Structures, Fall 2017, Homework 6 Solutions

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

For each statement below, state whether it is true or false. Then prove the statement if it is true, or its negation if it is false.

Remember, an example may only be used to prove that an existential statement is true or a universal statement is false. Any example or counter-example must include specific values for the variables and enough algebra and justification to illustrate that the example proves what you are claiming it proves.

You do not need to translate each statement into symbols first, though it is often useful to do so.

1. For any integers a and b, if $a \mid b$, then $a \mid (a+b)$.

   Solution: This is a true statement.

   Proof:

   Suppose a and b are arbitrarily chosen integers.

   Assume $a \mid b$.

   By the definition of divides, we know there exists an integer k such that $ak = b$.

   [We're trying to show $a \mid (a+b)$, so we want to get an equation that looks like $a + b = a * (\text{an integer})$]

   $a + b = a + ak = a(1 + k)$ by algebra and substitution.

   $1 + k$ is an integer by closure of the integers under addition.

   Therefore, because $a+b = a(1+k)$ and $1+k \in \mathbb{Z}$, we know $a \mid (a+b)$ by the definition of divides.

2. If n is an odd integer, then $n^2 - 1$ is divisible by 4.

   Solution: This is a true statement.

   Proof:

   Suppose n is an arbitrary odd integer. *[Or, "Suppose n is an arbitrary integer and assume n is odd." Either one is fine; they mean the same thing.]*

   By the definition of odd, there exists an integer k such that $x = 2k + 1$.

   *[We're trying to show $n^2 - 1$ is divisible by 4, so we want to get an equation that looks like $n^2 - 1 = 4$ * (an integer)]*

   $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k$ by algebra.

   $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4(k^2 + k)$ by algebra.

   Let $p = k^2 + k$. We know $p \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under multiplication and addition.

   Therefore, $n^2 - 1 = 4p$, and therefore $4 \mid n^2 - 1$ by the definition of divides.

3. $\forall a, b, c \in \mathbb{Z}\ [(a \mid c) \wedge (b \mid c)] \rightarrow [(a \mid b) \vee (b \mid a)]$.

   Solution: This is a false statement.

   Counter-example: *[Negation would be $\exists a, b, c \in \mathbb{Z}\ [(a \mid c) \wedge (b \mid c)] \wedge [(a \nmid b) \wedge (b \nmid a)]$.]*

   Let $a = 2$, $b = 3$, and $c = 6$. Then it is true that $a \mid c$ and $b \mid c$ (clearly $2 \mid 6$ and $3 \mid 6$), but it is not true that $a \mid b$ or $b \mid a$ (because 2 does not divide 3 and 3 does not divide 2).

4. The product of any two consecutive integers is even.

   Hints: Use only one universally-quantified variable, not two. Use the quotient-remainder theorem with $d = 2$.

   Symbols: $\forall x \in \mathbb{Z}\ \exists k \in \mathbb{Z}\ x(x+1) = 2k$.

   Solution:

   Suppose x is any arbitrary integer.

   By the quotient remainder theorem, there exists an integer q such that $x = 2q$ or $x = 2q + 1$.

   Case 1: Assume $x = 2q$.

   $x(x+1) = (2q)(2q+1) = 2[q(2q+1)]$ by algebra.

   Choose $k = q(2q+1)$. $k \in \mathbb{Z}$ by closure of the integers under addition and multiplication, and so therefore we know $x(x+1) = 2k$ for some integer k.

   Therefore, $x(x+1)$ is even by the definition of even.

   Case 2: Assume $x = 2q + 1$.

   $x(x+1) = (2q+1)(2q+2) = (2q+1)(2)(q+1) = 2[(2q+1)(q+1)]$.

   Choose $k = (2q+1)(q+1)$. $k \in \mathbb{Z}$ by closure of the integers under addition and multiplication, and so therefore we know $x(x+1) = 2k$ for some integer k.

   Therefore, $x(x+1)$ is even by the definition of even.

   Because in both Case 1 and 2 we reach the same conclusion (that $x(x+1)$ is even), and we know the quotient-remainder theorem tells us one of the two cases must be true, we can conclude that $x(x+1)$ must be even in general.

5. For any integers m, $m^2 - m$ can be written as either $3k$ or $3k + 2$ for some integer k.

   Hints: Use the quotient-remainder theorem on m. See if you can determine on your own what d should be.

   Symbols: $\forall m \in \mathbb{Z}\ [\exists k \in \mathbb{Z}\ m^2 - m = 3k \vee m^2 - m = 3k + 2]$

   Proof:

   Let m be an arbitrary integer.

By the quotient-remainder theorem, there exists an integer q such that $m = 3q$, $m = 3q + 1$, or $m = 3q + 2$.

Case 1: Assume $m = 3q$

$m^2 - m = (3q)^2 - 3q = 9q^2 - 3q = 3(3q^2 - q)$.

Let $k = 3q^2 - q$. $k \in \mathbb{Z}$ by the closure of the integers under multiplication and addition.

So $m^2 - m = 3k$.

Case 2: Assume $m = 3q + 1$

$$\begin{aligned} m^2 - m &= (3q + 1)^2 - (3q + 1) \\ &= 9q^2 + 6q + 1 - 3q - 1 \\ &= 9q^2 + 3q \\ &= 3(3q^2 + q) \end{aligned}$$

Let $k = 3q^2 + q$. $k \in \mathbb{Z}$ by the closure of the integers under multiplication and addition.

So $m^2 - m = 3k$.

Case 3: Assume $m = 3q + 2$

$$\begin{aligned} m^2 - m &= (3q + 2)^2 - (3q + 2) \\ &= 9q^2 + 12q + 4 - 3q - 2 \\ &= 9q^2 + 9q + 2 \\ &= 3(3q^2 + 3q) + 2 \end{aligned}$$

Let $k = 3q^2 + 3q$. $k \in \mathbb{Z}$ by the closure of the integers under multiplication and addition.

So $m^2 - m = 3k + 2$.

In all three cases, we reach the conclusion that $m^2 - m = 3k$ or $m^2 - m = 3k + 2$ for some integer k. Because we know one of the three cases must be true by the quotient-remainder theorem, we may conclude that $m^2 - m = 3k$ or $m^2 - m = 3k + 2$ for some integer k.

Hints: Remember, not all of these are necessarily true! Use the step-by-step example proofs from the handouts last week as a guide for these.

Do the following problems about sequences and series. Refer to section 5.1.

6. Write out the first four terms for each of the following sequences. List the name of the variable, the subscript, and the number itself. For example, for "$\forall n \in \mathbb{Z}^{\geq 1}\ d_n = 2n$" you would write "$d_1 = 2,\ d_2 = 4,\ d_3 = 6,\ d_4 = 8.$"

   (a) $\forall i \in \mathbb{Z}^{\geq 2}\ a_i = i(i-1)$

   Solution: $a_2 = 2, a_3 = 6, a_4 = 12, a_5 = 20.$

   (b) $\forall j \in \mathbb{Z}^{\geq 0}\ s_j = \dfrac{j}{j!}$

   Solution: $s_0 = 0, s_1 = 1, s_2 = 1, s_3 = 1/2.$

   (c) $\forall k \in \mathbb{Z}^{+}\ z_k = (1-k)(k-1)$

   Solution: $z_1 = 0, z_2 = -1, z_3 = -4, z_4 = -9.$

7. Write the following sums using sigma (Σ) notation.

   (a) $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n}{(n+1)!} = \displaystyle\sum_{i=1}^{n} \frac{i}{(i+1)!}$

   (b) $\dfrac{n}{1} + \dfrac{n-1}{2} + \dfrac{n-2}{3} + \cdots + \dfrac{1}{n} = \displaystyle\sum_{i=1}^{n} \frac{n-i+1}{i}$

   (c) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots = \displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$

8. Rewrite each of the following summations as an equivalent expression by separating off the last term. For reference, the first one has been done for you (we did this in class).

   (a) (Example:) $\displaystyle\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k+1)^2$

   (b) $\displaystyle\sum_{i=1}^{m+1} \frac{1}{2^{i-1}}$

   Solution: $\displaystyle\sum_{i=0}^{m+1} \frac{1}{2^{i-1}} = \sum_{i=0}^{m} \frac{1}{2^{i-1}} + \frac{1}{2^m}$

   (c) $\displaystyle\sum_{i=0}^{k} \frac{i+1}{i+2}$

   Solution: $\displaystyle\sum_{i=0}^{k} \frac{i+1}{i+2} = \sum_{i=0}^{k-1} \frac{i+1}{i+2} + \frac{k+1}{k+2}$