Prove that the product of any three consecutive integers is divisible by 3.

Restatement in symbols: $\forall n \in \mathbb{Z}\ \ 3 \mid n(n+1)(n+2)$.

Proof:

Suppose n is an arbitrary integer.

By the quotient-remainder theorem, there exists an integer q such that $n = 3q$ or $n = 3q + 1$ or $n = 3q + 2$.

Case 1: Assume $n = 3q$.

$n(n+1)(n+2) = 3q(3q+1)(3q+2) = 3\,[q(3q+1)(3q+2)]$ by algebra.

Let $k = q(3q+1)(3q+2)$. k is an integer by closure of the integers under multiplication and addition.

Therefore, $n(n+1)(n+2) = 3k$,

and therefore, $3 \mid n(n+1)(n+2)$ by the definition of divides.

Case 2: Assume $n = 3q + 1$.

$n(n+1)(n+2) = (3q+1)(3q+2)(3q+3) = (3q+1)(3q+2)(3(q+1)) = 3\,[(3q+1)(3q+2)(q+1)]$ by algebra.

Let $k = (3q+1)(3q+2)(q+1)$. k is an integer by closure of the integers under multiplication and addition.

Therefore, $n(n+1)(n+2) = 3k$,

and therefore, $3 \mid n(n+1)(n+2)$ by the definition of divides.

Case 3: Assume $n = 3q + 2$.

$n(n+1)(n+2) = (3q+2)(3q+3)(3q+4) = (3q+2)(3(q+1))(3q+4) = 3\,[(3q+2)(q+1)(3q+4)]$ by algebra.

Let $k = (3q+2)(q+1)(3q+4)$. k is an integer by closure of the integers under multiplication and addition.

Therefore, $n(n+1)(n+2) = 3k$,

and therefore, $3 \mid n(n+1)(n+2)$ by the definition of divides.

Because one of Cases 1, 2, and 3 must apply by the QRT, we can conclude that $n(n+1)(n+2)$ is divisible by 3.