Discrete Structures, Fall 2017, Homework 4 Solutions

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

1. Consider the following statement:

$$\exists x \in \mathbb{Z}\ x^2 = 4$$

Which of the following are correct translations of this statement? **Write down all the letters which are correct.** Note that some of the sentences below may be factually correct, but not correct translations of the original statement.

(a) The square of each integer is 4.

(b) Some integers have squares of 4.

(c) The number x, when squared, is equal to 4, for some integer x.

(d) If x is an integer, then $x^2 = 4$. (Hint: read page 70.)

(e) Some integer has a square of 4.

(f) There is at least one integer whose square is 4.

Solution: B, C, E, F.

2. Consider the following statement:

$$\forall n \in \mathbb{Z}\ \text{Even}(n) \rightarrow \text{Even}(n^2)$$

(You may interpret Even(x) to mean "x is an even number.")

Which of the following are correct translations of this statement? **Write down all the letters which are correct.** Note that some of the sentences below may be factually correct, but not correct translations of the original statement.

(a) All integers are even and have even squares.

(b) Given any integer that is even, the square of that integer is also even.

(c) For all integers, there are some whose square is even.

(d) Any integer that is even has an even square.

(e) If an integer is even, then its square is even.

(f) All even integers have even squares.

Solution: B, D, E, F

3. Let $E(x, y)$ mean "person x enjoys class y," let S be the set of all students, and let C be the set of all computer science classes.

   Translate each of the following into English statements. Make your sentences as natural-sounding as possible, while still being precise in meaning.

   (a) $\forall x \in S\ \forall y \in C\ E(x, y)$

   Solution: Every student enjoys every class.

   (b) $\exists x \in S\ \exists y \in C\ E(x, y)$

   Solution: There is a student who enjoys some class.

   (c) $\forall x \in S\ \exists y \in C\ E(x, y)$

   Solution: Every student has a class that they enjoy. / Every student enjoys some class (possibly a different class for each student!).

   (d) $\exists x \in S\ \forall y \in C\ E(x, y)$

   Solution: There is a student who enjoys all classes.

   (e) $\forall y \in C\ \exists x \in S\ E(x, y)$

   Solution: Every class has a student who enjoys it.

   (f) $\exists y \in C\ \forall x \in S\ E(x, y)$

   Solution: There is a class that is enjoyed by every student. / Every student enjoys some class (the same class for all students!).

4. Complete the following proofs using the method described in class (line numbers, rule justifications, etc).

   (a) P1: $\exists w \in D\ {\sim}M(w) \to N(w)$
   P2: $\forall x \in D\ {\sim}M(x) \vee R(x)$
   P3: $\forall y \in D\ {\sim}N(y) \to {\sim}R(y)$
   Prove: $\exists z \in D\ N(z)$

   Solution 1:

Line	Statement	Rule	Lines Used
1	${\sim}M(a) \to N(a)$	Existential instantiation	P1
2	${\sim}M(a) \vee R(a)$	Universal instantiation	P2
3	${\sim}N(a) \to {\sim}R(a)$	Universal instantiation	P3
4	$R(a) \to N(a)$	Contrapositive	3
5	$N(a)$	Dilemma	2, 1, 4
6	$\exists z \in D\ N(z)$	Existential generalization	5

   Solution 2:

Line	Statement	Rule	Lines Used
1	$\sim M(a) \to N(a)$	Existential instantiation	P1
2	$\quad \sim N(a)$	Assume	—
3	$\quad M(a)$	Modus tollens	1, 2
4	$\quad \sim M(a) \vee R(a)$	Universal instantiation	P2
5	$\quad R(a)$	Disjunctive syllogism	3, 4
6	$\quad N(a)$	Universal modus tollens	P3, 5
7	$\quad N(a) \wedge \sim N(a)$	Conjunctive addition	6, 2
8	$N(a)$	Closing cond world w/ contra	2–7
9	$\exists z \in D\ N(z)$	Existential generalization	8

(b) P1: $\forall w \in D \sim R(w) \wedge Q(w)$
P2: $\forall x \in D\ Q(x) \to \sim(P(x) \wedge S(x))$
P3: $\forall y \in D\ (T(y) \to R(y)) \to P(y)$
Prove: $\forall z \in D\ S(z) \to T(z)$

Solution:

Line	Statement	Rule	Lines Used
1	$\sim R(a) \wedge Q(a)$	Universal instantiation	P1
2	$Q(a)$	Conjunctive simplification	1
3	$\sim(P(a) \wedge S(a))$	Universal modus ponens	2, P2
4	$\sim P(a) \vee \sim S(a))$	DeMorgan's law	3
5	$\quad S(a)$	Assume	—
6	$\quad \sim P(a)$	Disjunctive syllogism	5, 4
7	$\quad \sim(T(a) \to R(a))$	Universal modus tollens	P3, 6
8	$\quad T(a) \wedge \sim R(a)$	Definition of implication	7
9	$\quad T(a)$	Conjunctive simplification	8
10	$S(a) \to T(a)$	Closing cond world w/out contra	5–9
11	$\forall z \in D\ S(z) \to T(z)$	Universal generalization	10

(c) P1: $\forall w \in D \sim B(w)$
P2: $\forall x \in D\ Q(x) \to (R(x) \wedge T(x))$
P3: $\forall y \in D\ [B(y) \to \sim Q(y)] \to [R(y) \to B(y)]$
Prove: $\forall z \in D \sim Q(z)$

Solution 1:

Line	Statement	Rule	Lines Used
1	$\sim B(a)$	Universal instantiation	P1
2	$\sim B(a) \vee \sim Q(a)$	Disjunctive addition	1
3	$B(a) \to \sim Q(a)$	Definition of implication	2
4	$R(a) \to B(a)$	Universal modus ponens	P3, 3
5	$\sim R(a)$	Modus tollens	4, 1
6	$\sim R(a) \vee \sim T(a)$	Disjunctive addition	5
7	$\sim(R(a) \wedge T(a))$	DeMorgan's law	6
8	$\sim Q(a)$	Universal modus tollens	P2, 7
9	$\forall z \in D \sim Q(z)$	Universal generalization	8

Solution 2:

Line	Statement	Rule	Lines Used
1	$\sim B(a)$	Universal instantiation	P1
2	$Q(a) \to (R(a) \wedge T(a))$	Universal instantiation	P2
3	$[B(a) \to \sim Q(a)] \to [R(a) \to B(a)]$	Universal instantiation	P3
4	$\sim[B(a) \to \sim Q(a)] \vee [R(a) \to B(a)]$	Definition of implication	3
5	$[B(a) \wedge Q(a)] \vee [\sim R(a) \vee B(a)]$	Definition of implication	4
6	$[[B(a) \wedge Q(a)] \vee \sim R(a)] \vee B(a)$	Associative law	5
7	$[B(a) \wedge Q(a)] \vee \sim R(a)$	Disjunctive syllogism	6, 1
8	$\quad Q(a)$	Assume	—
9	$\quad R(a) \wedge T(a)$	Modus ponens	2, 8
10	$\quad R(a)$	Conjunctive simplification	9
11	$\quad B(a) \wedge Q(a)$	Disjunctive syllogism	7, 10
12	$\quad B(a)$	Conjunctive simplification	11
13	$\quad B(a) \wedge \sim B(a)$	Conjunctive addition	12
14	$\sim Q(a)$	Closing cond world with contra	8–13
15	$\forall z \in D \sim Q(z)$	Universal generalization	14

5. In this problem, you are given a number of statements in English about people and musical instruments. You are also given a number of statements in predicate logic. For each of the English statements, you must decide which predicate logic statements are true for the English statement in question.

   Here are the predicate logic statements you can pick from:

   Let L be the set of people "Kate, Lisa, John;" let M be the set of musical instruments "piano, trumpet, accordion;" and let the predicate $P(x, y)$ mean "person x plays instrument y."

   1. $\forall x \in L\ \exists y \in M\ P(x, y)$
   2. $\exists x \in L\ \forall y \in M\ P(x, y)$
   3. $\forall y \in M\ \exists x \in L\ P(x, y)$
   4. $\exists y \in M\ \forall x \in L\ P(x, y)$

   You may assume that in each situation, each person plays only the instruments listed for him or her, and no others. In other words, if its not listed, they don't play it!

   Here are the English statements. For each statement, write down the corresponding numbers of all the predicate logic statements above that are true for the English statement.

   (a) John plays piano, Kate plays trumpet, and Lisa plays accordian.
   Solution: 1, 3

   (b) John plays piano, Kate plays piano and trumpet, and Lisa plays piano and accordian.
   Solution: 1, 3, 4

   (c) John plays trumpet, Kate plays piano, trumpet, and accordian, and Lisa doesn't play anything.
   Solution: 2, 3

   (d) John plays trumpet, Kate plays piano and trumpet, and Lisa plays trumpet.
   Solution: 1, 4

   (e) John plays trumpet, Kate doesn't play anything, and Lisa plays piano and accordian.
   Solution: 3

   (f) John plays accordian, Kate plays piano and accordian, and Lisa plays piano.
   Solution: 1

   (g) John plays piano, trumpet, and accordian, Kate plays trumpet and accordian, and Lisa plays accordian.
   Solution: 1, 2, 3, 4

(h) John plays piano and trumpet, Kate plays piano and accordian, and Lisa plays piano, trumpet, and accordian.

Solution: 1, 2, 3, 4