Discrete Structures, Fall 2016, Homework 7 Solutions

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

Prove each of the following statements using "regular" or "weak" induction.

1. $\forall n \in \mathbb{Z}^{\geq 0} \; \sum_{i=0}^{n}(3i^2 - i) = n^2(n+1)$

   Solution:

   Define $P(n)$ as $\sum_{i=0}^{n}(3i^2 - i) = n^2(n+1)$.

   Base Case:

   Prove $P(0)$, which means prove $\sum_{i=0}^{0}(3i^2 - i) = 0^2(0+1)$.

   LHS $= \sum_{i=0}^{0}(3i^2 - i) = 3(0)^2 - 0 = 0$.

   RHS $= 0^2(0+1) = 0$.

   LHS = RHS *(base case proved)*

   Inductive Case:

   Suppose k is an arbitrary integer in $\mathbb{Z}^{\geq 0}$.

   Inductive Hypothesis: Assume that $P(k)$ is true;
   in other words, assume $\sum_{i=0}^{k}(3i^2 - i) = k^2(k+1)$.

   Prove that $P(k+1)$ is true;
   in other words, prove $\sum_{i=0}^{k+1}(3i^2 - i) = (k+1)^2((k+1)+1)$

   To prove this equation is true, we will start with the left side and transform it into the right side.

$$
\begin{aligned}
\text{LHS} = \sum_{i=0}^{k+1}(3i^2 - i) &= \sum_{i=0}^{k}(3i^2 - i) + 3(k+1)^2 - (k+1) && \text{manipulation of a summation} \\
&= k^2(k+1) + 3(k+1)^2 - (k+1) && \text{by IH} \\
&= (k+1)[k^2 + 3(k+1) - 1] && \text{algebra: factor out } k+1 \\
&= (k+1)[k^2 + 3k + 3 - 1] && \text{algebra} \\
&= (k+1)[k^2 + 3k + 2] && \text{algebra} \\
&= (k+1)[(k+1)(k+2)] && \text{algebra} \\
&= (k+1)^2[k+2] && \text{algebra} \\
&= (k+1)^2[(k+1)+1] = \text{RHS} && \text{algebra}
\end{aligned}
$$

(inductive case proved)

2. Prove $\forall n \in \mathbb{Z}^+ \ \prod_{i=1}^{n} i(i+1) = (n+1)(n!)^2$

   Hint: Recall that $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, with $0!$ defined to be 1. However, an alternate formula involving recursion is the following:

   $$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

   This recursive definition will be useful during the inductive step.

   Solution:

   Define $P(n)$ as $\prod_{i=1}^{n} i(i+1) = (n+1)(n!)^2$.

   Base Case:

   Prove $P(1)$, which means prove $\prod_{i=1}^{1} i(i+1) = (1+1)(1!)^2$.

   LHS $= \prod_{i=1}^{1} i(i+1) = 1(1+1) = 2$.

   RHS $= (1+1)(1!)^2 = 2(1)^2 = 2$.

   LHS = RHS *(base case proved)*

   Inductive Case:

   Suppose k is an arbitrary integer in $\mathbb{Z}^+$.

   Inductive Hypothesis: Assume that $P(k)$ is true;
   in other words, assume $\prod_{i=1}^{k} i(i+1) = (k+1)(k!)^2$.

   Prove that $P(k+1)$ is true;
   in other words, prove $\prod_{i=1}^{k+1} i(i+1) = ((k+1)+1)((k+1)!)^2$.

   To prove this equation is true, we will start with the left side and transform it into the right side.

   $$\begin{aligned} \text{LHS} = \prod_{i=1}^{k+1} i(i+1) &= \prod_{i=1}^{k} i(i+1) \cdot (k+1)((k+1)+1) && \text{manipulation of a product} \\ &= (k+1)(k!)^2 \cdot (k+1)(k+2) && \text{by IH} \\ &= (k+1)^2 \cdot (k!)^2 \cdot (k+2) && \text{algebra} \\ &= [(k+1) \cdot (k!)]^2 \cdot (k+2) && \text{algebra} \end{aligned}$$

Now note that by the recursive definition of factorial, $(k+1)! = (k+1) \cdot k!$

$$
\begin{aligned}
&= [(k+1)!]^2 \cdot (k+2) && \text{algebra} \\
&= [(k+1)!]^2 \cdot ((k+1)+1) = \text{RHS} && \text{algebra}
\end{aligned}
$$

(inductive case proved)

3. $\forall n \in \mathbb{Z}^{\geq 0}$ $n(n+1)$ is even.

   Note: This is the same problem as question 1 on the last homework (prove that the product of any two consecutive integers is even). In homework 7, you did this with the QRT. On this homework, you should use induction (do not use the QRT here).

   Solution:

   Define $P(n)$ as "$n(n+1)$ is even".

   Base Case:

   Prove $P(0)$, which means prove $0(0+1)$ is even.

   We need to prove $0 \cdot 1 = 0$ is even. Zero is clearly even. *(base case proved)*

   Inductive Case:

   Suppose k is an arbitrary integer in $\mathbb{Z}^{\geq 0}$.

   Inductive Hypothesis: Assume that $P(k)$ is true;
   in other words, assume $k(k+1)$ is even.

   Prove that $P(k+1)$ is true;
   in other words, prove $(k+1)((k+1)+1)$ is even.

   To prove this is true, we will prove $(k+1)(k+2)$ is even by using the definition of even — we will try to get an equation that looks like $(k+1)(k+2) = 2 * (\textit{some integer})$.

$$
\begin{aligned}
(k+1)((k+1)+1) &= (k+1)(k+2) && \text{(by algebra)} \\
&= k^2 + k + 2k + 2 && \text{(by algebra)}
\end{aligned}
$$

   By the IH, we know $k(k+1)$ is even. Therefore, by the definition of even, there exists some integer p such that $k(k+1) = 2p$. By algebra, $k(k+1) = k^2 + k = 2p$.

$$
\begin{aligned}
(k+1)((k+1)+1) &= k^2 + k + 2k + 2 && \text{(from above)} \\
&= 2p + 2k + 2 && \text{(substitution)} \\
&= 2(p + k + 1) && \text{(algebra)}
\end{aligned}
$$

   $p+k+1$ is an integer by closure of the integers under addition. Therefore, $(k+1)((k+1)+1)$ is even by the definition of even.

   (inductive case proved)

4. $\forall n \in \mathbb{Z}^{\geq 0} \;\; 5 \mid 7^n - 2^n$.

Solution:

Define $P(n)$ as $5 \mid 7^n - 2^n$.

Base Case:

Prove $P(0)$, which means prove $5 \mid 7^0 - 2^0$

$7^0 - 2^0 = 1 - 1 = 0$. $5 \mid 0$. $\checkmark$ *(base case proved)*

Inductive Case:

Suppose k is an arbitrary integer in $\mathbb{Z}^{\geq 0}$.

Inductive Hypothesis: Assume that $P(k)$ is true;
in other words, assume $5 \mid 7^k - 2^k$.

Prove that $P(k+1)$ is true;
in other words, prove $5 \mid 7^{k+1} - 2^{k+1}$.

[To prove this is true, we will prove $5 \mid 7^{k+1} - 2^{k+1}$ *by using the definition of divides — we will try to get an equation that looks like* $7^{k+1} - 2^{k+1} = 5$**(some integer).]*

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k \qquad \text{(by algebra)}$$

By the IH, we know $5 \mid 7^k - 2^k$. Therefore, by the definition of divides, there exists some integer p such that $7^k - 2^k = 5p$. By algebra, $7^k = 5p + 2^k$.

$$\begin{aligned}
7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k && \text{(from above)} \\
&= 7 \cdot (5p + 2^k) - 2 \cdot 2^k && \text{(substitution)} \\
&= 35p + 7 \cdot 2^k - 2 \cdot 2^k && \text{(algebra)} \\
&= 35p + 2^k(7 - 2) && \text{(algebra)} \\
&= 35p + 2^k(5) && \text{(algebra)} \\
&= 5(7p + 2^k) && \text{(algebra)}
\end{aligned}$$

$7p + 2^k$ is an integer by closure of the integers under addition and multiplication. Therefore, $5 \mid 7^{k+1} - 2^{k+1}$ by the definition of divides.

(inductive case proved)