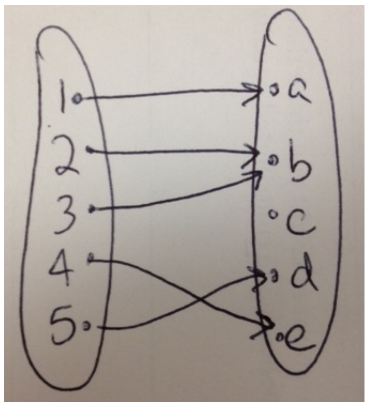
**Discrete Structures, Fall 2017, Homework 11 Solutions**

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

1. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d, e\}$. Define $f : X \to Y$ as follows: $f(1) = a$, $f(2) = b$, $f(3) = b$, $f(4) = e$, and $f(5) = d$.

   (a) Draw an arrow diagram for $f$.

   **Solution:**

   (b) Let $A = \{1, 2, 3\}$, $S = \{a\}$, $T = \{b, c, d\}$, and $W = \{c\}$. Find $f(A)$, $f(X)$, $f^{-1}(S)$, $f^{-1}(T)$, $f^{-1}(W)$, and $f^{-1}(Y)$. [Remember that images and pre-images/inverse images are sets!]

   **Solution:**
   $f(A) = \{a, b\}$
   $f(X) = \{a, b, d, e\}$
   $f^{-1}(S) = \{1\}$
   $f^{-1}(T) = \{2, 3, 5\}$
   $f^{-1}(W) = \{\} = \emptyset$ (note, not $\{\emptyset\}$)
   $f^{-1}(Y) = \{1, 2, 3, 4, 5\} = X$

2. Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = x^3 - 1$.

   (a) Is $f$ 1-1? Prove or give a counterexample.

   **Solution:**
   $f$ is 1-1.
   Proof:
   Let $x_1$ and $x_2$ be arbitrary elements in $\mathbb{R}$. Assume that $f(x_1) = f(x_2)$.
   By the definition of $f$, we know $x_1^3 - 1 = x_2^3 - 1$.
   Add 1 both sides to get $x_1^3 = x_2^3$.
   We can take the cube root of each side to get $x_1 = x_2$.
   [*Note: We can take the cube root because if* $a^3 = b^3$, *a must equal* $b$. *This is not true for even powers.*]

By closing the conditional world and universal generalization, we know
$\forall x_1, x_2 \in \mathbb{R}\ f(x_1) = f(x_2) \to x_1 = x_2$.
Therefore, $f$ is 1-1 by the definition of 1-1.

(b) Is $f$ onto? Prove or give a counterexample.

**Solution:**

$f$ is onto.

Proof:

Let $y$ be an arbitrary element in $\mathbb{R}$.
Define $x$ to be the real number $\sqrt[3]{y-1}$.
Then $f(x) = f(\sqrt[3]{y-1}) = (\sqrt[3]{y-1})^3 + 1 = (y-1) + 1 = y$.
Because we defined an $x$ existentially, we can say $\exists x \in \mathbb{R}\ f(x) = y$.
Because $y$ was chosen arbitrarily, we can say $\forall y \in R\ \exists x \in \mathbb{R}\ f(x) = y$.
Therefore, $f$ is onto by the definition of onto.

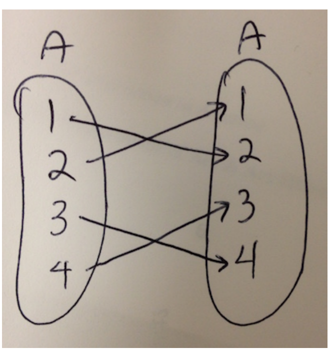
3. Let $A = \{1, 2, 3, 4\}$. Define a function $f : A \to A$ using an arrow diagram such that $f$ is 1-1 and onto, $f$ **is not** the identity function, but $f \circ f$ **is** the identity function.

**Solution:**

[*Note: There are lots of solutions to this problem. Here's one.*]

(Imagine the following function as an arrow diagram.)

Choose $f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 3$. Clearly $f$ is 1-1 and onto, and $f$ is not the identity function. However, note that $(f \circ f)(x) = f(f(x)) = x$ for all $x \in A$, so $f \circ f$ is the identity function.




4. Let $X$, $Y$, and $Z$ be any sets. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. If $g \circ f$ is 1-1, must it be true that $f$ is 1-1? Prove or give a counter-example.

**Solution:**

It is true that $f$ must be 1-1 if you know $g \circ f$ is 1-1.

Proof:

[*Note: We want to prove that $f$ is 1-1. We will put the definition of $f$ here (not in the proof, but off to the side) so we know what we're aiming for: $\forall x_1, x_2 \in X\ f(x_1) = f(x_2) \to x_1 = x_2$.*]

Suppose $x_1$ and $x_2$ are arbitrary elements in $X$. Assume that $f(x_1) = f(x_2)$.

We know that $g \circ f$ is 1-1, so let's invoke that definition:
We know $\forall x_1, x_2 \in X \ (g \circ f)(x_1) = (g \circ f)(x_2) \to x_1 = x_2$.

Apply $g$ to both sides of $f(x_1) = f(x_2)$ to get $g(f(x_1)) = g(f(x_2))$.

Use the definition of function composition to get $(g \circ f)(x_1) = (g \circ f)(x_2)$.

Therefore, by universal modus ponens, we know $x_1 = x_2$. [*We are referencing the earlier fact that $g \circ f$ is 1-1.*]

By closing the conditional world and universal generalization, we know
$\forall x_1, x_2 \in X \ f(x_1) = f(x_2) \to x_1 = x_2$.

Therefore, $f$ is 1-1 by the definition of 1-1.

5. Let $X$, $Y$, and $Z$ be any sets. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. If $g \circ f$ is 1-1, must it be true that $g$ is 1-1? Prove or give a counter-example.

**Solution:**

It is **not** true that $g$ must be 1-1 if you know $g \circ f$ is 1-1. In other words, it is possible for $g$ to not be 1-1.

Counterexample:

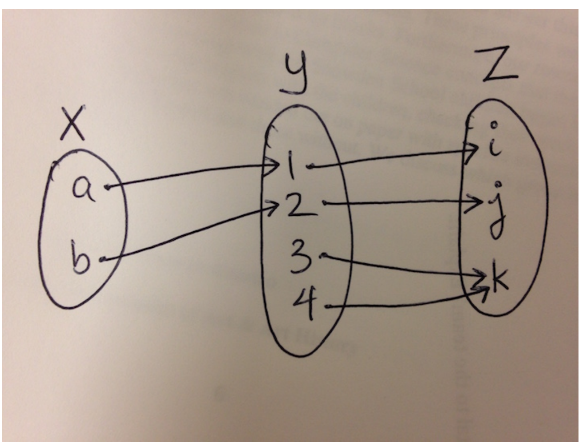
Define sets $X = \{a, b\}, Y = \{1, 2, 3, 4\}, Z = \{i, j, k\}$.

Define $f : X \to Y$ as $f(a) = 1$ and $f(b) = 2$.

Define $g : Y \to Z$ as $g(1) = i$, $g(2) = j$, and $g(3) = g(4) = k$.

Then $g \circ f$ can be computed as $(g \circ f)(a) = g(f(a)) = g(1) = i$, and $(g \circ f)(b) = g(f(b)) = g(2) = j$.

Clearly, $g \circ f$ is 1-1, but $g$ is not 1-1.




[*Note: The trick in this counterexample is to make sure that the elements of $Y$ that break the one-to-one-ness of $g$ are not elements that are mapped to by any elements in $X$.*]