Discrete Structures, Fall 2017, Homework 5 Solutions

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

For each statement below, state whether it is true or false. Then prove the statement if it is true, or its negation if it is false.

Remember, an example may only be used to prove that an existential statement is true or a universal statement is false. Any example or counter-example must include specific values for the variables and enough algebra and justification to illustrate that the example proves what you are claiming it proves.

You do not need to translate each statement into symbols first, though it is often useful to do so.

1. The product of any two odd integers is odd.

   Symbols: $\forall a, b \in \mathbb{Z}^{\text{odd}}\ \text{Odd}(a \cdot b)$

   This statement is true.

   Proof:

   Suppose a and b are two arbitrarily chosen odd integers.

   By the definition of odd, there exist integers k and p such that $a = 2k+1$ and $b = 2p+1$.

   $a \cdot b = (2k+1)(2p+1) = 4kp + 2p + 2k + 1 = 2(2kp + p + k) + 1$ by substitution and algebra.

   Let $s = 2kp+p+k$. $s \in \mathbb{Z}$ by closure of the integers under multiplication and addition.

   Because $a \cdot b = 2s + 1$, we can say that $a \cdot b$ is odd by the definition of odd.

2. For any integers a and b, if $a - b$ is even, then a and b are both even.

   Symbols: $\forall a, b \in \mathbb{Z}\ \text{Even}(a - b) \rightarrow (\text{Even}(a) \wedge \text{Even}(b))$

   This statement is false.

   Counterexample: Let $a = 5$ and $b = 3$. Then $a - b = 5 - 3 = 2$. 2 is clearly even, but a and b are odd.

3. If n is an even integer, then $n^2 - 2$ is even.

   Symbols: $\forall n \in \mathbb{Z}\ \text{Even}(n) \rightarrow \text{Even}(n^2 - 2)$

   This statement is true.

   Proof:

   Suppose n is an arbitrarily-chosen integer.

   Assume n is even.

   By the definition of even, there exists an integer k such that $n = 2k$.

$n^2 - 2 = (2k)^2 - 2 = 4k^2 - 2 = 2(2k^2 - 1)$ by algebra/substitution.

Let $m = 2k^2 - 1$. m is an integer by closure of the integers under multiplication and addition.

Therefore, $n^2 - 2 = 2m$, and so $n^2 - 2$ is even by the definition of even.

4. The sum of an integer and a rational number is rational.

   Symbols: $\forall n \in \mathbb{Z} \ \ \forall r \in \mathbb{Q} \ \ \text{Rational}(n + r)$

   This statement is true.

   Proof:

   Suppose n is an arbitrarily-chosen integer, and that r is an arbitrarily-chosen rational number.

   By the definition of rational, there exist integers a and b such that $r = a/b$ and $b \neq 0$.

   $n + r = n + \dfrac{a}{b} = \dfrac{nb + a}{b}$ by substitution and algebra.

   Let $p = nb + a$.

   p is an integer because the integers are closed under multiplication and addition.

   Because $n + r = p/b$, and we know $p \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $b \neq 0$, we can say that $n + r$ is rational by the definition of rational.

5. If n and m are rational numbers, then n/m is a rational number.

   Symbols: $\forall n, m \in \mathbb{Q} \ \ \text{Rational}(n/m)$

   This statement is false.

   Counterexample: Let $n = 2$ and $m = 0$. n is a rational number because it can be written as $2/1$, and m is rational because it can be written as $0/1$. *[Alternatively, you can just say n and m are rational because they are integers and all integers are rational.]*

   $n/m = 2/0$ which is not rational because the denominator is zero (and therefore n/m is undefined).