Discrete Structures, Fall 2017, Homework 8 Solutions

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

Prove the following statements by strong induction. Make sure to follow the form from class: explicitly define $P(n)$, label the basis step(s), inductive step, where you define the inductive hypothesis, where you define what you want to prove and where you use the inductive hypothesis.

1. Suppose we define a sequence as follows:

$$a_1 = 3;\ a_2 = 5;\ \text{and for all integers } i \geq 3,\ a_i = 4a_{i-1} - 3a_{i-2}.$$

Prove $\forall n \in \mathbb{Z}^{\geq 1}\ a_n = 3^{n-1} + 2$.

Solution:

Define $P(n)$ as "$a_n = 3^{n-1} + 2$." We are trying to prove $\forall n \in \mathbb{Z}^{\geq 1}\ P(n)$.

Base Cases:

Prove $P(1)$ and $P(2)$. That is, prove that $a_1 = 3^{1-1} + 2$ and $a_2 = 3^{2-1} + 2$.

LHS $= a_1 = 3$. RHS $= 3^{1-1} + 2 = 3^0 + 2 = 1 + 2 = 3$. LHS = RHS. ✓

LHS $= a_2 = 5$. RHS $= 3^{2-1} + 2 = 3^1 + 2 = 3 + 2 = 5$. LHS = RHS. ✓

(base cases proved)

Inductive Case:

Suppose k is an arbitrary integer ≥ 2.

Inductive hypothesis: Assume for all integers i, $1 \leq i \leq k$, that $P(i)$ is true.
In other words, assume $P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)$.
In other words, assume $a_1 = 3^{1-1} + 2$ and $a_2 = 3^{2-1} + 2$ and $a_3 = 3^{3-1} + 2$ and ... and $a_k = 3^{k-1} + 2$.

Inductive step: Prove $P(k+1)$ is true. In other words, prove that $a_{k+1} = 3^{k+1-1} + 2$.

$$a_{k+1} = 4a_k - 3a_{k-1} \qquad \text{definition of the sequence}$$

By the inductive hypothesis, $P(k-1)$ and $P(k)$ are both true. Therefore, $a_{k-1} = 3^{k-1-1} + 2$ and $a_k = 3^{k-1} + 2$. Substitute these equations into our equation above:

$$\begin{aligned}
a_{k+1} &= 4a_k - 3a_{k-1} && \text{from above}\\
&= 4(3^{k-1} + 2) - 3(3^{k-1-1} + 2) && \text{substitution}\\
&= 4 \cdot 3^{k-1} + 8 - 3 \cdot 3^{k-2} - 6 && \text{algebra}\\
&= 4 \cdot 3^{k-1} - 3 \cdot 3^{k-2} + 2 && \text{algebra}\\
&= 4 \cdot 3^{k-1} - 3^{k-1} + 2 && \text{algebra}\\
&= 3 \cdot 3^{k-1} + 2 = 3^k + 2 = 3^{k+1-1} + 2 && \text{algebra}
\end{aligned}$$

[Inductive case proved.]

2. Suppose we define a sequence as follows:

$$b_0 = 3;\ b_1 = 1;\ b_2 = 3;\ \text{and for all integers } i \geq 3,\ b_i = b_{i-3} + b_{i-2} + b_{i-1}.$$

Prove that every term in the sequence is odd.

Solution:

Define $P(n)$ as "b_n is odd." We are trying to prove $\forall n \in \mathbb{Z}^{\geq 0}\ P(n)$.

Base Cases:

Prove $P(0)$, $P(1)$, and $P(2)$. That is, prove that b_0, b_1, and b_2 are all odd.

$b_0 = 3$. 3 is odd. ✓

$b_1 = 1$. 1 is odd. ✓

$b_2 = 3$. 3 is odd. ✓

(base cases proved)

Inductive Case:

Suppose k is an arbitrary integer ≥ 2.

Inductive hypothesis: Assume for all integers i, $0 \leq i \leq k$, that $P(i)$ is true.
In other words, assume $P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)$.
In other words, assume b_1 is odd and b_2 is odd and b_3 is odd and ... and b_k is odd.

Inductive step: Prove $P(k+1)$ is true. In other words, prove that b_{k+1} is odd.

[We will prove b_{k+1} is odd using the def'n of odd. That is, we will try to get an equation that looks like $b_{k+1} = 2 * (\textit{some integer}) + 1$].

$$b_{k+1} = b_{k-2} + b_{k-1} + b_k \qquad \text{definition of the sequence}$$

By the inductive hypothesis, $P(k-2)$, $P(k-1)$ and $P(k)$ are all true. Therefore, b_{k-2}, b_{k-1} and b_k are all odd.

Therefore, by the definition of odd, there exist integers p, q, r such that $b_{k-2} = 2p + 1$, $b_{k-1} = 2q + 1$, and $b_k = 2r + 1$. Let's substitute these in:

$$\begin{aligned}
b_{k+1} &= b_{k-2} + b_{k-1} + b_k && \text{from above} \\
&= (2p+1) + (2q+1) + (2r+1) && \text{definition of the sequence} \\
&= 2p + 2q + 2r + 2 + 1 && \text{algebra} \\
&= 2(p+q+r+1) + 1 && \text{algebra}
\end{aligned}$$

$p + q + r + 1$ is an integer by closure of the integers under addition. Therefore, b_{k+1} is odd by the definition of odd. *[Inductive case done.]*

3. Suppose we define a sequence as follows:

$$c_0 = 2;\ c_1 = 7;\ \text{and for all integers } i \geq 2,\ c_i = 3c_{i-1} - 2c_{i-2}.$$

Prove $\forall n \in \mathbb{Z}^{\geq 0}\ 5 \mid (c_n - 2)$.

Solution:

Define $P(n)$ as "$5 \mid (c_n - 2)$." We are trying to prove $\forall n \in \mathbb{Z}^{\geq 0}\ P(n)$.

Base Cases:

Prove $P(0)$ and $P(1)$. That is, prove that $5 \mid (c_0 - 2)$ and $5 \mid (c_1 - 2)$.

$c_0 - 2 = 2 - 2 = 0$. $5 \mid 0$. ✓

$c_1 - 2 = 7 - 2 = 5$. $5 \mid 5$. ✓

(base cases proved)

Inductive Case:

Suppose k is an arbitrary integer ≥ 1.

Inductive hypothesis: Assume for all integers i, $0 \leq i \leq k$, that $P(i)$ is true.
In other words, assume $P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)$.
In other words, assume $5 \mid (c_0 - 2)$ and $5 \mid (c_1 - 2)$ and $5 \mid (c_2 - 2)$ and ... and $5 \mid (c_k - 2)$.

Inductive step: Prove $P(k+1)$ is true. In other words, prove that $5 \mid (c_{k+1} - 2)$.

[We will prove $5 \mid (c_{k+1} - 2)$ using the def'n of divides. That is, we will try to get an equation that looks like $c_{k+1} - 2 = 5 * (\text{some integer})$].

$$c_{k+1} = 3c_k - 2c_{k-1} \qquad \text{definition of the sequence}$$

By the inductive hypothesis, $P(k-1)$ and $P(k)$ are both true. Therefore, $5 \mid (c_{k-1} - 2)$ and $5 \mid (c_k - 2)$.

Therefore, by the definition of divides, there exist integers $q, r \in \mathbb{Z}$ such that $(c_{k-1} - 2) = 5q$ and $(c_k - 2) = 5r$.

Let's add 2 to both sides of both equations. We get $c_{k-1} = 5q + 2$ and $c_k = 5r + 2$. Substitute these into our earlier equation:

$$\begin{aligned} c_{k+1} &= 3c_k - 2c_{k-1} = 3(5q+2) - 2(5r+2) && \text{substitution} \\ &= 15q + 6 - 10r - 4 = 5(3q - 2r) + 2 && \text{all by algebra} \end{aligned}$$

Now subtract 2 from both sides:

$$c_{k+1} - 2 = 5(3q - 2r) + 2 \qquad \text{algebra}$$

Because the integers are closed under multiplication and addition, we know $3q - 2r \in \mathbb{Z}$. Therefore, by the definition of divides, we know $5 \mid c_{k+1} - 2$. *[Inductive case done.]*