Relations

* Introduce the idea of a function being written as a set of ordered pairs.
  + Define set S = {1, 2, 3}. T = {1, 2, 3..9}
  + Define f : S -> T by f(x) = x^2.
  + Draw arrow diagram.
  + We can also represent a function by the set of ordered pairs (x, y).
    - List out the set f = {(1, 1), (2, 4), (3, 9)}.
  + Where do ordered pairs usually come from (what set operation?) [Cartesian product]
  + So this set of ordered pairs is a subset of a Cartesian product of the domain and codomain!
    - What makes it a function? [the fact that the first element in each ordered pair appears exactly once in the set]
  + If we remove this restriction, we get another kind of mathematical object, called a relation.
* A relation is a subset of a Cartesian product.
  + Relational databases are built around relations.
* We will spend most of our time talking about binary relations, which are subsets of Cartesian products of 2 sets.
  + Because binary relations are sets (subsets of Cartesian products), a nice conceptualization of a binary relation is I give you an ordered pair, and you have to say YES if it’s in the relation, and NO if it’s not in the relation.
  + 3 ways to do this:
    - You can do this by looking at the entire relation as a set of ordered pairs,
    - or often there’s a rule for a relation you can use (like functions often have rules).
    - Arrow diagram
* Example:
  + A = {2, 4, 6} B = {1, 3, 5}.
  + Define relation R1 = {(2, 5), (4, 1), (2, 3)}
  + Define relation R2 by the rule “a is related to b if a = b+1”.
  + Define relation R1 by arrow diagram.
* Definition: If we have a relation R subseteq A times B, then we say:
  + x is related to y iff
  + (x,y) in A times B iff
  + x R y.
* Relations correspond to things you already know, but haven’t studied formally.
* Define a relation L subset of R times R by the rule x L y iff x < y.
  + 3 L 4? 4 L 3?
  + (6.2, 10) in L (-33, -100) in L?
* Define a relation D subset Z x Z by the rule (a, b) in D iff a | b.

* ~~Both functions and binary relations can be defined by ordered pairs, so sometimes we use the same language (define a relation from one set to another set). A binary relation is a generalization of a function. (can have more than one arrow emanating from something in the domain).~~
  + ~~Can draw arrow diagrams for binary relations just like functions.~~
  + Draw arrow diagram for A = 1,2,3 B = 2,6,8 and x R y <-> x | y
  + Relation ON a set is where the two sets involved in a binary relation are the same set.
  + Still can draw an arrow diagram, but usually **you collapse the two sides of the diagram into one. (so you get a directed graph)**
  + Example: D from above
    - Example: draw arrow (directed graph) diagram for S={1, 2, 3, 4, 5}
    - Relation R subset S x S is “(x, y) in R iff x and y have the same remainder when divided by 3.
  + Define a set X = {a,b,c}. Define a relation S subseteq P(X) times P(X) by the rule  
    A S B iff N(A) > N(B).
* Inverse relation:
  + Suppose you have a relation R subset A x B
  + R^-1 = {(y, x) in B x A such that (x, y) in R}
  + Or equivalently, (x, y) in r <-> (y, x) in R^-1.
  + Switches direction of arrows in arrow diagram.

9.2

* Review:
  + Relation: subset of a Cartesian product (of any number of sets).
  + Binary relation: subset of a Cartesian product of 2 sets.
  + Ways to express binary relations:
    - List items in the ordered pair.
    - Give a rule.
    - Draw an arrow diagram.
  + Suppose we have a two sets, A, and B, and a relation R subset of A x B.
    - “x is related to y” is equivalent to
    - x R y
    - (x, y) in R
  + Example: Define Z>=2 as ints >=2.
  + R subset A x B by rule
    - a R b <-> a and b have a prime factor in common
    - 15 R 25? Yes (5)
    - 2 R 5? No
    - 9 R 26? No
    - 9 R 12? Yes (3)
    - 44 R 10? Yes (2)
    - 6 R 6? Yes (2 or 3)
    - For all x in Z>=2, is x R x?
* R is reflexive iff for all x in A x R x.
  + reflexive = every element has a loop
  + Sample reflexive: R subset Z x Z, a R b <-> 3 | (a-b)
    - Called congreuence modulo 3.
* R is symmetric iff for all x,y in A x R y -> y R x.
  + Symmetric = if there is an arrow one way, must be arrow the other way.
  + Prove congruence mod 3 is symmetric.
* Transitive iff for all x,y,z in A, if x R y and y R z -> x R z.  
  + If there is an arrow from x to y, and an arrow from y to z, there must be an arrow from x to z.
  + Prove congruence mod 3 is transitive.

Proofs

a R b <-> 3 | (a-b)

This is reflexive, symmetric, transitive.

a R b <-> 3 | (a + b)

NOT reflexive (3 does not divide 2 + 2). YES symmetric. NOT transitive (3 | 2 + 4, 3 | 4 + 2, 3 does not divide 2 + 2).

(a, b) in R <-> |a| = |b|

This is reflexive, symmetric, transitive

Subset relation on U = {a, b, c}

(S, T) in R <-> S subseteq T.

Reflexive, NOT symmetric, transitive.

for all a,b in Z>=2 (a, b) in R <-> exists a prime number p such that p | a and p | b

Reflexive, symmetric, NOT transitive (2 R 6, 6 R 3, 2 not R 3)

* Relation that is reflexive, symmetric, and transitive is called an equivalence relation.
* Lots of relations are equivalence relations.
* They are useful because they express different notions of two items that are interchangeable with each other in a certain context.
* Imagine P = set of all computer programs.
  + Relation R subset P x P by the rule p1 R p2 <-> p1 and p2 produce the same output to the screen.
  + Compilers use equivalence relations to speed up programs by replacing one section of code with a faster section of code. (optimization)

9.3

* Equivalence relation = any relation that is reflexive, symmetric, and transitive.
* Every element in the original set from a relation has a set of things that it is related to. This is called it’s equivalence class.
* If R subset A x A, then for all a in A, [a] = { x in A | x R a}.
* Show diagram for how congruence mod 3 naturally partitions Z into 3 subsets.
* What are equivalence classes for

(a, b) in R <-> |a| = |b| infinite classes

(a, b) in R <-> 3 | (a^2 – b^2) 2 classes {0, 3, 6, 9..} and {1, 2, 4, 5}