

Study of Turing's Patterns

Using Reaction-Diffusion Model

Prakash Jha | 2101MC31

Prakhar Gupta | 2101MC32

Utkarsh Anavkar | 2101MC44

Table of contents

01

Introduction

04

Simulations

02

Mathematical
Model

05

Examples

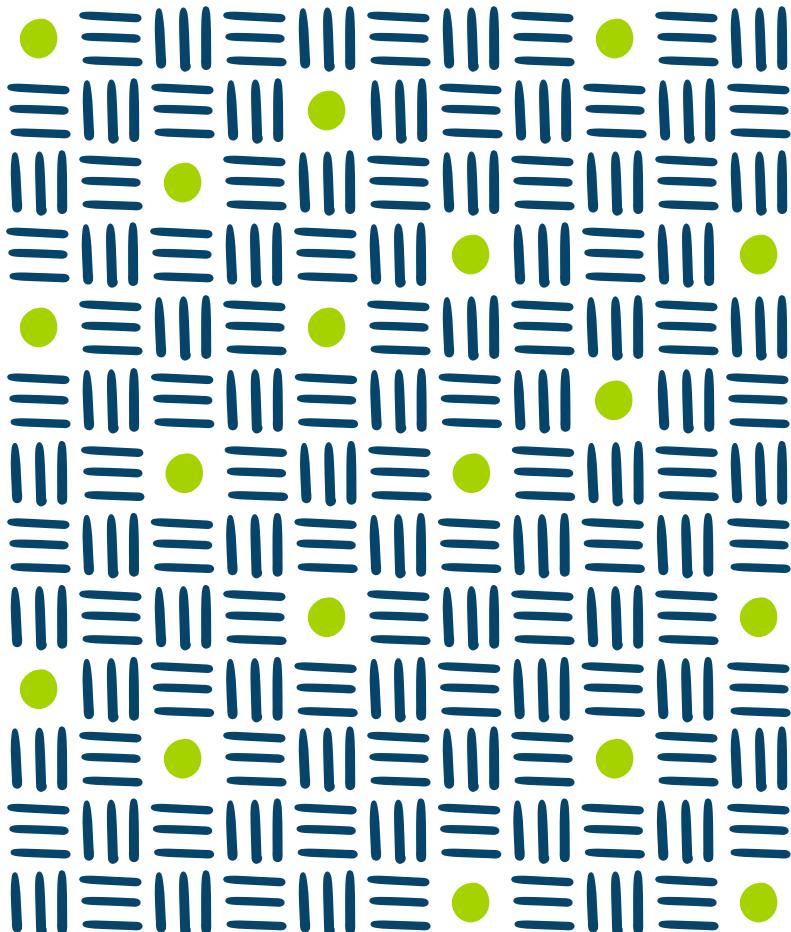
03

Stability
Analysis

06

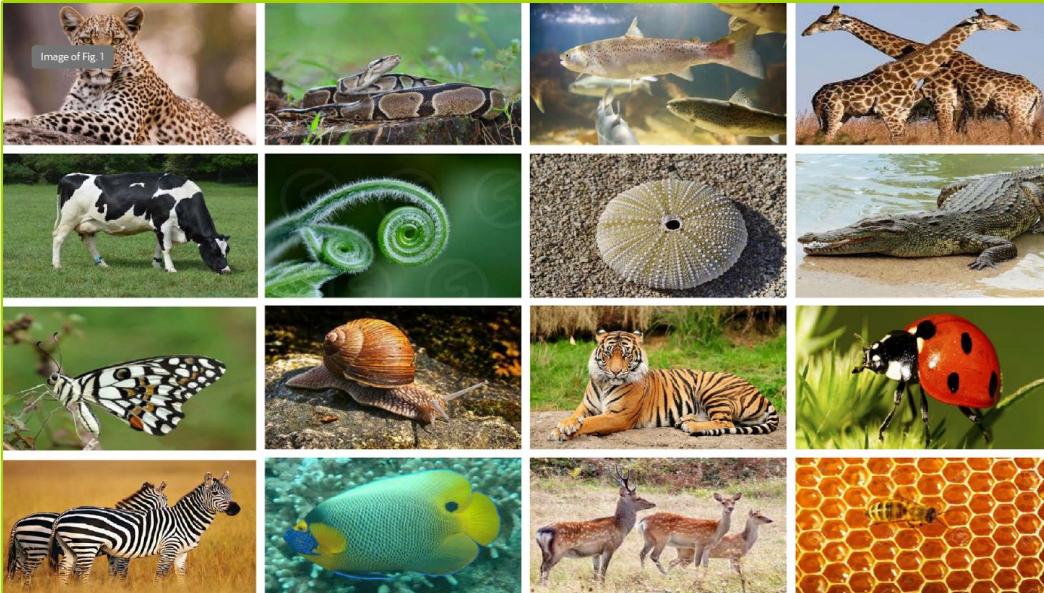
Conclusion





01

Introduction of Idea



Nobody could imagine these complex biological patterns can be explained by any mathematical model until Alan Turing proposed his hypothesis in 1952 by paper "The Chemical Basis of Morphogenesis".

- In recent decades, the studies of pattern formation in nature have become a fascinating subject for various fields of scientific research.
- The patterns occurring on animals in nature are incredibly vast and diverse. Nature has a remarkable ability to generate an immense array of patterns, colors, and shapes across various species.
- Examples of patterns in nature include the intricate spots and stripes of big cats like tigers and leopards, the dazzling displays of colors on birds like peacocks and parrots, the disruptive patterns on insects like butterflies and moths, and the cryptic patterns of many species that help them blend into their surroundings.



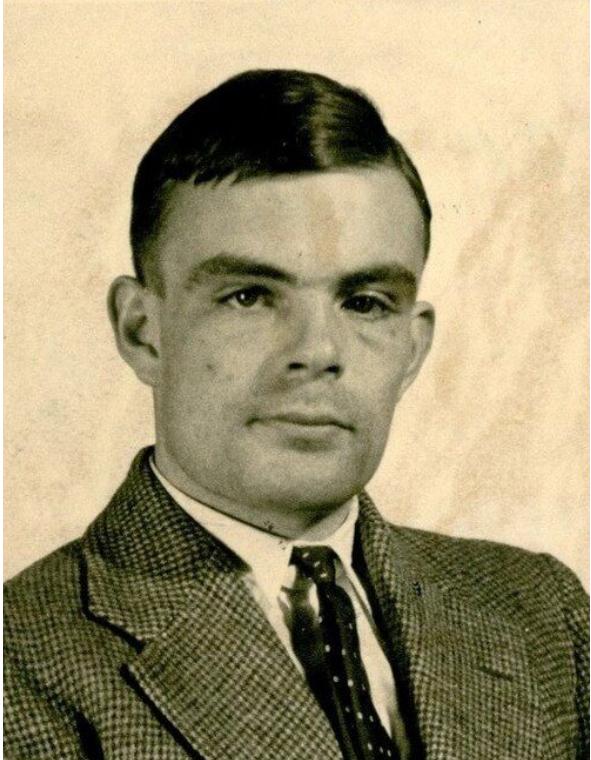
What do you think, which processes govern the formation of these patterns?

- Genetic Control
- Cellular Interactions
- Selective Pressures
- Developmental Plasticity
- Environmental Factors
- Epigenetic Mechanisms



During embryo development, cells communicate and interact with each other to create patterns. For example, in the case of Turing's hypothesis, interactions between diffusing chemicals can lead to the formation of stripes, spots, or other spatial patterns.





Alan Turing(1912-54)

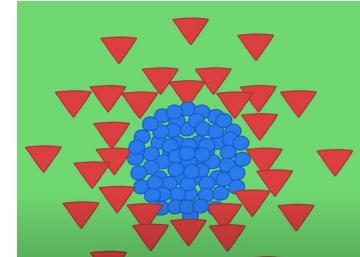
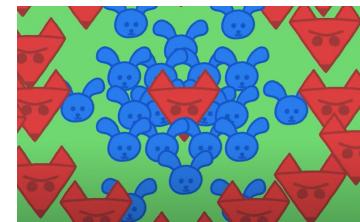
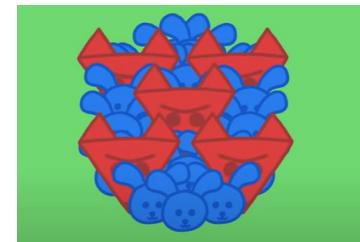
Mathematician and Computer Scientist

- Turing was the first to offer an explanation of morphogenesis through chemistry through his paper "The Chemical Basis of Morphogenesis" in 1952.
- He theorised that identical biological cells differentiate and change shape through a process called inter cellular reaction-diffusion.
- He proposed that two hypothetical chemicals, termed an **activator** and an **inhibitor**, could interact with each other and diffuse through a medium, such as a developing embryo.



What does activator and inhibitor do?

- Activator: Promotes its own production as well as the production of the inhibitor. Regions where the activator concentration is high tend to become even more active. Positive feedback loop.
- Inhibitor: Suppresses the production of both itself and the activator. Regions where the inhibitor concentration is high tend to inhibit both. Negative feedback loop.
- Interplay between the activator and inhibitor, diffusion through the medium, gives rise to spatial patterns through a process known as pattern formation.



Analogies

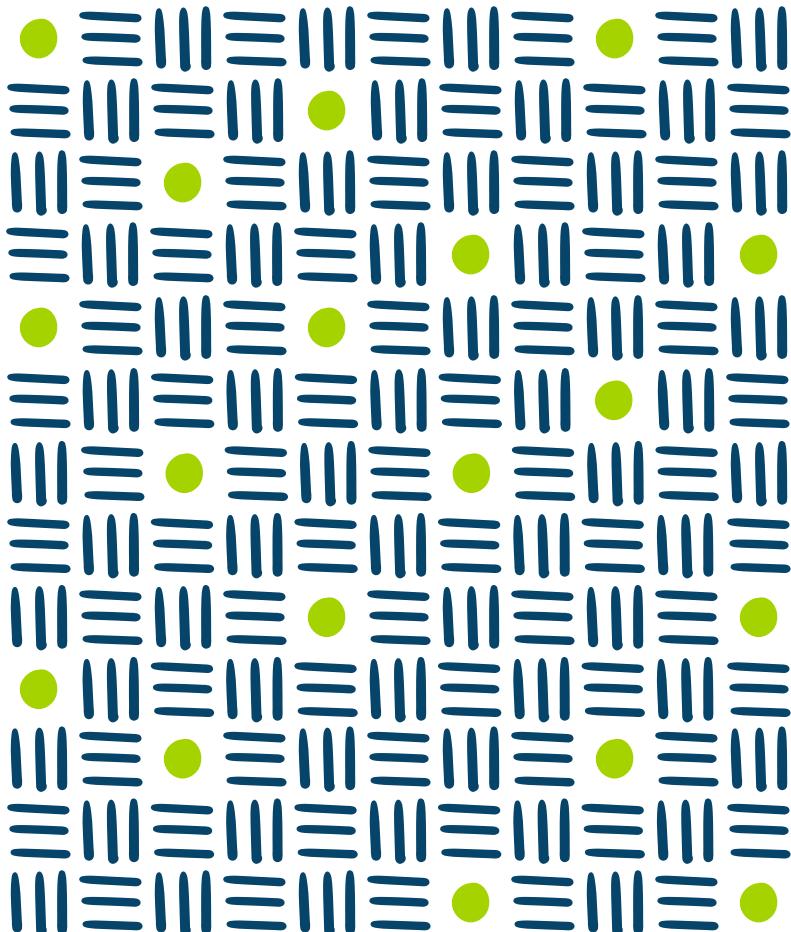
Fire and Grasshopper

- Grasshopper releases moisture which act as inhibitor. Fire acts as activator. When fire spreads, grasshopper prevents the grass from burning.
- In order to survive, grasshopper's diffusion rate(inhibitor) must be greater than fire's diffusion rate(activator).
- After certain time, pattern of burnt and unburnt grass is formed.
- Pattern becomes stationary after long period of time.

Foxes and Rabbits

- In this case, foxes act as inhibitor and rabbits as activator..
- In order to survive, rabbits (activator) must diffuse faster than foxes (inhibitor).
- After certain time, population of rabbits and foxes give rise to pattern formation.
- Pattern becomes stationary after long period of time.





02

Mathematical Model

General Reaction-Diffusion Model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v)$$

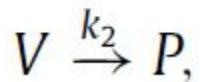
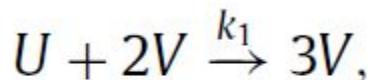


Diffusion



We shall discuss the Gray-Scott Model of reaction-diffusion.

- First introduced in 1984 by Gray and Scott.
- Describes the kinetics of cubic autocatalytic biochemical reactions involving two species U and V.



U is supplied with a constant feed rate 'F' and P is removed with a constant kill rate 'k'





Assumptions

- Reaction-Diffusion System: Substances react as well as diffuse in space.
- Local Activation and Long-Range Inhibition: Localized regions of high activator concentration and long-range inhibition that prevents the spread of activator.
- Homogeneous Initial Conditions: Concentrations of the activator and inhibitor are initially uniform throughout the medium.
- Nonlinear Kinetics: Rates of reaction depend nonlinearly on the concentrations of the reactants. Essential for generating the feedback loops necessary for pattern formation.
- Fixed Geometry: Two-dimensional surfaces or one-dimensional gradients.
- Stationary State: The model assumes that the system reaches a stationary state, where the spatial patterns stabilize and remain relatively unchanged over time.



Variables

- $u(x,t)$ be the concentration of activator.
 - $v(x,t)$ be the concentration of inhibitor.
 - D_1 be the diffusion rate of inhibitor.
 - D_2 be the diffusion rate of activator.
 - F is the Feed Rate of u , the activator.
 - K is the Kill Rate of v , the inhibitor.
 - Diffusion is considered in only x direction.
- 

Gray-Scott Reaction-Diffusion Model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1-u)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F + k)v$$

u : Activators
v : Inhibitors
 D_u : Diffusion rate of u
 D_v : Diffusion rate of v
F : Feed rate of u
k : Kill rate of v
 ∇^2 : Laplacian operator

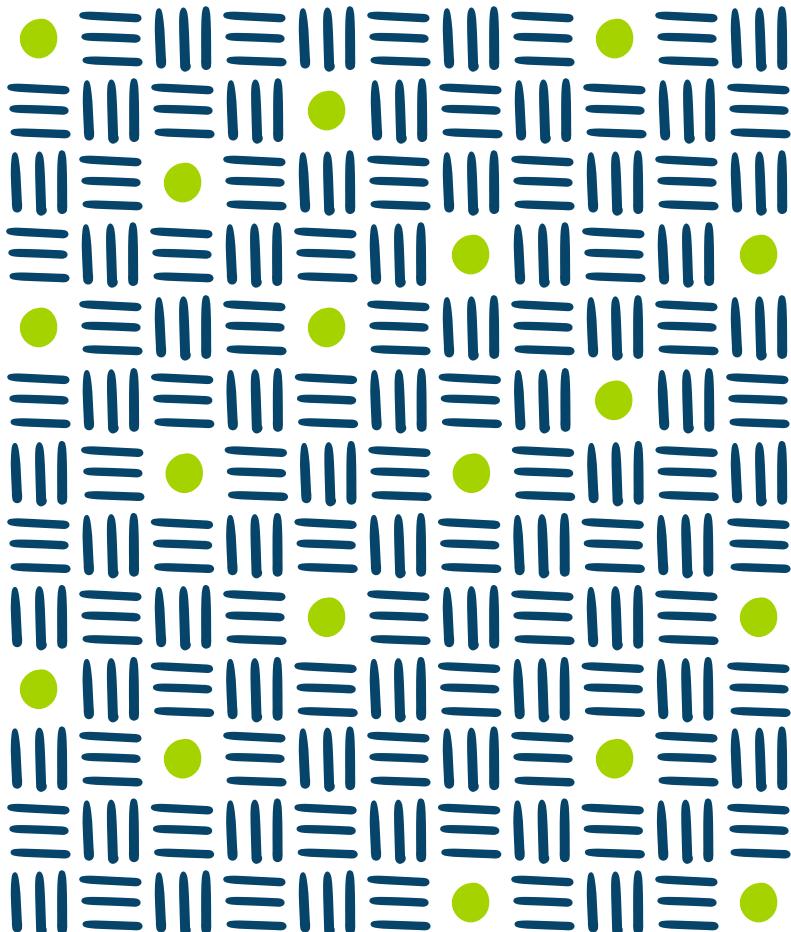
Explanation of The Model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1-u)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F + k)v$$

u	: Activators
v	: Inhibitors
D_u	: Diffusion rate of u
D_v	: Diffusion rate of v
F	: Feed rate of u
k	: Kill rate of v
∇^2	: Laplacian operator

- Why uv^2 ?
- Why -ve for $\frac{\partial u}{\partial t}$ and +ve for $\frac{\partial v}{\partial t}$?
- Why $F(1-u)$?
- Why $-(F+k)v$?



03

Stability

Analysis

Finding Critical Points

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + F(1-u) \quad (1)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F+k)v$$

For Critical Points,

$$-uv^2 + F(1-u) = 0 ; uv^2 - (F+k)v = 0$$

Critical Points :

$$A(1,0);$$

$$B\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4(F+k)^2}{F}}\right), \frac{1}{2}\frac{F}{(F+k)}\left(1-\sqrt{1-\frac{4(F+k)^2}{F}}\right)\right)$$

$$C\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4(F+k)^2}{F}}\right), \frac{1}{2}\frac{F}{(F+k)}\left(1+\sqrt{1-\frac{4(F+k)^2}{F}}\right)\right)$$

Jacobian without diffusion

Jacobian matrix without diffusion.

$$J = \begin{pmatrix} -v^2 - F & -2uv \\ v^2 & 2uv - (F+k) \end{pmatrix}$$

Stability Analysis at point A(1,0)

$$J_A = \begin{pmatrix} -F & 0 \\ 0 & -(F+k) \end{pmatrix}$$

eigen values, $\lambda_1 = -F$, $\lambda_2 = -(F+k)$

Point A is stable, without diffusion.

Stabilities at points A, B and C

For Point B and C,

$$v \neq 0.$$

$$\therefore uv = F + k.$$

$$J = \begin{pmatrix} -v^2 - F & -2(F+k) \\ v^2 & (F+k) \end{pmatrix}$$

$$\therefore Tr(J) = -v^2 + k$$

$$\therefore \det(J) = (v^2 - F)(F + k)$$

Stabilities at points A, B and C

$$-v^2 + k < 0$$

$$(v^2 - F)(F + k) > 0$$

Necessary Condition,

$$v > \frac{(F+k)^2}{F} ; \quad v > 2(F+k).$$

Jacobian with diffusion

For Jacobian matrix with diffusion,

$$J - D\omega^2 = \begin{pmatrix} -v^2 - F - D_u \omega^2 & -2(F+k) \\ v^2 & (F+k) - D_v \omega^2 \end{pmatrix}$$

$$\text{Tr}(J - D\omega^2) = -v^2 + k - D_u \omega^2 - D_v \omega^2 < 0$$

$$\det(J - D\omega^2) = \omega^4 D_u D_v + \omega^2 (D_v(v^2 + F) - D_u(F+k)) + (F+k)(v^2 - F)$$

Notice that Trace is always negative if Trace of Jacobian without diffusion is negative.

Jacobian with diffusion

$$\text{Let, } Q(\omega^2) = \omega^4 D_u D_v + \omega^2 (D_v(v^2 + F) - D_u(F + k)) \\ + (F + k)(v^2 - F)$$

For minimum point of $Q(\omega^2)$

$$\frac{dQ(\omega^2)}{d\omega^2} = 0 ; \quad \frac{d^2Q(\omega^2)}{d(\omega^2)^2} > 0$$

$$2\omega^2 D_u D_v + D_v(v^2 + F) - D_u(F + k) = 0$$

$$\omega_{\min}^2 = \frac{D_u(F+k) - D_v(v^2+F)}{2D_u D_v}$$

$$2D_u D_v > 0$$

Condition for Instability

Condition for Unstability is,

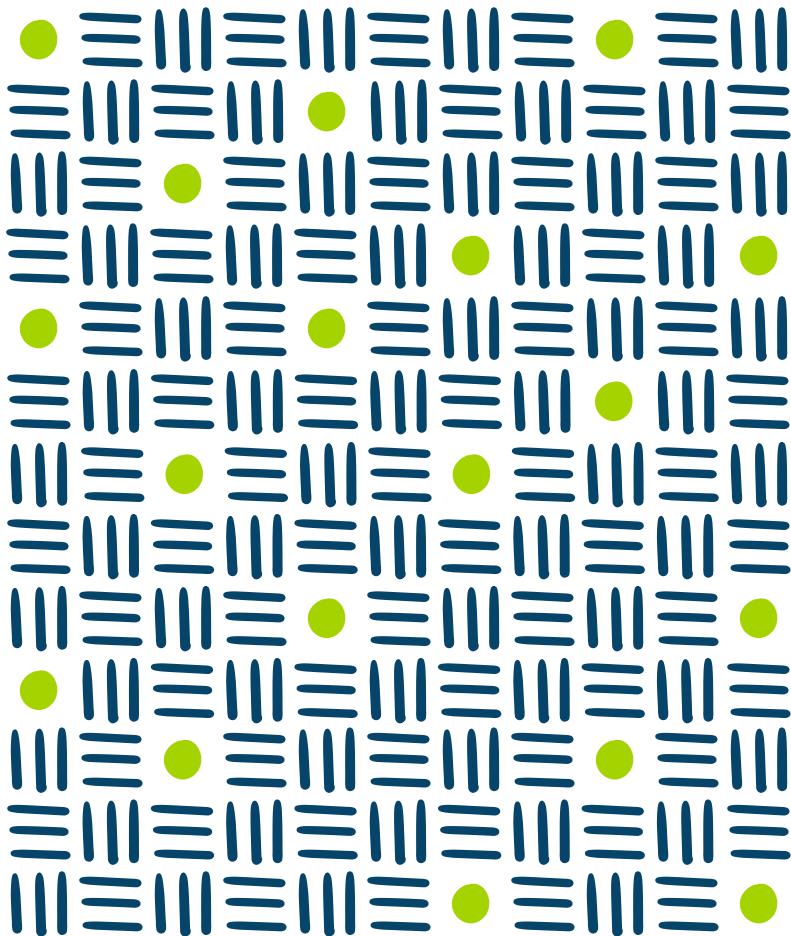
$$Q(\omega_{\min}^2) < 0$$

After substituting ω_{\min}^2 in $Q(\omega^2)$

$$\text{we get, } \left(\frac{D_v(v^2 + F) - D_u(F + k)}{2 D_u D_v} \right)^2 D_u D_v - \\ \left(\frac{D_v(v^2 + F) - D_u(F + k)}{2 D_u D_v} \right) (D_u(v^2 + F) - D_u(F + k)) + \\ (F + k)(v^2 - F) < 0$$

$$\boxed{\frac{(D_v(v^2 + F) - D_u(F + k))^2}{4 D_u D_v} + (F + k)(F - v^2) > 0}$$

Condition for instability, where v is
ordinate of
Critical Point.



04

Simulations

One-dimensional Gray Scott Patterns

- System of DEs were solved in MATLAB using pdepe solver.
- pdepe solver was used because our DEs are one dimensional.
- pdepe solve expects DEs in following form:

$$c\left(x, t, u, \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial t} = x^{-m} \frac{\partial}{\partial x} \left(x^m f\left(x, t, u, \frac{\partial u}{\partial x}\right) \right) + s\left(x, t, u, \frac{\partial u}{\partial x}\right).$$

- Here we have $m = 0$,
- $c = \text{diag}[1 \ 0; 0 \ 1]$
- $f = \text{diag}[d1 \ 0; 0 \ d2]$
- $s = [F(u, v); G(u, v)]$

One-dimensional Gray Scott Patterns

- We fixed initial condition for next 2 simulations as:

$$u(x, 0) = 1 - \frac{1}{2} \sin^{100} \left(\pi \frac{(x - 50)}{100} \right),$$

$$v(x, 0) = \frac{1}{4} \sin^{100} \left(\pi \frac{(x - 50)}{100} \right).$$

- We fixed boundary condition for next 2 simulations as homogeneous Dirichlet Condition.

```
N = 100;
T = 2500;
x = linspace(-50, 50, N);
t = linspace(0, T, N);

m = 0;
sol = pdepe(m,@pdefun,@pdeic,@pdebc,x,t);

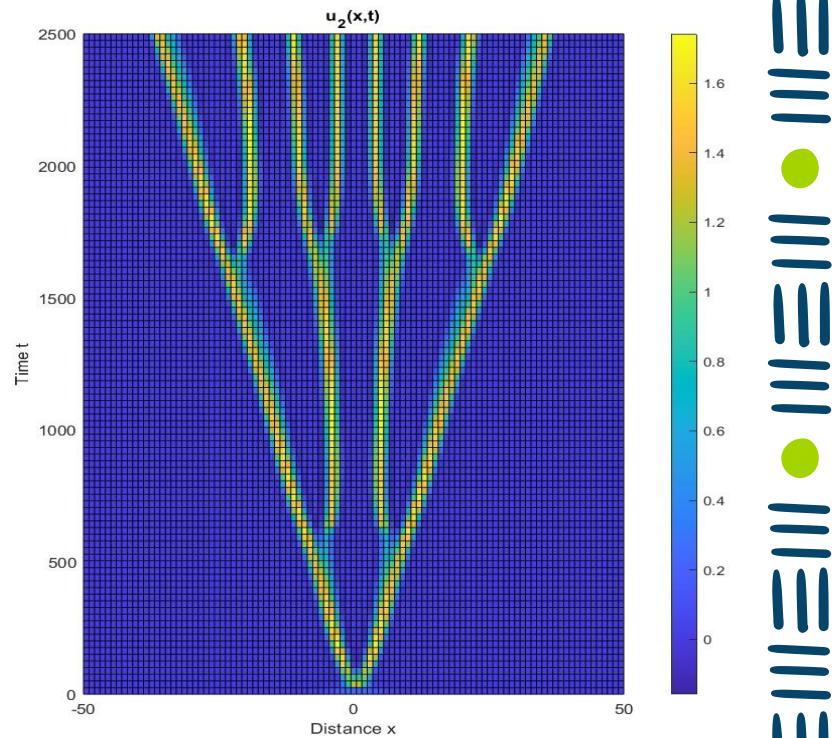
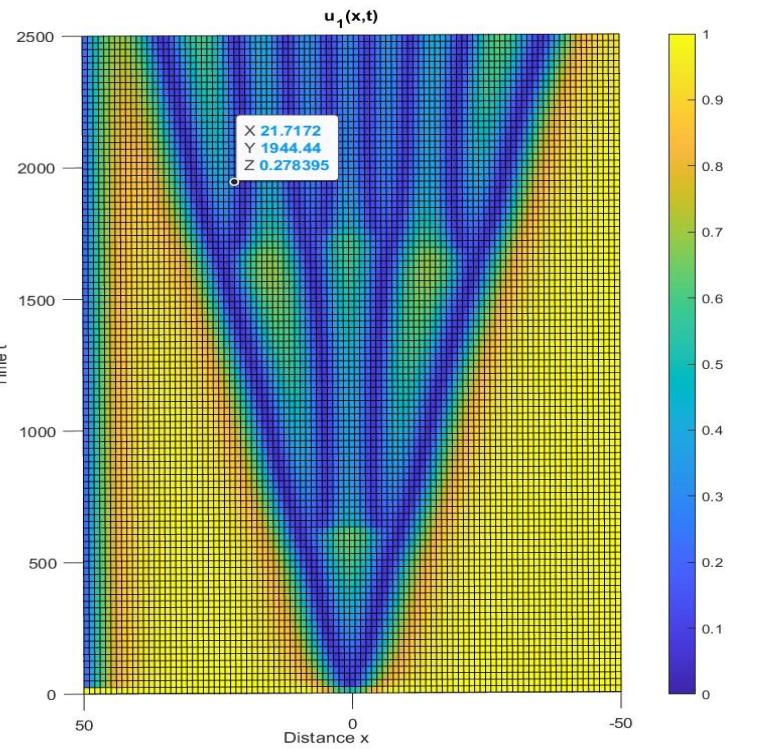
u1 = sol(:,:,1);
u2 = sol(:,:,2);

figure
subplot(1,2,1), surf(x,t,u1); colorbar;
title('u_1(x,t)')
xlabel('Distance x'); ylabel('Time t'); grid on
subplot(1,2,2), surf(x,t,u2); colorbar;
title('u_2(x,t)')
xlabel('Distance x'); ylabel('Time t'); grid on

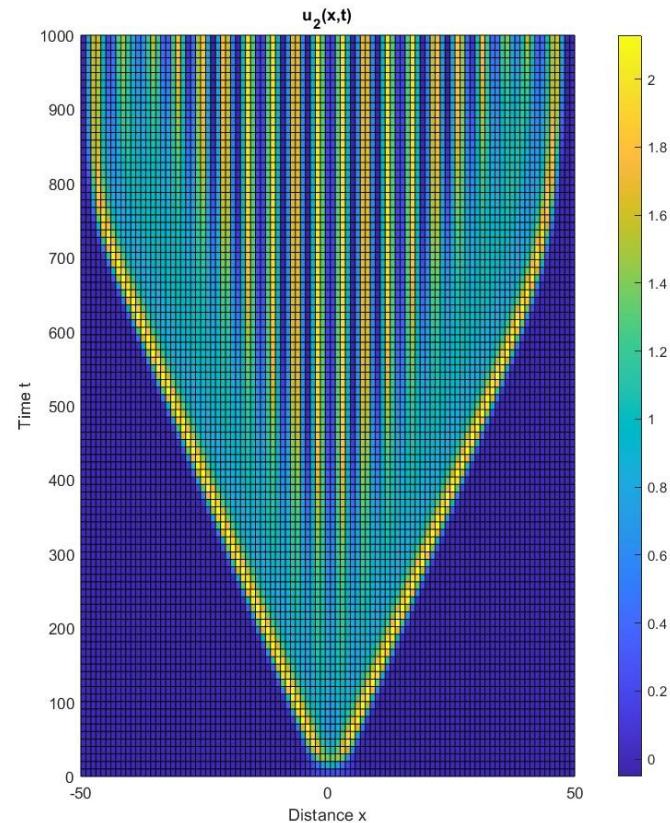
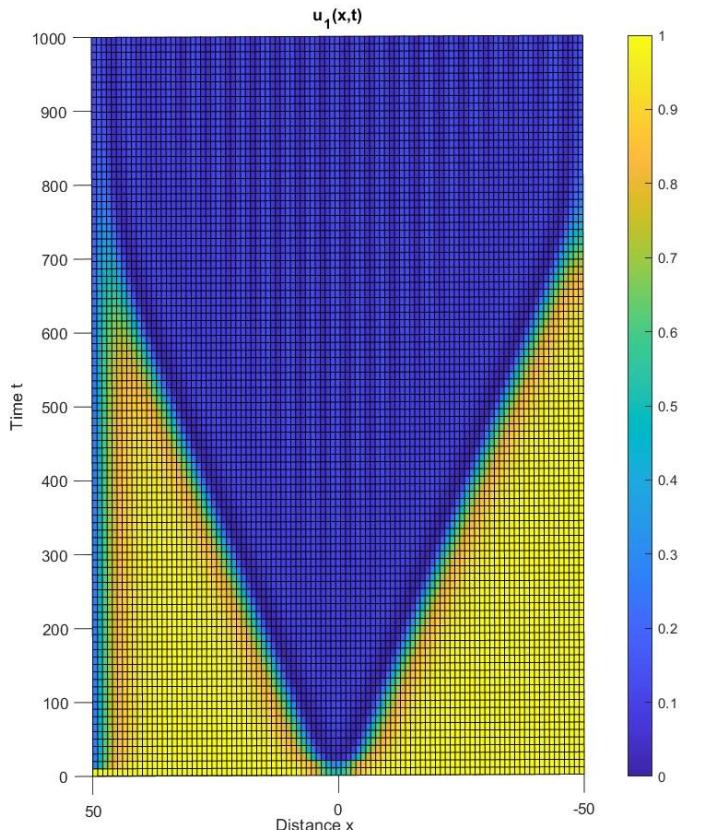
figure
surf(x,t,u1); title('u_1(x,t)'); colorbar
xlabel('Distance x'); ylabel('Time t'); grid on
hold on
surf(x,t,u2); title('u_1(x,t) and u_2(x,t)'); colorbar
xlabel('Distance x'); ylabel('Time t'); grid on
legend('u(x,t)', 'v(x,t)', Location='best');
```

```
function [c,f,s] = pdefun(x,t,u,dudx) % Equation to solve
    c = [1; 1];
    f = [1; 0.01] .* dudx;
    y1 = u(1);
    y2 = u(2);
    F = -y1*y2^2 + 0.062*(1-y1);
    G = y1*y2^2 - 0.126*y2;
    s = [F; G];
end
% -----
function u0 = pdeic(x) % Initial Conditions
u1 = 1 - 1/2*(sin(pi*(x-50)/100))^100;
u2 = 1/4*(sin(pi*(x-50)/100))^100;
u0 = [u1; u2];
end
% -----
function [pl,ql,pr,qr] = pdebc(xl,ul,xr,ur,t) % Boundary Conditions
pl = [0; ul(2)];
ql = [1; 0];
pr = [ur(1); 0];
qr = [0; 1];
end
```

$$D_1 = 1, D_2 = 0.01, F = 0.062, k = 0.064$$



D1 = 1, D2 = 0.01, F = 0.090, k = -0.0038





Some more patterns...

(Solving DEs using finite differences)

```
% Gray-Scott Reaction-Diffusion Model
% Implementation in MATLAB

% Parameters
L = 1000; % size of grid
Du = 0.16; % diffusion rate of u
Dv = 0.08; % diffusion rate of v
F = 0.04; % feed rate
k = 0.06; % kill rate
frame_rate = 10; % Set the frame rate (frames per second)

r=(1-((4*(F+k)*(F+k))/F))^0.5;
v1=(F/(2*(F+k)))*(1+r);
v2=(F/(2*(F+k)))*(1-r);
check1=(((Dv*(v1*v1+F)-Du*(F+k))^2)/(4*Du*Dv)) + (F+k)*(-1*v1*v1 +F);
check2=(((Dv*(v2*v2+F)-Du*(F+k))^2)/(4*Du*Dv)) + (F+k)*(-1*v2*v2 +F);

if(check1>0)
    disp("Point u1,v1 is unstable");
end
if(check2>0)
    disp("point u2,v2 is unstable");
end
```

```
% Initialize concentrations
[u, v] = meshgrid(linspace(0, 1, L), linspace(0, 1, L));

% Set initial conditions (stripes-like pattern)
stripe_width = 0.1; % adjust the width of the stripes as needed
stripe_frequency = 6; % adjust the frequency of the stripes as needed
u = 0.5 + 0.5 * sin(2 * pi * stripe_frequency * u);
v = 0.25 + 0.25 * sin(2 * pi * stripe_frequency * v);

% Create VideoWriter object with specified frame rate
writerObj = VideoWriter('dotted_animation.mp4', 'MPEG-4');
writerObj.FrameRate = frame_rate; % Set the frame rate
open(writerObj);
```

```
% Initialize figure
figure;

% Visualization
figure;
subplot(1,3,1);
initial_u_image = cat(3, u, zeros(size(u)), zeros(size(u)));
imshow(initial_u_image);
title('Initial Concentration of u');
subplot(1,3,2);
initial_v_image = cat(3, zeros(size(v)), v, zeros(size(v)));
imshow(initial_v_image);
title('Initial Concentration of v');
```

```
% Simulation
dt = 1; % time step
T = 80; % total simulation time
for t = 1:T
    % Compute Laplacian
    laplacian_u = del2(u);
    laplacian_v = del2(v);

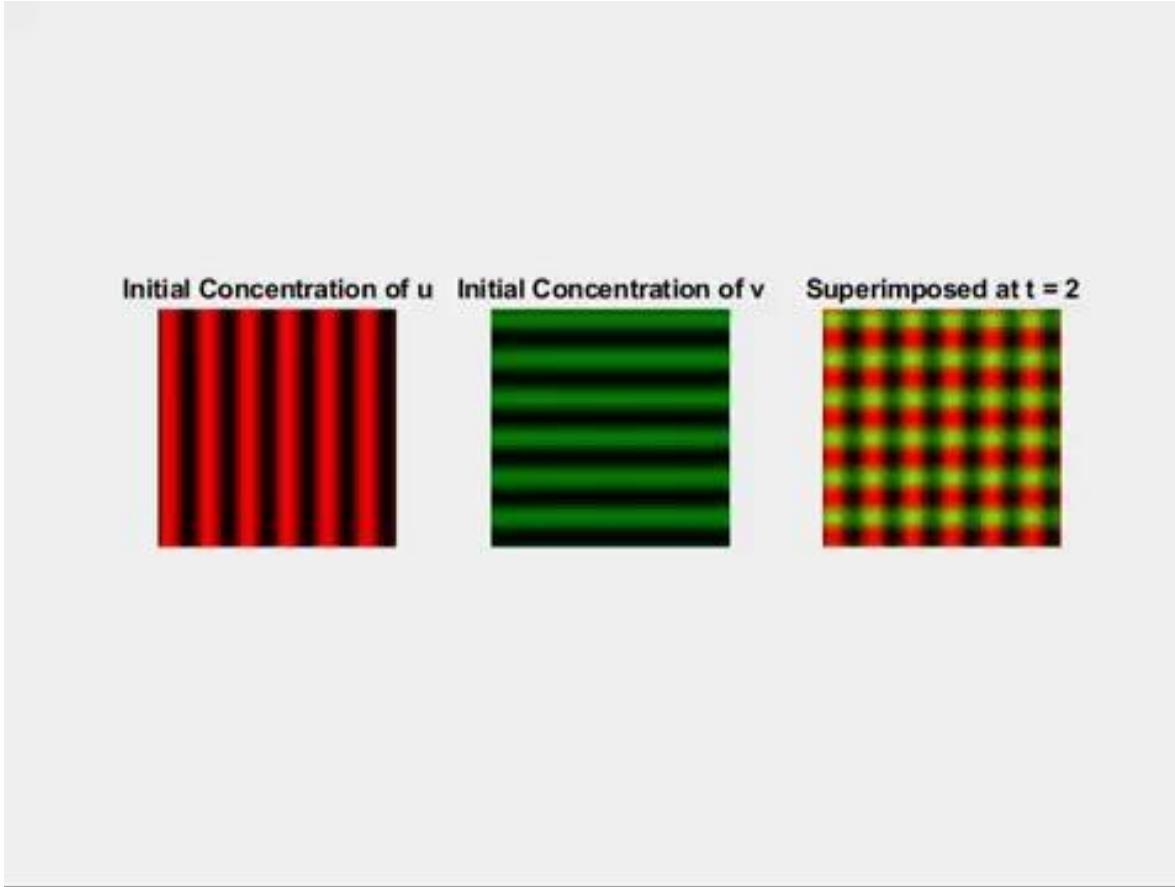
    % Update concentrations
    u_next = u + (Du * laplacian_u - u .* v.^2 + F * (1 - u)) * dt;
    v_next = v + (Dv * laplacian_v + u .* v.^2 - (F + k) * v) * dt;

    % Boundary conditions (periodic boundary conditions)
    u_next(:,1) = u_next(:,end-1);
    u_next(:,end) = u_next(:,2);
    u_next(1,:) = u_next(end-1,:);
    u_next(end,:) = u_next(2,:);

    v_next(:,1) = v_next(:,end-1);
    v_next(:,end) = v_next(:,2);
    v_next(1,:) = v_next(end-1,:);
    v_next(end,:) = v_next(2,:);

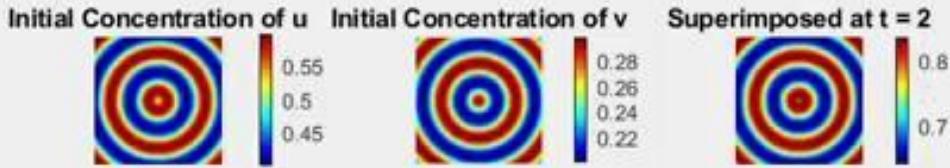
    % Update concentrations for next iteration
    u = u_next;
    v = v_next;
```

```
% Display current iteration every 1000 steps
if mod(t,1) == 0
    disp(['Iteration: ', num2str(t)]);
    subplot(1,3,3);
    combined_image = cat(3, u, v, zeros(size(u)));
    imshow(combined_image);
    title(['Superimposed at t = ', num2str(t)]);
    drawnow;
end
% Write frame to video
writeVideo(writerObj, getframe(gcf));
end
% Close VideoWriter object
close(writerObj);
```



By changing initial condition...

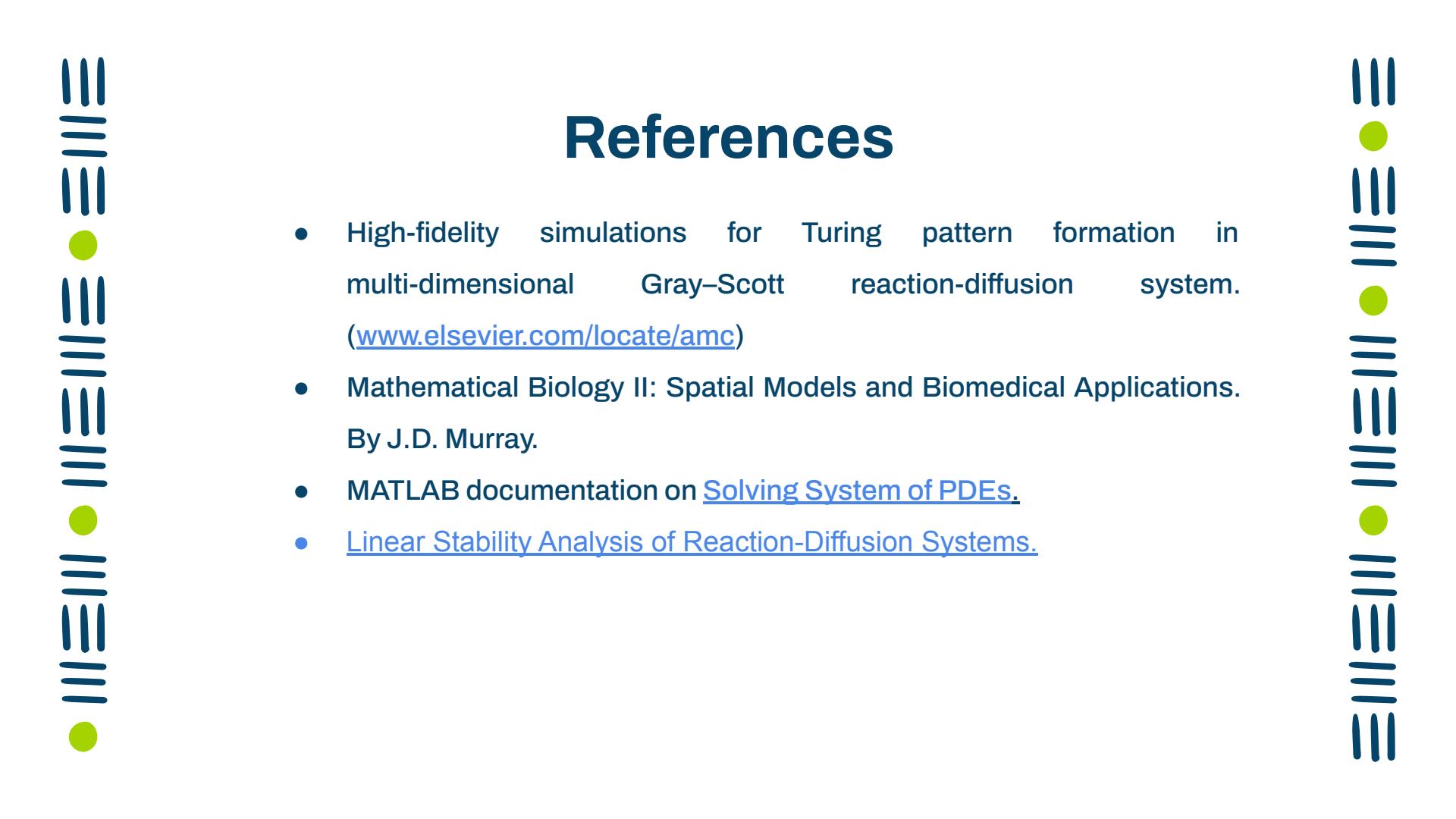
```
% Set initial conditions (spiral pattern)
center_x = 0.5;
center_y = 0.5;
r_max = 0.3;
u_init = 0.5 + 0.1 * (sin(2 * pi * sqrt((u -
center_x).^2 + (v - center_y).^2) / r_max));
v_init = 0.25 + 0.05 * (cos(2 * pi * sqrt((u
- center_x).^2 + (v - center_y).^2) /
r_max));
u = u_init;
v = v_init;
```





Conclusion

- Beauty of this model lies in its simplicity yet the ability to handle and mimic complex biological phenomena elegantly.
- By adjusting the reaction terms, and varying the parameters, one can obtain vast range of patterns found in nature.
- However, Turing's hypothesis, when proposed initially in 1952, faced backlash and skepticism .
- This was primarily due to novelty, complexity and lack of experimental validation.
- Later on, results were verified by various scientists by proposing different systems based on different conditions.
- Today, Turing's work on morphogenesis is the ground for scientific study about patterns in living creatures.



References

- High-fidelity simulations for Turing pattern formation in multi-dimensional Gray–Scott reaction-diffusion system.
(www.elsevier.com/locate/amc)
- Mathematical Biology II: Spatial Models and Biomedical Applications.
By J.D. Murray.
- MATLAB documentation on [Solving System of PDEs](#).
- [Linear Stability Analysis of Reaction-Diffusion Systems](#).

Thank You