MTH696A Project Report

FOURIER-SPECTRAL METHODS FOR NAVIER STOKES EQUATIONS

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1 Introduction

Fourier-Spectral method is a topic that has been discussed in this course. Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve differential equations, potentially involving the use of the Fast Fourier transform. The idea is to write the solution of the differential equation as a sum of certain "basis functions" (for example, as a Fourier series which is a sum of sinusoids) and then to choose the coefficients in the sum in order to satisfy the differential equation as well as possible.
We have implemented this method on a Partial differential equation in two dimensional. We have used some basic theory from fluid Meachnics to get Navier-Stokes Equation (which is a PDE in 2D): Total Derivative, Equation of Continuity, Momentum, Potential function, Vorticity, Steam function, viscosity, incompressible fluids etc. and some Mathematical Inequalities from Vector Differential Calculus.

2 Abstract

We have implemented Fourier-Spectral method to solve Navier-Stokes Equations on two dimensional flat torus with Crank-Nicolson method for time stepping. We have added some small random perturbation to better understand the dependency on initial condition by Navier-Stokes Equations and see the difference in evolution as well as evolution of uniform random initial data.

3 Motivation

We have studied formulation of Navier Stokes equation in fluid mechanics course and How to solve Partial differential equation analytically in PDE course. We want to use our prior knowledge to do this project which we hope, will give us better understanding of core and technical concepts. Although, there are many methods to solve Partial differential equation using Numerical Technique but in this project, we have decided to solve the problem using Fourier - Spectral method.
Thus, our motivation while doing this project has been to not only understand the inner workings of the Algorithm but also to get good results. If we manage to solve this problem thoroughly, we may be able to employ this methods in almost all partial differential equation in two dimensional with some boundary condition.

4 Literature Review

We have gone through the previous work mentioned in the references section. We will also implement some methods and mechanisms that have been discussed in these works as well as possible extensions that we could think of. In this section, we will discuss the approaches that have been presented in these works that we will use in our project as well.

4.1 Theory

For this problem, we will consider incompressible ($\nabla.v = 0$), Newtonian fluid (μ is constant). We will use 2-D form of Navier-Strokes equation.

$$\frac{\partial u}{\partial t} + u(\nabla.u) + \nabla p = \mu \triangle u \tag{1}$$

$$\nabla.u = 0$$

where μ is dynamic viscosity, p is pressure and u is velocity vector

Vorticity is defined as:

$$w = \nabla \times u \tag{2}$$

We derive vorticity stream function formulation of Navier Stokes Equation in two dimension by applying curl to the Navier Stokes equation. The following is a common way of deriving vorticity equation. We will use the following identity to get the Navier Stokes equation in Vorticity stream function:

$$\frac{1}{2}\nabla(u.u) = (u.\nabla)u + u \times (\nabla \times u) \tag{3}$$

$$\nabla \times \nabla \phi = 0 \tag{4}$$

$$\nabla \times (u \times w) = (w\nabla)u - (u.\nabla)w + u\nabla w - w\nabla u \tag{5}$$

From equation of continuity, $\nabla.u = 0$ and by using equation (4) we got, $\nabla w = 0$
Now using equation(3), we can transform (1) into

$$\frac{\partial u}{\partial t} + \frac{1}{2}\nabla(u.u) - u \times (\nabla \times u) + \nabla p = \mu \triangle u \tag{6}$$

We take the curl on both sides of the equation(6). Using the previously mentioned identities, we got the following **Vorticity Equation**

$$\frac{\partial w}{\partial t} + (u.\nabla)w - (w.\nabla)u = \mu \triangle w$$

Total derivative is defined as $\frac{Dw}{Dt} = \frac{\partial u}{\partial t} + (u.\nabla)w$. The Vorticity equation shows that the rate of change of the vorticity is controlled by the term reffered as vorticity stretching term $(w.\nabla)u$ and by diffusion term $\mu \triangle w$. Note that in two dimensions $u = v_1 e_x + v_2 e_y$ and $w = w(x, y)e_z$ and thus $(w.\nabla)u = 0$. This gives us **Two dimensional Vorticity Equation**

$$\frac{\partial w}{\partial t} = \mu \triangle w - (u.\nabla)w \tag{7}$$

Above is the main equation that we want to consider. This equation is a nonlinear advection diffusion equation. Once we can successfully solve for vorticity we solve for stream function ψ defined as

$$w = -\triangle \psi$$

$$u = v_1 e_x + v_2 e_y$$

$$v_1 = \delta_y \psi$$

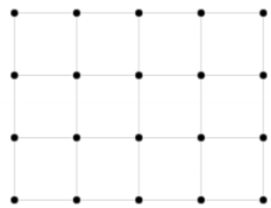
$$v_2 = -\delta_x \psi$$

Scaling arguments show that in the limit of very high viscosity or zero Reynolds number the stream function essentially reduces to biharmonic equation of the following form

$$\triangle^2 \psi = 0$$

In this project, we will focus mainly on two dimensional vorticity equation on T^2. The vorticity streamfunction formulation is easier to implement than more primitive variable formulation velocity.

4.2 Discretization and Implementation



First of all, we will discretize both space and time. For space discretization of T^2,we use equidistant square grid identifying both top and bottom and right and left sides as show in the above figure.
We will use Fourier - Spectral method for differentiation. So we take Fourier transform of the vorticity equation which gives the following equation:

$$\partial_t \hat{w} = -\mu(\xi_x^2 + \xi_y^2)\hat{w} - \widehat{u. \bigtriangledown w} \tag{8}$$

$$\partial_t \hat{w} = -\mu(\xi_x^2 + \xi_y^2)\hat{w} - \hat{v}_1 * \xi_x \hat{w} - \hat{v}_2 * \xi_y \hat{w} \tag{9}$$

The right hand side of the equation (9) can be solved using discrete Fourier transform on the grid points. We will use Fast Fourier Transform (FFT) algorithm to solve the right hand side.

For time stepping, we will use the Crank-Nicolson method. For linear evolution PDE's this method is unconditionally stable hence also thought to be good method for some non-linear PDE's. Crank-Nicolson method is an average of Forward Euler and Backward Euler methods after long algebra one can write the method in the explicit form

$$\hat{w}_{i,j}^{n+1} = \frac{1}{\frac{1}{\triangle t} - \frac{1}{2}\mu(\xi_x^2 + \xi_y^2)}((\frac{1}{\triangle t} + \frac{1}{2}\mu(\xi_x^2 + \xi_y^2))\hat{w}_{i,j}^n - \widehat{u_{i,j}^n. \bigtriangledown w_{i,j}^n}) \tag{10}$$

We also have to take care of the aliasing problem (misidentification of a signal frequency) by throwing out the frequencies that are higher than 2/3 times the grid size in the convolution.

We were able to successfully implement above method and obtain a physically feasible answer.

5 Results

We implemented FFT with different initial conditions like smooth, uniform random noise and combination of the two. Lets take this smooth initial vorticity.

$$
\begin{aligned}
\tilde{w}|_{t=0} = &\exp{(-\frac{(x-\pi+\pi/5)^2+(y-\pi+\pi/5)^2}{0.3})} \\
&-\exp{(-\frac{(x-\pi-\pi/5)^2+(y-\pi+\pi/5)^2}{0.2})} \\
&+\exp{(-\frac{(x-\pi-\pi/5)^2+(y-\pi-\pi/5)^2}{0.4})}
\end{aligned}
$$

The solution is of the following initial condition is given by Figure 1. The figure shows evolution of vorticity field with parameters μ = 0.005, T = 50 with $\triangle t = 0.1$.

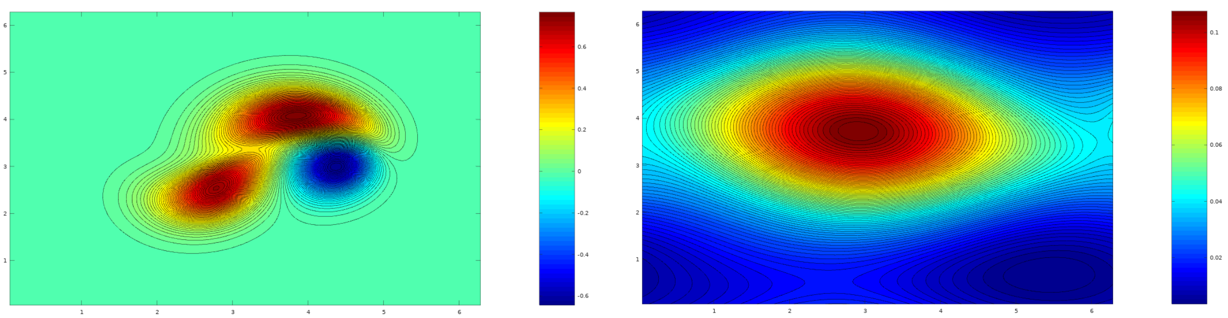


Figure 1: Initial & Final vorticity with no noise and $\mu = 0.005$

Next we added random normal noise with mean 0 & variance 1 to the vorticity and we are getting the following results.

$$w|_{t=0} = \tilde{w} + \epsilon N$$

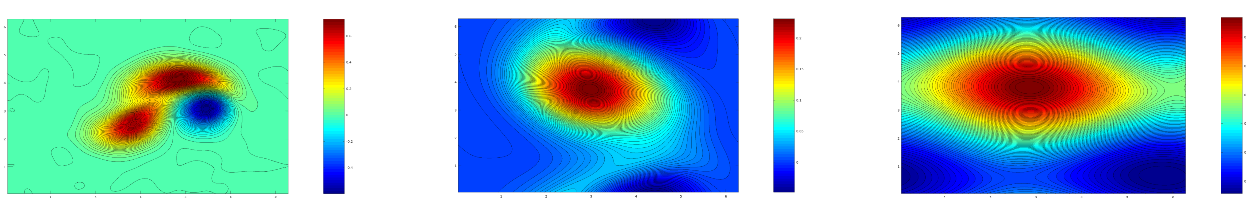


Figure 2: Initial, Intermediate state & Final vorticity with normal noise and $\mu = 0.005$

From figure 2 we can observe that the final evolution of w is not very different from that of $\tilde{}$w.

Just for the sake of completeness, we took vorticity as random noise with mean 0 & variance as 1 and we observed results which is given by figure4. Also to check the limit of our implementation we took random uniform velocity field with very low viscosity $= 0.005$ and we observed the results which is shown in figure5 as what we expect theoritically.

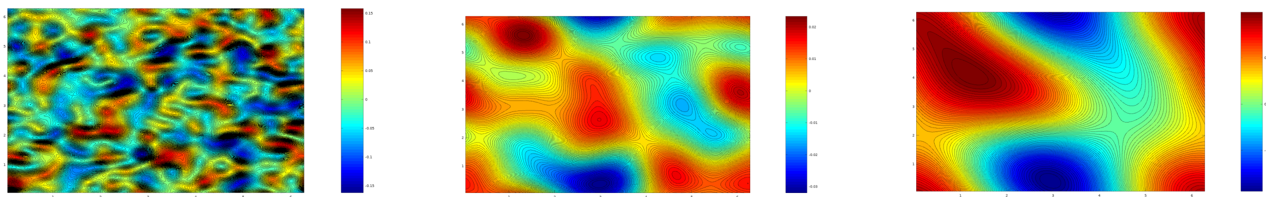


Figure 3: Initial, Intermediate state & Final vorticity with normal noise and $\mu = 0.01$

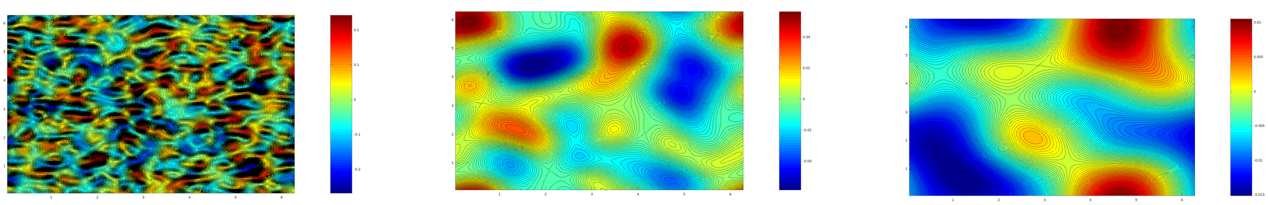


Figure 4: Initial, Intermediate state & Final vorticity with normal noise and $\mu = 0.005$

References

[1] http://www.math.mcgill.ca/gantumur/math595f14/NSMashbat.pdf

[2] Fundamental Mechanics of Fluids by I. G. Currie

[3] https://en.wikipedia.org/wiki/Spectral method