Self Referencing Sequences

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1 Introduction

2 Definitions and Notations

In this section we introduce

3 Prefixes

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Definition 1. *We say that two sequences $\{a_i\}$ and $\{b_i\}$ differ by a prefix of length n if*

$$a_{n+i} = b_i,$$

for i any positive integer. if no such n exists then we say that the two sequences are independent.

Note that a pair of sequences can have prefixes of different length. For example if we have

$$\{a_i\} = \{1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, \ldots\}$$
$$\{b_i\} = \{7, 8, 9, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, \ldots\},$$

then a_i and b_i differ by prefixes of length $3 + 6k$ for k a nonnegative integer. Because of this we make the following definition.

Definition 2. *We say that two sequences $\{a_i\}$ and $\{b_i\}$ differ by a minimal prefix of length n if they differ by a prefix of length n, but do not differ by a prefix of length m for all $0 < m < n$.*

We note that if a pair of sequences differs by a prefix they differ by a minimal prefix. In our above example a_i and b_i differ by a minimal prefix of length 3.

Theorem 1. *The self-referencing sequence beginning with a 1 generated over $\{1, c_1, c_2, \ldots, c_n\}$ differs from the self-referencing sequence beginning with c_1 generated over $\{1, c_1, c_2, \ldots, c_n\}$ by a minimal prefix of length 1.*

Proof. Let a_i refer to the ith term of the sequence beginning with a 1 and b_i refer to the ith term of the sequence beginning with c_1. We know that $a_1 = 1$, this means that the first block in a_i must be of length 1. In other words we must have $a_2 = c_1$, but since a_2 is now part of the second block of a_i the value of a_3 is not specified by a_1. Thus we could specify the rest of a_i only knowing the value of a_2, but $b_1 = c_1$, so this is exactly how the values of b_i is specified. Therefore

$$a_{1+i} = b_i,$$

so a_i and b_i differ by a prefix of length 1, but $a_1 \neq b_1$ so it is a minimal prefix. □

Theorem 2. *If $1 < a < b$ then the self-referencing sequence beginning with a generated over $\{1, a, b\}$ and the self-referencing sequence beginning with c_2 generated over $\{1, a, b\}$ are independent.*

Proof. We proceed by contradiction. If the two sequences are not independent then they must differ by some minimal prefix of length n. If we let a_i be the sequence beginning with a and b_i be the sequence beginning with b the

$$a_{n+k} = b_k.$$

We know that the first b terms of the sequence which begins with a b are b's. Because $b > a > 1$ no block can be longer then b. Thus the terms immediately preceding and following this block in the sequence a_i must not be b's. Because of this the first block of b_i is the $m + 1$th block of a_i, where m is some positive integer.

Since the value of the sequence at position k is also the length of the kth block this means that $a_{m+1} = b_1$. In fact the kth block of b_i is the $(m+k)$th block of a_i and so

$$a_{m+k} = b_k$$

We now claim that the prefix has length strictly larger than m. Assume otherwise so $n \leq m$. This means the entry in the a_i which gives the length of the $m+1$th block occurs at or after the beginning of the $m+1$th block. By our proof that self-referential sequences are well defined we know that the second case cannot occur. Therefore the only other option would be for the $m+1$th block to start at position $m+1$, but this would require every block to have length exactly 1, but this is a contradiction because $a_1 = a > 1$, that is the first block has length greater than one. Therefore the prefix must have length strictly larger than m.

We note that based on a previous step the two sequences differ by a prefix starting at m. Since $m < n$ we have produced a prefix which starts before n which is a contradiction. Therefore the two sequences are independent. $\square$