Wake Potential With Finite Electron Sheath

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Theory notebook for the flat beam regime.

1 Elliptical coordinates

1.1 Definition

Ellipse equation:

$$\frac{x^2}{x_p^2} + \frac{y^2}{y_p^2} = 1 \tag{1.1.1}$$

Hypobola equation:

$$\frac{x^2}{x_p^2} - \frac{y^2}{y_p^2} = 1 \tag{1.1.2}$$

The elliptical coordinates:

$$x = a \cosh \mu \cos \upsilon \tag{1.1.3}$$
$$y = a \sinh \mu \sin \upsilon \tag{1.1.4}$$

The back transformation:
Use the identity

$$\cosh^2 r - \sinh^2 r = 1 \tag{1.1.5}$$
$$\cos^2 \theta + \sin^2 \theta = 1 \tag{1.1.6}$$

Obtain

$$\sinh^2 r = \frac{1}{2a^2}\left((x^2+y^2-a^2) + \sqrt{(x^2+y^2-a^2)^2 + 4a^2y^2}\right) \tag{1.1.7}$$
$$\cos^2 \theta = \frac{1}{2a^2}\left((x^2+y^2+a^2) - \sqrt{(x^2+y^2+a^2)^2 - 4a^2x^2}\right) \tag{1.1.8}$$

Where

$$a = \sqrt{x_p^2 - y_p^2} \tag{1.1.9}$$

For constant μ, it is the ellipse equation in x-y coordinates

$$\frac{x^2}{a^2 \cosh^2 \mu} + \frac{y^2}{a^2 \sinh^2 \mu} = 1 \tag{1.1.10}$$

For constant υ, it is the hypobola equation in x-y coordinates

$$\frac{x^2}{a^2 \cos^2 \upsilon} - \frac{y^2}{a^2 \sin^2 \upsilon} = 1 \tag{1.1.11}$$

In this coordinate, μ is like the r in polar coordinate, and υ is like the θ in polar coordinate. By adding z direction, it becomes the 3-d elliptical coordinate which is suitable for the flat beam regime where each transverse slice is relatively independent.

1.2 scale factors

Gradient of a scalar field $$\nabla\phi = \frac{\hat{e}_k}{h_k}\frac{\partial \phi}{\partial q^k} \tag{1.2.1}$$

Divergence of a vector field $$\nabla \cdot \mathbf{F} = \frac{1}{J}\frac{\partial}{\partial q^k}\left(\frac{J}{h_k}F_k\right) \tag{1.2.2}$$

Curl of a vector field (3D only) $$\nabla \times \mathbf{F} = \frac{\hat{e}_k}{J}\varepsilon_{ijk}\frac{\partial}{\partial q^i}(h_j F_j) \tag{1.2.3}$$

Laplacian of a scalar field $$\nabla^2\phi = \frac{1}{J}\frac{\partial}{\partial q^k}\left(\frac{J}{h_k^2}\frac{\partial \phi}{\partial q^k}\right) \tag{1.2.4}$$

Operator	Expression
Gradient of a scalar field	$\nabla\phi = \hat{e}_1 \frac{1}{h_1}\frac{\partial\phi}{\partial q^1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial\phi}{\partial q^2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial\phi}{\partial q^3}$
Divergence of a vector field	$\nabla\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}(F_1h_2h_3) + \frac{\partial}{\partial q^2}(F_2h_3h_1) + \frac{\partial}{\partial q^3}(F_3h_1h_2)\right]$
Curl of a vector field	$\nabla\times\mathbf{F} = \frac{\hat{e}_1}{h_2h_3}\left[\frac{\partial}{\partial q^2}(h_3F_2) - \frac{\partial}{\partial q^3}(h_2F_3)\right] + \frac{\hat{e}_2}{h_3h_1}\left[\frac{\partial}{\partial q^3}(h_1F_1) - \frac{\partial}{\partial q^1}(h_3F_3)\right] + \frac{\hat{e}_3}{h_1h_2}\left[\frac{\partial}{\partial q^1}(h_2F_1) - \frac{\partial}{\partial q^2}(h_1F_2)\right]$
Laplacian of a scalar field	$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q^2}\right) + \frac{\partial}{\partial q^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q^3}\right)\right]$

Table 1: Differential operators in generalized coordinates

As it is an orthogonal coordinate system, scale factors can be used to obtain all kinds of differentials:

$$h_\mu = h_\nu = a\sqrt{\sinh^2\mu + \sin^2\nu} = a\sqrt{\cosh^2\mu - \cos^2\nu}. \tag{1.2.5}$$

$$h_z = 1 \tag{1.2.6}$$

Using the double argument identities for hyperbolic functions and trigonometric functions, the scale factors can be equivalently expressed as

$$h_\mu = h_\nu = a\sqrt{\frac{1}{2}(\cosh 2\mu - \cos 2\nu)} = h \tag{1.2.7}$$

Also one useful thing is its derivative:

$$\frac{\partial h}{\partial\mu} = \frac{a^2\sinh 2\mu}{2h} \tag{1.2.8}$$

$$\frac{\partial h}{\partial\nu} = \frac{a^2\sin 2\nu}{2h} \tag{1.2.9}$$

Consequently, an infinitesimal element of area equals

$$dA = h_\mu h_\nu d\mu d\nu = J d\mu d\nu \tag{1.2.10}$$

where

$$J = h_\mu h_\upsilon = h^2 = a^2(\sinh^2\mu + \sin^2\nu) = a^2(\cosh^2\mu - \cos^2\nu) = \frac{a^2}{2}(\cosh 2\mu - \cos 2\nu) \tag{1.2.11}$$

and the Laplacian reads

$$\nabla\cdot\vec{\mathbf{F}} = \frac{1}{h^2}\left(\frac{\partial}{\partial\mu}(hF_\mu) + \frac{\partial}{\partial\nu}(hF_\nu)\right) + \frac{\partial F_z}{\partial z} \tag{1.2.12}$$

$$\nabla\times\vec{\mathbf{F}} = \frac{\hat{\mathbf{e}}_\mu}{h}\left(\frac{\partial}{\partial\nu}F_z - \frac{\partial}{\partial z}hF_\nu\right) + \frac{\hat{\mathbf{e}}_\nu}{h}\left(\frac{\partial}{\partial z}hF_\mu - \frac{\partial}{\partial\mu}F_z\right) + \frac{\hat{\mathbf{e}}_z}{h^2}\left(\frac{\partial}{\partial\mu}hF_\nu - \frac{\partial}{\partial\nu}hF_\mu\right) \tag{1.2.13}$$

$$\nabla^2\phi = \frac{1}{h_\mu h_\nu}\left(\frac{\partial^2\phi}{\partial\mu^2} + \frac{\partial^2\phi}{\partial\nu^2}\right) + \frac{\partial^2\phi}{\partial z^2} \tag{1.2.14}$$

$$\tag{1.2.15}$$

1.3 Coordinate Transformation

$$\cosh 2\mu = 2\cosh^2\mu - 1 = \frac{1}{c^2}(x^2 + y^2 + z^2) \tag{1.3.1}$$

$$\cos 2\nu = 2\cos^2\nu - 1 = \frac{1}{c^2}(x^2 + y^2 - z^2) \tag{1.3.2}$$

where

$$z^2 = \sqrt{(x^2 - y^2 - c^2)^2 + 4x^2y^2} \tag{1.3.3}$$

2 Elliptical blowout

We have the following equation for the potentials:

$$-\nabla_\perp^2\begin{bmatrix}A\\\phi\end{bmatrix} = \begin{bmatrix}J\\\rho\end{bmatrix} \tag{2.0.1}$$

$$\nabla_\perp \cdot A_\perp = -\frac{\partial \psi}{\partial \xi} \tag{2.0.2}$$

Following can also be shown to describe the plasma electron evolution:

$$\frac{d}{d\xi} P_\perp = \frac{1}{1 - v_z} \left[- \left(E_\perp + (V \times B)_\perp \right) \right] \tag{2.0.3}$$

Where the field is given:

$$E_z = \frac{\partial \psi}{\partial \xi} \tag{2.0.4}$$

$$\begin{aligned} B_z &= \nabla_\perp \times \mathbf{A}_\perp && (2.0.5)\\ &= \frac{1}{h^2}\left(\frac{\partial}{\partial \mu} h A_\nu - \frac{\partial}{\partial \nu} h A_\mu\right) && (2.0.6)\\ &= \frac{1}{h^2}\left(\frac{a^2 \cosh\mu \sinh\mu}{h} A_\nu + h\frac{\partial A_\nu}{\partial \mu} - \frac{a^2 \cos\nu \sin\nu}{h} A_\mu - h\frac{\partial A_\mu}{\partial \nu}\right) && (2.0.7) \end{aligned}$$

$$E_\perp = -\nabla_\perp \phi - \frac{\partial \mathbf{A}_\perp}{\partial \xi} \tag{2.0.8}$$

$$B_\perp = \nabla_\perp \times (A_z \hat{z}) + \nabla_z \times \mathbf{A}_\perp \tag{2.0.9}$$

Then:

$$\mathbf{E}_\perp + (\mathbf{v} \times \mathbf{B})_\perp = -\nabla_\perp \phi - \frac{\partial \mathbf{A}_\perp}{\partial \xi} + (v_z \hat{\mathbf{z}} \times \mathbf{B}_\perp) + \mathbf{v}_\perp \times \mathbf{B}_z \tag{2.0.10}$$

$$= -\nabla_\perp \phi - \frac{\partial \mathbf{A}_\perp}{\partial \xi} + v_z \times \nabla_\perp \times (\mathbf{A}_z \hat{z}) + v_z \times \nabla_z \times \mathbf{A}_\perp + v_\perp \times \nabla_\perp \times A_\perp \tag{2.0.11}$$

From the constant of motion $\gamma - P_z = 1 + \psi$, the following can be obtained:

$$\begin{aligned} P_z &= \frac{1 + P_\perp^2 - (1+\psi)^2}{2(1+\psi)} && (2.0.12)\\ \hat{\gamma} &= \frac{1 + P_\perp^2 + (1+\psi)^2}{2(1+\psi)} && (2.0.13)\\ 1 - v_z &= \frac{2(1+\psi)^2}{1 + P_\perp^2 + (1+\psi)^2} && (2.0.14) \end{aligned}$$

2.1 Solution for electron beam inside of cofocal vacuum chamber

There are numerous academic resources available on solving potentials in elliptical coordinates. One key reference is the solution provided in this document: POTENTIAL AND FIELD PRODUCED BY A UNIFORM OR NON-UNIFORM ELLIPTICAL BEAM INSIDE A CONFOCAL ELLIPTIC VACUUM CHAMBER.

We define two boundaries, one for the vacuum chamber and one for the beam at $\mu = \mu_1$ and $\mu = \mu_2$.

$$\begin{aligned} 2\mu &= \xi && (2.1.1)\\ 2\nu &= \phi && (2.1.2)\\ \Phi_L &= \frac{\rho C^2}{8\epsilon_0}\left[\xi_1 - \xi - \frac{\sinh(\xi_1 - \xi)}{\cosh \xi_1}\cos\phi\right]\sinh\xi_2 && (2.1.3)\\ \Phi_P &= \frac{\rho C^2}{8\epsilon_0}\left(\left[\frac{\cosh(\xi_1 - \xi_2)}{\cosh \xi_1}\cosh\xi - 1\right]\cos\phi + \cosh\xi_2 - \cosh\xi + (\xi_1 - \xi_2)\sinh\xi_2\right) && (2.1.4) \end{aligned}$$

ϕ_L is defined for $\mu_2 < \mu < \mu_1$, outside the beam but inside the chamber, and ϕ_P is defined for $mu < \mu_2$, inside the beam.

2.2 Isolated beam ($\xi_1 \to \infty$)

Taking ξ_1 to be at infinite and ignore the constant term that contributes to infinity, we can recognize that:

$$\lim_{\xi_1 \to 0} \frac{\sinh(\xi_1 - \xi)}{\cosh \xi_1} = \cosh \xi - \sinh \xi \tag{2.2.1}$$

$$\lim_{\xi_1 \to 0} \frac{\cosh(\xi_1 - \xi_2)}{\cosh \xi_1} = \cosh \xi_2 - \sinh \xi_2 \tag{2.2.2}$$

$$\tag{2.2.3}$$

then:

$$\Phi_L = \frac{\rho c^2}{8\epsilon_0} \left[-\xi - (\cosh \xi - \sinh \xi) \cos \phi\right] \sinh \xi_2 \tag{2.2.4}$$

$$\Phi_P = \frac{\rho c^2}{8\epsilon_0} \left(\left[(\cosh \xi_2 - \sinh \xi_2) \cosh \xi - 1\right] \cos \phi + \cosh \xi_2 - \cosh \xi - \xi_2 \sinh \xi_2\right) \tag{2.2.5}$$

where ξ_2 is the boundary of the electron beam.

2.3 Isolated beam with electron sheet

Let's consider the wake potential ψ whose source term is $\rho - j_z$, which is conserved in each individual plane.

We can construct the source term using two large confocal ellipse source overlapping each other ($\xi_1 = \xi_2 + \Delta\xi$). The ion column should have density of -1 and let the electron sheet have thickness of $\Delta\xi$.

By superposition, we know:

$$\rho_1 = \rho_\delta \tag{2.3.1}$$

$$\rho_2 = \rho_i - \rho_\delta \tag{2.3.2}$$

By conservation of "source charge", we know:

$$Area = \frac{\pi c^2}{2} \sinh 2\mu = \frac{\pi c^2}{2} \sinh \xi \tag{2.3.3}$$

$$Qi = \rho_i \frac{\pi c^2}{2} \sinh \xi_2 \tag{2.3.4}$$

$$Q_\delta = \rho_\delta \frac{\pi c^2}{2} (\sinh(\xi_2 + \xi_\delta) - \sinh \xi_2) \tag{2.3.5}$$

$$Q_i = -Q_\delta \tag{2.3.6}$$

$$\rho_\delta = -\rho_i \frac{\sinh \xi_2}{\sinh(\xi_2 + \xi_\delta) - \sinh \xi_2} \tag{2.3.7}$$

Represent ρ_1 and ρ_2 with ρ_i:

$$\rho_1 = -\rho_i \frac{\sinh \xi_2}{\sinh(\xi_2 + \xi_\delta) - \sinh \xi_2} \tag{2.3.8}$$

$$\rho_2 = \rho_i \frac{\sinh(\xi_2 + \xi_\delta)}{\sinh(\xi_2 + \xi_\delta) - \sinh \xi_2} \tag{2.3.9}$$

We can easily obtained the pseudo-potential outside the electron sheet ($\xi > \xi_1$), by superposing ϕ_L of both ρ_1 and ρ_2:

$$\Phi(\xi > \xi_1) = (\rho_1 \sinh \xi_1 + \rho_2 \sinh \xi_2) \left[-\xi - (\cosh \xi - \sinh \xi)\right] \tag{2.3.10}$$

$$= (-\rho_i \frac{\sinh \xi_2}{\sinh(\xi_2 + \xi_\delta) - \sinh \xi_2} \sinh(\xi_2 + \Delta\xi) + \rho_i (\frac{\sinh(\xi_2 + \xi_\delta)}{\sinh(\xi_2 + \xi_\delta) - \sinh \xi_2}) \sinh(\xi_2) \tag{2.3.11}$$

$$\left[-\xi - (\cosh \xi - \sinh \xi)\right]) \tag{2.3.12}$$

$$= 0 \tag{2.3.13}$$

The conclusion is that as long as the shielding is confocal, Gauss's law would work like it is in the spherical coordinate. It corresponds to the observation that outside the blowout bubble, there is no electromagnetic force outside, which means the psi should be constant.

$$E_x = -\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial \xi} = 0 \tag{2.3.14}$$
$$E_y = -\frac{\partial \phi}{\partial y} - \frac{\partial A_y}{\partial \xi} = 0 \tag{2.3.15}$$
$$E_z = \frac{\partial \psi}{\partial \xi} = 0 \tag{2.3.16}$$
$$B_x = \frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial \xi} = 0 \tag{2.3.17}$$
$$B_y = -\frac{\partial A_x}{\partial \xi} - \frac{\partial A_z}{\partial x} = 0 \tag{2.3.18}$$

Consider $E_x + B_y$, $E_y + B_x$, and E_z, which represent the force experienced by particle traveled at the speed of the light, we can obtain that ψ is constant of x, y, ξ.

Now let us consider the messy part when $\xi < \xi_2$.

Therefore:

$$\begin{aligned}
\Phi(\xi < \xi_2) &= \frac{\rho_2 c^2}{8\epsilon_0}\left([(\cosh\xi_2 - \sinh\xi_2)\cosh\xi - 1]\cos\phi + \cosh\xi_2 - \cosh\xi - \xi_2\sinh\xi_2\right) && (2.3.19)\\
&+ \frac{\rho_1 c^2}{8\epsilon_0}\left([(\cosh\xi_1 - \sinh\xi_1)\cosh\xi - 1]\cos\phi + \cosh\xi_1 - \cosh\xi - \xi_1\sinh\xi_1\right) && (2.3.20)\\
&= -\frac{\rho_i c^2}{8\epsilon_0}\frac{\sinh\xi_2}{\sinh\xi_1 - \sinh\xi_2}\left([(\cosh\xi_1 - \sinh\xi_1)\cosh\xi - 1]\cos\phi + \cosh\xi_1 - \cosh\xi - \xi_1\sinh\xi_1\right) && (2.3.21)\\
&+ \frac{\rho_i c^2}{8\epsilon_0}\frac{\sinh\xi_1}{\sinh\xi_1 - \sinh\xi_2}\left([(\cosh\xi_2 - \sinh\xi_2)\cosh\xi - 1]\cos\phi + \cosh\xi_2 - \cosh\xi - \xi_2\sinh\xi_2\right) && (2.3.22)\\
&= \Phi_i + \Phi_e && (2.3.23)
\end{aligned}$$

Here comes the mathy part:

$$\begin{aligned}
\Phi(\xi < \xi_2) = &\frac{\rho_i c^2}{8\epsilon_0}\left[\frac{\sinh\xi_1 - (\cosh\Delta\xi - \sinh\Delta\xi)\sinh\xi_2}{\sinh\xi_1 - \sinh\xi_2}\right](\cosh\xi_2 - \sinh\xi_2)\cosh\xi\cos\phi && (2.3.24)\\
&- \frac{\rho_i c^2}{8\epsilon_0}\cos\phi && (2.3.25)\\
&+ \frac{\rho_i c^2}{8\epsilon_0}\left[\frac{\sinh\xi_1 - \cosh\Delta\xi\sinh\xi_2}{\sinh\xi_1 - \sinh\xi_2}\right]\cosh\xi_2 && (2.3.26)\\
&- \frac{\rho_i c^2}{8\epsilon_0}\frac{\sinh\xi_2}{\sinh\xi_1 - \sinh\xi_2}\sinh\xi_2\sinh\Delta\xi && (2.3.27)\\
&- \frac{\rho_i c^2}{8\epsilon_0}\cosh\xi && (2.3.28)\\
&- \frac{\rho_i c^2}{8\epsilon_0}\frac{\sinh\xi_1 - \cosh\Delta\xi\sinh\xi_2}{\sinh\xi_1 - \sinh\xi_2}\xi_2\sinh\xi_2 && (2.3.29)\\
&+ \frac{\rho_i c^2}{8\epsilon_0}\frac{\sinh\xi_2}{\sinh\xi_1 - \sinh\xi_2}(\xi_2\cosh\xi_2\sinh\Delta\xi + \Delta\xi\sinh\xi_2\cosh\Delta\xi + \Delta\xi\cosh\xi_2\sinh\Delta\xi)) && (2.3.30)
\end{aligned}$$

2.3.1 infinite thin electron sheet

We can take the limit that $\Delta\xi \to 0$ to obtain the formula which can be used to estimate the wake potential

$$\Phi(\xi < \xi_2) = \frac{\rho_i c^2}{8\epsilon_0}\left[\left(\frac{\cosh\xi}{\cosh\xi_2} - 1\right)\cos\phi + \cosh\xi_2 - \cosh\xi\right] \tag{2.3.31}$$

Let's consider another approach:

$$\Phi = \Phi_i + \Phi_e \tag{2.3.32}$$
$$\Phi_i = \frac{\rho c^2}{8\epsilon_0}\left([(\cosh\xi_2 - \sinh\xi_2)\cosh\xi - 1]\cos\phi + \cosh\xi_2 - \cosh\xi - \xi_2\sinh\xi_2\right) \tag{2.3.33}$$
$$\Phi_e = \frac{c^2}{8}\frac{Q_e}{Area}\left(\cos(\phi)\cosh(\xi)\left[\sinh(\xi_2)\tanh(\xi_2) - \sinh(\xi_2)\right] - \xi_2\sinh(\xi_2)\right) \tag{2.3.34}$$

Where

$$\frac{Q_e}{Area} = -\rho_i \tag{2.3.35}$$

if charge is conserved. However on the transverse slice, the ρ is not conserved.

$$\Phi = \frac{\rho c^2}{8\epsilon_0}\left(\left[(\frac{1}{\cosh\xi_2})\cosh\xi - 1\right]\cos\phi + \cosh\xi_2 - \cosh\xi\right) \tag{2.3.36}$$

2.3.2 Expansion respect to $\Delta\xi$

Now let's taking only the linear order of Δ_ξ.

$$\Phi(\xi < \xi_2) = \left[\frac{\cosh\xi}{\cosh\xi_2}\left(1 - \frac{\Delta\xi\tanh\xi_2}{2}\right) - 1\right]\cos\phi \tag{2.3.37}$$

$$+ \cosh\xi_2 - \cosh\xi - \frac{1}{2}\Delta\xi\sinh\xi_2 \tag{2.3.38}$$

$$+ \Delta\xi_2\sinh\xi_2 \tag{2.3.39}$$

$$= \left[\frac{\cosh\xi}{\cosh\xi_2}\left(1 - \frac{\Delta\xi\tanh\xi_2}{2}\right) - 1\right]\cos\phi + \cosh\xi_2 - \cosh\xi + \frac{1}{2}\Delta\xi\sinh\xi_2 \tag{2.3.40}$$

The following is the numerical verification of the analytical solution using FFT

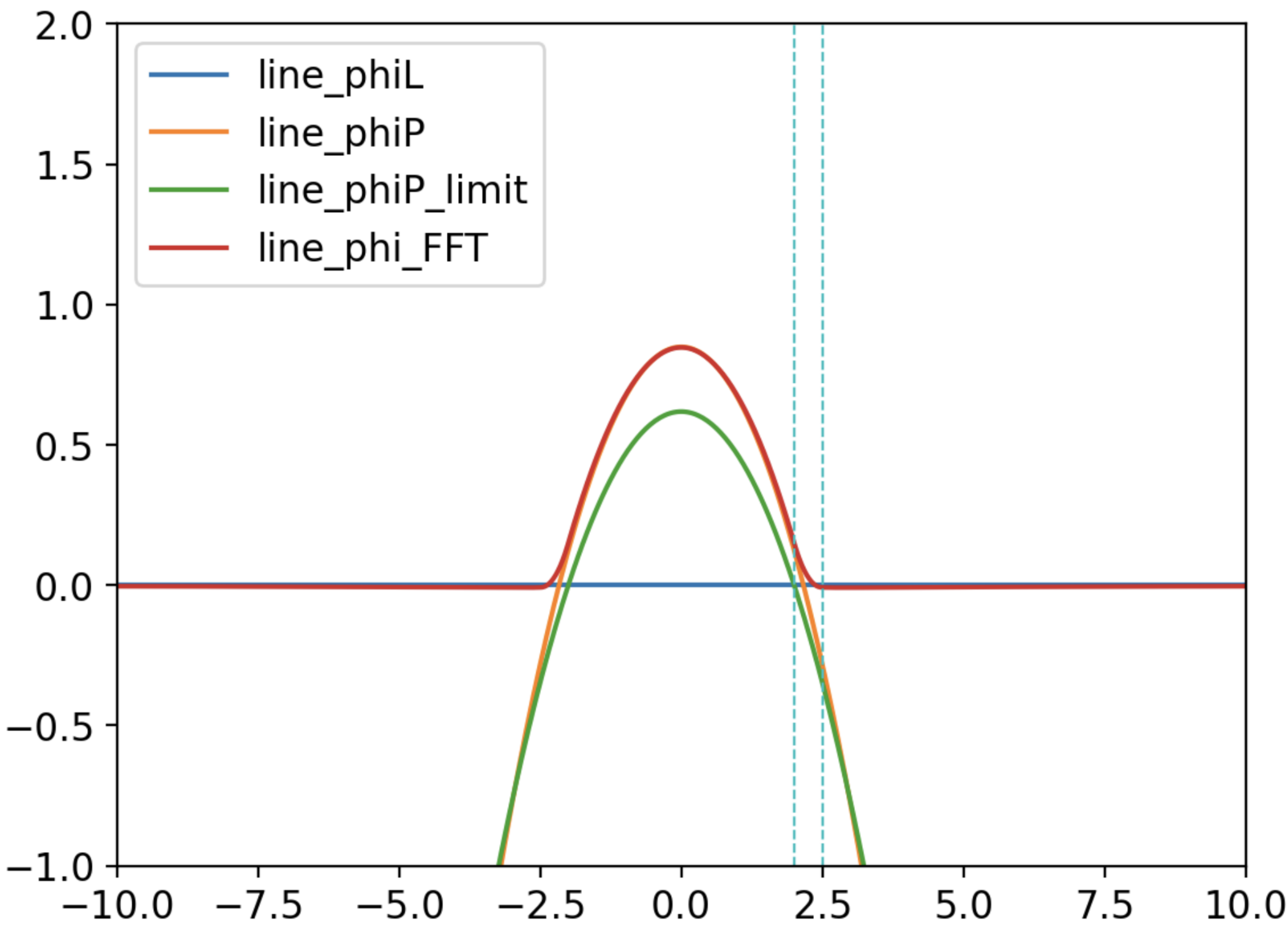

Figure 1: Numerical Verification

The Wake potential is in the similar form by assuming the sheath thickness is small compare to the blowout boundary. It can be transformed into the cartesian coordinate:

$$\psi = -\frac{x^2 b_p^2 + y^2 a_p^2 - a_p^2 b_p^2}{2(a_p^2 + b_p^2)} - \frac{\Delta\mu a_p b_p}{2(a_p^2 + b_p^2)^2}\left[(x^2 - y^2)(a_p^2 - b_p^2) + a_p^4 + b_p^4\right] \tag{2.3.41}$$

$$= \psi_0 + \Delta\mu\psi_{sheath} \tag{2.3.42}$$

The wake field can be calculated as:

$$W_x = -\frac{\partial\psi}{\partial x} = x\left[\frac{b_p^2}{a_p^2 + b_p^2} + \Delta\mu\frac{a_p b_p (a_p^2 - b_p^2)}{(a_p^2 + b_p^2)^2}\right] \tag{2.3.43}$$

$$W_y = -\frac{\partial\psi}{\partial y} = y\left[\frac{a_p^2}{a_p^2 + b_p^2} + \Delta\mu\frac{a_p b_p (b_p^2 - a_p^2)}{(a_p^2 + b_p^2)^2}\right] \tag{2.3.44}$$

$$W_z = \frac{\partial\psi}{\partial\xi} = \frac{a_p b_p}{(a_p^2 + b_p^2)^2}\Big((x^2 - y^2 + b_p^2) b_p a_p' + (-x^2 + y^2 + a_p^2) a_p b_p'\Big) \tag{2.3.45}$$

$$+ \frac{\Delta\mu}{2\left(a_p^2 + b_p^2\right)^3}\Big[a_p' b_p \left(a_p^6 + a_p^4\left(x^2 - y^2 + 5b_p^2\right) + b_p^4\left(x^2 - y^2 + b_p^2\right) - 3a_p^2 b_p^2\left(2x^2 - 2y^2 + b_p^2\right)\right) \tag{2.3.46}$$

$$+ a_p b_p' \left(b_p^6 + a_p^4\left(-x^2 + y^2 + a_p^2\right) + b_p^4\left(-x^2 + y^2 + 5a_p^2\right) - 3a_p^2 b_p^2\left(-2x^2 + 2y^2 + a_p^2\right)\right)\Big] \tag{2.3.47}$$