B555 HW 1

Chapter 1

5) $\Omega = \{x_1 \cdots x_n : x_n = H,\ x_i = H \text{ some } i < n,\ x_j = T\ j \neq i, j \neq n\}$

Let X be the flip giving the 2nd H.

$$P(X = x) = (x-1)\left(\tfrac{1}{2}\right)^x \quad x = 2, 3, \ldots$$

6) The uniform dist must satisfy $f(x) = c$ for $x = 0, 1, \ldots$

If $c = 0$ then $\sum f(x) = 0$

If $c > 0$ then $\sum f(x) = \infty$

So can't satisfy $\sum_x f(x) = 1$.

9) (1) $P(A \mid B) = \dfrac{P(A, B)}{P(B)} > 0$

(2) $P(\Omega \mid B) = \dfrac{P(\Omega, B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(3) Suppose $A_1, A_2, \ldots$ are disjoint events. then

$$P\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

$$= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i \mid B)$$

11) Let A, B be independent. Then since $A^c \cap B^c = (A \cup B)^c$

$$P(A^c \cap B^c) = 1 - [P(A) + P(B) - P(A)P(B)] = (1 - P(A))(1 - P(B))$$
$$= P(A^c)P(B^c)$$

12) Let $C \in \{1,2,3\}$ be chosen card and $S \in \{\text{Red}, \text{Green}\}$ be the color we see. $P(C = i) = 1/3 \quad i = 1, 2, 3$.

$$P(S = \text{Green} \mid C = i) = \begin{cases} 1 & i = 1 \\ 0 & i = 2 \\ 1/2 & i = 3 \end{cases}$$

$$P(C = i \mid S = \text{Green}) = \frac{P(C = i, S = \text{Green})}{P(S = \text{Green})} = \begin{cases} \frac{1/3}{1/2} & i = 1 \\ 0 & i = 2 \\ \frac{1/6}{1/2} & i = 3 \end{cases}$$

$$P(\text{other side Green} \mid S = \text{Green}) = P(C = 1 \mid S = \text{Green}) = 2/3.$$

15) a) Let N = # children with blue eyes $N \sim \text{Binomial}(3, 1/4)$

$$P(N \geq 2 \mid N \geq 1) = 1 - P(N < 2 \mid N \geq 1) = \frac{P(N = 1)}{P(N \geq 1)} = \frac{3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2}{1 - \left(\frac{3}{4}\right)^3}$$

b) The number of blue-eyed children from the remaining 2 children is $\text{Binomial}(2, 1/4)$.

$$P(\text{desired event}) = 1 - (3/4)^2$$

17)

Write $P_C(\cdot)$ for the prob. given C.

$$P(A,B,C) = P(A,B|C)\,P(C) = P_C(A|B)\,P_C(B)\,P(C)$$
$$= P(A|B,C)\,P(B|C)\,P(C).$$

19) $P(W) = .5$; $P(M) = .3$; $P(L) = .2$

$P(V|W) = .82$; $P(V|M) = .65$; $P(V|L) = .5$

$$P(W|V) = \frac{P(W)\,P(V|W)}{P(W)P(V|W) + P(M)P(V|M) + P(L)P(V|L)}$$

$$= \frac{.41}{.41 + .195 + .1} = \frac{.41}{.705}$$

20) a) $$P(H) = \sum_{i=1}^{5} P(C_i)\,P(H|C_i) = \frac{1}{5}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1\right) = 1/2$$

$$P(C_i|H) = \frac{P(C_i)\,P(H|C_i)}{P(H)} = \begin{cases} 0 & i=1 \\ \frac{1}{10} & i=2 \\ \frac{1}{5} & i=3 \\ \frac{3}{10} & i=4 \\ \frac{4}{10} & i=5 \end{cases}$$

b) $$P(H_2|H_1) = 0\cdot 0 + \frac{1}{10}\frac{1}{4} + \frac{2}{10}\frac{1}{2} + \frac{3}{10}\frac{3}{4} + \frac{4}{10}\,1 = 3/4$$

c)
$$P(C_i|B_4) = \frac{P(C_i, B_4)}{P(B_4)} \qquad P(C_i|B_4) = \begin{cases} 0 & i=1 \\ \dfrac{\frac{1}{5}\left(\frac{3}{4}\right)^3\frac{1}{4}}{41/80} & i=2 \\ \dfrac{\frac{1}{5}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)}{41/80} & i=3 \end{cases}$$

$$P(B_4) = \frac{1}{5}\,0 + \frac{1}{5}\left(\frac{3}{4}\right)^3\frac{1}{4} + \frac{1}{5}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + \frac{1}{5}\left(\frac{1}{4}\right)^3\frac{3}{4} + \frac{1}{5}\,0 = \frac{41}{80}$$

7) $P(Z > z) = P(X > z, Y > z) = (1-z)^2$

$$f_Z(z) = \frac{d P(Z \le z)}{dz} = \frac{d\; 1-(1-z)^2}{dz} = 2(1-z) \quad 0 < z < 1$$

11) a) $P(X=0, Y=0) = 0$ yet $P(X=0) = P(Y=0) = 1-p$.

b) Note that $X+Y = N$ so

$$P(X=x, Y=y) = P(X=x, Y=y, N=x+y) = P(X=x \mid N=x+y)\, P(N=x+y)$$

$$= \frac{e^{-\lambda}\lambda^{x+y}}{(x+y)!} \binom{x+y}{x} p^x (1-p)^y$$

$$= \frac{e^{-\lambda}\lambda^x p^x}{x!} \frac{(1-p)^y}{y!}$$

The joint pmf factors into $g(x)h(y)$ so must be independent.
However, can show more directly by computing marginals.

$$P(X=x) = \sum_{n=0}^{\infty} P(X=x \mid N=n) P(N=n) = \sum_{n=0}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \sum_{n=x}^{\infty} \frac{p^x (1-p)^{n-x} e^{-\lambda}\lambda^n}{x!\,(n-x)!}$$

$$= \sum_{n=0}^{\infty} \frac{p^x (1-p)^n e^{-\lambda}\lambda^{x+n}}{x!\, n!}$$

$$= \frac{e^{-p\lambda}(p\lambda)^x}{x!} \underbrace{\sum_{n=0}^{\infty} \frac{e^{-(1-p)\lambda}((1-p)\lambda)^n}{n!}}_{1} = \frac{e^{-p\lambda}(p\lambda)^x}{x!}$$

$X \sim \text{Poisson}(p\lambda)$ and by symmetry $Y \sim \text{Poisson}((1-p)\lambda)$

Indep follows by direct comparison of $P(X=x, Y=y)$ and $P(X=x)P(Y=y)$

14) $F_R(r) = P(R \le r) = r^2 \implies f_R(r) = 2r \quad 0 < r < 1$

15) $Y = F(X)$. Note $0 \le Y < 1$

$P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = y$

(By defn $F^{-1}(y)$ is # s.t. $P(X \le F^{-1}(y)) = y$)

$\implies Y = F(X) \sim Unif(0,1)$

Calculation shows if $X \sim F$ then $F(X) \sim Unif(0,1)$

Reverse calc shows $U \sim Unif(0,1) \implies F^{-1}(U) \sim F$

For $Exp(\beta)$ dist.

$f(x) = \frac{1}{\beta} e^{-x/\beta} \implies F(x) = 1 - e^{-x/\beta}$

$\implies F^{-1}(y) = -\beta \log(1-y)$

So if $U \sim Unif(0,1)$ then $-\beta \log(1-U) \sim Exp(\beta)$

21) Let $Y = \max(X_1 \cdots X_n)$ where $X_1 \cdots X_n \overset{iid}{\sim} Exp(\beta)$

$F_Y(y) = P(Y \le y) = P(X_1 \le y, \cdots X_n \le y) = (1 - e^{-y/\beta})^n$

$\implies f_Y(y) = \frac{n}{\beta}(1 - e^{-y/\beta})^{n-1} e^{-y/\beta} \quad y \geq 0$