Measure Theory Notes

Parker Knight

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1 Introduction

This document contains a collection of notes on measures and integrals, mostly inspired by the text *Elements of Integration* by Bartle. This is primarily a learning tool for myself; I'm currently learning this material in preparation for my PhD program which begins this fall.

In this section I provide some background definitions which may be referenced throughout the notes.

1.1 Background

2 σ-Algebras and Measurable Functions

Loosely speaking, a measure is a nonnegative, countably additive real-valued function defined on a collection of well-behaved sets. Before we can give a more rigorous definition of a measure, we need to examine what is precisely meant by *well-behaved*. This section explores the concept of the σ-algebra, a collection of sets which holds certain properties required for the formal definition of measure. I will also discuss measurable functions, and give examples of both.

2.1 σ-Algebras

Definition 2.1 (σ-algebra)**.** *Let X be any set, and let Σ be a collection of subsets of X. We say that Σ is a σ-algebra if:*

1. $\emptyset, X \in \Sigma$
2. *If $A \in \Sigma$, then $A^c \in \Sigma$*
3. *If (A_n) is a sequence of sets in Σ, then $\bigcup A_n \in \Sigma$*

Defintion 2.1 states that a σ-algebra is closed under taking complements and unions. It follows by De Morgan's laws that every σ-algebra is also closed under taking intersections. Below provides some simple examples.

Example 2.2. *If X is any set, then $\Sigma = \{X, \emptyset\}$ is the trivial σ-algebra.*

Example 2.3. *For any X, the power set of X (the set of all subsets of X) is a σ-algebra.*

Example 2.4. *For any X and $A \subset X$, the set $\Sigma = \{\emptyset, A, A^c, X\}$ is a σ-algebra.*

A useful result is that the intersection of two σ-algebras is itself a σ-algebra.

Proposition 2.5. *Let X be an arbitrary set, and let Σ_1 and Σ_2 be σ-algebras of X. Then $\Sigma_3 = \Sigma_1 \cap \Sigma_2$ is a σ-algeba.*

Proof. The proof is simple. First, we know that $\emptyset, X \in \Sigma_1$ and $\emptyset, X \in \Sigma_2$, so we have $\emptyset, X \in \Sigma_1 \cap \Sigma_2$. Now let $A \in \Sigma_1 \cap \Sigma_2$ be arbitrary. Then $A \in \Sigma_1$ and $A \in \Sigma_2$, and so we have $A^c \in \Sigma_1$ and $A^c \in \Sigma_2$ by Definition 2.1. So $A^c \in \Sigma_1 \cap \Sigma_2$. Finally, if (A_n) is a sequence in $\Sigma_1 \cap \Sigma_2$, then (A_n) is a sequence in Σ_1 and Σ_2 as well. So $\cup An \in \Sigma_1$ and $\cup A_n \in \Sigma_2$, so $\cup A_n \in \Sigma_1 \cap \Sigma_2$. $\square$

3 Measures

4 The Lebesgue Integral