# MEASURE THEORY NOTES

PARKER KNIGHT

## Contents

## 1. $\sigma$-Algebras and Measurable Functions

Loosely speaking, a measure is a nonnegative, countably additive real-valued function defined on a collection of well-behaved sets. Before we can give a more rigorous definition of a measure, we need to examine what is precisely meant by *well-behaved.* This section explores the concept of the $\sigma$-algebra, a collection of sets which holds certain properties required for the formal definition of measure. I will also discuss measurable functions, and give examples of both.

### 1.1. $\sigma$-Algebras.

**Definition 1.1** ($\sigma$-algebra)**.** *Let $X$ be any set, and let $\Sigma$ be a collection of subsets of $X$. We say that $\Sigma$ is a $\sigma$-algebra if:*

1. $\emptyset, X \in \Sigma$
2. *If $A \in \Sigma$, then $A^c \in \Sigma$*
3. *If $(A_n)$ is a sequence of sets in $\Sigma$, then $\bigcup A_n \in \Sigma$*

Defintion 1.1 states that a $\sigma$-algebra is closed under taking complements and unions. It follows by De Morgan's laws that every $\sigma$-algebra is also closed under taking intersections. For a set $X$ and $\sigma$-algebra $\Sigma$, the pair $(X, \Sigma)$ is called a *measurable space.* Below provides some simple examples.

**Example 1.2.** *If $X$ is any set, then $\Sigma = \{X, \emptyset\}$ is the trivial $\sigma$-algebra.*

**Example 1.3.** *For any $X$, the power set of $X$ (the set of all subsets of $X$) is a $\sigma$-algebra.*

**Example 1.4.** *For any $X$ and $A \subset X$, the set $\Sigma = \{\emptyset, A, A^c, X\}$ is a $\sigma$-algebra.*

A useful result is that the intersection of two $\sigma$-algebras is itself a $\sigma$-algebra.

**Proposition 1.5.** *Let $X$ be an arbitrary set, and let $\Sigma_1$ and $\Sigma_2$ be $\sigma$-algebras of $X$. Then $\Sigma_3 = \Sigma_1 \cap \Sigma_2$ is a $\sigma$-algeba.*

*Proof.* The proof is simple. First, we know that $\emptyset, X \in \Sigma_1$ and $\emptyset, X \in \Sigma_2$, so we have $\emptyset, X \in \Sigma_1 \cap \Sigma_2$. Now let $A \in \Sigma_1 \cap \Sigma_2$ be arbitrary. Then $A \in \Sigma_1$ and $A \in \Sigma_2$, and so we have $A^c \in \Sigma_1$ and $A^c \in \Sigma_2$ by Definition 1.1. So $A^c \in \Sigma_1 \cap \Sigma_2$. Finally, if $(A_n)$ is a sequence in $\Sigma_1 \cap \Sigma_2$, then $(A_n)$ is a sequence in $\Sigma_1$ and $\Sigma_2$ as well. So $\cup A_n \in \Sigma_1$ and $\cup A_n \in \Sigma_2$, so $\cup A_n \in \Sigma_1 \cap \Sigma_2$. $\square$

Proposition 1.5 yields the following definition.

**Definition 1.6.** *For any set $X$ and a collection of subsets $\mathcal{A}$ of $X$, the $\sigma$-algebra generated by $\mathcal{A}$ is the intersection of all $\sigma$-algebras that contain $\mathcal{A}$. By Proposition 1.5, this intersection is itself a $\sigma$-algebra.*

We can also think of the $\sigma$-algebra generated by a collection of subsets $\mathcal{A}$ as the *smallest* $\sigma$-algebra containing $\mathcal{A}$.

**Proposition 1.7.** *If $\mathcal{A}$ is a collection of subsets of $X$ and $\Sigma$ is the $\sigma$-algebra generated by $\mathcal{A}$, then every element of $\Sigma$ can be constructed as a countable union, intersection, or complement of elements of $\mathcal{A}$.*

The proof of this proposition is clear: the set of countable unions, intersections, and complements of $\mathcal{A}$ is $\sigma$-algebra that contains $\mathcal{A}$ as long as it also contains $X$ and $\emptyset$, and so the $\sigma$-algebra generated by $\mathcal{A}$ must be a subset of this set by definition.

**Definition 1.8** (Borel algebra)**.** *Let $X = \mathbb{R}$. The Borel $\sigma$-algebra $\mathcal{B}$ is the $\sigma$-algebra generated by all open intervals of the form $(a, b) \subset \mathbb{R}$. The elements of $\mathcal{B}$ are called Borel sets.*

We can show that $\mathcal{B}$ is also generated by all closed intervals, as well as half-open intervals $(a, b]$ and half-rays $(a, \infty)$.

**Proposition 1.9.** *The Borel algebra $\mathcal{B}$ is also generated by*

(1) *The closed intervals* $[a, b]$
(2) *The half-open intervals* $(a, b]$
(3) *The half-rays* $(a, \infty)$

*Proof.* For the proof of (1), let $\mathcal{B}'$ denote the $\sigma$-algebra generated by closed intervals. Notice that we can write any closed interval $[a, b]$ as $[a, b] = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$. So $[a, b]$ is an intersection of open intervals, and thus $[a, b] \in \mathcal{B}$. Since every element of $\mathcal{B}'$ can be constructed by countable union, intersection, or complement of closed intervals by Proposition 1.7 and $\mathcal{B}$ is closed under these operations, it follows that every element of $\mathcal{B}'$ is in $\mathcal{B}$, and thus $\mathcal{B}' \subseteq \mathcal{B}$. Similarly, we show that $\mathcal{B} \subseteq \mathcal{B}'$ by noting that $(a, b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$ and applying the same logic. Thus $\mathcal{B}' = \mathcal{B}$, and so the Borel algebra is generated by closed intervals.

The proofs of (2) and (3) follow similarly. □

## 1.2. **Measurable functions.**

**Definition 1.10** (Measurable function)**.** *Let $(X, \Sigma_1)$ and $(Y, \Sigma_2)$ be measurable spaces, and let $f : X \to Y$ be a function. We say that $f$ is a measurable function if*

$$f^{-1}(E) = \{x \in X : f(x) \in E\}$$

*is in $\Sigma_1$ for every $E \in \Sigma_2$.*

In other words, measurable functions *pull back* measurable sets in $\Sigma_2$ to sets in $\Sigma_1$. Note the similarity to the definition of continuous functions between metric spaces. We can give a simpler characterization of measurable functions when $(Y, \Sigma_2) = (\mathbb{R}, \mathcal{B})$.

**Proposition 1.11.** *Let $(X, \Sigma)$ be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if for every $\alpha > 0$, the set*

$$\{x \in X : f(x) > \alpha\}$$

*is in $\Sigma$.*

## 2. Measures

## 3. The Lebesgue Integral