MEASURE THEORY NOTES

PARKER KNIGHT

Contents

1. σ-Algebras and Measurable Functions

Loosely speaking, a measure is a nonnegative, countably additive real-valued function defined on a collection of well-behaved sets. Before we can give a more rigorous definition of a measure, we need to examine what is precisely meant by *well-behaved.* This section explores the concept of the σ-algebra, a collection of sets which holds certain properties required for the formal definition of measure. I will also discuss measurable functions, and give examples of both.

1.1. σ-Algebras.

Definition 1.1 (σ-algebra)**.** *Let X be any set, and let Σ be a collection of subsets of X. We say that Σ is a σ-algebra if:*

(1) $\emptyset, X \in \Sigma$
(2) *If $A \in \Sigma$, then $A^c \in \Sigma$*
(3) *If (A_n) is a sequence of sets in Σ, then $\bigcup A_n \in \Sigma$*

Defintion 1.1 states that a σ-algebra is closed under taking complements and unions. It follows by De Morgan's laws that every σ-algebra is also closed under taking intersections. For a set X and σ-algebra Σ, the pair (X, Σ) is called a *measurable space.* Below provides some simple examples.

Example 1.2. *If X is any set, then $\Sigma = \{X, \emptyset\}$ is the trivial σ-algebra.*

Example 1.3. *For any X, the power set of X (the set of all subsets of X) is a σ-algebra.*

Example 1.4. *For any X and $A \subset X$, the set $\Sigma = \{\emptyset, A, A^c, X\}$ is a σ-algebra.*

A useful result is that the intersection of two σ-algebras is itself a σ-algebra.

Proposition 1.5. *Let X be an arbitrary set, and let Σ_1 and Σ_2 be σ-algebras of X. Then $\Sigma_3 = \Sigma_1 \cap \Sigma_2$ is a σ-algeba.*

Proof. The proof is simple. First, we know that $\emptyset, X \in \Sigma_1$ and $\emptyset, X \in \Sigma_2$, so we have $\emptyset, X \in \Sigma_1 \cap \Sigma_2$. Now let $A \in \Sigma_1 \cap \Sigma_2$ be arbitrary. Then $A \in \Sigma_1$ and $A \in \Sigma_2$, and so we have $A^c \in \Sigma_1$ and $A^c \in \Sigma_2$ by Definition 1.1. So $A^c \in \Sigma_1 \cap \Sigma_2$. Finally, if (A_n) is a sequence in $\Sigma_1 \cap \Sigma_2$, then (A_n) is a sequence in Σ_1 and Σ_2 as well. So $\cup A_n \in \Sigma_1$ and $\cup A_n \in \Sigma_2$, so $\cup A_n \in \Sigma_1 \cap \Sigma_2$. $\square$

Proposition 1.5 yields the following definition.

Definition 1.6. *For any set X and a collection of subsets $\mathcal{A}$ of X, the σ-algebra generated by $\mathcal{A}$ is the intersection of all σ-algebras that contain $\mathcal{A}$. By Proposition 1.5, this intersection is itself a σ-algebra.*

We can also think of the σ-algebra generated by a collection of subsets $\mathcal{A}$ as the *smallest* σ-algebra containing $\mathcal{A}$.

Proposition 1.7. *If $\mathcal{A}$ is a collection of subsets of X and Σ is the σ-algebra generated by $\mathcal{A}$, then every element of Σ can be constructed as a countable union, intersection, or complement of elements of $\mathcal{A}$.*

The proof of this proposition is clear: the set of countable unions, intersections, and complements of $\mathcal{A}$ is σ-algebra that contains $\mathcal{A}$ as long as it also contains X and $\emptyset$, and so the σ-algebra generated by $\mathcal{A}$ must be a subset of this set by definition.

Definition 1.8 (Borel algebra)**.** *Let $X = \mathbb{R}$. The Borel σ-algebra $\mathcal{B}$ is the σ-algebra generated by all open intervals of the form $(a, b) \subset \mathbb{R}$. The elements of $\mathcal{B}$ are called Borel sets.*

We can show that $\mathcal{B}$ is also generated by all closed intervals, as well as half-open intervals $(a, b]$ and half-rays (a, ∞).

Proposition 1.9. *The Borel algebra $\mathcal{B}$ is also generated by*

1. *The closed intervals $[a, b]$*
2. *The half-open intervals $(a, b]$*
3. *The half-rays (a, ∞)*

Proof. For the proof of (1), let $\mathcal{B}'$ denote the σ-algebra generated by closed intervals. Notice that we can write any closed interval $[a, b]$ as $[a, b] = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$. So $[a, b]$ is an intersection of open intervals, and thus $[a, b] \in \mathcal{B}$. Since every element of $\mathcal{B}'$ can be constructed by countable union, intersection, or complement of closed intervals by Proposition 1.7 and $\mathcal{B}$ is closed under these operations, it follows that every element of $\mathcal{B}'$ is in $\mathcal{B}$, and thus $\mathcal{B}' \subseteq \mathcal{B}$. Similarly, we show that $\mathcal{B} \subseteq \mathcal{B}'$ by noting that $(a, b) = \cup_{n=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$ and applying the same logic. Thus $\mathcal{B}' = \mathcal{B}$, and so the Borel algebra is generated by closed intervals.

The proofs of (2) and (3) follow similarly. $\square$

1.2. **Measurable functions.**

Definition 1.10 (Measurable function)**.** *Let (X, Σ_1) and (Y, Σ_2) be measurable spaces, and let $f : X \to Y$ be a function. We say that f is a measurable function if*

$$f^{-1}(E) = \{x \in X : f(x) \in E\}$$

is in Σ_1 for every $E \in \Sigma_2$.

In other words, measurable functions *pull back* measurable sets in Σ_2 to sets in Σ_1. Note the similarity to the definition of continuous functions between metric spaces. We can give a simpler characterization of measurable functions when $(Y, \Sigma_2) = (\mathbb{R}, \mathcal{B})$, but we first need a pair of preliminary results.

Lemma 1.11. *Let (X, Σ) be a measurable space and let $f : X \to Y$. Then $T = \{E \subseteq Y : f^{-1}(E) \in \Sigma\}$ is a σ-algebra.*

Proof. Clearly $\emptyset, Y \in T$ since $f^{-1}(\emptyset) = \emptyset \in \Sigma$ and $f^{-1}(Y) = X \in \Sigma$. Now let $E \in T$ be arbitrary. Then $f^{-1}(E) \in \Sigma$, so $f^{-1}(E)^c = \{x \in X : f(x) \notin E\} = f^{-1}(E^c) \in \Sigma$. So $E^c \in T$. Finally let (E_n) be a sequence of sets in T. Then $(f^{-1}(E_n))$ is a sequence in Σ, and so $\cup f^{-1}(E_n) \in \Sigma$. Notice that

$$\bigcup f^{-1}(E_n) = \{x \in X : f(x) \in E_n \text{ for some n}\} = f^{-1}(\cup E_n)$$

So $f^{-1}(\cup E_n) \in \Sigma$, so $\cup E_n \in T$. So T is a σ-algebra.

$\square$

Lemma 1.12. *Let (X, Σ) be a measurable space and let $f : X \to Y$. Let $\mathcal{A}$ be collection of subsets of Y such that $f^{-1}(E) \in \Sigma$ for every $E \in \mathcal{A}$. Then $f^{-1}(F) \in \Sigma$ for every F in the σ-algebra generated by $\mathcal{A}$.*

Proof. Let $\Sigma_{\mathcal{A}}$ denote the σ-algebra generated by $\mathcal{A}$ and let $T = \{E \subseteq Y : f^{-1}(E) \in \Sigma\}$. Then $\mathcal{A} \subset T$ by hypothesis. Since we know by the previous lemma that T is a σ-algebra, then $\Sigma_{\mathcal{A}} \subset T$. So for every $F \in \Sigma_{\mathcal{A}}$, we know $F \in T$, and so $f^{-1}(F) \in \Sigma$. □

Now we can state the characterization of measurable functions from X to $\mathbb{R}$.

Proposition 1.13. *Let (X, Σ) be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if for every $\alpha > 0$, the set*

$$\{x \in X : f(x) > \alpha\}$$

is in Σ.

Proof. Recalling that the half-rays of the form (α, ∞) generate the Borel sets $\mathcal{B}$, it follows that $f^{-1}(E) \in \Sigma$ for every Borel set E by Lemma 1.12. So f is measurable by definition. □

In fact, the condition in Proposition 1.13 can be generalized to any set that generates $\mathcal{B}$, such as the open intervals. Now we provide some examples of measurable functions.

Example 1.14. *A constant function $f : (X, \Sigma_1) \to (Y, \Sigma_2)$ is measurable. To see this for say $f(x) = y$, notice for any $E \in \Sigma_2$ that $f^{-1}(E) = X$ if $y \in E$ and $f^{-1}(E) = \emptyset$ if $y \notin E$. Clearly $X, \emptyset \in \Sigma_1$, so f is measurable.*

Example 1.15. *If $E \in \Sigma$, then define 1_E to be the indicator (or characteristic) function of E; that is, $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise. The function $1_E : (X, \Sigma) \to (\mathbb{R}, \mathcal{B})$ is measurable.*

Example 1.16. *Any continuous function $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ is measurable: recall that the preimage of any open set under a continuous function is open.*

Lemma 1.17. *If f and g are measurable real-valued functions and $c \in \mathbb{R}$, then the following functions are measurable:*

(1) fc
(2) $f + g$
(3) fg
(4) $|f|$

For convenience, let $M(X, \Sigma)$ denote the set of measurable functions $f : X \to \bar{\mathbb{R}}$. The consideration of extended-real-valued functions is useful when dealing with limits and suprema of sequences, as in the next lemma.

Lemma 1.18. *Let (f_n) be a sequence in $M(X, \Sigma)$. Then the following functions are measurable:*

(1) $f(x) = \inf f_n(x)$
(2) $F(x) = \sup f_n(x)$
(3) $f^*(x) = \liminf f_n(x)$
(4) $F^*(x) = \limsup f_n(x)$

Proof. □

2. Measures

3. The Lebesgue Integral