# SPECTRAL PERTURBATION THEORY

NOTES BY PARKER KNIGHT

## 1. Introduction

This set of notes will closely follow the second chapter of [1]. My aim is to describe some useful matrix perturbation theorems, give their proofs, and describe an application of the theory to a problem in statistics.

The general problem is defined as follows: suppose there exists some matrix of interest $M^*$, and we observe a perturbed version $M = M^* + E$ where $E$ is a perturbation matrix. We are interested in characterizing how the spectral properties of $M^*$ (i.e. the eigenspace or singular subspace) change in light of the perturbation $E$. To do so, we will describe the classic Davis-Kahan $\sin \Theta$ theorem for symmetric matrices, and Wedin's extension to general matrices. But first, we describe various metrics for describing the distance between subspaces.

Throughout the note set, we let $\mathcal{U}^*$ and $\mathcal{U}$ denote two $r$ dimensional subspaces in $\mathbb{R}^n$. Let $U^*$ and $U$ be matrices in $\mathbb{R}^{n \times r}$ whose columns form an orthonormal basis for $\mathcal{U}^*$ and $\mathcal{U}$ respectively. For any matrix $A$, let $A_\perp$ denote its orthogonal compliment. Let $\mathcal{O}^{r \times r}$ denote the set of orthogonal matrices in $\mathbb{R}^{r \times r}$.

## 2. Distance between subspaces

A key challenge in describing the distance between subspaces is the notion of rotational ambiguity, namely for for any rotation matrix $R \in R^{r \times r}$, we have that $UR$ is also an orthogonal basis for $\mathcal{U}$. So, even when $\mathcal{U}^* = \mathcal{U}$, we may have $|||U - U^*||| \neq 0$ for our matrix norm of choice $|||.|||$, depending on how the bases are rotated. Any useful distance metric on subspaces much account for this rotational ambiguity. Following the approach of [1], we describe a few difference useful choices of metric.

### 2.1. Distance with optimal rotation.

A natural approach to addressing the rotational invariance problem is to simply choose the rotation of $U$ which is closest in norm to $U^*$. This yields the following distance metric:

$$\text{dist}(U, U^*) := \min_{R \in \mathcal{O}^{r \times r}} |||UR - U^*|||$$

### 2.2. Distance between projections.

Recall that the projection onto $\mathcal{U}$ is given by $UU^T$. A useful fact is that this projection matrix is unchanged by its rotation: for any $R \in \mathcal{O}^{r \times r}$, we have $UR(UR)^T = URR^TU^T = UU^T$. This motivates the following metric between subspaces:

$$\text{dist}_p(U, U^*) := |||UU^T - U^*U^{*T}|||$$

### 2.3. Distance via principal angles.

## 3. Davis-Kahan

## 4. Wedin

## 5. An application

## References

[1] Y. Chen, Y. Chi, J. Fan, and C. Ma, "Spectral methods for data science: A statistical perspective," vol. 14, no. 5, pp. 566–806.