On Reduced Rank Regression

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1 Introduction

Consider the multivariate linear model

$$\mathbb{Y} = \mathbb{X}B + E$$

where $\mathbb{Y}, E \in \mathbb{R}^{n \times q}$, $\mathbb{X} \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{p \times q}$. In other words, we have n samples, q distinct outcomes, and p features or covariates. For now, we assume $n \geq p$ and $\mathbb{X}$ is full rank. We can estimate B via least squares:

$$\begin{aligned}\hat{B}_{OLS} &= \arg\min_B \|\mathbb{Y} - \mathbb{X}B\|_F^2 \\ &= (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{Y}\end{aligned}$$

Note that this is equivalent to running q separate univariate regression, and combining the coefficient estimates into a matrix. This fails to share information across each of the q regression problems.

2 Reduced Rank Regression

One possible hypothesis is that the signal in B is driven by a small number of latent factors; i.e., B is low rank. This motivates *reduced rank regression*, in which we estimate B by solving the following:

$$\hat{B}_{RRR}(k) = \arg\min_{B:\text{rank}(B)\leq k} \|\mathbb{Y} - \mathbb{X}B\|_F^2$$

We will see that a convenient, closed form solution exists.