Fixed Income Derivatives:
Basis Spreads / Dual Curve Stripping

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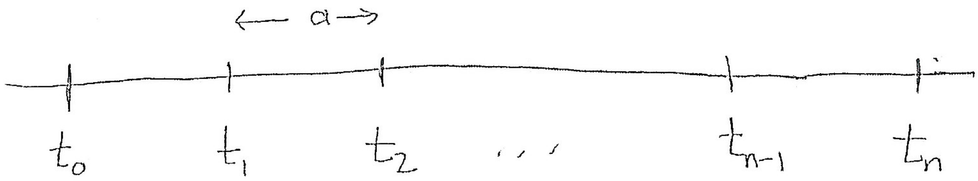
Arbitrage



- $R^{Libor} \approx R^{Libid} + 12.5\text{bps}$

$$\text{deposit:} \quad \left.\begin{array}{ll} \alpha_j R_j^{Libid} & \text{at } t_j \\ 1 + \alpha_n R_n^{Libid} & \text{at } t_n \end{array}\right\} \quad \text{worth \$1?}$$

$$\text{borrow:} \quad \left.\begin{array}{ll} \alpha_j R_j^{Libor} & \text{at } t_j \\ 1 + \alpha_n R_n^{Libor} & \text{at } t_n \end{array}\right\} \quad \text{worth \$1?}$$

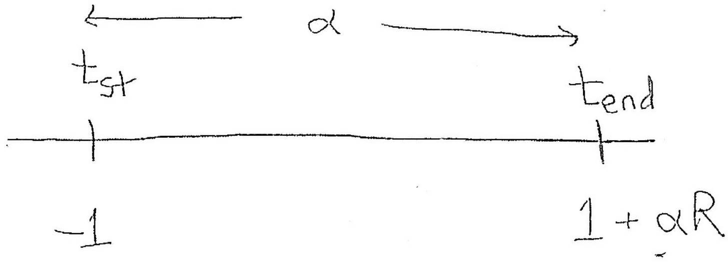
- Both can't be right !!

  bid/ask spread

$$R^{flt} = R^{Libid} + m \qquad \text{sums to \$1,}$$

$$\text{for some } 0 \leq m \leq 12.5\text{bps}$$

Basis spreads



- *Fair* or *true* rate:

$$(1+\alpha R)Z(t;t_{end}) - Z(t;t_{st}) = 0$$

$$R^{tr}(t;t_{st},t_{end}) = \frac{Z(t;t_{st}) - Z(t;t_{end})}{\alpha Z(t;t_{end})} = \textit{true rate} \text{ at } t$$

$$R_0^{tr}(t_{st},t_{end}) = \frac{D(t_{st}) - D(t_{end})}{\alpha D(t_{end})} = \textit{today's true rate}$$

- Floating rates R^A *(3m USD Libor, EONIA, ...)*

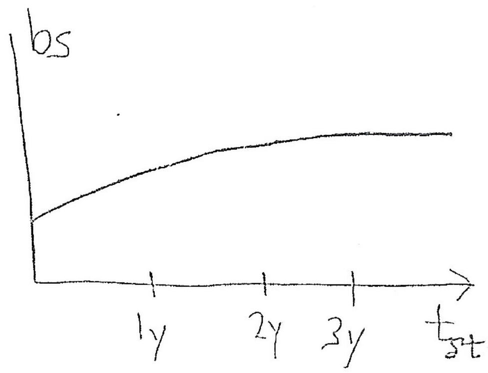
$$R^A(t;t_{st},t_{end}) = R^{tr}(t;t_{st},t_{end}) + bs^A(t_{st})$$

– $bs^A(t_{st})$ = *forward basis spread curve* for flt rate A

– t_{end} is unneeded

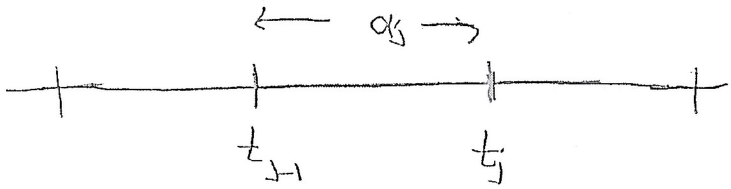
– basis spreads curves usually small

– treated as static



Valuing floating legs

$$\left.\begin{array}{ll} \alpha_j R_j^A & \text{at } t_j \\ 1+\alpha_n R_n^A & \text{at } t_n \end{array}\right\}$$

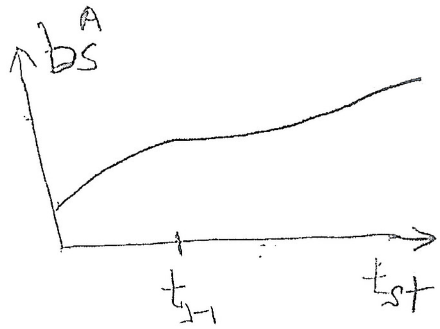


$$V(t) = \sum_{j=1}^{n} \alpha_j R^A(t; t_{j-1}, t_j) Z(t; t_j) + Z(t; t_n)$$

$$= \sum_{j=1}^{n} \alpha_j R^{tr}(t; t_{j-1}, t_j) Z(t; t_j) + Z(t; t_n) + \sum_{j=1}^{n} \alpha_j bs^A(t_{j-1}) Z(t; t_j)$$

$$= \sum_{j=1}^{n} [Z(t; t_{j-1}) - Z(t; t_j)] + Z(t; t_n) + \sum_{j=1}^{n} \alpha_j bs^A(t_{j-1}) Z(t; t_j)$$

$$= Z(t; t_0) + \sum_{j=1}^{n} \alpha_j bs^A(t_{j-1}) Z(t; t_j)$$



\$1 at t_0

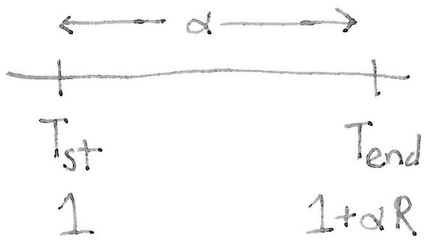
- $V(0) = D(t_0) + \sum_{j=1}^{n} \alpha_j bs^A(t_{j-1}) D(t_j)$

- Effective margins: most deals pay $R_j^A + m$

$$R_j^A + m = R_j^{true} + m_j^{eff},$$

$$m_j^{eff} = m + bs^A(t_{j-1})$$

Projection curves



- Need to handle basis spreads industrially

– discount curves / projection curves

– $Z_d(t;T)$ = discount curve

$$R_{tr}(t;T_{st},T_{end}) = \frac{Z_d(t;T_{st}) - Z_d(t;T_{end})}{\alpha Z_d(t;T_{end})}$$

– $Z_A(t;T)$ = projection curve for rate "A"
(EONIA, 1m USD Libor, ...)

$$R_A(t;T_{st},T_{end}) = \frac{Z_A(t;T_{st}) - Z_A(t;T_{end})}{\alpha Z_A(t;T_{end})}$$

- Model: $Z_A(t;T) = Z_d(t;T) \cdot Y_A(T)$

today:

$$Z_d(0;T) = D_d(T) = e^{-\int_0^T f_0(T')dT'}$$

$$Z_d(0;T) = D_A(T) = D_d(T)Y_A(T) = e^{-\int_0^T f_0(T')dT'} e^{-\int_0^T y_A(T')dT'}$$

Stripping basis swaps

• Basis swaps:

flt leg $R_A \Leftrightarrow$ flt leg $R_B + M$ for N yrs

USD 1m Libor ⇔ USD 3m Libor + M

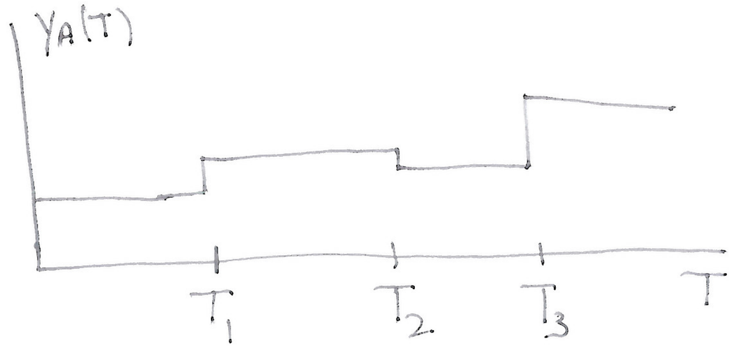
tenor	spread	
1yr	10 bp	T_1
2yr	13 bp	T_2
3yr	14 bp	T_3
5yr	14.5 bp	T_4
10yr	14 bp	T_5

$$\sum_{j=1}^{m} \alpha_j R_A(t_j) D_d(t_j) = \sum_{j=1}^{m} \tilde{\alpha}_j [R_B(\tilde{t}_j) + M_k] D_d(\tilde{t}_j)$$

$$R_A(t_j) = \frac{D_A(t_{j-1}) - D_A(t_j)}{\alpha_j D_A(t_j)} = \frac{D_d(t_{j-1}) E_A(t_{j-1}) - D_d(t_j) E_A(t_j)}{\alpha_j D_d(t_j) E_A(t_j)}$$

– if we know $E_B(T)$, can strip basis swaps to find $E_A(T)$

– $E_A(T) = e^{-\int_0^T y_A(T')dT'}$



– same code !!

Neutral curve

$$D_A(T) = e^{-\int_0^T f(T')dT'} e^{-\int_0^T y_A(T')dT'}$$

- Which rateA has $y_A(T) \equiv 0$? Which circumstances?

- *Old era:* USD 3m Libor has $y(T) \equiv 0$

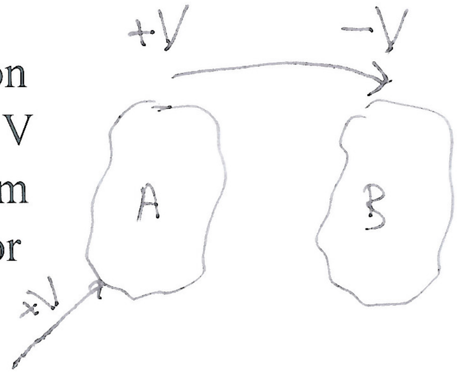
  USD 3m Libor is the neutral or funding curve

– use x-ccy basis swaps

  3m USD Libor $\Leftrightarrow$ *3m Ccy X-ibor* $+ m$ for N yrs

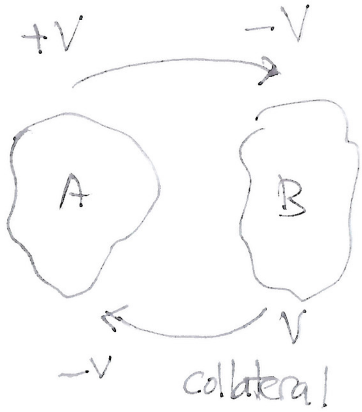
– then same ccy basis swaps

  n month Ccy X-ibor $\Leftrightarrow$ *3m Ccy X ibor* $+ \tilde{m}$ for N yrs



- *Old era:* Suppose bank A has +V position with Bank B. Then bank A has to borrow V to support position. Cost of financing is 3m Libor, so discount factor must be 3m Libor

  risk of liquidity squeezes / rising cost of funds borne by bank A, which has effectively loaned V to bank B



- *New era:* All deals between banks and between CCHs are collateralized. If bank A has +V position with Bank B, then bank B provides collateral (cash) to bank A. Bank A has to pay interest on collateral at overnight rate to bank B, so we must discount at OIS rates

Dual curve discounting

$$D_O^X(T) = e^{-\int_0^T f_0(T')dT'}$$ = dis. factor, ccy X, collateralized

$$D_{3m}^X(T) = e^{-\int_0^T f_0(T')dT} e^{-\int_0^T y_{3m}(T')dT'}$$ = proj. curve, 3m Libor

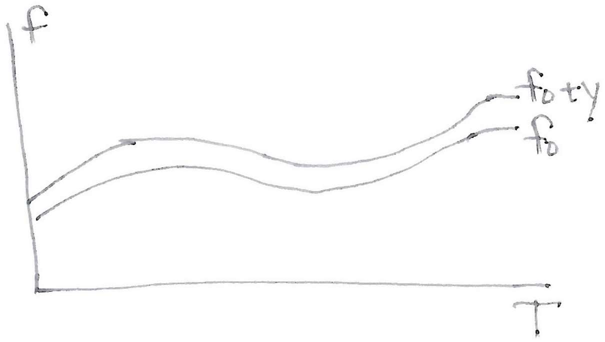
3m Libor + s_{OIS} $\Leftrightarrow$ compnded OIS, 3m Libor $\Leftrightarrow$ fixed

T_k	s_{OIS}	R_{fix}
1y	–25.0 bps	2.03%
2y	–30.0 bps	2.12%
3y	–32.5 bps	2.14%
5y	–34.0 bps	2.13%
7y	–35.0 bps	2.15%
10y	–36.0 bps	2.10%

$$\underbrace{\sum_{j=1}^{m} D_O^X(t_j)\alpha_j[R_{3m}(t_j) + s_{OIS}]}_{L_{3m}+s_{OIS}\text{ leg}} = \underbrace{D_O^X(t_0) - D_O^X(t_m)}_{\textit{compounded OIS leg}}$$

$$\underbrace{\sum_{j=1}^{m} D_O^X(t_j)\alpha_j R_{3m}(t_j)}_{L_{3m}\text{ leg}} = \underbrace{R_{fix}\sum_{j=1}^{m} \alpha_j D_O^X(t_j)}_{\textit{fixed leg}}$$

$$R_{3m}(t_j) = \frac{D_{3m}^X(t_{j-1}) - D_{3m}^X(t_j)}{\alpha_j D_{3m}^X(t_j)}$$



- Obtain $y_B(T)$ for other rates B by stripping $R_{3m} + s_B \Leftrightarrow$ rate B swaps

xCcy Discounting / Basis spreads

- x-ccy basis swap

  3m Libor (ccy X) $\Leftrightarrow$ 3m Libor + $s_{X,Y}$ (ccy Y) for N years

  also exchange initial & final notional

tenor	$s^{X,Y}$	
1 yr	10 bp	T_1
2 yr	13 bp	T_2
$\vdots$	$\vdots$	$\vdots$
20 yr	14.5 bp	T_4

- Let $D^{Y,X}(T)$ = dis. fact. in ccy Y when collateral in ccy X

$$\left\{-D^{Y,X}(t_0) + \sum_{j=1}^{m} \alpha_j [R_{3m}^{Y}(t_j) + s_{Y,X}] D^{X,Y}(t_j) + D^{Y,X}(t_m)\right\} FX(T_0)$$

$$= -D_O^X(t_j) + \sum_{j=1}^{m} \alpha_j R_{3m}^X(t_j) D_O^X(t_j) + D_O^X(t_m)$$

- Strip x-ccy basis spreads to obtain

$$D^{Y,X}(T) = D_O^Y(T) e^{-\int_0^T y_{X,Y}(T')dT'}$$ = df for ccy Y with collateral in ccy X

- FX fwds: $FX^{Y,X}(T)$ = value of 1 unit of ccy Y in ccy X, with collateral in ccy X

$$FX^{Y,X}(T) \cdot \frac{D_O^X(T)}{D_O^X(T_0)} = FX^{Y,X}(T_0) \cdot \frac{D^{Y,X}(T)}{D^{Y,X}(T_0)}$$

$$\frac{FX^{Y,X}(T)}{FX^{Y,X}(T_0)} = \frac{D^{Y,X}(T)}{D^{Y,X}(T_0)} \Big/ \frac{D_O^X(T)}{D_O^X(T_0)}$$