Homework 1

Patryk Kozlowski
Ph 12c

August 21, 2023

Problem 1: Sharply-peaked functions

(a)

We begin by considering:

$$\Gamma(1) = \int_0^\infty \frac{dx}{x} x^1 e^{-x} = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 0 - (-1) = 1$$

So, we have shown that $\Gamma(1) = 1 = 0!$. Next, we consider:

$$\Gamma(n+1) = \int_0^\infty \frac{dx}{x} x^{n+1} e^{-x} = \int_0^\infty x^n e^{-x} dx$$

Using integration by parts with

$$u = x^n \rightarrow du = nx^{n-1}$$

$$dv = e^{-x} \rightarrow v = -e^{-x}$$

we get that:

$$\Gamma(n+1) = -x^n e^{-x}\Big|_0^\infty + \int_0^\infty nx^{n-1} e^{-x} dx$$

In the first term

$$\lim_{x\to\infty} -x^n e^{-x} = 0$$

because the exponential term dominates the polynomial term as $x \to \infty$ for a finite n and, more trivially,

$$\lim_{x\to 0} -x^n e^{-x} = 0$$

Thus, $-x^n e^{-x}\Big|_0^\infty = 0$ and simplify to get that:

$$\Gamma(n+1) = \int_0^\infty n x^{n-1} e^{-x} dx = n \int_0^\infty \frac{dx}{x} x^n e^{-x} = n\Gamma(n)$$

By the previous identity and the previously proven base case with n=0, we have shown that $n! = \Gamma(n+1)$ with $\Gamma(n) \equiv \int_0^\infty \frac{dx}{x} x^n e^{-x}$.

(b)

We start by maximizing the integrand with respect to x to find the point x_0:

$$0 = \frac{\partial}{\partial x_0}[x_0^n e^{-x_0}] = n x_0^{n-1} e^{-x_0} - x_0^n e^{-x_0} = e^{-x_0}(n x_0^{n-1} - x_0^n)$$

It follows that

$$n x_0^{n-1} - x_0^n = 0 \to x_0 = n$$

To verify that this x_0 is indeed maximizing, we consider second-order conditions and using Mathematica, we find that:

$$\frac{\partial^2}{\partial x^2}[x^n e^{-x}]\Big|_n = -e^{-n} n^{n-1}$$

Since $n!$ is defined $\forall n \in \mathbb{Z}^+$, both e^{-n} and n^{n-1} are always positive and so we get that

$$\frac{\partial^2}{\partial x^2}[x^n e^{-x}]\Big|_n < 0$$

enabling us to conclude that the point $x_0 = n$ maximizes the integrand $\forall n \in \mathbb{Z}^+$. Now, expanding the quantity in the exponential around $x_0 = n$ using Mathematica, we get that:

$$n\log(x) - x = -n + n\log(n) - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} - \frac{y^6}{6n^5} + ...$$

where $y = x - x_0 = x - n$ and we have $a_2 = \frac{1}{2n} > 0$. Inserting this expansion into the integrand, we get that:

$$\Gamma(n+1) = \int_{-n}^\infty exp(-n + n\log(n) - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} - \frac{y^6}{6n^5} + ...)dy$$

We note that since we have changed the variable of integration from $x \rightarrow y = x - n$ we must change the bounds of the integral as follows:

$$Upper : \infty \rightarrow \infty - n$$

$$Lower : 0 \rightarrow \; -n$$

However, because for finite n, $\infty - n \approx \infty$, ∞ remains the upper bound for the integral following the change of variables.

(c)

Because of the 2nd order (and higher) powers in n present in the denominators of the $a_k y^k$ terms for $k > 2$, in the large-n limit, these terms are highly suppressed. Therefore, we Taylor expand the part of the exponential containing these terms. Setting $m = \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} - \frac{y^6}{6n^5} + ...$, we get that:

$$\Gamma(n+1) = \int_{-n}^{\infty} dy \exp\left(-n + n\log(n) - \frac{y^2}{2n}\right)(1 + m + \frac{m^2}{2!} + ...)$$

$$\Gamma(n+1) = \int_{-n}^{\infty} dy e^{a_0 - a_2 y^2}(1 + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} + (\frac{1}{6n^5} + (\frac{1}{3n^2})^2/2)y^6 + O(y^7))$$

(d)

Now, when n is large, we concluded earlier that a_k terms for $k > 2$ are highly suppressed. It is these terms, however, that would, for small n, make the largest contributions to the integrand for values of y far from the integrand's maximum at $x_0 = n$. Because these terms are suppressed and the 0th, 1st, and 2nd order terms in y do not make large contributions to the integrand for large deviations from $x_0 = n$, at values of $y < x_0$, the exponential function as a whole is highly suppressed. Therefore, the lower bound of the integrand can be changed from x_0 to $-\infty$ yielding only small errors.
Performing the resulting Gaussian integral in Mathematica, we arrive back to Stirling's approximation:

$$\Gamma(n+1) = (2\pi n)^{1/2} n^n e^{-n}(1 + \frac{1}{12n} + O(1/n^2))$$

1 sharply piqued functions

1.1 a

we begin by considering:

$$\Gamma = \int_0^\infty \frac{dx}{x} x^1 e^{-x} = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 0 - (-1) = 1 \tag{1}$$

so come we have shown that $\Gamma = 1 = 0!$. next, we consider:

$$x \to \infty \times \tag{2}$$