

# Homework 1

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## Problem 1: Sharply-peaked functions

(a)

We begin by considering:

$$\Gamma(1) = \int_0^\infty \frac{dx}{x} x^1 e^{-x} = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 0 - (-1) = 1$$

So, we have shown that  $\Gamma(1) = 1 = 0!$ . Next, we consider:

$$\Gamma(n+1) = \int_0^\infty \frac{dx}{x} x^{n+1} e^{-x} = \int_0^\infty x^n e^{-x} dx$$

Using integration by parts with

$$u = x^n \rightarrow du = nx^{n-1}$$

$$dv = e^{-x} \rightarrow v = -e^{-x}$$

we get that:

$$\Gamma(n+1) = -x^n e^{-x} \Big|_0^\infty + \int_0^\infty nx^{n-1} e^{-x} dx$$

In the first term

$$\lim_{x \rightarrow \infty} -x^n e^{-x} = 0$$

because the exponential term dominates the polynomial term as  $x \rightarrow \infty$  for a finite  $n$  and, more trivially,

$$\lim_{x \rightarrow 0} -x^n e^{-x} = 0$$

Thus,  $-x^n e^{-x} \Big|_0^\infty = 0$  and simplify to get that:

$$\Gamma(n+1) = \int_0^\infty n x^{n-1} e^{-x} dx = n \int_0^\infty \frac{dx}{x} x^n e^{-x} = n \Gamma(n)$$

By the previous identity and the previously proven base case with  $n=0$ , we have shown that  $n! = \Gamma(n+1)$  with  $\Gamma(n) \equiv \int_0^\infty \frac{dx}{x} x^n e^{-x}$ .

**(b)**

We start by maximizing the integrand with respect to  $x$  to find the point  $x_0$ :

$$0 = \frac{\partial}{\partial x_0} [x_0^n e^{-x_0}] = n x_0^{n-1} e^{-x_0} - x_0^n e^{-x_0} = e^{-x_0} (n x_0^{n-1} - x_0^n)$$

It follows that

$$n x_0^{n-1} - x_0^n = 0 \rightarrow x_0 = n$$

To verify that this  $x_0$  is indeed maximizing, we consider second-order conditions and using Mathematica, we find that:

$$\left. \frac{\partial^2}{\partial x^2} [x^n e^{-x}] \right|_n = -e^{-n} n^{n-1}$$

Since  $n!$  is defined  $\forall n \in \mathbb{Z}^+$ , both  $e^{-n}$  and  $n^{n-1}$  are always positive and so we get that

$$\left. \frac{\partial^2}{\partial x^2} [x^n e^{-x}] \right|_n < 0$$

enabling us to conclude that the point  $x_0 = n$  maximizes the integrand  $\forall n \in \mathbb{Z}^+$ . Now, expanding the quantity in the exponential around  $x_0 = n$  using Mathematica, we get that:

$$n \log(x) - x = -n + n \log(n) - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} - \frac{y^6}{6n^5} + \dots$$

where  $y = x - x_0 = x - n$  and we have  $a_2 = \frac{1}{2n} > 0$ . Inserting this expansion into the integrand, we get that:

$$\Gamma(n+1) = \int_{-n}^\infty \exp(-n + n \log(n) - \frac{y^2}{2n} + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} - \frac{y^6}{6n^5} + \dots) dy$$

We note that since we have changed the variable of integration from  $x \rightarrow y = x - n$  we must change the bounds of the integral as follows:

$$Upper : \infty \rightarrow \infty - n$$

$$Lower : 0 \rightarrow -n$$

However, because for finite  $n$ ,  $\infty - n \approx \infty$ ,  $\infty$  remains the upper bound for the integral following the change of variables.

(c)

Because of the 2nd order (and higher) powers in  $n$  present in the denominators of the  $a_k y^k$  terms for  $k > 2$ , in the large- $n$  limit, these terms are highly suppressed. Therefore, we Taylor expand the part of the exponential containing these terms. Setting  $m = \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} - \frac{y^6}{6n^5} + \dots$ , we get that:

$$\Gamma(n+1) = \int_{-n}^{\infty} dy \exp\left(-n + n \log(n) - \frac{y^2}{2n}\right) \left(1 + m + \frac{m^2}{2!} + \dots\right)$$

$$\Gamma(n+1) = \int_{-n}^{\infty} dy e^{a_0 - a_2 y^2} \left(1 + \frac{y^3}{3n^2} - \frac{y^4}{4n^3} + \frac{y^5}{5n^4} + \left(\frac{1}{6n^5} + \left(\frac{1}{3n^2}\right)^2/2\right)y^6 + O(y^7)\right)$$

(d)

Now, when  $n$  is large, we concluded earlier that  $a_k$  terms for  $k > 2$  are highly suppressed. It is these terms, however, that would, for small  $n$ , make the largest contributions to the integrand for values of  $y$  far from the integrand's maximum at  $x_0 = n$ . Because these terms are suppressed and the 0th, 1st, and 2nd order terms in  $y$  do not make large contributions to the integrand for large deviations from  $x_0 = n$ , at values of  $y < x_0$ , the exponential function as a whole is highly suppressed. Therefore, the lower bound of the integrand can be changed from  $x_0$  to  $-\infty$  yielding only small errors.

Performing the resulting Gaussian integral in Mathematica, we arrive back to Stirling's approximation:

$$\Gamma(n+1) = (2\pi n)^{1/2} n^n e^{-n} \left(1 + \frac{1}{12n} + O(1/n^2)\right)$$

# 1 sharply piqued functions

## 1.1 a

we begin by considering:

$$\Gamma = \int_0^\infty \frac{dx}{x} x^1 e^{-x} = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 0 - (-1) = 1 \quad (1)$$

so come we have shown that  $\Gamma = 1 = 0!$ . next, we consider:

$$x \rightarrow \infty \times \quad (2)$$