Physics 125b
Problem set number 2
Due midnight Wednesday, January 17, 2024

READING: Sections 17.1 and 17.2 in Shankar on stationary state (aka time-independent) perturbation theory.

PROBLEMS:

5. Now let's confront our estimates in problem 4 (problem set 1) with experiment.

   (a) Make a simple table comparing your variational bounds with the observed ground state energies for lithium, beryllium, and nitrogen. Note that a simple web search for "ionization potentials" will get you a multitude of tables of observed values, or you can look at a reference such as the CRC Press's *Handbook of Chemistry and Physics*. The table entries are typically of the form:

   $$B(Z,N) - B(Z,N-1).$$

   (b) Do your results make sense? If not, can you figure out what is wrong, and whether the calculation we did for He is to be trusted?

6. When we discussed the WKB method, we used the fact that if the potential was a constant, the phase of the wave function depended on x according to $px = \sqrt{2m(E-V)}x$ (if $\phi(x=0)=0$). Allowing the potential to slowly vary with x, we computed the change in phase from $x = x_0$ to x as

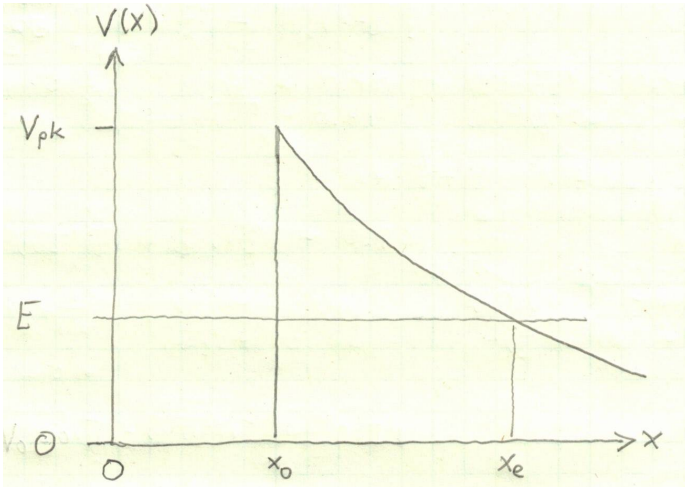
   $$\phi(x) = \int_{x_0}^{x} \sqrt{2m(E-V)}dx. \tag{1}$$

   Thus, if the potential is slowly varying (such that the main variation of the wave function is due to the phase variation), we can write the the wave function at x in terms of the wave function at x_0 approximately by

   $$\psi(x) \approx \psi(x_0) \exp\left[i \int_{x_0}^{x} \sqrt{2m(E-V(x'))}dx'\right]. \tag{2}$$

   The WKB approximation may be used to estimate tunneling rates through a barrier. That is, there may be a region of potential greater than the energy between two regions where the potential is smaller than the energy. If the particle is in one of the "allowed" regions, it cannot classically penetrate the barrier, but it can according to quantum mechanics. Let us consider an example.

   We know that the ^{238}U nucleus is unstable, emitting a 4.267 MeV α particle (kinetic energy) with a half-life of 1.41×10^{17} s, according to Wikipedia. We'll assume spherical symmetry, and model the equivalent one-dimensional potential for the α as in the figure below. We'll assume the α carries off no orbital angular momentum as well. The model assumes that "inside" ($x < x_0$) the nucleus, the α feels no force, but for $x > x_0$ the α particle feels the repulsive Coulomb force due to the remaining nucleus. Let's approximate the nuclear size as $x_0 = 10$ fm.



We use the WKB method to estimate the tunneling rate as follows: We imagine the α is bound in the nucleus, between $x = 0$ and $x = x_0$. Sometimes it bounces up against the potential barrier at $x = x_0$. When this occurs, there is some probability that it will escape the nucleus, where this probability is just $|\psi(x_e)/\psi(x_0)|^2$, where the wave function is estimated according to the WKB method in Eq. 2.

In the classically forbidden region between $x = x_0$ and $x = x_e$, the "phase" is imaginary, and the exponential factor damps the wave function according to $e^{-\Delta/2}$, where

$$\Delta/2 = \int_{x_0}^{x_e} \sqrt{2m\,[V(x') - E]}dx'. \tag{3}$$

(a) Find an expression for Δ by evaluating this integral. It will be convenient to use the ratio $\rho \equiv x_0/x_e$. Try to simplify as much as you can. Note that you may find the discussion in the text on pages 444-445 helpful.

(b) Find an expression for x_e in terms of known quantities.

(c) To estimate the decay rate, we must multiply the tunneling probability by the rate at which the α strikes the potential barrier at x_0. Find an expression for this rate to strike the barrier by estimating the speed of the α and using the distance traveled between collisions.

(d) Finally put it all together and obtain a numerical prediction for the uranium decay rate, Γ. How does your result compare with the measured rate?

7. Let's do a very simple calculation using the Ritz method, in order to make sure we understand the idea. Suppose we are interested in the energy levels of a particle of mass m in the one dimensional potential:

$$V(x) = \begin{cases} 0 & x \in (-L/2, L/2) \\ \infty & \text{otherwise} \end{cases} \tag{4}$$

Of course, we know the exact answer to both the eigenstates and eigenvalues for this system. However, let us pretend that we don't, and try using the Ritz method to estimate the two lowest energy levels. Thus, let us pick two trial wave functions.

We might by accident in this simple case actually pick the exact functions, but we'll avoid that. Let us make an "educated guess", and choose:

$$|1\rangle = A_1 \left[(L/2)^2 - x^2\right] \tag{5}$$
$$|2\rangle = A_2 x \left[(L/2)^2 - x^2\right], \tag{6}$$

where A_1 and A_2 are normalization constants and it is understood that these functions are taken to be zero when $|x| \geq L/2$. I encourage you to consider why this might be a good guess, even if we avoided the exact answers.

(a) Carry out the Ritz procedure using these two trial wave functions, and estimate the two lowest energy levels. Along the way, try to note how we really did make some good choices.

(b) Compare your results with the exact eigenvalues for the two lowest levels.

8. Consider an "anharmonic" oscillator, in which we add a perturbation

$$V = -\rho\omega\hat{X}^3. \tag{7}$$

to the one-dimension harmonic oscillator Hamiltonian. Here, we define $\hat{X} \equiv \sqrt{m\omega}X$ and the unperturbed Hamiltonian is

$$H_0 = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}. \tag{8}$$

The parameter $0 < \rho \ll 1$ is a small dimensionless parameter giving the strength of the perturbation. [We assume that the perturbation is sufficiently small that we needn't worry about what happens at very large values of x.]

(a) Write the perturbation in terms of the raising and lowering operators a and $a^\dagger$. Compute all of the non-zero matrix elements of V in the basis of the unperturbed eigenstates.

(b) Compute the perturbed energy levels to the lowest non-trivial order in the perturbation.

(c) Compute the perturbed energy eigenfunctions to the lowest non-trivial order in the perturbation. Express in terms of the unperturbed energy eigenfunctions.