Physics 125a
Problem set number 1 – Solutions to Problem set number 1
Due midnight Wednesday, October 4, 2023

PROBLEMS:

1. Part of the point of this problem is to give you some practice (in I hope an interesting context) making calculations with our simple $\hbar = c = 1$ units, so I hope you will attempt your calculations in this spirit.

   Shortly before quantum mechanics was invented, Bohr invented a semi-classical model for the atom that had some success in "understanding" atomic energy levels. While incorrect (for one thing, it would seem that the orbiting electron should radiate its energy away), it is a handy model for estimating (in agreement with experiment!) some general features. We consider here the Bohr model for the 1-electron atom, with nuclear charge Ze. Let m_e be the mass of the electron and m_A be the mass of the nucleus. The features are:

   - The electron orbits in the Coulomb potential (Gaussian units, same as in the textbook):
   $$V(r) = -\frac{Ze^2}{r}, \tag{1}$$
   where $r \equiv |\mathbf{x} - \mathbf{x}_A|$, $\mathbf{x}$ is the electron position, and $\mathbf{x}_A$ is the nucleus position. Assuming the virial theorem is valid,
   $$\langle T \rangle = \frac{1}{2}\left\langle r\frac{\partial V}{\partial r}\right\rangle = \left\langle \frac{Ze^2}{2r}\right\rangle = \frac{1}{2}\langle mv^2\rangle, \tag{2}$$
   where we are assuming that the motion is non-relativistic. We have expressed the kinetic energy, T, in terms of the reduced mass,
   $$m = \frac{m_e m_A}{m_e + m_A} \approx m_e, \tag{3}$$
   and the relative speed of the electron with respect to the nucleus,
   $$v = |\mathbf{v}_e - \mathbf{v}_A|. \tag{4}$$
   Alternatively, we could remember the force equation for circular motion:
   $$F = \frac{Ze^2}{r^2} = \frac{mv^2}{r}, \tag{5}$$
   which leads to the same result. We assume circular orbits, with constant r, and hence $\langle T \rangle = T$.
   - The big step (beyond classical mechanics) by Bohr is to assume that angular momentum is quantized:
   $$L = |\mathbf{r} \times \mathbf{p}| = n, \quad n = 1, 2, 3, \ldots \tag{6}$$
   This restricts the possible energy levels. Setting $mvr = n$, and
   $$E = T + V = -\frac{1}{2}\frac{Ze^2}{r} = -\frac{1}{2}mv^2, \tag{7}$$

we find $v = Ze^2/n = Z\alpha/n$. The non-relativistic approximation is self-consistent if

$$Z\frac{\alpha}{n} \ll 1. \tag{8}$$

In particular, the ground state energy for hydrogen in this model is

$$\begin{aligned} E &= -\frac{1}{2}(0.511 \times 10^6 \,\mathrm{eV})/(137)^2 \\ &= -13.6 \text{ eV}. \end{aligned} \tag{9}$$

This agrees nicely with experiment! Also, $v = \alpha = 1/137$ in the ground state, and $r = n^2/m\alpha$ leads to a ground state radius of

$$\begin{aligned} r &= \frac{1}{m\alpha} = \frac{137}{m} \\ &= \frac{137}{0.511\,\mathrm{MeV}} 200\,\mathrm{MeV\text{-}fm}\,10^{-5}\,\text{Å}/\mathrm{fm} \approx 0.5\,\mathrm{Angstrom}. \end{aligned} \tag{10}$$

However, the orbital angular momentum in the ground state is $L = 1$ in this model, and that is ultimately not correct.

We notice that the quantities of interest can be described in terms of the "coupling strength" $\alpha = e^2$ and the reduced mass $m \approx m_e$. In other words, we have only one "scale" in the problem: m_e. In addition to this scale, much of atomic physics is then describable in terms of three dimensionless parameters:

(a) α (strength of the interaction)

(b) Z (number of protons in the nucleus)

(c) m_e/m_A (corrections for finite mass of nucleus).

Note that, if m_N is the nucleon (proton, neutron) mass, $m_e/m_N \approx 0.511/940 \sim 1/2000$ This is a small number. Hence, the nucleus is essentially at rest in the atom's rest frame, since $\mathbf{p}_A = -\mathbf{p}_e$ gives $m_A\mathbf{v}_A = -m_e\mathbf{v}_e$, and thus

$$|\mathbf{v}_A| = \frac{m_e}{m_A}|\mathbf{v}_e|. \tag{11}$$

This atom is held together by electromagnetism. We may consider the very similar problem of a gravitationally bound "Bohr atom". The gravitational potential between two objects of masses m_1 and m_2 is

$$V = -\frac{Gm_1m_2}{r},$$

where $r = |\mathbf{r}_1 - \mathbf{r}_2|$, and

$$G \equiv G_N = 6.67 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}, \tag{12}$$
$$= 6.71 \times 10^{-39}\,\mathrm{GeV}^{-2}, \tag{13}$$

is Newton's constant.

(a) With the same quantization condition as for Bohr's atom, find the formulas for the energy levels, relative velocities, and relative separations, for a "gravity atom". Let n be the quantum level (*i.e.*, $n = 1, 2, 3, \ldots$ is the orbital angular momentum).

Solution: First, let us write:

$$V = \frac{-GmM}{r},$$

where $M = m_1 + m_2$ and m is the reduced mass. The virial theorem argument gives, for a circular orbit,

$$T = -\frac{1}{2}V = \frac{1}{2}\frac{GmM}{r} = \frac{1}{2}mv^2,$$

where v is the relative speed and non-relativistic motion is assumed (but should be checked!). Now we set $mvr = n$, and see what this implies. The energy is:

$$E = T + V = -\frac{1}{2}\frac{GMm}{r} = -\frac{1}{2}mv^2,$$

and thus

$$\begin{aligned} r &= \frac{n^2}{GMm}\frac{1}{m} && (14) \\ v &= \frac{GMm}{n} && (15) \\ E &= -\frac{1}{2}\left(\frac{GMm}{n}\right)^2 m. && (16) \end{aligned}$$

(b) Find n for the earth-sun system. Is quantum mechanics important here?

Solution: The period is approximately $T = 2\pi r/v = \pi 10^7$ s, and the scale m is approximately $m \approx m_{oplus} = 6 \times 10^{27}$ g. The orbit radius is $r = 1.5 \times 10^{13}$ cm. Thus,

$$rv = \frac{2\pi r^2}{T} = \frac{n}{m}, \qquad (17)$$

or

$$\begin{aligned} n &= \frac{2\pi m r^2}{T} \\ &= 2\pi \frac{6 \times 10^{27}(\text{g})[0.511(\text{MeV})/(9.11 \times 10^{-28}(\text{g})]2.25 \times 10^{52}(\text{fm-fm})}{\pi 10^7(\text{s})3 \times 10^{23}(\text{fm/s})200(\text{MeV-fm})} \\ &= \frac{2 \times 6 \times .511 \times 2.25}{9.11 \times 3 \times 2} 10^{27+28+52-7-23-2} \\ &\approx 2 \times 10^{74}. && (18) \end{aligned}$$

The levels for such high values of n are extremely closely spaced, hence the "quantumness" is essentially invisible.

(c) What would r be for the ground state of the earth-sun system (give answer in meters)?

Solution: Since $r \propto n^2$, and $r = 1.5 \times 10^{26}$ fm for $n = 2 \times 10^{74}$, we have for $n = 1$:

$$r_1 = \frac{1.5 \times 10^{26}}{4 \times 10^{148}} \text{ fm} \approx 4 \times 10^{-123} \text{ fm}, \tag{19}$$

a very small distance indeed!

(d) Consider a gravitationally bound system of two neutrons, where we suppose that we have "turned off" other potentially important interactions (such as the strong interaction). What is the ground state energy (in GeV) and the ground state separation (in meters) for this system? Is gravity important compared with other forces for a real system of two neutrons in a real ground state?

Solution: We'll approximate the mass of the neutron as $m_n = 1$ GeV for this exercise. The reduced mass is $m = 1/2m_n$ and the total mass is $M = 2m_n$. The ground state energy corresponds to $n = 1$ in this model:

$$E_1 = -\frac{1}{2}G^2(2m)^2\frac{m}{2}^2\frac{m}{2} = -\frac{1}{4}G^2m^5. \tag{20}$$

Newton's constant is, approximately,

$$G = 6.7 \times 10^{-39} \text{ GeV}^{-2}. \tag{21}$$

Thus,

$$E_1 \approx -10^{-77} \text{ GeV}. \tag{22}$$

The ground state separation between the two neutrons in this model is

$$r_1 \approx \frac{2}{Gm^3} \approx \frac{2 \times (200 \text{ MeV-fm}) \times (10^{-15} \text{ m/fm}}{(6.7 \times 10^{-39} \text{ GeV}) \times (10^3 \text{ MeV/GeV})} \approx 6 \times 10^{22} \text{ m}. \tag{23}$$

Nuclear sizes are of order 1 fm, and nuclear binding energies are of order MeV (*e.g.*, the binding energy of the deuteron is approximately 2 MeV). It appears that gravity is unimportant compared with the strong interaction (at least) for a system of two neutrons which are not far apart.

(e) Physicists have invented quite a few systems of units for measurement. One notion of at least theoretical value is the notion of "natural units". Our use of $\hbar = c = 1$ units is such a system. Around 1900, Planck proposed such a system based on his quantum of action (now $\hbar$). Thus, he proposed to measure masses in units such that the gravitational strength is unity, i.e., $G = 1$. We set $Gm_P^2 = 1$ to find this unit of mass, m_P. In SI units, we would write this equation as $Gm_P^2 = \hbar c$, as you can check that the units match. Determine the mass m_P in GeV.

Solution: We wish to find the mass M_P such that

$$|V(r)| = \frac{1}{r} = \frac{Gm_P^2}{r}. \tag{24}$$

Hence, the "Planck mass" is

$$m_P = \frac{1}{\sqrt{G}} = \frac{1}{\sqrt{6.7 \times 10^{-39} \text{ GeV}^{-2}}} \approx 10^{19} \text{ GeV}. \tag{25}$$

2. Let us consider the action of Gallilean transformations on a quantum mechanical wave function. We restrict ourselves here to the "proper" Gallilean Transformations: (i) translations; (ii) velocity boosts; (iii) rotations. We shall consider a transformation to be acting on the state (not on the observer). Thus, a translation by $\mathbf{x}_0$ on a state localized at $\mathbf{x}_1$ produces a new state, localized at $\mathbf{x}_1 + \mathbf{x}_0$. In "configuration space", we have a wave function of the form $\psi(\mathbf{x}, t)$. A translation $T(\mathbf{x}_0)$ by $\mathbf{x}_0$ of this state yields a new state (please don't confuse this translation operator with the time reversal operator, which is also often denoted by T, but without an argument):

$$\psi'(\mathbf{x}) = T(\mathbf{x}_0)\psi(\mathbf{x}, t) = \psi(\mathbf{x} - \mathbf{x}_0, t). \tag{26}$$

Note that we might have attempted a definition of this transformation with an additional introduction of some overall phase factor. However, it is our interest to define such operators as simply as possible, consistent with what should give a valid classical correspondence. Whether we have succeeded in preserving the appropriate classical limit must be checked, of course.

Consider a free particle of mass m. The momentum space wave function is

$$\hat{\psi}(\mathbf{p}, t) = \hat{f}(\mathbf{p}) \exp\left(-\frac{itp^2}{2m}\right), \tag{27}$$

where $p = |\mathbf{p}|$. If this isn't clear, check that it satisfies the Schrödinger equation. The "hats" here denote functions in momentum space (not operators). The configuration space wave function is related by the (inverse) Fourier transform:

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{(\infty)} d^3(\mathbf{p}) e^{i\mathbf{x}\cdot\mathbf{p}} \hat{\psi}(\mathbf{p}, t). \tag{28}$$

Obtain simple transformation laws, on both the momentum and configuration space wave functions, for each of the following proper Gallilean transformations:

(a) Translation by $\mathbf{x}_0$: $T(\mathbf{x}_0)$ (note that we have already seen the result in configuration space).

Solution: [Should draw a figure...] In configuration space, the result was:

$$\psi'(\mathbf{x}, t) = T(\mathbf{x}_0)\psi(\mathbf{x}, t) = \psi(\mathbf{x} - \mathbf{x}_0, t). \tag{29}$$

Thus, in momentum space, we have:

$$\begin{aligned}
\hat{\psi}'(\mathbf{p}, t) &= T(\mathbf{x}_0)\hat{\psi}(\mathbf{p}, t) && (30)\\
&= \frac{1}{(2\pi)^{3/2}} \int_{(\infty)} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi'(\mathbf{x}, t) d^3(\mathbf{x}) \\
&= \frac{1}{(2\pi)^{3/2}} \int_{(\infty)} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{x} - \mathbf{x}_0, t) d^3(\mathbf{x}) \\
&= \frac{1}{(2\pi)^{3/2}} \int_{(\infty)} e^{-i\mathbf{p}\cdot(\mathbf{x}+\mathbf{x}_0)} \psi(\mathbf{x}, t) d^3(\mathbf{x}) \\
&= e^{-i\mathbf{p}\cdot\mathbf{x}_0} \hat{\psi}(\mathbf{p}, t). && (31)
\end{aligned}$$

(b) Translation by time t_0: $M(t_0)$.

Solution: Let's take the time translation to act on a configuration space wave function in the obvious way:

$$M(t_0)\psi(\mathbf{x},t) = \psi(\mathbf{x},t-t_0). \tag{32}$$

Likewise,

$$\begin{aligned} M(t_0)\hat{\psi}(\mathbf{p},t) &= \hat{\psi}(\mathbf{p},t-t_0) && (33)\\ &= \exp\left(\frac{it_0p^2}{2m}\right)\hat{\psi}(\mathbf{p},t). && (34)\end{aligned}$$

(c) Velocity boost by $\mathbf{v}_0$: $V(\mathbf{v}_0)$. (Hint: first find

$$\hat{\psi}'(\mathbf{p},0) = \hat{f}'(\mathbf{p}) = V(\mathbf{v}_0)\hat{f}(\mathbf{p}), \tag{35}$$

then

$$\hat{\psi}'(\mathbf{p},t) = \hat{f}'(\mathbf{p})e^{-itp^2/2m}, \tag{36}$$

etc.)

Solution: Under a velocity transformation, the space coordinates and momentum transform according to

$$\begin{aligned} \mathbf{x}' &= \mathbf{x}+\mathbf{v}_0t && (37)\\ \mathbf{p}' &= \mathbf{p}+m\mathbf{v_0}. && (38)\end{aligned}$$

Thus, it appears reasonable to take

$$V(\mathbf{v}_0)\hat{f}(\mathbf{p}) = \hat{f}(\mathbf{p}-m\mathbf{v_0}). \tag{39}$$

Thus,

$$\begin{aligned} \hat{\psi}'(\mathbf{p},t) &= \hat{f}'(\mathbf{p})e^{-itp^2/2m} && (40)\\ &= \hat{f}(\mathbf{p}-m\mathbf{v_0})e^{-itp^2/2m}\\ &= \hat{\psi}(\mathbf{p}-m\mathbf{v}_0,t)\exp\left[\frac{it(\mathbf{p}-m\mathbf{v}_0)^2}{2m}\right]e^{-itp^2/2m}\\ &= \exp\left[-it\left(\mathbf{p}\cdot\mathbf{v}_0-\frac{1}{2}m\mathbf{v}_0^2\right)\right]\hat{\psi}(\mathbf{p}-m\mathbf{v}_0,t). && (41)\end{aligned}$$

In configuration space, this becomes

$$\begin{aligned} \psi'(\mathbf{x},t) &= \frac{1}{(2\pi)^{3/2}}\int_{(\infty)} d^3(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\hat{\psi}'(\mathbf{p},t)\\ &= \frac{1}{(2\pi)^{3/2}}\int_{(\infty)} d^3(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}\exp\left[-it\left(\mathbf{p}\cdot\mathbf{v}_0-\frac{1}{2}m\mathbf{v}_0^2\right)\right]\hat{\psi}(\mathbf{p}-m\mathbf{v}_0,t)\\ &= \frac{1}{(2\pi)^{3/2}}\int_{(\infty)} d^3(\mathbf{q})e^{i(\mathbf{q}+m\mathbf{v}_0)\cdot\mathbf{x}}e^{\left[-it\left((\mathbf{q}+m\mathbf{v}_0)\cdot\mathbf{v}_0-\frac{1}{2}m\mathbf{v}_0^2\right)\right]}\hat{\psi}(\mathbf{q},t)\\ &= e^{im\mathbf{v}_0\cdot\mathbf{x}}e^{\left[-it\left(mv_0^2-\frac{1}{2}m\mathbf{v}_0^2\right)\right]}\int_{(\infty)}\frac{d^3}{(2\pi)^{3/2}}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\exp\left(-it\mathbf{q}\cdot\mathbf{v}_0\right)\hat{\psi}(\mathbf{q},t)\\ &= e^{\left[im\mathbf{v}_0\cdot\mathbf{x}-it\left(\frac{1}{2}m\mathbf{v}_0^2\right)\right]}\int_{(\infty)}\frac{d^3}{(2\pi)^{3/2}}(\mathbf{q})\exp\left(i\mathbf{q}\cdot\mathbf{x}-it\mathbf{q}\cdot\mathbf{v}_0\right)\hat{\psi}(\mathbf{q},t)\\ &= \exp\left[im\mathbf{v}_0\cdot\mathbf{x}-it\left(\frac{1}{2}m\mathbf{v}_0^2\right)\right]\psi(\mathbf{x}-\mathbf{v}_0t,t) && (42)\end{aligned}$$

(d) Rotation about the origin given by 3×3 matrix R: $U(R)$.

Solution: For a rotation on a vector $\mathbf{x}$, rotating it to a new vector $\mathbf{x}'$, $x' = R\mathbf{x}$. Acting on a wave function, the rotated wave function ψ', when evaluated at a rotated point $\mathbf{x}'$, is the same as the unrotated wave function evaluated at the unrotated point $\mathbf{x}$:

$$\psi'(\mathbf{x}', t) = \psi'(R\mathbf{x}, t) = \psi(\mathbf{x}, t). \tag{43}$$

Thus,

$$\psi'(\mathbf{x}, t) = R_{\mathrm{op}}\psi(\mathbf{x}, t) = \psi(R^{-1}\mathbf{x}, t). \tag{44}$$

I have used the notation R_{op} to distinguish the operator on the Hilbert space from the 3×3 matrix R.

In momentum space, we may note that the same argument holds for momenta, or we may Fourier transform the configuration space result. In either event, we obtain

$$R_{\mathrm{op}}\hat{\psi}(\mathbf{p}, t) = \hat{\psi}(R^{-1}\mathbf{p}, t). \tag{45}$$

Make sure your answers make sense to you, at least in terms of classical correspondence.

3. Consider the (real) vector space of real continuous functions with continuous first derivatives in the closed interval $[0, 1]$. Do the following define scalar products?

(a) $\langle f|g\rangle = \int_0^1 f'(x)g'(x)dx + f(0)g(0)$

(b) $\langle f|g\rangle = \int_0^1 f'(x)g'(x)dx$

Solution: A scalar product must satisfy, for any $f, g, h \in V$, the conditions:

(a) $\langle f|f\rangle \geq 0, \quad$ with $\langle f|f\rangle = 0$ iff $f = 0$.

(b) $\langle f|g\rangle = \langle g|f\rangle^*$.

(c) $\langle f|cg\rangle = c\langle f|g\rangle$, where c is any complex number.

(d) $\langle f|g + h\rangle = \langle f|g\rangle + \langle f|h\rangle$.

In the present case, we are dealing with real vector spaces, hence the second condition becomes $\langle f|g\rangle = \langle g|f\rangle$ and the constant in the third condition is restricted to be a real number.

It may be readily checked that the scalar product defined in part (a):

$$\langle f|g\rangle = \int_0^1 f'(x)g'(x)\,dx + f(0)g(0), \tag{46}$$

satisfies all of the properties. However, the product defined in (b):

$$\langle f|g\rangle = \int_0^1 f'(x)g'(x)\,dx, \tag{47}$$

does not. The property that fails is the first – this product will yield zero for $\langle f|f\rangle$ if f is any constant, *i.e.*, not only $f = 0$.

4. Show that, with a suitable measure, any summation over discrete indices may be written as a Lebesgue integral:

$$\sum_{n=1}^{\infty} f(x_n) = \int_{\{x\}} f(x)\mu(dx). \tag{48}$$

Be careful, as always, that you get the units right.

Note: I use the language of measure theory here, but all we need is some "function" $\mu(dx)$ that gives the "size" of the interval $(x, x + dx)$ (I've shown it as open, does that matter?) that will give the desired result. Note that this size might depend on x, unlike the familiar "Lebesgue measure", $\mu(dx) = dx$.

Solution:

$$\mu(dx) = \sum_{n=1}^{\infty} \delta(x - x_n)dx. \tag{49}$$