Ch/ChE 164 Winter 2024

Homework Problem Set #4

Due Date: Thursday, February 15, 2024 @ 11:59pm PT

For all problems, please consider reasonable simplifications of your final results.

1. (15 pts.) (Adapted from Callen). Consider a mixture of two non-identical monatomic ideal gases.

- Starting from the expression for the grand canonical partition function and taking the limit of small fugacity, show that the canonical partition function Z is factorizable and

$$Z = Z_1 Z_2 = \frac{1}{N_1!} q_1^{N_1} \frac{1}{N_2!} q_2^{N_2} \tag{1}$$

(You may wish to use the occupancy representation $\mid n_1 m_1, n_2 m_2 \ldots)$, where n_1 denotes occupancy of energy level 1 of gas 1 , and m_1 denotes occupancy of energy level 1 of gas 2 , etc.).

- Compute the entropy and show that (comparing to the entropy of the two separate gases) there is an entropy of mixing of the form

$$S_{\text{mixing}} = \left(-x_1 \log x_1 - x_2 \log x_2\right) Nk \tag{2}$$

where N is the total number of particles.

2. In class we derived the heat capacity of the Fermi gas at low temperature by an intuitive argument, which $C_v \sim NkO\left(T/T_F\right)$. Here we will derive the precise form and constants (adapted from Callen).

Denote the Fermi-Dirac distribution at temperature T as $f(\epsilon, T)$ and the (temperature dependent) chemical potential by μ (note this is not the Fermi energy ϵ_F except when $T=0$). We will first derive a general result for an integral of the form (Sommerfeld expansion)

$$I \equiv \int_0^\infty \phi(\epsilon) f(\epsilon, T) d\epsilon = \int_0^\mu \phi(\epsilon) d\epsilon + \frac{\pi^2}{6}(kT)^2 \phi'(\mu) + \frac{7\pi^4}{360}(kT)^4 \phi'''(\mu) + \ldots \tag{3}$$

a) (10 pts.) Integrate I by parts, and let $\Phi \equiv \int_0^\epsilon \phi(\epsilon')\, d\epsilon'$. Then expanding $\Phi(\epsilon)$ in a power series in $\epsilon - \mu$ to third order, deduce

$$I = -\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Phi(\mu)}{d\mu^m} I_m \tag{4}$$

where $I_m = \int_0^\infty (\epsilon - \mu)^m \frac{df}{d\epsilon} d\epsilon = -\beta^{-m} \int_{-\beta\mu}^{\infty} \frac{e^x}{(e^x+1)^2} x^m dx$

b) (5 pts.) Show that only an exponentially small error is made by taking the lower limit of integration as $-\infty$, and that then all terms with m odd vanish.

c) (5 pts.) Evaluate the first two non-vanishing terms and show that this agrees with the expansion of I.

d) (10 pts.) Using the result for I, express N in the form of such an integral and obtain an expansion for $N(V, T, \mu)$ in terms of kT/μ (to second order). Verify that $T \to 0$ yields the relation between N and ϵ_F derived in class.

e) (10 pts.) Invert this relationship to obtain $\mu(T)$ as a function of kT/ϵ_F (to second order) for fixed N.

f) (5 pts.) Similarly obtain an expansion for the internal energy E as a function of kT/μ (to second order).

g) (5 pts.) Substituting in $\mu(T)$ into the energy expansion, obtain an expansion of E in kT/ϵ_F to second order, and thus C_v. Hence see why we skipped the detailed computation in class.

3. (20 pts.) Show that for the Bose-Einstein and Fermi-Dirac gas at low density and/or high temperature the equation of state is given by

$$p = kT\rho \left(1 \mp \frac{\rho\Lambda^3}{2^{5/2}} + \dots\right) \tag{5}$$

(1) $v_1 T_{\mu_2} 2$ non-interacting gases

(i)

$$\text{gas 1: } n_1, n_2, n_3 \dots N_1 = \sum_{i=1}^{\infty} n_i$$

$$\text{gas 2: } m_1, m_2, \cdots N_2 = \sum_{i=1}^{\infty} m_i$$

$E_v = E_v' + E_v^2 \longleftarrow$ sum b/c non interacting otherwise need interaction potential term

$$E' = \sum_{i=1}^{\infty} n_i \epsilon_i^{(1)} + \sum_{i=1}^{\infty} m_i \epsilon_i^{(2)}$$

- grand canonical , N, E can fluctuate

$$\Xi = \sum_v e^{-\beta E_v + \beta\mu^{(1)}N_1 + \beta\mu^{(2)}N_2}$$

$$= \sum_v \exp\left(-\beta\left(\sum_{i=1}^{\infty} n_i\epsilon_i^{(1)} + \sum_{i=1}^{\infty} m_i t_i^{(2)}\right) + \beta\mu\sum_{i=1}^{(1)} n_i^{-} + \beta\mu^{(2)}\sum_{i=1}^{\infty} m_i\right)$$

$$\Xi = \sum_{\mathrm{sn}\,3}\sum_{\{m\}} \exp\left(-\beta\left[\sum_i^{\infty} n\epsilon_i^{(1)} + \sum_i^{\infty} m_i\epsilon^{(2)} - \sum_i^{\infty}\mu_i^{(1)}n_i - \sum_i^{\infty}\mu_i^{(2)}m\right]\right)$$

- move sums from exp to product

$$\Xi = \prod_i \sum_{n_i}\sum_{m_i} \exp\left(-\beta\left[n_i t_i^{(1)} + m_i t_i^{(2)} - \mu_i^{(1)}n_i - \mu_i^{(2)}m_i\right]\right)$$

$$= \underbrace{}_{i}\sum_{n_i} \exp\left(\beta n_i\left(\mu_i^{(1)} - \epsilon_i^{(1)}\right)\right)\pi_i \sum_{m_i} \frac{\exp\left(\beta m_i\left[\mu_i^{(2)} - \epsilon_i^{(2)}\right]\right)}{\Xi_2}$$

$$\Xi = \Xi_1\Xi_2$$

each obey FD, BE stats: $BE: n_i, m_i$ can be any any
FD: ni, mi can be o, 1 only

$$= \prod_i e^{(\beta\mu_i - \epsilon_i)^{n_i}} \prod_i e^{(\beta\mu_i - \epsilon_i)^m}$$

$BE: \Xi = \prod_i \left(1 - e^{\beta\mu - \beta t_i}\right)^{-1}, FD: \Pi_i\left(1 + e^{\beta\mu - \beta t_i}\right)$

$$\Xi = \prod_i \left(1 \mp e^{\beta\mu - \beta\epsilon_i}\right) \mp 1$$

$$\ln\Xi = \mp\ln\prod_i\left[1 \mp e^{\beta\mu - \epsilon_i}\right] = \mp\sum_i \ln\left[1 \mp e^{\beta\mu - c_i}\right]$$

Chow fugacity $e^{\beta\mu} \ll 1, T.E.$
$\ln(1+x)$ for small $x :\longrightarrow x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$$\ln \sqsubseteq = \mp\sum_i\left(\mp e^{\beta\mu - t_i}\right) = +e^{\beta\mu}\sum_i e^{\epsilon_i} = e^{\beta\mu}q$$

q is single particle partition f $\times$ n

$$\Xi = \exp\left(e^{\beta\mu} q\right), e^x = \sum_{N=0}^{\infty} \frac{1}{N!} x^N$$

$$\Xi = \sum_{N=0}^{\infty} \frac{1}{N!} q^N e^{\beta\mu N} = \Xi_1 \Xi_2$$

$$\Xi = \sum_{N=0}^{\infty} \frac{1}{N!} q^N e^{\beta M^{(1)} N} \sum_{M}^{\infty} \frac{1}{M!} q^M e^{\beta M^{(2)} M} = \sum_{N}^{\infty} z e^{\beta\mu N}$$

$$\rightarrow Z = \left(\frac{1}{N}! q^N\right)\left(\frac{1}{M!} q^M\right)$$

$$Z = Z_1 Z_2 = \frac{1}{N_1!} q_1^N \frac{1}{N_2} q_2^{N_2}$$

(ii) find entropy (thermos approach)
Smixing $= S_{\text{mix }} - S_1 \cdot S_2$

$$F = -kT \ln z$$

$$S = -\left(\frac{\delta F}{\delta T}\right)_{N_1, N_2, v}, q = \frac{v}{\Lambda^3} = \left(\frac{2mkT}{n^2}\right)^{3/2} v$$

$$\left.S_1 = -kN_1 \ln \rho_1^* \Lambda^3 + \frac{5}{2} kN_1\right\} \rho_1^*, \rho_2^* b/c$$

occupy $v_1, v_2 \neq v$

$$\rho_2 = -kN_2 \ln \rho_2^* \Lambda^3 + \frac{5}{2} kN_2 = \frac{N}{V_1}, \frac{V_1}{v} = x_1$$

$$F_{\text{mix}} = -kT \ln\left(z_1 2_2\right) = -kT \ln\left(\frac{1}{N_1!} q_1^{N_1} \frac{1}{N_2}! q_2^{N_2}\right)$$

$$= -kT\left[N_1 \ln N_1 - N_1 + N_1 \ln q_1 + N_2 \ln N_2 - N_2 + N_2 \ln q_2\right]$$

$$
\begin{aligned}
S_{\text{mix }} & =-\left(\frac{\delta F_{\text{mix}}}{\delta T}\right)_{N_{1} N_{2}, v}=-k \ln \frac{1}{N_{1}!} \frac{1}{N_{2}!}-\frac{\delta}{\delta T} k T \ln q_{1}^{N_{1}} q_{2}^{N_{2}} \\
& =-k \ln \frac{1}{N_{1}!} \frac{1}{N_{2}!}-\frac{\delta}{\delta T} k T\left(N_{1} \ln \left(\frac{2 m k}{n^{2}}\right)^{3 / 2} V+N_{2} \ln \left(\frac{2 m k}{\hbar^{2}}\right)^{3 / 2} V\right. \\
& \left.+\frac{3}{2}\left(N_{1}+N_{2}\right) \ln T\right) \\
& =-k\left[\ln \frac{1}{N_{1}}!\frac{1}{N_{2}}!+\ln \left(\frac{2 m k}{\hbar}\right)^{3 / 2} V\left(N_{1}+N_{2}\right)\right]-k \ln T\left(N_{1}+N_{2}\right) \\
& +\frac{2}{2} \frac{1}{T}\left(N_{1}+N_{2}\right) k T \\
& =k\left(N_{1}+N_{2}-N_{1} \ln \left(\frac{2 m k}{\hbar^{2}}\right)^{3 / 2} \rho_{1}-N_{2} \ln \left(\frac{2 m k}{\hbar^{2}}\right)^{3 / 2} \rho_{2}\right. \\
& \left.-\frac{3}{2}\left(N_{1}+N_{2}\right)(\ln T-1)\right)=k\left(-N_{1} \ln \rho_{1} \Lambda^{3}-N_{2} \ln \rho_{2} \Lambda^{3}+\frac{5}{2}\left(N_{1}+N_{2}\right)\right] \\
S_{\text{mixing }} & =S_{\text{mix }}-S_{1}-S_{2}=\frac{S}{2} k N_{1}+\frac{S}{2} k N_{2}-N_{1} k \ln \rho_{1} \wedge^{3} \\
& -N_{2} \ln \rho_{2} \Lambda^{3}-\frac{5}{2} K N_{1}+K N_{1} \ln \rho_{1}^{*} \Lambda^{3} \\
& +k N_{2} \ln _{2}^{*} \cap^{3} \\
\text{mixing } & =N k\left(-x_{1} \ln \rho_{1}-x_{1} \pi \pi \wedge^{3}-x_{2} \ln \rho_{2}\right. \\
& -x_{2} \operatorname{tn} \Lambda^{3}+x_{1} \ln \rho_{1}^{*}-x_{1} \operatorname{tn} \Lambda^{3} \\
& \left.+x_{2} \ln \rho_{2}^{*}-x_{2}+n \Lambda^{3}\right)
\end{aligned}
$$

- no variation in Λ, but variation in $\rho(v)$

$$
\begin{aligned}
\text{Smixing } & =N_{k}\left(-x_{1} \ln \left(\frac{N_{1}}{V} \cdot \frac{V_{1}}{N_{1}}\right)-x_{2} \ln \left(\frac{N_{2}}{V} \cdot \frac{V_{2}}{N_{2}}\right)\right) \\
\frac{V_{1}}{V} & =\frac{N_{1}}{N_{1}+N_{2}}=x_{1}, \frac{V_{2}}{V}=\frac{N_{2}}{N_{1}+N_{2}}=x_{2} \\
\text{Smixing } & =N_{k}\left(-x_{1} \ln x_{1}-x_{2} \ln x_{2}\right)
\end{aligned}
$$

(2) Cr of Fermions

$$
f(t, T)=\left(e^{\beta(t-\mu)}+1\right)^{-1}, I \equiv \int_{0}^{\infty} \phi(t) f(t, T) d t
$$

(a) integrate by parts gives arg of ϕ

$$
\begin{aligned}
&u = f(t,T), du = \frac{\delta f}{\delta t} dt \\
&dv = \phi(t) \quad v = \int \phi(\epsilon) dt \\
&I = uv - \int vdu = \ f(t,T) \int_0^t \phi\left(t'\right) dt' \bigg|_0^\infty \quad f(\infty) = 0 \\
&\quad - \int_0^0 \frac{\delta f}{\delta t} \left[\int_0^t \phi\left(t'\right) dt' \right] dt \\
&I \pm - \int_0^\infty \frac{\delta f}{\delta t} \Phi(t) dt
\end{aligned}
$$

- Taylor Expand around μ

$$
\Phi(\mu) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\delta^m \Phi}{\delta \epsilon^m} \right) (t-\mu)^m
$$
$$
I = - \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\delta^m \Phi}{\delta \epsilon^m} \right) \int_0^\infty \frac{\delta f}{\delta t} (t-\mu)^m dt
$$
$$
I = - \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\delta^m \Phi(\mu)}{\delta \epsilon^m} \right) I_m
$$

(b)

$$
\begin{aligned}
&\text{(b) } I_m = \int_{-\beta\mu}^{\infty} \frac{e^x x^m}{\left(e^x+1\right)^2} dx, \text{ argue } \int_{-\beta\mu} \to \int_{-\infty} \\
&I_m = \int_0^\infty \frac{\delta f}{\delta t} (\epsilon - \mu)^m dt = \int_0^\infty \frac{\beta e^{\beta(t-\mu)}}{\left(e^{\beta(t-\mu)+1)^2}\right.} (t-\mu)^m dt \\
&x = t - \mu \quad = -\beta^{-m} \int_{-\beta\mu}^{\infty} \frac{e^x}{\left(e^x+1\right)^2} x^m dx
\end{aligned}
$$

need to show $\int_{-\infty}^{-\beta\mu} \frac{e^x x^m}{(e^x+1)^2} dx$ is exponentially $\sim \int_{-\infty}^{-\beta\mu} e^x x^m \leq \int_{-\infty}^{-\beta\mu} \frac{e^x x^m}{(e^x+1)^2} dx \quad b/c$ smallest value of $\rightsquigarrow \int_{-\infty}^{-\beta\mu} e^x x^m, e^x$ donimates $x^m \$ e^x$ is exponentially small for all $x < 0$ and $\beta\mu >> 1$

$$
\begin{aligned}
&I_m = -\beta^{-m} \int_{-\infty}^{\infty} \frac{e^x x^m}{\left(e^x+1\right)^2} dx + \beta^m \int_{-\infty}^{-\beta\mu} \frac{e^x x^m}{\left(e^x+1\right)^2} dx \\
&I_m \simeq -\beta^{-m} \int_{-\infty}^{\infty} \frac{e^x x^m}{\left(e^x+1\right)^2} dx
\end{aligned}
$$

- when m is odd $\to x^m$ is oddfxn $\left(x, x^3, x^s \text{ even } f_{xn} : \int_{-\infty}^0 \cdots = \int_0^\infty \text{ v. } x^2, x^4\right)$

odd $f \times n$: $\quad \int_{-\infty}^{0} = -\int_{0}^{\infty} \rightarrow \int_{-\infty}^{\infty} = 0$

because we are taking $\int_{-\infty}^{\infty}, x^m$ where m is odd cancel ($e^x/\left(e^x+1\right)^2$ is even)

(C) (5 pts.) Evaluate the first two non-vanishing terms and show that this agrees with the expansion of I.

$$
\begin{aligned}
& I_0=(1) \Phi(\mu) \int_{-\infty}^{\infty} \frac{e^x}{\left(e^x+1\right)^2} d x=\Phi(\mu)\left[\left(e^x-1\right)\right]_{-\infty}^1 \\
& I_0=\underline{P}(\mu)=\int_0^t \phi\left(t^{\prime}\right) d t^{\prime} \\
& I_2=+\frac{1}{2!}\left(\frac{\delta^2 \Phi(\mu)}{\delta t^2}\right) \int_{-\infty}^{\infty} \frac{x^2 \cdot e^x}{\left(e^x+1\right)^2} d x \\
& \Phi \equiv \int_0^\epsilon \phi\left(\epsilon^{\prime}\right) d \epsilon^{\prime}
\end{aligned}
$$

mathematica $\rightarrow$ $I_2 = \frac{1}{2!}\beta^{-2}\frac{\delta^2 \Phi(\mu)}{\delta \epsilon^2} \cdot \frac{\pi^2}{3} = kT^2 \frac{\pi^2}{6} \Phi'' = (kT)^2 \frac{\pi^2}{6} \phi'(\mu)$

$$
I \equiv \int_0^{\infty} \phi(\epsilon) f(\epsilon, T) d \epsilon=\underbrace{\int_0^\mu \phi(\epsilon) d \epsilon}_{\checkmark}+\underbrace{\frac{\pi^2}{6}(k T)^2 \phi^{\prime}(\mu)}_{\checkmark}+\frac{7 \pi^4}{360}(k T)^4 \phi^{\prime \prime \prime}(\mu)+\ldots
$$

(d)

d) (10 pts.) Using the result for I, express N in the form of such an integral and obtain an expansion for $N(V, T, \mu)$ in terms of $k T / \mu$ (to second order). Verify that $T \rightarrow 0$ yields the relation between N and ϵ_F derived in class.

$$
N=\sum_\alpha\left\langle n_\alpha\right\rangle=\int_0^{\infty} \phi(\varepsilon) f(t, T) d t
$$

L density of states for particle #

(a) 89

eq

$$\phi(t) \to \rho(t) = \frac{V}{2\pi^2} \cdot \frac{(2m)^{3/2}}{\hbar^3} t^{1/2}$$

$$\phi'(t) \to \rho'(\mu) = \frac{V}{4\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \mu^{-1/2}$$

$$N = \int_0^\mu \rho(t) dt + \frac{\pi^2}{6} (kT)^2 \frac{V}{4\pi^2} \left[\frac{(2m)^{3/2}}{\hbar^3} t^{-1/2} \right] + \ldots$$

$$N = \frac{2}{3} \cdot \frac{V}{2\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \mu^{3/2} + \frac{\pi^2}{6} (kT)^2 \frac{V}{4\pi^2} \left[\frac{(2m)^{3/2}}{\hbar^3} \mu^{-1/2} \right] + \cdots$$

$$N\left(V_1 T, \mu\right) = \frac{V}{3\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right]$$

$$kT \to 0, \quad \mu = \epsilon_F$$

$$\left[N\left(V, T, \epsilon_F\right) = \frac{V}{3\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{3/2} \right]$$

(e) 10 pts.) Invert this relationship to obtain $\mu(T)$ as a function of kT/ϵ_F (to second order) for fixed N.

$$N(V, T, \mu) = N\left(V, T, \epsilon_F\right)$$

$$\frac{v}{\beta\pi^2} \left(\frac{2m}{\pi^2} \right)^{3/2} \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right] = \frac{V}{\beta\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{3/2}$$

$$\left(\frac{\mu}{\epsilon_F} \right)^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right] = 1$$

$$\left(\frac{\mu}{\epsilon_F} \right)^{3/2} = 1 - \frac{\pi^2}{8} \left(\frac{kT}{\epsilon_F} \right)^2 \left(\frac{\mu}{\epsilon_F} \right)^{-1/2} \quad \text{low } T$$

$$\frac{\mu}{\epsilon_F} \approx \left(1 - \frac{\pi^2}{8} \left(\frac{kT}{\epsilon_F} \right)^2 \right)^{2/3}$$

$$\mu \approx t_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 \right]$$

(f)

(5 pts.) Similarly obtain an expansion for the internal energy E as a function of kT/μ (to second order).

$$E = \int_0^\infty \phi(\varepsilon) f(t, T) dt$$

density of states for energy

$$\phi(t) \to \int \in (t) = \int \frac{V}{2\pi^2} \cdot \frac{(2m)^{3/2}}{\hbar^3} t^{3/2} \to \frac{2}{S} \cdot \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{5/2} \Big|_0^{\mu}$$

$$\phi'(t) \to \epsilon \rho'(t) = \frac{V}{2\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \cdot \frac{3}{2} \epsilon'^{/2} \quad \text{change to } \mu$$

$$E = \frac{V}{S\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{5/2} + \frac{\pi^2}{6} (kT)^2 - \frac{2}{5} \cdot \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{1/2}$$

$$E = \frac{V}{S\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{s/2} \left[1 + \frac{\pi^2}{6} \left(\frac{kT}{\mu}\right)^2\right]$$

(9)

g) (5 pts.) Substituting in $\mu(T)$ into the energy expansion, obtain an expansion of E in kT/ϵ_F to second order, and thus C_v. Hence see why we skipped the detailed computation in class.

$$(v, T, \mu) = \frac{v}{3\pi^2} \left(\frac{2m}{\pi^2}\right)^{3/2} \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu}\right)^2\right]$$

$$E = \frac{3}{5} N \epsilon_F \left[\left(\frac{\mu}{t_F}\right)^{5/2} + \frac{5\pi^2}{24} \left(\frac{kT}{\epsilon_F}\right)^2 \left(\frac{\mu}{\epsilon_F}\right)^{1/2}\right]$$

$$E = \frac{3}{5} N t_F \left[\left(1 - \frac{\pi^2}{12} \left(\frac{kF}{\epsilon_F}\right)^2\right]^{5/2} + \frac{\pi^2}{24} \left(\frac{kT}{\epsilon_F}\right)^2 \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F}\right)^2\right]^{1/2}\right]$$

$$\longrightarrow C_V = \frac{\delta E}{\delta T} = 2 \left(\frac{3}{2} t_F + \frac{\pi^2}{4} N \epsilon_F \left(\frac{kT}{\epsilon_F}\right)^2 N t_F \left(\frac{k}{\epsilon_F}\right)^2 T\right.$$

$$C_V = \frac{\pi^2}{2} N \frac{k^2}{t_F} T$$

I used these,

https://bingweb.binghamton.edu/~suzuki/SolidStatePhysics/

10-4 Sommerfeld formula.:pdf

https://farside.ph.utexas.edu/teaching/sm1/Thermalhtml/node107.html#e8.127d

(3) (20 pts.) Show that for the Bose-Einstein and Fermi-Dirac gas at low density and/or high fem

$$p = kTp \left(1 + \frac{\alpha^3 z^3}{+z^3} + \dots\right).$$

$\longrightarrow$ online lecture 8 notes
alow density limit $e^{\beta\mu} \ll 1, e^{\beta\mu} = z''$ 'fugacity'

- FD -BE partition fan

$$\begin{aligned}
\ln \Xi &= \sum_i \ln \left(1 \mp e^{\beta\mu - \beta\varepsilon_i}\right)^{\mp 1} \\
&= \frac{1}{8} \cdot \frac{V}{4\pi^3\hbar^3} \int_0^\infty dp 4\pi p^2 \ln \left(1 \mp e^{\beta\mu - \beta p^2/2m}\right)^{\mp 1} \\
&= \frac{V}{1^3} \frac{4}{\sqrt{\pi}} \int_0^\infty dx \cdot x^2 \ln \left(1 \mp \left\{ e^{-x^2}\right)^{\mp 1}
\end{aligned}$$

$\{ \rightarrow 0$, expand series in ξ and integrate

$$= \frac{V}{\Lambda^3} \left(\frac{4}{\sqrt{\pi}}\right) \left(\frac{\sqrt{\pi}}{4}\right) \left[\sum_{n=1}^{\infty} (F \mid)^{n+1} \frac{\xi^n}{n^{s/2}}\right] = \frac{V}{\Lambda^3} f_{s/2}(\xi)$$

$$dW = -SdT - PdV - Nd\mu$$

$$W = -kT \ln \bar{U}$$

$$P = -\left(\frac{\delta W}{\delta V}\right)_{T,\mu} = kT \left(\frac{1}{\Lambda^3} f_{s/2}(\xi)\right)$$

$$N = -\left(\frac{\delta W}{\delta \beta\mu}\right)_{T,V} = \left(\frac{kT\delta \ln \Xi}{\delta\mu}\right) = \xi kT \left(\frac{V}{\Lambda^3} f_{s/2}(\xi)\right)$$

$$\beta P(T,\mu) = \frac{f_{s/2}(\xi)}{\Lambda^3}, \beta N = \frac{V}{\Lambda^3} f_{3/2}(\xi)$$

expand in

$$\begin{aligned}
{}^n f_{s/2}(\xi) &= \left\{ F \frac{\xi^2}{2^{s/2}} \cdots \right. \\
\beta P &= \frac{1}{1^3} \left(\xi^1 + \frac{\xi^2}{2^{5/2}} \cdots\right) \\
\frac{N}{V} = \rho &= \frac{1}{1^3} \left(\xi + \xi^2/2^{3/2} \cdots\right)
\end{aligned}$$

$\beta P = \rho$ ideal gas
need to get $\beta P = \rho f(\xi) = \rho \left(\Lambda^3 \rho\right)$ viral expansion $\rightarrow$ eliminate $\{$ from ρ
plug $\{$ in

$$\Lambda^3 \rho = \left\{ \mp \frac{\gamma^2}{2^{3/2}} \rightarrow \right\{ = \Lambda^3 \rho \mp \frac{\xi^2}{2^{3/2}}$$

to βP

$$\xi = \Lambda^3 p \mp \frac{\left(\Lambda^3 p\right)^2}{2^{3/2}}$$

$$P = kT\frac{1}{\Lambda_3}\left(\Lambda^3 p \mp \frac{\left(\Lambda^3 p\right)^2}{2^{2/3}} \mp\right.$$

$$P = kTp\left(1 \mp \frac{\Lambda^3 p}{2^{2/5} \cdots)}\right.$$

$$P = kT\frac{1}{\Lambda_3}\left(\Lambda^3 p \mp \frac{\left(\Lambda^3 p\right)^2}{2^{2/3}} \mp \frac{\left(\Lambda^3 p\right)^2}{2^{5/2}} \cdots\right)$$