

# Homework 3

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## 1 Understanding the simple harmonic oscillator intuitively in a spatial basis

### 1.1 How we got $a$ and $a^\dagger$

This was done when we attempted to complete the square in our Hamiltonian, which went like  $X^2 + P^2$ . We neglect the constant terms here. Thus, we got that our Hamiltonian factors to something like  $(X + iP)(X - iP)$ . We associated  $a$  with  $X + iP$  and  $a^\dagger$  with  $X - iP$ .

### 1.2 Define $X$ and $P$ in terms of $a$ and $a^\dagger$

We define  $X$  and  $P$ , keeping track of the constants now, as follows:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad (1)$$

$$P = i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) \quad (2)$$

### 1.3 Calculation of $\langle X^2 \rangle$ and $\langle P^2 \rangle$

We define the expectation value of an operator as:

$$\langle A \rangle = \langle \Psi | A | \Psi \rangle \quad (3)$$

Working in terms of the states of the simple harmonic oscillator, we first calculate  $\langle X^2 \rangle$ :

$$\langle X^2 \rangle = \langle n | X^2 | n \rangle \quad (4)$$

Substituting in our expression for  $X$  from (1):

$$= \frac{\hbar}{2m\omega} \langle n | (a + a^\dagger)^2 | n \rangle \quad (5)$$

Expanding the square:

$$= \frac{\hbar}{2m\omega} \langle n | (a^2 + a^\dagger a + a a^\dagger + (a^\dagger)^2) | n \rangle \quad (6)$$

The first and the last term will vanish due to orthonormality, so we just have to consider the middle terms:

$$= \frac{\hbar}{2m\omega} \langle n | (a^\dagger a + a a^\dagger) | n \rangle \quad (7)$$

Now, using our definitions for the action of  $a$  and  $a^\dagger$  on the states of the simple harmonic oscillator:

$$a | n \rangle = \sqrt{n} | n - 1 \rangle \quad (8)$$

$$a^\dagger | n \rangle = \sqrt{n + 1} | n + 1 \rangle \quad (9)$$

First we consider the first term of equation 7:

$$\langle n | a^\dagger a | n \rangle = \langle n | a^\dagger \sqrt{n} | n - 1 \rangle = \sqrt{n} \sqrt{n} \langle n | n \rangle = n \langle n | n \rangle = n \quad (10)$$

Now we consider the second term of equation 7:

$$\langle n | a a^\dagger | n \rangle = \langle n | \sqrt{n + 1} | n + 1 \rangle = \sqrt{n + 1} \sqrt{n + 1} \langle n | n \rangle = (n + 1) \langle n | n \rangle = n + 1 \quad (11)$$

Now we can substitute these results back into equation 7:

$$= \frac{\hbar}{2m\omega} (n + n + 1) = \frac{\hbar}{2m\omega} (2n + 1) \quad (12)$$

Now we move on to the calculation of  $\langle P^2 \rangle$ :

$$\langle P^2 \rangle = \langle n | P^2 | n \rangle \quad (13)$$

Since

$$P = i \sqrt{\frac{m\omega\hbar}{2}} (a - a^\dagger) \quad (14)$$

$$\langle P^2 \rangle = -\frac{m\omega\hbar}{2} \langle n | (a - a^\dagger)^2 | n \rangle \quad (15)$$

Expanding the square:

$$= -\frac{m\omega\hbar}{2} \langle n | (a^2 - a^\dagger a - aa^\dagger + (a^\dagger)^2) | n \rangle \quad (16)$$

The first and the last term will vanish due to orthonormality, so we just have to consider the middle terms:

$$= -\frac{m\omega\hbar}{2} \langle n | (-a^\dagger a - aa^\dagger) | n \rangle \quad (17)$$

Now, using our definitions for the action of  $a$  and  $a^\dagger$  on the states of the simple harmonic oscillator:

$$a | n \rangle = \sqrt{n} | n - 1 \rangle \quad (18)$$

$$a^\dagger | n \rangle = \sqrt{n+1} | n + 1 \rangle \quad (19)$$

First we consider the first term of equation 17:

$$\langle n | a^\dagger a | n \rangle = n \quad (20)$$

Now we consider the second term of equation 17:

$$\langle n | aa^\dagger | n \rangle = n + 1 \quad (21)$$

Now we can substitute these results back into equation 17:

$$= +\frac{m\omega\hbar}{2} (n + n + 1) = \boxed{\frac{m\omega\hbar}{2} (2n + 1)} \quad (22)$$

## 1.4 Uncertainty of Operators

We start by calculating  $\Delta X$ :

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \quad (23)$$

We already have  $\langle X^2 \rangle$  from part 3, so we just need to calculate  $\langle X \rangle$ :

$$\langle X \rangle = \langle n | X | n \rangle \quad (24)$$

Substituting in our expression for  $X$ :

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle n | (a + a^\dagger) | n \rangle \quad (25)$$

Due to the orthonormality of the states, both terms will vanish to give:

$$\langle X \rangle = 0 \rightarrow \langle X \rangle^2 = 0 \quad (26)$$

So we have that:

$$\Delta X = \sqrt{\left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(2n+1)} = \boxed{\left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{4}} \sqrt{2n+1}} \quad (27)$$

Next, we consider  $\Delta P$ :

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} \quad (28)$$

We will have a similar argument why the second term vanishes, so we can just plug in our answer for the first term to get:

$$\Delta P = \sqrt{\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{2}}(2n+1)} = \boxed{\left(\frac{m\omega\hbar}{2}\right)^{\frac{1}{4}} \sqrt{2n+1}} \quad (29)$$

#### 1.4.1 Question

Intuitively, what does this let us say about any wavefunction that satisfies the SHO solution to the perturbation of a potential minimum?

*Hint:* Does the wavefunction become more or less spread in  $X$ , ...

#### 1.4.2 Answer

As we go up the rungs of the ladder of the SHO, due to the  $\sqrt{2n+1}$  dependents of the uncertainty in position, the wave function will become more spread out in  $X$ .

## 2 Quantizing the Simple Harmonic Oscillator

*Objective:* To explore if quantizing the simple harmonic oscillator actually changes the kinetics, and if commutators of operators can really predict the time dependence of physical variables.

### 2.1 Time Dependent State

#### 2.1.1 Question

A time dependent state satisfies  $i\hbar \frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$ . Let  $\Omega$  be any time-independent operator. Show that

$$\frac{d}{dt}\langle\Psi(t)|\Omega|\Psi(t)\rangle = \frac{i}{\hbar}\langle\Psi(t)|[H, \Omega]|\Psi(t)\rangle.$$

#### 2.1.2 Solution

We start by distributing the time derivative into all three elements of the left side of the equation:

$$\frac{d}{dt}\langle\Psi(t)|\Omega|\Psi(t)\rangle = \langle\Psi(t)|\Omega\left|\frac{d}{dt}\Psi(t)\right\rangle + \langle\Psi(t)|\frac{d}{dt}\Omega|\Psi(t)\rangle + \left\langle\frac{d}{dt}\Psi(t)\right|\Omega|\Psi(t)\rangle \quad (30)$$

We recognize:

$$\frac{d}{dt}|\Psi(t)\rangle = \frac{1}{i\hbar}H|\Psi(t)\rangle \quad (31)$$

Furthermore, we recognize that the adjoint of this is:

$$\frac{d}{dt}\langle\Psi(t)| = -\frac{1}{i\hbar}\langle\Psi(t)|H \quad (32)$$

Substituting these into equation 30:

$$\frac{d}{dt}\langle\Psi(t)|\Omega|\Psi(t)\rangle = \langle\Psi(t)|\Omega\left(\frac{1}{i\hbar}H|\Psi(t)\rangle\right) + \langle\Psi(t)|\frac{d}{dt}\Omega|\Psi(t)\rangle - \left(\frac{1}{i\hbar}\langle\Psi(t)|H\right)\Omega|\Psi(t)\rangle \quad (33)$$

We can confidently conclude that the middle term vanishes because  $\Omega$  is time independent e.g.  $\frac{d}{dt}\Omega = 0$ . Pulling out the constance from the first and last terms, we get:

$$\frac{d}{dt}\langle\Psi(t)|\Omega|\Psi(t)\rangle = \frac{1}{i\hbar}\langle\Psi(t)|\Omega H|\Psi(t)\rangle - \frac{1}{i\hbar}\langle\Psi(t)|H\Omega|\Psi(t)\rangle \quad (34)$$

We recognize the commutator:

$$\frac{d}{dt} \langle \Psi(t) | \Omega | \Psi(t) \rangle = \frac{1}{i\hbar} \langle \Psi(t) | \Omega H | \Psi(t) \rangle - \frac{1}{i\hbar} \langle \Psi(t) | H \Omega | \Psi(t) \rangle = \boxed{\frac{i}{\hbar} \langle \Psi(t) | [H, \Omega] | \Psi(t) \rangle} \quad (35)$$

## 2.2 Harmonic Oscillator Specialization

### 2.2.1 Question

Now specialize using part 1 to the harmonic oscillator. You will need to use the fact that

$$\frac{i}{\hbar} [H, X] = \frac{1}{m} P \quad , \quad \frac{i}{\hbar} [H, P] = -m\omega^2 X.$$

For the harmonic oscillator, show that the expectation value of the position satisfies the classical equation of motion

$$\left[ \frac{d^2}{dt^2} + \omega^2 \right] \langle \Psi(t) | X | \Psi(t) \rangle = 0.$$

### 2.2.2 Solution

Since both the position and momentum operators are time independent we can substitute them for  $\Omega$  in the result from part 1. Starting from the left most term, we can make a substitution:

$$\frac{d}{dt} \left( \frac{d}{dt} \langle \Psi(t) | X | \Psi(t) \rangle \right) = \frac{d}{dt} \left( \langle \Psi(t) | \frac{i}{\hbar} [H, X] | \Psi(t) \rangle \right) \quad (36)$$

Now we can substitute in the commutator from the problem statement:

$$\frac{d}{dt} \left( \langle \Psi(t) | \frac{i}{\hbar} [H, X] | \Psi(t) \rangle \right) = \frac{d}{dt} \left( \langle \Psi(t) | \left( \frac{1}{m} P \right) | \Psi(t) \rangle \right) = \frac{1}{m} \frac{d}{dt} (\langle \Psi(t) | P | \Psi(t) \rangle) \quad (37)$$

Now we can substitute in the result from part 1:

$$= \frac{1}{m} \frac{i}{\hbar} \langle \Psi(t) | [H, P] | \Psi(t) \rangle = \frac{1}{m} \langle \Psi(t) | (-m\omega^2 X) | \Psi(t) \rangle = -\omega^2 \langle \Psi(t) | X | \Psi(t) \rangle \quad (38)$$

So, we have:

$$\boxed{[-\omega^2 + \omega^2] \langle \Psi(t) | X | \Psi(t) \rangle = 0} \quad (39)$$

## 2.3 Time Derivation and Operator Commutation

### 2.3.1 Question

Show that for the harmonic oscillator

$$\frac{d}{dt}\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{1}{m}\langle\Psi(t)|PX + XP|\Psi(t)\rangle.$$

You will need  $[H, AB] = [H, A]B + A[H, B]$ .

### 2.3.2 Solution

Since the position squared is a time independent operator, we can use the relation from part 1:

$$\frac{d}{dt}\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{i}{\hbar}\langle\Psi(t)|[H, X^2]|\Psi(t)\rangle \quad (40)$$

Now, from the problem statement, we have that:

$$[H, X^2] = [H, X]X + X[H, X] = \frac{\hbar}{im}(PX + XP) \quad (41)$$

So, we can get our desired result:

$$\frac{d}{dt}\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{i}{\hbar}\langle\Psi(t)|\frac{\hbar}{im}(PX + XP)|\Psi(t)\rangle = \boxed{\frac{1}{m}\langle\Psi(t)|PX + XP|\Psi(t)\rangle} \quad (42)$$

### 2.3.3 Question

Then take another time derivative to show that

$$\left[\frac{d^2}{dt^2} + 4\omega^2\right]\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{4}{m}\langle\Psi(t)|H|\Psi(t)\rangle.$$

## 2.4 Solution

Will start by taking another time derivative of our previous result:

$$\frac{d}{dt}\left(\frac{d}{dt}\langle\Psi(t)|X^2|\Psi(t)\rangle\right) = \frac{1}{m}\left(\frac{d}{dt}\langle\Psi(t)|PX|\Psi(t)\rangle + \frac{d}{dt}\langle\Psi(t)|XP|\Psi(t)\rangle\right) \quad (43)$$

We will first focus on the first term on the right side of the equality:

$$\frac{d}{dt}\langle\Psi(t)|PX|\Psi(t)\rangle \quad (44)$$

in part 1, we found that:

$$\frac{d}{dt}\langle\Psi(t)|PX|\Psi(t)\rangle = \frac{i}{\hbar}\langle\Psi(t)|[H, PX]|\Psi(t)\rangle \quad (45)$$

Now,

$$\frac{i}{\hbar}[H, PX] = \frac{i}{\hbar}[H, P]X + \frac{i}{\hbar}P[H, X] = -m\omega X^2 + \frac{1}{m}P^2 \quad (46)$$

Likewise, for the other order, we have:

$$\frac{i}{\hbar}[H, XP] = \frac{i}{\hbar}[H, X]P + \frac{i}{\hbar}X[H, P] = -m\omega X^2 + \frac{1}{m}P^2 \quad (47)$$

So, we have that

$$\frac{d}{dt}\left(\frac{d}{dt}\langle\Psi(t)|X^2|\Psi(t)\rangle\right) = \frac{2}{m}\left(\langle\Psi(t)|\frac{P^2}{m} - m\omega X^2|\Psi(t)\rangle\right) = -2\omega^2\langle\Psi(t)|X^2|\Psi(t)\rangle + \frac{2}{m^2}\langle\Psi(t)|P^2|\Psi(t)\rangle \quad (48)$$

So we have cotton that:

$$\left[\frac{d^2}{dt^2} + 4\omega^2\right]\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{2}{m^2}\langle\Psi(t)|P^2|\Psi(t)\rangle + 2\omega^2\langle\Psi(t)|X^2|\Psi(t)\rangle \quad (49)$$

Now we consider that:

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 \rightarrow \frac{4}{m}H = \frac{2}{m^2}P^2 + 2\omega^2 X^2 \quad (50)$$

This matches what we got above, so:

$$\left[\frac{d^2}{dt^2} + 4\omega^2\right]\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{2}{m^2}\langle\Psi(t)|P^2|\Psi(t)\rangle + 2\omega^2\langle\Psi(t)|X^2|\Psi(t)\rangle = \frac{4}{m}\langle\Psi(t)|H|\Psi(t)\rangle$$

(51)

## 2.5 Classical Harmonic Oscillator

### 2.5.1 Question

The solution to the classical harmonic oscillator is  $x_{Cl} = x_0 \cos(\omega t + \phi)$  and the energy is  $E_{Cl} = \frac{1}{2}m\omega^2 x_0^2$ . Show that

$$\left[\frac{d^2}{dt^2} + 4\omega^2\right]x_{Cl}^2 = \frac{4}{m}E_{Cl}.$$



### 2.5.2 Solution

We square the solution to the classical harmonic isolator:

$$x_{Cl}^2 = x_0^2 \cos^2(\omega t + \phi) \quad (52)$$

First, we take one derivative with respect to time, invoking they chain role. Treating  $x_0^2$  as a constant, we get on to the nasty algebra

$$\frac{d}{dt} x_{Cl}^2 = -x_0^2 (2 \cos(\omega t + \phi) (-\sin(\omega t + \phi) \omega)) = -\omega x_0^2 \sin(2(\omega t + \phi)) \quad (53)$$

Now we take another derivative with respect to time:

$$\frac{d}{dt} \frac{d}{dt} x_{Cl}^2 = -\omega x_0^2 (2 \cos(\omega t + \phi) \omega) = -2\omega^2 x_0^2 \cos(2(\omega t + \phi)) \quad (54)$$

Now, plugging in the equation from the problem statement:

$$\left[ \frac{d^2}{dt^2} + 4\omega^2 \right] x_{Cl}^2 = -2\omega^2 x_0^2 \cos(2(\omega t + \phi)) + 4\omega^2 (x_0^2 \cos^2(\omega t + \phi)) \quad (55)$$

Now, we have the relation

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \rightarrow \cos^2(\omega t + \phi) = \frac{1}{2}(1 + \cos(2(\omega t + \phi))) \quad (56)$$

So, we can substitute this in to get:

$$\left[ \frac{d^2}{dt^2} + 4\omega^2 \right] x_{Cl}^2 = -2\omega^2 x_0^2 \cos(2(\omega t + \phi)) + 4\omega^2 x_0^2 \left( \frac{1}{2}(1 + \cos(2(\omega t + \phi))) \right) \quad (57)$$

$$= -2\omega^2 x_0^2 \cos(2(\omega t + \phi)) + 2\omega^2 x_0^2 + 2\omega^2 x_0^2 \cos(2(\omega t + \phi)) = 2\omega^2 x_0^2 \quad (58)$$

Now, for the right and side we have

$$\frac{4}{m} E_{Cl} = \frac{4}{m} \frac{1}{2} m \omega^2 x_0^2 = 2\omega^2 x_0^2 \quad (59)$$

So, we have that:

$$\boxed{\left[ \frac{d^2}{dt^2} + 4\omega^2 \right] x_{Cl}^2 = 2\omega^2 x_0^2 = \frac{4}{m} E_{Cl}} \quad (60)$$

**2.5.3 In other words, intuitively, does it make any change if we solve for our dynamics using only operator commutations?**

Know, it doesn't and the algebra is much simpler :)

### 3 Problem 3

For this problem, let's explore raising and lowering operators in something that's not quite a simple harmonic oscillator (SHO). The kinetic energy is the same as the SHO, but let's define our potential as a quartic:

$$V(x) = \frac{k}{2}x^4$$

We will assume that  $\omega = \sqrt{\frac{k}{m}}$  still holds.

**3.1 Write the Hamiltonian of this system. (Hint: think kinetic + potential)**

$$H = \frac{p^2}{2m} + \frac{k}{2}x^4 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^4 = \frac{1}{2m} (p^2 + m^2\omega^2x^4) = \frac{1}{2m} (p^2 + (m\omega x^2)^2) \quad (61)$$

**3.2 'Factor' the Hamiltonian of the quartic potential into the product of two terms. Define the “+” factor as  $b$  and the “-” factor as  $b^\dagger$ . (Hint: you should get something that looks like the SHO, but with  $x^2$  instead of  $x$ )**

In the previous section, we got for the hamiltonian:

$$H = \frac{1}{2m} (p^2 + (m\omega x^2)^2) \quad (62)$$

Completing the square, defining  $p = c$  and  $m\omega x^2 = d$ , we get:

$$H = \frac{1}{2m} (c^2 + d^2) = \frac{1}{2m} ((c + id)(c - id)) \quad (63)$$

Substituting in for  $c$  and  $d$ , we find

$$H = \frac{1}{2m} ((p + im\omega x^2)(p - im\omega x^2)) \quad (64)$$

Now we are in a place to define  $b$  and  $b^\dagger$ :

$$b = \sqrt{\frac{m\omega}{2\hbar}}x^2 + \frac{i}{\sqrt{2m\omega\hbar}}p \quad (65)$$

$$b^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x^2 - \frac{i}{\sqrt{2m\omega\hbar}}p \quad (66)$$

### 3.3 What is $[b^\dagger, b]$ ? (Hint: calculate the commutator of $x^2$ with $\hat{p}$ )

We begin by considering that

$$[b^\dagger, b] = b^\dagger b - bb^\dagger \quad (67)$$

We will first look at the first term:

$$b^\dagger b = \left( \sqrt{\frac{m\omega}{2\hbar}}x^2 - \frac{i}{\sqrt{2m\omega\hbar}}p \right) \left( \sqrt{\frac{m\omega}{2\hbar}}x^2 + \frac{i}{\sqrt{2m\omega\hbar}}p \right) \quad (68)$$

multiplying out, we get

$$b^\dagger b = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 + \frac{i}{2\hbar}(x^2p - px^2) = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 + \frac{i}{2\hbar}([x^2, p]) \quad (69)$$

We can use the relation

$$[x^2, p] = x[x, p] + [x, p]x = x(i\hbar) + (i\hbar)x = 2i\hbar x \quad (70)$$

So, we can substitute this in to get:

$$b^\dagger b = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 + \frac{i}{2\hbar}(2i\hbar x) = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 - x \quad (71)$$

Now, we consider the second term:

$$bb^\dagger = \left( \sqrt{\frac{m\omega}{2\hbar}}x^2 + \frac{i}{\sqrt{2m\omega\hbar}}p \right) \left( \sqrt{\frac{m\omega}{2\hbar}}x^2 - \frac{i}{\sqrt{2m\omega\hbar}}p \right) \quad (72)$$

multiplying out, we get

$$bb^\dagger = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 - \frac{i}{2\hbar}(x^2p - px^2) = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 - \frac{i}{2\hbar}([x^2, p]) \quad (73)$$

We can use the relation

$$[x^2, p] = x[x, p] + [x, p]x = x(i\hbar) + (i\hbar)x = 2i\hbar x \quad (74)$$

So, we can substitute this in to get:

$$bb^\dagger = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 - \frac{i}{2\hbar}(2i\hbar x) = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 + x \quad (75)$$

Now, we can substitute these results back into equation 59:

$$[b^\dagger, b] = \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 - x - \left( \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 + x \right) = \boxed{-2x} \quad (76)$$

**3.4 Rewrite the Hamiltonian of the quartic oscillator in terms of the product  $b^\dagger b$  that you calculated in the previous step. How is energy in a quartic oscillator different from an SHO?**

$$H = \hbar\omega \left( \frac{m\omega}{2\hbar}x^4 + \frac{1}{2m\omega\hbar}p^2 - x + x \right) = \hbar\omega (b^\dagger b + x) \quad (77)$$

In the SHO, we had:

$$H = \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) \quad (78)$$

So, instead of the energy depending on a constant term, it depends on the position of the particle. We get an energy spacing is still separated by  $\hbar\omega$ , but depends on the position, as shown in this image:

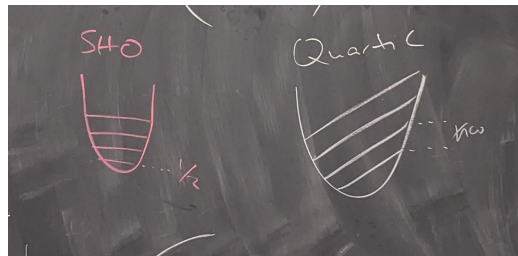


Figure 1: Energy levels of the quartic oscillator