

READING: Section 19.1-19.3 in Shankar on scattering and the Born approximation. PROBLEMS:

## 1

21. Demonstrate our claim in the long solenoid discussion that the solution to the Schrödinger equation when there is a flux  $\Phi$  in the solenoid is:

$$\psi(\mathbf{x}, t) = \psi_L(\mathbf{x}, t)e^{iqg_L(\mathbf{x})} + \psi_R(\mathbf{x}, t)e^{iqg_R(\mathbf{x})}$$

where

$$\begin{aligned} g_L(\mathbf{x}) &\equiv \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x}' \cdot \mathbf{A}(\mathbf{x}) && \text{along a left path} \\ g_R(\mathbf{x}) &\equiv \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x}' \cdot \mathbf{A}(\mathbf{x}) && \text{along a right path} \end{aligned}$$

and  $\psi_{L,R}(\mathbf{x}, t)$  satisfy the Schrödinger equation when  $\Phi = 0$ .

### 1.1

In case, the Hamiltonian we wanted to solve was:

$$H = \frac{1}{2m} [\mathbf{p} - q\mathbf{A}(\mathbf{x}, t)]^2 + q\Phi(\mathbf{x}, t) + U(\mathbf{x}, t) \quad (1)$$

So, we want to plug the given solution in, which is:

$$\psi(\mathbf{x}, t) = \psi_L(\mathbf{x}, t)e^{iqg_L(\mathbf{x})} + \psi_R(\mathbf{x}, t)e^{iqg_R(\mathbf{x})} \quad (2)$$

We want to show that the show enter equation sat since this relation for the the and vote sites independently: We are kippen:

$$\Psi = \Phi_L + \Phi_R \quad (3)$$

Where:

$$\Phi_L = \psi_L(\mathbf{x}, t)e^{iqg_L(\mathbf{x})} \quad (4)$$

We not that this set size the time dependent short ensure equation:

$$i\partial_t \Phi_L = H\Phi_L \quad (5)$$

22. Prove the theorem (essentially exercise 18.4.4 in text):

Theorem: Let the Hamiltonian for a charged particle interacting with an electromagnetic field be  $H$  :

$$H = \frac{1}{2m} [\mathbf{p} - q\mathbf{A}(\mathbf{x}, t)]^2 + q\Phi(\mathbf{x}, t) + U(\mathbf{x}, t)$$

Let  $H'$  be the Hamiltonian obtained from  $H$  by a gauge transformation:

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) &\rightarrow \mathbf{A}'(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t) + \nabla\chi(\mathbf{x}, t) \\ \Phi(\mathbf{x}, t) &\rightarrow \Phi'(\mathbf{x}, t) = \Phi(\mathbf{x}, t) - \partial_t\chi(\mathbf{x}, t)\end{aligned}$$

If  $i\partial_t\psi = H\psi$  and  $i\partial_t\psi' = H'\psi'$ , then

$$\psi'(\mathbf{x}, t) = e^{iq\chi(\mathbf{x}, t)}\psi(\mathbf{x}, t)$$

We will start by making the ansatz that  $\Psi' = e^{iq\chi(\mathbf{x}, t)}\Psi$ . We will call the first time dependent short ensure equation,  $A$ , and the second,  $B$ . and if we have guessed  $\Psi'$  correctly, then  $B$  is true. We want to consider the relationship:

$$e^{-iq\chi(\mathbf{x}, t)}B - A \tag{6}$$

## 1.2

The hambletonian with the gauge transformation is given by the same, but with the prime notation:

$$H' = \frac{1}{2m} [\mathbf{p} - q\mathbf{A}'(\mathbf{x}, t)]^2 + q\Phi'(\mathbf{x}, t) + U(\mathbf{x}, t) \tag{7}$$

Plugging in the relations for the crimes:

$$H' = \frac{1}{2m} [\mathbf{p} - q\mathbf{A}(\mathbf{x}, t) - q\nabla\chi(\mathbf{x}, t)]^2 + q\Phi(\mathbf{x}, t) - q\partial_t\chi(\mathbf{x}, t) + U(\mathbf{x}, t) \tag{8}$$

We assume that our solution is like  $e^{iq\chi(\mathbf{x}, t)}\psi(\mathbf{x}, t)$ , so we can plug this in to the Hamiltonian:

## 2

23. Exercise 18.5.2 in the text, on the photoelectric effect.

## 2.1

Actually, we have:

$$\sigma = \frac{128a_0^3\pi e^2 p_f^3}{3m\hbar^3\omega c [1 + p_f^2 a_0^2 / \hbar^2]^4} \quad (9)$$

We aval with this to:

$$\sigma = 1.31204968079106 \cdot 10^{-34} \quad (10)$$

```
1 from sympy import symbols, sqrt, pi, latex
2
3 # Constants
4 eV_to_J = 1.602e-19 # Conversion from eV to Joules
5 Ry_to_eV = 13.6 # Conversion from Rydbergs to eV
6 a_0 = 0.529e-10 # Bohr radius in meters
7 m = 9.11e-31 # Electron mass in kg
8 e = 1.6e-19 # Elementary charge in Coulombs
9 c = 3e8 # Speed of light in m/s
10 hbar = 1.054e-34 # Reduced Planck's constant in Js
11
12 # Kinetic energy in Rydbergs
13 KE_Ry = 10
14 # Convert Ry to eV to J
15 KE_J = KE_Ry * Ry_to_eV * eV_to_J
16
17 # Calculate the momentum p_f using the relation E = p^2/2
18 # m (non-relativistic kinetic energy)
19 # p_f = sqrt(2*m*E)
20 p_f_value = sqrt(2 * m * KE_J)
21
22 # Calculate the second expression
23 omega = (KE_Ry * Ry_to_eV * eV_to_J) / hbar # Angular
24 # frequency
25 second_expression = (128 * a_0**3 * pi * e**2 * p_f_value
26 **3) / (3 * m * hbar**3 * omega * c * (1 + p_f_value
27 **2 * a_0**2 / hbar**2)**4)
28
29 # Print the second result in LaTeX
30 print(latex(second_expression.evalf()))
```

That of the atom is much smaller:

```

1 # compute the atoms geometric cross section comma which
  is given by a_0^2\pi
2 pi = 3.14159265358979323846
3 atoms_cross_section = a_0**2 * pi
4 print(latex(atoms_cross_section))

```

$$\sigma_{\text{atom}} = 8.79146429773221 \cdot 10^{-21} \quad (11)$$

## 2.2

First, we want to check the condition of  $\frac{p_f a_0}{\hbar}$  for a kinetic energy of 10Ry, where  $a_0$  is the Bohr radius. We have:

```

1 from sympy import *
2 # estimate the value of  $\frac{p_f a_0}{\hbar}$  for a
  kinetic energy of 10 \text{Ry} in a hydrogen atom
3 # were  $p_f$  is the momentum of the electron and  $a_0$  is
  the Bohr radius
4 p_f, a_0, hbar = symbols('p_f a_0 hbar')
5 p_f = sqrt(2*10*13.6)
6 a_0 = 0.529e-10
7 hbar = 1.054e-34
8 print(latex(p_f*a_0/hbar))

```

giving the result:

$$\frac{p_f a_0}{\hbar} = 8.27750617059485 \cdot 10^{24} \quad (12)$$

which is much greater than 1. So, we can use the approximation for the differential cross section given in the text:

$$\frac{d\sigma}{d\Omega} = \frac{32e^2 \hbar^5 \cos^2 \theta}{mc\omega p_f^5 a_0^5} \quad (13)$$

We bring the solid angle over to the other side, bring the constance out in front and integrate:

$$\sigma = \frac{64\pi e^2 \hbar^5}{mc\omega p_f^5 a_0^5} \int_0^\pi \cos^2 \theta \sin \theta d\theta \quad (14)$$

### 3

24. The text discusses the photoelectric effect using the "dipole approximation". While an electron is ejected from the atom in the photoelectric effect, this approximation is useful for lower energy interactions as well. We thus investigate induction of an atomic dipole moment due to an external electromagnetic field. We'll set up some things in this problem, and continue the discussion in the next problem set. As we have discussed, the Hamiltonian for an atomic electron in an external electromagnetic field may be written, in the Coulomb gauge:

18.5.30

$$H = H_0 + H_1,$$

where

$$H_0 = \frac{P^2}{2m} + V(R)$$

and

$$H_1 = -\frac{q}{m}\mathbf{P} \cdot \mathbf{A}(\mathbf{x}, t) + \frac{q^2}{2m}\mathbf{A}(\mathbf{x}, t)^2 - \frac{q}{m}\mathbf{S} \cdot \mathbf{B}(\mathbf{x}, t)$$

We have included here the possibility of an interaction of the spin magnetic moment with the magnetic field.

We could discuss the relative strength of the different terms in a given situation, but here we'll assume that the  $\mathbf{P} \cdot \mathbf{A}(\mathbf{x}, t)$  term dominates.

(a) With this assumption, and also assuming that the wave is traveling in the  $y$  direction and the polarization is in the  $z$  direction, write  $H_1$  in terms of the wavenumber  $k$  mode of the plane wave expansion for the field.

(b) The dipole approximation consists in assuming that the external field varies slowly over the relevant distance scale of the problem. Thus, the  $e^{\pm i\mathbf{k} \cdot \mathbf{x}}$  factors are expanded in Taylor series and only the first term kept. In this approximation, write  $H_D = H_1$  in terms of the strength  $E_0 = |\mathbf{E}|$  of the electric field, where the  $D$  subscript indicates the dipole approximation. To simplify the algebra, make a choice of phase of pure imaginary for the relevant expansion coefficient of the vector potential.