Homework 4 – CH125A

The purpose of this homework is to get you comfortable with angular momentum for next week when we work on spin-orbit coupling and angular momentum addition. The other problem introduces an interesting aspect of spin, which we will further treat in class next week.

Problem 1: Evaluate the following commutators and angular momentum relations

A. Calculate the following matrix element: $\langle l, m+2|(L_x)^2|l, m\rangle$

B. A particle moves in a three-dimensional potential $V(x, y, z)$ that satisfies $[L_y, V] = 0$ and $[L_z, V] = 0$. Prove that $[L_x, V] = 0$. What does this mean intuitively about the potential?

C. The matrix representation of L_x for spin one is

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the eigenvector of L_x that corresponds to the zero eigenvalue. For an angular momentum 1 particle with $|\Psi\rangle = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ what is the probability that a measurement of L_x will give zero, as in what is $|\langle L_x = 0|\Psi\rangle|^2$?

Problem 2: Outer products

A. What is the outer product $L \otimes S$ for $L = 2,\ S = \frac{3}{2}$? (In other words, what are the possible values of J_{total} for $J = L + S$?)

B. Describe each of the possible states with spectroscopic notation.

C. Now, imagine that we apply a strong magnetic field to our particle. Magnetic fields couple with angular momentum, allowing for degeneracy to be fully broken (i.e., each possible state has a different energy). In this case, we will assume that <u>lower values of J correspond to lower energies</u>. Draw an energy level diagram for this system in the presence of a magnetic field and specify each state with its spectroscopic label.

Problem 3: This problem will show that there is not a coordinate space representation of spin ½. In other words, there are no spherical harmonics corresponding to spin ½. This is why spin is talked about as an internal degree of freedom, with the total angular momentum being $J = L + S$.

A. Show that $L_+\left|\frac{1}{2}, \frac{1}{2}\right\rangle = 0$ and $L_-\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = 0$.

B. From the coordinate space representation of the raising and lower operators (Equation 12.5.27 Shankar), we expect that

$$L_+\left|\frac{1}{2},\frac{1}{2}\right\rangle = 0 \ \rightarrow \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right)Y_{\frac{1}{2}}^{\frac{1}{2}}(\theta,\phi) = 0$$

$$L_-\left|\frac{1}{2},-\frac{1}{2}\right\rangle = 0 \ \rightarrow \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot(\theta)\frac{\partial}{\partial\phi}\right)Y_{\frac{1}{2}}^{-\frac{1}{2}}(\theta,\phi) = 0$$

To be eigenfunctions of L$_z$ the states must have a ϕ dependence of $e^{\mp\frac{i\phi}{2}}$. Show that the solutions are $Y_{\frac{1}{2}}^{\frac{1}{2}}(\theta,\phi) = A(sin\theta)^{1/2}e^{\frac{i\phi}{2}}$ and $Y_{\frac{1}{2}}^{-\frac{1}{2}}(\theta,\phi) = B(sin\theta)^{1/2}e^{\frac{-i\phi}{2}}$ where A and B are normalization constants.

C. The answers in (a) look fine, but unfortunately they are not consistent with two more results, namely that $L_+\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \hbar\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $L_-\left|\frac{1}{2},\frac{1}{2}\right\rangle = \hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle$. Starting from $Y_{\frac{1}{2}}^{\frac{1}{2}}(\theta,\phi)$, the lowering operator should give $Y_{\frac{1}{2}}^{-\frac{1}{2}}(\theta,\phi)$. Show that

$$L_-Y_{\frac{1}{2}}^{\frac{1}{2}} = -\hbar A\frac{cos\theta}{\sqrt{sin\theta}}e^{-\frac{i\phi}{2}}$$

The right hand side is not equal to $Y_{\frac{1}{2}}^{-\frac{1}{2}}$; in fact it is infinite at $\theta = 0$ and π. The same can be shown for the raising operator. In other words, no set of spherical harmonics exist for spin ½.

This is a good example, again, of how we started with arbitrary definitions of raising and lowering operators, then found eigenvectors including real-space descriptions, which then lead to new science. In particular, spin ½ is a different degree of freedom that has no real space definition.