1 Problem 1: Adding special relativity to the Schrödinger equation

1.1

We start with the original deferential equation:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{1}$$

first, we consider the plain wave solution:

$$E(x,t) = E_0 \exp(i(kx - \omega t)) \tag{2}$$

We take the second partial derivative with respect to x:

$$\frac{\partial^2 E}{\partial x^2} = -k^2 E(x,t) \tag{3}$$

We then take the second partial derivative with respect to t:

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2 E(x,t) \tag{4}$$

We then plug in the second partial derivatives into the original differential equation:

$$-k^2 E(x,t) - \frac{1}{c^2}(-\omega^2 E(x,t)) = 0 \tag{5}$$

Dividing through by the electric field:

$$-k^2 - \frac{1}{c^2}(-\omega^2) = 0 \tag{6}$$

We then multiply through by c^2 and bring the k^2 term to the right hand side:

$$\omega^2 = c^2 k^2 \tag{7}$$

Now, the relations are defined as:

$$\mathcal{E} = \hbar\omega \quad \text{and} \quad p = \hbar k \tag{8}$$

So, we have:

$$\mathcal{E}^2 = \hbar^2\omega^2 \rightarrow \omega^2 = \frac{\mathcal{E}^2}{\hbar^2} \tag{9}$$

Next, we also have:

$$p^2 = \hbar^2 k^2 \rightarrow k^2 = \frac{p^2}{\hbar^2} \tag{10}$$

We then plug in the relations into the original equation:

$$\omega^2 = c^2 k^2 \rightarrow \frac{\mathcal{E}^2}{\hbar^2} = c^2\frac{p^2}{\hbar^2} \tag{11}$$

We then multiply through by $\hbar^2$:

$$\boxed{\mathcal{E}^2 = c^2 p^2} \tag{12}$$

1.2

We start by using again the relations:

$$\mathcal{E} = \hbar\omega \quad \text{and} \quad p = \hbar k \tag{13}$$

We then plug in the relations into the original equation:

$$\mathcal{E}^2 = p^2c^2 + m^2c^4 \to \hbar^2\omega^2 = \hbar^2k^2c^2 + m^2c^4 \tag{14}$$

We then divide through by $\hbar^2$:

$$\omega^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2} \tag{15}$$

Multiplying through by negative Ψ:

$$-\omega^2\Psi = -k^2c^2\Psi - \frac{m^2c^4}{\hbar^2}\Psi \tag{16}$$

Again, the plain wave solution is defined as:

$$\Psi(x,t) = \Psi_0 \exp(i(kx - \omega t)) \tag{17}$$

So, the Laplacian for the plan wave solution is:

$$\nabla^2\Psi = -k^2\Psi(x,t) \tag{18}$$

Similarly the second derivative with respect to time of the plane wave solution is:

$$\frac{\partial^2\Psi}{\partial t^2} = -\omega^2\Psi(x,t) \tag{19}$$

Recognizing the right hand sites in our equation 16, we plug in to get:

$$\frac{\partial^2\Psi}{\partial t^2} = c^2\nabla^2\Psi - \frac{m^2c^4}{\hbar^2}\Psi \tag{20}$$

dividing through by the sweet of light squared:

$$\boxed{\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \nabla^2\Psi - \frac{m^2c^2}{\hbar^2}\Psi} \tag{21}$$

2 Problem 2

This problem is a good practice on Dirac notation. The math here is nothing but simple addition / multiplication, but when tied into Dirac notation, it adds a level of hidden sub-text that is confusing.

The Hermitian operator H acts in a two-dimensional space with orthonormal basis vectors $|1\rangle$ and $|2\rangle$. The matrix elements are

$$\begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \tag{1}$$

The eigenvalues are 5 and -5. The column vectors representation of the eigenvalues $|A\rangle$ and $|B\rangle$ is

$$\begin{pmatrix} \langle 1|A\rangle \\ \langle 2|A\rangle \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \langle 1|B\rangle \\ \langle 2|B\rangle \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2}$$

H can be diagonalized by a unitary operator U (with $U^\dagger U = I$), i.e. $U^\dagger H U = D$ where

$$\begin{pmatrix} \langle 1|U|1\rangle & \langle 1|U|2\rangle \\ \langle 2|U|1\rangle & \langle 2|U|2\rangle \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{3}$$

and

$$\begin{pmatrix} \langle 1|D|1\rangle & \langle 1|D|2\rangle \\ \langle 2|D|1\rangle & \langle 2|D|2\rangle \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \tag{4}$$

2.1 Show that the column vectors in (2) are the eigenvectors of (1).

We start by plugging in the column vectors one at a time into the matrix in (1):

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5 \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) \tag{22}$$

Next, we plug in the second column vector:

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 5 \\ -10 \end{pmatrix} = -5 \left(\frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right) \tag{23}$$

So, they are eigen vectors with eigen values of 5 and negative 5, respectively.

2.2 Show that $U^\dagger H U = D$. If we think of our kets as unit vectors, what would this operation physically represent? As in, what if H was initially x-hat, and U made it y-hat.

We start by carrying out the matrix multiplication:

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \tag{24}$$

Consolidating the constants and carrying out the right-hand side matrix multiplication first, we simplify to:

$$\frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 10 & 5 \\ 5 & -10 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \tag{25}$$

So, we have shown that $U^\dagger H U = D$. If we think of our kets as unit vectors, this operation would represent a rotation of basis. The choice of x-hat and y-hat is typically used for column vectors, but the same idea applies. If H was initially x-hat, and U made it y-hat, then the matrix U would be a rotation matrix.

3. Since $H = UDU^\dagger$, it also follows that $H^2 = UD^2U^\dagger$ and in general that $H^n = UD^nU^\dagger$. The exponential of H is therefore given by

$$\begin{aligned} e^H &= \sum_{n=0}^{\infty} \frac{1}{n!} H^n \\ &= U \left[\sum_{n=0}^{\infty} \frac{1}{n!} D^n \right] U^\dagger \\ &= U e^D U^\dagger \\ &= U \begin{pmatrix} e^5 & 0 \\ 0 & e^{-5} \end{pmatrix} U^\dagger \end{aligned}$$

2.3 Perform the matrix multiplication on the above right to obtain the values of the four matrix elements of e^H in the $|1\rangle$, $|2\rangle$ basis

First, we will perform the matrix multiplication on the left hand side:

$$U \begin{pmatrix} e^5 & 0 \\ 0 & e^{-5} \end{pmatrix} U^\dagger = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^5 & 0 \\ 0 & e^{-5} \end{pmatrix} U^\dagger = \frac{1}{\sqrt{5}} \begin{pmatrix} 2e^5 & -e^{-5} \\ e^5 & 2e^{-5} \end{pmatrix} U^\dagger \tag{26}$$

next we plug in four $U^\dagger$:

$$\frac{1}{5} \begin{pmatrix} 2e^5 & -e^{-5} \\ e^5 & 2e^{-5} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \boxed{\frac{1}{5} \begin{pmatrix} 4e^5 + e^{-5} & 2e^5 - 2e^{-5} \\ 2e^5 - 2e^{-5} & e^5 + 4e^{-5} \end{pmatrix}} \tag{27}$$

2.4 compute the four matrix elements of e^H in the $|1\rangle$, $|2\rangle$ basis to show it is the same as above.

We start with the first aliment, which is:

$$\begin{aligned} \langle 1| e^H |1\rangle &= e^5 \langle 1|A\rangle \langle A|1\rangle + e^{-5} \langle 1|B\rangle \langle B|1\rangle && (28) \\ &= \frac{4}{5} e^5 + \frac{1}{5} e^{-5} && (29) \end{aligned}$$

Next, we have:

$$\begin{aligned} \langle 1| e^H |2\rangle &= e^5 \langle 1|A\rangle \langle A|2\rangle + e^{-5} \langle 1|B\rangle \langle B|2\rangle && (30) \\ &= \frac{2}{5} e^5 - \frac{2}{5} e^{-5} && (31) \end{aligned}$$

Next, we have:

$$\langle 2| e^H |1\rangle = e^5 \langle 2|A\rangle \langle A|1\rangle + e^{-5} \langle 2|B\rangle \langle B|1\rangle \tag{32}$$

$$= \frac{2}{5}e^5 - \frac{2}{5}e^{-5} \tag{33}$$

Finally, we have:

$$\langle 2| e^H |2\rangle = e^5 \langle 2|A\rangle \langle A|2\rangle + e^{-5} \langle 2|B\rangle \langle B|2\rangle \tag{34}$$

$$= \frac{1}{5}e^5 + \frac{4}{5}e^{-5} \tag{35}$$

Taking out a factor of $\frac{1}{5}$ and consolidating everything into a matrix format:

$$\boxed{\frac{1}{5}\begin{pmatrix} 4e^5 + e^{-5} & 2e^5 - 2e^{-5} \\ 2e^5 - 2e^{-5} & e^5 + 4e^{-5} \end{pmatrix}} \tag{36}$$

This is the same as the matrix we got in part 3.