Extension: 72 hours (end of Saturday)

29. We continue with our discussion of the "dipole approximation" for the interaction of an atom with a non-ionizing external electromagnetic field.

I hope you obtained in problem 24 an interaction Hamiltonian that looked something like

$$H_D = \frac{qE}{m\omega} P_z \sin\omega t, \tag{1}$$

where q is the electron charge, m is the electron mass, E is the electric field strength, ω is the field oscillation angular frequency, and P_z is the z-component momentum operator.

Using first order time-dependent perturbation theory, determine the wave function at time t as an expansion in the eigenstates of H_0. Denote the orthonormal eigenstates of H_0 according to $H_0|n\rangle = \omega_n|n\rangle$ (interaction representation). Assume that the atom is in the ground state $|0\rangle$ at time $t = 0$. Leave your expression in terms of the matrix elements of P_z. You may define the difference in energies $\omega_{n0} \equiv \omega_n - \omega_0$. Try to express your answer using ω_{n0} and ω.

In order to do the next problem, we are going to need our kets expressed in the Schrödinger representaion, so go ahead and do this (you needn't do anything with the matrix element of P_z).

$$H_0 = \frac{P^2}{2m} + V(r)$$

In the interaction representation, the perturbed potential is expressed as:

$$V(t) = e^{iH_0t/\hbar} V_t \, e^{-iH_0t/\hbar}, \quad V_t = \frac{qE}{m\omega} P_z \sin\omega t$$

From the notes, we have

$$\langle n|\Psi(t)\rangle = \langle n|0\rangle + \frac{1}{i}\int_{t_0}^{t} dt_1 \langle n| e^{iH_0t/\hbar} V_t \, e^{-iH_0t/\hbar}|0\rangle$$

natural units

$$= \delta_{n0} + \frac{1}{i}\frac{qE}{m\omega}\int_{t_0}^{t} \langle n| e^{i\omega_n t_1} P_z e^{-i\omega_0 t}|0\rangle \sin\omega t_1$$

$$= \delta_{n0} - i\frac{qE}{m\omega}\int_{t_0}^{t} \langle n|P_z|0\rangle e^{i\omega_{n0}t} \sin\omega t_1 \, dt_1$$

$(\omega_{n0} = \omega_n - \omega_0)$

Setting $t_0 = 0$, we evaluate the integral:

$$\int_0^t e^{i\omega_{n0}t_1}\sin(\omega t_1)\,dt_1 = \frac{1}{2i}\int_0^t e^{i(\omega_{n0}+\omega)t_1} - e^{i(\omega_{n0}-\omega)t_1}\,dt_1$$

$$\int_0^{t} e^{i\omega_{n0}t_1}\sin(\omega t_1)\,dt_1 = \frac{1}{2i}\int_0^t e^{i(\omega_{n0}+\omega)t_1} - e^{i(\omega_{n0}-\omega)t_1}\,dt_1$$

$$= \frac{1}{2i}\left[\frac{1}{i(\omega_{n0}+\omega)} e^{i(\omega_{n0}+\omega)t_1}\Big|_{t_1=0}^{t} - \frac{1}{i(\omega_{n0}-\omega)} e^{t_1 i(\omega_{n0}-\omega)}\Big|_{t_1=0}^{t}\right]$$

$$= -\frac{1}{2}\left(\frac{e^{i(\omega_{n0}+\omega)t}-1}{\omega_{n0}+\omega} - \frac{e^{i(\omega_{n0}-\omega)t}-1}{\omega_{n0}-\omega}\right)$$

Thus, we have

$$\langle n|\Psi(t)\rangle = \delta_{n0} + \frac{i}{2}\frac{qE}{m\omega}\langle n|P_z|0\rangle\left(\frac{e^{i(\omega_{n0}+\omega)t}-1}{\omega_{n0}+\omega} - \frac{e^{i(\omega_{n0}-\omega)t}-1}{\omega_{n0}-\omega}\right)$$

Since

$$|\Psi(t)\rangle = \sum_n |n\rangle\langle n|\Psi(t)\rangle,$$

we have

$$|\Psi(t)\rangle = |0\rangle + \frac{iqE}{2m\omega}\sum_n |n\rangle\langle n|P_z|0\rangle\underbrace{\left(\frac{e^{i(\omega_{n0}+\omega)t}-1}{\omega_{n0}+\omega} - \frac{e^{i(\omega_{n0}-\omega)t}-1}{\omega_{n0}-\omega}\right)}_{\equiv x,}$$

since $\sum_n \delta_{n0}|n\rangle = |0\rangle$.

However, notice that this form is represented in the Heisenberg/Interaction picture. To convert it to the Shrodinger picture, we use

$$|\psi_S(t)\rangle \to U_S^{0\dagger}(t,t_0)|\psi_S(t)\rangle = |\psi_I(t)\rangle$$

(I = interaction, S = shrodinger) where

$U_S^0(t,t_0) = e^{-iH_S^0(t-t_0)/\hbar}$

Thus, we have that the shrodinger representation of the wavefunction is

$$|\psi_s(t)\rangle = e^{-i\omega_0 t}|0\rangle + \frac{iqE}{2m\omega}\sum_n e^{-i\omega_n t}|n\rangle\langle n|P_z|0\rangle \cdot x,$$

$$= e^{-i\omega_0 t}|0\rangle + \frac{iqE}{2m\omega}\sum_n |n\rangle\langle n|P_z|0\rangle\left(\frac{e^{i(\omega_0+\omega)t}-1}{\omega_{n0}+\omega} - \frac{e^{i(\omega_0-\omega)t}-1}{\omega_{n0}-\omega}\right)$$

$$\Rightarrow \boxed{e^{i\omega_0 t}|\psi_s(t)\rangle = |0\rangle + \frac{iqE}{2m\omega}\sum_n |n\rangle\langle n|P_z|0\rangle\left(\frac{e^{i\omega t}-1}{\omega_{n0}+\omega} - \frac{e^{-i\omega t}-1}{\omega_{n0}-\omega}\right)}$$

Since $|\psi_s(t)\rangle = U_s(t,t_0)|\psi_I(t)\rangle$ (U is unitary)

and $U_s(t,t_0)|n\rangle = e^{iH_s^0 t}|n\rangle = e^{iH_0 t}|n\rangle = e^{i\omega_n t}|n\rangle$

and $U_s(t,t_0)$ is linear.

Notice

$$\frac{e^{i\omega t}-1}{\omega_{n0}+\omega} + \frac{-e^{-i\omega t}+1}{\omega_{n0}-\omega} = \frac{(e^{i\omega t}-1)(\omega_{n0}-\omega) + (-e^{-i\omega t}+1)(\omega_{n0}+\omega)}{\omega_{n0}^2-\omega^2}$$

$$= \frac{2i\omega_{n0}\sin(\omega t) - 2\omega\cos(\omega t) + 2\omega}{\omega_{n0}^2-\omega^2}$$

30. We continue from problem 29 with our dipole approximation for atomic transitions in an external electromagnetic field. You have computed the wave function to first order in perturbation theory. Now use it to compute the induced dipole moment $\langle D_z\rangle(t)$:

$$\langle D_z\rangle(t) = \langle\psi(t)|qZ|\psi(t)\rangle. \tag{2}$$

Compute this to first order in the external field. You may assume that the natural oscillator (ω_{n0}) time dependence damps out (even though we haven't put in a damping term) and may be neglected, so only the forced oscillation from the external field is left. The discussion on Shankar page 459 may help to deal with the zeroth order term. Your homework from last week can be useful to express matrix elements of P_z in terms of matrix elements of Z. Note the appearance of the oscillator strengths (up to a factor of m) you found a sum rule for last week.

We have derived a result that can be used in providing a model for the dielectric constant of a material.

Notice $\langle D_z\rangle(t) = \langle \psi_s(t)|qZ|\psi_s(t)\rangle$

$= \langle e^{i\omega_0 t}\psi_s(t)|qZ|e^{i\omega_0 t}\psi_s(t)\rangle$

Substituting in our answer for $|\psi_s(t)\rangle$ in the previous problem, we have

$$\langle D_z\rangle(t) = \left(\langle 0| + c_1^*\sum_n \langle n|\langle 0|P_z|n\rangle C_n^*(t)\right) qZ$$
$$\left(|0\rangle + c_1\sum_n |n\rangle\langle n|P_z|0\rangle C_n(t)\right)$$

where $c_1 = \dfrac{iqE}{2m\omega}$, $C_n(t) = \dfrac{e^{i\omega t}-1}{\omega_{n0}+\omega} - \dfrac{e^{-i\omega t}-1}{\omega_{n0}-\omega}$.

Since we are finding the solution first-order in E, we neglect any cross-terms of the sum. Therefore, we have

$$\langle D_z\rangle(t) = \langle 0|qZ|0\rangle + q\sum_n c_1 C_n(t)\langle n|P_z|0\rangle\langle 0|Z|n\rangle$$
$$+ c_1^* C_n^*(t)\langle 0|P_z|n\rangle\langle n|Z|0\rangle$$

We know that

$$\langle 0|P_z|n\rangle = a_n \langle 0|z|n\rangle$$

where

$$a_n = -im\omega_{n0}$$

Thus,

$$\langle D_z\rangle(t) = q\langle 0|z|0\rangle + q\sum_n c_1 c_n(t) a_n \langle n|z|0\rangle\langle n|z|0\rangle^*$$

$$+ c_1^* c_n^*(t) a_n^* \langle n|z|0\rangle^*\langle n|z|0\rangle$$

$$= q\langle 0|z|0\rangle + q\sum_n 2c_1 a_n \left(c_n(t) - c_n^*(t)\right) |\langle n|z|0\rangle|^2$$

Notice

$$c_n(t) - c_n^*(t) = \frac{e^{i\omega t}-1}{\omega_{n0}+\omega} - \frac{e^{-i\omega t}-1}{\omega_{n0}-\omega} - \frac{e^{-i\omega t}-1}{\omega_{n0}+\omega} + \frac{e^{i\omega t}-1}{\omega_{n0}-\omega}$$

and

$$\frac{e^{i\omega t}-1}{\omega_{n0}+\omega} + \frac{e^{i\omega t}-1}{\omega_{n0}-\omega} = \frac{(e^{i\omega t}-1)((\omega_{n0}+\omega)+(\omega_{n0}-\omega))}{\omega_{n0}^2-\omega_n^2}$$

$$= \frac{2\omega_{n0}(e^{i\omega t}-1)}{\omega_{n0}-\omega^2}$$

Thus,

$$\Rightarrow c_n(t) - c_n^*(t) = \frac{2\omega_{n0}[(e^{i\omega t}-1)-(e^{-i\omega t}-1)]}{\omega_{n0}^2-\omega^2}$$

$$= \frac{2\omega_{n0}\cdot 2i\sin(\omega t)}{\omega_{n0}^2-\omega^2}$$

We know from Shankar p. 459 that

$$\langle \alpha_2 l_2 m_2 | Z | \alpha_1 l_1 m_1 \rangle = 0 \quad \text{unless} \quad \begin{cases} l_2 = l_1 \pm 1 \\ m_2 = m_1 \end{cases}$$

$$\langle \alpha_2 l_2 m_2 | X \text{ or } Y | \alpha_1 l_1 m_1 \rangle = 0 \quad \text{unless} \quad \begin{cases} l_2 = l_1 \pm 1 \\ m_2 = m_1 \pm 1 \end{cases} \qquad (17.2.21)$$

Since l=m=0 and n=1 for $|0\rangle$, we have

$$\langle 0 | z | 0 \rangle = 0$$

Thus

$$\langle p_z \rangle(t) = 2q \sum_{n \neq 0} c_1 \alpha_n \left(c_n(t) - c_n^*(t) \right) \left| \langle n | z | 0 \rangle \right|^2$$

Notice

$$2\alpha_n \left| \langle n | z | 0 \rangle \right|^2 = -i f_{n0}$$

Moreover,

$$-i q c_1 = \frac{q^2 E}{2m\omega}$$

Thus,

$$\boxed{\langle p_z \rangle(t) = 2i \frac{q^2 E}{m\omega} \sin(\omega t) \sum_{n \neq 0} \frac{\omega_{n0}}{\omega_{n0}^2 - \omega^2} f_{n0}}$$

(to first order in E)

31. In our discussion of scattering theory, we supposed we had a beam of particles from some ensemble of wave packets, and obtained an "effective" (observed) differential cross-section:

$$\sigma_{\text{eff}}(\mathbf{u}) = \int_{\{\alpha\}} f(\alpha)d\alpha \int_{|\mathbf{x}|\leq R} d^2(\mathbf{x})P(\mu;\alpha;\mathbf{x}) \tag{3}$$

1

As discussed in class, $P(\mu;\alpha;\mathbf{x})$ is the probability that a particle with beam parameters $(\alpha;\mathbf{x})$ scatters into solid angle element $d\Omega_{\mathbf{u}}$ about direction $\mathbf{u}$. The probability density function for α, which could be a multi-dimensional quantity, is $f(\alpha)$. This formula assumes that the beam particles are distributed uniformly in a disk of radius R centered at the origin in the $\hat{e}_1 - \hat{e}_2$ plane, and that the distribution of the shape parameter α is uncorrelated with position in this disk. The purpose of this problem is to see that this can be generalized further, and in the process to give you a chance to think about what we are doing.

(a) Try to obtain an expression for $\sigma_{\text{eff}}(\mathbf{u})$ without making these assumptions. You will likely find it useful to define a notion of an "effective area" of the beam.

(b) Using part (a), write down an expression for $\sigma_{\text{eff}}(\mathbf{u})$ appropriate to the case where the beam particles are distributed according to a Gaussian of standard deviation ρ in radial distance from the origin (in the $\hat{e}_2 - \hat{e}_3$ plane), and where the wave packets are also drawn from a Gaussian distribution in the expectation value of the magnitude of the momentum. Let the standard deviation of this momentum distribution be $\alpha = \alpha(\mathbf{x})$, for beam position $\mathbf{x}$. Your answer will likely have an intuitive feel to it.

(c) For your generalized result of part (a), try to repeat our limiting case argument to obtain the "fundamental" cross section.

a.) Let $A_{eff}(\vec{u})$ be the "effective area" for direction $\vec{u}$.

(i.e. $\frac{1}{A_{eff}(\vec{u})} \int_{(\infty)} d^2\vec{x}\, P(\vec{u}, \alpha, \vec{x})$ is the average of $P(\vec{u}, \alpha, \vec{x})$

over $\vec{x}$).

We know that $f(\alpha, \vec{x})$ now has an $\vec{x}$-dependance.
Since we are not making the aforementioned assumption. It is still normalized.

$$1 = \int_{(\infty)} d\alpha \int_{(\infty)} d^2\vec{x}\, f(\alpha, \vec{x})$$

Thus, we have

$$\frac{d\sigma_{eff}}{d\Omega} = A_{eff} \int_{(\infty)} d\alpha \int_{(\infty)} d^2\vec{x}\, f(\alpha, \vec{x})\, P(\vec{n}, \alpha, \vec{x})$$

Integrating w.r.t. Ω, we have

$$\boxed{\sigma_{eff} = A_{eff} \int_{(\infty)} d\alpha \int_{(\infty)} d^2\vec{x}\, f(\alpha, \vec{x})}$$

since

$$1 = \int d\Omega\, P(n) = \frac{1}{A_{eff}} \int d\Omega \int d\alpha \int d^2\vec{x}\, P(n, \alpha, \vec{x})$$

Let the total cross-section be "a"

$$\sigma_{T,eff}(\vec{n}) = a$$

for $|\vec{x}| < \sqrt{\frac{\pi}{a}}$. ← b/c this is the radius of a circle with area "a"!

Then, we have

$$A_{eff} \approx \frac{a}{\int_{(\infty)} d\alpha \int_{|\vec{x}| < \sqrt{\frac{a}{\pi}}} d^2\vec{x}\, f(\alpha, \vec{x})}$$

The effective area is the RHS in the limit as $a \to 0$. Thus,

a

$$A_{eff} = \lim_{a \to 0} \frac{a}{\int_{(\infty)} d\sigma \int_{|\vec{x}| < \sqrt{\frac{a}{\pi}}} d^2\vec{x}\, f(\sigma, \vec{x})}$$

b.) We know from part (a) that

$$A_{eff} = \lim_{a \to 0} a \cdot \left[\int_{(\infty)} d\rho \int_{|x| \le \sqrt{\frac{a}{\pi}}} f(\sigma(\vec{x}), \vec{x})\, d^2\vec{x} \right]^{-1}$$

Moreover, we know that

$$f(\sigma(\vec{x}), \vec{x}) = \frac{1}{\sqrt{2\pi}\rho} e^{-r^2/2\rho^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma(\vec{x})} e^{-\frac{(p^2 - p_0^2)}{2\sigma^2(x)}}$$

Notice

$$\int_{(\infty)} dp \frac{1}{\sqrt{2\pi}\,\sigma(x)} e^{-(p^2 - p_0^2)/2\sigma^2(\vec{x})} = 1$$

Thus,

$$A_{eff} = \lim_{a \to 0} a \cdot \left[\int_{|x| < \sqrt{\frac{a}{\pi}}} \frac{1}{\sqrt{2\pi}\rho} e^{-|x|^2/2\rho^2} \right]^{-1}$$

in the limit as $a \to 0$, we may approximate

$$f(x, \sigma) \approx f(0, \sigma).$$

Thus,

$$A_{eff} = \lim a \cdot \left[a \cdot \frac{1}{\sqrt{2\pi}\rho} e^0 \right]^{-1}$$

$$A_{eff} = \lim_{a \to 0} a \cdot \left[a \cdot \frac{1}{\sqrt{2\pi}\rho} e^{0} \right]^{-1}$$

$$= \left[\frac{1}{\sqrt{2\pi}\rho} \right]^{-1} = \boxed{\sqrt{2\pi}\rho = A_{eff}}$$

This is an intuitive answer because the effective area should be proportional to how much the particles in the beam are "spread out" (i.e., their standard deviation).

c.)

Using our eqn from part (a):

$$\frac{d\sigma_{eff}}{d\Omega} = A_{eff} \int_{(\alpha)} \int_{(\infty)} f(\alpha(\vec{x}); \vec{x})\, d^2\vec{x}\, P(\vec{u}, \alpha(\vec{x}), \vec{x})\, d^2\vec{x}$$

$$P(\mathbf{u}; \cancel{\alpha}^{\alpha(\vec{x})}, \mathbf{x}) = \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}') q^2 \delta(q - q') \qquad (32)$$

$$T(q\mathbf{u}, \mathbf{q}) T^*(q'\mathbf{u}, \mathbf{q}') \phi_0(\mathbf{q}; \cancel{\alpha}^{\alpha(\vec{x})}) \phi_0^*(\mathbf{q}'; \cancel{\alpha}^{\alpha(\vec{x})}) e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')}, \qquad (33)$$

As in the notes, to derive the fundamental cross section, we assume that the beam is sufficiently large compared to the area where it matters and that the momentum-space wavefunctions are sharply peaked about the nominal momentum, $p_i \hat{e}_3$. Since the beam is sufficiently large, we can approximate

$\alpha(\vec{x})$

as a constant. Moreover, since the momentum-space wave-function is sharply peaked, we know that

$$\phi_0(\mathbf{p};\alpha) \approx 0, \quad \text{unless } \mathbf{p} \approx \mathbf{p}_i = p_i \hat{\mathbf{e}}_3.$$

Thus, the product

$$\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha) \approx 0, \quad \text{unless both } \mathbf{q} \approx \mathbf{p}_i \text{ and } \mathbf{q}' \approx \mathbf{p}_i.$$

In particular, this product is small unless $\mathbf{q} \approx \mathbf{q}'$, and $q_3 \approx q_3'$.

as stated in the lecture notes. Moreover, our formula for $P(\vec{u}, \alpha(\vec{x}), \vec{x})$ reduces to the orignal formula:

$$P(\mathbf{u};\alpha;\mathbf{x}) = \int_{(\infty)} d^3(\mathbf{q}) \int_{(\infty)} d^3(\mathbf{q}') q^2 \delta(q-q') \tag{32}$$

$$T(q\mathbf{u},\mathbf{q})T^*(q'\mathbf{u},\mathbf{q}')\phi_0(\mathbf{q};\alpha)\phi_0^*(\mathbf{q}';\alpha)e^{-i\mathbf{x}\cdot(\mathbf{q}-\mathbf{q}')}, \tag{33}$$

Moreover, since the beam is sufficiently large, we may assume that $f(\alpha, \vec{x})$ is constant over the domain of interest. In order for f to be normalized, we must have

$$\int_{A_{eff}} d^2\vec{x}\, f(\alpha, \vec{x}) \approx \int_{A_{eff}} f(\alpha)\, d^2\vec{x} = A_{eff} \cdot f(\alpha)$$

$$= 1$$

Therefore, we have

$$f(\alpha, \vec{x}) \approx \frac{f(\alpha)}{A_{eff}}.$$

Therefore, we may follow the same logic as in the lecture notes to simplify $P(\vec{u}, \alpha(\vec{x}), \vec{x})$, except with an extra factor of $1/A_{eff}$:

$$\Rightarrow \quad P(\vec{u}, \alpha, \vec{x}) = |T(q\vec{u}, \vec{q})|^2 \cdot 2\pi^2 \cdot \frac{1}{A_{eff}}$$

Thus we have

Thus, we have

$$\sigma_{eff} = A_{eff}\,|T(q\hat{n},\bar{q})|^2 \cdot 2\pi^2 \cdot \frac{1}{A_{eff}}$$

$$\rightarrow \boxed{\sigma = 2\pi^2\,|T(q\hat{n},\bar{q})|^2}$$

which is identical to the expression derived in the notes where the assumptions were made.

32. We discussed the Ahronov-Bohm effect. Let us make some further interesting observations.

   (a) Consider again the path integral in the vicinity of the long thin solenoid. In particular, consider a path which starts at $\boldsymbol{x}$, loops around the solenoid, and returns to $\boldsymbol{x}$. Since the $\boldsymbol{B}$ and $\boldsymbol{E}$ fields are zero everywhere in the region of the path, the only effect on the particle's wave function in traversing this path is a phase shift, and the amount of phase shift depends on the magnetic flux in the solenoid, as we discussed in class. Suppose we are interested in a particle with charge of magnitude e (*e.g.*, an electron). Show that the magnetic flux Φ in the solenoid must be quantized, and give the possible values that Φ can have.

   (b) The BCS theory for superconductivity assumes that the basic "charge carrier" in a superconductor is a pair of electrons (a "Cooper pair"). The Meissner effect for a (Type I) superconductor is that when such a material is placed in a magnetic field, and then cooled below a critical temperature, the magnetic field is excluded from the superconductor. Suppose that there is a small non-superconducting region traversing the superconductor, in which magnetic flux may be "trapped" as the material is cooled below the critical temperature. What values do you expect to be possible for the trapped flux? [This effect has been experimentally observed.]

   (c) So far, no one has observed (at least not convincingly) a magnetic "charge", analogous to the electric charge. But there is nothing fundamental that seems to prevent us from modifying Maxwell's equations to accommodate the existence of such a "magnetic monopole". In particular, we may alter the divergence equation to (in CGS gaussian units):

$$\nabla \cdot \boldsymbol{B} = 4\pi\rho_M,$$

where ρ_M is the magnetic charge density.
Consider a magnetic monopole of strength e_M located at the origin. The $\boldsymbol{B}$-field due to this charge is simply:

where ρ_M is the magnetic charge density.

Consider a magnetic monopole of strength e_M located at the origin. The $\boldsymbol{B}$-field due to this charge is simply:

$$\boldsymbol{B} = \frac{e_M}{r^2}\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction. The $\hat{\boldsymbol{r}}$-component of the curl of the vector potential is:

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}.$$

A solution, as you should quickly convince yourself, is a vector potential in the ϕ direction:

$$A_\phi = e_M \frac{1-\cos\theta}{r\sin\theta}.$$

Unfortunately(?), this is singular at $\theta = \pi$, *i.e.*, on the negative z-axis. We can fix this by using this form everywhere except in a cone about $\theta = \pi$, *i.e.*, for $\theta \leq \pi - \epsilon$, and use the alternate solution:

$$A'_\phi = e_M \frac{-1-\cos\theta}{r\sin\theta}$$

in the (overlapping) region $\theta \geq \epsilon$, thus covering the entire space. In the overlap region ($\epsilon \leq \theta \leq \pi - \epsilon$), either $\boldsymbol{A}$ or $\boldsymbol{A'}$ may be used, and must give the same result, *i.e.*, the two solutions are related by a gauge transformation – that is, they differ by the gradient of a scalar function.

Consider the effect of the vector potential on the wave function of an electron (charge $-e$). Invoke single-valuedness of the wave function, and determine the possible values of e_M that a magnetic charge can have. [This is sometimes called a "Dirac monopole".]

a.) We know from lecture that if a particle circles around a solenoid with vector potential A (and there is no B-field outside), its wavefunction undergoes the transformation below:

$$\psi \rightarrow e^{iq\oint_C \vec{A}\cdot d\vec{l}}\,\psi$$

Notice that

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{a} = \int_S \vec{B}\cdot d\vec{a} = \Phi$$

by Stokes theorem. (and because $B=0$ outside the Solenoid).

We must require that the transformed wavefunction is equal to the original wavefunction since it returns to its original position.

Therefore, we must have

$$e^{iq\Phi} = 0$$

Thus,

$$\Phi = \frac{2\pi n}{q}, \quad n \in \mathbb{Z}$$

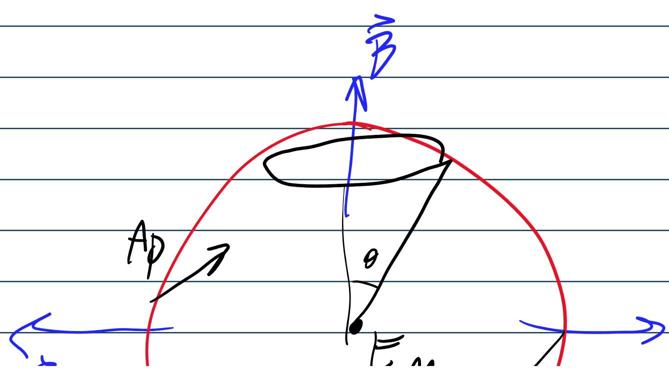
for a particle of charge "e", we must have a quantized flux given by

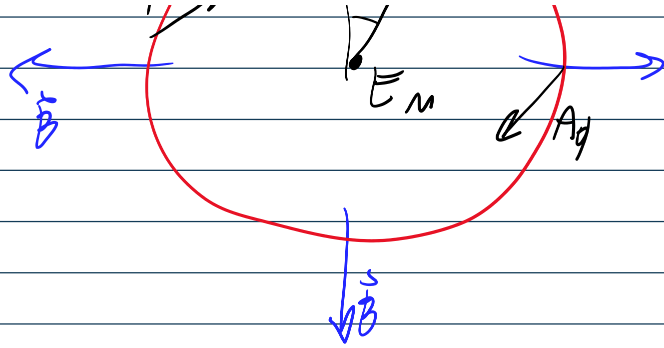
$$\boxed{\Phi = \frac{2\pi n}{e} \quad \text{for some } n \in \mathbb{Z}}$$

b.) In this case, the cavity is analogous to the solenoid and the cooper pairs are analogous to the electrons. However, in this case, cooper-pairs have a charge of 2e, instead of e. Therefore, the possible values for the trapped flux are

$$\boxed{\Phi = \frac{\pi n}{e}, \quad n \in \mathbb{Z}}$$

c.)





Consider the contour given by constant $\varepsilon \leq \theta \leq \pi - \varepsilon$, r
Then, the electron wavefunction is transformed according to

$$\psi \to e^{iq\oint_C \vec{A}\cdot d\vec{l}}\,\psi = e^{iq\oint \vec{A}'\cdot d\vec{l}}\,\psi$$

by gauge symmetry. Let

$$\nabla \chi = \vec{A} - \vec{A}' = 2e_M \cdot \frac{1}{r\sin\theta}\hat{\phi}$$

Then we must have

$$\exp\left(iq\oint_C \nabla\chi \cdot d\vec{l}\right) = 1 \qquad \text{since } e^{iq\oint_C \vec{A}\cdot d\vec{l}} = e^{iq\oint_C \vec{A}'\cdot d\vec{l}}$$

Thus, we must have

$$2\pi n = q\oint_C \nabla\vec{\chi}\cdot d\vec{l} = q\oint_0^{2\pi} r\sin\theta\, d\phi \cdot 2e_M \frac{1}{r\sin\theta}\hat{\phi}\cdot\hat{\phi}$$
$(n \in \mathbb{Z})$

$$= q\oint_0^{2\pi} 2e_M\, d\phi = 2\pi q \cdot 2e_M$$

Thus,

$|\mathrm{km}\rangle,$

$$2e_m \cdot g = n \rightarrow \boxed{e_m = \frac{n}{2e}, \; n \in \mathbb{Z}}$$