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Ch/ChE 164 Winter 2024
Homework Problem Set #4
Due Date: Thursday, February 15, 2024 @ 11:59pm PT
For all problems, please consider reasonable simplifications of your final results.

1. *(15 pts.)* (Adapted from Callen). Consider a mixture of two non-identical monatomic ideal gases.

- Starting from the expression for the grand canonical partition function and taking the limit of small fugacity, show that the canonical partition function Z is factorizable and

$$Z = Z_1 Z_2 = \frac{1}{N_1!} q_1^{N_1} \frac{1}{N_2!} q_2^{N_2} \tag{1}$$

(You may wish to use the occupancy representation $|n_1 m_1, n_2 m_2 \ldots\rangle$, where n_1 denotes occupancy of energy level 1 of gas 1, and m_1 denotes occupancy of energy level 1 of gas 2, etc.).

- Compute the entropy and show that (comparing to the entropy of the two separate gases) there is an entropy of mixing of the form

$$S_{\text{mixing}} = (-x_1 \log x_1 - x_2 \log x_2) N k \tag{2}$$

where N is the total number of particles.

Callen chp 18 393

2. In class we derived the heat capacity of the Fermi gas at low temperature by an intuitive argument, which $C_v \sim NkO(T/T_F)$. Here we will derive the precise form and constants (adapted from Callen).

Denote the Fermi-Dirac distribution at temperature T as $f(\epsilon, T)$ and the (temperature dependent) chemical potential by μ (note this is not the Fermi energy ϵ_F except when $T = 0$). We will first derive a general result for an integral of the form (Sommerfeld expansion)

$$I \equiv \int_0^\infty \phi(\epsilon) f(\epsilon, T) d\epsilon = \int_0^\mu \phi(\epsilon) d\epsilon + \frac{\pi^2}{6}(kT)^2 \phi'(\mu) + \frac{7\pi^4}{360}(kT)^4 \phi'''(\mu) + \ldots \tag{3}$$

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a) *(10 pts.)* Integrate I by parts, and let $\Phi \equiv \int_0^\epsilon \phi(\epsilon') d\epsilon'$. Then expanding $\Phi(\epsilon)$ in a power series in $\epsilon - \mu$ to third order, deduce

$$I = -\sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m \Phi(\mu)}{d\mu^m} I_m \tag{4}$$

where $I_m = \int_0^\infty (\epsilon - \mu)^m \frac{df}{d\epsilon} d\epsilon = -\beta^{-m} \int_{-\beta\mu}^{\infty} \frac{e^x}{(e^x+1)^2} x^m dx$

b) *(5 pts.)* Show that only an exponentially small error is made by taking the lower limit of integration as $-\infty$, and that then all terms with m odd vanish.

c) *(5 pts.)* Evaluate the first two non-vanishing terms and show that this agrees with the expansion of I.

d) *(10 pts.)* Using the result for I, express N in the form of such an integral and obtain an expansion for $N(V, T, \mu)$ in terms of kT/μ (to second order). Verify that $T \to 0$ yields the relation between N and ϵ_F derived in class.

e) *(10 pts.)* Invert this relationship to obtain $\mu(T)$ as a function of kT/ϵ_F (to second order) for fixed N.

f) *(5 pts.)* Similarly obtain an expansion for the internal energy E as a function of kT/μ (to second order).

g) *(5 pts.)* Substituting in $\mu(T)$ into the energy expansion, obtain an expansion of E in kT/ϵ_F to second order, and thus C_v. Hence see why we skipped the detailed computation in class.

3. *(20 pts.)* Show that for the Bose-Einstein and Fermi-Dirac gas at low density and/or high temperature the equation of state is given by

$$p = kT\rho \left(1 \mp \frac{\rho\Lambda^3}{2^{5/2}} + \dots\right). \tag{5}$$

① V, T

μ_1, μ_2

2 non-interacting gases

(i) gas 1 : $n_1, n_2, n_3 \ldots$ $\qquad N_1 = \sum_{i=1}^{\infty} n_i$

gas 2 : $m_1, m_2, \ldots$ $\qquad N_2 = \sum_{i=1}^{\infty} m_i$

$$E_\nu = E_\nu^1 + E_\nu^2$$ ← sum b/c non-interacting

- otherwise need interaction potential term

$$E^1 = \sum_{i=1}^{\infty} n_i \epsilon_i^{(1)} + \sum_{i=1}^{\infty} m_i \epsilon_i^{(2)}$$

- grand canonical, N, ϵ can fluctuate

$$\Xi = \sum_\nu e^{-\beta E_\nu + \beta\mu^{(1)} N_1 + \beta \mu^{(2)} N_2}$$

$$= \sum_\nu \exp\left(-\beta\left(\sum_{i=1}^{\infty} n_i \epsilon_i^{(1)} + \sum_{i=1}^{\infty} m_i \epsilon_i^{(2)}\right) + \beta\mu^{(1)} \sum_{i=1}^{\infty} n_i + \beta\mu^{(2)} \sum_{i=1}^{\infty} m_i\right)$$

$$\Xi = \sum_{\{n\}} \sum_{\{m\}} \exp\left(-\beta\left[\sum_i^{\infty} n \epsilon_i^{(1)} + \sum_i^{\infty} m_i \epsilon^{(2)} - \sum_i^{\infty} \mu_i^{(1)} n_i - \sum_i^{\infty} \mu_i^{(2)} m_i\right]\right)$$

- move sums from exp to product

$$\Xi = \prod_i \sum_{n_i} \sum_{m_i} \exp\left(-\beta\left[n_i \epsilon_i^{(1)} + m_i \epsilon_i^{(2)} - \mu_i^{(1)} n_i - \mu_i^{(2)} m_i\right]\right)$$

$$= \underbrace{\prod_i \sum_{n_i} \exp\left(\beta n_i \left(\mu_i^{(1)} - \epsilon_i^{(1)}\right)\right)}_{\Xi_1} \underbrace{\prod_i \sum_{m_i} \exp\left(\beta m_i \left[\mu_i^{(2)} - \epsilon_i^{(2)}\right]\right)}_{\Xi_2}$$

$$\rightsquigarrow \Xi = \Xi_1 \Xi_2$$

each obey FD, BE stats: BE: n_i, m_i can be any any

FD: n_i, m_i can be $0, 1$ only

$$= \prod_i e^{(\beta\mu_i - \epsilon_i)n_i} \prod_i e^{(\beta\mu_i - \epsilon_i)m_i}$$

BE: $\Xi = \prod_i \left(1 - e^{\beta\mu - \beta\epsilon_i}\right)^{-1}$, FD: $\prod_i \left(1 + e^{\beta\mu - \beta\epsilon_i}\right)$

$$\Xi = \prod_i \left(1 \mp e^{\beta\mu - \beta\epsilon_i}\right)^{\mp 1}$$

$$\ln \Xi = \mp \ln \prod_i \left[1 \mp e^{\beta\mu - \epsilon_i}\right] = \mp \sum_i \ln\left[1 \mp e^{\beta\mu - \epsilon_i}\right]$$

@ low fugacity $e^{\beta\mu} \ll 1$, T.E.

$\ln(1+x)$ for small x: $\longrightarrow x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$$\ln \Xi = \mp \sum_i \left(\mp e^{\beta\mu - \epsilon_i}\right) = \pm e^{\beta\mu} \sum_i e^{\epsilon_i} = e^{\beta\mu} q$$

q is single particle partition fxn

$$\Xi = \exp(e^{\beta\mu} q), \quad e^x = \sum_{N=0}^{\infty} \frac{1}{N!} x^N$$

$$\Xi = \sum_{N=0}^{\infty} \frac{1}{N!} q^N e^{\beta\mu N} = \Xi_1 \Xi_2$$

$$\Xi = \sum_{N=0}^{\infty} \frac{1}{N!} q^N e^{\beta\mu^{(1)} N} \sum_{M}^{\infty} \frac{1}{M!} q^M e^{\beta\mu^{(2)} M} = \sum_{N}^{\infty} Z e^{\beta\mu N}$$

$$\hookrightarrow \quad Z = \left(\frac{1}{N!} q^N\right)\left(\frac{1}{M!} q^M\right)$$

$$\boxed{Z = Z_1 Z_2 = \frac{1}{N_1!} q_1^{N_1} \frac{1}{N_2!} q_2^{N_2}}$$

(ii) find entropy (thermo approach)

$$S_{mixing} = S_{mix} - S_1 - S_2$$

$$F = -kT \ln Z$$

$$S = -\left(\frac{\delta F}{\delta T}\right)_{N_1, N_2, V}, \quad q = \frac{V}{\Lambda^3} = \left(\frac{2mkT}{h^2}\right)^{3/2} V$$

(lecture 8)

$$S_1 = -kN_1 \ln \rho_1^* \Lambda^3 + \frac{5}{2} k N_1$$

$$S_2 = -kN_2 \ln \rho_2^* \Lambda^3 + \frac{5}{2} k N_2$$

ρ_1^*, ρ_2^* b/c occupy $V_1, V_2 \neq V$

$$\rho_1^* = \frac{N}{V_1}, \quad \frac{V_1}{V} = x_1$$

$$F_{mix} = -kT \ln (Z_1 Z_2) = -kT \ln \left(\frac{1}{N!} q_1^{N_1} \frac{1}{N_2!} q_2^{N_2}\right)$$

$$= -kT \left[N_1 \ln N_1 - N_1 + N_1 \ln q_1 + N_2 \ln N_2 - N_2 + N_2 \ln q_2\right]$$

$$S_{mix} = -\left(\frac{\delta F_{mix}}{\delta T}\right)_{N_1, N_2, V} = -k \ln \frac{1}{N_1!}\frac{1}{N_2!} - \frac{\delta}{\delta T} kT \ln q_1^{N_1} q_2^{N_2}$$

$$= -k \ln \frac{1}{N_1!}\frac{1}{N_2!} - \frac{\delta}{\delta T} kT \left(N_1 \ln \left(\frac{2mk}{\hbar^2}\right)^{3/2} V + N_2 \ln \left(\frac{2mk}{\hbar^2}\right)^{3/2} V + \frac{3}{2}(N_1+N_2)\ln T\right)$$

$$= -k\left[\ln \frac{1}{N_1!}\frac{1}{N_2!} + \ln\left(\frac{2mk}{\hbar}\right)^{3/2} V \,(N_1+N_2)\right] - k \ln T (N_1+N_2) + \frac{3}{2}\frac{1}{T}(N_1+N_2)kT$$

$$= k\left(N_1+N_2 - N_1 \ln\left(\frac{2mk}{\hbar^2}\right)^{3/2}\rho_1 - N_2 \ln\left(\frac{2mk}{\hbar^2}\right)^{3/2}\rho_2 - \frac{3}{2}(N_1+N_2)(\ln T - 1)\right) = k\left[-N_1 \ln \rho_1 \Lambda^3 - N_2 \ln \rho_2 \Lambda^3 + \frac{5}{2}(N_1+N_2)\right]$$

$$S_{mixing} = S_{mix} - S_1 - S_2 = \cancel{\tfrac{5}{2}kN_1} + \cancel{\tfrac{5}{2}kN_2} - N_1 k \ln \rho_1 \Lambda^3 - N_2 \ln \rho_2 \Lambda^3 - \cancel{\tfrac{5}{2}kN_1} + kN_1 \ln \rho_1^* \Lambda^3 + kN_2 \ln \rho_2^* \Lambda^3$$

$$S_{mixing} = Nk\left(-x_1 \ln \rho_1 - \cancel{x_1 \ln \Lambda^3} - x_2 \ln \rho_2 - \cancel{x_2 \ln \Lambda^3} + x_1 \ln \rho_1^* - \cancel{x_1 \ln \Lambda^3} + x_2 \ln \rho_2^* - \cancel{x_2 \ln \Lambda^3}\right)$$

· no variation in Λ, but variation in ρ (V)

$$S_{mixing} = Nk\left(-x_1 \ln\left(\frac{N_1}{V}\cdot\frac{V_1}{N_1}\right) - x_2 \ln\left(\frac{N_2}{V}\cdot\frac{V_2}{N_2}\right)\right)$$

$$\frac{V_1}{V} = \frac{N_1}{N_1+N_2} = x_1, \quad \frac{V_2}{V} = \frac{N_2}{N_1+N_2} = x_2$$

$$\boxed{S_{mixing} = Nk(-x_1 \ln x_1 - x_2 \ln x_2)}$$

② C_V of Fermions

$$f(\epsilon,T) = \left(e^{\beta(\epsilon-\mu)}+1\right)^{-1}, \quad I \equiv \int_0^\infty \phi(\epsilon) f(\epsilon,T)\, d\epsilon$$

gives avg of ϕ

ⓐ integrate by parts

$$u = f(\epsilon,T), \quad du = \frac{\delta f}{\delta \epsilon} d\epsilon$$

$$dv = \phi(\epsilon) \qquad v = \int \phi(\epsilon)\, d\epsilon$$

$f(\infty) = 0$

$\int_0^0 \cdots = 0$

$$I = uv - \int v\, du = \underbrace{f(\epsilon,T)\int_0^\epsilon \phi(\epsilon')\,d\epsilon' \Big|_0^\infty}_{0}$$

$$- \int_0^\infty \frac{\delta f}{\delta \epsilon}\left[\int_0^\epsilon \phi(\epsilon')\, d\epsilon'\right] d\epsilon$$

$$I = -\int_0^\infty \frac{\delta f}{\delta \epsilon}\, \Phi(\epsilon)\, d\epsilon$$

· Taylor Expand Φ around μ

$$\Phi(\mu) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\delta^m \Phi}{\delta \epsilon^m}\right)(\epsilon-\mu)^m$$

$$I = -\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\delta^m \Phi}{\delta \epsilon^m}\right) \underbrace{\int_0^\infty \frac{\delta f}{\delta \epsilon}(\epsilon-\mu)^m\, d\epsilon}_{I_m}$$

$$\boxed{I = -\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\delta^m \Phi(\mu)}{\delta \epsilon^m}\right) I_m}$$

b) *(5 pts.)* Show that only an exponentially small error is made by taking the lower limit of integration as $-\infty$, and that then all terms with m odd vanish.

(b) $I_m = \int_{-\beta\mu}^{\infty} \frac{e^x x^m}{(e^x+1)^2} dx$, argue $\int_{-\beta\mu} \to \int_{-\infty}$

$$I_m = \int_0^\infty \frac{\delta f}{\delta \epsilon}(\epsilon-\mu)^m d\epsilon = \int_0^\infty \frac{\beta e^{\beta(\epsilon-\mu)}}{(e^{\beta(\epsilon-\mu)}+1)^2}(\epsilon-\mu)^m d\epsilon$$

$x = \epsilon - \mu$

$$= -\beta^{-m} \int_{-\beta\mu}^{\infty} \frac{e^x}{(e^x+1)^2} x^m dx$$

need to show $\int_{-\infty}^{-\beta\mu} \frac{e^x x^m}{(e^x+1)^2} dx$ is exponentially small

$\rightsquigarrow \int_{-\infty}^{-\beta\mu} e^x x^m \leq \int_{-\infty}^{-\beta\mu} \frac{e^x x^m}{(e^x+1)^2} dx$ b/c smallest value of $(e^x+1)^2$ is 1

$\rightsquigarrow \int_{-\infty}^{-\beta\mu} e^x x^m$, e^x dominates x^m \$ e^x is exponentially small for all $x << 0$ and $\beta\mu >> 1$

$$I_m = -\beta^{-m}\int_{-\infty}^{\infty} \frac{e^x x^m}{(e^x+1)^2} dx + \cancel{\beta^m \int_{-\infty}^{-\beta\mu} \frac{e^x x^m}{(e^x+1)^2} dx} \; 0$$

$$\left[I_m \approx -\beta^{-m} \int_{-\infty}^{\infty} \frac{e^x x^m}{(e^x+1)^2} dx \right]$$

· when m is odd $\rightarrow$ x^m is odd fxn (x, x^3, x^5 v. x^2, x^4)

even fxn: $\int_{-\infty}^{0} \cdots = \int_{0}^{\infty}$

odd fxn: $\int_{-\infty}^{0} = -\int_{0}^{\infty} \rightarrow \int_{-\infty}^{\infty} = 0$

· because we are taking $\int_{-\infty}^{\infty}$, x^m where m is odd cancel ($e^x/(e^x+1)^2$ is even)

(c) *(5 pts.)* Evaluate the first two non-vanishing terms and show that this agrees with the expansion of I.

$$I_0 = (1)\,\Phi(\mu) \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2}\,dx = \Phi(\mu)\left[\cancel{(e^x+1)}\right]_{-\infty}^{\infty}$$

1

$$I_0 = \Phi(\mu) = \int_0^{\epsilon} \phi(\epsilon')\,d\epsilon'$$

$$\Phi \equiv \int_0^{\epsilon} \phi(\epsilon')d\epsilon'$$

$$I_2 = +\frac{1}{2!}\left(\frac{\delta^2 \Phi(\mu)}{\delta \epsilon^2}\right) \int_{-\infty}^{\infty} \frac{x^2 \cdot e^x}{(e^x+1)^2}\,dx$$

mathematica

$$I_2 = \frac{1}{2!}\beta^{-2}\frac{\delta^2 \Phi(\mu)}{\delta \epsilon^2} \cdot \frac{\pi^2}{3} = kT^2\frac{\pi^2}{6}\Phi'' = (kT)^2\frac{\pi^2}{6}\phi'(\mu)$$

$$I \equiv \int_0^\infty \phi(\epsilon) f(\epsilon, T) d\epsilon = \underbrace{\int_0^\mu \phi(\epsilon) d\epsilon}_{\checkmark} + \underbrace{\frac{\pi^2}{6}(kT)^2 \phi'(\mu)}_{\checkmark} + \frac{7\pi^4}{360}(kT)^4 \phi'''(\mu) + \dots$$

d

d) *(10 pts.)* Using the result for I, express N in the form of such an integral and obtain an expansion for $N(V, T, \mu)$ in terms of kT/μ (to second order). Verify that $T \to 0$ yields the relation between N and ϵ_F derived in class.

$$N = \sum_\alpha \langle n_\alpha \rangle = \int_0^\infty \phi(\varepsilon) f(\epsilon, T)\, d\epsilon$$

density of states for particle #

Callen eq 18.19

$$\phi(\epsilon) \rightarrow \rho(\epsilon) = \frac{V}{2\pi^2} \cdot \frac{(2m)^{3/2}}{\hbar^3} \epsilon^{1/2}$$

$$\phi'(\epsilon) \rightarrow \rho'(\mu) = \frac{V}{4\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \mu^{-1/2}$$

$$N = \int_0^\mu \rho(\epsilon) d\epsilon + \frac{\pi^2}{6}(kT)^2 \frac{V}{4\pi^2}\left[\frac{(2m)^{3/2}}{\hbar^3} \epsilon^{-1/2}\right] + \ldots$$

$$N = \frac{\cancel{2}}{3} \cdot \frac{V}{\cancel{2}\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \mu^{3/2} + \frac{\pi^2}{6}(kT)^2 \frac{V}{4\pi^2}\left[\frac{(2m)^{3/2}}{\hbar^3} \mu^{-1/2}\right] + \ldots$$

$$\boxed{N(V, T, \mu) = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]}$$

$$kT \rightarrow 0, \quad \mu = \epsilon_F$$

$$\left[N(V, T, \epsilon_F) = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon_F^{3/2}\right]$$

(e) *(10 pts.)* Invert this relationship to obtain $\mu(T)$ as a function of kT/ϵ_F (to second order) for fixed N.

$$N(V,T,\mu) = N(V,T,\epsilon_F)$$

$$\cancel{\frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}} \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] = \cancel{\frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}} \epsilon_F^{3/2}$$

$$\left(\frac{\mu}{\epsilon_F}\right)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] = 1$$

$$\left(\frac{\mu}{\epsilon_F}\right)^{3/2} = 1 - \frac{\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2 \cancel{\left(\frac{\mu}{\epsilon_F}\right)^{-1/2}}$$

~ 1 @ low T

$$\frac{\mu}{\epsilon_F} \approx \left(1 - \frac{\pi^2}{8}\left(\frac{kT}{\epsilon_F}\right)^2\right)^{2/3}$$

$$\boxed{\mu \approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right]}$$

(f) *(5 pts.)* Similarly obtain an expansion for the internal energy E as a function of kT/μ (to second order).

$$E = \int_0^\infty \phi(\epsilon) f(\epsilon,T)\, d\epsilon$$

density of states for energy

$$\phi(\epsilon) \rightarrow \int \epsilon \rho(\epsilon) = \int \frac{V}{2\pi^2} \cdot \frac{(2m)^{3/2}}{\hbar^3} \epsilon^{3/2} \rightarrow \frac{2}{5} \cdot \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{5/2}\Big|_0^\mu$$

$$\phi'(\epsilon) \rightarrow \epsilon\rho'(\epsilon) = \frac{V}{2\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \cdot \frac{3}{2} \epsilon^{1/2}$$

change to μ

$$E = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{5/2} + \frac{\pi^2}{6}(kT)^2 \cdot \frac{2}{5} \cdot \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{1/2}$$

$$\boxed{E = \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu^{5/2}\left[1+\frac{\pi^2}{6}\left(\frac{kT}{\mu}\right)^2\right]}$$

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g) *(5 pts.)* Substituting in $\mu(T)$ into the energy expansion, obtain an expansion of E in kT/ϵ_F to second order, and thus C_v. Hence see why we skipped the detailed computation in class.

$$N(V,T,\mu) = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\mu^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]$$

$$E = \frac{3}{5}N\epsilon_F\left[\left(\frac{\mu}{\epsilon_F}\right)^{5/2}+\frac{5\pi^2}{24}\left(\frac{kT}{\epsilon_F}\right)^2\left(\frac{\mu}{\epsilon_F}\right)^{1/2}\right]$$

$$\mu \approx \epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right]$$ only up to 2nd term

$$E = \frac{3}{5}N\epsilon_F\left[\left(1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right)^{5/2}+\frac{5\pi^2}{24}\left(\frac{kT}{\epsilon_F}\right)^2\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2\right]^{1/2}\right]$$

$$E = \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\epsilon_F\left(\frac{kT}{\epsilon_F}\right)^2$$

$$C_v = \frac{\delta E}{\delta T} = 2\left(\frac{\pi^2}{4}\right)N\epsilon_F\left(\frac{k}{\epsilon_F}\right)^2 T$$

$$\boxed{C_v = \frac{\pi^2}{2}N\frac{k^2}{\epsilon_F}T}$$

I used these links + callen!

https://bingweb.binghamton.edu/~suzuki/SolidStatePhysics/10-4_Sommerfeld_formula.pdf

https://farside.ph.utexas.edu/teaching/sm1/Thermalhtml/node107.html#e8.127d

③ *(20 pts.)* Show that for the Bose-Einstein and Fermi-Dirac gas at low density and/or high temperature the equation of state is given by

$$p = kT\rho\left(1 \mp \frac{\rho\Lambda^3}{2^{5/2}} + \dots\right). \tag{5}$$

$\rightarrow$ online lecture & notes

@ low density limit $e^{\beta\mu} << 1$, $e^{\beta\mu} = z$ "fugacity"

- FD-BE partition fxn

$$\ln \Xi = \sum_i \ln\left(1 \mp e^{\beta\mu - \beta\varepsilon_i}\right)^{\mp 1}$$

$$= \frac{1}{8} \cdot \frac{V}{4\pi^3\hbar^3} \int_0^\infty dp\, 4\pi p^2 \ln\left(1 \mp e^{\beta\mu - \beta p^2/2m}\right)^{\mp 1}$$

$x \equiv \sqrt{\frac{\beta}{2m}}\, p$

$$= \frac{V}{\Lambda^3} \frac{4}{\sqrt{\pi}} \int_0^\infty dx \cdot x^2 \ln\left(1 \mp \xi e^{-x^2}\right)^{\mp 1}$$

$\xi \rightarrow 0$, expand series in ξ and integrate

$$= \frac{V}{\Lambda^3}\left(\frac{4}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{4}\right)\left[\sum_{n=1}^{\infty} (\mp 1)^{n+1} \frac{\xi^n}{n^{5/2}}\right] = \frac{V}{\Lambda^3} f_{5/2}(\xi)$$

$$dW = -S\,dT - P\,dV - N\,d\mu$$

$$W = -kT \ln \Xi$$

$$P = -\left(\frac{\delta W}{\delta V}\right)_{T,\mu} = kT\left(\frac{1}{\Lambda^3} f_{5/2}(\xi)\right)$$

$$N = -\left(\frac{\delta W}{\delta \beta \mu}\right)_{T,V} = \left(\frac{kT\delta \ln \Xi}{\delta \mu}\right) = \xi kT\left(\frac{V}{\Lambda^3} f_{5/2}(\xi)\right)$$

$$\beta P(T,\mu) = \frac{f_{5/2}(\xi)}{\Lambda^3}, \quad \beta N = \frac{V}{\Lambda^3} f_{3/2}(\xi)$$

expand in ξ

$$f_{5/2}(\xi) = \xi \mp \frac{\xi^2}{2^{5/2}} \cdots$$

$$\beta P = \frac{1}{\Lambda^3}\left(\xi \mp \frac{\xi^2}{2^{5/2}} \cdots\right)$$

$$\frac{N}{V} = \rho = \frac{1}{\Lambda^3}\left(\xi \mp \xi^2/2^{3/2} \cdots\right)$$

$\beta P = \rho$ ideal gas

- need to get $\beta P = \rho f(\xi) = \rho(\Lambda^3 \rho)$
- virial expansion $\rightarrow$ eliminate ξ from ρ

$$\Lambda^3 \rho = \xi \mp \frac{\xi^2}{2^{3/2}} \rightarrow \xi = \Lambda^3 \rho \mp \frac{\xi^2}{2^{3/2}}$$

plug ξ into βP

$$\xi = \Lambda^3 \rho \mp \frac{(\Lambda^3 \rho)^2}{2^{3/2}}$$

$$P = kT\frac{1}{\Lambda_3}\left(\Lambda^3 \rho \mp \frac{(\Lambda^3 \rho)^2}{2^{2/3}} \mp \frac{(\Lambda^3 \rho)^2}{2^{5/2}} \cdots\right)$$

$$\boxed{P = kT\rho\left(1 \mp \frac{\Lambda^3 \rho}{2^{2/5}} \cdots\right)}$$