1. Show that, for the Dirichlet case, the equation for V in terms of the charge distribution and the Green Function, Equation 3.46 of the Lecture Notes, recovers the boundary condition on V. That is, prove that, if the LHS is evaluated for $\vec{r} \in \mathcal{S}$, the RHS reduces to $V(\vec{r} \in \mathcal{S})$.

We know that

$$V(\vec{r}) = \int_{\mathcal{V}} d\tau'\, \rho(\vec{r}\,')\, G(\vec{r}, \vec{r}\,') + \epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' \left[G(\vec{r}, \vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,') - V(\vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G(\vec{r}, \vec{r}\,') \right] \quad (3.45)$$

Choosing $\vec{r} \in S$, we find that

b/c no charge on S

$$V(\vec{r} \in S) = 0 + \epsilon_0 \oint_S da' \left[G\, \hat{n} \cdot \vec{\nabla}_{\vec{r}\,'} V - V\, \hat{n} \cdot \vec{\nabla}_{\vec{r}\,'} G \right]$$

Applying Green's Second identity, we see that

$$V(\vec{r} \in S) = \epsilon_0 \int_V d\tau' \left[G(\vec{r}, \vec{r}\,') \nabla^2_{\vec{r}\,'} V(\vec{r}\,') - V(\vec{r}\,') \nabla^2_{\vec{r}\,'} G(\vec{r}, \vec{r}\,') \right]$$

We see that

$$G(\vec{r}, \vec{r}\,') \nabla^2_{\vec{r}\,'} V(\vec{r}\,')$$

is zero inside b/c $G(\vec{r}, \vec{r}\,') = 0 \;\; \forall \vec{r}\,' \notin S$

and zero on the surface since

$$\nabla^2_{\vec{r}\,'} V(\vec{r}\,') = -\frac{\rho(\vec{r}\,' \in S)}{\epsilon_0} = 0 \;\; \forall \vec{r}\,' \in S.$$

Thus,

$$V(\vec{r} \in S) = -\epsilon_0 \int_V V(\vec{r}\,') \nabla^2_{\vec{r}\,'} G(\vec{r}, \vec{r}\,')\, d\tau'$$

by definition of G

$$= -\epsilon_0 \int_V V(\vec{r}\,') \left(-\frac{1}{\epsilon_0} \delta(\vec{r} - \vec{r}\,') \right) d\tau'$$

$$= \int_V V(\vec{r}\,') \underbrace{\delta(\vec{r} - \vec{r}\,')}_{1/m^3} \underbrace{d\tau'}_{m^3}$$

$$= V(\vec{r})$$

Thus, we have shown the that RHS reduces to $V(\vec{r} \in S)$ when evaluated for $\vec{r} \in$ S.

$\square$

2. A volume $\mathcal{V}$ is bounded by a surface $\mathcal{S}$ consisting of several separate surfaces (conductors) $\mathcal{S}_i$. Let all the conductors be held at zero potential except $\mathcal{S}_1$. Show that the potential $V(\vec{r})$ anywhere in the volume $\mathcal{V}$ *and* on any of the surfaces $\mathcal{S}_i$ can be written

$$V(\vec{r}) = \oint_{\mathcal{S}_1} da'\, \sigma_1(\vec{r}\,')\, G(\vec{r},\vec{r}\,') \tag{1}$$

where $\sigma_1(\vec{r}\,')$ is the surface charge density on $\mathcal{S}_1$ and $G(\vec{r},\vec{r}\,')$ is the Dirichlet Green function in the presence of all the surfaces that are held at zero potential but with $\mathcal{S}_1$ *absent.* Show also that the electrostatic potential energy is

$$U = \frac{1}{2}\oint_{\mathcal{S}_1} da \oint_{\mathcal{S}_1} da'\, \sigma_1(\vec{r})\, \sigma_1(\vec{r}\,')\, G(\vec{r},\vec{r}\,') \tag{2}$$

Note: When $\mathcal{S}_1$ is removed, an additional volume of empty space is necessarily added to $\mathcal{V}$. You may assume that the volume that is added does not include infinity. An example of such a situation would be if the surfaces $\mathcal{S}_i$ were the faces of a cube. Removing one face would then connect the interior of the cube to the exterior, including infinity. The theorem is provable for such cases, but we'll have you prove it for the less general case because the more general case is a distraction from the point we are trying to have you see with this problem. If you want a picture in your head, assume that one of the surfaces (not $\mathcal{S}_1$) encloses $\mathcal{V}$ and separates it from infinity, while the other surfaces are interior and cause $\mathcal{V}$ to deviate from being simply connected.

$$V(\vec{r}) = \oint_{\mathcal{S}_1} da'\, \sigma_1(\vec{r}\,')\, G(\vec{r},\vec{r}\,')$$

First, we apply $-\varepsilon_0 \nabla^2_{\vec{r}}$ to both sides:

$$-\varepsilon_0 \nabla^2_{\vec{r}} V(\vec{r}) = -\varepsilon_0 \nabla^2_{\vec{r}} \oint_{S_1} da'\, \sigma_1(\vec{r}\,') G(\vec{r},\vec{r}\,')$$

$$= \oint_{S_1} da'\, \sigma_1(\vec{r}) \cdot (-\varepsilon_0 \nabla^2_{\vec{r}} G(\vec{r},\vec{r}\,'))$$

$$= \oint da'\, \sigma_1(\vec{r})\, \delta^3(\vec{r}-\vec{r}\,')$$

where we used the fact that

$$-\varepsilon_0 \nabla^2 G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$

$$-\varepsilon_0 \nabla^2 G(r, r') = \delta(r - r')$$

by construction.

Thus,

$$\nabla^2 V = 0 \quad \text{if} \quad \vec{r} \notin S$$

Moreover, we find

$$\nabla^2 V(\vec{r}) = -\rho_1(\vec{r})/\varepsilon_0 \quad \text{for } \vec{r} \in S$$

(since $\delta^3(\vec{r} - \vec{r}')$ has units of $1/m^3$)

Finally, we see that

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} \to 0 \text{ as } |\vec{r} - \vec{r}'| \to \infty$$

Thus, $V(\vec{r}) \to 0$ as $\vec{r}$ gets far away from S.

Thus, we have shown the above formula for V satisfies poisson's equation and the B.C.s. at $|\vec{r}| = \infty$.

Thus we showed the formula is correct. $\square$

Next, we will show this formula below is the correct one for calculating electric potential energy.

$$U = \frac{1}{2} \oint_{\mathcal{S}_1} da \oint_{\mathcal{S}_1} da' \, \sigma_1(\vec{r}) \, \sigma_1(\vec{r}') \, G(\vec{r}, \vec{r}')$$

First, recall the equation for electric potential energy expressed in terms of V:

$$U = \frac{1}{8\pi\epsilon_0} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}} d\tau' \, \frac{\rho(\vec{r})\,\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{1}{2} \int_{\mathcal{V}} d\tau \rho(\vec{r}) \, V(\vec{r})$$

Since we only have surface charges, we may analogously write

$$U = \frac{1}{2}\oint_{S_1} \sigma_1(\vec{r})\, V(\vec{r}) \cdot da$$

$$= \frac{1}{2}\oint_{S_1} da\, \sigma_1(\vec{r}) \cdot \oint_{S_1} da'\, \sigma_1(r')\, G(\vec{r},\vec{r}')$$

$$= \frac{1}{2}\oint_{S_1} da \oint_{S_1} da'\, \sigma_1(\vec{r})\sigma_1(\vec{r}')\, G(\vec{r},\vec{r}')$$

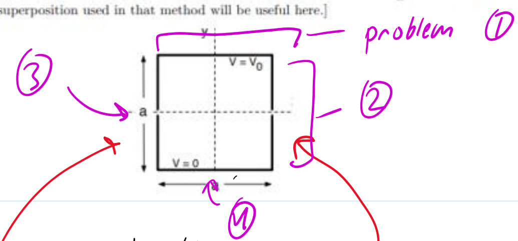
$\square$

3. A long conducting pipe of square cross-section (with side a) is split down the middle lengthwise (cuts in the middle of a pair opposite sides). One half is held at zero potential and the other at potential V_0. Assume the gap between the halves is infinitesimal. Use the method of separation of variables to find the electric potential everywhere inside. Choose the origin at the center of the square.

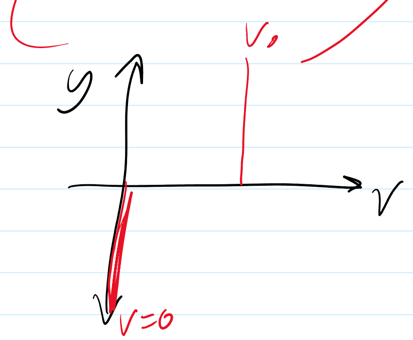
Solve this as a two-dimensional problem.

Note: There are at least two different ways to do this problem. They must yield the same result, but it may take some work to see that they are the same.

[*Hint:* Recall how we discussed in class how to extend our solution for the box with all but one face grounded to deal with the case of a box with fewer than five faces grounded. The concept of superposition used in that method will be useful here.]



The potential on the sides looks like this:



We will solve this problem by splitting it up into 4 mini problems:

- Notice that problem 4 (the one that solves the fourth boundary condition and is zero for the rest) is the trivial solution and thus need not be considered.
- Moreover, problems 2 and 3 are symmetric so we need only solve one.
- Problem one is just V=V0 on the top, with the rest of the sides set to zero potential (V=0)

As shown in the problem set sessions and in lecture, we can separate the potential in cartesian coordinates:

$$V(x,y) = X(x) * Y(y).$$

For problem one, we see that $Y(y) \propto \sinh\left(\frac{n\pi}{a}\left(y+\frac{a}{2}\right)\right)$ satisfies both the boundary conditions (zero at y=-a/2 and nonnegative at y=a/2) and its corresponding differential equation:

$\frac{1}{Y(y)}Y''(y) = \left(\frac{n\pi}{a}\right)^2 = const$.

Moreover, we see that $X(x) \propto \sin\left(\frac{n\pi}{a}\left(x+\frac{a}{2}\right)\right)$ also satisfies both the boudnary conditions (zero at |x|=a/2) and its differential equation (identical to that of y). Also notice that the constant on the RHS of X's diffeq is the negative of Y's so that they add to zero as required by Laplaces' equaition:

$$\nabla^2 V = \frac{1}{Y(y)}Y''(y) + \frac{1}{X(x)}X''(x) = \left(\frac{n\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2 = 0.$$

Therefore, the potential is the superposition of these solutions that satisfies $V_1(x,y) = V_0$, the coefficients of which are the fourier series of the funciton V=V0 expanded in the $\sin\left(\frac{n\pi}{a}\left(x+\frac{a}{2}\right)\right)$ basis:

$$V_1(x,y) = \frac{2}{a}\sum_n \sin\left(\alpha_n\left(x+\tfrac{a}{2}\right)\right) \frac{\sinh\left(\gamma_n\left(y+\frac{a}{2}\right)\right)}{\sinh\left(\gamma_n\left(\frac{a}{2}+\frac{a}{2}\right)\right)} \cdot \underbrace{\int_{-a/2}^{a/2} dx' \cdot V_0 \cdot \sin\left(\alpha_n\left(x'+\tfrac{a}{2}\right)\right)}$$

- $\alpha_n = \frac{n\pi}{a}$
- $\gamma_n = \sqrt{\alpha_n^2} = \pm\alpha_n$

$$= \int_0^a dx' \sin(\alpha_n x') \cdot V_0$$

$$= \left(1+(-1)^{n+1}\right)\cdot\frac{a}{n\pi}\cdot V_0$$

$$= \frac{2a}{n\pi}\cdot V_0 \quad \text{if } n \text{ odd}$$

$$\Rightarrow \quad V_1(x,y) = \sum_{n\ \text{odd}}^{\infty} \frac{4V_0}{n\pi}\sin\left(\alpha_n\left(x+\tfrac{a}{2}\right)\right)\frac{\sinh\left(\alpha_n\left(y+\frac{a}{2}\right)\right)}{\sinh(n\pi)}$$

We now solve problem 2 (and likewise problem 3) using an analogous approach and analogous basis functions:

$$V_2(x,y) = \frac{2}{a}\sum_n \sin\left(\alpha_n\left(y+\tfrac{a}{2}\right)\right)\frac{\sinh\left(\alpha_n\left(x+\frac{a}{2}\right)\right)}{\sin(n\pi)}$$

Heavyside function

$$\cdot \underbrace{\int_{-a/2}^{a/2} V_0 \cdot H(y')\sin\left(\alpha_n\left(y'+\tfrac{a}{2}\right)\right)dy'}$$

C.O.V. + use property of H

$$\Rightarrow V_0 \int^a \sin(\alpha_n u')\,du' = \frac{a}{\ldots}\left((-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right)\right)$$

$\to$ C.O.V. + use property of

$$\Rightarrow V_0 \int_{a/2}^{a} \sin(\alpha_n y')\,dy' = \frac{a}{n\pi}\left((-1)^{n+1} + \cos\left(\frac{n\pi}{2}\right)\right)$$

Mathematica

$$\Rightarrow V_2(x,y) = \sum_{n=1}^{\infty} \frac{4V_0}{n\pi} \sin^2\left(\frac{n\pi}{4}\right) \cdot \sin\left(\frac{n\pi}{a}\left(y+\frac{a}{2}\right)\right) \cdot \frac{\sinh\left(\alpha_n\left(x+\frac{a}{2}\right)\right)}{\sinh(n\pi)}$$

To get $V_3(x,y)$ simply take $x \to (-x)$.

Thus the total solution is

$$\boxed{\begin{aligned} V(x,y) &= (V_1 + V_2 + V_3 + 0)(x,y) \\ &= 4V_0 \sum_{n=1}^{\infty} \frac{1}{n\pi} \Big[\left(1 + (-1)^{n+1}\right) \sin\left(\alpha_n\left(x+\tfrac{a}{2}\right)\right) \sinh\left(\alpha_n\left(y+\tfrac{a}{2}\right)\right) \\ &\quad + \left((-1)^{n+1} + \cos(n\pi/2)\right) \sin\left(\alpha_n\left(y+\tfrac{a}{2}\right)\right) \left(\sinh\left(\alpha_n\left(x+\tfrac{a}{2}\right)\right) + \sinh\left(\alpha_n\left(-x+\tfrac{a}{2}\right)\right) \right) \Big] \\ &\quad \cdot \left(\sinh(n\pi)\right)^{-1} \end{aligned}}$$