# Ph125b Set

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# 1: Extension Note

An extension was granted for an additional 24 hours.

# 2: Problem 9

## 2.1. Part A

From relativity, we know that:

$$E^2 = P^2c^2 + m^2c^4 \tag{1}$$

$$E = c\sqrt{P^2 + m^2c^2} \tag{2}$$

Expanding in powers of $P$:

$$E = mc^2 + \frac{P^2}{2m} - \frac{3P^4}{4!c^2m^3} \tag{3}$$

Ignoring the constant rest mass, quantizing, and putting this in a potential, we have:

$$\hat{\mathrm{H}} = \frac{\hat{\mathbf{P}}^2}{2m} - \frac{\hat{\mathbf{P}}^4}{8c^2m^3} + \hat{\mathrm{V}} \tag{4}$$

Now we can use pertubation theory since $\hat{\mathrm{H}}_0 \coloneqq \hat{\mathrm{H}}_H$ is a well known problem. Thus we need only the $\hat{\mathbf{P}}^4$ term. We can also write:

$$\hat{\mathbf{P}}^4 = \left(\frac{\hat{\mathbf{P}}^2}{2m}\right)^2 (2m)^2 = (2m)^2\left(\hat{\mathrm{H}}_0 - \hat{\mathrm{V}}\right)^2 \tag{5}$$

$$\langle\psi|\hat{\mathbf{P}}^4|\psi\rangle = (2m)^2\langle\psi|\left(\hat{\mathrm{H}}_0 - \hat{\mathrm{V}}\right)^2|\psi\rangle \tag{6}$$

$$= (2m)^2\langle\psi|\hat{\mathrm{H}}_0^2 - \hat{\mathrm{H}}_0\hat{\mathrm{V}} - \hat{\mathrm{V}}\hat{\mathrm{H}}_0 + \hat{\mathrm{V}}^2|\psi\rangle \tag{7}$$

Letting $\psi$ be an energy eigenket,

$$= (2m)^2\langle\psi|\varepsilon_n^2 - \hat{\mathrm{H}}_0\hat{\mathrm{V}} - \hat{\mathrm{V}}\varepsilon_n + \hat{\mathrm{V}}^2|\psi\rangle \tag{8}$$

$$= (2m)^2\left(\varepsilon_n^2 + \langle\psi|\hat{\mathrm{V}}^2 - \left(\hat{\mathrm{H}}_0 + \varepsilon_n\right)\hat{\mathrm{V}}|\psi\rangle\right) \tag{9}$$

$$= (2m)^2\left(\varepsilon_n^2 + \left\langle\frac{e^4}{r^2}\right\rangle_{nlm} - \langle\psi|\left(\hat{\mathrm{H}}_0 + \varepsilon_n\right)\hat{\mathrm{V}}|\psi\rangle\right) \tag{10}$$

$$= (2m)^2\left(\varepsilon_n^2 + \left\langle\frac{e^4}{r^2}\right\rangle_{nlm} - \left\langle\left(\hat{\mathrm{H}}_0 + \varepsilon_n\right)^\dagger\psi\,\middle|\,\hat{\mathrm{V}}\,\middle|\,\psi\right\rangle\right) \tag{11}$$

$$= (2m)^2\left(\varepsilon_n^2 + \left\langle\frac{e^4}{r^2}\right\rangle_{nlm} + 2\varepsilon_n\left\langle\psi\,\middle|\,\hat{\mathrm{V}}\,\middle|\,\psi\right\rangle\right) \tag{12}$$

$$= (2m)^2\left(\varepsilon_n^2 + 2\varepsilon_n\left\langle\frac{e^2}{r}\right\rangle_{nlm} + \left\langle\left(\frac{e^2}{r}\right)^2\right\rangle_{nlm}\right) \tag{13}$$

Thus:

$$E_n^{(1)} = -\left\langle n \left| \frac{\hat{\mathbf{P}}^4}{8c^2m^3} \right| n \right\rangle \tag{14}$$

$$E_n^{(1)} = -\frac{(2m)^2}{8c^2m^3}\left(\varepsilon_n^2 + 2\varepsilon_n \left\langle \frac{e^2}{r} \right\rangle_{nlm} + \left\langle \left(\frac{e^2}{r}\right)^2 \right\rangle_{nlm}\right) \tag{15}$$

$$\boxed{E_n^{(1)} = -\frac{1}{2mc^2}\left(\varepsilon_n^2 + 2\varepsilon_n \left\langle \frac{e^2}{r} \right\rangle_{nlm} + \left\langle \left(\frac{e^2}{r}\right)^2 \right\rangle_{nlm}\right)} \tag{16}$$

## 2.2. Part B

Up to this point has been independent, but unfortunately I discovered pg. 467 on Shankar so I follow along from there:

Using Quantum Virial Theorem, we have:

$$\left\langle \frac{e^2}{r} \right\rangle = -2\varepsilon_n \tag{17}$$

And from problem 11, we'll find that

$$\left\langle \frac{e^4}{r^2} \right\rangle = \frac{e^4}{a_0^2 n^3 \left(l + \frac{1}{2}\right)} = \frac{4\varepsilon_0^2 n}{l + \frac{1}{2}} \tag{18}$$

Therefore we have:

$$E_n^{(1)} = -\frac{1}{2mc^2}\left(\varepsilon_n^2 + 2\varepsilon_n(-2\varepsilon_n) + \frac{4\varepsilon_0^2 n}{l + \frac{1}{2}}\right) \tag{19}$$

$$E_n^{(1)} = -\frac{\varepsilon_n^2}{2mc^2}\left(1 - 4 + \frac{4n}{l + \frac{1}{2}}\right) \tag{20}$$

$$E_n^{(1)} = -\frac{\varepsilon_n^2}{2mc^2}\left(-3 + \frac{4n}{l + \frac{1}{2}}\right) \tag{21}$$

$$E_n^{(1)} = -\frac{\frac{\left(mc^2\alpha^2\right)^2}{n^4}}{2mc^2}\left(-3 + \frac{4n}{l + \frac{1}{2}}\right) \tag{22}$$

$$\boxed{E_n^{(1)} = -\frac{1}{2}mc^2\alpha^4\left(-\frac{3}{n^4} + \frac{4}{n^3\left(l + \frac{1}{2}\right)}\right)} \tag{23}$$

# 3: Problem 10

## 3.1. Part A

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}_H + \hat{\mathrm{V}} \tag{1}$$

Consider a sphere of uniform charge. By Newton's Shell Theorem, it can be calculated that:

$$\Phi(r) = \begin{cases} \frac{KQ}{r} & ; \quad r \geq R \\ \frac{3KQ}{2R}\left(1 - \frac{r^2}{3R^2}\right) & ; \quad r \leq R \end{cases} \tag{2}$$

Thus the pertubation is

$$\hat{\mathrm{V}} = \frac{3KQ}{2R}\left(1 - \frac{r^2}{3R^2}\right) - \frac{KQ}{r} \tag{3}$$

for $r \leq R$, and 0 otherwise.

## 3.2. Part B

We wish to know:

$$\left\langle \hat{\mathrm{V}} \right\rangle = \langle \psi | \hat{\mathrm{V}} | \psi \rangle = \oiiint R_{nl}(r) Y_{lm}(\theta, \phi) \hat{\mathrm{V}} R_{nl}(r) Y_{lm}(\theta, \phi) \tag{4}$$

Assume that $R_{nl}$ is slowly varying and thus constant:

$$= R_{nl}(0)^2 \oiiint Y_{lm}(\theta, \phi) \hat{\mathrm{V}}(r) Y_{lm}(\theta, \phi) \tag{5}$$

$$= R_{nl}(0)^2 \oiiint Y_{lm}(\theta, \phi) \hat{\mathrm{V}}(r) Y_{lm}(\theta, \phi) \tag{6}$$

Recognize now that if $l \neq 0$, then $R_{nl} = 0$. So we may assume $l = m = 0$ and therefore, with $Y_{00} = \frac{1}{\sqrt{4\pi}}$

$$= \frac{R_{nl}(0)^2}{4\pi} \oiiint \hat{\mathrm{V}}(r) \tag{7}$$

$$= \frac{R_{nl}(0)^2}{4\pi} \oiiint \frac{3KQ}{2R}\left(1 - \frac{r^2}{3R^2}\right) - \frac{KQ}{r} \tag{8}$$

$$= \frac{R_{nl}(0)^2}{4\pi} \int_0^R \int_0^\pi \int_0^\tau \frac{3KQ}{2R}\left(1 - \frac{r^2}{3R^2}\right) - \frac{KQ}{r} \mathrm{d}\theta \sin(\phi) \mathrm{d}\phi r^2 \mathrm{d}r \tag{9}$$

$$= \left(R_{nl}(0)^2\right) \int_0^R \frac{3KQ}{2R}\left(r^2 - \frac{r^4}{3R^2}\right) - KQr\mathrm{d}r \tag{10}$$

$$= \left(R_{nl}(0)^2\right) \int_0^R \frac{3KQ}{2R}\left(r^2 - \frac{r^4}{3R^2} - \frac{2R}{3}r\right) \mathrm{d}r \tag{11}$$

$$= \frac{3KQ}{2R} R_{nl}(0)^2 \left( -\frac{R^3}{15} \right) \tag{12}$$

$$= \frac{KQ}{2} R_{nl}(0)^2 \left( -\frac{R^2}{5} \right) \tag{13}$$

$$= -\frac{KQR^2}{10} R_{nl}(0)^2 \tag{14}$$

Using the results on the set:

$$R_{nl}(0) = \left( \frac{2Z}{na_0} \right)^{\frac{3}{2}} \sqrt{\frac{1}{2n^2}} L^1_{n-1}(0) \tag{15}$$

$$= \left( \frac{2Z}{na_0} \right)^{\frac{3}{2}} \sqrt{\frac{1}{2n^2}} \binom{n}{n-1} \tag{16}$$

$$= 2 \left( \frac{Z}{na_0} \right)^{\frac{3}{2}} \tag{17}$$

Thus:

$$\left\langle \hat{\mathrm{V}} \right\rangle = -\frac{KQR^2}{10} \left( 2 \left( \frac{Z}{na_0} \right)^{\frac{3}{2}} \right)^2 \tag{18}$$

$$\left\langle \hat{\mathrm{V}} \right\rangle = -\frac{KQR^2}{10} 4 \left( \frac{Z}{na_0} \right)^3 \tag{19}$$

$$\left\langle \hat{\mathrm{V}} \right\rangle = -\frac{2KQR^2Z^3}{5n^3a_0^3} \tag{20}$$

## 3.3. Part C

We see that for $l \geq 0$, we find these states are unaffected and have 0 expected pertubation energy (to first order). $s$-orbitals however have:

$$\left\langle \hat{\mathrm{V}} \right\rangle = -\frac{2KQR^2Z^3}{5n^3a_0^3} \tag{21}$$

Now recognize that for every $n$, we have $n^2$ degeneracies (no spin). It's natural to wonder if we should be worried about our example such as in Example 5.2:1. **The answer is no.** The reason being that because $\hat{\mathrm{V}}$ is angularly-independent, that the inner product gives $\delta_{mm'}\delta_{ll'}$ in the answer. Therefore to get a non-zero result, we must have $l = 0$ and $m = 0$. This leaves us with $n$, but if $n \neq n'$, then they are not degenerate, showing we have nothing to worry about.

# 4: Problem 11

## 4.1. Part A

We follow along with pg 470 of Shankar. Observe that in the radial equation, we have:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(2\frac{\mu}{\hbar^2}\right)\left(E - \hat{\mathrm{V}}(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right)\right]U_{\mathrm{el}} = 0 \tag{1}$$

But taking out the pertubation term from $\hat{\mathrm{V}}(r)$, we may adjust the centrifugal term.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(2\frac{\mu}{\hbar^2}\right)\left(E - \hat{\mathrm{V}}_0(r) - \frac{\lambda}{r^2} - \frac{l(l+1)\hbar^2}{2\mu r^2}\right)\right]U_{\mathrm{el}} = 0 \tag{2}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(2\frac{\mu}{\hbar^2}\right)\left(E - \hat{\mathrm{V}}_0(r) - \frac{l'(l'+1)\hbar^2}{2\mu r^2}\right)\right]U_{\mathrm{el}} = 0 \tag{4.1:3}$$

Since we've only modified the differential equation, the energy from Shankar 13.1.14 must still hold:

$$E(l') = -\frac{me^4}{2\hbar^2(k+l'+1)^2} = E^{(0)} + E^{(1)} + \cdots \tag{4}$$

Where now $l'$ is derived from **Equation (3)**. We now simplify an equation for $l'(\lambda)$ implicitly:

$$\frac{\hbar^2 l(l+1)}{2mr^2} + \frac{\lambda}{r^2} = \frac{\hbar^2 l'(l'+1)}{2mr^2} \tag{5}$$

$$l'(l'+1) = l(l+1) + \frac{2m\lambda}{\hbar^2} \tag{6}$$

$$l'(l'+1) = l(l+1) + \frac{2m\lambda}{\hbar^2} \tag{4.1:7}$$

But now remembering that (In the Professor's formulation, not Shankar's):

$$\left\langle\frac{1}{r^2}\right\rangle = E^{(1)} = \left[\frac{\mathrm{d}E}{\mathrm{d}\lambda}\right|_{\lambda=0} = \left[\frac{\mathrm{d}E}{\mathrm{d}l'}\frac{\mathrm{d}l'}{\mathrm{d}\lambda}\right|_{\lambda=0} \tag{8}$$

Differentiating:

$$\frac{\mathrm{d}E}{\mathrm{d}l'} = \frac{me^4}{\hbar^2(k+l'+1)^3} \tag{9}$$

$$2\left(l' + \frac{1}{2}\right)\mathrm{d}l' = \frac{2m}{\hbar^2}\mathrm{d}\lambda \tag{10}$$

$$\frac{\mathrm{d}l'}{\mathrm{d}\lambda} = \frac{m}{\hbar^2\left(l' + \frac{1}{2}\right)} \tag{11}$$

Which if we evaluate **Equation (7)** with $\lambda = 0$, we see implies either:

$$l' = l \quad \text{or} \quad l' = -(l+1) \tag{12}$$

It is clear though that $l' = -(l+1)$ is unphysical so we have:

$$\left[\frac{\mathrm{d}E}{\mathrm{d}l'}\frac{\mathrm{d}l'}{\mathrm{d}\lambda}\right|_{\lambda=0} = \left(\frac{me^4}{\hbar^2(k+l'+1)^3}\frac{m}{\hbar^2\left(l'+\frac{1}{2}\right)}\right) \tag{13}$$

$$\left\langle\frac{1}{r^2}\right\rangle = E^{(1)} = \frac{1}{a_0^2(k+l'+1)^3\left(l'+\frac{1}{2}\right)} \tag{14}$$

$$\left\langle\frac{1}{r^2}\right\rangle = E^{(1)} = \frac{1}{a_0^2 n^3\left(l+\frac{1}{2}\right)} \tag{15}$$

## 4.2. Part B

We know from 12.6:3 that the radial part of the Hamiltonian is

$$\frac{-\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right) \tag{16}$$

Observe that with

$$\hat{\mathrm{P}}_r := -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \tag{17}$$

we have

$$\frac{\hat{\mathrm{P}}_r^2}{2m} = \frac{-\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right) \tag{18}$$

Likewise, observe that for an eigenstate $\psi$ and by Ehrenfest's Theorem, we have:

$$\frac{\mathrm{d}\left\langle\hat{\mathrm{P}}_r\right\rangle}{\mathrm{d}t} = 0 = \left\langle\left[\hat{\mathrm{H}},\hat{\mathrm{P}}_r\right]\right\rangle \tag{19}$$

Expliclty evaluating the commutator, we find, using the radial equation, that

$$\left[\hat{\mathrm{H}},\hat{\mathrm{P}}_r\right] = \hat{\mathrm{H}}\hat{\mathrm{P}}_r - \hat{\mathrm{P}}_r\hat{\mathrm{H}} \tag{20}$$

$$= \left[-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}\right)+\hat{\mathrm{V}}\right]\hat{\mathrm{P}}_r - \hat{\mathrm{P}}_r\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{l(l+1)}{r^2}\right)+\hat{\mathrm{V}}\right] \tag{21}$$

$$= \left[-\frac{\hbar}{2m}\frac{l(l+1)}{r^2}+\hat{\mathrm{V}}\right]\hat{\mathrm{P}}_r - \hat{\mathrm{P}}_r\left[-\frac{l(l+1)}{r^2}+\hat{\mathrm{V}}\right] \tag{22}$$

$$= -\frac{\hbar}{2m}\left[\frac{l(l+1)}{r^2},\hat{\mathrm{P}}_r\right]+\left[\hat{\mathrm{V}},\hat{\mathrm{P}}_r\right] \tag{23}$$

Breaking this up:

$$\left[\hat{\mathrm{V}},\hat{\mathrm{P}}_r\right] \propto \left[\frac{1}{r},\frac{\partial}{\partial r}\right] = \frac{1}{r}\psi' - \frac{\partial}{\partial r}\frac{\psi}{r} = \frac{1}{r^2} \tag{24}$$

$$\left[\frac{1}{r^2},\hat{\mathrm{P}}_r\right] \propto \left[\frac{1}{r^2},\frac{\partial}{\partial r}\right] = \frac{1}{r^2}\psi' - \frac{\partial}{\partial r}\frac{\psi}{r^2} = \frac{2}{r^3} \tag{25}$$

Therefore we have:

$$\left[\hat{\mathrm{H}}, \hat{\mathrm{P}}_r\right] = K\left(-\frac{\hbar^2}{2m}l(l+1)\left[\frac{1}{r^2}, \frac{\partial}{\partial r}\right] + e^2\left[\frac{1}{r}, \frac{\partial}{\partial r}\right]\right) \tag{26}$$

$$\left[\hat{\mathrm{H}}, \hat{\mathrm{P}}_r\right] = K\left(-\frac{\hbar^2}{2m}\frac{2l(l+1)}{r^3} + \frac{e^2}{r^2}\right) \tag{27}$$

But now we see that we must have:

$$\left\langle -\frac{\hbar^2}{m}\frac{l(l+1)}{r^3} + \frac{e^2}{r^2}\right\rangle = 0 \tag{28}$$

$$\left\langle \frac{\hbar^2}{m}\frac{l(l+1)}{r^3}\right\rangle = \left\langle \frac{e^2}{r^2}\right\rangle \tag{29}$$

$$\left\langle \frac{1}{r^3}\right\rangle = \frac{me^2}{\hbar^2 l(l+1)}\left\langle \frac{1}{r^2}\right\rangle \tag{30}$$

$$\left\langle \frac{1}{r^3}\right\rangle = \frac{1}{a_0 l(l+1)}\left\langle \frac{1}{r^2}\right\rangle \tag{31}$$

But combining our previous statement, we have:

$$\left\langle \frac{1}{r^3}\right\rangle = \frac{1}{a_0 l(l+1)}\frac{1}{a_0^2 n^3\left(l+\frac{1}{2}\right)} \tag{32}$$

$$\left\langle \frac{1}{r^3}\right\rangle = \frac{1}{a_0^3 n^3 l\left(l+\frac{1}{2}\right)(l+1)} \tag{33}$$

# 5: Problem 12

## 5.1. Part A

$$\hat{\mathrm{H}}_0 = \frac{\hat{\mathbf{P}}^2}{2m} - \frac{\alpha}{r} \tag{1}$$

$$\hat{\mathrm{V}} = -Eez \tag{2}$$

$$\hat{\mathrm{H}} = \frac{\hat{\mathbf{P}}^2}{2m} - \frac{\alpha}{r} - Eez \tag{3}$$

It is clear that with a homogenous electric field, we need only consider the relative radius, as did the original Hamiltonian. Consider now in the $\boldsymbol{r}$ basis:

$$\left[\hat{\mathrm{V}}, \hat{\mathrm{L}}_z\right] = \hat{\mathrm{V}}\left(\hat{\mathrm{X}}\hat{\mathrm{P}}_y - \hat{\mathrm{Y}}\hat{\mathrm{P}}_x\right) - \left(\hat{\mathrm{X}}\hat{\mathrm{P}}_y - \hat{\mathrm{Y}}\hat{\mathrm{P}}_x\right)\hat{\mathrm{V}} \tag{4}$$

$$= Ee\left(z\left(\hat{\mathrm{X}}\hat{\mathrm{P}}_y - \hat{\mathrm{Y}}\hat{\mathrm{P}}_x\right) - \left(\hat{\mathrm{X}}\hat{\mathrm{P}}_y - \hat{\mathrm{Y}}\hat{\mathrm{P}}_x\right)z\right) \tag{5}$$

But these commute so:

$$\left[\hat{\mathrm{V}}, \hat{\mathrm{L}}_z\right] = 0 \tag{6}$$

This gives us a selection rule in accordance with Eq 17.2.12 from Shankar. Namely, that the matrix elements between any states of different eigenvalues for $\hat{\mathrm{L}}_z$ will always be 0. Formally:

$$\left\langle \alpha_1 m_1 \middle| \hat{\mathrm{V}} \middle| \alpha_2 m_2 \right\rangle = 0 \tag{7}$$

if $m_1 \neq m_2$.

## 5.2. Part B

Since we're ignoring spin, and since we've removed worry about any degeneracies between different eigenvalues of $\hat{\mathrm{L}}_z$, we now only need to worry about when $n_1 = n_2$, $m_1 = m_2$, but $l_1 \neq l_2$. Recalling that $0 \leq m \leq l \leq n$, this implies we have $(n - m)$ degenerate states.

## 5.3. Part C

This leaves us with only 4 potentially non-zero elements for $n = 2$:

$$\left\langle 200 \middle| \hat{\mathrm{V}} \middle| 200 \right\rangle \quad ; \quad \left\langle 210 \middle| \hat{\mathrm{V}} \middle| 200 \right\rangle \quad ; \quad \left\langle 200 \middle| \hat{\mathrm{V}} \middle| 210 \right\rangle \quad ; \quad \left\langle 210 \middle| \hat{\mathrm{V}} \middle| 210 \right\rangle \tag{8}$$

We have:

$$|200\rangle = K_0\left(2 - \frac{r}{a_0}\right)e^{-\frac{r}{2a_0}} \tag{9}$$

$$|210\rangle = K_1 \cos(\theta)\frac{r}{a_0}e^{-\frac{r}{2a_0}} \tag{10}$$

To cut work for ourselves, first recognize that $z = r\cos(\phi)$ is odd parity, while combining any two psi's, that is, $\langle\psi \mid \psi\rangle$ is even. Thus $\langle\psi|z|\psi\rangle$ is odd and the integral over it must cancel to 0. This gives $\left\langle 200 \middle| \hat{\mathrm{V}} \middle| 200 \right\rangle = \left\langle 210 \middle| \hat{\mathrm{V}} \middle| 210 \right\rangle = 0$. We can also recognize that $\left\langle 200 \middle| \hat{\mathrm{V}} \middle| 210 \right\rangle = \left\langle 210 \middle| \hat{\mathrm{V}} \middle| 200 \right\rangle$ since $\hat{\mathrm{V}}$ commutes with all wave functions. Thus we have:

$$\langle 210 \,|\, z \,|\, 200 \rangle = \oiiint K_0 \left(2 - \frac{r}{a_0}\right) e^{-\frac{r}{2a_0}} r \cos(\phi) K_1 \cos(\phi) \frac{r}{a_0} e^{-\frac{r}{2a_0}} \tag{11}$$

$$= 4\pi K_0 K_1 \int_0^\infty \int_0^\pi \frac{r}{a_0}\left(2 - \frac{r}{a_0}\right) r^3 \cos(\phi)^2 \sin(\phi) e^{-\frac{r}{a_0}} \mathrm{d}\phi \mathrm{d}r \tag{12}$$

$$= 4\pi K_0 K_1 \int_0^\infty \frac{r}{a_0}\left(2 - \frac{r}{a_0}\right) r^3 e^{-\frac{r}{a_0}} \int_0^\pi \cos(\phi)^2 \sin(\phi) \mathrm{d}\phi \mathrm{d}r \tag{13}$$

$$= \frac{8\pi}{3} K_0 K_1 \int_0^\infty \frac{r}{a_0}\left(2 - \frac{r}{a_0}\right) r^3 e^{-\frac{r}{a_0}} \mathrm{d}r \tag{14}$$

$$= -\frac{8\pi}{3} K_0 K_1 (72 a_0^4) \tag{15}$$

$$= -\frac{8\pi}{3} \left(\frac{3}{4\pi} \frac{\sqrt{3}}{8\sqrt{12}}\right) (72 a_0) \tag{16}$$

$$= -3a_0 \tag{17}$$

Therefore:

$$\left\langle 210 \,\middle|\, \hat{\mathrm{V}} \,\middle|\, 200 \right\rangle = 3Eea_0 \tag{18}$$

## 5.4. Part D

For the constant electric field, we find:

$$\left\langle 210 \,\middle|\, \hat{\mathrm{V}} \,\middle|\, 200 \right\rangle = 3Eea_0 = 1.5\,\mathrm{m}e\mathrm{V} \tag{19}$$

For a Hydrogen atom, we find: $E = 5 \cdot 10^{11}\ \frac{\mathrm{N}}{\mathrm{m}}$ so:

$$\left\langle 210 \,\middle|\, \hat{\mathrm{V}} \,\middle|\, 200 \right\rangle = 3Eea_0 = 81.6\,e\mathrm{V} \tag{20}$$

So we see that for $E$ fields comparable to the hydrogen atom, it is unreasonable, but it is fine for fields comparable to $E = 100\ \frac{\mathrm{kV}}{\mathrm{cm}}$