Problem Set 2

Applied Math 205, Fall 2024

Due as a pdf file on Gradescope before 3:00 pm, Monday 30 September *I used ChatGPT to help me here.*

1

Low-rank approximation of Old Glory. On Canvas, you will find two files flag1.jpg and flag2.jpg, each with a color image of the United States flag. Adapt the code p6_iterativeGE.m from the Sept. 16 lecture, or its Python equivalent, to produce low-rank approximations of monochrome versions of these images by Gaussian elimination with complete pivoting. As always in AM205, be sure to list your code.

Run your code to produce a plot of 9 images corresponding to approximations of image 1 of ranks $k = 1, 2, 4, \ldots, 256$. Also, produce a plot of the root-mean-square error in the low-rank approximation as a function of k (all values of k, not just powers of 2). Then do all this again for image2. Comment on the results, and in particular on the comparison between the two images. Why does one series converge faster than the other?

1.1

The low-rank approximation for the first image converges faster; this is expected because it is simpler and has more structure, which are things that can be taken advantage of by a low-rank approximation, as indicated by the faster convergence in RMSE.

2

2. Least-squares fitting by ellipses. Suppose we have m data points $(x_1, y_1), \ldots, (x_m, y_m)$ in the $x - y$ plane and we want to find an ellipse that fits them well. Finding the geometrically closest fit is a nonlinear problem, but we can come close by a linear formulation. An equation for an ellipse centered at $(0, 0)$ is

$$bx^2 + cxy + dy^2 = 1$$

Let us view b, c and d as unknowns and find them by solving a linear least-squares problem.

2.1

(a) Write down in matrix form an $m \times 3$ least-squares problem whose unknown vector is $(b, c, d)^T$.

2.1.1

The least-squares problem is given by the following matrix equation:

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ x_2^2 & x_2 y_2 & y_2^2 \\ \vdots & \vdots & \vdots \\ x_m^2 & x_m y_m & y_m^2 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{1}$$

This matrix equation is merely an extension of the ellipse equation to the m data points.

2.2

(b) Write a function $[b, c, d] =$ ellipse (x, y) which uses " " to solve this problem. Write driver code to call ellipse for the data

$$(3,3), (1,-2), (0,3), (-1,2), (-2,-2), (0,-4), (-2,0), (2,0)$$

print b, c, d and also plot the data points and the fitting ellipse. (Hint: you may find it helpful to take a range of angles θ and work with corresponding ratios $y/x = \tan\theta$.)
For fun (but not to hand in) you might enjoy fitting ellipses to points input interactively with the mouse. In Matlab you can do it like this:

```
hold off, axis([-3 3 -3 3 3 $]$, axis manual, hold on, grid on
$\mathrm{x}=[] ; \mathrm{y}=[] ;$ button = 1;
disp('input points with mouse, button >= 2 for final point')
while button == 1
    [xx,yy,button] = ginput(1)
    $\mathrm{x}=[\mathrm{x} ; \mathrm{xx}] ; \mathrm{y}=[\mathrm{y} ; \mathrm{yy}] ; \mathrm
end
```

3

Least-squares solution of the Laplace equation. Consider the Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ on the unit disk $x^2 + y^2 \leq 1$, with boundary data $f(x, y)$.

Equivalently, one could think in polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Solve this problem numerically with the boundary data $f(x, y) = \sqrt{(x - 1.1)^2 + y^2}$ by the following method. Discretize the unit circle by $m = 1000$ equally spaced points, and let n be a small integer. Make an $m \times (2n + 1)$ matrix A whose columns correspond to discretizations of the functions (each of which satisfies the Laplace equation)

$$1, r\cos(\theta), r\sin(\theta), r^2\cos(2\theta), r^2\sin(2\theta), \ldots, r^n\cos(n\theta), r^n\sin(n\theta)$$

Now solve a least-squares problem involving A to fit the given function f on the unit circle by a linear combination of these functions, and evaluate the result to estimate the solution value $u(x = 0.8, y = 0.2)$. What numbers do you get for $n = 0, 2, 4, \ldots, 20$? What do you think is the exact solution $u(0.8, 0.2)$ to 4 digits of accuracy?

3.1

The numbers I get are shown by the print out of the notebook. I think that the exact solution to 4 digits of accuracy is 0.4957.

hw2_jupyter

September 29, 2024

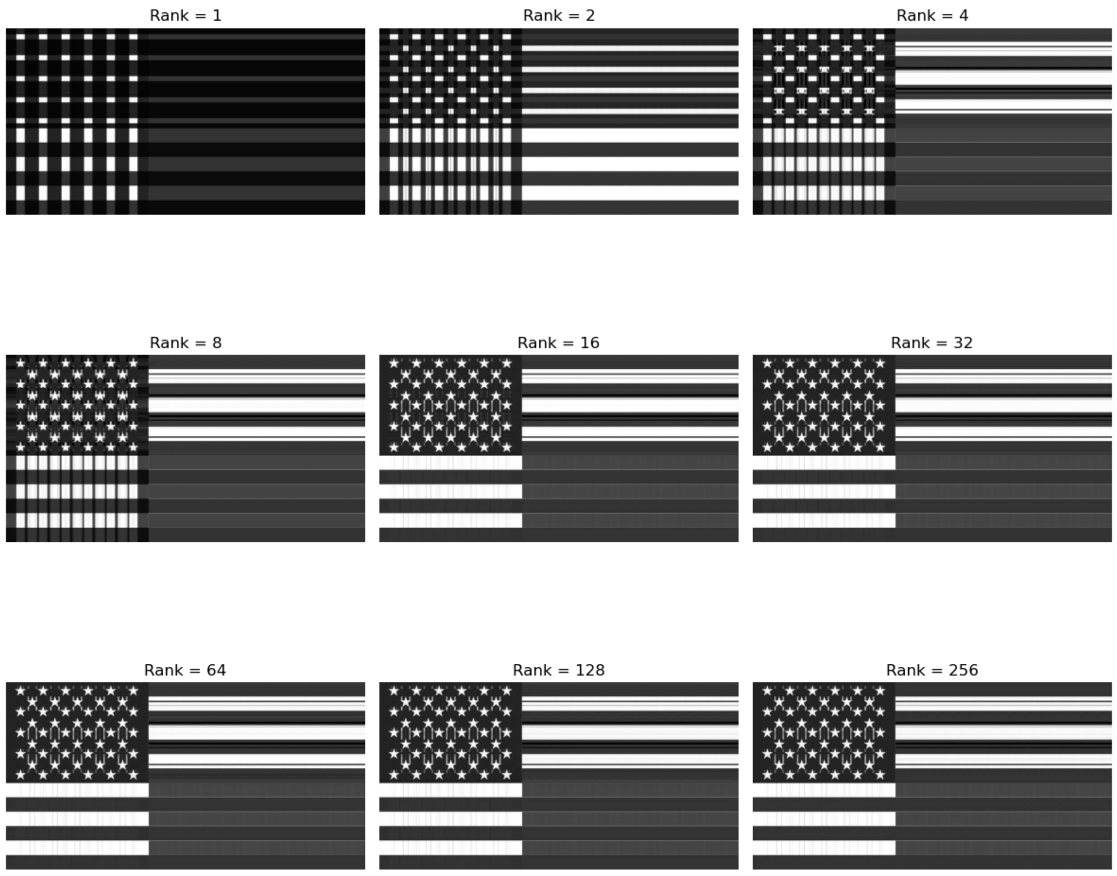
p1

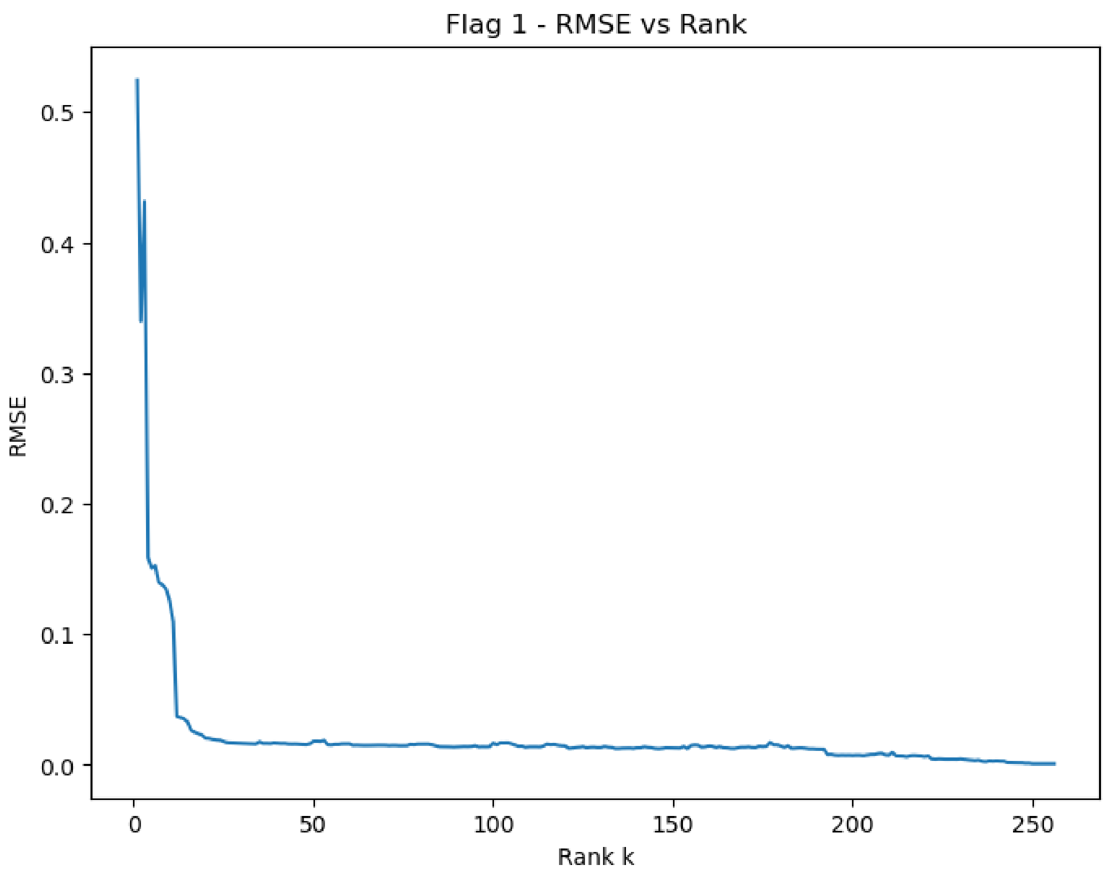
```
[7]: import numpy as np
import matplotlib.pyplot as plt
from skimage import io, color
from numpy.linalg import norm

def low_rank_approximation(image_path, max_rank, ranks_to_plot, title_prefix):
    # Load image and convert to grayscale
    A = io.imread(image_path)
    A = color.rgb2gray(A)
    # Convert image to float and normalize to [0, 1] if necessary
    if A.dtype == np.uint8:
        A = A.astype(float) / 255.0
    else:
        A = A.astype(float)

    # Initialize variables
    Aapprox = np.zeros_like(A)
    sq = np.sqrt(A.size)
    errors = []

    # Prepare to store images at specified ranks
    approx_images = {}

    # Copy of A for iterative approximation
    A_error = A.copy()

    for k in range(1, max_rank + 1):
        # Find indices of the maximum absolute error
        max_abs_err = np.max(np.abs(A_error))
        i, j = np.unravel_index(np.argmax(np.abs(A_error), axis=None), A_error.
 ↪shape)
        pivot = A_error[i, j]

        # Check if there's any error left to reduce
        if pivot == 0:
            break  # No more error to reduce
        # Construct the rank-1 matrix to reduce the maximum error
        Ak = np.outer(A_error[:, j], A_error[i, :] / pivot)

        # Update the approximation and residual error
        Aapprox += Ak
        A_error -= Ak

        # Clip Aapprox to maintain pixel value range
        Aapprox = np.clip(Aapprox, 0, 1)
        A_error = np.clip(A_error, -1, 1)  # Ensuring A_error stays within␣
↪reasonable bounds

        # Compute and store the RMSE
        err = norm(A_error, 'fro') / sq
        errors.append(err)

        # Save the approximation at specified ranks
        if k in ranks_to_plot:
            approx_images[k] = Aapprox.copy()
            print(f"At rank {k}, Aapprox min: {Aapprox.min()}, max: {Aapprox.
↪max()}")

    # Plot the images at specified ranks
    fig, axes = plt.subplots(3, 3, figsize=(12, 12))
    for idx, k in enumerate(ranks_to_plot):
        ax = axes.flatten()[idx]
        ax.imshow(approx_images[k], cmap='gray', vmin=0, vmax=1)
        ax.set_title(f'Rank = {k}')
        ax.axis('off')
    plt.suptitle(f'{title_prefix} - Low-Rank Approximations', fontsize=16)
    plt.tight_layout()
    plt.show()

    # Plot RMSE as a function of k
    plt.figure(figsize=(8, 6))
    plt.plot(range(1, len(errors)+1), errors)
    plt.xlabel('Rank k')
    plt.ylabel('RMSE')
    plt.title(f'{title_prefix} - RMSE vs Rank')
    plt.show()

# Parameters
max_rank = 256
ranks_to_plot = [1, 2, 4, 8, 16, 32, 64, 128, 256]

# Run for flag1.jpg
```

```python
low_rank_approximation('flag1.jpg', max_rank, ranks_to_plot, 'Flag 1')

# Run for flag2.jpg
low_rank_approximation('flag2.jpg', max_rank, ranks_to_plot, 'Flag 2')
```

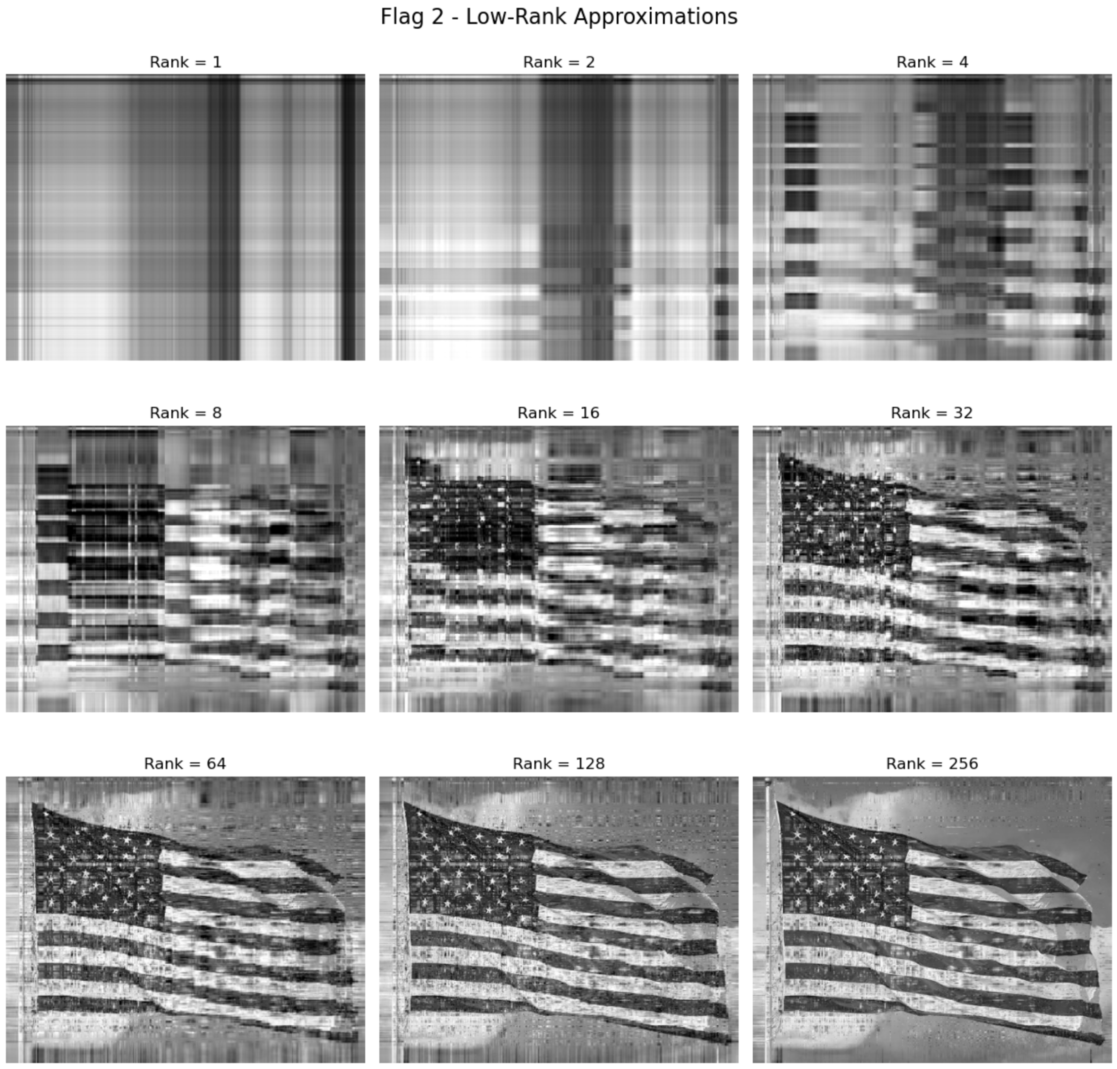
```
At rank 1, Aapprox min: 0.010631669658746637, max: 1.0
At rank 2, Aapprox min: 0.016667855100532585, max: 1.0
At rank 4, Aapprox min: 0.0, max: 1.0
At rank 8, Aapprox min: 0.0, max: 1.0
At rank 16, Aapprox min: 0.0, max: 1.0
At rank 32, Aapprox min: 0.0, max: 1.0
At rank 64, Aapprox min: 0.0, max: 1.0
At rank 128, Aapprox min: 0.0, max: 1.0
At rank 256, Aapprox min: 0.0, max: 1.0
```

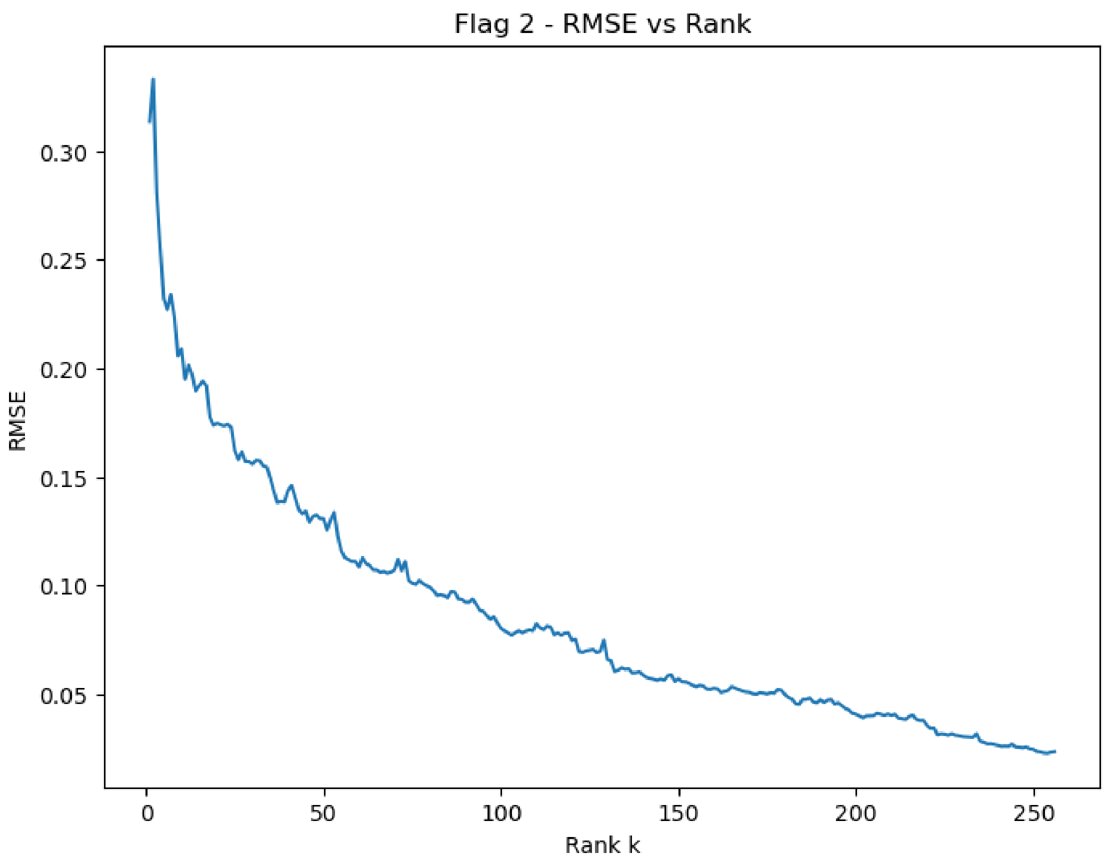
Flag 1 - Low-Rank Approximations





```
At rank 1, Aapprox min: 0.014447865476355248, max: 1.0
At rank 2, Aapprox min: 0.0, max: 1.0
At rank 4, Aapprox min: 0.0, max: 1.0
At rank 8, Aapprox min: 0.0, max: 1.0
At rank 16, Aapprox min: 0.0, max: 1.0
At rank 32, Aapprox min: 0.0, max: 1.0
At rank 64, Aapprox min: 0.0, max: 1.0
At rank 128, Aapprox min: 0.0, max: 1.0
At rank 256, Aapprox min: 0.0, max: 1.0
```


Flag 2 - Low-Rank Approximations
Rank = 1
Rank = 2
Rank = 4
Rank = 8
Rank = 16
Rank = 32
Rank = 64
Rank = 128
Rank = 256



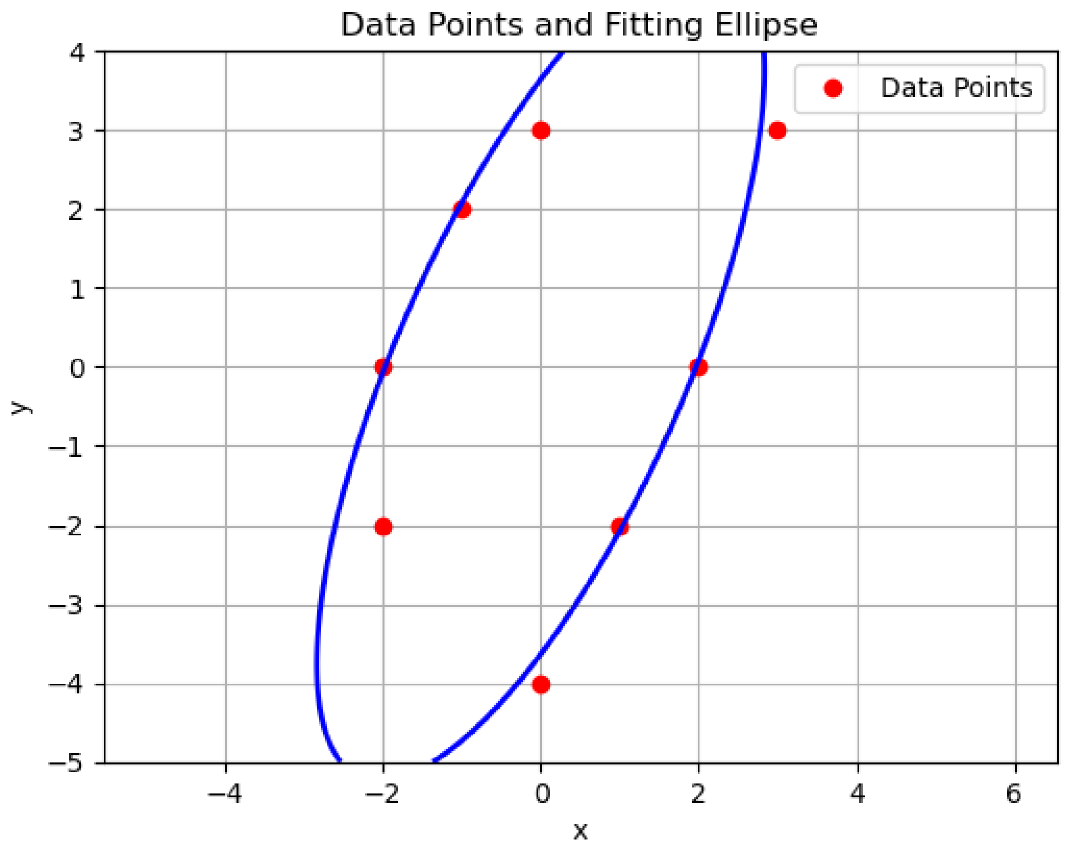
p2

```
[1]: import numpy as np
     import matplotlib.pyplot as plt

     def ellipse(x, y):
         # Number of data points
         m = len(x)

         # Construct matrix A
         A = np.column_stack((x**2, x * y, y**2))

         # Construct right-hand side vector s
         s = np.ones(m)

         # Solve the least-squares problem
         coeffs, residuals, rank, s_vals = np.linalg.lstsq(A, s, rcond=None)

         # Extract coefficients
         b, c, d = coeffs
```

```python
    return b, c, d

# Define the data points
x = np.array([3, 1, 0, -1, -2, 0, -2, 2])
y = np.array([3, -2, 3, 2, -2, -4, 0, 0])

# Call the ellipse function
b, c, d = ellipse(x, y)

# Print the coefficients
print('Coefficients:')
print(f'b = {b:.4f}')
print(f'c = {c:.4f}')
print(f'd = {d:.4f}')

# Plot the data points
plt.figure()
plt.plot(x, y, 'ro', label='Data Points')

# Define the range for x and y
x_range = np.linspace(min(x) - 1, max(x) + 1, 400)
y_range = np.linspace(min(y) - 1, max(y) + 1, 400)
X, Y = np.meshgrid(x_range, y_range)

# Compute the ellipse equation values
Z = b * X**2 + c * X * Y + d * Y**2

# Plot the contour where Z equals 1
plt.contour(X, Y, Z, levels=[1], colors='b', linewidths=2, label='Fitting␣
 ↪Ellipse')

# Add labels and title
plt.xlabel('x')
plt.ylabel('y')
plt.title('Data Points and Fitting Ellipse')
plt.legend()
plt.axis('equal')
plt.grid(True)
plt.show()
```

```
Coefficients:
b = 0.2574
c = -0.2005
d = 0.0756
```

```
/var/folders/q8/8t3lfc9d7t97bsq581_1r9780000gn/T/ipykernel_4486/2860856257.py:47
: UserWarning: The following kwargs were not used by contour: 'label'
  plt.contour(X, Y, Z, levels=[1], colors='b', linewidths=2, label='Fitting
Ellipse')
```



p3

[5]:
```python
import numpy as np

def construct_A(theta, n):
    m = len(theta)
    A = np.ones((m, 1))  # phi_0
    for k in range(1, n + 1):
        cos_k_theta = np.cos(k * theta).reshape(-1, 1)
        sin_k_theta = np.sin(k * theta).reshape(-1, 1)
        A = np.hstack((A, cos_k_theta, sin_k_theta))
    return A

def compute_f(x, y):
    f = np.sqrt((x - 1.1)**2 + y**2)
    return f

def solve_least_squares(A, f):
    c, residuals, rank, s = np.linalg.lstsq(A, f, rcond=None)
    return c

def evaluate_u(c, x, y):
    r = np.sqrt(x**2 + y**2)
    theta_point = np.arctan2(y, x)
    u = c[0]  # Start with c0 * phi0

    n_terms = (len(c) - 1) // 2
    for k in range(1, n_terms + 1):
        idx_cos = 2 * k - 1
        idx_sin = 2 * k
        u += c[idx_cos] * (r**k) * np.cos(k * theta_point)
        u += c[idx_sin] * (r**k) * np.sin(k * theta_point)
    return u

# Discretize the boundary
m = 1000
theta = np.linspace(0, 2 * np.pi, m, endpoint=False)
x_boundary = np.cos(theta)
y_boundary = np.sin(theta)

# Compute boundary data
f_boundary = compute_f(x_boundary, y_boundary)

# Points to evaluate
x_point = 0.8
y_point = 0.2

# Loop over n values
n_values = range(0, 21, 2)
u_values = []

print("n\t u(0.8, 0.2)")
for n in n_values:
    A = construct_A(theta, n)
    c = solve_least_squares(A, f_boundary)
    u = evaluate_u(c, x_point, y_point)
    u_values.append(u)
    print(f"{n}\t {u:.6f}")
```

```
n        u(0.8, 0.2)
0        1.342864
2        0.535095
4        0.496083
6        0.492680
8        0.493664
```

```
10        0.494705
12        0.495298
14        0.495565
16        0.495663
18        0.495686
20        0.495685
```

[ ]:

```
```