Non-Conjugate Models and Grid Approximations

Patrick Lam

Outline

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> triangle.prior <- function(x) {
+     if (x >= 0 && x < 0.25)
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There are two ways to get the normalized posterior in R.

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> unnormal.post.ord <- posterior.function(theta = grid.points,
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> post.ord <- unnormal.post.ord/normal.constant
```

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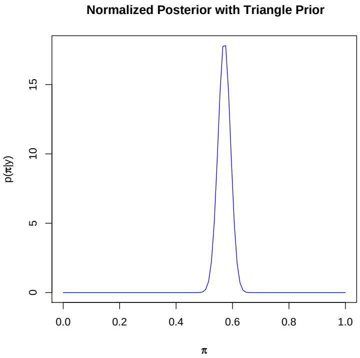
```
> plot(x = grid.points, y = post.ord, type = "l", col = "blue",
+     xlab = expression(pi), ylab = expression(paste("p(", pi,
+         "|y)")), main = "Normalized Posterior with Triangle Prior")
```



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Note that the normalized posterior ordinates are not probabilities, but rather the heights of the density. However, the `sample()` function in R will normalize the ordinates into probabilities.

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3. Sample grid points with unnormalized posterior ordinates as the probabilities (letting R normalize them for us).

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3. Sample grid points with unnormalized posterior ordinates as the probabilities (letting R normalize them for us).

```
> set.seed(12345)
> posterior.triangle.2 <- sample(grid.points, size = 10000, replace = T,
+     prob = unnormal.post.ord)
> all.equal(posterior.triangle.1, posterior.triangle.2)

[1] TRUE
```

Outline

The Binomial Model with a Non-Conjugate Prior

Bayesian Regression with Grid Approximations

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Let's use a couple of Normal priors on α and β.

$$\text{Posterior}(\alpha, \beta) \quad \propto \quad \prod_{i=1}^{n} \text{Poisson}(\lambda_i) \times \text{Normal}(\mu_\alpha, \sigma_\alpha^2) \times \text{Normal}(\mu_\beta, \sigma_\beta^2)$$

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p(\alpha, \beta | \mathbf{y}, \mathbf{x}) &\propto \prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!} \times \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp\left(-\frac{(\alpha - \mu_\alpha)^2}{2\sigma_\alpha^2}\right) \\
&\quad \times \frac{1}{\sqrt{2\pi\sigma_\beta^2}} \exp\left(-\frac{(\beta - \mu_\beta)^2}{2\sigma_\beta^2}\right)
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$$\lambda_i = \exp(\alpha + \beta x_i)$$

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Probit:

- Link: $\Phi^{-1}(\pi_i) = \alpha + \beta x_i$
- Inverse Link: $\pi_i = \Phi(\alpha + \beta x_i)$

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So our posterior is

$$p(\alpha, \beta | \mathbf{y}, \mathbf{x}) \propto \prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!} \times \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp\left(-\frac{(\alpha - \mu_\alpha)^2}{2\sigma_\alpha^2}\right) \times \frac{1}{\sqrt{2\pi\sigma_\beta^2}} \exp\left(-\frac{(\beta - \mu_\beta)^2}{2\sigma_\beta^2}\right)$$

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Using our inverse link function, we get

$$p(\alpha, \beta | \mathbf{y}, \mathbf{x}) \propto \prod_{i=1}^{n} \frac{e^{-\exp(\alpha + \beta x_i)} (\exp(\alpha + \beta x_i))^{y_i}}{y_i!} \times \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp\left(-\frac{(\alpha - \mu_\alpha)^2}{2\sigma_\alpha^2}\right) \times \frac{1}{\sqrt{2\pi\sigma_\beta^2}} \exp\left(-\frac{(\beta - \mu_\beta)^2}{2\sigma_\beta^2}\right)$$

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We can combine and simplify a bunch of terms ... or we can use some canned R functions.

$$\ln \text{Posterior}(\alpha, \beta) \quad \propto \quad \sum_{i=1}^{n} \ln \text{Poisson}(\exp(\alpha + \beta x_i)) + \ln \text{Normal}(\mu_\alpha, \sigma_\alpha^2)$$
$$+ \ln \text{Normal}(\mu_\beta, \sigma_\beta^2)$$

$$\ln \text{Posterior}(\alpha, \beta) \propto \sum_{i=1}^{n} \ln \text{Poisson}(\exp(\alpha + \beta x_i)) + \ln \text{Normal}(\mu_\alpha, \sigma_\alpha^2) + \ln \text{Normal}(\mu_\beta, \sigma_\beta^2)$$

```
> poisson.posterior <- function(theta, y, x, prior.mean.a, prior.var.a,
+     prior.mean.b, prior.var.b) {
+     a <- theta[1]
+     b <- theta[2]
+     lambda <- exp(a + b * x)
+     log.like <- sum(dpois(y, lambda = lambda, log = T))
+     log.prior.a <- dnorm(a, mean = prior.mean.a, sd = sqrt(prior.var.a),
+         log = T)
+     log.prior.b <- dnorm(b, mean = prior.mean.b, sd = sqrt(prior.var.b),
+         log = T)
+     log.post <- log.like + log.prior.a + log.prior.b
+     return(exp(log.post))
+ }
```

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> y.vec <- sanction$num
> x.vec <- sanction$coop
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> data(sanction)
```

Our dependent variable is the number of sanctions, and our covariate is level of cooperation.

```
> y.vec <- sanction$num
> x.vec <- sanction$coop
```

For the priors on α and β, let's use two Normal(0,20) distributions, which are relatively flat.

An Example

Let's do an example using the `sanction` dataset from the `Zelig` library.

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For the priors on α and β, let's use two Normal(0,20) distributions, which are relatively flat.

```
> mu.a <- mu.b <- 0
> sigma2.a <- sigma2.b <- 20
```

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One reasonable solution is to center our grid at the MLE.

```
> mle <- glm(num ~ coop, data = sanction, family = poisson)$coef
> mle.se <- summary(glm(num ~ coop, data = sanction,
+     family = poisson))$coef[, 2]
```

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```
> grid.a <- seq(from = mle[1] - 5 * mle.se[1], to = mle[1] + 5 *
+     mle.se[1], length.out = 200)
> grid.b <- seq(from = mle[2] - 5 * mle.se[2], to = mle[2] + 5 *
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> post.ord <- apply(grid.points, MARGIN = 1, FUN = poisson.posterior,
+     y = y.vec, x = x.vec, prior.mean.a = mu.a, prior.var.a = sigma2.a,
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```
> sample.indices <- sample(1:nrow(grid.points), size = 10000, replace = T,
+     prob = post.ord)
> sim.posterior <- grid.points[sample.indices, ]
```

Posterior Summary

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```
> posterior.sd <- apply(sim.posterior, MARGIN = 2, FUN = sd)
> posterior.sd

  Var1   Var2
0.1466 0.0453

> mle.se

(Intercept)        coop
    0.14602     0.04504
```