Delta Method and Generalized Linear Models

Patrick Lam

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What if g(X) isn't linear?

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We will use the **delta method**, which relies on a linear approximation to $g(X)$ near the mean of X.

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How good the approximations are depend on how nonlinear $g(X)$ is in the neighborhood of μ_X and on the size of Var(X).

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Steps to running a GLM:

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- Ordered Categories:

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- Unordered Categories: Multinomial

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$$X\beta = \beta_0 + x_1\beta_1 + x_2\beta_2 + \cdots + x_k\beta_k$$

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This is the systematic component that we've been talking about all along.

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Do it.

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2. Run $X\beta$ through inverse link function to get expected values.
3. Draw from distribution of Y for predicted values.

Binary Dependent Variable

Let our dependent variable be a binary random variable that can take on values of either 0 or 1.

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5. Simulate Quantities of Interest

Identification

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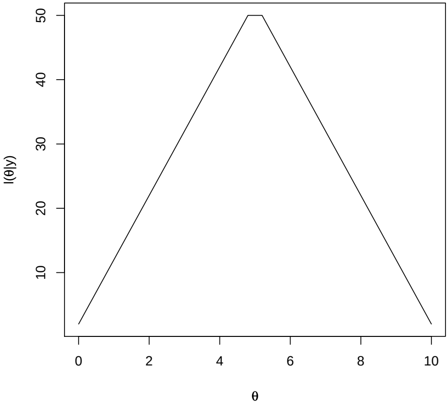
Suppose we have the following log-likelihood function:



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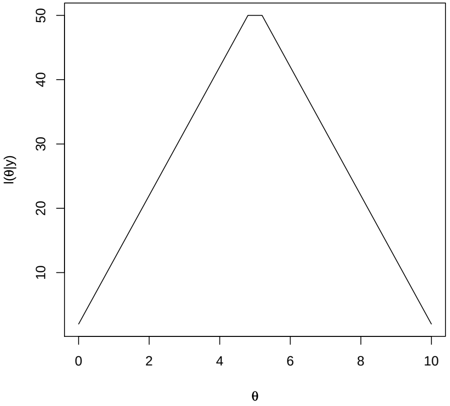

What's wrong with this?

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  - Fix $\tau_1 = 0$ and estimate an intercept.