Summarizing the Posterior

Patrick Lam

Outline

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We can sometimes do the integrals analytically, but more often than not we want to simulate from the marginal distribution.

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θ_1	θ_2	$p(\theta_1, \theta_2)$
0	0	0.4
0	0.5	0.7
0	1	0.5
0.5	0	0.6
0.5	0.5	0.8
0.5	1	0.8
1	0	0.7
1	0.5	0.6
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$$\begin{aligned} p(\theta_1 = 0) &= 0.4 + 0.7 + 0.5 = 1.6 \\ p(\theta_1 = 0.5) &= 0.6 + 0.8 + 0.8 = 2.2 \\ p(\theta_1 = 1) &= 0.7 + 0.6 + 0.5 = 1.8 \end{aligned}$$

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We can sample from the marginal grid to get simulations from the marginal distribution $p(\theta_1)$.

Integrating Simulations

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Now suppose that we have a bunch of simulated draws from the joint distribution (obtained either by sampling from the joint grid or other simulation methods).

Table: Joint Simulations

Draw #	θ_1	θ_2
1	0.5	1
2	1	0
3	1	0
4	0	1
5	0	0.5
6	0.5	0.5
7	1	0.5
8	0.5	0
9	0.5	1
10	0.5	0.5

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To obtain simulations from the marginal distribution, we simply drop the simulated values from the other parameter(s).

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Consider the previous example of the beta-binomial model with a Beta(1,1) prior (uniform) and a Beta$(y + \alpha, n - y + \beta)$ posterior, where $n = 500$, $y = 285$, $\alpha = 1$, and $\beta = 1$.

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Simulation:

```
> a <- 1
> b <- 1
> posterior.unif.prior <- rbeta(10000, shape1 = 285 + a, shape2 = 500 -
+     285 + b)
> mean(posterior.unif.prior)

[1] 0.5699
```

Posterior Variance

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$$\frac{(y+\alpha)(n-y+\beta)}{[(y+\alpha)+(n-y+\beta)]^2[(y+\alpha)+(n-y+\beta)+1]} \approx 0.00049$$

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```
> var(posterior.unif.prior)
[1] 0.0004832
```

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Analytically:

$$\frac{(y+\alpha)(n-y+\beta)}{[(y+\alpha)+(n-y+\beta)]^2[(y+\alpha)+(n-y+\beta)+1]} \approx 0.00049$$

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```
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[1] 0.0004832
```

Take the square root for standard deviation.

Posterior Probability $0.5 < \pi < 0.6$

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Analytically:

$$\int_{0.5}^{0.6} \frac{\Gamma((y+\alpha)+(n-y+\beta))}{\Gamma(y+\alpha)\Gamma(n-y+\beta)} \pi^{(y+\alpha-1)}(1-\pi)^{(n-y+\beta-1)} d\pi \approx 0.91$$

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Simulation:

```
> mean(posterior.unif.prior > 0.5 & posterior.unif.prior < 0.6)
[1] 0.9114
```

Central 95% Credible Interval

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Simulation:

```
> quantile(posterior.unif.prior, probs = c(0.025, 0.975))

  2.5%  97.5%
0.5269 0.6125
```

95% Highest Posterior Density Region

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Find the smallest interval(s) that contains 95% of the area of the posterior.

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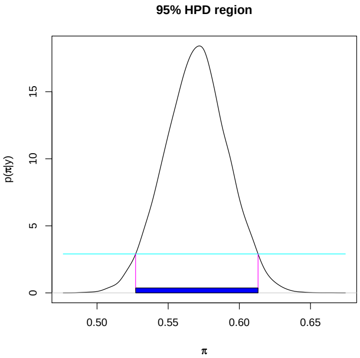
Simulation:

```
> library(hdrcde)
> hdr(posterior.unif.prior, prob = 95)$hdr

      [,1]   [,2]
95% 0.5271 0.6132
```

```
> hdr.den(posterior.unif.prior, prob = 95, main = "95% HPD region",
+     xlab = expression(pi), ylab = expression(paste("p(", pi,
+         "|y)")))
```



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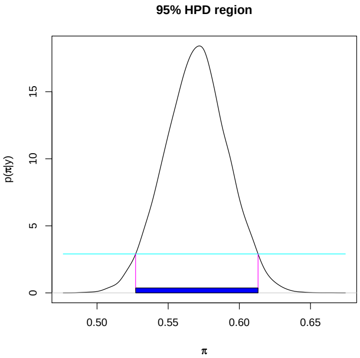

Very similar to central 95% credible interval for Beta posterior.

What about for this posterior?