Probability Theory

Patrick Lam

Outline

Most Basic Definition of Probability:

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$$\frac{\text{number of successes}}{\text{number of possible occurrences}}$$

Three Axioms of Probability

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- For any event A, $0 \leq P(A) \leq 1$.
- If $A \subset B$, then $P(A) \leq P(B)$.
- For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

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Law of Total Probability:

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

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Conditional Independence:

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Outline

Random Variables

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A **random variable** is a function that takes a random experiment and assigns a number to the outcome of the experiment.

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Outcome values are assigned probabilities by a probability mass function (for discrete RV) or probability density function (for continuous RV).

Probability Mass Function

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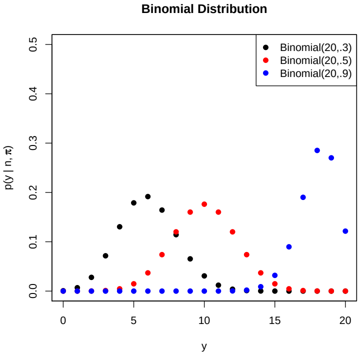

Probability Density Function

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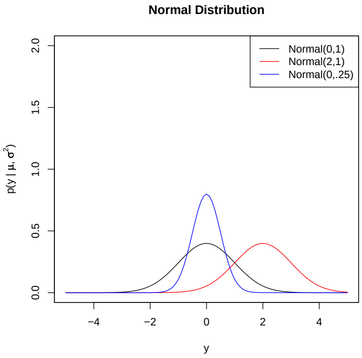

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- The support is all y's where $P(Y = y) > 0$.

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where $f(x)$ is the probability density function (PDF).

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Fancy way of saying we will simulate random draws to calculate quantities of interest.

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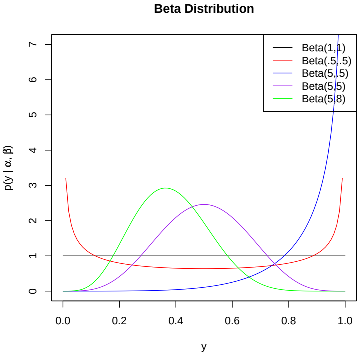

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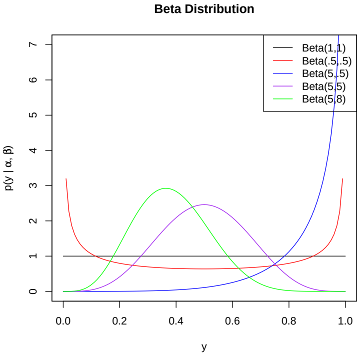

$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{(\alpha-1)}(1-x)^{(\beta-1)}$$

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```
> ex.beta.func <- function(x, alpha, beta) {
+     x * gamma(alpha + beta)/(gamma(alpha) * gamma(beta)) * x^(alpha -
+         1) * (1 - x)^(beta - 1)
+ }
> e.x <- integrate(Vectorize(ex.beta.func), lower = 0, upper = 1,
+     alpha = 2, beta = 3)$value
> e.x

[1] 0.4
```

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> x.draws <- rbeta(10000, shape1 = 2, shape2 = 3)
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Why does this work?

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By the Strong Law of Large Numbers, our estimate $\hat{I}_M$ is a simulation consistent estimator of I as $M \to \infty$ (our estimate gets better as we increase the number of simulations).

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Monte Carlo Integration tells us we need $E(g(X))$.

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This seems trivial but is one of the foundations of statistics, especially Bayesian statistics.

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General idea:

1. Suppose we have an experiment where we want to know the probability of success. Simulate from the population many times.
2. For each simulation, conduct the experiment and see whether there is success.
3. The proportion of simulations that achieve success is the probability of success.

An Example

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Suppose we have two urns containing marbles. The first urn contains 6 red marbles and 4 green marbles and the second urn contains 9 red marbles and 1 green marble. Take one marble from the first urn (without looking at it) and put it in the second urn. Then take one marble from the second urn (again without looking at it) and put it in the first urn. What is the probability of now drawing a red marble from the first urn?

An Example

Suppose we have two urns containing marbles. The first urn contains 6 red marbles and 4 green marbles and the second urn contains 9 red marbles and 1 green marble. Take one marble from the first urn (without looking at it) and put it in the second urn. Then take one marble from the second urn (again without looking at it) and put it in the first urn. What is the probability of now drawing a red marble from the first urn?

```
> urn.func <- function(n.sims, urn1, urn2) {
+     final.draws <- c()
+     for (i in 1:n.sims) {
+         draw1 <- sample(urn1, 1)
+         draw2 <- sample(c(urn2, draw1), 1)
+         final.draws[i] <- sample(c(urn1, draw2), 1)
+     }
+     prob <- mean(final.draws)
+     return(prob)
+ }
> urn.func(n.sims = 10000, urn1 = c(rep(1, 6), rep(0, 4)), urn2 = c(rep(1,
+     9), 0))

[1] 0.6293
```

An Example

Suppose we have two urns containing marbles. The first urn contains 6 red marbles and 4 green marbles and the second urn contains 9 red marbles and 1 green marble. Take one marble from the first urn (without looking at it) and put it in the second urn. Then take one marble from the second urn (again without looking at it) and put it in the first urn. What is the probability of now drawing a red marble from the first urn?

```
> urn.func <- function(n.sims, urn1, urn2) {
+     final.draws <- c()
+     for (i in 1:n.sims) {
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+         draw2 <- sample(c(urn2, draw1), 1)
+         final.draws[i] <- sample(c(urn1, draw2), 1)
+     }
+     prob <- mean(final.draws)
+     return(prob)
+ }
> urn.func(n.sims = 10000, urn1 = c(rep(1, 6), rep(0, 4)), urn2 = c(rep(1,
+     9), 0))

[1] 0.6293
```

$$\left(\frac{6}{10}\right)\left(\frac{10}{11}\right)\left(\frac{6}{10}\right)+\left(\frac{6}{10}\right)\left(\frac{1}{11}\right)\left(\frac{5}{10}\right)+\left(\frac{4}{10}\right)\left(\frac{9}{11}\right)\left(\frac{7}{10}\right)+\left(\frac{4}{10}\right)\left(\frac{2}{11}\right)\left(\frac{6}{10}\right)\approx 0.63$$

Another Example

Another Example

Suppose we have two urns containing marbles. The first urn contains $10 - g$ red marbles and g green marbles and the second urn contains 9 red marbles and 1 green marble. Take one marble from the first urn (without looking at it) and put it in the second urn. Then take one marble from the second urn (again without looking at it) and put it in the first urn. What is the minimum g such that the probability of now drawing a red marble is less than 0.5?

Another Example

Suppose we have two urns containing marbles. The first urn contains $10 - g$ red marbles and g green marbles and the second urn contains 9 red marbles and 1 green marble. Take one marble from the first urn (without looking at it) and put it in the second urn. Then take one marble from the second urn (again without looking at it) and put it in the first urn. What is the minimum g such that the probability of now drawing a red marble is less than 0.5?

```
> urn.func2 <- function(n.sims, urn2, p) {
+     final.draws <- c()
+     g <- 0
+     urn1 <- c(rep(1, 10 - g), rep(0, g))
+     prob <- 1
+     while (prob >= p) {
+         for (i in 1:n.sims) {
+             draw1 <- sample(urn1, 1)
+             draw2 <- sample(c(urn2, draw1), 1)
+             final.draws[i] <- sample(c(urn1, draw2), 1)
+         }
+         prob <- mean(final.draws)
+         g <- g + 1
+         urn1 <- c(rep(1, 10 - g), rep(0, g))
+     }
+     g <- g - 1
+     return(g)
+ }
> urn.func2(1000, urn2 = c(rep(1, 9), 0), p = 0.5)

[1] 6
```

Outline

Outline

The Bernoulli Distribution

The Bernoulli Distribution

$Y \sim \text{Bernoulli}(\pi)$

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$y = 0, 1$

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probability of success: $\pi \in [0, 1]$

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$p(y|\pi) = \pi^y (1 - \pi)^{(1-y)}$

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$p(y|\pi) = \pi^{y}(1 - \pi)^{(1-y)}$

$E(Y) = \pi$

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$E(Y) = \pi$

$\text{Var}(Y) = \pi(1-\pi)$

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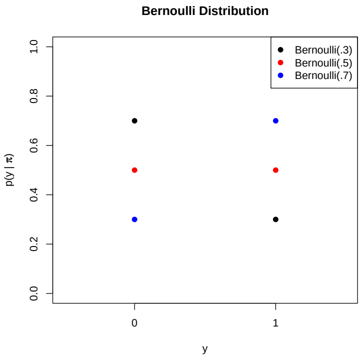

The Binomial Distribution

The Binomial Distribution

$Y \sim \text{Binomial}(n, \pi)$

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$y = 0, 1, \ldots, n$

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$y = 0, 1, \ldots, n$

number of trials: $n \in \{1, 2, \ldots\}$
probability of success: $\pi \in [0, 1]$

$p(y|\pi) = \binom{n}{y} \pi^y (1 - \pi)^{(n-y)}$

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$p(y|\pi) = \binom{n}{y} \pi^y (1 - \pi)^{(n-y)}$

$E(Y) = n\pi$

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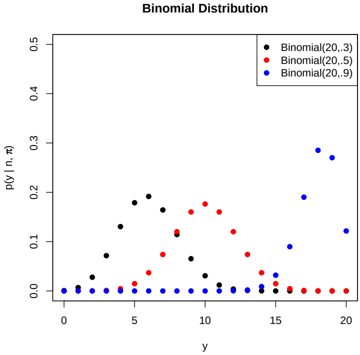

The Multinomial Distribution

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$Y \sim \text{Multinomial}(n, \pi_1, \ldots, \pi_k)$

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number of trials: $n \in \{1, 2, \ldots\}$
probability of success for j: $\pi_j \in [0, 1];\;\; \sum_{j=1}^{k} \pi_j = 1$

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$p(\mathbf{y}|n, \boldsymbol{\pi}) = \frac{n!}{y_1! y_2! \ldots y_k!} \pi_1^{y_1} \pi_2^{y_2} \ldots \pi_k^{y_k}$

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$E(Y_j) = n\pi_j$

$\text{Var}(Y_j) = n\pi_j(1 - \pi_j)$

$\text{Cov}(Y_i, Y_j) = -n\pi_i\pi_j$

The Poisson Distribution

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$Y \sim \text{Poisson}(\lambda)$

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$y = 0, 1, \ldots$

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expected number of occurrences: $\lambda > 0$

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$E(Y) = \lambda$

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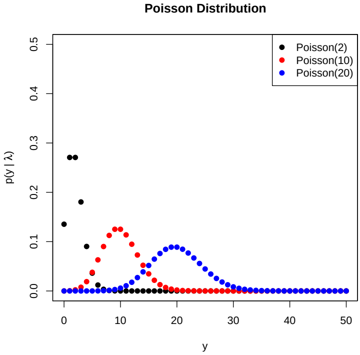

The Geometric Distribution

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How many Bernoulli trials until success?

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$E(Y) = \frac{1}{\pi}$

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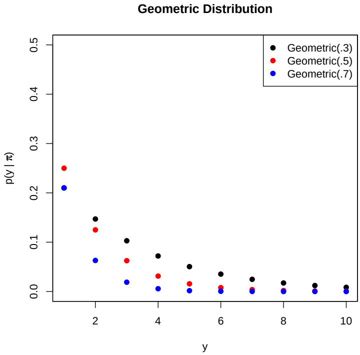

Outline

The Univariate Normal Distribution

The Univariate Normal Distribution

$$Y \sim \text{Normal}(\mu, \sigma^2)$$

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$y \in \mathbb{R}$

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mean: $\mu \in \mathbb{R}$

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$y \in \mathbb{R}$

mean: $\mu \in \mathbb{R}$
variance: $\sigma^2 > 0$

The Univariate Normal Distribution

$Y \sim \text{Normal}(\mu, \sigma^2)$

$y \in \mathbb{R}$

mean: $\mu \in \mathbb{R}$
variance: $\sigma^2 > 0$

$$p(y|\mu, \sigma^2) = \frac{\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}$$

The Univariate Normal Distribution

$Y \sim \text{Normal}(\mu, \sigma^2)$

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mean: $\mu \in \mathbb{R}$
variance: $\sigma^2 > 0$

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$E(Y) = \mu$

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$Y \sim \text{Normal}(\mu, \sigma^2)$

$y \in \mathbb{R}$

mean: $\mu \in \mathbb{R}$
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$E(Y) = \mu$

$\text{Var}(Y) = \sigma^2$

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mean: $\mu \in \mathbb{R}$
variance: $\sigma^2 > 0$

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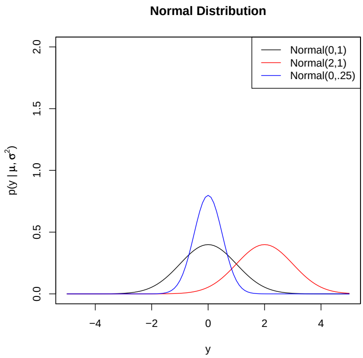

The Multivariate Normal Distribution

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$$Y \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

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mean vector: $\boldsymbol{\mu} \in \mathbb{R}^k$

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$\mathbf{y} \in \mathbb{R}^k$

mean vector: $\boldsymbol{\mu} \in \mathbb{R}^k$
variance-covariance matrix: $\boldsymbol{\Sigma}$ positive definite $k \times k$ matrix

The Multivariate Normal Distribution

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$\mathbf{y} \in \mathbb{R}^k$

mean vector: $\boldsymbol{\mu} \in \mathbb{R}^k$
variance-covariance matrix: $\boldsymbol{\Sigma}$ positive definite $k \times k$ matrix

$$p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\pi}) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{\mathbf{-1}}(\mathbf{y} - \boldsymbol{\mu})\right)$$

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$E(Y) = \boldsymbol{\mu}$

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$E(Y) = \boldsymbol{\mu}$

$\mathrm{Var}(Y) = \boldsymbol{\Sigma}$

The Uniform Distribution

The Uniform Distribution

$Y \sim \text{Uniform}(\alpha, \beta)$

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$y \in [\alpha, \beta]$

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$y \in [\alpha, \beta]$

Interval: $[\alpha, \beta]$; $\beta > \alpha$

The Uniform Distribution

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Interval: $[\alpha, \beta]$; $\beta > \alpha$

$p(y|\alpha, \beta) = \frac{1}{\beta - \alpha}$

The Uniform Distribution

$Y \sim \text{Uniform}(\alpha, \beta)$

$y \in [\alpha, \beta]$

Interval: $[\alpha, \beta]$; $\beta > \alpha$

$p(y|\alpha, \beta) = \frac{1}{\beta - \alpha}$

$E(Y) = \frac{\alpha + \beta}{2}$

The Uniform Distribution

$Y \sim \mathrm{Uniform}(\alpha, \beta)$

$y \in [\alpha, \beta]$

Interval: $[\alpha, \beta]$; $\beta > \alpha$

$p(y|\alpha, \beta) = \frac{1}{\beta - \alpha}$

$E(Y) = \frac{\alpha+\beta}{2}$

$\mathrm{Var}(Y) = \frac{(\beta-\alpha)^2}{12}$

The Beta Distribution

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$Y \sim \text{Beta}(\alpha, \beta)$

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$y \in [0, 1]$

The Beta Distribution

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$y \in [0, 1]$

shape parameters:
$\alpha > 0; \quad \beta > 0$

The Beta Distribution

$Y \sim \text{Beta}(\alpha, \beta)$

$y \in [0, 1]$

shape parameters:
$\alpha > 0; \;\; \beta > 0$

$p(y|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{(\alpha-1)}(1-y)^{(\beta-1)}$

The Beta Distribution

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$y \in [0, 1]$

shape parameters:
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The Beta Distribution

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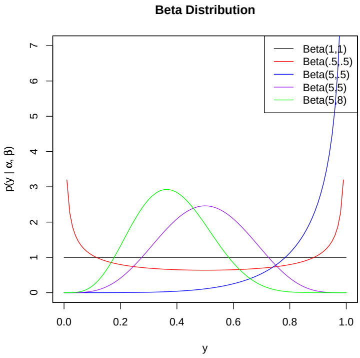
$$E(Y) = \frac{\alpha}{\alpha+\beta}$$

$$\text{Var}(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2)\alpha+\beta+1)}$$



The Gamma Distribution

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The Gamma Distribution

$Y \sim \mathsf{Gamma}(\alpha, \beta)$

$y > 0$

The Gamma Distribution

$Y \sim \text{Gamma}(\alpha, \beta)$

$y > 0$

shape parameter: $\alpha > 0$

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$Y \sim \text{Gamma}(\alpha, \beta)$

$y > 0$

shape parameter: $\alpha > 0$
inverse scale parameter: $\beta > 0$

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$y > 0$

shape parameter: $\alpha > 0$
inverse scale parameter: $\beta > 0$

$p(y|\alpha, \beta) =$
$\frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{(\alpha-1)} \exp\left(-\beta y\right)$

The Gamma Distribution

$Y \sim \text{Gamma}(\alpha, \beta)$

$y > 0$

shape parameter: $\alpha > 0$
inverse scale parameter: $\beta > 0$

$p(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{(\alpha-1)} \exp(-\beta y)$

$E(Y) = \frac{\alpha}{\beta}$

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$y > 0$

shape parameter: $\alpha > 0$
inverse scale parameter: $\beta > 0$

$p(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{(\alpha-1)} \exp\left(-\beta y\right)$

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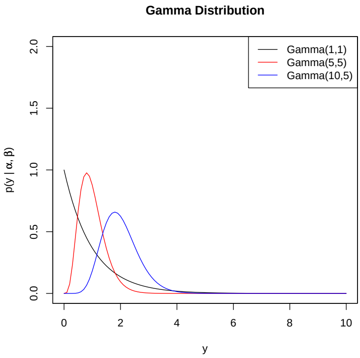

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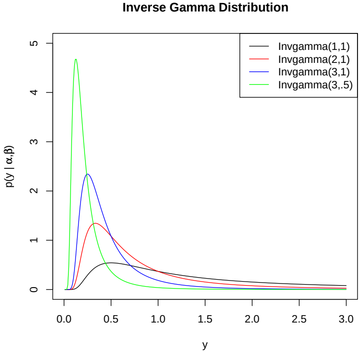

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