Bayesian Statistics in One Hour

Patrick Lam

Outline

Outline

References

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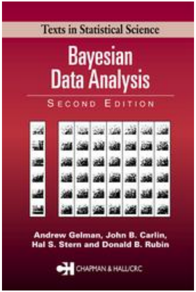
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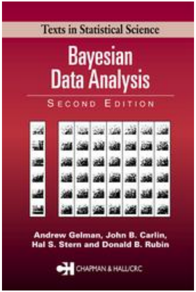
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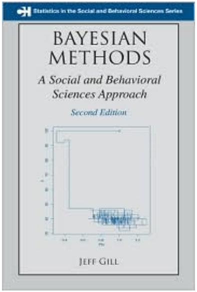
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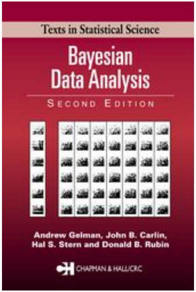
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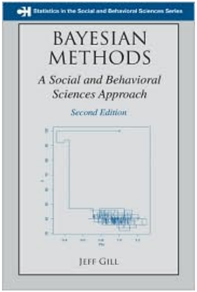
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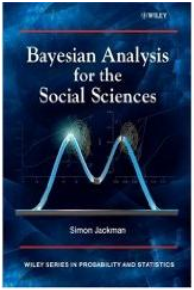
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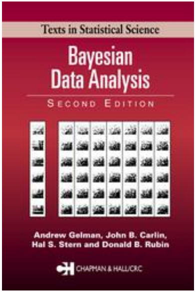

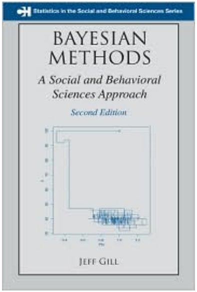

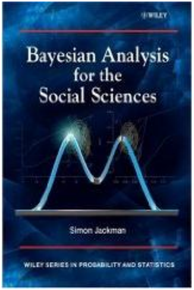

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Introduction

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Three general approaches to statistics:

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- frequentist (Neyman-Pearson, hypothesis testing)

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We'll go over some of the <u>Bayesian critiques of non-Bayesian analysis</u>

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We'll go over some of the Bayesian critiques of non-Bayesian analysis and *non-Bayesian critiques of Bayesian analysis*.

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<u>Bayesian statistics is convenient because it does not require repeated sampling or large n assumptions.</u>

Maximum Likelihood

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\begin{aligned}
p(\theta|y) &= \frac{p(y|\theta)p(\theta)}{p(y)} \\
&= p(y|\theta)k(y)
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There is a fixed, true value of θ, and we maximize the likelihood to estimate θ and make assumptions to generate uncertainty about our estimate of θ.

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- The posterior is a probability distribution that must integrate to 1.
- The prior is usually a probability distribution that integrates to 1 (proper prior).

θ as Fixed versus as a Random Variable

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NB: *Bayesian is too hard. Why use it?*
B: Bayesian methods allow us to easily estimate models that are too hard to estimate (cannot computationally find the MLE) or unidentified (no unique MLE exists) with non-Bayesian methods. Bayesian methods also allow us to incorporate prior/qualitative information into the model.

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<u>There is a Bayesian way to do any non-Bayesian parametric model.</u>

A Simple (Beta-Binomial) Model

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$$Y \sim \mathrm{Binomial}(n, \pi)$$

with $n = 82$.

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$$\begin{aligned} p(\pi|y) &\propto p(y|\pi)p(\pi) \\ &= \text{Binomial}(n, \pi) \times \text{Beta}(\alpha, \beta) \end{aligned}$$

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&= \text{Binomial}(n,\pi) \times \text{Beta}(\alpha,\beta) \\
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Bayesian priors are just adding pseudo observations to the data.

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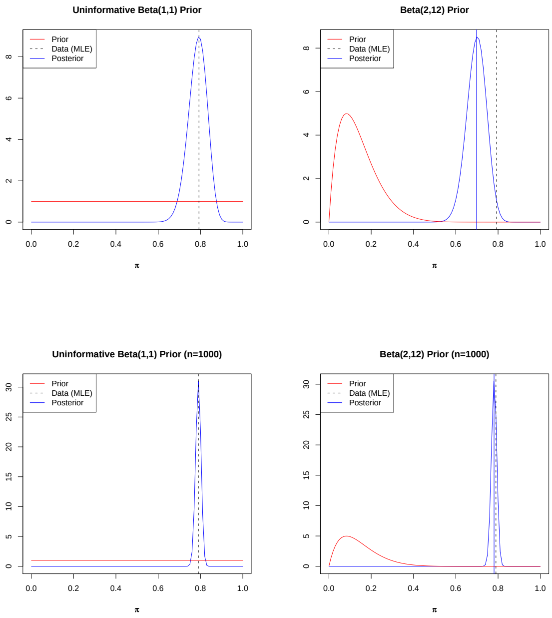
Uninformative Beta(1,1) Prior
Beta(2,12) Prior
Uninformative Beta(1,1) Prior (n=1000)
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Prior
Data (MLE)
Posterior
0.0 0.2 0.4 0.6 0.8 1.0
π
0 2 4 6 8
0 5 10 15 20 25 30

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Conjugate models are great because we can find the exact posterior, but we almost never have conjugacy except in very simple models.

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Likelihood $\times$ prior doesn't look like any distribution we know (non-conjugacy) and normalizing constant is too hard to find, so how do we find our posterior?

MCMC to the Rescue

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But there is no way to know for sure whether our chain has converged.

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M-H always works, but can be very slow.

Outline

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One non-Bayesian approach to dealing with missing data is **multiple imputation**.

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$$L(\phi|D_{obs}) = \prod_{i=1}^{n} \int p(D_{i,obs}|\phi) dD_{mis}$$

How do we do this integral?

Assume Normality since the marginals of a multivariate Normal are Normal.

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4. Draw m μ, Σ values, then use them to predict values for D_{mis}.

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We then run our regular analyses on the m datasets and combine the results using Rubin's rule.

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Use Gibbs Sampling or M-H to sample both D_{mis} and ϕ.

We don't have to assume normality of D to integrate over D_{mis}. We can just drop the draws of D_{mis}.

We can also incorporate both imputation and analyses in the same model.

$$p(\boldsymbol{\theta}, D_{mis}, \phi, \gamma | D_{obs}, M) \propto p(\mathbf{y}_{obs}, \mathbf{y}_{mis} | \boldsymbol{\theta}) p(D_{obs} | D_{mis}, \phi) p(D_{mis} | \phi) p(\phi) p(\boldsymbol{\theta})$$

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Moral: We can easily set up an application specific Bayesian model to incorporate missing data.

Multilevel Data

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We can have covariates on multiple levels. How do we deal with this type of data?

The Fixed Effects Model

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However, the fixed effects model involves estimating many parameters, and also cannot take into account group-level covariates.

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Instead of assuming a completely different intercept for each group, we can assume that the intercepts are drawn from a common (Normal) distribution.

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This is a relatively difficult model to estimate using non-Bayesian methods. The `lme4()` package in R can do it.

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We incorporate data with more than two levels easily as well.

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Cons:

- It's hard(er)
- Computationally intensive
- Need defense of priors
- No guarantee of MCMC convergence

Statistical packages for Bayesian are also less developed (`MCMCpack()` in R, WinBUGS, JAGS).