Sampling Methods

Patrick Lam

Outline

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If we stick in a value of x into $F(x)$, we get some value u in the interval [0,1] (which corresponds to $P(X \le x)$).

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Now we can sample using the inverse cdf method.

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```
> invcdf.func <- function(u) {
+     if (u >= 0 && u < 0.25)
+         sqrt(u)/2
+     else if (u >= 0.25 && u <= 1)
+         1 - sqrt(3 * (1 - u))/2
+ }
> x <- unlist(lapply(u, invcdf.func))
```

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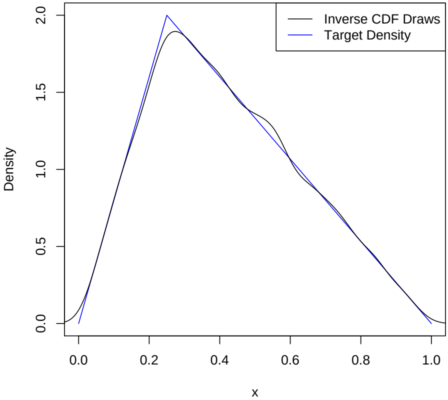

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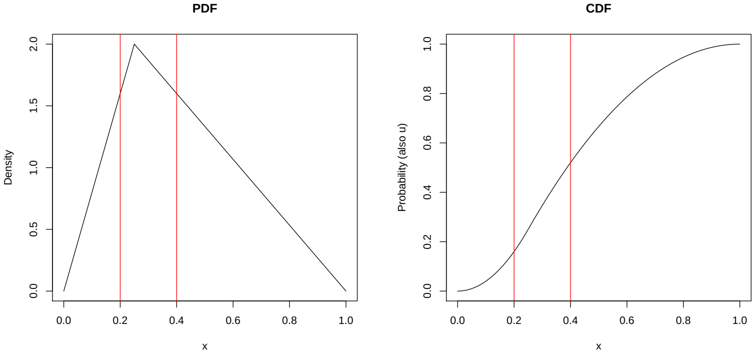

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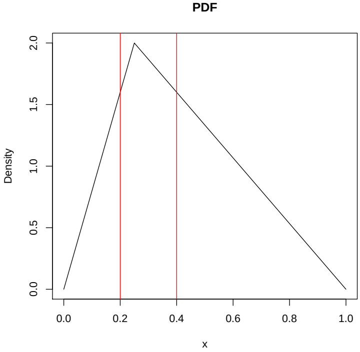

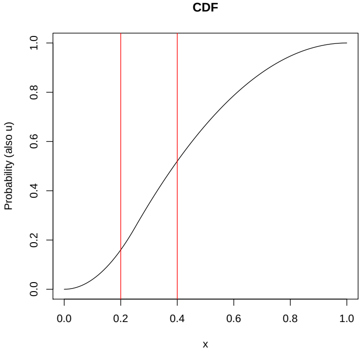

The areas with more density on the PDF (for example, the interval [0.2,0.4]) have a steeper "slope" on the CDF, so they cover more of the [0,1] space of u, and thus will be drawn more often.

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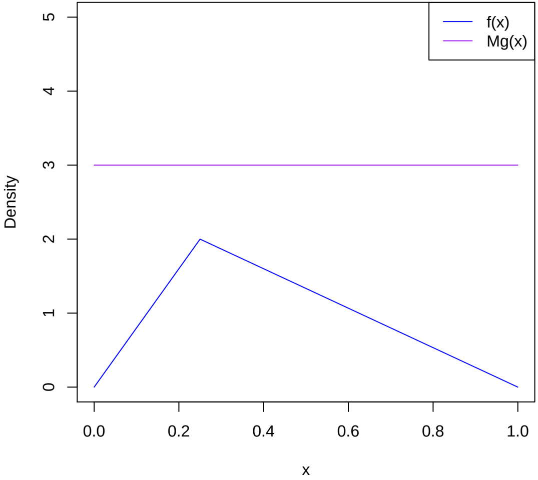
f(x)
Mg(x)
Density
x
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0

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```
> m <- 10000
> n.draws <- 0
> draws <- c()
> x.grid <- seq(0, 1, by = 0.01)
> while (n.draws < m) {
+     x.c <- runif(1, 0, 1)
+     accept.prob <- f.x(x.c)/(M * g.x(x.c))
+     u <- runif(1, 0, 1)
+     if (accept.prob >= u) {
+         draws <- c(draws, x.c)
+         n.draws <- n.draws + 1
+     }
+ }
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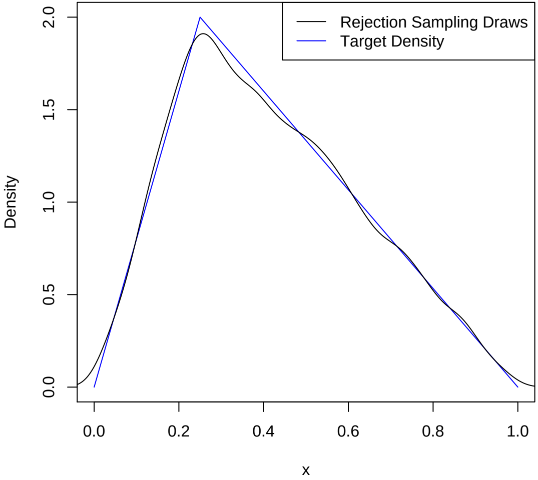

Rejection Sampling Draws
Target Density
Density
x
0.0
0.5
1.0
1.5
2.0
0.0
0.2
0.4
0.6
0.8
1.0

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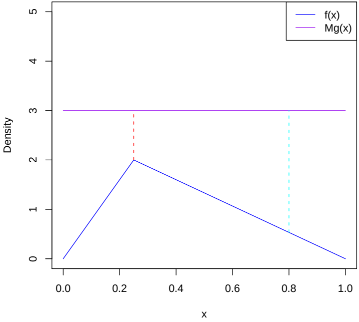

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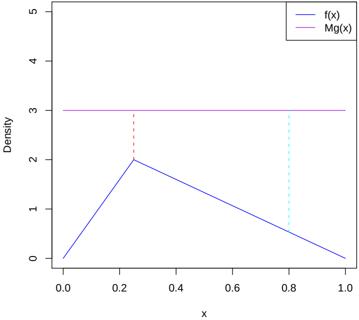

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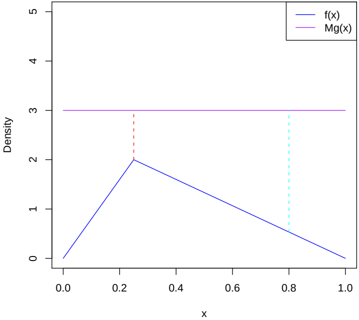

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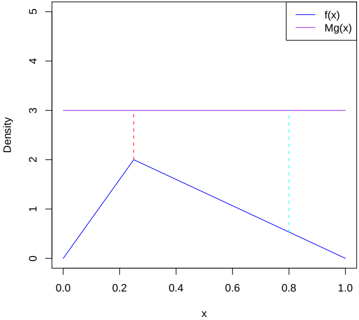

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A version of rejection sampling forms the basis for the Metropolis-Hastings algorithm that we will use later to sample from (possibly multivariate) posteriors without knowing the normalizing constant.