Maximum Likelihood

Patrick Lam

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Maximum likelihood is a way to find θ.

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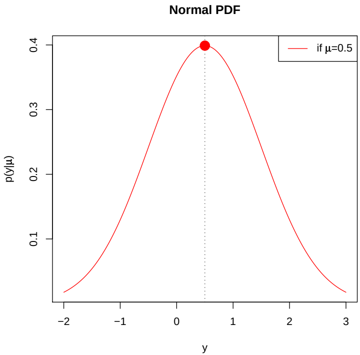
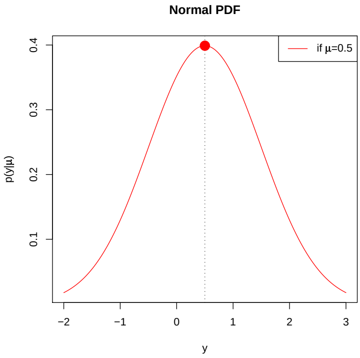
Estimate the mean μ of the distribution.

Obvious guess: $\mu = 0.5$


Normal PDF
if μ=0.5
p(y|μ)
0.1
0.2
0.3
0.4
−2
−1
0
1
2
3
y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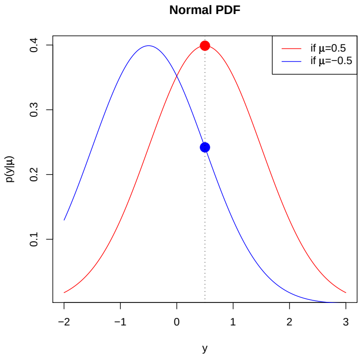
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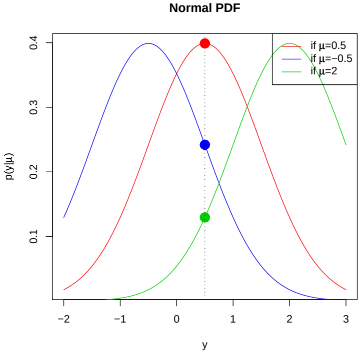

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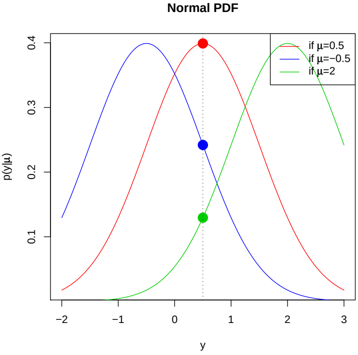
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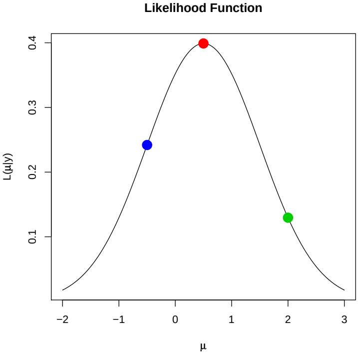
Likelihood Function
L(μ|y)
0.1
0.2
0.3
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−1
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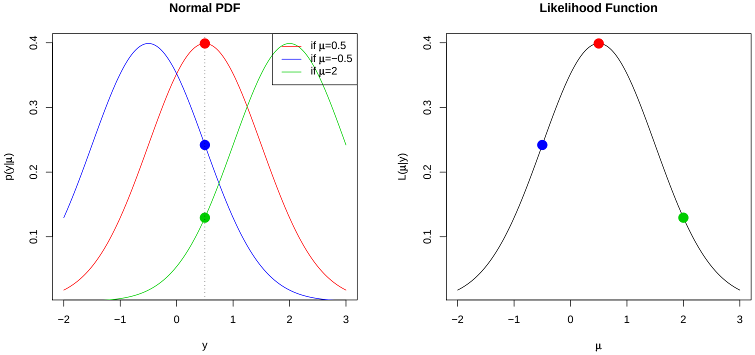
1
2
3
μ



Our best estimate of θ is the value of θ that maximizes $L(\theta|y)$ (MLE).

Example: Poisson Distribution

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We want to find λ, which is the mean of the Poisson distribution.

The PMF (discrete) for the data is

$$p(\mathbf{y}|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

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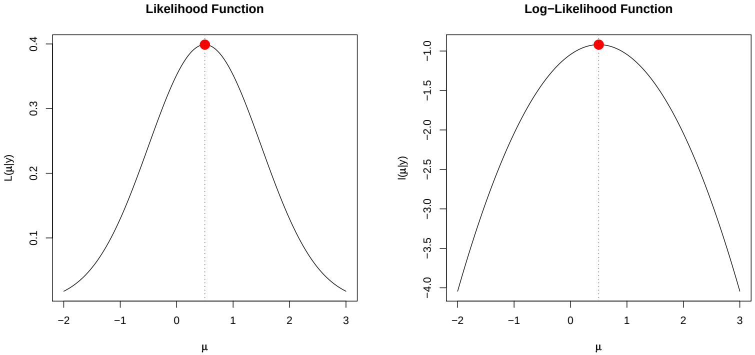

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What the exp()?

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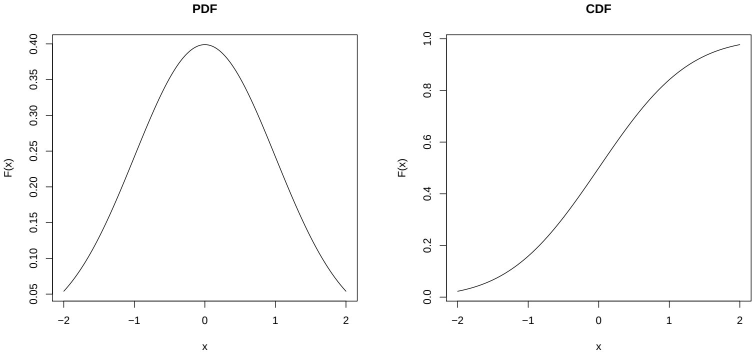
$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$$

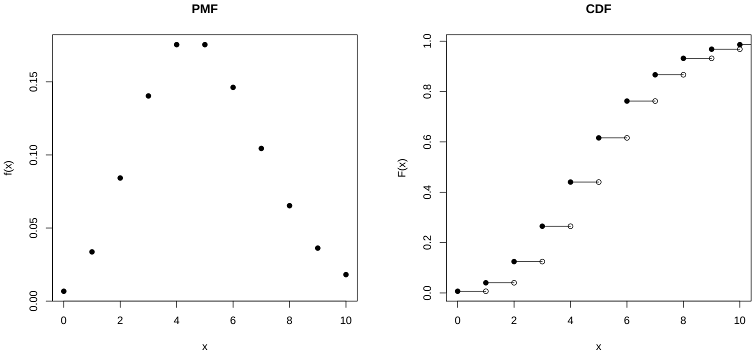
For continuous case:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Standard Normal PDF and CDF



Poisson(5) PMF and CDF



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Why shouldn't we use a CDF of a discrete variable?

What about the CDF of a bounded variable?

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How do we define the joint distribution for our data (and thus our likelihood)?

Use an indicator variable.

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Let D be an indicator variable such that $d_i = 1$ if i follows one distribution, and $d_i = 0$ if i follows another distribution.

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We can incorporate the indicator variable into our likelihood.

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d &= c(1,1,0,0,0) \\
p(\theta_1, \theta_2 | \mathbf{y}) &= \prod_{i=1}^{n} [p(y_i|\theta_1)]^{d_i} [p(y_i|\theta_2)]^{1-d_i}
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where $F(y)$ is the CDF for $p(y|\theta)$.

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$$F(z) = P(Y \leq z)$$