A Brief Review of Probability

Patrick Lam

Outline

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Expectation

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Continuous Case:

$$E(X) = \int_{-\infty}^{\infty} x p(x) dx$$

where $p(x)$ is the probability density function (PDF).

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This is sometimes known as the *Law of the Unconscious Statistician* (LOTUS).

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&= E(X^2) - 2[E(X)]^2 + [E(X)]^2 \\
&= \mathbf{E}(\mathbf{X}^2) - [\mathbf{E}(\mathbf{X})]^2
\end{aligned}$$

We can then find the first part with LOTUS.

Marginal, Conditional, and Joint Densities

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Expectation, Variance, and Densities

Important Distributions

Discrete Distributions

Continuous Distributions

The Bernoulli Distribution

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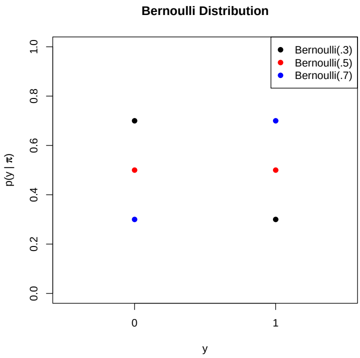

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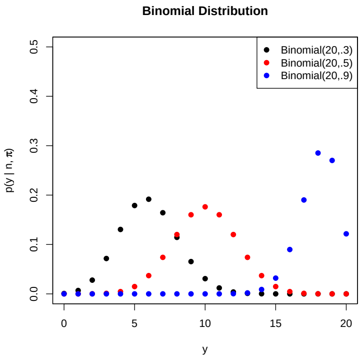

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$\text{Cov}(Y_i, Y_j) = -n\pi_i\pi_j$

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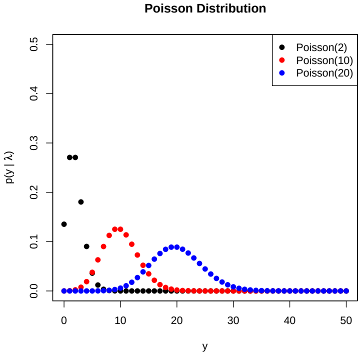

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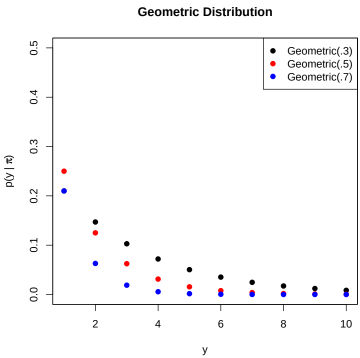

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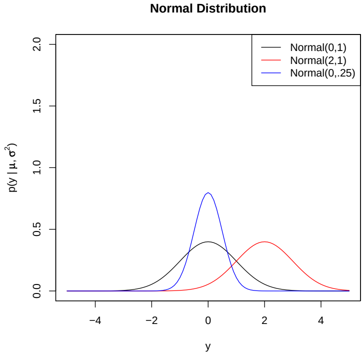

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variance-covariance matrix: $\boldsymbol{\Sigma}$ positive definite $k \times k$ matrix

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$$p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\pi}) = (2\pi)^{-k/2}|\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{\mathbf{-1}}(\mathbf{y} - \boldsymbol{\mu})\right)$$

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$E(Y) = \boldsymbol{\mu}$

$\mathrm{Var}(Y) = \boldsymbol{\Sigma}$

The Uniform Distribution

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Interval: $[\alpha, \beta]$; $\beta > \alpha$

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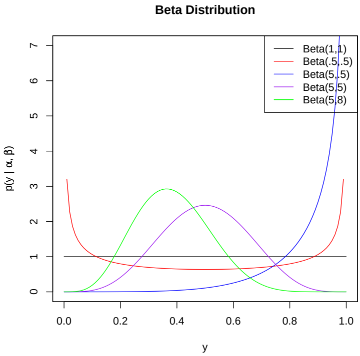

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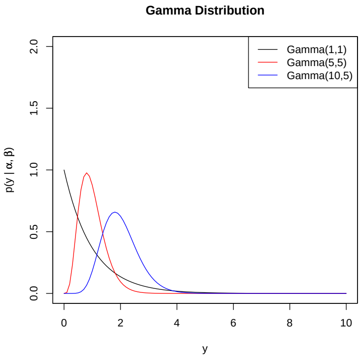

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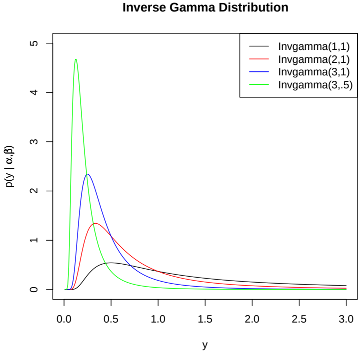

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$\text{Cov}(Y_i, Y_j) = -\frac{\alpha_i\alpha_j}{\alpha_0^2(\alpha_0+1)}$