Survival Models

Patrick Lam

February 1, 2008

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- Observations should be on the same clock time, but not necessarily calendar time.

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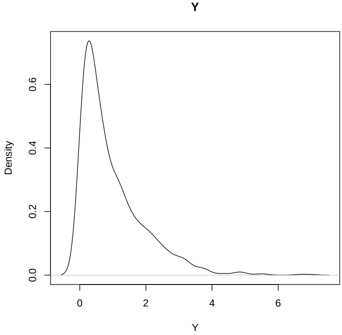

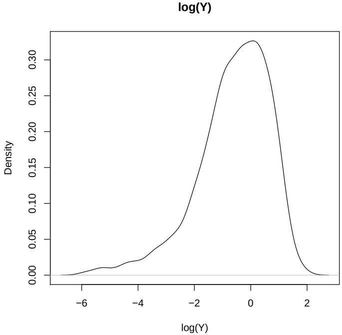

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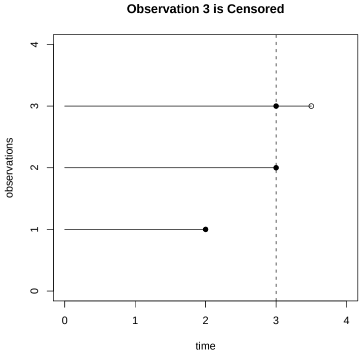

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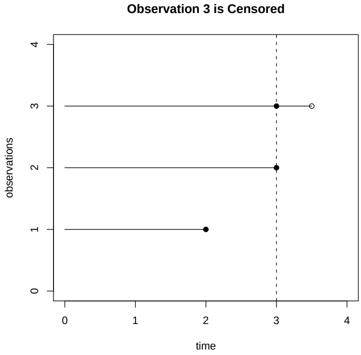

Observation 3 is censored in that it has not experienced the event at the time we stop collecting data, so we don't know its true duration.

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   - OLS cannot handle multiple values of GDP per observation.
   - You can set up data in a special way with survival models such that you can accomodate TVCs (not going to talk about this today).

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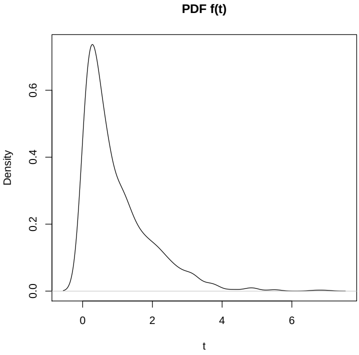

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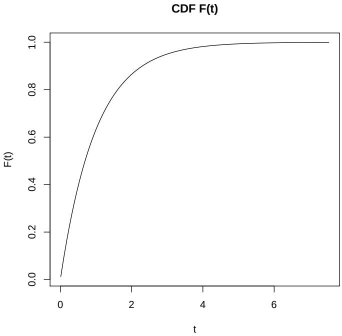

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- Parametric models usually assume some shape for the hazard rate (i.e. flat, monotonic, etc.).

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When all the covariates are 0, $h_i(t) = g(\beta_0)$. We call this the **baseline hazard**.

Estimation of the Parameters

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Answer: Use maximum likelihood estimation.

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$$P(y|\theta) = f(t)$$

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For each observation, create a censoring indicator c_i such that

$$c_i = \begin{cases} 1 & \text{if not censored} \\ 0 & \text{if censored} \end{cases}$$

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So uncensored observations contribute to the density function and censored observations contribute to the survivor function in the likelihood.

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3. Estimate via ML.
4. Interpret quantities of interest (hazard ratios, expected survival times).

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- Censoring occurs because of constitutionally mandated elections: governments fall apart in anticipation of such elections

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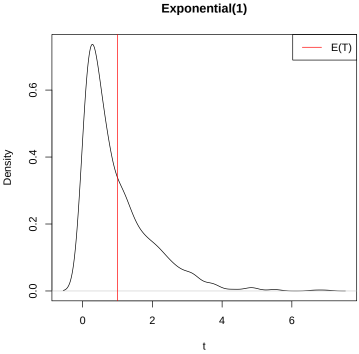
$\lambda_i > 0$ and is known as the rate parameter.



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\mathcal{L} &= \prod_{i=1}^{n} [f(t_i)]^{c_i} \; [S(t_i)]^{1-c_i} \\
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> X <- as.matrix(cbind(1, coalition[, c("invest", "fract", "polar",
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> expo.lik <- function(par, T, X, C) {
+     beta <- par
+     lambda <- exp(-(X %*% beta))
+     log.lik <- sum(C * (-(X %*% beta)) - (lambda * T))
+     return(log.lik)
+ }
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> my.coef <- optim(par = c(0, 0, 0, 0, 0, 0), fn = expo.lik, T = T,
+     X = X, C = C, method = "BFGS", control = list(fnscale = -1))$par
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(Intercept)      invest       fract       polar      numst2      crisis
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> expo.surv$loglik[2]

[1] -1046

> expo.lik(par = my.coef, X = X, T = T, C = C)

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```

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> x.min <- colMeans(model.matrix(expo.surv))
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```

Simulating Estimation Uncertainty:

```
> betas <- mvrnorm(1000, mu = expo.surv$coef, Sigma = vcov(expo.surv))
```

Hazard Ratios:

$$\mathrm{HR} = \frac{h(t|\mathbf{x}_{\mathrm{maj}})}{h(t|\mathbf{x}_{\mathrm{min}})}$$

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Constant hazard ratio across time is the *proportional hazards assumption*.

```
> hr.1 <- exp(-betas[, "numst2"])
```

```
> hr.1 <- exp(-betas[, "numst2"])

> hr.2 <- exp(-x.maj %*% t(betas))/exp(-x.min %*% t(betas))
```

```
> hr.1 <- exp(-betas[, "numst2"])
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> all.equal(hr.1, as.numeric(hr.2))
[1] TRUE
```

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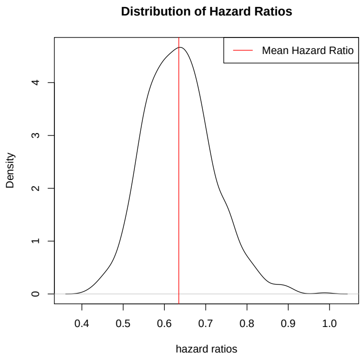

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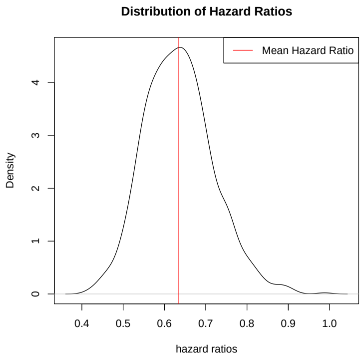

Majority governments survive longer than minority governments.

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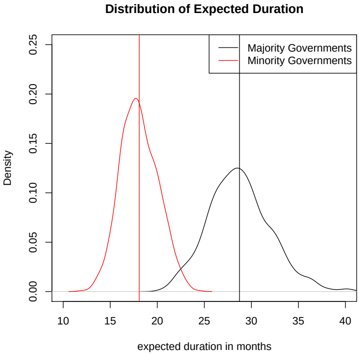

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Draw predicted values from the exponential distribution.

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> predict.maj <- apply(X = 1/expect.maj, MARGIN = 2, FUN = rexp,
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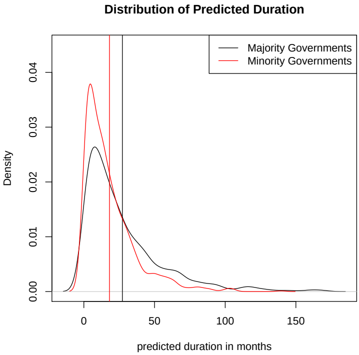

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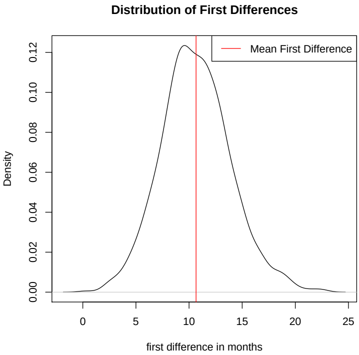

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What if we want to loosen that restriction by assuming a monotonic hazard?

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We can use the Weibull model.

The Weibull Model

Assume:

$$T_i \quad \sim \quad \mathrm{Weibull}(\lambda_i, p)$$

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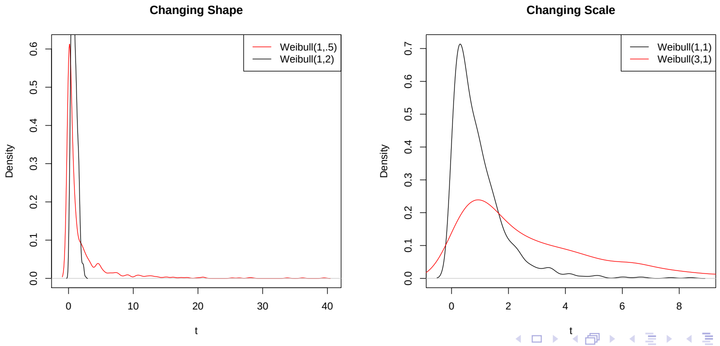

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Note that the link is different from the exponential model.

Modeling $h(t)$ with covariates:

$$h(t_i) \quad = \quad \left(\frac{p}{\lambda_i^p}\right) t_i^{p-1}$$

We want to constrain the $h(t)$ to be positive.

$$\lambda_i \quad = \quad e^{\mathbf{x}_i \beta}$$

Note that the link is different from the exponential model. Positive β implies that hazard decreases and average survival time increases as x increases.

$h(t_i)$ is modeled with both λ_i and p and is a function of t_i. Thus, the Weibull model assumes a **monotonic hazard**.

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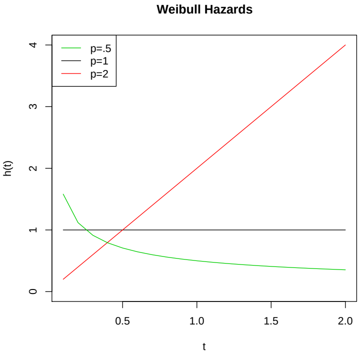

$$\mathcal{L} \;=\; \prod_{i=1}^{n} \left[f(t_i)\right]^{c_i} \; \left[S(t_i)\right]^{1-c_i}$$

$$
\begin{aligned}
\mathcal{L} &= \prod_{i=1}^{n} [f(t_i)]^{c_i} \; [S(t_i)]^{1-c_i} \\
&= \prod_{i=1}^{n} \left[\left(\frac{p}{\lambda}\right) \left(\frac{t}{\lambda}\right)^{p-1} e^{-(t/\lambda)^p} \right]^{c_i} \; \left[e^{-(t/\lambda)^p} \right]^{1-c_i}
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&= \sum_{i=1}^{n} c_i \left[\ln p - p \, \mathbf{x}_i \beta + (p-1) \ln t_i \right] - \left(\frac{t_i}{e^{\mathbf{x}_i \beta}}\right)^p
\end{aligned}
$$

Maximizing your own likelihood:

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```
> weib.lik <- function(par, T, X, C) {
+     beta <- par[1:ncol(X)]
+     p <- exp(par[(ncol(X) + 1)])
+     lambda <- exp((X %*% beta))
+     log.lik <- sum(C * (log(p) - p * log(lambda) + (p - 1) *
+         log(T)) - (T/lambda)^p)
+     return(log.lik)
+ }
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> my.max <- optim(par = c(0, 0, 0, 0, 0, 0, 0), fn = weib.lik,
+     T = T, X = X, C = C, method = "BFGS", control = list(fnscale = -1))$par
> my.coef <- my.max[1:ncol(X)]
> my.p <- exp(my.max[(ncol(X) + 1)])
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```

Using the survival package:

```
> weib.surv <- survreg(Surv(duration, ciep12) ~ invest + fract +
+     polar + numst2 + crisis, data = coalition, dist = "weibull")
```

```
> summary(weib.surv)

Call:
survreg(formula = Surv(duration, ciep12) ~ invest + fract + polar +
    numst2 + crisis, data = coalition, dist = "weibull")
               Value Std. Error     z        p
(Intercept)  4.75007    0.53072  8.95 3.55e-19
invest      -0.47160    0.11643 -4.05 5.11e-05
fract       -2.11762    0.75876 -2.79 5.26e-03
polar       -0.02792    0.00506 -5.52 3.33e-08
numst2       0.42746    0.11025  3.88 1.06e-04
crisis       0.00538    0.00183  2.94 3.28e-03
Log(scale)  -0.15644    0.04971 -3.15 1.65e-03

Scale= 0.855

Weibull distribution
Loglik(model)= -1042   Loglik(intercept only)= -1101
Chisq= 117.8 on 5 degrees of freedom, p= 0
Number of Newton-Raphson Iterations: 5
n= 314
```

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The scale parameter given by `survreg()` is NOT the same as the scale parameter in the Weibull distribution, which should be $\lambda_i = e^{\mathbf{x}_i \beta}$.

```
> rbind(weib.surv.coef = weib.surv$coef, my.coef)

               (Intercept)  invest  fract    polar numst2   crisis
weib.surv.coef       4.750 -0.4716 -2.118 -0.02792 0.4275 0.005377
my.coef              4.753 -0.4719 -2.122 -0.02792 0.4271 0.005405
```

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But what if we don't want to make an assumption about the shape of the hazard?

Outline

The Cox Proportional Hazards Model

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- Takes advantage of the proportional hazards assumption.

1. Reconceptualize each t_i as a discrete event time rather than a duration or survival time (non-censored observations only).

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    - No two events can occur at the same instant. It only seems that way because our unit of measurement is not precise enough.
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3. Assume no events can happen between event times.

We know that exactly one event occurred at each t_i for all non-censored i.

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All observations (censored and non-censored) j such that $t_j \geq t_i$

For example, if $t_i = 5$ months, then all observations that do not experience the event or are not censored before 5 months are at risk.

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&= \prod_{i=1}^{n} \left[\frac{h_0(t)h_i(t_i)}{\sum_{j \in R_i} h_0(t)h_j(t_j)}\right]^{c_i}
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$h_0(t)$ is the baseline hazard, which is the same for all observations, so it cancels out.

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There is no β_0 term estimated. This implies that the shape of the baseline hazard is left unmodeled.

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- Shape of hazard is unknown (although there are semi-parametric ways to derive the hazard and survivor functions)

How do I run a Cox proportional hazards model in R?

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Use the `coxph()` function in the `survival` package (also in the `Design` and `Zelig` packages).

Outline

How Do Survival Models Relate to Duration Dependence in a Logit Model?

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- Not all observations are independent. We may see some duration dependence.
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How can we account for this duration dependence in a logit model?

Think of the observations as grouped duration data:

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Year	t_k	Dyad	Y_i	T_i
1992	1	US-Iraq	0	
1993	2	US-Iraq	0	
1994	3	US-Iraq	0	
1995	4	US-Iraq	0	
1996	5	US-Iraq	0	
1997	6	US-Iraq	0	
1998	7	US-Iraq	0	12
1999	8	US-Iraq	0	
2000	9	US-Iraq	0	
2001	10	US-Iraq	0	
2002	11	US-Iraq	0	
2003	12	US-Iraq	1	

Then we end up with:

$$P(y_{i,t_k} = 1|\mathbf{x}_{i,t_k}) \quad = \quad h(t_k|\mathbf{x}_{i,t_k})$$

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$$\begin{aligned} P(y_{i,t_k} = 1 | \mathbf{x}_{i,t_k}) &= h(t_k | \mathbf{x}_{i,t_k}) \\ &= 1 - P(\text{surviving beyond } t_k | \text{survival up to } t_{k-1}) \end{aligned}$$

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It can be shown in general that

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So then we get

$$P(y_{i,t_k} = 1|\mathbf{x}_{i,t_k}) = 1 - e^{-\int_{t_{k-1}}^{t_k} h(u)du}$$

where we take the integral from t_{k-1} to t_k in order to get the conditional survival.

$$P(y_{i,t_k} = 1 | \mathbf{x}_{i,t_k}) \quad = \quad 1 - \exp\left(- \int_{t_{k-1}}^{t_k} h(u) du \right)$$

$$
\begin{aligned}
P(y_{i,t_k} = 1 | \mathbf{x}_{i,t_k}) &= 1 - \exp\left(-\int_{t_{k-1}}^{t_k} h(u)du\right) \\
&= 1 - \exp\left(-\int_{t_{k-1}}^{t_k} e^{\mathbf{x}_{i,t_k}\beta} h_0(u)du\right)
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P(y_{i,t_k} = 1 | \mathbf{x}_{i,t_k}) &= 1 - \exp\left(- \int_{t_{k-1}}^{t_k} h(u) du \right) \\
&= 1 - \exp\left(- \int_{t_{k-1}}^{t_k} e^{\mathbf{x}_{i,t_k}\beta} h_0(u) du \right) \\
&= 1 - \exp\left(- e^{\mathbf{x}_{i,t_k}\beta} \int_{t_{k-1}}^{t_k} h_0(u) du \right)
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This is equivalent to a model with a complementary log-log (cloglog) link and time dummies κ_{t_k}.

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- Using a variable such as "number of years at peace" instead of time dummies imposes a monotonic hazard.
- The use of time dummies may use up a lot of degrees of freedom, so BKT suggest using restricted cubic splines.

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- Variables that do not vary across units
  - May be collinear with time dummies.

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Go forth and learn.

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