Model Checking

Patrick Lam

Outline

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Through simulation, we can get a **posterior predictive distribution**.

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If y is a vector of n observations, then y^{rep} is also a vector of length n with covariates set at the observed (model checking) or hypothetical values (prediction) and $p(y^{\text{rep}}|y)$ can be thought of as an n-variate distribution.

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We can use the posterior predictive distribution to predict the future or assess model accuracy with posterior predictive checks.

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We can attempt to check specific model assumptions with **posterior predictive checks**.

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If our posterior predictive p-value is close to 0 or 1 (say 0.05 or 0.95), then it suggests that our observed data has an extreme test statistic and that something in our model may be inadequate.

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- It is not always clear how to correct the incorrect model assumptions.

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An Example

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> library(MCMCpack)
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The resulting posterior predictive distribution is an $n \times m$ matrix.

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What's wrong with this test statistic?

- Unclear what assumption are we testing.
- The fraction of 1s is explicitly being modeled in the logit model.
  - The test will never show anything is wrong regardless of how bad our model is.

A Better Test Statistic

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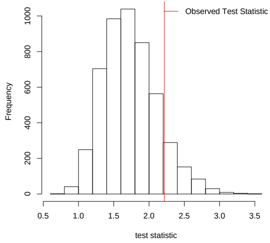

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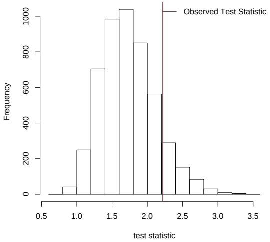

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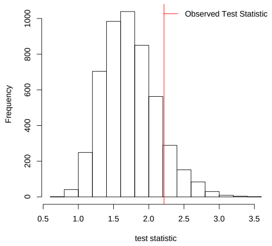

Does this mean our assumption is correct? Not necessarily (low power?)

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This is known as the Bayes factor approach and it is the Bayesian alternative to hypothesis testing in classical statistics.

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$$\text{posterior odds} = \text{Bayes factor} \times \text{prior odds}$$

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The individual terms that make up the Bayes factor, $p(\mathbf{y}|M_1)$ and $p(\mathbf{y}|M_2)$, are known as **marginal likelihoods**

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where $\boldsymbol{\theta}_k$ are the model parameters for model M_k.

Thus, the marginal likelihood is "marginal" because it is the likelihood of $\mathbf{y}$ under M_k *averaged* over the model parameters $\boldsymbol{\theta}_k$.

Note that $p(\boldsymbol{\theta}_k|M_k)$ is just the prior for $\boldsymbol{\theta}$ under M_k.

The marginal likelihood can be interpreted as

- the normalizing constant of the posterior $p(\boldsymbol{\theta}|\mathbf{y})$ given M_k
- the expected value of the likelihood function taken over the prior density

Note that the marginal likelihood is just

$$p(\mathbf{y}) = \int_{\boldsymbol{\Theta}} p(\mathbf{y}|\boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

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Therefore, except in simple cases (such as conjugacy), the marginal likelihood usually has to be approximated.

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- less than $\frac{1}{100}$ leads us to reject M_1
- greater than 1 leads us to accept M_1

One can also rely on tables such as the one given by Jeffreys for the Bayes factor of M_1 over M_2:

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Bayes Factor	Strength of Evidence for M_1
< 1	Negative (supports M_2)
1 to 3	Barely Worth Mentioning
3 to 10	Substantial
10 to 30	Strong
30 to 100	Very Strong
> 100	Decisive