A possible solution to assignment: basic mathematics for TLA+

Dr. Tianxiang Lu

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Note:

- There is no standard solution, this is one possible solution.
- The errors introduced to the assignment by intention are corrected in this solution.
- If you find further errors in this solution, please send an Email to contact@tiit.lu

1. Absolute Value

Definition 1. For a number x, the **absolute value function** $|x|$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Prove some properties of real function. (Hint: practice case analysis)

Theorem 1. *For every real number x, $|x| \geq 0$.*

Proof. As the absolute function is defined by cases, it is very natural to prove the theorem by cases on x.

Case 1: $x \geq 0$.

In this case, $x \geq 0$, and $|x| = x$, therefore $|x| \geq 0$.

Case 2: $x < 0$.

In this case, $|x| = -x$. As x is negative, $-x$ is positive, and thus, $-x > 0$. □

Theorem 2. *$|x| \geq x$ for all x.*

Proof. Here, we do cases, as above.

Case 1: $x \geq 0$.

In this case, $|x| = x$, and therefore $|x| \geq x$.

Case 2: $x < 0$.

In this case, $|x| = -x$, and as x is negative, $-x$ is positive. Positive numbers are always larger than negative numbers, and thus $|x| = -x > x$. □

Theorem 3 (Triangle Inequality)**.** *For every a, b real numbers, we have*

$$|a + b| \leq |a| + |b|$$

Proof 1. One idea may be to enumerate all the cases on how a and b are comparable to 0.

Case 1: $a, b \geq 0$

$|a+b| = a+b = |a|+|b|$.

Case 2: $a, b \leq 0$

$$|a+b| = -a-b = |a|+|b|$$

Case 3: $a \leq 0$ and $b \geq 0$.

Now, don't know what exactly $a+b$ is. Imagine however we are b. We know when we get a added to us, we get smaller, so we know $b > a+b$. We want to conclude that $|b| > |a+b|$, but we cannot because a might be very negative, and might have made this number very small. In a way, we see that our current path is stuck unless we know more about $a+b$. Therefore, we do more cases.

Case 3.1: $a+b > 0$

Now we have even more information. We have already seen $b > a+b$, and now we know that $a+b$ is itself positive. Therefore, $|b| > |a+b|$. Here, we are already done though, because adding something positive a quantity which is already larger than another quantity preserves that inequality; that is, since $|a| > 0$:

$$|a+b| < |b| < |b|+|a|$$

Case 3.2 $a+b < 0$

The roles of a and b switch in this case. Now, making the same observations in the beginning of case 3 we see that $a < a+b$. But of course, when you take the absolute value a smaller negative becomes a larger positive, so $|a| > |a+b|$. Then we have

$$|a+b| < |a| < |b|+|a|$$

Case 3.3 $a+b = 0$

This is an easy case to forget, but without it the proof would not be complete. If $a+b=0$, then $|a+b| = 0$. $0 \leq p$ for any positive number p, so as $|a|+|b| > 0$

$$|a+b| = 0 \leq |a|+|b|$$

Case 4: $a > 0$ and $b < 0$

Notice, we will be doing the same work here as we did in case 3.1 and 3.2. As an exercise, you should do this case.

□

Proof. We can shorten the above and really think about what cases we need to get the job done.

Case 1: $a+b \geq 0$

If $a+b \geq 0$ then $|a+b| = a+b$. By the theorem above, we have $a \leq |a|$ and $b \leq |b|$. But then we have $a+b \leq |a|+|b|$. Chaining these together, we get

$$|a+b| = a+b \leq |a|+|b|$$

Case 2: $a + b < 0$

Here, we nice that $-a - b > 0$, and moreover $|a + b| = |-a - b|$. Therefore, we have that

$$|a + b| = |-a - b| = -a - b \leq |-a| + |-b| = |a| + |b|$$

$\square$

2. Proof by contradiction

Theorem 4. *For all positive real numbers x, y, If $x > y$ then $\sqrt{x} > \sqrt{y}$*

Proof. Suppose that we were wrong about this theorem being correct. Then we would have an x and y that **witness** this.

Given $x > y$, assuming the negation of the hypothesis, namely $\sqrt{x} \leq \sqrt{y}$. We know that squaring is monotone over the positive reals, so if $\sqrt{x} \leq \sqrt{y}$ we can square both sides and get $x \leq y$. This contradict to our given fact $x > y$. So, the assumption is wrong, therefore, the original hypothesis is correct. $\square$

The above proof is called a **proof by contradiction**.

3. Set construction:

(1) If a set A has n members, how many members does the power set of A has?

Answer. $|\wp(A)| = 2^{|A|} = 2^n$

(2) Can you build a set of all sets? How does it look like?

Theorem 5 (Bertrand Russell 1901)**.** *If one is able to do comprehension (set builder notation) to form new sets, there cannot be a set of all sets.*

Proof. For sake of contradiction, let us suppose that this theorem is false. Then we know that there is a set of all sets, and we are able to do comprehension. Let Ω be the set of all set. Now, we want to get a contradiction, and we suspect (given the way the theorem was stated) it has something to do with comprehension. So define a set B as follows:

$$B = \{\, x \in \Omega \mid x \notin x \,\}$$

That is, B is the set of all sets which do not contain themselves as members. We don't know much about B; maybe it is empty, maybe it is very large. But as B is a set, we are allowed to ask questions to B about who are its members. B itself is either a member of B, or not.

Case 1: $B \in B$.
We defined B by comprehension, so something gets inside of B only when it satifies the above formula. Therefore, we know $B \notin B$, which is a contradiction.
Case 2: $B \notin B$.
Look at the definition of B. We put a set x in B exactly when $x \notin x$. As $B \notin B$ we know therefore that $B \in B$, which of course is a contradiction.

As B is a set, one of these two cases must hold, but they both lead to contradictions. Therefore, we can conclude that our initial assumption was incorrect: Ω is not a set. $\square$

4. Prove the following theory

Theorem 6. *If* $f : A \to B$ *and* $X, Y \subseteq A$ *then*

- $f[X \cup Y] = f[X] \cup f[Y]$
- $f[X \cap Y] \subseteq f[X] \cap f[Y]$

Proof. Here I only show the proof of the first theory, the proof of the second one is similar and therefore omitted.

- '$\subseteq$': Let $a \in A$ and $b \in B$. By definition of f, for every $b \in f[X \cup Y]$ there exists an $a \in (X \cup Y) \subseteq A$ such that $f(a) = b$.
  By definition of $\cup$: $a \in X \vee a \in Y$.
  Case 1: $a \in X$: by definition of f, we have $b \in f[X]$,
  Case 2: $a \in Y$: by definition of f, $b \in f[Y]$.
  By definition of $\cup$ again, we have $b \in f[X] \cup f[Y]$.
- '$\supseteq$': Let $a \in A$ and $b \in B$. Assume $b \in f[X] \cup f[Y]$.
  By definition of $\cup$: $b \in f[X] \vee b \in f[Y]$.
  Case 1: $b \in f[X]$: by definition of f, there must exist an $a \in X$, such that $f(a) = b$. By definition of $\cup$, $a \in X \cup Y$. By definition of f, $b \in f[X] \cup f[Y]$.
  Case 2: $b \in f[Y]$: by definition of f, there must exist an $a \in Y$, such that $f(a) = b$. By definition of $\cup$, $a \in X \cup Y$. By definition of f, $b \in f[X] \cup f[Y]$.

$\square$