Assignment solution: Basic Math for TLA+: 2

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Due: May 18, 2015, except modeling projects

1. Propositional logic

   Definition 1. For given boolean variables x and y, the **de morgan rule** can be expressed as

   - $\neg(x \wedge y) \equiv (\neg x) \vee (\neg y)$
   - $\neg(x \vee y) \equiv (\neg x) \wedge (\neg y)$

   One can rewrite the expression $x \rightarrow y$ to $\neg x \vee y$. The unary boolean operator $\neg$ binds stronger than binary operators like $\wedge$ or *vee*.

   A formula is in *Negation Normal Form (NNF)* if the negation operator ($\neg$, not) is only applied to variables and the only other allowed Boolean operators are conjunction ($\wedge$, and) and disjunction ($\vee$, or).

   Transform the following boolean expression to Negation Normal Form

   (a) $\neg(x \rightarrow (x \rightarrow y))$

   Solution:

   - Define R1: $x \rightarrow y \equiv \neg x \vee y$
   - Define DM1: $\neg(x \wedge y) \equiv (\neg x) \vee (\neg y)$
   - Define DM2: $\neg(x \vee y) \equiv (\neg x) \wedge (\neg y)$
   - Define KT1: $x \wedge (y \wedge z) \equiv (x \wedge y) \wedge z$
   - Define ID1: $x \wedge x \equiv x$
   - We have:

   $$\begin{aligned} & \neg(x \rightarrow (x \rightarrow y)) \\ \overset{\text{R1}}{\equiv} \quad & \neg(\neg x \vee (\neg x \vee y)) \\ \overset{\text{DM2}}{\equiv} \quad & x \wedge \neg(\neg x \vee y)) \\ \overset{\text{DM2}}{\equiv} \quad & x \wedge (x \wedge \neg y) \\ \overset{\text{KT1}}{\equiv} \quad & (x \wedge x) \wedge \neg y \\ \overset{\text{ID1}}{\equiv} \quad & x \wedge \neg y \textit{(full point result)} \end{aligned}$$

   (b) $\neg(x \wedge \neg y) \rightarrow \neg(x \leftrightarrow \neg y)$

   Solution:

   - Define R1: $x \rightarrow y \equiv \neg x \vee y$

- Define DM1: $\neg(x \wedge y) \equiv (\neg x) \vee (\neg y)$
- Define DM2: $\neg(x \vee y) \equiv (\neg x) \wedge (\neg y)$
- Define KT2: $x \vee (y \vee z) \equiv (x \vee y) \vee z$
- Define Def1: $x \leftrightarrow y \equiv (x \rightarrow y) \wedge (y \rightarrow x)$
- Define DT: $x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$
- We have:

$$
\begin{aligned}
& \neg(x \wedge \neg y) \rightarrow \neg(x \leftrightarrow \neg y) \\
\overset{\text{R1}}{\equiv}\ & (x \wedge \neg y) \vee \neg(x \leftrightarrow \neg y) \\
\overset{\text{Def1}}{\equiv}\ & (x \wedge \neg y) \vee \neg((x \rightarrow \neg y) \wedge (\neg y \rightarrow x)) \\
\overset{\text{DM1}}{\equiv}\ & (x \wedge \neg y) \vee (\neg(x \rightarrow \neg y) \vee \neg(\neg y \rightarrow x)) \\
\overset{\text{KT2}}{\equiv}\ & (x \wedge \neg y) \vee \neg(x \rightarrow \neg y) \vee \neg(\neg y \rightarrow x) \\
\overset{\text{R1}}{\equiv}\ & (x \wedge \neg y) \vee \neg(\neg x \vee \neg y) \vee \neg(y \vee x) \\
\overset{\text{DM2}}{\equiv}\ & (x \wedge \neg y) \vee (x \wedge y) \vee (\neg y \wedge \neg x) \textit{(full point result)} \\
\overset{\text{DT}}{\equiv}\ & x \vee \neg y \textit{(optional result, no bonus point)}
\end{aligned}
$$

2. First order logic

Definition 2. $\forall$ is a conjunction over the universe of objects and $\exists$ is a disjunction over the universe of objects. Therefore, de morgan rule can be applied on first order quantifiers $\forall$, $\exists$.

Please transform the following first order expression to Negation Normal Form Let $f : X \mapsto X$, and $p : X \mapsto \{\text{TRUE}, \text{FALSE}\}$

- $\neg(\forall x \in X.p(f(x)) \rightarrow p(x))$

  Solution:

  - Define R1: $x \rightarrow y \equiv \neg x \vee y$
  - Define DM2: $\neg(x \vee y) \equiv (\neg x) \wedge (\neg y)$
  - Define DM3: $\neg(\forall x \in X.e) \equiv \exists x \in X.\neg e$
  - We have:

$$
\begin{aligned}
& \neg(\forall x \in X.p(f(x)) \rightarrow p(x)) \\
\overset{\text{DM3}}{\equiv}\ & \exists x \in X.\neg(p(f(x)) \rightarrow p(x)) \\
\overset{\text{R1}}{\equiv}\ & \exists x \in X.\neg(\neg p(f(x)) \vee p(x)) \\
\overset{\text{DM2}}{\equiv}\ & \exists x \in X.p(f(x)) \wedge \neg p(x)
\end{aligned}
$$

- $\neg(\exists x \in X.p(x) \rightarrow (p(x) \rightarrow p(x)))$

  Solution:

  - Define R1: $x \rightarrow y \equiv \neg x \vee y$
  - Define DM2: $\neg(x \vee y) \equiv (\neg x) \wedge (\neg y)$
  - Define DM4: $\neg(\exists x \in X.e) \equiv \forall x \in X.\neg e$
  - Define ID1: $x \wedge x \equiv x$
  - Define KT2: $x \vee (y \vee z) \equiv (x \vee y) \vee z$

- Define T1: $x \vee \neg x \equiv \text{TRUE}$
- We have:

$$\begin{aligned}
&\neg(\exists x \in X.p(x) \rightarrow (p(x) \rightarrow p(x))) \\
\overset{\text{R1}}{\equiv}\ &\neg(\exists x \in X.p(x) \rightarrow (\neg p(x) \vee p(x)) \\
\overset{\text{R1}}{\equiv}\ &\neg(\exists x \in X.\neg p(x) \vee (\neg p(x) \vee p(x)) \\
\overset{\text{KT2}}{\equiv}\ &\neg(\exists x \in X.\neg p(x) \vee \neg p(x) \vee p(x) \\
\overset{\text{ID1}}{\equiv}\ &\neg(\exists x \in X.\neg p(x) \vee p(x)) \\
\overset{\text{T1}}{\equiv}\ &\neg(\exists x \in X.\text{TRUE}) \\
\overset{\text{DM4}}{\equiv}\ &\forall x \in X.\text{FALSE}
\end{aligned}$$

3. Syntax

- Please write down the first order logic expression as semantic of the syntactic sugar "IF p then x ELSE y ".

  Solution: following options are all acceptable:

  - In TLA+: $\text{CHOOSE } v : (p \rightarrow (v = x)) \wedge (\neg p \rightarrow (v = y))$
  - In first order logic, we must define to which value will this x and y be assigned to, if we don't restrict them to be boolean:
    $\forall x \in X, y \in Y, z \in Z.X \subseteq Z \wedge Y \subseteq Z \wedge (p \rightarrow (z = x)) \wedge (\neg p \rightarrow (z = y))$
  - In first order logic, if we restrict x and y to boolean, then we actually treat it as propositional logic formula, which is also fine.
    $(p \rightarrow x) \wedge (\neg p \rightarrow y)$
  - The above formula can be written to
    $(p \wedge x) \vee (\neg p \wedge y) \vee (x \wedge y)$
  - A simple expression, sharing the same truth table as above:
    $(p \wedge x) \vee (\neg p \wedge y)$

- Please write down the semantic of a function in TLA$^+$
  $[f \text{ EXCEPT } ![e_1] = e_2]$

  Solution: following options are all acceptable:

  - $[x \in \text{DOMAIN} f \mapsto \text{IF } x = e_1 \text{ THEN } e_2 \text{ ELSE } f[x]]$
  - In words, it is the function $\hat{f}$ that is the same as f, except that $\hat{f}[e_1] = e_2$

- Please write down the equivalent logic expression using bracket for the following TLA+ expression.

$$\begin{aligned}
&\vee\ d_1 \\
&\vee\ \wedge\ b_1 \\
&\quad\ \ \wedge\ b_2
\end{aligned}$$

  Solution: $d_1 \vee (b_1 \wedge b_2)$

4. TLA modeling: specify your own clock showing the actions of hour, minute and second, use "PrintT" from module TLC to print out each variable changes.

   One possible solution: to be found in separate TLA+ code package

5. (Optional) Mathematical puzzle modeling

   A farmer has a 40 pound stone and a balance scale. How can he break the stone into 4 pieces so that, using those pieces and the balance scale, he can weigh out any integral number of pounds of corn from 1 pound through 40 pounds.

   Try to use TLA+ to specify the problem and use TLC to find the solution. Student who can successfully finish this task on his own and handed in the hand written model will get one extra point for the final exam.

   One possible solution: to be found in attached TLA+ model.