Fachbereich Mathematik
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TECHNISCHE UNIVERSITÄT DARMSTADT

Winter Term 2015/16

Exercise Sheet no. 2 – Formal Foundations of Computer Science

(E2.1) [warm-up: equivalence relations]

For a natural number $k \geq 2$, define the relation $\equiv_k$ on $\mathbb{Z}$ by

$$a \equiv_k b \quad \text{iff} \quad (b - a) \text{ is an integer multiple of } k.$$

(i) Show that this is an equivalence relation.

(ii) Recall that the *index* of an equivalence relation is the number of equivalence classes. What is the index of $\equiv_k$? Describe the equivalence classes of $\equiv_k$. Give a system of representatives for the equivalence classes of $\equiv_k$.

(iii) Is the relation

$$a \sim_k b \quad \text{iff} \quad |b - a| \leq k$$

an equivalence relation?

(E2.2) [the equivalence relation $\sim_L$]

Recall that for a language $L \subseteq \Sigma^*$ we defined the equivalence relation $\sim_L$ on Σ^* by

$$w \sim_L w' \quad \text{iff} \quad (wx \in L \leftrightarrow w'x \in L \text{ for all } x \in \Sigma^*).$$

(i) Let $L \subseteq \{0,1\}^*$ be the language

$$\{w \mid |w|_1 \text{ is a multiple of } 3\}.$$

What is the index of $\sim_L$ for this L? Give a system of representatives. Construct the automaton with this system of representatives as states, as in the lecture.

(ii) Let $L' \subseteq \{a\}^*$ be the language

$$\{a^{n^2} \mid n \geq 1\} = \{1, 1111, 111111111, \ldots\}.$$

Show that $\text{index}(\sim_{L'}) = \infty$ by giving an infinite set of words $\{w_1, w_2, \ldots\}$ such that no pair satisfies $w_i \sim_{L'} w_j$ for $i \neq j$. Can you find a system of representatives for $\sim_{L'}$? Is L' a regular language?

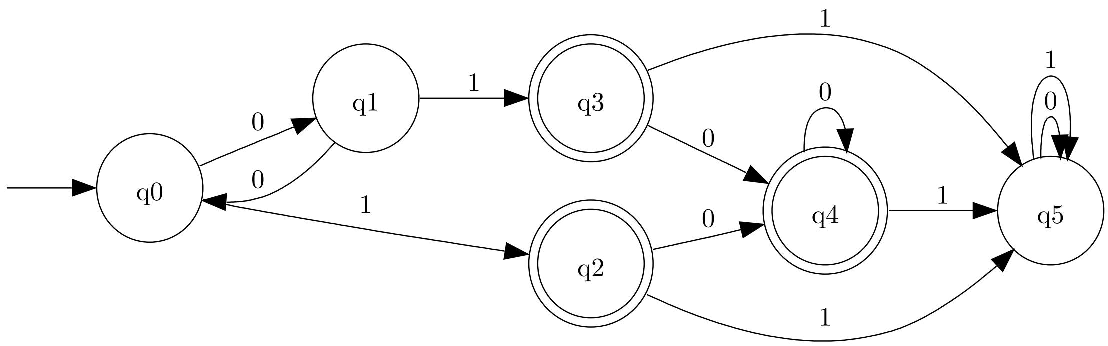
(iii) We know that $\sim_L$ is *right-invariant*, i.e. if $w \sim_L w'$ and $x \in \Sigma^*$, then $wx \sim_L w'x$. Make sure you understand the proof of this.

Now consider the language $L'' = L(a^*b^*) \subseteq \{a,b\}^*$. Show that $\sim_{L''}$ is not left-invariant, i.e. there are words $u, v, x \in \Sigma^*$ such that

$$u \sim_{L''} v \qquad \text{but} \qquad xu \not\sim_{L''} xv.$$

(E2.3) [the equivalence relation $\sim_A$]

Let A be the following DFA:



Recall that we defined $\sim_A$ on Σ^* by

$$w \sim_A w' \quad \text{iff} \quad \hat{\delta}(q_0, w) = \hat{\delta}(q_0, w').$$

(i) What is the index of $\sim_A$ for this automaton? Give a system of representatives of $\sim_A$.

(ii) Let $L = L(A)$ be the language accepted by A. What is the index of $\sim_L$? Give a system of representatives of $\sim_L$.

Hint: $L = L(0{*}10{*}) = \{w \in \{0,1\}^* \mid |w|_1 = 1\}$.

(iii) Convince yourself that $\sim_A$ refines $\sim_L$. Which equivalence classes of $\sim_A$ get merged to equivalence classes of $\sim_L$? What does the automaton based on $\sim_L$ (as in the previous exercise) for this language look like?