Zad. 1

$$P(A|B\cap C) = \frac{P(A\cap B\cap C)}{P(B\cap C)}$$

$$P(B|A) = \frac{P(B\cap A)}{P(A)}$$

$$P(A\cap B) = P(B|A)P(A)$$

$$P(C|A\cap B) = \frac{P(A\cap B\cap C)}{P(A\cap B)}$$

$$P(A\cap B\cap C) = P(C|A\cap B)P(A\cap B)$$

$$P(A\cap B\cap C) = P(C|A\cap B)P(B|A)P(A)$$

$$P(A\cap B\cap C) = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{2}{5} = \frac{2}{40} = \frac{1}{20}$$

$$P(C|B) = \frac{P(B\cap C)}{P(B)}$$

$$P(B\cap C) = P(C|B)P(B)$$

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

$$P(B) = P(A\cup B)-P(A)+P(A\cap B)$$

$$P(B) = P(A\cup B)-P(A)+P(B|A)P(A)$$

$$P(B) = \frac{6}{10}-\frac{4}{10}+\frac{1}{4}\cdot\frac{2}{5} = \frac{2}{10}+\frac{1}{10} = \frac{3}{10}$$

$$P(B\cap C) = \frac{1}{3}\cdot\frac{3}{10} = \frac{1}{10}$$

$$P(A|B\cap C) = \frac{1}{20}\cdot 10 = \frac{1}{2}$$

Zad. 2

$$\mathrm{corr}(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{EXY-EXEY}{\sigma_X\sigma_Y}$$

$$X_i \sim U(0,1)$$

$$U_i = (b-a)X_i + a \sim U(a,b)$$

$$X = \min(U_1, \dots, U_n) = (b-a)\min(X_1, \dots, X_n) + a$$

$$Y = \max(U_1, \dots, U_n) = (b-a)\max(X_1, \dots, X_n) + a$$

$$\operatorname{Corr}((b-a)\min(X_1, \dots, X_n) + a, (b-a)\max(X_1, \dots, X_n) + a) =$$

$$= \operatorname{Corr}(\min(X_1, \dots, X_n), \max(X_1, \dots, X_n))$$

$$X_m = \min(X_1, \dots, X_n) \sim \text{Beta}(1, n)$$

$$Y_m = \max(X_1, \dots, X_n) \sim \text{Beta}(n, 1)$$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

$$f_{X_m Y_m}(x, y) = n! \frac{(y-x)^{n-2}}{(n-2)!} \quad \text{dla} \quad x < y \quad (\text{wzór z teorii})$$

$$EX_m Y_m = \int_0^1 \int_0^{1-x} xy \cdot \frac{n!}{(n-2)!} (y-x)^{n-2} \, dy \, dx =$$

$$= \int_0^1 \int_x^1 x \frac{n!}{(n-2)!} y (y-x)^{n-2} \, dy \, dx = \left| \begin{array}{ll} y - x = t & y = t + x \\ dy = dt & \end{array} \right| =$$

$$= \frac{n!}{(n-2)!} \int_0^1 x \int_0^{1-x} (t+x) t^{n-2} \, dt \, dx = \frac{n!}{(n-2)!} \int_0^1 x \int_0^{1-x} t^{n-1} + x t^{n-2} \, dt \, dx =$$

$$= \frac{n!}{(n-2)!} \int_0^1 x \left[ \frac{t^n}{n} + \frac{x t^{n-1}}{n-1} \right]_0^{1-x} dx =$$

$$= \frac{n!}{(n-2)!} \int_0^1 \frac{x(1-x)^n}{n} + \frac{x^2 (1-x)^{n-1}}{n-1} \, dx = \left| \begin{array}{ll} 1 - x = t & x = 1 - t \\ -dx = dt & \end{array} \right| =$$

$$= \frac{n!}{(n-2)!} \int_0^1 \frac{(1-t) t^n}{n} + \frac{(1 - 2t + t^2) t^{n-1}}{n-1} \, dt =$$

$$= \frac{n!}{(n-2)!} \int_0^1 \frac{t^n}{n} - \frac{t^{n+1}}{n} + \frac{t^{n-1}}{n-1} - \frac{2t^n}{n-1} + \frac{t^{n+1}}{n-1} \, dt =$$

$$= \frac{n!}{(n-2)!} \left[ \frac{t^{n+1}}{n(n+1)} - \frac{t^{n+2}}{n(n+2)} + \frac{t^n}{(n-1)n} - \frac{2t^{n+1}}{(n-1)(n+1)} + \frac{t^{n+2}}{(n-1)(n+2)} \right]_0^1 =$$

$$= \frac{\cancel{n!}}{\cancel{(n-2)!(n-1)n}} \left[ \frac{n-1}{n+1} - \frac{n-1}{n+2} + 1 - \frac{2n}{n+1} + \frac{n}{n+2} \right] =$$

$$= \frac{n-1-2n}{n+1} + \frac{n-n+1}{n+2} + 1 = \frac{-n-1}{n+1} + \frac{1}{n+2} + \frac{(n+1)(n+2)}{(n+1)(n+2)} =$$

$$= \frac{-\cancel{(n+1)(n+2)} + \cancel{n+1}^{1} + \cancel{(n+1)(n+2)}}{\cancel{(n+1)}_{1}(n+2)} = \frac{1}{n+2}$$

$$EX_n = \frac{1}{n+1} \qquad Var(X_n) = \frac{n}{(1+n)^2(n+2)}$$

$$EY_n = \frac{n}{n+1} \qquad Var(Y_n) = \frac{n}{(1+n)^2(n+2)}$$

$$Cov(X,Y) = \frac{1}{n+2} - \frac{n}{(n+1)^2} = \frac{(n+1)^2 - n(n+2)}{(1+n)^2(n+2)}$$

$$corr = \frac{(n+1)^2 - n(n+2)}{\cancel{(1+n)^2(n+2)}} \cdot \frac{\cancel{(1+n)^2(n+2)}}{n} = \frac{\cancel{n^2} + \cancel{2n} + 1 - \cancel{n^2} - \cancel{2n}}{n} = \frac{1}{n}$$

zad. 3

$$X = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu_X, \tfrac{1}{n}) \qquad Y = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N(\mu_Y, \tfrac{1}{n})$$

Trzeba obliczyć $E\min(X,Y)$.

Aby estymator $\min(X,Y)$ był nieobciążony to $E\min(X,Y) = \mu_X$ przy założeniu $Y > X$ (założenie $X > Y$ jest analogiczne).

$$corr = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

$$cov(X,Y) = cov\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{j=1}^{n} Y_j\right) = \left| \text{niezależność zm. } X_i \text{ i } Y_j \right| =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} cov(X_i, Y_i) = \frac{1}{n} cov(X_1, Y_1) = \frac{\rho}{n}$$

$$corr = \frac{\frac{\rho}{n}}{\frac{1}{n}} = \rho$$

$$\begin{cases} X = \sigma_X Z_1 + \mu_X \\ Y = \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2) + \mu_Y \end{cases}, \text{ gdzie } Z_1, Z_2 \sim N(0,1)$$

$$Y = \sigma_Y\left(\rho \frac{X-\mu_X}{\sigma_X} + \sqrt{1-\rho^2}\, Z_2\right) + \mu_Y$$

$$Y = \rho(X-\mu_X) + \sqrt{\frac{1-\rho^2}{n}}\, Z_2 + \mu_Y$$

$$\sqrt{\frac{1-\rho^2}{n}} = 0$$

$$1-\rho^2 = 0$$

$$\rho^2 = 1$$

$$\rho = 1 \lor \rho = -1$$

Dla $\rho = -1$

$$Y = -X + \mu_X + \mu_Y$$

przy założeniu, że $Y > X$ mamy $\mu_Y > \mu_X$ stąd $\mu_X + \mu_Y > 0$ czyli

$Y = \mu_X + \mu_Y - X < X$, nie spełnia warunków.

Dla $\rho = 1$

$$Y = X - \mu_X + \mu_Y$$

przy założeniu, że $Y > X$ mamy $\mu_Y > \mu_X$ stąd $\mu_Y - \mu_X > 0$ czyli

$Y = X + \mu_Y - \mu_X > X$.

Zad. 4

$X \sim LN(a, b)$

$a = \mu$

$b = 1$

$EX = \exp\{\mu + \frac{1}{2}\}$

$$\begin{cases} P(X \le q) = \Phi(\ln(q) - \mu) = 0{,}6 \\ P(X \le r) = \Phi(\ln(r) - \mu) = 0{,}4 \end{cases}$$

$$\begin{cases} \ln(q) = \Phi^{-1}(0{,}6) + \mu \\ \ln(r) = \Phi^{-1}(0{,}4) + \mu \quad \Big| + \end{cases}$$

$$\ln(qr) = 2\mu + 0{,}253 - 0{,}253$$

$$\mu = \frac{\ln(q\nu)}{2}$$

$$EX = \exp\left\{\frac{\ln(q\nu)}{2} + \frac{1}{2}\right\} = \exp\left\{\ln(q\nu)^{\frac{1}{2}}\right\} \exp\left\{\frac{1}{2}\right\} = \sqrt{q\nu e}$$

zad. 5

$X_i \sim Exp(1)$

$P(N=k) = (1-q)q^k \quad k = 0, 1, \ldots$

$$S = \sum_{i=1}^{N} X_i$$

$S \mid N = n \sim \text{Gamma}(n, 1)$

$$P(N=n \mid S=s) = c \cdot f(S=s \mid N=n) P(N=n) = c \cdot \frac{1}{\Gamma(n)} s^{n-1} e^{-s} (1-q) q^n =$$

$$= c \cdot \frac{1}{(n-1)!} s^{n-1} q^n = c \cdot \frac{(sq)^{n-1}}{(n-1)!}$$

Funkcja p-stwa wygląda jak przesunięty rozkład Poiss(1) czyli mamy zm. los. $Y = X+1$ gdzie $X \sim Poiss(1)$

$E[N \mid S=s] = sq + 1$

$Var(N \mid S=s) = sq$

$Var(N \mid S=s) = E[N \mid S=s] - 1$

zad. 6

$$P(X_1^2 - 5X_2^2 < 5X_3^2 - X_4^2) = P(X_1^2 + X_4^2 < 5X_3^2 + 5X_2^2) =$$

$$= P\left(\frac{X_1^2 + X_4^2}{X_2^2 + X_3^2} < 5\right) = \left| \begin{array}{l} X_i \sim N(0, \sigma^2) \\ \frac{X_i}{\sigma} \sim N(0,1) \end{array} \right| = P\left(\frac{\sigma^2\left(\frac{X_1^2}{\sigma^2} + \frac{X_4^2}{\sigma^2}\right)}{\sigma^2\left(\frac{X_2^2}{\sigma^2} + \frac{X_3^2}{\sigma^2}\right)} < 5\right) =$$

$$= \left| \frac{X_1^2}{\sigma^2} \sim \chi^2(1) ; \ \frac{X_1^2}{\sigma^2} + \frac{X_4^2}{\sigma^2} = Y_1 \sim \chi^2(2) \right| = P\left(\frac{Y_1}{Y_2} < 5\right) =$$

$$= \left| \frac{Y_1/2}{Y_2/2} = W \sim F(2,2) \right|$$

Zm. los. W ma rozkład F(2,2), gęstość:

$$f_W(x) = \frac{\sqrt{\frac{(2x)^2\, 2^2}{(2x+2)^4}}}{x\, B(1,1)} = \frac{\frac{4x}{(2x+2)^2}}{x} = \frac{4}{4(x+1)^2} = \frac{1}{(x+1)^2}$$

$$P(W<5) = \int_0^5 \frac{1}{(x+1)^2}\, dx = \left| \begin{matrix} t = x+1 \\ dt = dx \end{matrix} \right| = \int_1^6 t^{-2}\, dt = -\frac{1}{t}\Big|_1^6 = -\frac{1}{6} + 1 = \frac{5}{6}$$

Zad. 7

$X_i \sim \exp(\frac{1}{\mu})$

$$L = \prod_{i=1}^{20} \frac{1}{\mu} e^{-\frac{x_i}{\mu}} \cdot \left(e^{-\frac{3}{\mu}}\right)^{20} = \frac{1}{\mu^{20}} e^{-\frac{1}{\mu}\sum_{i=1}^{20} x_i} \cdot e^{-\frac{60}{\mu}} =$$

$$= \frac{1}{\mu^{20}} e^{-\frac{160}{\mu} - \frac{60}{\mu}} = \frac{1}{\mu^{20}} e^{-\frac{220}{\mu}}$$

$$\ln L = \ln\left(\frac{1}{\mu^{20}}\right) - \frac{220}{\mu}$$

$$\ln' L = \frac{-20\mu^{-21}}{\mu^{-20}} + \frac{220}{\mu^2} = 0$$

$$\frac{220}{\mu^2} = \frac{20}{\mu} \quad \Big| \cdot \mu^2$$

$$220 = 20\mu$$

$$\hat{\mu} = \frac{11}{4}$$

Zad. 8

$p_2 > p_1$

$$\frac{\left(\frac{\lambda^{p_2}}{\Gamma(p_2)}\right)^n \left(\prod_{i=1}^n x_i\right)^{p_2-1} e^{-\lambda \sum_{i=1}^n x_i}}{\left(\frac{\lambda^{p_1}}{\Gamma(p_1)}\right)^n \left(\prod_{i=1}^n x_i\right)^{p_1-1} e^{-\lambda \sum_{i=1}^n x_i}} = \left(\frac{\lambda^{p_2}}{\lambda^{p_1}}\right)^n \left(\frac{\Gamma(p_1)}{\Gamma(p_2)}\right)^n \left(\prod_{i=1}^n x_i\right)^{p_2-p_1}$$

funkcja rosnąca statystyki $\prod_{i=1}^n X_i$

$$\prod_{i=1}^n X_i > k \Rightarrow \ln \prod_{i=1}^n X_i > \ln k \Rightarrow \sum_{i=1}^n \ln X_i > k$$

$$\ln X_1 + \dots + \ln X_n > k$$

$$P\left(\sum_{i=1}^n \ln X_i > k\right) = \alpha$$

$$P(\ln X < t) = P(X < e^t) = \int_0^{e^t} \lambda^2 x e^{-\lambda x} = 1 - e^{-\lambda e^t} - \lambda e^t e^{-\lambda e^t}$$

$f_{\ln(x)} = \lambda e^{2t} e^{-\lambda e^t}$ gęstość zależy od $\lambda$ więc suma też czyli zależy od parametru $\lambda$ i liczby $\alpha$

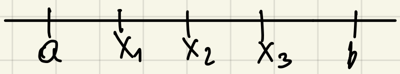
Zad. 9

$X_i \sim U(a, b) \qquad 0 \leq a < b$

$H_0: \quad a = 0$

$H_1: \quad a > 0$

$f(x) = \frac{1}{b-a}, \quad a < x < b$




$f(x_1, \dots, x_n) = \left(\frac{1}{b-a}\right)^n \qquad \begin{matrix} b \geq \max(x_i) \\ a \leq \min(x_i) \end{matrix}$

$M = \max(X_1, \dots, X_n)$

$m = \min(X_1, \dots, X_n)$

$$\frac{\sup(a;X)}{\sup(b;X)} = \frac{\left(\frac{1}{M-m}\right)^n}{\left(\frac{1}{M}\right)^n} = \left(\frac{M}{M-m}\right)^n > k$$

Wszystkie wyrazy, które nie zawierają informacji o $X_i$ wrzucam do stałej $k$.

$$\frac{M}{M-m} > k \Rightarrow \frac{M-m}{M} > \frac{1}{k} \Rightarrow 1 - \frac{m}{M} > \frac{1}{k}$$

$$\frac{m}{M} > k$$

Zad. 10

Trzeba obliczyć $P^{100}$:

$$P^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0{,}5 & 0{,}2 & 0 & 0{,}3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0{,}5 & 0{,}2 & 0 & 0{,}3 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0{,}2 & 0{,}3 & 0 & 0{,}5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0{,}2 & 0{,}3 & 0 & 0{,}5 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0{,}5 & 0{,}2 & 0 & 0{,}3 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0{,}3 & 0{,}5 & 0 & 0{,}2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^4 = P^3 \cdot P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0{,}5 & 0{,}2 & 0 & 0{,}3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P^4 = P$$

$$P^{100} = P$$

$$\begin{bmatrix} 0{,}5 & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0{,}5 & 0{,}2 & 0 & 0{,}3 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0{,}25 & 0{,}1 & 0 & 0{,}65 \end{bmatrix}$$

Odp. 0,65