The Central Limit Theorem and the Law of Large Numbers

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Moment Generating Functions

Definition

Let Y be a continuous random variable. Then we define the *kth moment about the origin* of Y to be the quantity

$$\mu'_k = E\left[Y^k\right] \text{ for } k = 1, 2, \ldots$$

We define the *kth moment about the mean*, or the *kth central moment* of Y to be the quantity

$$\mu_k = E\left[(Y - \mu)^k\right] \text{ for } k = 1, 2, \ldots$$

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Let Y be a random variable. Then we define the *moment generating function*, $m_Y(t)$, of Y to be the quantity

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Moment Generating Functions

Theorem

Let Y and $Y_1, Y_2, \ldots$ be random variables with moment generating fucntions $m(t)$ and $m_1(t), m_2, (t), \ldots$, respectively. If

$$\lim_{n \to \infty} m_n(t) = m(t) \text{ for all } t \in \mathbb{R}$$

then the distribution funcion of Y_n converges to the distribuition function of Y as $n \to \infty$.

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The Central Limit Theorem

Theorem

Let $Y_1, \ldots, Y_n$ be independent, identically distributed random variables with $E[Y_i] = \mu$ and $Var(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n \to \infty} P(U_n \leq u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt \text{ for all } u.$$

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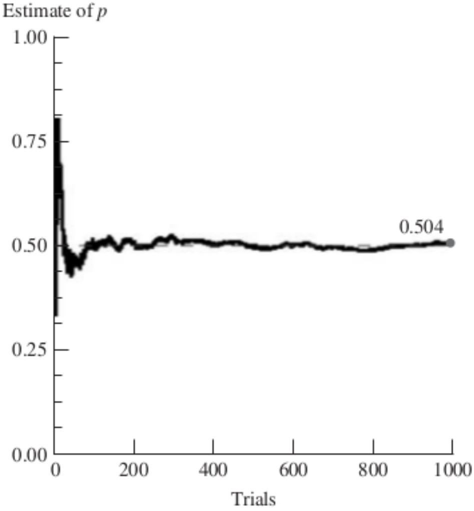
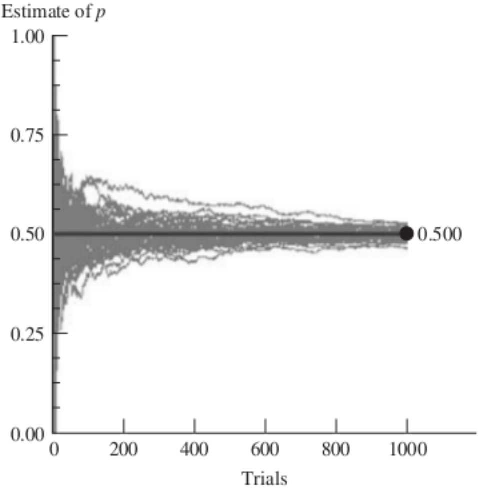
The Law of Large Numbers

FIGURE **9.1**
Values of $\hat{p} = Y/n$ for a single sequence of 1000 Bernoulli trials, $p = 0.5$



The Law of Large Numbers

FIGURE 9.2
Values of $\hat{p} = Y/n$ for 50 sequences of 1000 Bernoulli trials, $p = 0.5$



The Law of Large Numbers

1. The previous two graphs suggest that for any $\varepsilon > 0$ there is a large n so that
$$P\left(\left|\frac{Y}{n} - p\right| \leq \varepsilon\right)$$
should be close to 1.

Definition

The estimator $\hat{\theta}_n$ is said to be a *consistent estimator* of θ if, for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \leq \varepsilon) = 1$$

or equivalently

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| > \varepsilon) = 0$$

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An unbiased estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if

$$\lim_{n\to\infty} V(\hat{\theta}_n) = 0$$

Theorem

The Law of Large Numbers: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a distribution with mean μ and variance $\sigma^2 < \infty$. Then

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

is a consistent estimator of μ, i.e.

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