Assignment

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1 Probability

2 Assignment-2

(**Answer-1**) Let X and Y are discrete. Then,

$$\begin{aligned}
E[X+Y] &= \sum_{x\in X}\sum_{y\in Y}(x+y)p_{X,Y}(x,y) \\
&= \sum_{x\in X}\sum_{y\in Y}xp_{X,Y}(x,y) + \sum_{x\in X}\sum_{y\in Y}yp_{X,Y}(x,y) \\
&= \sum_{x\in X}x\sum_{y\in Y}p_{X,Y}(x,y) + \sum_{y\in Y}y\sum_{x\in X}p_{X,Y}(x,y) \\
&= \sum_{x\in X}xp_X(x) + \sum_{y\in Y}yp_Y(y) \\
&= E[X] + E[Y].
\end{aligned}$$

Similarly, we can show $E[X+Y] = E[X] + E[Y]$, when X and Y are continuous.

(**Answer-2**) Variance of a random variable X is defined as

$$Var(X) = E[(X - E[X])^2].$$

Therefore,

$$\begin{aligned}
Var(X+Y) &= E[(X+Y-E[X+Y])^2] \\
&= E[(X+Y-E[X+Y])^2] \\
&= E[((X-E[X]) + (Y-E[Y]))^2] \\
&= Var(X) + Var(Y) + 2E[(X-E[X])(Y-E[Y])] \\
&= Var(X) + Var(Y) + 2(E[XY] - E[X]E[Y]) \\
&= Var(X) + Var(Y) + 2Cov(X,Y).
\end{aligned}$$

(**Answer-3**) $Cov(X,Y) = 0$ implies

$$E[XY] = E[X]E[Y]$$

This implies X and Y are linearly independent but not necessarily functionally independent.

Let's take a closer look:

$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$, therefore, if $Cov(X,Y) > 0$, we expect that both variables X and Y increase or decrease together (since the product $(X - E[X])(Y - E[Y]) > 0$). Similarly, if the covariance is negative, we expect that if one variable increases other decreases. In both the above cases, X and Y has either positive or negative linear relationship. But if $Cov(X,Y) = 0$, there is no linear relationship between X and Y. Although, this does not mean that there are no non-linear relationship either between X and Y. Let us consider the following example:

For $\theta \in [0, 2\pi]$, let us take a random variable X and Y defined as

$$X = Sin(\theta)$$

$$Y = Cos(\theta).$$

Note that $E[X] = 0$ and $E[Y] = 0$. Also $Sin(\theta)Cos(\theta) = \frac{1}{2}Sin(2\theta)$, therefore $E[XY] = 0$ and hence $Cov(X,Y) = 0$. But $X = \sqrt{1 - Y^2}$. This shows that X and Y are not functionally independent.

(**Answer-4**) Let $X_1, X_2, ..., X_n$ are *i.i.d.* random variable each having mean $E[X_i] = \mu$. Weak law of large numbers states that

$$\lim_{n \mapsto \infty} P\left(\left|\frac{\sum_i X_i}{n} - \mu\right| > \varepsilon\right) = 0.$$

Chebyshev's Inequality: Let X be a random variable with finite mean μ and finite variance σ^2 and let $k > 0$ be a positive integer. Then

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Let

$$\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then,

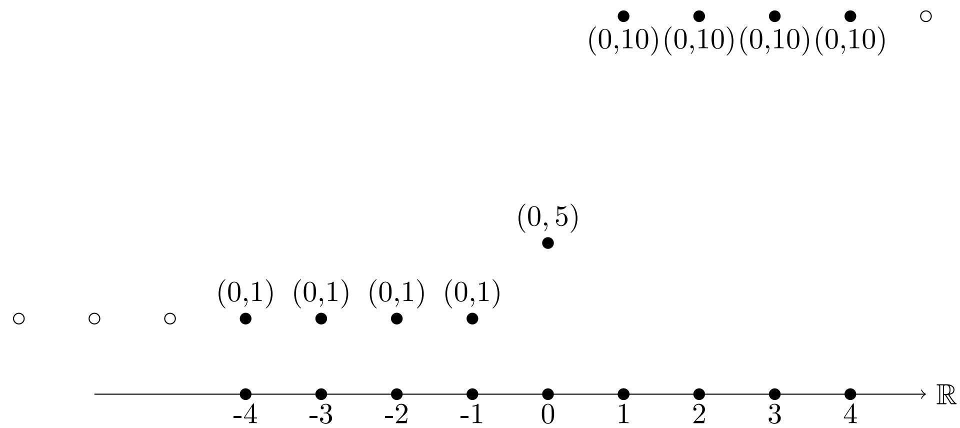
$$E[\bar{X}] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}n\mu = \mu$$

$$Var(\bar{X}) = \sum_{i=1}^{n} \frac{1}{n^2} Var(X_i) = \frac{\sigma^2}{n}.$$

Now, the Chebyshev's inequality for the random variable $\bar{X}$ gives:

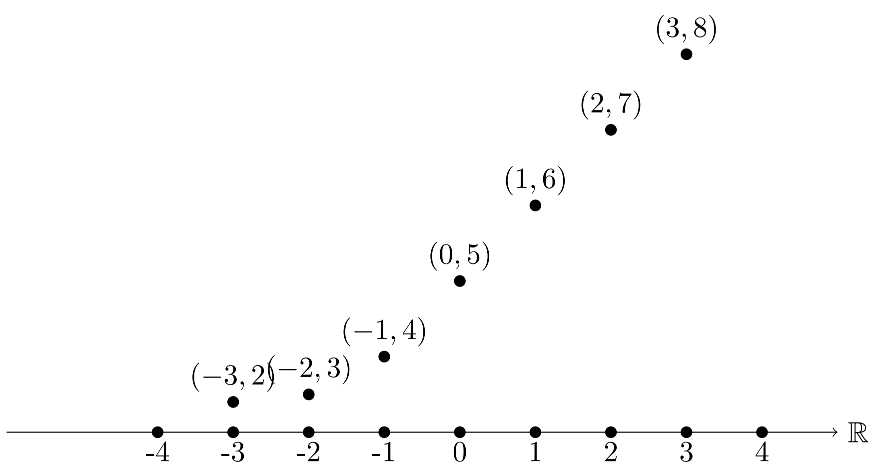
$$\begin{aligned} P(|\bar{X}-\mu| \geq \varepsilon) &\leq \frac{\sigma^2}{n\varepsilon^2} \\ \Rightarrow \lim_{n \mapsto \infty} P(|\bar{X}-\mu| \geq \varepsilon) &= 0. \qquad (\text{Since } \sigma^2 < \infty) \end{aligned}$$

(**Answer-5**) Let $X_1, ..., X_n$ are *i.i.d* random variable with distribution as shown in the following figure.



It is clear that the the distribution has finite mean (approximately 5), but the variance is infinite. Actually not, the above also has finite variance. We need to modify a bit:

The following distribution has mean 5 and infinite variance. Note that, by making small adjustment to the graph, it can have infinite mean as well.



Let X_i has distribution as above, if we randomly choose numbers from X_i, it is clear that the sum $\frac{1}{n}\sum_i X_i$ will not converge to $\mu = 5$. This illustrate that the law of large number does not hold, in general, if we have infinite variance.

(**Answer-6 & 7**) Let X be a random variable the Moment generating function, denoted ϕ_X and Characteristic function, denoted Φ are defined as follows:

$$\phi_X(t) := E[\exp^{tX}]$$
$$\Phi_X(t) := E[\exp^{itX}]$$

Note that $\phi_X(t)$ exist only if Taylor series expansion of $\exp^{tX}$ around $t = 0$ exist, whereas $\Phi_X(t)$ always exist.

Now, let $X_1, ..., X_n$ are *i.i.d* random variable with mean μ and variance σ. The *Central Limit Theorem* states

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \sim N(0,1) =: Y$$

Assuming that $\phi_{\bar{X}}(t)$ exist, we will show it to be equal to $\phi_Y(t) = E[\exp^{tY}] = \frac{t^2}{2}$.

Now,

$$\begin{aligned}
\phi_{\bar{X}}(t) &= E[\exp^{t\sum_{i=1}^{n}\frac{(X_i-\mu)}{\sigma\sqrt{n}}}] \\
&= E[\prod e^{t\frac{(X_i-\mu)}{\sigma\sqrt{n}}}] \\
&= \prod_{i=1}^{n} E[e^{t\frac{(X_i-\mu)}{\sigma\sqrt{n}}}] \\
&= \left(E[e^{t\frac{(X_1-\mu)}{\sigma\sqrt{n}}}]\right)^n \\
&= \left(1 + t\frac{(E[X_1]-\mu)}{\sigma\sqrt{n}} + t^2\frac{E[(X_1-\mu)^2]}{2(\sigma\sqrt{n})^2} + \ldots\right)^n \\
&\approx \left(1 + t^2\frac{\sigma^2}{2(\sigma\sqrt{n})^2}\right)^n \\
&= \left(1 + \frac{t^2}{2n}\right)^n
\end{aligned}$$

Taking $n \mapsto \infty$, we get $\exp^{t^2/2}$.

If we would have used $\Phi_{\bar{X}}(t)$ instead, we would have got $\left(1 - \frac{t^2}{2n}\right)^n$ and further taking $n \mapsto \infty$, we get the limit to be $\exp^{-t^2/2}$ which is $\Phi_Y(t)$.