Assignment

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1 Probability

2 Assignment-3

(**Answer-1**) Let $\langle S^N \rangle = \frac{b-a}{N} \sum\limits_{i=0}^{N} f(x_i)$, where x_i are randomly chosen points from the interval $[a, b]$. Let X be a uniform distributed random variable on $[a, b]$. The expected value of $\langle S^N \rangle$ is as follows:

$$\begin{aligned} E(\langle S^N \rangle) &= E(\frac{b-a}{N} \sum_{i=0}^{N} f(x_i)) \\ &= \frac{b-a}{N} \sum_{i=0}^{N} E(f(x_i)) \\ &= \frac{1}{N} \sum_{i=0}^{N} \int_a^b f(x)dx \end{aligned}$$

By law of large number we have

$$P\left(\lim_{N \mapsto \infty} \langle S^N \rangle = \int_a^b f(x)dx \right) = 1.$$

(**Answer-2**)

Let $f(x) = \frac{4}{1+x^2}$ and $\langle S^N \rangle = E(\frac{1}{N} \sum\limits_{i=0}^{N} f(x_i))$. Then as seen in the above exercise,

$$E(\langle S^N \rangle) = \int_0^1 f(x)dx.$$

The error in the estimation is given by the standard deviation σ, computed as follows:

$$\begin{aligned}
\sigma^2 &= \sigma^2\left(\frac{1}{N}\sum_{i=0}^{N} f(x_i)\right) \\
&= \frac{1}{N^2}\sum_{i=0}^{N}\sigma^2(f(x_i)) \\
&= \frac{1}{N^2}\sum_{i=0}^{N}(E[f(x)^2]-E[f(x)]^2) \\
&= \frac{E[f(x)^2]-E[f(x)]^2}{N} \\
\Rightarrow \sigma &= \frac{\sqrt{E[f(x)^2]-E[f(x)]^2}}{\sqrt{N}}.
\end{aligned}$$

(**Answer-5**)

- Let $dX = \mu dt + \sigma dB(t)$ where $B(t)$ is the Canonical Brownian Motion.

  (a) Let $f(X_t, t) = X_t^2$. In this case, we have $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial X_t} = 2X_t$, and $\frac{\partial^2 f}{\partial X_t^2} = 2$. Thus, by Ito's lemma, we obtain

  $$\begin{aligned}
  df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 dt \\
  &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\mu\right)dt + \frac{\partial f}{\partial X_t}\sigma dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 dt \\
  &= (2X_t\mu + \sigma^2)dt + 2X_t\sigma dB(t).
  \end{aligned}$$

  (b) Let $f(X_t, t) = X_t^3$. In this case, we have $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial X_t} = 3X_t^2$, and $\frac{\partial^2 f}{\partial X_t^2} = 6X_t$. Thus, by Ito's lemma, we obtain

  $$\begin{aligned}
  df &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\mu\right)dt + \frac{\partial f}{\partial X_t}\sigma dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 dt \\
  &= 3X_t(X_t\mu + \sigma^2)dt + 3X_t^2\sigma dB(t).
  \end{aligned}$$

  (c) Let $f(X_t, t) = log(X_t)$. Then $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial X_t} = \frac{1}{X_t}$, and $\frac{\partial^2 f}{\partial X_t^2} = \frac{-1}{X_t^2}$. Thus, by Ito's lemma, we obtain

  $$\begin{aligned}
  df &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\mu\right)dt + \frac{\partial f}{\partial X_t}\sigma dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 dt \\
  &= \frac{1}{X_t}\left(\mu - \frac{1}{2X_t}\sigma^2\right)dt + \frac{1}{X_t}\sigma dB(t).
  \end{aligned}$$

- Let $\frac{dS}{S} = \mu dt + \sigma dB(t)$ where $B(t)$ is the Canonical Brownian Motion. For $f(X_t, t) \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$, Ito's lemma states

$$\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}dX_t^2 \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}(\mu X_t dt + \sigma X_t dB(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(\mu X_t dt + \sigma X_t dB(t))^2 \\
&= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}(\mu X_t dt + \sigma X_t dB(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 X_t^2 dt \\
&= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_t}\mu X_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 X_t^2\right)dt + \frac{\partial f}{\partial X_t}\sigma X_t dB(t)
\end{aligned}$$

(a) Let $f(S_t, t) = S_t^2$. In this case, we have $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial S_t} = 2S_t$, and $\frac{\partial^2 f}{\partial S_t^2} = 2$. Substituting these values in the above equation we get

$$\begin{aligned}
df &= (2\mu S_t^2 + \sigma^2 S_t^2)dt + 2\sigma S_t^2 dB(t) \\
&= S_t^2((2\mu + \sigma^2)dt + 2\sigma dB(t)).
\end{aligned}$$

3 Assignment-4

(**Answer-1**)

Let $M(t)$ denote the amount of money at time $t \in [0, T]$ after continuously compounding at the rate of $r(t)$. Then it will follow the following differential equation:

$$\frac{dM(t)}{dt} = r(t)M(t).$$

Solving this we get

$$\begin{aligned}
\frac{dM(t)}{M(t)} &= r(t)dt \\
\int_{M}^{M(T)} \frac{dM(t)}{M(t)} &= \int_0^T r(t)dt \\
\ln\left(\frac{M(T)}{M}\right) &= \int_0^T r(t)dt \\
M(T) &= Me^{\int_0^T r(t)dt}.
\end{aligned}$$