Math 3551 - Fall 2024
Sylvan Franklin - Homework 7

1. Let G_1 and G_2 be groups, $\varphi : G_1 \to G_2$ be a homomorphism, and H be any subgroup of G_2. Define:

$$\varphi^{-1}(H) = \{g \in G_1 : \varphi(g) \in H\}$$

   (a) Prove that $\varphi^{-1}(H)$ is a subgroup of G_1.

   Proof: Non empty: Since H is a subgroup it contains the indentity, and since φ is a homomorphism and $\varphi\left(e_{G_1}\right) = e_{G_2}$, the identity is in $\varphi^{-1}(H)$. Closure: take any $a, b \in \varphi^{-1}(H)$, now $\varphi(a) \in H$ and $\varphi(b) \in H$, Since H is a group $\varphi(a)\varphi(b) \in H$. Since φ is a homomorphism we have $\varphi(ab) \in H$, which implies $ab \in \varphi^{-1}(H)$. A similar argument holds for the inverse: for $g \in \varphi^{-1}(H)$, there is $\varphi(g) \in H$ and since H is a group and φ is a homomorphism, there exists an inverse and the means to map it back $\varphi(g) \in H \iff \varphi(g)^{-1} \in H \implies \varphi(g^{-1})$ which by definition means that $g^{-1} \in \varphi^{-1}(H)$

   $\square$

   (b) Prove that if H is normal in G_2, then $\varphi^{-1}(H)$ is normal in G_1.

   Proof: $g_2 h g_2^{-1} \in H \implies g_1 a g_2 \in G_2$ forall $g_1, g_2, h, a \in G_1, G_2, H, \varphi^{-1}(H)$, so now we have $g_2 h g_2^{-1} \in H \implies \varphi^{-1}(g_2 h g_2^{-1}) \in G_1 \implies \varphi^{-1}(g_2)\varphi^{-1}(h)\varphi^{-1}(g_2^{-1}) \in G$

   $\square$

2. Let H be a subgroup of G, and define a relation $\sim$ on G by $a \sim b$ if and only if $b^{-1}a \in H$

   (a) Prove that $\sim$ is an equivalence relation.

   Proof: Reflexive: $g^{-1}g \in H$, since $g^{-1}g = e \in H$, Symmetric: $b^{-1}a \in H$, since H is a subgroup, it has both inverses, and all compositions of all elements, therefore $a^{-1}b \in H$. Transitivity: Given $a \sim b : b^{-1}a \in H$ and $b \sim c : c^{-1}b \in H$. Want to show $a \sim c : c^{-1}a \in H$. Since H is closed $b^{-1}, a, c^{-1}, b \in H \implies c^{-1}a \in H$

   $\square$

   (b) Fix $g \in G$, and describe the equivalence class of this relation. We have defined this set in class, what was it called?

   Rearranging $b^{-1}a \in H$ you get $a \in bH$, or the left coset.

3. Let H be a subgroup of G **The right coset of** $H \in G$ **represented by** g, is the set:

$$Hg = \{hg : h \in H\}$$

   and the set of right cosets of $H \in G$ is denoted $H \setminus G$.

   (a) For any right coset of Hg_1 of H, show that the map $x \to x^{-1}$ is injective from Hg_1 onto some left coset $g_2 H$ of H.

   Proof: If you have some hg, in the right coset, and you invert it $((hg))^{-1}$, that would give you $(g)^{-1}(h)^{-1}$, which is definitionally some left coset, with $g_2 = (g)^{-1}$ and $(h)^{-1} \in H$ since H is closed under the inverse. Also $(a)^{-1} = (b)^{-1} \implies a = b$, since a and b are in the right coset, which means that they are both related by some g, which can be expressed ag and bg, now by inverting these we get $((a, g))^{-1} = (g)^{-1}(a)^{-1}$ and $((b, g))^{-1} = (g)^{-1}(b)^{-1}$, now we have that $((a, g))^{-1} = ((b, g))^{-1} \implies (g)^{-1}(a)^{-1} = (g)^{-1} = (b)^{-1}$, and by left cancelation $a = b$.

$\square$

(b) Show that the map $x \to x^{-1}$ induces a bijection between the sets $G \setminus H$ and $H \setminus G$

Proof: Simliar to the last proof, proving both that:

$G \setminus H \mapsto H \setminus G$:

For any $(ga), (g, b) \in G \setminus H \left[((ga))^{-1} = ((gb) \Longrightarrow ga = gb)^{-1}\right]$ $((ga))^{-1} = (a)^{-1}(g)^{-1}$, and also $((gb))^{-1} = (b)^{-1}(g)^{-1}$, and by cancellation, we get $a = b$

A similar argument holds for $H \setminus G \mapsto G \setminus H$, using the fact that H is a subgroup, which means that it is closed, and in particular closed under inverses.

$\square$