Model Formulation suited for semi-Lagrangian integration

We describe in some detail the reformulation of the Global Eulerian Spectral model of CPTEC. This new formulation is suited for the implementation of a semi-Lagrangian scheme. Moreover, it already includes several other benefits, as a potential increase in efficiency, due to the reduction on the number of Legendre Transforms involved over a time-step. We will also change the horizontal diffusion process to an implicit method, what possibly will provide better stability properties at higher resolutions. We will start the description of the new formulation by the Primitive Equations in the form we will employ. The main modification with respect to the current CPTEC's model is in the formulation of the momentum equations, to which we give a U, V formulation, instead of a Vorticity-Divergence form. Moreover, we will make the advection terms explicit in the equations, leaving the model ready to a Lagrangian treatment of advection, replacing the actual Eulerian formulation (which, however, shall remain as an option in the code). The equations are described in the next section.

0.1 Primitive Equations

The current Eulerian model is based on the following formulation of the momentum equations:

$$\frac{\partial U}{\partial t} - (\xi + f)V + \dot{\sigma}\frac{\partial U}{\partial \sigma} + \frac{1}{a}\left(\frac{\partial(\Phi + E)}{\partial \lambda} + RT\frac{\partial \ln ps}{\partial \lambda}\right) = F_u \tag{1}$$

$$\frac{\partial V}{\partial t} + (\xi + f)U + \dot{\sigma}\frac{\partial V}{\partial \sigma} + \frac{\cos\varphi}{a}\left(\frac{\partial(\Phi + E)}{\partial \varphi} + RT\frac{\partial \ln ps}{\partial \varphi}\right) = F_v \tag{2}$$

where $E = (U^2 + V^2)/(2\cos^2\varphi)$. This formulation is not suited for a semi-Lagrangian scheme, since the advection terms are mixed with other contributions. We will rather adopt the following form of the momentum equations, in which the advection terms are isolated:

$$\frac{\partial U}{\partial t} + \frac{1}{a\cos^2\varphi}\left(U\frac{\partial U}{\partial \lambda} + V\cos\varphi\frac{\partial U}{\partial \varphi}\right) + \dot{\sigma}\frac{\partial U}{\partial \sigma} - fV + \frac{1}{a}\left(\frac{\partial \Phi}{\partial \lambda} + RT\frac{\partial \ln ps}{\partial \lambda}\right) = F_u \tag{3}$$

$$\frac{\partial V}{\partial t}+\frac{1}{a\cos^2\varphi}(U\frac{\partial V}{\partial\lambda}+V\cos\varphi\frac{\partial V}{\partial\varphi})+\dot{\sigma}\frac{\partial V}{\partial\sigma}+fU+\frac{\cos\varphi}{a}(\frac{\partial\Phi}{\partial\varphi}+RT\frac{\partial\ln ps}{\partial\varphi})+\frac{\sin\varphi}{a\cos^2\varphi}(U^2+V^2)=F_v \quad (4)$$

Thermodynamic equation:

$$\frac{\partial T}{\partial t}+\frac{1}{a\cos^2\varphi}(U\frac{\partial T}{\partial\lambda}+V\cos\varphi\frac{\partial T}{\partial\varphi})+\dot{\sigma}\frac{\partial T}{\partial\sigma}-\theta\dot{\sigma}\frac{\partial\Pi}{\partial\sigma}=\kappa T(\frac{\partial}{\partial t}+\vec{V}.\nabla)\ln ps+F_T \quad (5)$$

Moisture equation:

$$\frac{\partial q}{\partial t}+\frac{1}{a\cos^2\varphi}(U\frac{\partial q}{\partial\lambda}+V\cos\varphi\frac{\partial q}{\partial\varphi})+\dot{\sigma}\frac{\partial q}{\partial\sigma}=F_q \quad (6)$$

Surface Pressure equation:

$$\frac{\partial\ln ps}{\partial t}+\int_0^1(\vec{V}.\nabla\ln ps)d\sigma+\int_0^1 D\ d\sigma=0 \quad (7)$$

Hydrostatic equation (diagnostic):

$$\frac{\partial\phi}{\partial\sigma}+\frac{RT}{\sigma}=0 \quad (8)$$

Vertical velocity equation (diagnostic - comes from continuity equation):

$$\sigma\frac{\partial ps}{\partial t}+\int_0^\sigma\nabla.(ps\vec{V})d\sigma=-ps\dot{\sigma} \quad (9)$$

Variables:
$U=u\cos\varphi$, $V=v\cos\varphi$, $\vec{V}=(U,V)$

T - virtual temperature, θ - potential temperature, $T = \Pi\theta$
ps - surface pressure, $\sigma = p/ps$ - vertical coordinate, $\dot{\sigma}$ - vertical velocity, f - Coriolis, q - specific humidity, D - horizontal divergence. F_U, F_V, F_T and F_q are forcing terms due to the physical parameterization processes.

0.2 Prognostic Variables and Spectral discretization

The prognostic variables

- D - Divergence field
- ξ - Vorticity field
- The velocities U and V will be derived from ξ and D.
- T - Virtual temperature
- q - specific humidity
- $\ln ps$ - log of surface pressure (this is a two-dimensional field).

will be stored in spectral space. At each level k, we store the coefficients F_n^m of an expansion like $F(\lambda, \varphi) = \sum_{m=-M}^{M} \sum_{n=|m|}^{M} F_n^m P_n^m(\sin\varphi) e^{im\lambda}$, where we have employed a triangular truncation, for each prognostic field F.

0.3 The main aspects of the reformulation

We present here a general view of the principles of the current Eulerian model and discuss the necessary changes to adapt the model formulation for a semi-Lagrangian treatment of advection. These changes will also lead to a new Eulerian formulation. Let W be the vector of the prognostic variables for the primitive equations and let us write the system of equations as:

$$\frac{\partial W}{\partial t} + L(W) + N(W) = 0 \tag{10}$$

where the differential operator of the system is separeted into an operator N including the non-linearities of the system and a linear operator L. In the current Eulerian model, system (10) is discretized in time as:

$$(I + \Delta t L)(W(t + \Delta t)) = (I - \Delta t L)(W(t - \Delta t)) - 2\Delta t N(W(t)) \tag{11}$$

The non-linear operator $N(W)$ is evaluated with the help of Spectral transforms. It includes the horizontal advection terms $\vec{V}.\nabla F$ (where F is U, V, T or q), which are written as $\nabla.(\vec{V}F) - FD$, where $D = \nabla.\vec{V}$ is the horizontal divergence. The products are computed in grid-point space and the derivatives are obtained through spectral transforms. The solution of the system is entirely done spectrally. (We won't describe all the details here, since the COLA documentation does it to a great extent. We shall however point out were some of the computations are done in the code, when mapping algorithms to routines.)

For the new formulation, the primitive equations are reformulated as already described. System (10) will assume a form like:

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + A(W) = -\tilde{L}(W) - \tilde{N}(W) \tag{12}$$

where we have isolated the advection terms in $A(W)$, $\tilde{L}$ is again a linear part to be treated implicitly and $\tilde{N}$ contains the remaining non-linear contributions. The semi-Lagrangian discretization of (12) will have the form:

$$(I + \Delta t\tilde{L})(W((t + \Delta t)) = (I - \Delta t\tilde{L})(W((t - \Delta t))_* - 2\Delta t\tilde{N}(W(t))_M \tag{13}$$

where $*$ denotes the departure point (at time $t - \Delta t$) of a particle arriving at a Gaussian grid-point at time $t + \Delta t$. The subscript M denotes the mid-point (at time t) of this trajectory. The main difference, apart from the absorption of the advection in the Lagrangian trajectories, is that the operators $\tilde{L}$ and $\tilde{N}$ will have to be completely evaluated in grid-point space at the points determined by the Lagrangian trajectories. For this, these operators will be evaluated on the gaussian grid at the corresponding time instants, and then the values at the trajectory points will be obtained by interpolation. (Notice that we could not talk about trajectory points in spectral space.) Only after completing evaluation of the right-hand-sides of the system (11), including tendencies due to physical parameterizations, we can transform the system to spectral space and solve it there. This new scheme will have an Eulerian discretization as an option. In this case the equations have the form:

$$(I + \Delta t\tilde{L})(W((t + \Delta t)) = (I - \Delta t\tilde{L})(W((t - \Delta t)) - 2\Delta t(A(W(t)) + \tilde{N}(W(t))) \tag{14}$$

We observe that it is very simple to have both options in the code. For the Eulerian version the advection terms are added to the tendencies, and all terms from the right-hand-side are computed at the Gaussian grid, with no need for interpolation. In the semi-Lagrangian version we don't add the advection contributions to the tendencies. Instead, we compute the trajectories and interpolate the contribuitons to the right-hand-sides, computed on the Gaussian grid, to the trajectory points. Besides that, the only other modification is that also the metric term in the momentum equations ($\sin\varphi(U^2+V^2)/a\cos^2\varphi$) is removed from $\tilde{N}$ and absorbed in the semi-Lagrangian treatment of the momentum equations, written in vector form.

0.4 Semi-implicit Eulerian discretization

For the time discretization of the equations we employ a three-time-level semi-implicit scheme. In the following description the upper indices refer to the time-levels. The variable T is split into $T = T' + T_0$, with T_0 constant on each vertical layer. We have:

Momentum Equations:

$$U^{n+1} + \frac{\Delta t}{a}(\frac{\partial \Phi}{\partial \lambda} + RT_0 \frac{\partial \ln ps}{\partial \lambda})^{n+1} = U^{n-1} + 2\Delta_t T_U \qquad (15)$$

where

$$\begin{aligned} T_U &= F_u^n - \frac{1}{a\cos^2\varphi}(U\frac{\partial U}{\partial \lambda} + V\cos\varphi\frac{\partial U}{\partial \varphi})^n - (\dot\sigma\frac{\partial U}{\partial \sigma})^n + fV^n - \frac{1}{a}(RT'\frac{\partial \ln ps}{\partial \lambda})^n \\ &\quad -\frac{1}{2a}(\frac{\partial \Phi}{\partial \lambda} + RT_0\frac{\partial \ln ps}{\partial \lambda})^{n-1} \end{aligned}$$

$$V^{n+1} + \frac{\cos\varphi}{2a}(\frac{\partial \Phi}{\partial \varphi} + RT_0 \frac{\partial \ln ps}{\partial \varphi})^{n+-1} = V^{n-1} + 2\Delta_t T_V \qquad (16)$$

where

$$\begin{aligned} T_V &= F_v^n - \frac{1}{a\cos^2\varphi}(U\frac{\partial V}{\partial \lambda} + V\cos\varphi\frac{\partial V}{\partial \varphi})^n - (\dot\sigma\frac{\partial V}{\partial \sigma})^n - fU^n - \frac{\cos\varphi}{a}(RT'\frac{\partial \ln ps}{\partial \varphi})^n \\ &\quad -\frac{\sin\varphi}{a\cos^2\varphi}(U^2+V^2)^n - \frac{\cos\varphi}{2a}(\frac{\partial \Phi}{\partial \varphi} + RT_0\frac{\partial \ln ps}{\partial \varphi})^{n-1} \end{aligned}$$

Thermodynamic equation:

The treatment of the thermodynamic equation requires a more detailed description, involving the vertical discretization, in order to be able to describe which terms will be discretized implicitly or explicitly. We will leave this description for the tendency calculations part. Here we just say that $\kappa T_0 \int_0^1 D d\sigma$ and part of the vertical advection term will be treated implicitly.

Specific humidity equation:

This equation, describing advection of moisture, is treated explicitly:

$$q^{n+1} = q^{n-1} + 2\Delta t T_q \tag{17}$$

where

$$T_q = F_q^n - \frac{1}{a\cos^2\varphi}(U\frac{\partial q}{\partial \lambda} + V\cos\varphi\frac{\partial q}{\partial \varphi})^n - (\dot{\sigma}\frac{\partial q}{\partial \sigma})^n$$

Surface Pressure equation:

$$\ln ps^{n+1} + \Delta t(\int_0^1 D\ d\sigma)^{n+1} = \ln ps^{n-1} + 2\Delta t T_{ps} \tag{18}$$

where

$$T_{ps} = -(\int_0^1 (\vec{V}.\nabla \ln ps) d\sigma)^n - \frac{1}{2}(\int_0^1 D\ d\sigma)^{n-1}$$

0.5 Semi-implicit Semi-Lagrangian discretization

The semi-implicit method for the semi-Lagrangian scheme follows the same lines of the Eulerian method described in last section. The horizontal and vertical advection terms won't be included in the tendencies, as well as the metric term in the V-momentum equation. The tendencies will be separated in parts related to time step t and $t - \Delta t$, and interpolated to the corresponding trajectory locations (determined in the semi-Lagrangian trajectory computations, to be performed in grid-point space, before or after the tendencies evaluation on the Gaussian grid. The remaining of the algorithm follow exactly the same lines of the Eulerian method we describe in the following sections.

Remark: In the surface pressure equation we have a two dimensional field being advected by the mean wind $\int_0^1 \vec{V} d\sigma$, which should give the Lagrangian trajectory for $\ln ps$. There are alternative treatments in the literature and the best option is still to be investigated.

0.6 Time-step Evolution

A time-step begins with the variables at time t stored in spectral space. At this point we also assume that we have all variables and their necessary derivatives at time $t - \Delta t$ stored in grid-point-space. The time-step will then begin by the evaluation of the necessary variables at time t on the grid through spectral to grid transforms. Then, on the grid, the right-hand-side of the discretized equations will be computed. Also in grid-point-space we will carry out the time filtering process. Finished the grid-point computations we will transform the right-hand-side of the equations to spectral space. In spectral space we will then be able to solve the system of equations, to obtain the new variables at time $t+\Delta t$. This solution process will also entail the semi-implicit computations. After obtaining the updated variables, these will still be modified by horizontal diffusion, and in case of necessity, we apply at the end of the time-step a selective enhanced diffusion, depending on the maximum wind speeds at each layer. In the following sections we will detail the several parts which compose the time-step.

0.7 Spectral-to-grid transforms

The first step is to derive the coefficients of some fields which need to be transformed. This is the case for the velocities U and V, as well as for meridional components of the gradients of T, q and $\ln ps$. Coefficients of U and V are obtained through the relations:

$$U_n^m = \frac{1}{a}\left(im\chi_n^m + (n-1)\varepsilon_n^m\psi_{n-1}^m - (n+2)\varepsilon_{n+1}^m\psi_{n+1}^m\right) \tag{19}$$

$$V_n^m = \frac{1}{a}\left(im\psi_n^m - (n-1)\varepsilon_n^m\chi_{n-1}^m + (n+2)\varepsilon_{n+1}^m\chi_{n+1}^m\right) \tag{20}$$

where $\chi_n^m = -(a^2/n(n+1))D_n^m$, $\psi_n^m = -(a^2/(n(n+1))\xi_n^m$ and $\chi_0^0 = \psi_0^0 = 0$. The coefficients for the meridional component of the gradient of any field F are given by:

$$\left(\cos\varphi\frac{\partial F}{\partial\varphi}\right)_n^m = -(n-1)\varepsilon_n^m F_{n-1}^m + (n+2)\varepsilon_{n+1}^m F_{n+1}^m \tag{21}$$

After this we are ready to perform the Legendre transforms:

$$F^m(\varphi) = \sum_{n=|m|}^{M} F_n^m P_n^m(\sin\varphi) \tag{22}$$

for the following fields: U, V, D, ξ, q, T, $\ln ps$, $\cos\varphi\partial T/\partial\varphi$, $cos\varphi\partial q/\partial\varphi$ and $\cos\varphi\partial\ln ps/\partial\varphi$ (in a total of 8 three-dimensional fields and 2 two-dimensional fields transformed). These are followed by Fourier transforms ($\sum_{m=-M}^{M} F^m(\varphi)e^{im\lambda}$) in order to get the following fields in grid-point-space: U, V, D, ξ, q, T, $\ln ps$, $\cos\varphi\partial T/\partial\varphi$, $\cos\varphi\partial q/\partial\varphi$, $\cos\varphi\partial\ln ps/\partial\varphi$, $\partial U/\partial\lambda$,

$\partial V/\partial\lambda$, $\partial T/\partial\lambda$, $\partial q/\partial\lambda$ and $\partial \ln ps/\partial\lambda$. We point out, that for computing the values of a field F and of its derivative $\partial F/\partial\lambda$, we only need to perform one Legendre transform (getting $F^m(\varphi)$) followed by two Fourier transforms (one for the field itself and one for the λ-derivative - after multiplying $F^m(\varphi)$ by im). In the total we have Fourier transforms of 12 three-dimensional and of 2 two-dimensional fields. (In the semi-Lagrangian scheme we don't need to transform ξ, $\cos\varphi\partial q/\partial\varphi$, $\partial U/\partial\lambda$, $\partial V/\partial\lambda$ and $\partial q/\partial\lambda$. This amounts to saving two 3-D Legendre transforms and 5 3-D Fourier transforms.)

0.8 Computations in grid-point-space

Complete wind gradients (only for Eulerian scheme: We only compute the zonal derivatives of the winds through spectral transforms. In order to get the meridional derivatives, instead of performing two more Legendre transforms, followed by the corresponding Fourier transforms, we will employ the following relations (on the grid):

$$\cos\varphi\frac{\partial U}{\partial\varphi} = \frac{\partial V}{\partial\lambda} - a\cos^2\varphi\xi \tag{23}$$

$$\cos\varphi\frac{\partial V}{\partial\varphi} = -\frac{\partial U}{\partial\lambda} + a\cos^2\varphi D \tag{24}$$

Vertical integration of the Divergence:

$$\int_0^1 D d\sigma = \int_0^1 D d\hat{\sigma} = \sum_{l=1}^{K} D_l \Delta_l \tag{25}$$

where $\Delta_l = \Delta\hat{\sigma}_l$, and $\hat{\sigma}_l = 1 - \sigma_l$. We have K vertical layers, on which the prognostic variables are defined. The variable $\sigma = p/ps$ varies from $\sigma_1 = 1$ at the surface, until $\sigma_{K+1} = 0$ on the top of the atmosphere. A layer k is delimited by $\hat{\sigma}_k$ and $\hat{\sigma}_{k+1}$ and the width of the layer is $\Delta\hat{\sigma}_k = \hat{\sigma}_{k+1} - \hat{\sigma}_k$.

Vertical integration of "advection" of $\ln ps$:

$$\int_0^1 \vec{V}.\nabla \ln ps d\sigma = \sum_{l=1}^{K} (\vec{V}.\nabla \ln ps)_l \Delta_l \tag{26}$$

Computation of $\dot{\hat{\sigma}}$: For the computation of $\dot{\hat{\sigma}}$ we employ the continuity equation and integrate it to get:

$$\hat{\sigma}\partial \ln ps/\partial t + \int_0^{\hat{\sigma}} (D + \vec{V}.\nabla \ln ps) d\hat{\sigma} + \dot{\hat{\sigma}} = 0$$

Using then the surface pressure equation (7) we obtain that

$$\dot{\hat{\sigma}}_k = \hat{\sigma}_k \int_0^1 (D + \vec{V}.\nabla \ln ps) d\hat{\sigma} - \int_0^{\hat{\sigma}_k} (D + \vec{V}.\nabla \ln ps) d\hat{\sigma}$$

From this equation, and using that $\dot{\hat{\sigma}}_1 = \dot{\hat{\sigma}}_{K+1} = 0$ we can compute $\dot{\hat{\sigma}}$ on all levels, as:

$$\dot{\hat{\sigma}}_{l+1} = \dot{\hat{\sigma}}_l + \Delta_l \sum_{j=1}^{K} (D_j + \vec{V}_j.\nabla \ln ps)\Delta_j - \Delta_l (D_l + \vec{V}_l.\nabla \ln ps) \qquad (27)$$

Computation of $\omega = dp/dt$ The vertical velocity ω will be needed in the physics. It's computation will be based on the equation

$$\omega = ps(\dot{\sigma} + \sigma(\partial \ln ps/\partial t + \vec{V}.\nabla \ln ps))$$

whose discretization takes the form:

$$\omega_k = ps_k + (\frac{\dot{\hat{\sigma}}_{k+1} + \dot{\hat{\sigma}}_k}{2} + \hat{\sigma}_k(\vec{V}_k.\nabla \ln ps - \int_0^1 (\vec{V}.\nabla \ln ps + D) d\hat{\sigma})) \qquad (28)$$

Computation of the gradient of the Geopotential: An expression for the geopotential will be derived, with the help of the hydrostatic equation:

$$\partial \phi/\partial \sigma + \alpha ps = 0$$

with $\alpha = 1/\rho = RT/p$. We get that $\partial \phi/\partial \sigma + RT/\sigma = 0$. Therefore:

$$\sigma \partial \phi/\partial \sigma + RT = 0 \qquad (29)$$

or, in an alternative form:

$$\partial \phi/\partial \sigma + c_p \theta \partial \pi/\partial \sigma = 0 \qquad (30)$$

where $T = \pi\theta$, $\pi = (p/p_0)^\kappa$, $\kappa = R/c_p$. We can work out that $\partial \pi/\partial \sigma = \kappa \pi/\sigma$. The geopotential computation from COLA's model is based on Arakawa's discretization, which leads to the equations:

$$\phi_{k-1} - \phi_k = \frac{c_p}{2}\left[T_{k-1}(\frac{\pi_k}{\pi_{k-1}} - 1) + T_k(1 - \frac{\pi_{k-1}}{\pi_k})\right] \ , k = 2, .., K \qquad (31)$$

and

$$\sum_{j=1}^{K} \phi_j \Delta_j = \hat{\phi}_1 + R \sum_{j=1}^{K} T_j \Delta_j \quad , \tag{32}$$

where $\hat{\phi}_1$ is the surface geopotential. Equation (31) may be written as:

$$\frac{\phi_k - \phi_{k-1}}{\Delta_{k-1}} = \frac{c_p}{2}(\frac{T_{k-1}}{\pi_{k-1}} + \frac{T_k}{\pi_k})(\frac{\pi_k - \pi_{k-1}}{\Delta_{k-1}}) = \frac{c_p}{2}(\theta_{k-1} + \theta_k)(\frac{\pi_k - \pi_{k-1}}{\Delta_{k-1}}) \tag{33}$$

which is a second order central-difference form of the hydrostatic equation (30), discretized over a given layer. Equation (32) is a discrete integration of the hydrostatic equation written in form (29), after integration by parts of the first term $(\sigma \partial \phi / \partial \sigma = \partial / \partial \sigma (\sigma \phi) - \phi)$. Equations (31) and (32) lead to a linear system of the form:

$$\begin{bmatrix} 1 & -1 & 0 & . & 0 \\ 0 & 1 & -1 & 0 & . \\ . & 0 & . & . & \\ 0 & . & . & 1 & -1 \\ \Delta_1 & \Delta_2 & . & . & \Delta_K \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ . \\ . \\ \phi_K \end{bmatrix} = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ \hat{\phi}_1 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 & 0 & . & 0 \\ 0 & a_2 & b_2 & 0 & . \\ . & . & . & . & . \\ . & . & 0 & a_{K-1} & b_{K-1} \\ R\Delta_1 & R\Delta_2 & . & . & R\Delta_k \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ . \\ . \\ T_K \end{bmatrix} \tag{34}$$

where

$$a_i = \frac{c_p}{2}(\frac{\pi_{i+1}}{\pi_i} - 1), \quad b_i = \frac{c_p}{2}(1 - \frac{\pi_i}{\pi_{i+1}}) \quad .$$

Therefore, we have the equation for the geopotential:

$$\phi = AT + \tilde{\phi}_1 \tag{35}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 & . & 0 \\ 0 & 1 & -1 & 0 & . \\ . & 0 & . & . & \\ 0 & . & . & 1 & -1 \\ \Delta_1 & \Delta_2 & . & . & \Delta_K \end{bmatrix}^{-1} . \begin{bmatrix} a_1 & b_1 & 0 & . & 0 \\ 0 & a_2 & b_2 & 0 & . \\ . & . & . & . & . \\ . & . & 0 & a_{K-1} & b_{K-1} \\ R\Delta_1 & R\Delta_2 & . & . & R\Delta_k \end{bmatrix}$$

and

$$\tilde{\phi}_1 = \begin{bmatrix} 1 & -1 & 0 & . & 0 \\ 0 & 1 & -1 & 0 & . \\ . & 0 & . & . & \\ 0 & . & . & 1 & -1 \\ \Delta_1 & \Delta_2 & . & . & \Delta_K \end{bmatrix}^{-1} . \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ \hat{\phi}_1 \end{bmatrix} = \begin{bmatrix} \hat{\phi}_1 \\ . \\ . \\ \hat{\phi}_1 \\ \hat{\phi}_1 \end{bmatrix}$$

Remark: We observe that the last column of the inverse of the matrix used in last equation is identically 1, leading to the expression obtained for

$\tilde{\phi}_1$. To see this, note that the first $(K-1)$ rows of the matrix show that all entries in this last column must be equal. The fact that $\sum_{j=1}^{K}\Delta_j = 1$ shows the rest.

Equation (35) for the geopotential couples the values of geopotential and temperature at different vertical levels. It may be used either in spectral space or in grid-point space. We employ it here, during grid-point computations, in order to get the gradient of the geopotential through the relation:

$$\nabla\phi = A\nabla T + \nabla\tilde{\phi}_1 \tag{36}$$

where the gradients in the right-hand-side have been computed through spectral transforms (the surface geopotential gradient has to be evaluated just once, at the beginning, since it does not change during integration).

Computation of tendencies: momentum equations We will assume the tendencies T_U and T_V for the U and V momentum equations, initially set to 0. They will be composed by the terms from time-step $t_{n-1} = t - \Delta t$, denoted by T_U^{n-1} and T_V^{n-1}, and the terms from time-step $t_n = t$ (T_U^n andT_V^n). The contributions from time $t - \Delta t$ will be added to the tendencies:

$$T_U^{n-1} = T_U^{n-1} - \frac{1}{2a}\left(\frac{\partial\phi}{\partial\lambda} + RT_0\frac{\partial \ln ps}{\partial\lambda}\right)^{t_{n-1}} \tag{37}$$

$$T_V^{n-1} = T_V^{n-1} - \frac{\cos\varphi}{2a}\left(\frac{\partial\phi}{\partial\varphi} + RT_0\frac{\partial \ln ps}{\partial\varphi}\right)^{t_{n-1}} \tag{38}$$

We observe that all variables and their derivatives have already been computed through spectral transforms. These values from the previous time-step are stored until needed. The contributions to the tendencies from time t are as follows. First we have the advection terms, which are only computed in case of the Eulerian model. In this case we have the horizontal advection terms

$$T_U^n = T_U^n - \frac{1}{a\cos^2\varphi}\left(U\frac{\partial U}{\partial\lambda} + V\cos\varphi\frac{\partial U}{\partial\varphi}\right)^{t_n} \tag{39}$$

$$T_V^n = T_V^n - \frac{1}{a\cos^2\varphi}\left(U\frac{\partial V}{\partial\lambda} + V\cos\varphi\frac{\partial V}{\partial\varphi}\right)^{t_n} \tag{40}$$

and the vertical advection contributions (at level k) are given by

$$T_U^n = T_U^n - \frac{1}{2\Delta_k}\left(\dot{\sigma}_{k+1}(U_{k+1} - U_k) + \dot{\sigma}_k(U_k - U_{k-1})\right)^{t_n} \tag{41}$$

$$T_V^n = T_V^n - \frac{1}{2\Delta_k}\left(\dot{\sigma}_{k+1}(V_{k+1} - V_k) + \dot{\sigma}_k(V_k - V_{k-1})\right)^{t_n} \tag{42}$$

The coriolis terms add to the tendencies as:

$$T_U^n = T_U^n + fV^{t_n} \tag{43}$$

$$T_V^n = T_V^n - fU^{t_n} \tag{44}$$

and the metric term contribution (computed only in the Eulerian model) is

$$T_V^n = T_V^n - \frac{\sin\varphi}{a\cos^2\varphi}(U^2+V^2)^{t_n} \tag{45}$$

The tendencies for the momentum equations are completed through the non-linear part of the pressure gradient:

$$T_U^n = T_U^n - \left(\frac{RT'}{a}\frac{\partial \ln ps}{\partial \lambda}\right)^{t_n} \tag{46}$$

$$T_V^n = T_V^n - \left(\frac{\cos\varphi RT'}{a}\frac{\partial \ln ps}{\partial \varphi}\right)^{t_n} \tag{47}$$

where $T' = T - T_0$. The final tendencies are the sum of the contributions from time t_{n-1} and t_n. In case of a semi-Lagrangian integration these contributions will first have to be interpolated respectively to the departure points and to the mid-point of the Lagrangian trajectories, before they can be added. The forcings F_U and F_V (from the physics) will also be added to the tendencies.

Computation of tendencies: thermodynamic equation The tendency for the thermodynamic equation gets the following contribution from time $t - \Delta t$:

$$T_T^{n-1} = T_T^{n-1} - \frac{1}{2}\left(\kappa T_0 \int_0^1 D d\hat{\sigma}\right)^{t_{n-1}} \tag{48}$$

where the integral of divergence is computed as already described. We now pass to describe how the vertical advection is handled in the thermodynamic equation. This term can be put in the form:

$$\dot{\sigma}\frac{\partial T}{\partial \sigma} - \theta\dot{\sigma}\frac{\partial \pi}{\partial \sigma} = \pi\dot{\sigma}\frac{\partial \theta}{\partial \sigma} \tag{49}$$

making explicit the temperature vertical advection term $\dot{\sigma}\partial T/\partial\sigma$. In the current Eulerian formulation equation (49) is discretized (at level k) in two parts as follows:

$$\frac{1}{2\Delta_k}\left[\dot{\hat{\sigma}}_{k+1}\left(\frac{\pi_k}{\pi_{k+1}}T'_{k+1} - T'_k\right) + \dot{\hat{\sigma}}_k\left(T'_k - \frac{\pi_k}{\pi_{k-1}}T'_{k-1}\right)\right] \tag{50}$$

and

$$\frac{1}{2\Delta_k}\left[\dot{\hat{\sigma}}_{k+1}\left(\frac{\pi_k}{\pi_{k+1}}T_0(k+1)-T_0(k)\right)+\dot{\hat{\sigma}}_k\left(T_0(k)-\frac{\pi_k}{\pi_{k-1}}T_0(k-1)\right)\right] \quad (51)$$

Equation (50) is treated explicitly (at time t) and added to the tendency T_T^n. In the second part (equation (51)) we substitute the expression for the vertical velocity:

$$\dot{\hat{\sigma}}_k = \hat{\sigma}\sum_{j=1}^{K}\left(D_j+V_j.\nabla\ln ps\right)\Delta_j - \sum_{j=1}^{k-1}\left(D_j+V_j.\nabla\ln ps\right)\Delta_j \quad (52)$$

Naming

$$H_k^1 = \frac{\pi_k}{\pi_{k+1}}T_0(k+1)-T_0(k) \text{ , and } H_k^2 = T_0(k)-\frac{\pi_k}{\pi_{k-1}}T_0(k-1)$$

we will also decompose the term $1/(2\Delta_k)(\dot{\hat{\sigma}}_{k+1}H_k^1+\dot{\hat{\sigma}}_k H_k^2)$ in the following two expressions:

$$\begin{aligned}&\frac{1}{2\Delta_k}\left\{\left[\hat{\sigma}_{k+1}\sum_{j=1}^{K}\left(V_j.\nabla\ln ps\right)\Delta_j-\sum_{j=1}^{k}\left(V_j.\nabla\ln ps\right)\Delta_j\right]H_k^1\right\}\\ &+\frac{1}{2\Delta_k}\left\{\left[\hat{\sigma}_{k}\sum_{j=1}^{K}\left(V_j.\nabla\ln ps\right)\Delta_j-\sum_{j=1}^{k-1}\left(V_j.\nabla\ln ps\right)\Delta_j\right]H_k^2\right\}\end{aligned} \quad (53)$$

and

$$\frac{1}{2\Delta_k}\left\{\left[\hat{\sigma}_{k+1}\sum_{j=1}^{K}D_j\Delta_j-\sum_{j=1}^{k}D_j\Delta_j\right]H_k^1+\left[\hat{\sigma}_{k}\sum_{j=1}^{K}D_j\Delta_j-\sum_{j=1}^{k-1}D_j\Delta_j\right]H_k^2\right\} \quad (54)$$

The first part (equation (53)) will be treated explicitly, at time t, and its contribution added to the temperature tendency T_T^n. The term from equation (54) will be treated implicitly, leading to another contribution (at time $t-\Delta t$) to tendency T_T^{n-1}. The two implicity terms - the one just mentioned and the term $\kappa T_0(k)\int_0^1 D d\sigma = \kappa T_0(k)\sum_{j=1}^{K}D_j\Delta_j$ - will be compounded and lead to the following term in the equation for temperature

$$\Delta t B\tilde{D}$$

where $\tilde{D}$ is the column vector $(D_1, D_2, ..., D_K)$ of the divergence on all levels and B is the $K\times K$ matrix, whose lines are $(B_{i,1}, ..., B_{i,K})$, where

$$B_{i,j} = -\kappa T_0(i)\Delta_j + \frac{1}{2\Delta_i}(\hat{\sigma}_{i+1}H_i^1+\hat{\sigma}_i H_i^2)\Delta_j - \frac{\Delta_j H_i^1}{2\Delta_i} - \frac{\Delta_j H_i^2}{2\Delta_i} \quad (55)$$

where the second last term only exists if $j\le i$ and the last one only for $j<i$.

Comments about the semi-Lagrangian option In a semi-Lagrangian scheme, the term $\dot{\sigma}\partial T/\partial\sigma$ should be absorbed in the semi-Lagrangian treatment of advection. The term $-\theta\dot{\sigma}\partial\pi/\partial\sigma$ (of equation (49)) remains to be discretized. In order to do this, we can follow the same lines of the treatment of the term $\pi\dot{\sigma}\partial\theta/\partial\sigma$. It's discretized version would be:

$$\frac{T'_k + T_0(k)}{2\Delta_k}\left[\dot{\hat{\sigma}}_{k+1}\left(\frac{\pi_{k+1}}{\pi_k} - 1\right) + \dot{\hat{\sigma}}_k\left(1 - \frac{\pi_{k-1}}{\pi_k}\right)\right] \tag{56}$$

Again we treat the non-linear part (involving T'_k) explicitly. The part refering to $T_0(k)$ would be broken in two as before. If we name:

$$H_k^1 = T_0(k)\left(\frac{\pi_{k+1}}{\pi_k} - 1\right) \quad \text{and} \quad H_k^2 = T_0(k)\left(1 - \frac{\pi_{k-1}}{\pi_k}\right)$$

we get to the same discretization as before.

Other terms of thermodynamic equation We still need to add some other explicit terms (at time t) to the tendencies. These are the horizontal advection of temperature:

$$T_T^n = T_T^n - \frac{1}{a\cos^2\varphi}\left(U\frac{\partial T}{\partial\lambda} + V\cos\varphi\frac{\partial T}{\partial\varphi}\right)^{t_n} \tag{57}$$

and the integrals (discretized as described)

$$T_T^n = T_T^n - \kappa(T'_k + T_0(k))\left(\int_0^1 V.\nabla\ln ps d\sigma - V_k.\nabla\ln ps\right) - \kappa T'_k\int_0^1 D d\sigma \tag{58}$$

Again, the final tendency T_T is the sum of T_T^{n-1} and T^n computed at the appropriate locations, depending on the model option (Eulerian or semi-Lagrangian). Physics contributions shall also be added to the final tendency.

Computation of tendencies: specific humidity equation This equation is treated fully explicitly. The contributions to the tendency come only from horizontal advection

$$T_q^n = T_q^n - \frac{1}{a\cos^2\varphi}\left(U\frac{\partial q}{\partial\lambda} + V\cos\varphi\frac{\partial q}{\partial\varphi}\right)^{t_n} \tag{59}$$

and from vertical advection

$$T_q^n = T_q^n - \frac{1}{2\Delta_k}\left(\dot{\hat{\sigma}}_{k+1}(q_{k+1} - q_k) + \dot{\hat{\sigma}}_k(q_k - q_{k-1})\right)^{t_n} \tag{60}$$

For the semi-Lagrangian option, we have a full Lagrangian treatment of this equation.

Computation of tendencies: surface pressure equation The surface pressure equation has two tendency terms, a contribution from time $t - \Delta t$

$$T_{ps}^{n-1} = T_{ps}^{n-1} - \frac{1}{2}\left(\int_0^1 D d\sigma\right)^{t_{n-1}} \tag{61}$$

and one from time t (only for the Eulerian version):

$$T_{ps}^{n} = T_{ps}^{n} - \left(\int_0^1 \vec{V}.\nabla \ln ps d\sigma\right)^{t_n} \tag{62}$$

This two integrals are discretized as already shown in this section.

0.9 Transforms from grid-point to spectral space

After completing the evaluation of the tendencies (these shall also include the contributions from the physics parametrization processes) the implicit system has still to be solved. For this we will first complete the right-hand-sides of each equation. (For a prognostic variable F we build the right-hand-side as $R_F = F^{t_{n-1}} + 2\Delta t T_F$. Notice that in the semi-Lagrangian scheme the field value at time t_{n-1} has to be interpolated to the departure points locations. We will then rather add it to the tendency contributions at this time-level before interpolating.) Then, the RHS's of the equations for T, q and $\ln ps$ will be transformed to spectral space as they are. The momentum equations will be changed into equations for Vorticity and Divergence (by taking curl and divergence of the equations).

Before describing how this is done, we first remember how a grid-point field $R(\lambda, \varphi)$ is transformed to spectral space (or in other words, how we obtain the spectral coefficients R_n^m of R from its grid-point values. Knowing the function R, its spherical harmonics coefficients are given by:

$$R_n^m = \frac{1}{2\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} R(\lambda, \mu) P_n^m(\mu) e^{-im\lambda} d\lambda d\mu \tag{63}$$

where $\mu = \sin(\varphi)$. This integral is evaluated in two steps, the first one consists of a Fourier transform (this is the integral with respect to λ through a multiple trapezoidal rule):

$$R_n(\mu_j) = \frac{1}{N_\lambda}\sum_{l=1}^{N_\lambda} R(\lambda_l, \mu_j) e^{-im\lambda_l} \tag{64}$$

where N_λ is usually taken as approximately $3M$ (where M is the spectral truncation). The second step is a Legendre transform, actually consisting of a gaussian integration for the integral with respect to μ, given by:

$$R_n^m = \sum_{j=1}^{N_\varphi} \omega_j R_n(\mu_j) P_n^m(\mu_j) \tag{65}$$

where the ω_j's are the Gaussian weights from the integration formula and N_φ is the number of latitudes ($N_\varphi = N_\lambda/2$).

We now proceed to the transformation of the momentum equations:

$$U^{t_{n+1}} + \frac{\Delta t}{a}\left(\frac{\partial \phi}{\partial \lambda} + RT_0 \frac{\partial \ln ps}{\partial \lambda}\right)^{t_{n+1}} = R_U \tag{66}$$

$$V^{t_{n+1}} + \frac{\Delta t \cos\varphi}{a}\left(\frac{\partial \phi}{\partial \varphi} + RT_0 \frac{\partial \ln ps}{\partial \varphi}\right)^{t_{n+1}} = R_V \tag{67}$$

into the equations for vorticity and divergence:

$$\xi^{t_{n+1}} = \frac{1}{a\cos^2\varphi}\left(\frac{\partial R_V}{\partial \lambda} - \cos\varphi \frac{\partial R_U}{\partial \varphi}\right) = R_\xi \tag{68}$$

$$D^{t_{n+1}} + \Delta t \nabla^2 \left(\phi + RT_0 \ln ps\right)^{t_{n+1}} = \frac{1}{a\cos^2\varphi}\left(\frac{\partial R_U}{\partial \lambda} + \cos\varphi \frac{\partial R_V}{\partial \varphi}\right) = R_D \tag{69}$$

The transfomed right-hand-sides R_ξ and R_D are given by the expressions:

$$R_{\xi}{}_n^m = \frac{1}{a}\int_{-\pi/2}^{\pi/2} \left(im\tilde{V}^m(\varphi) P_n^m(\sin\varphi) + \tilde{U}^m(\varphi) H_n^m(\sin\varphi)\right)\cos\varphi d\varphi \tag{70}$$

$$R_{D}{}_n^m = \frac{1}{a}\int_{-\pi/2}^{\pi/2} \left(im\tilde{U}^m(\varphi) P_n^m(\sin\varphi) - \tilde{V}^m(\varphi) H_n^m(\sin\varphi)\right)\cos\varphi d\varphi \tag{71}$$

which are obtained by the transforms (63), with application of integration by parts. In these formulas we have that

$$\tilde{U}^m(\varphi) = \frac{1}{2\pi\cos^2\varphi}\int_{-\pi}^{\pi} R_U(\varphi,\lambda) e^{-im\lambda} d\lambda \tag{72}$$

$$\tilde{V}^m(\varphi) = \frac{1}{2\pi\cos^2\varphi}\int_{-\pi}^{\pi} R_V(\varphi,\lambda) e^{-im\lambda} d\lambda \tag{73}$$

(which are computed through FFT's). In the present implementation of the Eulerian model, the transformed RHS's for the vorticity and divergence are evaluated directly from equations (70) and (71), leading to four Legendre transforms (two with the Legendre Polynomials P_n^m and two with their derivatives H_n^m). We will instead first compute only two Legendre transforms to get

$$\tilde{U}_n^m = \int_{-\pi/2}^{\pi/2} \tilde{U}^m(\varphi) P_n^m(\sin\varphi)\cos\varphi d\varphi \tag{74}$$

and

$$\tilde{V}_n^m = \int_{-\pi/2}^{\pi/2} \tilde{V}^m(\varphi) P_n^m(\sin\varphi)\cos\varphi d\varphi \tag{75}$$

Then we use that

$$H_n^m(\sin\varphi) = -n\varepsilon_{n+1}^m P_{n+1}^m(\sin\varphi) + (n+1)\varepsilon_n^m P_{n-1}^m(\sin\varphi) \tag{76}$$

in order to get

$$R_{\xi_n}^{\ m} = \frac{1}{a}\left(im\tilde{V}_n^m - n\varepsilon_{n+1}^m \tilde{U}_{n+1}^m + (n+1)\varepsilon_n^m \tilde{U}_{n-1}^m\right) \tag{77}$$

and

$$R_{D_n}^{\ m} = \frac{1}{a}\left(im\tilde{U}_n^m + n\varepsilon_{n+1}^m \tilde{V}_{n+1}^m - (n+1)\varepsilon_n^m \tilde{V}_{n-1}^m\right) \tag{78}$$

for $|m| \leq n \leq M+1$.

Summarizing, we transform the RHS's R_U, R_V, R_T, R_q and R_{ps} from grid-point space to spectral space, amounting to Fourier and Legendre transforms of 4 three-dimensional fields and 1 two-dimensional. After this we already have the new value of vorticity through equation (68) and of the specific humidity through equation (6). The remaining prognostic variables will be determined in the semi-implicit computations.

0.10 Semi-implicit computations

It remains to solve the system which couples the values of T, D and $\ln ps$ for the time-step $t + \Delta t$ in the different vertical layers. This system has the form:

$$\begin{aligned} D + \Delta t \nabla^2(\phi + RT_0 \ln ps) &= R_D \\ T + \Delta t \left(\kappa T_0(k) + \frac{\sigma_{k+1}H_k^1 + \sigma_k H_k^2}{2\Delta_k}\right)\sum_{j=1}^{K} D_j\Delta_j - & \\ \frac{\Delta t}{2\Delta_k}\left(H_k^1 \sum_{j=1}^{k} D_j\Delta_j + H_k^2 \sum_{j=1}^{k-1} D_j\Delta_j\right) &= R_T \\ \ln ps + \Delta t \sum_{j=1}^{K} D_j\Delta_j &= R_{ps} \end{aligned} \tag{79}$$

We will solve the system in spectral space, using the matrices A and B from the equations (35) and (55). The system is coupled only in the vertical, being independent for each wave-number (n, m). It assumes the following form:

$$\begin{aligned} \tilde{D}_n^m + \bar{A}\tilde{T}_n^m + \tilde{\beta}\ln ps_n^m &= \tilde{\tilde{R}}_{D_n}^{\ m} = \tilde{R}_{D_n}^{\ m} + \Delta t \frac{n(n+1)}{a^2}\tilde{\phi}_{1_n}^{\ m} \\ \tilde{T}_n^m + \bar{B}\tilde{D}_n^m &= \tilde{R}_{T_n}^{\ m} \\ \ln ps_n^m + \tilde{\delta}.\tilde{D}_n^m &= R_{ps_n}^{\ m} \end{aligned} \tag{80}$$

where $\bar{A} = -(\Delta t n(n+1)/a^2)A$, $\bar{B} = \Delta t B$, $\tilde{\beta} = -(\Delta t n(n+1)/a^2)R(T_0(1), ..., T_0(K))$, $\tilde{\delta} = \Delta t(\Delta_1, ..., \Delta_K)$ and the $\tilde{}$'s denote vertical vectors.

The solution of this system for each pair (n, m) is obtained by first substituting the values of T and $\ln ps$ from the second and third equation into the first, to get:

$$\tilde{D}_n^m + \bar{A}(\tilde{R}_{T_n}^{\ m} - \bar{B}\tilde{D}_n^m) + \tilde{\beta}(R_{ps_n}^{\ m} - \tilde{\delta}.\tilde{D}_n^m) = \tilde{\tilde{R}}_{D_n}^{\ m}$$

which leads to:

$$\tilde{D}_n^m = (I - \bar{A}\bar{B} - \tilde{\beta}.\tilde{\delta}^T)^{-1}(\tilde{\tilde{R}}_{D_n}^{\ m} - \tilde{\beta}R_{ps_n}^{\ m} - \bar{A}\tilde{R}_{T_n}^{\ m})$$

The matrix $(I - \bar{A}\bar{B} - \tilde{\beta}.\tilde{\delta}^T)$ is explicitly computed and inverted. Since it is independent on the pair (n, m) this is not very expensive. Once $\tilde{D}_n^m$ has been computed, one obtains $\tilde{T}_n^m = \tilde{R}_{T_n}^{\ m} - \bar{B}\tilde{D}_n^m$ and $\ln ps_n^m = R_{ps_n}^{\ m} - \tilde{\delta}.\tilde{D}_n^m$.

0.11 Horizontal Diffusion

At the moment, horizontal diffusion is computed through an explicit method with the potential of leading to more severe stability restrictions at higher resolutions. The horizontal diffusion for a variable F employed in the model is of the form:

$$\frac{\partial F}{\partial t} = T_F - c(-1)^p \nabla^{2p} F \tag{81}$$

In the explicit method this equation is integrated in spectral space as:

$$\frac{\tilde{F}_n^m(t + \Delta t) - \tilde{F}_n^m(t - \Delta t)}{2\Delta t} = T_F - c\left(\frac{n(n+1)}{a^2}\right)^p \tilde{F}_n^m(t - \Delta t) \tag{82}$$

So, the term $-c\left(\frac{n(n+1)}{a^2}\right)^p \tilde{F}_n^m(t - \Delta t)$ is added to the tendency of a prognostic variable F (still before the semi-implicit computations). For the temperature the term added is: $-c\left(\frac{n(n+1)}{a^2}\right)^p (\tilde{T}_n^m - \tilde{C}_T \ln ps_n^m)(t - \Delta t)$, with

$\tilde{C}_T = (\sigma_1(\partial T/\partial\sigma)_1, ..., \sigma_K(\partial T/\partial\sigma)_K)$. Specific humidity is treated similarly, with $\tilde{C}_T$ changed by $\tilde{C}_q$. The diffusion coefficient c may vary for each variable, currently is employed one coefficient (stronger) for Divergence and one for the other prognostic variables. A choice of $p = 2$ is normally made in the operational forecast model.

Implicit diffusion We will change the method to an implicit diffusion scheme. In this case, the diffusion equation (81) will be discretized spectrally by a fractional step method as:

$$\tilde{F}_n^m(t + \Delta t) = \bar{F}_n^m(t + \Delta t) - 2c\Delta t \left(\frac{n(n+1)}{a^2}\right)^p \tilde{F}_n^m(t + \Delta t) \qquad (83)$$

where $\bar{F}_n^m$ is a temporary value for $\tilde{F}_n^m$ at the new time-step, obtained after the semi-implicit computations (the implicit horizontal diffusion will be applied at the end of the time-step). For the temperature and specific humidity, the corresponding modification leads to:

$$\tilde{F}_n^m(t + \Delta t) = \left(1 + 2c\Delta t \left(\frac{n(n+1)}{a^2}\right)^p\right)^{-1} \left(\bar{F}_n^m(t + \Delta t) + 2c\Delta t \left(\frac{n(n+1)}{a^2}\right)^p \tilde{C} \ln ps_n^m(t + \Delta t)\right)$$

0.12 Enhanced Diffusion

An extra diffusion term is applied at a given level k when the maximum wind speed at this level $V_{max}(k)$, computed during the grid-point computations, exceeds a critical value V_{crit} (taken as $85m/s$). We define a factor $\beta = V_{crit} n_0 \Delta t_0/\Delta t$, with $\Delta t_0 = 900s$, $n_0 = 63$ (Eulerian time-step for a reference resolution n_0) and Δt the time-step of the model in use. We obtain a critical wave-number $n_{crit} = \beta/V_{max}$. The spectral coefficients of the prognostic variables at level k will be selectively dumped, but only for wave-numbers above the critical value. The dumping factor for each $n > n_{crit}$ is defined as $cdump_n = (1 + \alpha\Delta t V_{max}(n - n_{crit})/a)^{-1}$, with $\alpha = 2.5$, and each spectral coefficient at this wave-number multiplied by $cdump$. (These values are chosen such that an advection equation with velocity V_{max} would be integrated stably with an Eulerian explicit scheme)

0.13 Time-Filter

A Robert-Asselin time filter is applied (for a prognostic variable F) in the form:

$$\bar{F}(t) = F(t) + \gamma\left(\bar{F}(t-\Delta t) - 2F(t) + F(t+\Delta t)\right) \tag{84}$$

where the overbar denotes filtered quantities. This filter will be applied in grid-point space in two steps:

$$\tilde{F}(t) = F(t) + \gamma\left(\bar{F}(t-\Delta t) - 2F(t)\right)$$

at the end of a time-step and

$$\bar{F}(t-\Delta t) = \tilde{F}(t-\Delta t) + \gamma F(t)$$

when $F(t)$ becomes available in grid-point space. $\tilde{F}$ will be stored until the next step, when it is used in the computation of $\bar{F}$. These filtered values at time $t - \Delta t$ (and unfiltered at time t) will be used in the computations of the right-hand-sides of the equations.

0.14 Spherical Harmonics

For completeness we include here a general description about expansions in Spherical Harmonics and its basic properties. Spherical harmonics are defined on the sphere, for each (integer) zonal wavenumber m and total wavenumber n ($n \geq |m|$), as:

$$Y_n^m(\lambda, \varphi) = e^{im\lambda} P_n^m(sin\varphi) \tag{85}$$

where P_n^m is an associated Legendre Polynomial of order m and degree n. They are given by Rodrigues formula as:

$$P_n^m(\mu) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} \frac{(1-\mu^2)^{|m|/2}}{2^n n!} \frac{d^{n+|m|}(1-\mu^2)^n}{d\mu^{n+|m|}} \tag{86}$$

The Legendre polynomials in the above formula are normalized such that, for each order m, they build an orthonormal family with respect to the inner product $< f, g >= \int_{-1}^{1} f(\mu)g(\mu)d\mu$. On the other hand, the spherical harmonics constitute a complete orthonormal family with respect to integration over the sphere (taking $\mu = sin\varphi$):

$$\frac{1}{2\pi} \int_{-1}^{1} \int -\pi^{\pi} Y_n^m \bar{Y}_{n'}^{m'} d\lambda d\mu = \delta_{n,n'} \delta_{m,m'} \tag{87}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

Another important property of the spherical harmonics is that they are eingefunctions of the Laplace operator on the two-dimensional sphere of radius a. We have:

$$\nabla^2 Y_n^m = \frac{-n(n+1)}{a^2} Y_n^m \tag{88}$$

This is a very important property, used throughout the model. Furthermore we have that $\partial Y_n^m / \partial \lambda = imY_n^m$, meaning that the spherical harmonics are also eigenfunctions of the zonal differentiation operator. The same is not true for differentiation in the meridional direction. In this case we will make use of the relation

$$(\mu^2 - 1)\frac{dP_n^m(\mu)}{d\mu} = n\varepsilon_{n+1}^m P_{n+1}^m(\mu) - (n+1)\varepsilon_n^m P_{n-1}^m(\mu) \tag{89}$$

where

$$\varepsilon_n^m = \sqrt{\frac{n^2 - m^2}{4n^2 - 1}}$$

In the global spectral model we develop the meteorological fields in spherical harmonic expansions, for a given truncation M. Our choice is to use a triangular truncation, which provides an isotropic resolution over the globe (in other words, the resolution of a triangular truncation would be invariant under any rotation of the coordinate system). Any field F will therefore have a truncated expansion like

$$F(\lambda, \varphi) = \sum_{m=-M}^{M} \sum_{n=|m|}^{M} F_n^m Y_n^m(\lambda, \varphi) = \sum_{m=-M}^{M} \sum_{n=|m|}^{M} F_n^m e^{im\lambda} P_n^m(\sin\varphi) \tag{90}$$

Given a field F on the sphere, its spectral coeffcients F_n^m can be obtained (using (87)) as

$$F_n^m = \frac{1}{2\pi} \int_{-1}^{1} \int -\pi^{\pi} F(\lambda, \mu) \bar{Y}_n^m(\lambda, \mu) d\lambda d\mu \tag{91}$$

where $\mu = \sin\varphi$.

This computation is done numerically in two steps (for a field F known on a spherical grid of N_λ by N_φ points). The integral with respect to λ is computed as

$$F_n(\mu_j) = \frac{1}{N_\lambda} \sum_{l=1}^{N_\lambda} F(\lambda_l, \mu_j) e^{-im\lambda_l} \tag{92}$$

which is a discrete Fourier transform. It is followed by a numerical integration with respect to μ, done as

$$F_n^m = \sum_{j=1}^{N_\varphi} \omega_j F_n(\mu_j) P_n^m(\mu_j) \tag{93}$$

which is a Gaussian integration formula with weights ω_j. In order to have optimal precision in the numerical evaluation of the integral with respect to μ, the latitudes on the grid are chosen such that $\mu_j = \sin\varphi_j$ are the zeros of a Legendre Polynomial of order zero and degree N_φ (number of latitudes on the grid). N_φ and N_λ are chosen such that the spectral coefficients of quadratic non-linear products are obtained with no errors (alias free). For this, N_φ has to be at least $(3M+1)/2$ and $N_\lambda = 2N_\varphi$.

In the evaluation of the Legendre transform (93) one should use the fact that $P_n^m(-\mu) = (-1)^{n+|m|}P_n^m(\mu)$ in order to save computations.

The computation of the grid values of a function given by its spectral coefficients is also done in two steps, an inverse Legendre transform

$$F^m(\varphi) = \sum_{n=|m|}^{M} F_n^m P_n^m(\sin\varphi) \tag{94}$$

followed by an inverse Fourier transform

$$F(\lambda,\varphi) = \sum_{m=-M}^{M} F^m(\varphi)e^{im\lambda} \tag{95}$$

In the inverse transforms it is used that $P_n^m(\mu) = P_n^{-m}(\mu)$ and that $\bar{F}_n^m = F_n^{-m}$ (for a real field) in order to save computations.