8. INNER PRODUCT SPACES

On first meeting vector spaces, it is quite natural to think of $\mathbb{R}^n$ as a typical example. However, as has already been commented, it is important to appreciate that $\mathbb{R}^n$ has a lot of structure beyond being just a real vector space. It has coordinates already assigned (and so a canonical basis) and distances and angles can be measured, for example using the dot (or scalar) product. Vector spaces, in general, have none of this extra structure.

The dot product is an example of an inner product; an inner product is a means of measuring distance and angles within a vector space. A vector space together with an inner product is called an *inner product space.* Initially we will consider inner products only on real vector spaces, but we will later discuss complex inner product spaces. Inner products appear in many areas of mathematics and they have particular importance in Fourier series and in quantum theory.

8.1 Bilinear forms

Definition 193 *Let V be a vector space over $\mathbb{F}$. A* ***bilinear form*** *B on V is a function $B\colon V \times V \to \mathbb{F}$, such that*

(a) $B(\alpha_1 v_1 + \alpha_2 v_2, v_3) = \alpha_1(v_1, v_3) + \alpha_2(v_2, v_3)$ for all v_1, v_2, $v_3 \in V$ and α_1, $\alpha_2 \in \mathbb{F}$;

(b) $B(v_1, \alpha_2 v_2 + \alpha_3 v_3) = \alpha_2(v_1, v_2) + \alpha_3(v_1, v_3)$ for all v_1, v_2, $v_3 \in V$ and α_2, $\alpha_3 \in \mathbb{F}$.

(a) says that B is linear in the first variable (when we fix the second variable), and (b) says the same for the second variable.

Example 194 *For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}^n$, we define*

$$B(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_n y_n.$$

This gives a bilinear form. In $\mathbb{R}^n$ this is the familiar dot product, or scalar product, often written $\mathbf{x} \cdot \mathbf{y}$.

Example 195 *Take $A \in \mathcal{M}_{n\times n}(\mathbb{F})$. Note for $\mathbf{x}$, $\mathbf{y} \in \mathbb{F}^n$ that $B(\mathbf{x}, \mathbf{y}) = \mathbf{x}A\mathbf{y}^T$ defines a bilinear form on $\mathbb{F}^n$.*

Note that the usual scalar product is an example of this in the special case that $A = I_n$, because $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}\mathbf{y}^T$. (Officially, $\mathbf{x}A\mathbf{y}^T$ is a 1×1 matrix, not an element of $\mathbb{F}$, but it is completely natural to identify 1×1 matrices with scalars).

And for $\mathbf{x}$, $\mathbf{y} \in \mathbb{F}^n_{\mathrm{col}}$ then $B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$ defines a bilinear form on $\mathbb{F}^n_{\mathrm{col}}$.

Definition 196 *Let V be a vector space over $\mathbb{F}$ and let B be a bilinear form on V. Take $v_1, \ldots, v_n \in V$. The **Gram matrix** of B with respect to $v_1, \ldots, v_n$ is the $n \times n$ matrix $(B(v_i, v_j)) \in \mathcal{M}_{n\times n}(\mathbb{F})$.*

Proposition 197 *Let V be a finite-dimensional vector space over $\mathbb{F}$ and let $v_1, \ldots, v_n$ be a basis for V. Let B be a bilinear form on V and let $A \in \mathcal{M}_{n\times n}(\mathbb{F})$ be the associated Gram matrix. For X, $Y \in V$, let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{F}^n$ be the unique coordinate vectors such that*

$$X = x_1v_1 + \cdots + x_nv_n \qquad \textit{and} \qquad Y = y_1v_1 + \cdots + y_nv_n.$$

Then $B(X, Y) = \mathbf{x}A\mathbf{y}^T$.

Remark 198 *Consequently the bilinear form in Example 195 essentially describes **all** bilinear forms on V. Note that if A is the Gram matrix of a bilinear form, then any other Gram matrix of B (that is, a Gram matrix with respect to a different basis) equals P^TAP where $P \in \mathcal{M}_{n\times n}(\mathbb{F})$ is invertible.*

Proof. We have

$$\begin{aligned}
B(X,Y) &= B\left(\sum_{i=1}^n x_iv_i, \sum_{j=1}^n y_jv_j\right) \\
&= \sum_{i=1}^n x_iB\left(v_i, \sum_{j=1}^n y_jv_j\right) \qquad \text{[using linearity in the first entry]} \\
&= \sum_{i=1}^n x_i \sum_{j=1}^n y_jB(v_i, v_j) \qquad \text{[using linearity in the second entry]} \\
&= \sum_{i=1}^n \sum_{j=1}^n x_ia_{ij}y_j \\
&= xAy^T.
\end{aligned}$$

■

Definition 199 *We say that a bilinear form $B \colon V \times V \to \mathbb{F}$ is **symmetric** if*

$$B(v_1, v_2) = B(v_2, v_1) \qquad \textit{for all} \qquad v_1, v_2 \in V.$$

Note that a bilinear form is symmetric if and only if any Gram matrix of the bilinear form is symmetric.

8.2 Inner product spaces

Definition 200 *Let V be a real vector space. We say that a bilinear form $B \colon V \times V \to \mathbb{R}$ is **positive definite** if $B(v, v) \geqslant 0$ for all $v \in V$, with $B(v, v) = 0$ if and only if $v = 0$. N.B. we are defining **real** inner product spaces here; the requirement that $B(v, v) \geqslant 0$ does not make sense in a general field.*

Definition 201 *An **inner product** on a real vector space V is a positive definite, symmetric, bilinear form on V. Inner products are usually denoted $\langle x, y\rangle$ rather than $B(x, y)$.*

*We say that a real vector space is an **inner product space** if it is equipped with an inner product. Unless otherwise specified, $\langle -, -\rangle$ will denote an inner product, rather than a general bilinear form.*

Example 202 *The dot product on $\mathbb{R}^n$ is an inner product. We noted earlier that it is a bilinear form, and it is clearly symmetric. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{x} \neq \mathbf{0}$, then*

$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + \cdots + x_n^2 > 0,$$

*so the dot product is also positive definite. The inner product space consisting of $\mathbb{R}^n$ equipped with the dot product is known as n-dimensional **Euclidean space**. The dot product also turns $\mathbb{R}^n_{\mathrm{col}}$ into an inner product space.*

Example 203 *Let $V = \mathbb{R}_n[x]$, the vector space of polynomials of degree $\leqslant n$. For f, $g \in V$, define*

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$$

where $a < b$. Then $\langle -, -\rangle$ is bilinear – as integration is linear – and symmetric – as the integrand is symmetric in f and g.

If $f \in V$ and $f \neq 0$, then $f(x) = 0$ for only finitely many x in $[a, b]$, and $(f(x))^2 > 0$ at other x, and we find that

$$\langle f, f\rangle = \int_a^b f(x)^2\,\mathrm{d}x > 0.$$

So $\langle -, -\rangle$ is positive definite.

Hence $\langle -, -\rangle$ is an inner product on V. In fact, more generally, $\langle -, -\rangle$ defines an inner product on the space $C[a, b]$ of continuous real-valued functions on the interval $[a, b]$.

Importantly, inner products allow us to define length and angle, something which is not possible with the structure of a vector space alone.

Definition 204 *Let V be an inner product space. For $v \in V$, we define the **norm** (or **magnitude** or **length**) of v to be*

$$\|v\| := \sqrt{\langle v, v\rangle}.$$

The distance between two vectors v, $w \in V$ is defined to be

$$d(v, w) = \|v - w\|.$$

Proposition 205 *The norm $\| - \|$ has the following properties; for $v, w \in V$ and $\alpha \in \mathbb{R}$,*

(a) $\|v\| \geqslant 0$ and $\|v\| = 0$ if and only if $v = 0_V$.

(b) $\|\alpha v\| = |\alpha|\,\|v\|$.

*(c) $\|v + w\| \leqslant \|v\| + \|w\|$. This is known as the **triangle inequality**.*

Proof. (a) and (b) are straightforward. To prove (c) we will first prove the *Cauchy-Schwarz inequality*. ■

Proposition 206 ***(Cauchy-Schwarz Inequality)*** *For v, w in an inner product space V, then*

$$|\langle v, w\rangle| \leqslant \|v\| \, \|w\| \, .$$

Equality holds if and only if v and w are linearly dependent.

Proof. If $w = 0$ then the result is immediate, so assume that $w \neq 0$. For $t \in \mathbb{R}$, note that

$$\begin{aligned} 0 &\leqslant \|v + tw\|^2 \\ &= \langle v + tw, v + tw\rangle \\ &= \langle v, v\rangle + 2t\langle v, w\rangle + t^2\langle w, w\rangle \qquad \text{[by linearity and symmetry]} \\ &= \|v\|^2 + 2t\langle v, w\rangle + t^2 \|w\|^2 \, . \end{aligned}$$

As $\|w\| \neq 0$, the last line is a quadratic in t which is always non-negative. So it either has complex roots or a repeated real root, meaning its discriminant is non-positive. So

$$\text{discriminant} = 4\langle v, w\rangle^2 - 4 \|w\|^2 \|v\|^2 \leqslant 0$$

and the Cauchy-Schwarz inequality follows. For equality, the discriminant has to be zero which means there is a repeated real root $t = t_0$. But then $\|v + t_0 w\| = 0$ and hence $v + t_0 w = 0_V$ showing that v and w are linearly dependent. The converse is immediate. ■

Proof. Continuing the proof of Proposition 205 (c): we now see

$$\begin{aligned} \|v + w\|^2 &= \langle v + w, v + w\rangle \\ &= \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \\ &\leqslant \|v\|^2 + 2\,|\langle v, w\rangle| + \|w\|^2 \\ &\leqslant \|v\|^2 + 2\,\|v\|\,\|w\| + \|w\|^2 \qquad \text{[by the Cauchy-Schwarz inequality]} \\ &= (\|v\| + \|w\|)^2 \end{aligned}$$

and the triangle inequality follows. ■

Proposition 207 *The distance function $d(v, w) = \|v - w\|$ satisfies the following properties: for $u, v, w \in V$ we have*

(a) $d(v, w) \geqslant 0$ and $d(v, w) = 0$ if and only if $v = w$.
(b) $d(v, w) = d(w, v)$.
(c) $d(u, w) \leqslant d(u, v) + d(v, w)$.

Here (a), (b), (c) show d has the properties of a ***metric****.*

Proof. These properties follow straightforwardly from the properties of the norm. ■

You may recall that in $\mathbb{R}^2$ or $\mathbb{R}^3$ we have

$$\mathbf{x} \cdot \mathbf{y} = \|x\|\|y\| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. In general, we can use this idea to *define* a notion of angle in an abstract inner product space V: we define the **angle** between nonzero vectors x, $y \in V$ to be

$$\cos^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right).$$

Note by the Cauchy-Schwarz inequality that this is well-defined as

$$\left|\frac{\langle x, y\rangle}{\|x\|\|y\|}\right| \leqslant 1$$

for any nonzero vectors x, y in an inner product space.

Example 208 *Let m, n be integers. Show that $\sin mx$ and $\cos nx$ are perpendicular in $C[-\pi, \pi]$ with the inner product from Example 203. Show also that $\cos mx$ perpendicular to $\cos nx$ when $m \neq n$ and find $\|\cos mx\|$.*

Solution. Note that

$$\langle \sin mx, \cos nx\rangle = \int_{-\pi}^{\pi} \sin mx \cos nx \,\mathrm{d}x = 0$$

as the integrand is odd. For the second part, recall the trigonometric identity

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x].$$

So if $m \neq n$ then

$$\begin{aligned}
\langle \cos mx, \cos nx\rangle &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)x \,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}$$

If $m = n \neq 0$ then

$$\|\cos mx\|^2 = \langle \cos mx, \cos mx\rangle = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 2mx + 1)\,\mathrm{d}x = \pi,$$

and if $m = n = 0$ then

$$\|1\|^2 = \langle 1, 1\rangle = \int_{-\pi}^{\pi} \mathrm{d}x = 2\pi.$$

■

Remark 209 *The above orthogonality relations are crucial in the study of* ***Fourier series****. If we can represent a function on $-\pi < x < \pi$ as a Fourier series*

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

then, provided the integration and infinite sum can be interchanged, we would have

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\sin lx \,\mathrm{d}x &= \int_{-\pi}^{\pi} \left(\frac{1}{2}a_0 \sin lx + \sum_{k=1}^{\infty} (a_k \cos kx \sin lx + b_k \sin kx \sin lx) \right) \mathrm{d}x \\
&= \left(\int_{-\pi}^{\pi} \frac{1}{2}a_0 \sin lx \,\mathrm{d}x \right) + \sum_{k=1}^{\infty} a_k \left(\int_{-\pi}^{\pi} \cos kx \sin lx \,\mathrm{d}x \right) + \sum_{k=1}^{\infty} b_k \left(\int_{-\pi}^{\pi} \sin kx \sin lx \,\mathrm{d}x \right) \\
&= 0 + \sum_{k=1}^{\infty} a_k \times 0 + \sum_{k=1}^{\infty} b_k \pi \delta_{kl} \\
&= \pi b_l.
\end{aligned}$$

Hence

$$b_l = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin lx \,\mathrm{d}x \qquad \textit{for} \qquad l \geqslant 1.$$

Similar calculations show that

$$a_l = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos lx \,\mathrm{d}x \qquad \textit{for} \qquad l \geqslant 0.$$

These are the Fourier coefficients of $f(x)$. Validating convergency issues and interchanging the integration and infinite sum are difficult matters of analysis.

8.3 Orthogonal Maps

Definition 210 *A linear map $\alpha\colon V \to V$ of an inner product space V is said to be* ***orthogonal*** *if*

$$\langle \alpha(v), \alpha(w) \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

Proposition 211 *Let $\alpha\colon \mathbb{R}^n_{\mathrm{col}} \to \mathbb{R}^n_{\mathrm{col}}$ be a linear map and let A denote the matrix of α with respect to the standard basis. Then α is orthogonal with respect to the dot product if and only if A is an orthogonal matrix.*

Proof. Suppose that α is orthogonal with respect to the dot product. Denote the standard basis as $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then

$$\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \alpha(\mathbf{e}_i) \cdot \alpha(\mathbf{e}_j) = (A\mathbf{e}_i)^T (A\mathbf{e}_j) = \mathbf{e}_i^T A^T A \mathbf{e}_j.$$

Now $\mathbf{e}_i^T A^T A \mathbf{e}_j$ is the (i,j)th entry of $A^T A$, and this equals δ_{ij} which is the (i,j)th entry of I_n. Hence $A^T A = I_n$ as this is true for all i, j.

Reversing the implications of the above argument takes us from $A^T A = I_n$ to $\alpha(\mathbf{e}_i)\cdot\alpha(\mathbf{e}_j) = \delta_{ij}$. But then by linearity

$$\begin{aligned}
\alpha\left(\sum_i u_i \mathbf{e}_i\right)\cdot\alpha\left(\sum_j v_j \mathbf{e}_j\right) &= \sum_i\sum_j u_i v_j \alpha(\mathbf{e}_i)\cdot\alpha(\mathbf{e}_j) \\
&= \sum_i\sum_j u_i v_j \delta_{ij} \\
&= \sum_i u_i v_i \\
&= \left(\sum_i u_i \mathbf{e}_i\right)\cdot\left(\sum_j v_j \mathbf{e}_j\right)
\end{aligned}$$

and so α is orthogonal. ■

Proposition 212 *An orthogonal map is an isometry of an inner product space.*

Proof. Say α is orthogonal. Then, by linearity,

$$d(\alpha(v),\alpha(w))^2 = \|\alpha(v-w)\|^2 = \langle\alpha(v-w),\alpha(v-w)\rangle = \langle v-w, v-w\rangle = \|v-w\|^2 = d(v,w)^2.$$

Hence α is an isometry. In fact, it can be shown that any linear isometry of a finite-dimensional vector space is orthogonal. (This is proven in the Geometry course for $\mathbb{R}^n$.) ■

Definition 213 *Let V be an inner product space. We say that $\{v_1,\ldots,v_k\}\subseteq V$ is an* ***orthonormal set*** *if for all i, j we have*

$$\langle v_i, v_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

So the vectors are of unit length and are mutually perpendicular.

Lemma 214 *In an inner product space V, an orthonormal set is linearly independent.*

Proof. Say $\{v_1,\ldots,v_k\}$ is orthonormal and $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$ such that $\alpha_1 v_1+\cdots+\alpha_k v_k = 0_V$. Then for $1\leqslant i\leqslant k$ we have

$$\begin{aligned}
0 = \langle 0_V, v_i\rangle &= \langle\alpha_1 v_1+\cdots+\alpha_k v_k, v_i\rangle \\
&= \alpha_1\langle v_1, v_i\rangle+\cdots+\alpha_k\langle v_k, v_i\rangle \\
&= \alpha_i
\end{aligned}$$

so $\alpha_1=\cdots=\alpha_k=0$. ■

Remark 215 *Note then that n orthonormal vectors in an n-dimensional inner product space is an orthonormal basis. It is the case that every finite-dimensional inner product space has an orthonormal basis, but this result will be proved in Linear Algebra II.*

Recall that a matrix $X \in \mathcal{M}_{n\times n}(\mathbb{R})$ is **orthogonal** if $XX^T = I_n = X^T X$. Equivalently, X is orthogonal if X is invertible and $X^{-1} = X^T$.

Proposition 216 *Take $X \in \mathcal{M}_{n\times n}(\mathbb{R})$. Consider $\mathbb{R}^n$ (or $\mathbb{R}^n_{\text{col}}$) equipped with the usual inner product $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y}$. The following are equivalent:*
(a) $XX^T = I_n$;
(b) $X^T X = I_n$;
(c) the rows of X form an orthonormal basis of $\mathbb{R}^n$;
(d) the columns of X form an orthonormal basis of $\mathbb{R}^n_{\text{col}}$;
(e) for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n_{\text{col}}$, we have $X\mathbf{x} \cdot X\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

Proof. (a) $\Leftrightarrow$ (b): For any A, $B \in \mathcal{M}_{n\times n}(\mathbb{R})$, we have $AB = I_n$ if and only if $BA = I_n$.
(a) $\Leftrightarrow$ (c): Say the rows of X are $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Note that the (i,j)th entry of XX^T is $\mathbf{x}_i \cdot \mathbf{x}_j$. But $XX^T = I_n$ if and only if the (i,j) entry of XX^T is δ_{ij}, i.e. if and only if the rows are orthonormal. As there are n rows then the rows further form an orthonormal basis.
(b) $\Leftrightarrow$ (d): Say the columns of X are $\mathbf{y}_1, \ldots, \mathbf{y}_n$. We see that the (i,j)th entry of $X^T X$ is $\mathbf{y}_i \cdot \mathbf{y}_j$. The remainder of the argument is as given above.
(b) $\Rightarrow$ (e): Recall that we can identify $\mathbf{x} \cdot \mathbf{y}$ with $\mathbf{x}^T \mathbf{y}$. Assume that $X^T X = I_n$ and take $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n_{\text{col}}$. Then

$$\begin{aligned}(X\mathbf{x}) \cdot (X\mathbf{y}) &= (X\mathbf{x})^T (X\mathbf{y}) \\ &= (\mathbf{x}^T X^T)(X\mathbf{y}) \\ &= \mathbf{x}^T (X^T X)\mathbf{y} \\ &= \mathbf{x}^T I_n \mathbf{y} \\ &= \mathbf{x}^T \mathbf{y} \\ &= \mathbf{x} \cdot \mathbf{y}.\end{aligned}$$

(e) $\Rightarrow$ (d): Assume that $X\mathbf{x} \cdot X\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n_{\text{col}}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n_{\text{col}}$. But then

$$\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = X\mathbf{e}_i \cdot X\mathbf{e}_j.$$

And so $X\mathbf{e}_1, \ldots, X\mathbf{e}_n$, which are the columns of X, form an orthonormal basis. ∎

Remark 217 *Condition (e) says that the map $R_X \colon \mathbb{R}^n \to \mathbb{R}^n$ sending $\mathbf{x}$ to $\mathbf{x}X$ preserves the inner product, and hence preserves length and angle. Such a map is called an **isometry** of the Euclidean space $\mathbb{R}^n$. So the previous proposition says that X is orthogonal if and only if the map R_X is an isometry.*

8.4 Complex inner product spaces

Whilst the above theory of inner products applies very much to real vector spaces, rather than to vector spaces over a general field, the theory can be adapted and extended to vector spaces over $\mathbb{C}$. However, the usual dot product on $\mathbb{C}^n$ isn't an inner product: it is bilinear and symmetric but we'd find in $\mathbb{C}^2$ that

$$\|(1,i)\|^2 = 1^2 + i^2 = 0$$

even though $(1,i) \neq (0,0)$. We can avoid this problem by defining the standard inner product on $\mathbb{C}^n$ to be

$$(z_1, \ldots, z_n) \cdot (w_1, \ldots, w_n) = \sum_{i=1}^{n} z_i \overline{w_i}.$$

We then have that

$$(z_1, \ldots, z_n) \cdot (z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i \overline{z_i} = \sum_{i=1}^{n} |z_i|^2$$

which is non-negative and zero if and only if $(z_1, \ldots, z_n) = \mathbf{0}$.

Note that this form is linear in the first variable, positive definite, but "conjugate symmetric" in the sense that

$$\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}.$$

Definition 218 *Let V be a complex vector space. A function $\langle -,- \rangle \colon V \times V \to \mathbb{C}$ is a* ***sesquilinear form*** *if*

(a) $\langle \alpha_1 v_1 + \alpha_2 v_2, v_3 \rangle = \alpha_1 \langle v_1, v_3 \rangle + \alpha_2 \langle v_2, v_3 \rangle$ *for all v_1, v_2, $v_3 \in V$ and α_1, $\alpha_2 \in \mathbb{C}$; and*$\|\|$

(b) $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ *for all v_1, $v_2 \in V$.*

In particular, we have $\langle v, v \rangle \in \mathbb{R}$ for all $v \in V$. We say that a sesquilinear form is ***positive definite*** *if $\langle v, v \rangle \geqslant 0$ for all $v \in V$, with $\langle v, v \rangle = 0$ if and only if $v = 0$.*

A ***complex inner product space*** *is a complex vector space equipped with a positive definite, sesquilinear form.*

Remark 219 *The prefix* sesqui- *relates to "$1\frac{1}{2}$ times"; for example a sesquicentennary is 150 years.*

Remark 220 *Positive definite sesquilinear forms are often called* ***Hermitian forms****, and complex inner product spaces are often called* ***Hermitian spaces****.*

Remark 221 *The equivalent of orthogonal maps for real inner product spaces are the* ***unitary*** *maps. That is, a linear map $U \colon V \to V$ of a complex inner product space V is unitary if $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x, y \in V$. A unitary matrix is a square matrix such that $UU^* = I = U^*U$ where $U^* = \overline{U}^T$.*

Should you study quantum theory later, then you will see that the theory is generally set within complex inner product spaces. The wave function ψ of a particle is complex-valued and its norm-squared $\|\psi\|^2 = \psi\overline{\psi}$ is a probability density function.

You will explore inner product spaces further in Linear Algebra II and Part A Linear Algebra.