4. BASES

One key goal of this section is to develop a sensible notion of the 'dimension' of a vector space. In order to do this, we need to develop some theory that is in itself both important and interesting.

4.1 Spans and Spanning sets

Lemma 87 *Let V be a vector space over a field $\mathbb{F}$, and take $S = \{u_1, u_2, \dots, u_m\} \subseteq V$. Define*

$$U := \{\alpha_1 u_1 + \cdots + \alpha_m u_m : \alpha_1, \dots, \alpha_m \in \mathbb{F}\}.$$

Then $U \leqslant V$.

Proof. Applying the subspace test, we note.

- $0_V \in U$: have $0_V = 0u_1 + \cdots + 0u_m \in U$.
- $\lambda v_1 + v_2 \in U$: take $v_1, v_2 \in U$, say $v_1 = \alpha_1 u_1 + \cdots + \alpha_m u_m$ and $v_2 = \beta_1 u_1 + \cdots + \beta_m u_m$, where $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{F}$. Take $\lambda \in \mathbb{F}$. Then

$$\lambda v_1 + v_2 = (\lambda\alpha_1 + \beta_1)u_1 + \cdots + (\lambda\alpha_m + \beta_m)u_m \in U.$$

So, by the subspace test, $U \leqslant V$. ■

Definition 88 *Let V be a vector space over $\mathbb{F}$, take $u_1, u_2, \dots, u_m \in V$. A* ***linear combination*** *of $u_1, \dots, u_m$ is a vector $\alpha_1 u_1 + \cdots + \alpha_m u_m$ for some $\alpha_1, \dots, \alpha_m \in \mathbb{F}$. We define the* ***span*** *of $u_1, \dots, u_m$ to be*

$$\langle u_1, \dots, u_m \rangle := \{\alpha_1 u_1 + \cdots + \alpha_m u_m : \alpha_1, \dots, \alpha_m \in \mathbb{F}\}.$$

This is the smallest subspace of V that contains $u_1, \dots, u_m$.

More generally, we can define the span of any set $S \subseteq V$ (even a potentially infinite set S) as

$$\langle S \rangle := \{\alpha_1 s_1 + \cdots + \alpha_m s_m : m \geqslant 0, s_1, \dots, s_m \in S, \alpha_1, \dots, \alpha_m \in \mathbb{F}\}.$$

Note that a linear combination only ever involves finitely many elements of S, even if S is infinite. There isn't enough structure in a vector space to be able to define infinite sums. By convention the span of the empty set in $\{0_V\}$.

Definition 89 *Let V be a vector space over $\mathbb{F}$. If $S \subseteq V$ is such that $V = \langle S \rangle$, then we say that S* ***spans*** *V, and that S is a* ***spanning set*** *for V.*

Example 90 $\{(1,1),(2,-1)\}$ *spans* $\mathbb{R}^2$ *as every* (x,y) *can be written*

$$(x,y)=\left(\frac{x+2y}{3}\right)(1,1)+\left(\frac{x-y}{3}\right)(2,-1)\,.$$

Whilst the span of $\{(2,2),(-1,-1)\}$ *is the line* $y=x$ *in* $\mathbb{R}^2$.

Example 91 $\{(1,1,2),(2,-1,3)\}$ *spans the plane given parametrically as*

$$\mathbf{r}=\alpha(1,1,2)+\beta(2,-1,3) \qquad \alpha,\beta\in\mathbb{R}.$$

By eliminating α,β *from the expressions*

$$x=\alpha+2\beta, \qquad y=\alpha-\beta, \qquad z=2\alpha+3\beta,$$

then we can see this is the plane with equation

$$5x+y-3z=0.$$

Example 92 *The three vectors* $\{(1,1,2),(2,-1,3),(3,0,5)\}$ *span the same plane* $5x+y-3z=0$. *This is because*

$$(3,0,5)=(1,1,2)+(2,-1,3)$$

and so the third vector is itself a linear combination of the first two. Note that any point in the plane can be written in many different ways as a linear combination of the three vectors. For example

$$\begin{aligned}(0,3,1) &= 2(1,1,2)-1(2,-1,3)+0(3,0,5)\\ &= 1(1,1,2)-2(2,-1,3)+1(3,0,5)\\ &= 3(1,1,2)+0(2,-1,3)-1(3,0,5).\end{aligned}$$

This third vector means there is redundancy in the set. Any two of the three vectors are sufficient to span the plan. The issue here is that the three vectors are not linearly independent.

Definition 93 *Given a matrix, its* ***row space*** *is the span of its rows and its* ***column space*** *is the span of its column. For an* $m\times n$ *matrix* A, *we write* $\mathrm{Row}(A)\leqslant\mathbb{R}^n$ *for its row space and* $\mathrm{Col}(A)\leqslant\mathbb{R}^m_{\mathrm{col}}$ *for its column space.*

Example 94 *In Example 7 we met the matrix on the left below, and the matrix on the right is its RRE form.*

$$\left(\begin{array}{cccc|c}1&-1&1&3&2\\2&-1&1&2&4\\4&-3&3&8&8\end{array}\right), \qquad \left(\begin{array}{cccc|c}1&0&0&-1&2\\0&1&-1&-4&0\\0&0&0&0&0\end{array}\right).$$

A check will show that these two matrices have the same row space – we will see in Proposition 117 that EROs don't change row space. However it is clear that $(1,2,4)^T$ *is in the column space of the first matrix and not of the second – so EROs do change column space.*

4.2 Linear independence

Definition 95 *Let V be a vector space over $\mathbb{F}$. We say that $v_1,\ \ldots,\ v_m \in V$ are* ***linearly independent*** *if the only solution to the equation*

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0_V \qquad \text{where} \quad \alpha_1, \ldots, \alpha_m \in \mathbb{F}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0.$$

Otherwise $v_1,\ \ldots,\ v_m$ are said to be linearly dependent, which means there is a non-trivial linear combination of $v_1,\ \ldots,\ v_m$ which adds to 0_V.

We say that $S \subseteq V$ is linearly independent if every finite subset of S is linearly independent.

Example 96 $\{(1,1,2),(2,-1,3)\} \subseteq \mathbb{R}^2$ *is linearly independent. To check this, we see that comparing the x- and y-coordinates in*

$$\alpha(1,1,2) + \beta(2,-1,3) = (0,0,0),$$

implies

$$\alpha + \beta = 0, \qquad 2\alpha - \beta = 0.$$

These equations alone are enough to show $\alpha = \beta = 0$. Note though that these two vectors do not span $\mathbb{R}^3$.

Example 97 $\{(1,1,2),(2,-1,3),(3,0,5)\}$ *is linearly dependent. We previously noted that*

$$(1,1,2) + (2,-1,3) = (3,0,5)$$

so that

$$1(1,1,2) + 1(2,-1,3) + (-1)(3,0,5) = (0,0,0).$$

This is a non-trivial linear combination which adds up to $\mathbf{0}$.

Example 98 *Let V denote the vector space of differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. Then the set*

$$S = \{\sin x, \cos x, \sin 2x\}$$

is linearly independent. Say that

$$\alpha \sin x + \beta \cos x + \gamma \sin 2x = 0_V,$$

noting 0_V denotes the zero function, so that the above is an identity of functions. If we set $x = 0$ then this gives $\beta = 0$. If we set $x = \pi/2$ then $\alpha = 0$. Hence $\gamma = 0$ also.

Proposition 99 *Let $S = \{v_1,\ \ldots,\ v_m\}$ be a linearly independent subset of a vector space V. Then*

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = \beta_1 v_1 + \cdots + \beta_m v_m$$

if and only if $\alpha_i = \beta_i$ for all $1 \leqslant i \leqslant m$. Hence we may 'compare coefficients'.

Proof. If $\alpha_i = \beta_i$ for all $1 \leqslant i \leqslant m$ then the result clearly follows. Conversely, we can rearrange the above equation as

$$(\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_m - \beta_m)v_m = 0_V.$$

As S is linearly independent then $\alpha_i - \beta_i = 0$ for all i as required. ■

Example 100 *Let $V = \mathbb{C}$, considered as a real vector space. Then $\{1, i\}$ is linearly independent for if*

$$x + yi = 0_{\mathbb{C}}$$

then $x = \operatorname{Re} 0_{\mathbb{C}} = 0$ and $y = \operatorname{Im} 0_{\mathbb{C}} = 0$. Hence by the previous proposition 'comparing real and imaginary parts' is valid.

Example 101 *Let $V = \mathbb{R}[x]$, the vector space of polynomials with real coefficients. Then the set $S = \{1, x, x^2, \ldots\}$ is linearly independent. Recall that an infinite set is linearly independent if every finite subset is linearly independent. So say that*

$$a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0_{\mathbb{R}[x]}$$

for some coefficients $a_0, a_1, a_2, \ldots, a_n$. Recall that the above is an identity of functions. We can see that $a_0 = 0$ by setting $x = 0$. We can then see that $a_1 = 0$ by differentiating and setting $x = 0$. In a similar fashion we can see that all the coefficients are zero and that S is linearly independent.

Lemma 102 *Let $v_1, \ldots, v_m$ be linearly independent elements of a vector space V. Let $v_{m+1} \in V$. Then $v_1, v_2, \ldots, v_m, v_{m+1}$ are linearly independent are linearly independent if and only if*

$$v_{m+1} \notin \langle v_1, \ldots, v_m \rangle.$$

Proof. ($\Longleftarrow$) Suppose that $v_{m+1} \notin \langle v_1, \ldots, v_m \rangle$. Take $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{F}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_{m+1} v_{m+1} = 0_V.$$

We aim to show that the α_i are all 0. If $\alpha_{m+1} \neq 0$, then we have

$$v_{m+1} = -\frac{1}{\alpha_{m+1}}(\alpha_1 v_1 + \cdots + \alpha_m v_m) \in \langle v_1, \ldots, v_m \rangle,$$

which is a contradiction. So $\alpha_{m+1} = 0$ and hence $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0_V$. But $v_1, \ldots, v_m$ are linearly independent, so this means that $\alpha_1 = \cdots = \alpha_m = 0$ as well.

($\Longleftarrow$) Conversely say that $v_1, v_2, \ldots, v_m, v_{m+1}$ are linearly independent. If $v_{m+1} \in \langle v_1, \ldots, v_m \rangle$ then there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that

$$v_{m+1} = \alpha_1 v_1 + \cdots + \alpha_m v_m$$

so that $\alpha_1 v_1 + \cdots + \alpha_m v_m - v_{m+1} = 0_V$ which contradicts the linear independence of $v_1, v_2, \ldots, v_m, v_{m+1}$. Hence $v_{m+1} \notin \langle v_1, \ldots, v_m \rangle$. ■

4.3 Bases

Definition 103 *Let V be a vector space. A* ***basis*** *of V is a linearly independent, spanning set. (The plural is 'bases', pronounced 'bay-seas'.)*

If V has a finite basis, then we say that V is ***finite-dimensional.***

Remark 104 *It is important to note the language here. We can talk about 'a' basis of a vector space. Typically, vector spaces have many bases so we should not talk about 'the' basis. Some vector spaces have a 'standard' or 'canonical' basis though.*

Remark 105 *Not every vector space is finite-dimensional. For example, the space of real polynomials or the space of real sequences do not have finite bases. But in this course we'll generally study finite-dimensional vector spaces. The courses on Functional Analysis in Parts B and C (third and fourth year) explore the theory of infinite-dimensional vector spaces which have further analytical structure. Note in a vector space that only finite sums are well-defined. To meaningfully form an infinite sum, a notion of convergence is needed which is why further structure is needed.*

Where possible, we will work with general vector spaces, but sometimes we'll need to specialise to the finite-dimensional case.

Example 106 *In $\mathbb{R}^n$, for $1 \leqslant i \leqslant n$, let $\mathbf{e}_i$ be the row vector with coordinate 1 in the ith entry and 0 elsewhere. Then $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent: if*

$$\alpha_1\mathbf{e}_1 + \cdots + \alpha_n\mathbf{e}_n = \mathbf{0}$$

then by looking at the ith entry we see that $\alpha_i = 0$ for all i. Also, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span $\mathbb{R}^n$, because

$$(a_1, \ldots, a_n) = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n.$$

So $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a basis of $\mathbb{R}^n$. We call it the ***standard basis*** *or* ***canonical basis*** *of $\mathbb{R}^n$.*

Example 107 *Let $V = M_{m\times n}(\mathbb{F})$ denote the vector space of $m \times n$ matrices over a field $\mathbb{F}$. Then the standard basis of V is the set*

$$\{E_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$$

which has entry of 1 for the (i, j)th entry, and all other entries are zero. Note that a matrix $A = (a_{ij})$ can be written

$$A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}$$

and this is the unique expression of A as a linear combination of the standard basis.

Example 108 *Let $V = \{(x.y.z) \in \mathbb{R}^3 \mid x + 2y + z = 0\} \leqslant \mathbb{R}^3$. Then a basis for V is*

$$\{(1, 0, -1), (0, 2, -1)\}.$$

To see this note that x and y can be used to parameterize V and a general vector can be written uniquely as

$$(x, y, -x - 2y) = x(1, 0, -1) + y(0, 2, -1).$$

Example 109 *Let $V \leqslant \mathbb{R}^5$ be the space of vectors $(x_1, x_2, x_3, x_4, x_5)$ satisfying the three equations*

$$\begin{aligned} x_1 + x_2 - x_3 + x_5 &= 0; \\ x_1 + 2x_2 + x_4 + 3x_5 &= 0; \\ x_2 + x_3 + x_4 + 2x_5 &= 0. \end{aligned}$$

We can represent these equations as

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \overset{\text{RRE}}{\longrightarrow} \begin{pmatrix} 1 & 0 & -2 & -1 & -1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we assign parameters to the last three columns (as there are no leading $1s$ in these columns) by setting $x_3 = \alpha, x_4 = \beta, x_5 = \gamma$ then

$$x_1 = 2\alpha + \beta + \gamma, \qquad x_2 = -\alpha - \beta - 2\gamma$$

and hence

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (2\alpha + \beta + \gamma, -\alpha - \beta - 2\gamma, \alpha, \beta, \gamma) \\ &= \alpha(2, -1, 1, 0, 0) + \beta(1, -1, 0, 1, 0) + \gamma(1, -2, 0, 0, 1). \end{aligned}$$

So a basis for V is

$$\{(2, -1, 1, 0, 0), (1, -1, 0, 1, 0), (1, -2, 0, 0, 1)\}.$$

Example 110 *The space $\mathbb{F}[x]$ of polynomials over a field $\mathbb{F}$ (that is, with coefficients from the field $\mathbb{F}$) has standard basis*

$$\{1, x, x^2, x^3, \ldots\}.$$

Every polynomial can be uniquely written as a finite linear combination of this basis.

Proposition 111 *Let V be a vector space over $\mathbb{F}$, let $S = \{v_1, \ldots, v_n\} \subseteq V$. Then S is a basis of V if and only if every vector in V has a unique expression as a linear combination of elements of S.*

Proof. ($\Rightarrow$) Let S be a basis of V. Take $v \in V$. Since S spans V, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Further as S is linearly independent, then by Proposition 99 these scalars $\alpha_1, \ldots, \alpha_n$ are unique.

($\Leftarrow$) Conversely, suppose that every vector in V has a unique expression as a linear combination of elements of S.

- S spanning set: for any $v \in V$ we can write v as a linear combination of elements of S. So $\operatorname{span}(S) = V$.
- S linearly independent: for $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, if $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 = 0v_1 + \cdots + 0v_n$, then by uniqueness we have $\alpha_i = 0$ for all i.

So S is a basis for V. ■

Propostion 111 allows us to define:

Definition 112 *Given a basis $\{v_1, \ldots, v_n\}$ of V then every $v \in V$ can be uniquely written*

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

and the scalars $\alpha_1, \ldots, \alpha_n$ are known as the ***coordinates*** *of v with respect to the basis $\{v_1, \ldots, v_n\}$.*

Remark 113 *Thus choosing a basis $\{v_1, \ldots, v_n\}$ for a finite-dimensional vector space V identifies V with $\mathbb{R}^n$. To a vector v can be associated a coordinate vector $\mathbf{v} = (\alpha_1, \ldots, \alpha_n)$.*

A vector space has an origin, but no axes. Choosing a basis of V introduces α_i-axes into V and identifies a vector v with a coordinate vector $\mathbf{v}$. I will denote coordinate vectors in bold, or underline them when writing by hand. It is important to note that a coordinate vector is meaningless without the context of a basis as we can see in the following example.

Example 114 *Let $V = \{f : \mathbb{R} \to \mathbb{R}, f''(x) = 4f(x)\}$. Then the general solution of the differential equation can be written uniquely as*

$$f(x) = Ae^{2x} + Be^{-2x}$$

or as

$$f(x) = C \sinh 2x + D \cosh 2x.$$

So $\{e^{2x}, e^{-2x}\}$ is a basis of V as is $\{\sinh 2x, \cosh 2x\}$. Note that the same vector e^{2x} has coordinates $(A, B) = (1, 0)$ using the first basis and has coordinates $(C, D) = (1, 1)$ with respect to the second basis as

$$e^{2x} = \sinh 2x + \cosh 2x,$$

Similarly the same coordinate vector $(1, 0)$ represents the vector e^{2x} with respect to the first basis, but a different vector $\sinh 2x$ with respect to the second basis.

Remark 115 *The above, of course, raises the question of whether there is a best way to coordinatize a vector space – or equivalently a best way to choose a basis.*

Remark 116 *The question of whether all vector spaces have a basis is an important foundational one. Every vector space does have a basis provided we assume the so-called 'axiom of choice', which is not a standard axiom of set theory. However, it can be shown that a basis of a space like l^∞, the space of bounded real sequences, is necessarily uncountable. So the structure of vector spaces, solely, is not well suited to working with some infinite-dimensional vector spaces which explains why the topic of infinite-dimensional space is more one of 'functional analysis' where infinite linear combinations can be well-defined.*

We now turn to the question of how we determine whether a set of vectors is linearly independent or spanning. Recall that we write $\mathrm{Row}(M)$ for the row space of a matrix M, that is the span of the rows of M.

Proposition 117 *Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be a $k \times m$ matrix. Let $R = (r_{ij})$ be a matrix in RRE form which can be obtained by EROs from A.*

(a) The non-zero rows of R are independent.

(b) The rows of R are linear combinations of the rows of A.

(c) $\mathrm{Row}(BA)$ *is contained in* $\mathrm{Row}(A)$.

(d) If $k = m$ and B is invertible then $\mathrm{Row}(BA) = \mathrm{Row}(A)$.

(e) $\mathrm{Row}(R) = \mathrm{Row}(A)$.

Proof. (a) Denote the non-zero rows of R as $\mathbf{r}_1, \ldots, \mathbf{r}_r$ and suppose that $c_1\mathbf{r}_1 + \cdots + c_r\mathbf{r}_r = \mathbf{0}$. Say the leading 1 of $\mathbf{r}_1$ appears in the jth column. Then

$$c_1 + c_2 r_{2j} + c_3 r_{3j} + \cdots + c_r r_{rj} = 0.$$

But as R is in RRE form each of $r_{2j}, r_{3j}, \ldots, r_{rj}$ is zero, being entries under a leading 1. It follows that $c_1 = 0$. By focusing on the column which contains the leading 1 of $\mathbf{r}_2$ we can likewise show that $c_2 = 0$ and so on. As $c_i = 0$ for each i then the non-zero rows $\mathbf{r}_i$ are independent.

We shall prove (c) first and then (b) follows from it. Recall that

$$(i,j)\,\text{th entry of } BA = \sum_{s=1}^{m} b_{is}a_{sj} \qquad (1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n).$$

Thus the ith row of BA is the row vector

$$\left(\sum_{s=1}^{m} b_{is}a_{s1}, \ldots, \sum_{s=1}^{m} b_{is}a_{sn}\right) \quad = \quad \sum_{s=1}^{m} b_{is}\underbrace{(a_{s1}, a_{s2}, \cdots, a_{sn})}_{s\text{th row of } A}, \tag{4.1}$$

which is a linear combination of the rows of A. So every row of BA is in $\mathrm{Row}(A)$. A vector in the row space $\mathrm{Row}(BA)$ is a linear combination of BA's rows which, in turn, are linear combinations of A's rows. Hence $\mathrm{Row}(BA)$ is contained in $\mathrm{Row}(A)$. Because $R = E_k \cdots E_1 A$ for some elementary matrices $E_1, E_2, \ldots, E_k$ then (b) follows from (c) with $B = E_k \cdots E_1$. Now (d) also follows from (c). We know $\mathrm{Row}(BA)$ is contained in $\mathrm{Row}(A)$ and likewise $\mathrm{Row}(A) = \mathrm{Row}(B^{-1}(BA))$ is contained in $\mathrm{Row}(BA)$. Finally (e) follows from (d) by taking $B = E_k \cdots E_1$ which is invertible as elementary matrices are invertible. ■

Corollary 118 ***(Test for Independence)*** *Let A be an $m \times n$ matrix. Then* $\mathrm{RRE}(A)$ *contains a zero row if and only if the rows of A are dependent.*

Proof. We have that $\mathrm{RRE}(A) = BA$ where B is a product of elementary matrices and so invertible. Say the ith row of BA is $\mathbf{0}$. By (4.1)

$$\mathbf{0} = \sum_{s=1}^{m} b_{is}(s\text{th row of } A).$$

Now b_{is} are the entries of the ith row of B which, as B is invertible, cannot all be zero. The above then shows the rows of A are linearly dependent.

Conversely suppose that the rows of A are linearly dependent. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ denote the rows of A and, without any loss of generality, assume that $\mathbf{r}_m = c_1\mathbf{r}_1 + \cdots + c_{m-1}\mathbf{r}_{m-1}$ for real numbers $c_1, \ldots, c_{m-1}$. By performing the EROs $A_{1m}(-c_1), \ldots, A_{(m-1)m}(-c_{m-1})$ we arrive at a matrix whose mth row is zero. We can continue to perform EROs on the top $m-1$ rows, leaving the bottom row untouched, until we arrive at a matrix in RRE form. Once we have shown RRE form is unique (to follow) then we have that RRE(A) has a zero row. ■

Corollary 119 ***(Test for a Spanning Set)*** *Let A be an $m \times n$ matrix. Then the rows of A span $\mathbb{R}^n$ if and only if*

$$\mathrm{RRE}\,(A) = \begin{pmatrix} I_n \\ 0_{(m-n)n} \end{pmatrix}.$$

Proof. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ be the rows of A in $\mathbb{R}^n$ and suppose they span $\mathbb{R}^n$. Now row space is invariant under EROs. If it were the case that the ith column of RRE$\,(A)$ does not contain a leading 1 then $\mathbf{e}_i$ would not be in the row space. Consequently every column contains a leading 1 and so

$$\mathrm{RRE}\,(A) = \begin{pmatrix} I_n \\ 0_{(k-n)n} \end{pmatrix}.$$

Conversely if RRE$\,(A)$ has the above form then the rows of RRE$\,(A)$ are spanning and hence so are the original rows $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. ■

Remark 120 *The above corollaries show that if vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent in $\mathbb{R}^n$ then $k \leqslant n$. (For if $k > n$ then the RRE form will necessarily have a zero row.) They further show that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are spanning then there must be n leading 1s and hence we must have $k \geqslant n$. This then shows that a basis, any basis, of $\mathbb{R}^n$ contains n vectors.*

The above takes a coordinate approach, and relies on some results we are yet to prove – especially uniqueness of the RRE form. We will shortly prove this result more formally, without making use of coordinates, but we will see that this is generally true of finite-dimensional vector spaces. This common cardinality of all bases is called the ***dimension*** *of the vector space.*

Example 121 *(a) The vectors $\mathbf{v}_1 = (1, 2, -1, 0)$, $\mathbf{v}_2 = (2, 1, 0, 3)$, $\mathbf{v}_3 = (0, 1, 1, 1)$ in $\mathbb{R}^4$ are linearly independent. If we row reduce the matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ we get*

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{\mathrm{RRE}} \begin{pmatrix} 1 & 0 & 0 & 1.6 \\ 0 & 1 & 0 & -0.2 \\ 0 & 0 & 1 & 1.2 \end{pmatrix}$$

and hence the three vectors are independent because there is no zero row.

(b) A vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ if and only if $8x_1 + 6x_3 = x_2 + 5x_4$. One way to see this is to row reduce the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 1.6 \\ 0 & 1 & 0 & -0.2 \\ 0 & 0 & 1 & 1.2 \\ 0 & 0 & 0 & x_4 - 1.6x_1 + 0.2x_2 - 1.2x_3 \end{pmatrix}$$

which has a zero row if and only $8x_1 + 6x_3 = x_2 + 5x_4$.