



A advecção regime não-linear

Ondas de Gravidade



Modelagem gravity waves



Considere as equações da água rasa uni-dimensional , com A força de Coriolis desprezadas, e linearizado sobre um estado de repouso

$$U = 0, \Phi = \Phi_o = gH$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x} \quad \text{Dinâmica} \quad (53)$$

$$\frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x} \quad \text{Massa} \quad (54)$$

Estas equações dão suporte para as soluções da equação

$$u(x, t) = R_e [\hat{u} e^{-i(kx-vt)}] \quad h(x, t) = R_e [\hat{h} e^{-i(kx-vt)}]$$

Estas ondas são Não-dispersivas e todas elas têm a mesma velocidade de fase. Por exemplo considere a solução para as condições iniciais de $h(x,0)=f(x)$ e $h(x,0)=0$, e o domínio periódico



CTCS as solução ondas de gravidade



Discretize utilizando equações da água rasa de segunda ordem com diferenças centrada no espaço e no tempo.

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + g \frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} = 0 \quad (55)$$

$$\frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} + H \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \quad (56)$$

Dado

$$u_j^{n+1} = u_j^{n-1} - \frac{\Delta t}{\Delta x} g (h_{j+1}^n - h_{j-1}^n) \quad (57)$$

$$h_j^{n+1} = h_j^{n-1} - \frac{H\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n) \quad (58)$$

Como de costume, podemos analisar as propriedades do presente regime, a Von Neumann. Olhar para as soluções de forma

$$U_j^n = A_n e^{ikj\delta x} \quad \Phi_j^n = BA_n e^{ikj\delta x}$$

Onde ambas **A** e **B** podem ser constantes Complexas.



CTCS as solução ondas de gravidade



$$u_j^{n+1} = u_j^{n-1} - \frac{\Delta t}{\Delta x} g(h_{j+1}^n - h_{j-1}^n)$$

$$h_j^{n+1} = h_j^{n-1} - \frac{H\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n)$$

$$u_j^{n+1} = A^{n+1} e^{ikj\Delta x}$$

$$h_{j+1}^n = A^n e^{ik(j+1)\Delta x}$$

$$u_j^{n-1} = A^{n-1} e^{ikj\Delta x}$$

$$h_{j-1}^n = BA^n e^{ik(j-1)\Delta x}$$

$$A^{n+1} e^{ikj\Delta x} = A^{n-1} e^{ikj\Delta x} - \frac{gB\Delta t}{\Delta x} g(A^n e^{ik(j+1)\Delta x} - A^n e^{ik(j-1)\Delta x})$$

$$BA^{n+1} e^{ikj\Delta x} = BA^{n-1} e^{ikj\Delta x} - \frac{H\Delta t}{\Delta x} (A^n e^{ik(j+1)\Delta x} - A^n e^{ik(j-1)\Delta x})$$



CTCS as solução ondas de gravidade

$$A^n A e^{ikj\Delta x} = A^n A^{-1} e^{ikj\Delta x} - \frac{gB\Delta t}{\Delta x} (A^n e^{ik(j)\Delta x} e^{ik\Delta x} - A^n e^{ik(j)\Delta x} e^{-ik\Delta x})$$

$$A^n A B e^{ikj\Delta x} = A^n A^{-1} B e^{ikj\Delta x} - \frac{H\Delta t}{\Delta x} (A^n e^{ik(j)\Delta x} e^{ik\Delta x} - A^n e^{ik(j)\Delta x} e^{-ik\Delta x})$$

$$A^n A e^{ikj\Delta x} = A^n A^{-1} e^{ikj\Delta x} - \frac{gB\Delta t}{\Delta x} A^n e^{ikj\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x})$$

$$A^n A B e^{ikj\Delta x} = A^n A^{-1} B e^{ikj\Delta x} - \frac{H\Delta t}{\Delta x} A^n e^{ikj\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x})$$

$$A = A^{-1} - \frac{gB\Delta t}{\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x})$$

$$AB = BA^{-1} - \frac{H\Delta t}{\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x})$$



CTCS as solução ondas de gravidade

$$A^2 = 1 - \frac{gAB\Delta t}{\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x})$$

$$BA^2 = B - \frac{HA\Delta t}{\Delta x} (e^{ik\Delta x} - e^{-ik\Delta x})$$

$$A^2 + A \frac{gB\Delta t}{\Delta x} 2i \sin(k\Delta x) - 1 = 0$$

$$A^2 + A \frac{H\Delta t}{B\Delta x} 2i \sin(k\Delta x) - 1 = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



CTCS as solução ondas de gravidade

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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$A = \frac{-\frac{gB\Delta t}{\Delta x} 2i \sin(k\Delta x) \pm \sqrt{\left(\frac{gB\Delta t}{\Delta x} 2i \sin(k\Delta x)\right)^2 + 4}}{2}$$

$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{\frac{4}{4} \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x)\right)^2 + 1}$$

$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}$$



CTCS as solução ondas de gravidade



$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$|A|^2$$



CTCS as solução ondas de gravidade

$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$AA^* = \left(-\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right) \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right)$$

$$|A|^2 = \left(-\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right) \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right)$$

$$|A|^2 = \left(\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 - \frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1} + \frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1} + \left(\sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right)^2\right)$$



CTCS as solução ondas de gravidade

$$|A +|^2 = \left(\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 - \frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1} \right) + \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1} \right) + \left(\sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1} \right)^2$$

$$|A +|^2 = \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 - \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1$$

$$|A +|^2 = +1$$



CTCS as solução ondas de gravidade

$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$|A|^2$$



CTCS as solução ondas de gravidade

$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$AA^* = \left(-\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right) \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right)$$

$$|A -|^2 = \left(-\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) - \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right) \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) - \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right)$$

$$\begin{aligned} & |A -|^2 \\ &= \left(\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + \frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right) - \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right) + \left(\sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}\right)^2 \end{aligned}$$



CTCS as solução ondas de gravidade

$$\begin{aligned} & |A -|^2 \\ &= \left(\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + \frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{- \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1} \right) \\ &\quad - \left(\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \sqrt{- \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1} \right) + \left(\sqrt{- \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1} \right)^2 \end{aligned}$$

$$|A -|^2 = \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 - \left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x) \right)^2 + 1$$

$$|A -|^2 = +1$$



CTCS as solução ondas de gravidade



$$A^2 + A \frac{H\Delta t}{B\Delta x} 2i \sin(k\Delta x) - 1 = 0$$



CTCS as solução ondas de gravidade

$$A^2 + A \frac{H\Delta t}{B\Delta x} 2i \sin(k\Delta x) - 1 = 0$$

$$A = \frac{-\frac{H\Delta t}{B\Delta x} 2i \sin(k\Delta x) \pm \sqrt{\left(\frac{H\Delta t}{B\Delta x} 2i \sin(k\Delta x)\right)^2 + 4}}{2}$$

$$A = -\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



CTCS as solução ondas de gravidade



$$A = -\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$|A +|^2$$



CTCS as solução ondas de gravidade

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$$|A|^2 = \left(-\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) + \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1} \right) \left(\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) + \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1} \right)$$

$$|A|^2 = \left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 - \left(\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1} \right) + \left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1} \right) + \left(\sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x)\right)^2 + 1} \right)^2$$



CTCS as solução ondas de gravidade



$$|A +|^2 = \left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 - \left(\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 + 1} \right) + \left(\frac{H\Delta t}{B\Delta x} i \sin(k\Delta x) \sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 + 1} \right) + \left(\sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 + 1} \right)^2$$

$$|A +|^2 = \left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 + \left(\sqrt{-\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 + 1} \right)^2$$

$$|A +|^2 = -\left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 - \left(\frac{H\Delta t}{B\Delta x} \sin(k\Delta x) \right)^2 + 1$$

$$|A +|^2 = +1$$



Critério de Estabilidade para a CTC esquema

Resultando em

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x}$$

$$A = -\frac{gB\Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{gB\Delta t}{\Delta x} \sin(k\Delta x)\right)^2 + 1}$$

$$\frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x}$$

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$$u_j^{n+1} = u_j^{n-1} - \frac{\Delta t}{\Delta x} g(h_{j+1}^n - h_{j-1}^n)$$

$$h_j^{n+1} = h_j^{n-1} - \frac{H\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n)$$



Critério de Estabilidade para a CTC esquema





Gravidade onda: Exercício



Resolver numericamente a onda de gravidade das equações 55 e 56 no domínio $0 \leq X \leq 1000\text{m}$. Defina $\Delta x = 1\text{ m}$, assumir Condições de limite periódica. Suponha que a altura média dos Sistema é tal que $gH = 1 \frac{\text{m}^2}{\text{s}^2}$ ou $H = 1/g$. Defina o estado inicial Para h como sendo um triângulo

$$h(x,0) = \begin{cases} 0 & \text{Para } X < 400 \\ 0.001 (X - 400) & \text{Para } 400 \leq X \leq 500 \\ 0,2 - 0,001 (x - 400) & \text{Para } 500 \leq X \leq 600 \\ 0 & \text{Para } X > 600 \end{cases}$$

A condição inicial para o velocity é $u(x, 0) = 0$. Escolha o Intervalo de tempo tal que o sistema seja estável. Integrar para o $t=2000$,

E mostrar as soluções (Φ e U Para $T = 0\text{s}$, $T = 200\text{s}$, $T = 400\text{s}$, $T = 600\text{s}$, $T = 800\text{s}$, $T = 1000\text{s}$, $T = 1200\text{s}$, $T = 1400\text{s}$, $T = 1600\text{s}$, $T = 1800\text{s}$, $T = 2000\text{s}$.



Gravidade onda: Exercício



$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x} + K \frac{\partial^2 u}{\partial^2 x}$$

$$\frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x}$$

$$u_j^{n+1} = u_j^{n-1} - \frac{\Delta t}{\Delta x} g (h_{j+1}^n - h_{j-1}^n) + K \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$h_j^{n+1} = h_j^{n-1} - \frac{H \Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n)$$