第二节　导数的运算法则

习题　2-2

1.　推导余切函数及余割函数的导数公式:

(1)　$(\cot x)' = -\csc^2 x$;　　　(2)　$(\csc x)' = -\csc x \cot x$.

解　(1)　$(\cot x)' = (\dfrac{\cos x}{\sin x})' = \dfrac{(\cos x)'\sin x - (\sin x)'\cos x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$.

(2)　$(\csc x)' = (\dfrac{1}{\sin x})' = -\dfrac{1}{\sin^2 x}(\sin x)' = -\csc x \cot x$.

2.　求下列函数的导数:

(1)　$y = 2x^3 - \dfrac{3}{x^2} + 7$;　　　(2)　$y = \ln 2x + 2^x + x$;

(3)　$y = 2\csc x + \cot x$;　　　(4)　$y = \mathrm{e}^x \arccos x$;

(5)　$y = x^3 \log_2^x$;　　　(6)　$y = \dfrac{\ln x}{x}$;

(7)　$y = x^2 \ln x \cos x$;　　　(8)　$y = \dfrac{1+x^2}{1-x^2}$;

(9)　$y = x^a a^x \quad (a > 0)$;　　　(10)　$s = \dfrac{1+\sin t}{1+\cos t}$.

解　(1)　$y' = (2x^3)' - (\dfrac{3}{x^2})' + (7)' = 6x^2 + \dfrac{6}{x^3}$.

(2)　$y' = (\ln 2x)' + (2^x)' + (x)' = \dfrac{(2x)'}{2x} + 2^x \ln 2 + 1 = \dfrac{1}{x} + 2^x \ln 2 + 1$.

(3)　$y' = (2\csc x)' + (\cot x)' = -2\csc x \cot x - \csc^2 x$.

(4)　$y' = (\mathrm{e}^x)' \arccos x + \mathrm{e}^x (\arccos x)'$

$$= \mathrm{e}^x \arccos x - \mathrm{e}^x \frac{1}{\sqrt{1-x^2}} = \mathrm{e}^x (\arccos x - \frac{1}{\sqrt{1-x^2}}) .$$

(5)　$y' = (x^3)' \log_2^x + x^3 (\log_2^x)' = 3x^2 \log_2^x + x^3 \dfrac{1}{x \ln 2} = x^2 (3\log_2^x + \dfrac{1}{\ln 2})$.

(6) $y' = \dfrac{(\ln x)'x - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$.

(7) $y' = (x^2)'\ln x\cos x + x^2(\ln x)'\cos x + x^2\ln x(\cos x)'$

$= 2x\ln x\cos x + x\cos x - x^2\ln x\sin x$.

(8) $y' = \dfrac{(1+x^2)'(1-x^2)-(1+x^2)(1-x^2)'}{(1-x^2)^2} = \dfrac{4x}{(1-x^2)^2}$.

(9) $y' = (x^a)'a^x + x^a(a^x)' = ax^{a-1}a^x + x^a a^x\ln a = x^{a-1}a^x(x\ln a + a)$.

(10) $s' = \dfrac{(1+\sin t)'(1+\cos t)-(1+\sin t)(1+\cos t)'}{(1+\cos t)^2} = \dfrac{1+\cos t+\sin t}{(1+\cos t)^2}$.

3. 以初速 v_0 竖直上抛的物体, 其上升高度 s 与时间 t 的关系是 $s = v_0 t - \dfrac{1}{2}gt^2$, 求:

(1) 该物体的速度 $v(t)$;

(2) 该物体到达最高点的时刻.

解 (1) $v(t) = s' = v_0 - gt$.

(2) 物体到达最高点时, $v(t)=0$, 即 $v_0 - gt = 0$, 从而 $t = \dfrac{v_0}{g}$.

4. 求曲线 $y = x(\ln x - 1)$ 上横坐标为 $x = \mathrm{e}$ 的点处的切线方程和法线方程.

解 该点为 $(\mathrm{e}, 0)$, 所求切线的斜率为 $y'\big|_{x=\mathrm{e}} = (\ln x - 1 + 1)\big|_{x=\mathrm{e}} = \ln x\big|_{x=\mathrm{e}} = 1$, 从而切线方程为: $y = x - \mathrm{e}$, 法线方程为: $y = -x + \mathrm{e}$.

5. 求下列函数的导数:

(1) $y = \mathrm{e}^{-3x^2}$;

(2) $y = \cos(4-3x^2)$;

(3) $y = \arctan(\mathrm{e}^x)$;

(4) $y = (\arcsin x)^2$;

(5) $y = a^{\tan x^2}$ $(a>0)$;

(6) $y = \cos^2(\tan^3 x)$;

(7) $y = 2^{\sin^2\frac{1}{x}}$;

(8) $y = \sqrt[x]{x}$.

解 (1) $y' = \mathrm{e}^{-3x^2}(-3x^2)' = -6x\mathrm{e}^{-3x^2}$.

(2) $y' = -\sin(4-3x^2)(4-3x^2)' = 6x\sin(4-3x^2)$.

(3) $y' = \dfrac{(e^x)'}{1+e^{2x}} = \dfrac{e^x}{1+e^{2x}}$.

(4) $y' = 2(\arcsin x)(\arcsin x)' = \dfrac{2\arcsin x}{\sqrt{1-x^2}}$.

(5) $y' = a^{\tan x^2}\ln a(\tan x^2)' = a^{\tan x^2}\ln a\sec^2 x^2(x^2)' = 2\ln a\cdot x\sec^2 x^2 a^{\tan x^2}$.

(6) $y' = 2\cos(\tan^3 x)[\cos(\tan^3 x)]' = -2\cos(\tan^3 x)\sin(\tan^3 x)(\tan^3 x)'$

$= -3\sin(2\tan^3 x)\tan^2 x(\tan x)' = -3\tan^2 x\sec^2 x\sin(2\tan^3 x)$.

(7) $y' = 2^{\sin^2\frac{1}{x}}\ln 2(\sin^2\dfrac{1}{x})' = 2^{\sin^2\frac{1}{x}}2\ln 2\sin\dfrac{1}{x}(\sin\dfrac{1}{x})'$

$= 2^{\sin^2\frac{1}{x}}2\ln 2\sin\dfrac{1}{x}\cos\dfrac{1}{x}(\dfrac{1}{x})' = -\dfrac{\ln 2}{x^2}2^{\sin^2\frac{1}{x}}\sin\dfrac{2}{x}$.

(8) $y' = (e^{\frac{1}{x}\ln x})' = e^{\frac{1}{x}\ln x}(\dfrac{1}{x}\ln x)' = \sqrt[x]{x}(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x^2}) = \dfrac{1-\ln x}{x^2}\sqrt[x]{x}$.

6. 求下列函数在指定点处的导数值:

(1) $f(\varphi) = \sin 3\varphi + \dfrac{\varphi}{1-\varphi^2}$，$\varphi = 0$;

(2) $y = \dfrac{1}{x}\arcsin 2x,\ x = \dfrac{\sqrt{3}}{4}$;

(3) $y = 3e^{-5x} - 5(1-x),\ x = -1$.

解 (1) $f'(\varphi) = 3\cos 3\varphi + \dfrac{1-\varphi^2+2\varphi^2}{(1-\varphi^2)^2} = 3\cos 3\varphi + \dfrac{1+\varphi^2}{(1-\varphi^2)^2}$，$f'(0) = 4$.

(2) $y' = -\dfrac{1}{x^2}\arcsin 2x + \dfrac{2}{x\sqrt{1-4x^2}}$，$y'\big|_{x=\frac{\sqrt{3}}{4}} = \dfrac{16}{9}(3\sqrt{3}-\pi)$.

(3) $y' = -15e^{-5x} + 5,\ \ y'\big|_{x=-1} = 5(1-3e^5)$.

7.　求下列函数的导数:

(1)　$y=\ln(\sec x+\tan x)$;　　(2)　$y=\ln\tan\frac{x}{2}+\arctan(\frac{1}{2}\tan\frac{x}{2})$;

(3)　$y=\sin^n x\cos nx$;　　(4)　$y=\arcsin\sqrt{\frac{1-x}{1+x}}$;

(5)　$y=\frac{\sin x^2}{\sin^2 x}$;　　(6)　$y=\frac{1}{x+\sqrt{1+x^2}}$;

(7)　$y=x\arcsin\frac{x}{2}+\sqrt{4-x^2}$;　　(8)　$y=\mathrm{sh}\frac{2}{x}\mathrm{ch}3x$;

(9)　$y=\ln \mathrm{ch}x+\frac{1}{2\mathrm{ch}^2 x}$;　　(10)　$y=a^{a^x}+x^{a^a}+a^{x^a}$.

解　(1)　$y'=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\sec x$.

(2)　$y'=\frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}+\frac{\frac{1}{4}\sec^2\frac{x}{2}}{1+\frac{1}{4}\tan^2\frac{x}{2}}=\csc x+\frac{1}{1+3\cos^2\frac{x}{2}}$.

(3)　$y'=n\sin^{n-1}xcox\cos nx-n\sin^n x\sin nx=n\sin^{n-1}x\cos(n+1)x$.

(4)　$y'=\frac{1}{\sqrt{1-\frac{1-x}{1+x}}}\frac{1}{2\sqrt{\frac{1-x}{1+x}}}\frac{-(1+x)-(1-x)}{(1+x)^2}=-\frac{1}{(1+x)\sqrt{2x(1-x)}}$.

(5)　$y'=\frac{2x\cos x^2\sin^2 x-\sin x^2 2\sin x\cos x}{\sin^4 x}=\frac{2x\cos x^2\sin x-2\sin x^2\cos x}{\sin^3 x}$.

(6)　$y'=-\frac{1}{(x+\sqrt{1+x^2})^2}(1+\frac{2x}{2\sqrt{1+x^2}})=-\frac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})}$.

(7)　$y'=\arcsin\frac{x}{2}+x\frac{1}{2}\frac{1}{\sqrt{1-\frac{x^2}{4}}}+\frac{-2x}{2\sqrt{4-x^2}}=\arcsin\frac{x}{2}$.

(8)　$y'=\mathrm{ch}\frac{2}{x}(-\frac{2}{x^2})\mathrm{ch}3x+3\mathrm{sh}\frac{2}{x}\mathrm{sh}3x=-\frac{2}{x^2}\mathrm{ch}\frac{2}{x}\mathrm{ch}3x+3\mathrm{sh}\frac{2}{x}\mathrm{sh}3x$.

(9)　$y'=\frac{\mathrm{sh}x}{\mathrm{ch}x}-2\frac{\mathrm{sh}x}{2\mathrm{ch}^3 x}=\mathrm{th}x(1-\frac{1}{\mathrm{ch}^2 x})=\mathrm{th}^3 x$.

(10)　$y'=a^{a^x}\ln a(a^x)'+a^a x^{a^a-1}+a^{x^a}\ln a(x^a)'$

$$=\ln^2 a a^x a^{a^x}+a^a x^{a^a-1}+a\ln a x^{a-1}a^{x^a}.$$

8. 设 $f(x)$ 和 $g(x)$ 都可导，求下列函数 y 的导数 $\dfrac{dy}{dx}$.

(1) $y=f(e^x)e^{f(x)}$；　　(2) $y=f(\sin^2 x)+f(\cos^2 x)$；

(3) $y=\ln f(\sqrt{x})+\arctan g(x^2)$；　　(4) $y=\sqrt{f^2(x)+\sqrt{g(x)}}$.

解 (1) $y'=f'(e^x)e^x e^{f(x)}+f(e^x)e^{f(x)}f'(x)=f'(e^x)e^{f(x)+x}+f(e^x)e^{f(x)}f'(x)$.

(2) $y'=f'(\sin^2 x)2\sin x\cos x-f'(\cos^2 x)2\sin x\cos x$

$$=\sin 2x[f'(\sin^2 x)-f'(\cos^2 x)].$$

(3) $y'=\dfrac{f'(\sqrt{x})}{f(\sqrt{x})}\dfrac{1}{2\sqrt{x}}+\dfrac{g'(x^2)2x}{1+g^2(x^2)}=\dfrac{f'(\sqrt{x})}{2\sqrt{x}f(\sqrt{x})}+\dfrac{2xg'(x^2)}{1+g^2(x^2)}$.

(4) $y'=\dfrac{2f(x)f'(x)+\dfrac{g'(x)}{2\sqrt{g(x)}}}{2\sqrt{f^2(x)+\sqrt{g(x)}}}=\dfrac{4f(x)f'(x)\sqrt{g(x)}+g'(x)}{4\sqrt{f^2(x)+\sqrt{g(x)}}\sqrt{g(x)}}$.

9. 设 $f(x)$ 在 $(-l,l)$ 内可导，证明：如果 $f(x)$ 是偶函数，则 $f'(x)$ 是奇函数；如果 $f(x)$ 是奇函数，则 $f'(x)$ 是偶函数.

证 如果 $f(x)$ 是偶函数，则有 $f(-x)=f(x)$，对等式两边对 x 求导，有，$-f'(-x)=f'(x)$，从而 $f'(-x)=-f'(x)$，即 $f'(x)$ 是奇函数.

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