第二节

不定积分的换元积分法(1)

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一、主要内容

(一) 定理(第一类换元积分法) 设 $f(u)$具有原函数，

$u=\varphi(x)$可导，

$$\int f[\varphi(x)]\underline{\varphi'(x)\mathrm{d}x}=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

令$u=\varphi(x)$

即
$$\boxed{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f[\varphi(x)]\mathrm{d}\varphi(x)}$$

—— 换元公式

换元思想: 设变换 $u=\varphi(x)$,

化积分为易于求解的形式.

关键: 如何选择 $u=\varphi(x)$?

注　一般地，如何选择 $u=\varphi(x)$?

1°　需要熟悉一些常见函数的微分形式，

直接配元用公式：$\int f(x)\,\mathrm{d}x = F(x)+C$

$$\boxed{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f[\varphi(x)]\mathrm{d}\varphi(x) = F[\varphi(x)]+C}$$

2° 对于不易观察的情形，可从被积函数中拿出某个因式求导数，若这个导数恰是剩下的其他因式(最多相差一个常数)，则这个因式可作为 $\varphi(x)$.

(二) 常见的选 $u=\varphi(x)$ 规律

(1) $\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\left[\int f(u)\,\mathrm{d}u\right]_{u=ax+b}$

(2) $\int f(x^{\mu+1})x^{\mu}\,\mathrm{d}x$ $\quad (u=x^{\mu+1}, \mu\neq -1)$

(3) $\int \frac{f(\ln x)}{x}\,\mathrm{d}x$ $\quad (u=\ln x)$

(4) $\int f(\cos x)\sin x\,\mathrm{d}x$ $\quad (u=\cos x)$

(5) $\int f(\sin x)\cos x\,\mathrm{d}x$ $\quad (u=\sin x)$

(6) $\int \frac{f(\arcsin x)}{\sqrt{1-x^2}}\,\mathrm{d}x$ $\quad(u = \arcsin x)$

(7) $\int \frac{f(\arctan x)}{1+x^2}\,\mathrm{d}x$ $\quad(u = \arctan x)$

(8) $\int f(\tan x)\sec^2 x\,\mathrm{d}x$ $\quad(u = \tan x)$

(9) $\int f(\sec x)\sec x\tan x\,\mathrm{d}x$ $\quad(u = \sec x)$

$\vdots$

常见的选 $u=\varphi(x)$ 规律(续1)

(10) $\int \sin^m x\cos^n x\mathrm{d}x$

$$\begin{cases} m(\text{或}\,n):\text{奇数，}\quad \text{设}\,u=\cos x(\text{或}\,u=\sin x) \\ m,n\,\text{均为偶数，}\ \text{用倍角公式} \\ \text{特别地，当}\ m=n\ \text{时，}\ \text{设}\,u=\sin 2x \end{cases}$$

常见的选 $u=\varphi(x)$ 规律(续2)

(11) $\int \sin mx \cos nx \, \mathrm{d}x$

$$\int \cos mx \cos nx \, \mathrm{d}x$$

$$\int \sin mx \sin nx \, \mathrm{d}x$$

当 $m=n$ 时，用倍角公式;

当 $m \neq n$ 时，用积化和差公式 .

常见的选 $u=\varphi(x)$ 规律(续3)

(12) $\int \tan^m x \sec^n x \, \mathrm{d}x$

$$\begin{cases} n: \text{偶数，设 } u=\tan x \\ m: \text{奇数，设 } u=\sec x \end{cases}$$

$$\int \cot^m x \csc^n x \, \mathrm{d}x = \cdots$$

(三)基本积分公式的补充

(9) $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$, $\int \cot x\mathrm{d}x = \ln|\sin x| + C$

(10) $\int \sec x\mathrm{d}x = \ln|\sec x + \tan x| + C$

$$\int \csc x\mathrm{d}x = \ln|\csc x - \cot x| + C$$

(11) $\int \frac{1}{a^2 + x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$

(12) $\int \frac{1}{x^2 - a^2}\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(13) $\int \frac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x = \arcsin\frac{x}{a} + C$

二、典型例题

例1 求 $\int (ax+b)^n \mathrm{d}x \quad (a \neq 0,\ n$为自然数$)$

验证: $\left[\dfrac{(ax+b)^{n+1}}{a(n+1)}\right]' = (ax+b)^n$

解 $\int (ax+b)^n \mathrm{d}x$

$$= \frac{1}{a}\int \underbrace{(ax+b)^n}_{u} \underbrace{(ax+b)' \,\mathrm{d}x}_{du\ ?}$$

$$= \frac{1}{a}\int (ax+b)^n \mathrm{d}(ax+b) = \frac{1}{a}\int u^n \mathrm{d}u \quad (\text{令} u = ax+b)$$

$$= \frac{1}{a}\cdot\frac{u^{n+1}}{n+1} + C = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (u = ax+b \text{代入})$$

例2 $\int \cos(3x+2)\,\mathrm{d}x$

解

$$\boxed{\text{联想公式}\quad \int \cos u\,\mathrm{d}u = \sin u + C}$$

$$\text{原式} = \frac{1}{3}\int \cos(3x+2)\,\mathrm{d}(3x+2) \qquad u = 3x+2$$

$$= \frac{1}{3}\int \cos u\,\mathrm{d}u = \frac{1}{3}\sin u + C$$

$$\overset{u=3x+2}{=\!=\!=} \frac{1}{3}\sin(3x+2) + C.$$

例3 求下列不定积分:

(1) $\int xe^{x^2}\,\mathrm{d}x$

解 联想公式: $\int e^u\,\mathrm{d}u = e^u + C$

$$\therefore \int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}\int e^{x^2}\,\mathrm{d}x^2$$

$$\underline{\underline{u = x^2}}\ \frac{1}{2}\int e^u\,\mathrm{d}u = \frac{1}{2}e^{x^2} + C$$

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(2) $\int x\sqrt{1-x^2}\,\mathrm{d}x$

解 $\boxed{\text{联想公式: } \int u^{\alpha}\,\mathrm{d}x = \dfrac{u^{\alpha+1}}{\alpha+1} + C}$

$$\text{原式} = -\frac{1}{2}\int\sqrt{1-x^2}\,\mathrm{d}(1-x^2)$$

$$\underline{\underline{u=1-x^2}}\; -\frac{1}{2}\int\sqrt{u}\,\mathrm{d}u = -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$$

$$\underline{\underline{u=1-x^2}}\; -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$$

例4 (1) $\displaystyle\int \frac{\log_a x}{x}\,\mathrm{d}x$

$$(\log_a x)' = \frac{1}{\ln a \cdot x}$$

$$= \ln a \int \log_a x \,\mathrm{d}(\log_a x)$$

$$u = \log_a x$$

$$= \frac{(\log_a x)^2}{2} \cdot \ln a + C$$

(2) $\displaystyle\int \frac{1}{x(1+\ln x)^2}\,\mathrm{d}x$

$$(1+\ln x)' = \frac{1}{x}$$

$$u = 1 + \ln x$$

$$= \int \frac{\mathrm{d}(1+\ln x)}{(1+\ln x)^2} = -\frac{1}{1+\ln x} + C$$

例5 求下列不定积分:

$$\boxed{\int f(x^{\mu+1})x^{\mu}\,\mathrm{d}x \quad (u = x^{\mu+1})}$$

(1) $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,\mathrm{d}x$

$$\boxed{x^{-\frac{1}{2}}}$$

$(u=\sqrt{x})$

$$= \int \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\int \frac{1}{1+(\sqrt{x})^2}\,\mathrm{d}\sqrt{x}$$

$$= 2\arctan\sqrt{x} + C$$

(2) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x = 2\int \sin\sqrt{x}\,\mathrm{d}\sqrt{x}$

$$= -2\cos\sqrt{x} + C$$

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例6 求 $\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}\,\mathrm{d}x.$

$$\int \frac{f(\arcsin x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$(u = \arcsin x)$

解 原式 $= \int \frac{1}{\arcsin\frac{x}{2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\,\mathrm{d}x$

$$\overset{u=\frac{x}{2}}{=\!=\!=\!=} \int \frac{1}{\arcsin u}\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u$$

$$= \int \frac{1}{\arcsin u}\,\mathrm{d}(\arcsin u)$$

$$= \ln\left|\arcsin u\right| + C = \ln\left|\arcsin\frac{x}{2}\right| + C$$

例7 $\int \csc x \,\mathrm{d}x$

$$\int f(\cos x)\sin x \,\mathrm{d}x \quad (u=\cos x)$$

解 (方法1)

$$\int \csc x \,\mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x = \int \frac{1}{\sin^2 x}\cdot \sin x \,\mathrm{d}x$$

$$= -\int \frac{1}{1-\cos^2 x}\,\mathrm{d}(\cos x) \quad = -\int \frac{1}{1-u^2}\mathrm{d}u \quad (u=\cos x)$$

$$= -\frac{1}{2}\int \left(\frac{1}{1-u}+\frac{1}{1+u}\right)\mathrm{d}u = -\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C$$

$$= -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + C.$$

(方法2) $\int \csc x \,\mathrm{d}x = \int \frac{\csc x \cdot (\csc x + \cot x)}{\csc x + \cot x} \mathrm{d}x$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \mathrm{d}x$$

$$= -\int \frac{1}{\csc x + \cot x} \mathrm{d}(\cot x + \csc x)$$

$$= -\ln\left|\csc x + \cot x\right| + C \overset{\text{或}}{=} \ln\left|\csc x - \cot x\right| + C$$

类似地， $\int \sec x \,\mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$

$$(\cot x + \csc x)' = -\csc^2 x - \csc x \cot x$$

(方法3) $\int \csc x \,\mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}}\mathrm{d}x = \frac{1}{2}\int \frac{1}{\tan\frac{x}{2}} \cdot \sec^2\frac{x}{2}\mathrm{d}x$$

$$= \int \frac{1}{\tan\frac{x}{2}}\mathrm{d}(\tan\frac{x}{2}) = \ln\left|\tan\frac{x}{2}\right| + C.$$

$$\int f(\tan x)\sec^2 x\,\mathrm{d}x$$
$(u = \tan x)$

注　三种方法，积分结果形式上各不相同，但它们最多相差一个常数.

例8 求下列不定积分:

$$\int f(\cos x)\sin x\,\mathrm{d}x \quad (u=\cos x)$$

(1) $\int \sin^3 x\cos^2 x\,\mathrm{d}x$

$$= \int (1-\cos^2 x)\cos^2 x\cdot\sin x\,\mathrm{d}x$$

$$= -\int (1-\cos^2 x)\cos^2 x\,\mathrm{d}(\cos x)$$

$$= -\int (\cos^2 x-\cos^4 x)\,\mathrm{d}(\cos x)$$

$$= -\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+C.$$

$$(2)\int \cos^4 x\,\mathrm{d}x$$

$$=\int \left(\frac{1+\cos 2x}{2}\right)^2 \mathrm{d}x=\cdots(\text{降次})$$

$$=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

$$(3)\int \sin^2 x\cos^2 x\,\mathrm{d}x=\frac{1}{4}\int \sin^2 2x\,\mathrm{d}x$$

$$=\frac{1}{4}\int \frac{1-\cos 4x}{2}\mathrm{d}x=\frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C.$$

例9 求 $\int \cos 3x \cos 2x \, dx$.

分析 当被积函数为 $\sin ax \cos bx$, $\sin ax \sin bx$, 或 $\cos ax \cos bx$ 的形式时，常用积化和差公式将被积函数化简后再积分.

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$,

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x \, dx = \frac{1}{2}\int (\cos x + \cos 5x) \, dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

例10 (1) $\int \tan^2 x \sec^6 x \,\mathrm{d}x$

$$\int f(\tan x)\sec^2 x \,\mathrm{d}x \quad (u = \tan x)$$

$= \int \tan^2 x(1+\tan^2 x)^2 \cdot \sec^2 x \,\mathrm{d}x$

$= \int \tan^2 x\,(1+\tan^2 x)^2 \mathrm{d}\tan x = \cdots$

(2) $\int \tan^5 x \sec^3 x \,\mathrm{d}x$

$$\int f(\sec x)\sec x \tan x \,\mathrm{d}x \quad (u = \sec x)$$

$= \int \tan^4 x \sec^2 x \,\mathrm{d}\sec x$

$= \int (\sec^2 x - 1)^2 \sec^2 x \,\mathrm{d}\sec x$

$= \cdots$

例11 $\int \frac{\mathrm{d}x}{\sqrt{1-2x-x^2}}$

$$= \int \frac{\mathrm{d}(x+1)}{\sqrt{2-(x+1)^2}}$$

$$= \arcsin\left(\frac{x+1}{\sqrt{2}}\right) + C$$

例12 (1) $\int \frac{2x}{1+x^2}\,\mathrm{d}x = \int \frac{\mathrm{d}(x^2+1)}{x^2+1} = \ln(x^2+1)+C$

(2) $\int \frac{e^{2x}}{e^{2x}+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(e^{2x}+1)}{e^{2x}+1} = \frac{1}{2}\ln(e^{2x}+1)+C$

(3) $\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{1+e^x-e^x}{1+e^x}\,\mathrm{d}x$

分解

$$= \int \left(1-\frac{e^x}{1+e^x}\right)\mathrm{d}x$$

$$= x-\ln(1+e^x)+C$$

$$\int \frac{\mathrm{d}u}{u} = \ln u + C$$

例13 $\int \frac{x^3}{(x^2+a^2)^{3/2}}\mathrm{d}x = \frac{1}{2}\int \frac{x^2\,\mathrm{d}(x^2+a^2)}{(x^2+a^2)^{3/2}}$

$$= \frac{1}{2}\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\mathrm{d}(x^2+a^2)$$

$$\boxed{u = x^2+a^2}$$

$$= \frac{1}{2}\int (x^2+a^2)^{-1/2}\mathrm{d}(x^2+a^2) - \frac{a^2}{2}\int (x^2+a^2)^{-3/2}\mathrm{d}(x^2+a^2)$$

$$= \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}} + C$$

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例14 $\displaystyle\int \frac{x+1}{x(1+xe^{x})}\mathrm{d}x$

$(1+xe^{x})' = (1+x)\ e^{x}$

$$=\int \frac{(x+1)e^{x}}{xe^{x}(1+xe^{x})}\mathrm{d}x = \int \frac{\mathrm{d}(1+xe^{x})}{xe^{x}(1+xe^{x})}$$ (令 $u=1+xe^{x}$)

$$=\int \frac{\mathrm{d}u}{(u-1)u} = \int \left(\frac{1}{u-1}-\frac{1}{u}\right)\mathrm{d}u$$

$$=\ln|u-1|-\ln|u|+C$$

$$=\ln\left|xe^{x}\right|-\ln\left|1+xe^{x}\right|+C$$

例15 求 $\int\left[\frac{f(x)}{f'(x)}-\frac{f''(x)f^2(x)}{f'^3(x)}\right]\mathrm{d}\,x.$

解　原式 $=\int\frac{f(x)}{f'(x)}\left[1-\frac{f''(x)f(x)}{f'^2(x)}\right]\mathrm{d}x$

$$=\int\frac{f(x)}{f'(x)}\cdot\underline{\frac{f'^2(x)-f''(x)f(x)}{f'^2(x)}}\,\mathrm{d}x$$

$$=\int\frac{f(x)}{f'(x)}\mathrm{d}\left(\frac{f(x)}{f'(x)}\right)=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$$

例16 求 $\int \frac{1}{1+\cos x} dx.$

解 (方法1) $\int \frac{1}{1+\cos x} \mathrm{d}x = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} \mathrm{d}x$

$$= \int \frac{1-\cos x}{1-\cos^2 x} \mathrm{d}x = \int \frac{1-\cos x}{\sin^2 x} \mathrm{d}x$$

$$= \int \frac{1}{\sin^2 x} \mathrm{d}x - \int \frac{1}{\sin^2 x} \mathrm{d}(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

(方法2) $\int \frac{1}{1+\cos x}\mathrm{d}x = \int \frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x$

$$= \int \sec^2\frac{x}{2}\mathrm{d}\left(\frac{x}{2}\right)$$

$$= \tan\frac{x}{2} + C.$$

例17 求 $\displaystyle\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-1}}\,\mathrm{d}x.$

解 原式 $\displaystyle=\int \frac{\sqrt{2x+3}-\sqrt{2x-1}}{(\sqrt{2x+3}+\sqrt{2x-1})(\sqrt{2x+3}-\sqrt{2x-1})}\,\mathrm{d}x$

$$=\frac{1}{4}\int\sqrt{2x+3}\,\mathrm{d}x-\frac{1}{4}\int\sqrt{2x-1}\,\mathrm{d}x$$

$$=\frac{1}{8}\int\sqrt{2x+3}\,\mathrm{d}(2x+3)-\frac{1}{8}\int\sqrt{2x-1}\,\mathrm{d}(2x-1)$$

$$=\frac{1}{12}\left(\sqrt{2x+3}\right)^3-\frac{1}{12}\left(\sqrt{2x-1}\right)^3+C.$$

例18 求 $\int\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x=\int\frac{1-x}{\sqrt{1-x^2}}\,\mathrm{d}x$

$$=\int\left(\frac{1}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}\right)\mathrm{d}x$$

$$=\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x-\int\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$=\arcsin x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\,\mathrm{d}(1-x^2)$$

$$=\arcsin x+\sqrt{1-x^2}+C$$

例19 $\int \frac{1+\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x \quad (u=1+\arctan\sqrt{x})$

$$=2\int(1+\arctan\sqrt{x})\mathrm{d}(1+\arctan\sqrt{x})$$

$$=(1+\arctan\sqrt{x})^2+C$$

$$(1+\arctan\sqrt{x})'=\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}(1+x)}$$

例20 $\int \frac{x}{\sqrt{1-4x^4}}\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{1-4x^4}}\mathrm{d}x^2$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1-(2x^2)^2}}\mathrm{d}(2x^2)$$

$$= \frac{1}{4}\arcsin(2x^2) + C$$

三、同步练习

1. $\int \mathrm{e}^{ax+b}\,\mathrm{d}x$

2. $\int \frac{\mathrm{d}x}{x^2+2x+3}$

3. (1) $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$ (2) $\int \frac{12x-16}{3x^2-8x+4}\mathrm{d}x$

4. $\int \sec x\,\mathrm{d}x$

5. $\int \sin x\, \mathrm{e}^{\cos x}\,\mathrm{d}x$

6. $\int \sin^4 x\,\mathrm{d}x$

四、同步练习解答

1. 求 $\int \mathrm{e}^{ax+b}\,\mathrm{d}x$

解 令 $u = ax + b$，则 $\mathrm{d}u = a\,\mathrm{d}x$，

$$\therefore \int \mathrm{e}^{ax+b}\,\mathrm{d}x = \frac{1}{a}\int \mathrm{e}^{u}\,\mathrm{d}u$$

$$= \frac{1}{a}\mathrm{e}^{u} + C$$

$$= \frac{1}{a}\mathrm{e}^{ax+b} + C$$

2. $\int \frac{\mathrm{d}x}{x^2+2x+3}$

$$= \int \frac{\mathrm{d}(x+1)}{2+(x+1)^2}$$

$$\int \frac{\mathrm{d}u}{1+u^2} = \arctan u + C$$

$$= \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}(\frac{x+1}{\sqrt{2}})}{1+(\frac{x+1}{\sqrt{2}})^2}$$

$$u = \frac{x+1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \arctan(\frac{x+1}{\sqrt{2}}) + C$$

3. 求下列不定积分:

(1) $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\mathrm{d}x = 2\int e^{3\sqrt{x}}\mathrm{d}\sqrt{x}$

$$\boxed{u = 3\sqrt{x}}$$

$= \frac{2}{3}\int e^{3\sqrt{x}}\mathrm{d}(3\sqrt{x}) = \frac{2}{3}e^{3\sqrt{x}} + C$

(2) $\int \frac{12x - 16}{3x^2 - 8x + 4}\mathrm{d}x = 2\int \frac{\mathrm{d}(3x^2 - 8x + 4)}{3x^2 - 8x + 4}$

$= 2\ln\left|3x^2 - 8x + 4\right| + C$

4. $\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$

$= \frac{1}{2}\int \left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right]\mathrm{d}\sin x$

$= \frac{1}{2}\left[\ln|1+\sin x| - \ln|1-\sin x|\right] + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$

解 2 $\int \sec x \,\mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\mathrm{d}x$

$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C$

$\boxed{\int \frac{\mathrm{d}u}{u} = \ln u + C}$

类似 $\boxed{\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C}$

5. 求 $\int \sin x\, \mathrm{e}^{\cos x}\, \mathrm{d}x$

解 被积函数中的一个因子为 $\mathrm{e}^{\cos x} = \mathrm{e}^{u}, u = \cos x$

剩下的因子 $\sin x$，恰巧是中间变量 $u = \cos x$ 的

导数，于是有

$$\int \sin x\, \mathrm{e}^{\cos x}\, \mathrm{d}x = -\int \mathrm{e}^{\cos x}\, \mathrm{d}\cos x$$

$$= -\int \mathrm{e}^{u}\, \mathrm{d}u = -\mathrm{e}^{u} + C = -\mathrm{e}^{\cos x} + C$$

6. 求$\int \sin^4 x\,\mathrm{d}x$

解 $\int \sin^4 x\,\mathrm{d}x = \int \left(\frac{1-\cos 2x}{2}\right)^2 \mathrm{d}x$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\mathrm{d}x$

$= \frac{1}{8}\int \left(3 - 4\cos 2x + \cos 4x\right)\mathrm{d}x$

$= \frac{1}{8}\left(3x - 2\sin 2x + \frac{\sin 4x}{4}\right) + C$