第四节　高阶导数

习题　2-4

1. 求下列函数的二阶导数:

(1) $y = 3x^2 + e^{2x} + \ln x$;　　(2) $y = x\cos x$;

(3) $y = e^{-t}\sin t$;　　(4) $y = (1+x^2)\arctan x$.

解 (1) $y' = 6x + 2e^{2x} + \dfrac{1}{x}$, $y'' = 6 + 4e^{2x} - \dfrac{1}{x^2}$.

(2) $y' = \cos x - x\sin x$, $y'' = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x$.

(3) $y' = -e^{-t}\sin t + e^{-t}\cos t$,

$$y'' = e^{-t}\sin t - e^{-t}\cos t - e^{-t}\cos t - e^{-t}\sin t = -2e^{-t}\cos t.$$

(4) $y' = 2x\arctan x + 1$, $y'' = 2\arctan x + \dfrac{2x}{1+x^2}$.

2. 设 $f(x) = (x+10)^6$, 求 $f'''(2)$.

解 由于 $f'(x) = 6(x+10)^5$, $f''(x) = 30(x+10)^4$, $f'''(x) = 120(x+10)^3$, 所以

$$f'''(2) = 12^3 \cdot 120 = 207360.$$

3. 设 $f''(x)$ 存在, 求下列函数 y 的二阶导数 $\dfrac{d^2y}{dx^2}$:

(1) $y = f(e^{-x})$;　　(2) $y = \ln f(x)$.

解 (1) $y' = -f'(e^{-x})e^{-x}$, $y'' = e^{-2x}f''(e^{-x}) + e^{-x}f'(e^{-x})$.

(2) $y' = \dfrac{f'(x)}{f(x)}$, $y'' = \dfrac{f''(x)f(x) - f'^2(x)}{f^2(x)}$.

4. 试从 $\dfrac{dx}{dy} = \dfrac{1}{y'}$ 导出:

(1) $\dfrac{d^2x}{dy^2}=-\dfrac{y''}{(y')^3}$; (2) $\dfrac{d^3x}{dy^3}=\dfrac{3(y'')^2-y'y'''}{(y')^5}$.

证 (1) $\dfrac{d^2x}{dy^2}=\dfrac{d}{dy}(\dfrac{1}{y'})=\dfrac{d}{dx}(\dfrac{1}{y'})\dfrac{dx}{dy}=-\dfrac{y''}{(y')^2}\dfrac{1}{y'}=-\dfrac{y''}{(y')^3}$.

(2) $\dfrac{d^3x}{dy^3}=\dfrac{d}{dy}(\dfrac{d^2x}{dy^2})=\dfrac{d}{dx}(-\dfrac{y''}{(y')^3})\dfrac{dx}{dy}=\dfrac{3(y')^2(y'')^2-y'''(y')^3}{(y')^6}\dfrac{1}{y'}=\dfrac{3(y'')^2-y'y'''}{(y')^5}$.

5. 验证函数 $y=e^x\sin x$ 满足关系式

$$y''-2y'+2y=0.$$

解 因为

$$y'=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x),$$

$$y''=e^x(\sin x+\cos x)+e^x(\cos x-\sin x)=2e^x\cos x,$$

故而

$$y''-2y'+2y=2e^x\cos x-2e^x(\sin x+\cos x)+2e^x\sin x=0.$$

6. 求下列函数的 n 阶导数的表达式:

(1) $y=\sin^2 x$; (2) $y=x\ln x$;

(3) $y=xe^x$; (4) $y=\dfrac{1}{x^2-a^2}$;

(5) $\ln\dfrac{1+x}{1-x}$.

解 (1) $y=\sin^2 x=\dfrac{1}{2}(1-\cos 2x)$.

$$y^{(n)}=-\frac{2^n}{2}\cos(2x+n\frac{\pi}{2})=-2^{n-1}\cos(2x+n\frac{\pi}{2})=2^{n-1}\sin[2x+(n-1)\frac{\pi}{2}].$$

(2) $y'=\ln x+1,\ y''=\dfrac{1}{x}$,

$$y^{(n)}=(\frac{1}{x})^{(n-2)}=(-1)^{n-2}(n-2)!\frac{1}{x^{n-1}}=(-1)^n(n-2)!\frac{1}{x^{n-1}}\,(n\geq 2).$$

(3) $y'=e^x+xe^x=(x+1)e^x$, $y''=e^x+(x+1)e^x=(x+2)e^x$, $y^{(n)}=(x+n)e^x$.

(4) $y=\dfrac{1}{2a}(\dfrac{1}{x-a}-\dfrac{1}{x+a})$,

$$y^{(n)}=\frac{1}{2a}[(\frac{1}{x-a})^{(n)}-(\frac{1}{x+a})^{(n)}]=\frac{1}{2a}[\frac{(-1)^n n!}{(x-a)^{n+1}}-\frac{(-1)^n n!}{(x+a)^{n+1}}]$$

$$=\frac{(-1)^n n!}{2a}[\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}].$$

(5) $y'=[\ln(1+x)-\ln(1-x)]'=\frac{1}{x+1}-\frac{1}{x-1}$,

$$y^{(n)}=(\frac{1}{x+1}-\frac{1}{x-1})^{(n-1)}=\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}-\frac{(-1)^{n-1}(n-1)!}{(x-1)^n}$$

$$=(-1)^{n-1}(n-1)![\frac{1}{(x+1)^n}-\frac{1}{(x-1)^n}].$$

7. 求下列函数的指定阶的导数:

(1) $y=\mathrm{e}^x\cos x$，求 $y^{(4)}$； (2) $y=x^2\sin 2x$，求 $y^{(50)}$.

解 (1) $y'=\mathrm{e}^x\cos x-\mathrm{e}^x\sin x=\mathrm{e}^x(\cos x-\sin x)$,

$$y''=\mathrm{e}^x(\cos x-\sin x)-\mathrm{e}^x(\sin x+\cos x)=-2\mathrm{e}^x\sin x,$$

$$y'''=-2\mathrm{e}^x\cos x-2\mathrm{e}^x\sin x=-2\mathrm{e}^x(\cos x+\sin x),$$

$$y^{(4)}=-2\mathrm{e}^x(\cos x+\sin x)-2\mathrm{e}^x(-\sin x+\cos x)=-4\mathrm{e}^x\cos x.$$

(2) $y^{(50)}=(x^2\sin 2x)^{(50)}=x^2(\sin 2x)^{(50)}+50\cdot 2x(\sin 2x)^{(49)}+\frac{50\cdot 49}{2}\cdot 2(\sin 2x)^{(48)}$

$$=x^2 2^{50}\sin(2x+25\pi)+100x2^{49}\sin(2x+\frac{49}{2}\pi)+2450\cdot 2^{48}\sin(2x+24\pi)$$

$$=2^{50}(-x^2\sin 2x+50x\cos 2x+\frac{1225}{2}\sin 2x).$$

8. 求下列方程所确定的隐函数 y 的二阶导数 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$:

(1) $y=\tan(x+y)$； (2) $xy=\mathrm{e}^{x+y}$.

解 (1) 对方程两边对 x 求导，得

$$y'=\sec^2(x+y)(1+y'),\quad y'=-\csc^2(x+y),$$

对 $y'=-\csc^2(x+y)$ 两边对 x 求导， 得

$$y'' = 2\csc^2(x+y)\cot(x+y)(1+y') = -2\csc^2(x+y)\cot^3(x+y) .$$

(2) 对方程两边对 x 求导，得

$$y + xy' = \mathrm{e}^{x+y}(1+y'),\ \ y' = \frac{\mathrm{e}^{x+y} - y}{x - \mathrm{e}^{x+y}} = \frac{xy - y}{x - xy},$$

对 $y + xy' = \mathrm{e}^{x+y}(1+y')$ 两边对 x 继续求导,

$$2y' + xy'' = \mathrm{e}^{x+y}(1+y')^2 + \mathrm{e}^{x+y}y'',$$

$$y'' = \frac{\mathrm{e}^{x+y}(1+y')^2 - 2y'}{x - \mathrm{e}^{x+y}} = \frac{xy(1+y')^2 - 2y'}{x - xy} = \frac{y[(x-1)^2 + (y-1)^2]}{x^2(1-y)^3} .$$

9. 求下列参数方程所确定的函数 y 的二阶导数 $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$:

(1) $\begin{cases} x = a\cos t, \\ y = b\sin t; \end{cases}$ (2) $\begin{cases} x = f'(t), \\ y = tf'(t) - f(t). \end{cases}$ 其中 $f''(t)$ 存在且不为零.

解 (1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{b\cos t}{-a\sin t} = -\dfrac{b}{a}\cot t$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}(-\frac{b}{a}\cot t) = \frac{\mathrm{d}}{\mathrm{d}t}(-\frac{b}{a}\cot t)\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{b}{a}\csc^2 t\,\frac{1}{-a\sin t} = -\frac{b}{a^2\sin^3 t} .$$

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{tf''(t)}{f''(t)} = t,\ \ \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}t}{\mathrm{d}x} = \dfrac{1}{f''(t)}$.