总复习(五)

积分学

- 一、主要内容
- 二、典型例题

一、主要内容

1. 积分法

A. 不定积分法

(1) 性质

(2) 基本积分公式

(3) 第一换元法

(凑微分法)

(4) 第二换元法

5种代换

(5) 分部积分法

选u的原则

B. 定积分法

(1) 定义

(2) 性质　① 线性性；　⑤ 估值性；

　　　　　② 可加性；　⑥ 积分中值定理

　　　　　③ 保号性；　⑦ 奇偶性；

　　　　　④ 单调性；　⑧ 周期性.

(3) 牛顿—莱布尼茨公式

(4) 换元法　换元一定要换限

上限 ↔ 上限

(下)　　(下)

(5) 分部积分法

(6) *Wallis* 公式:

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}\,x$$

$$=\begin{cases}\dfrac{(n-1)!!}{n!!}\cdot\dfrac{\pi}{2}, & n\text{为偶数}\ (n\geq 2);\\[2ex] \dfrac{(n-1)!!}{n!!}\cdot 1, & n\text{为奇数}\ (n\geq 3).\end{cases}$$

(7) 几个重要关系

① $\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x$

② $\int_0^{\pi} f(\sin x)\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$

③ $\int_0^{\pi} x\, f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$

④ $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_0^{a} [f(x) + f(-x)]\,\mathrm{d}x$

C. 广义积分法: 利用定义

2. 有特殊技巧的积分

A. 不定积分

(1) $$\int \frac{\mathrm{d}\,x}{a\sin x+b\cos x}=\frac{1}{\sqrt{a^2+b^2}}\int\frac{1}{\sin(x+\varphi)}\mathrm{d}\,x$$

$$(\tan\varphi=\frac{b}{a})$$

(2) $$\int\frac{c\sin x+d\cos x}{a\sin x+b\cos x}\mathrm{d}\,x=Ax+B\ln\left|a\sin x+b\cos x\right|+C$$

(3) $$\int\frac{x^2+1}{x^4+1}\mathrm{d}\,x=\int\frac{1}{(x-\frac{1}{x})^2+(\sqrt{2})^2}\mathrm{d}(x-\frac{1}{x})$$

B. 定积分

(1) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \frac{\pi^2}{4}$ 利用关系③，或令 $x=\pi-t$.

(2) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,\mathrm{d}x = \frac{\pi}{8}\ln 2$ 令 $x=\frac{\pi}{4}-t$

(3) $\int_0^{1} \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{8}\ln 2$ 令 $x=\tan u$，转化成(2)

(4) $\int_0^{1} \frac{\arctan x}{1+x}\,\mathrm{d}x = \frac{\pi}{8}\ln 2$ 分部积分，转化成(3)

(5) $\int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x$

$$= \int_0^{\frac{\pi}{4}} \ln(2\sin x\cos x)\ \mathrm{d}x$$ 倍角公式，分解

$$= \int_0^{\frac{\pi}{4}} \ln 2\ \mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln\sin x\ \mathrm{d}x + \int_0^{\frac{\pi}{4}} \ln\cos x\ \mathrm{d}x$$

$$\int_0^{\frac{\pi}{4}} \ln\cos x\ \mathrm{d}x \overset{x=\frac{\pi}{2}-t}{=} -\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\sin t\ \mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin t\ \mathrm{d}t$$

$$\therefore \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x = \frac{\pi}{4}\ln 2 + \int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x$$

令 $x = 2u$

$$= \frac{\pi}{4}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln(\sin 2u)\,\mathrm{d}u$$

$$\therefore \int_0^{\frac{\pi}{4}} \ln(\sin 2x)\,\mathrm{d}x = -\frac{\pi}{4}\ln 2$$

二、典型例题

例1 填空题

1. 已知 $f(x)$连续，且

$$\int_0^x t\,f(2x-t)\,\mathrm{d}t=\frac{1}{2}\arctan x^2,\ f(1)=1,$$

则 $\int_1^2 f(t)\,\mathrm{d}t=$ $\underline{\quad\frac{3}{4}\quad}$.

解 $\int_0^x t\,f(2x-t)\,\mathrm{d}t \overset{令u=2x-t}{=\!=\!=} \int_{2x}^x (2x-u)\,f(u)(-\mathrm{d}u)$

$$= \int_{x}^{2x} (2x - u) f(u) \mathrm{d} u$$

$$= 2x \int_{x}^{2x} f(u) \mathrm{d} u - \int_{x}^{2x} u f(u) \mathrm{d} u$$

$$\therefore \quad 2x \int_{x}^{2x} f(u) \mathrm{d} u - \int_{x}^{2x} u f(u) \mathrm{d} u = \frac{1}{2} \arctan x^{2}$$

两边对x求导：

$$2\{1 \cdot \int_{x}^{2x} f(u) \mathrm{d} u + x \cdot [\cancel{2 f(2x)} - f(x)]\}$$

$$-[\cancel{2x f(2x)} \cdot 2 - x f(x)] = \frac{x}{1 + x^{4}}$$

$$2\int_x^{2x} f(u)\,\mathrm{d}u - xf(x) = \frac{x}{1+x^4}$$

$$\int_x^{2x} f(u)\,\mathrm{d}u = \frac{1}{2}[xf(x) + \frac{x}{1+x^4}]$$

令 $x = 1$，得

$$\int_1^{2} f(u)\,\mathrm{d}u = \frac{1}{2}[f(1) + \frac{1}{2}] = \frac{1}{2}(1 + \frac{1}{2}) = \frac{3}{4}.$$

2. $\int_0^2 x\sqrt{2x-x^2}\,\mathrm{d}x =$ $\underline{\quad \frac{\pi}{2} \quad}$.

解 $\int_0^2 x\sqrt{2x-x^2}\,\mathrm{d}x = \int_0^2 x\sqrt{1-(x-1)^2}\,\mathrm{d}x$

$$\xlongequal{令 t=x-1} \int_{-1}^{1}(t+1)\sqrt{1-t^2}\,\mathrm{d}t$$

$$= \int_{-1}^{1} t\sqrt{1-t^2}\,\mathrm{d}t + \int_{-1}^{1}\sqrt{1-t^2}\,\mathrm{d}t$$

$$= 0 + \frac{1^2\cdot\pi}{2} = \frac{\pi}{2}.$$

3. 设 $f(x+\frac{1}{x})=\frac{x+x^3}{1+x^4}$，则 $\int_2^{2\sqrt{2}} f(x)\mathrm{d}x=$ $\underline{\frac{1}{2}\ln 3}$.

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解 $f(x+\frac{1}{x})=\dfrac{\frac{1}{x}+x}{\frac{1}{x^2}+x^2}=\dfrac{x+\frac{1}{x}}{(x+\frac{1}{x})^2-2}$

令 $u=x+\frac{1}{x}$，则 $f(u)=\frac{u}{u^2-2}$.

$\therefore \quad \int_2^{2\sqrt{2}} f(x)\mathrm{d}x=\int_2^{2\sqrt{2}}\frac{x}{x^2-2}\mathrm{d}x$

$$=\frac{1}{2}\int_2^{2\sqrt{2}}\frac{1}{x^2-2}\mathrm{d}(x^2-2)=\frac{1}{2}\ln(x^2-2)\Big|_2^{2\sqrt{2}}=\frac{1}{2}\ln 3.$$

4. $\lim\limits_{x\to 0}\dfrac{\int_0^x[\int_0^{u^2}\arctan(1+t)\,\mathrm{d}t]\,\mathrm{d}u}{x(1-\cos x)}=$ $\underline{\quad\dfrac{\pi}{6}\quad}$.　　$(\dfrac{0}{0})$

解　原式 $=\lim\limits_{x\to 0}\dfrac{\int_0^x[\int_0^{u^2}\arctan(1+t)\,\mathrm{d}t]\,\mathrm{d}u}{x\cdot\dfrac{x^2}{2}}$

$$=2\lim_{x\to 0}\frac{\int_0^x[\int_0^{u^2}\arctan(1+t)\,\mathrm{d}t]\,\mathrm{d}u}{x^3}$$

$$= 2\lim_{x\to 0}\frac{\{\int_0^x[\int_0^{u^2}\arctan(1+t)\,\mathrm{d}t]\,\mathrm{d}u\}'}{(x^3)'}$$

$$= 2\lim_{x\to 0}\frac{\int_0^{x^2}\arctan(1+t)\,\mathrm{d}t}{3x^2} \qquad (\frac{0}{0})$$

$$= 2\lim_{x\to 0}\frac{2\cancel{x}\cdot\arctan(1+x^2)}{6\cancel{x}} = \frac{2}{3}\cdot\frac{\pi}{4} = \frac{\pi}{6}.$$

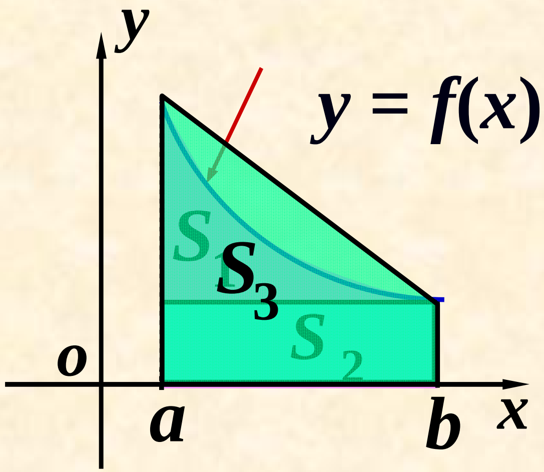
例2 选择题

(1) 设在$[a,b]$上，$f(x)>0,\ f'(x)<0,\ f''(x)>0,$

令 $S_1=\int_a^b f(x)\,\mathrm{d}x,$

$S_2=f(b)(b-a)$

$S_3=\frac{1}{2}[f(b)+f(a)](b-a)$



则 (**B**).

(A) $S_1<S_2<S_3$　　(B) $S_2<S_1<S_3$

(C) $S_3<S_1<S_2$　　(D) $S_2<S_3<S_1$

(2) 设 $F(x)=\int_{x}^{x+2\pi} e^{\sin t}\sin t\,dt$, 则 $F(x)$(A).

(A) 为正常数； (B) 为负常数；

(C) 恒为零； (D) 不为常数.

解(方法1) 令 $f(t)=e^{\sin t}\sin t$, 则

$$f(t+2\pi)=f(t)$$

若 $f(x)$连续，$f(x+T)=f(x)$,

则 $\int_{a}^{a+T} f(x)dx=\int_{0}^{T} f(x)dx.$

$$F(x)=\int_{0}^{2\pi} e^{\sin t}\sin t\,dt$$

$$=\int_{-\pi}^{\pi} e^{\sin t}\sin t\,dt$$ 为常数

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$$=\int_0^{\pi}(e^{\sin t}\sin t-e^{-\sin t}\sin t)\,dt$$

$$\int_{-a}^{a}f(x)\,dx=\int_0^{a}[f(x)+f(-x)]\,dx$$

$$=\int_0^{\pi}(e^{\sin t}-e^{-\sin t})\sin t\,dt$$

$\because$ 当$t\in(0,\pi)$时, $\sin t>0$

$\therefore$ $e^{\sin t}-e^{-\sin t}>0$, $(e^{\sin t}-e^{-\sin t})\sin t>0$,

$\therefore$ $F(x)>0$且$F(x)$为常数. 选(A)

(方法2) $F(x)=\int_0^{2\pi}\mathrm{e}^{\sin t}\sin t\,\mathrm{d}t$

$$=-\int_0^{2\pi}\mathrm{e}^{\sin t}\,\mathrm{d}(\cos t)$$

$$=-[\mathrm{e}^{\sin t}\cos t\Big|_0^{2\pi}-\int_0^{2\pi}\mathrm{e}^{\sin t}\cos^2 t\,\mathrm{d}t]$$

$$=\int_0^{2\pi}\mathrm{e}^{\sin t}\cos^2 t\,\mathrm{d}t>0$$

类似题 证明： $\int_0^{\sqrt{2\pi}} \sin x^2 \,\mathrm{d}x > 0.$

提示： $\int_0^{\sqrt{2\pi}} \sin x^2 \,\mathrm{d}x \overset{t=x^2}{=} \int_0^{2\pi} (\sin t)\cdot \frac{1}{2\sqrt{t}} \mathrm{d}t$

$$\overset{u=\pi-t}{=} \frac{1}{2}\int_{\pi}^{-\pi} \frac{\sin u}{\sqrt{\pi-u}}(-\mathrm{d}u) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin u}{\sqrt{\pi-u}} \mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\pi} \left(\frac{\sin u}{\sqrt{\pi-u}} - \frac{\sin u}{\sqrt{\pi+u}}\right) \mathrm{d}u$$

$$= \frac{1}{2}\int_0^{\pi} \left(\frac{1}{\sqrt{\pi-u}} - \frac{1}{\sqrt{\pi+u}}\right) \sin u \,\mathrm{d}u > 0$$

例3 计算下列积分：

(1) $\displaystyle\int \frac{\mathrm{d}\,x}{(2x^2+1)\sqrt{x^2+1}}$

解法1 原式 $\displaystyle\overset{x=\tan t}{=} \int \frac{1}{(2\tan^2 t+1)\sqrt{\tan^2 t+1}}\cdot \sec^2 t\,\mathrm{d}\,t$

$$=\int \frac{1}{(2\tan^2 t+1)\sec t}\cdot \sec^2 t\,\mathrm{d}\,t$$

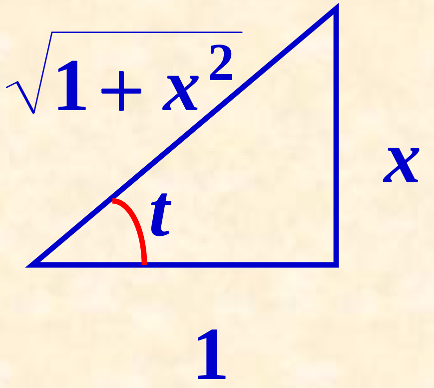
$$=\int \frac{1}{(2\tan^2 t+1)\cos t}\,\mathrm{d}t=\int \frac{\cos t}{2\sin^2 t+\cos^2 t}\,\mathrm{d}\,t$$

$$= \int \frac{\cos t}{1+\sin^2 t}\,\mathrm{d}t = \int \frac{1}{1+\sin^2 t}\,\mathrm{d}(\sin t)$$

$$= \arctan(\sin t) + C$$

$$= \arctan \frac{x}{\sqrt{1+x^2}} + C.$$

$x = \tan t$



解法2 原式 $\overset{x=\frac{1}{t}}{=} \displaystyle\int \frac{t^3}{(2+t^2)\sqrt{1+t^2}} \cdot \left(-\frac{1}{t^2}\right)\mathrm{d}t$

$$= -\int \frac{t}{(2+t^2)\sqrt{1+t^2}}\,\mathrm{d}t$$

$$\overset{u=1+t^2}{=}\ -\frac{1}{2}\int\frac{1}{(1+u)\sqrt{u}}\,\mathrm{d}u$$

$$=-\int\frac{1}{1+(\sqrt{u})^2}\,\mathrm{d}(\sqrt{u})$$

$$=-\arctan(\sqrt{u})+C$$

$$=-\arctan\left(\frac{\sqrt{1+x^2}}{x}\right)+C$$

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(2) $\int_0^1 \frac{x^2 \arcsin x}{\sqrt{1-x^2}} \mathrm{d}x$

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化为定积分

解 这是广义积分，瑕点：$x = 1$.

$$\text{原式} \overset{\text{令}\, x=\sin t}{=\!=\!=} \int_0^{\frac{\pi}{2}} \frac{t\sin^2 t}{\cancel{\cos t}} \cdot \cancel{\cos t}\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} t\sin^2 t\,\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} t\left(\frac{1-\cos 2t}{2}\right)\mathrm{d}t = \frac{t^2}{4}\Big|_0^{\frac{\pi}{2}} - \frac{1}{4}\int_0^{\frac{\pi}{2}} t\,\mathrm{d}(\sin 2t)$$

$$= \frac{\pi^2}{16} - \frac{1}{4}\left[t(\sin 2t)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin 2t\,\mathrm{d}t\right] = \frac{\pi^2}{16} - \frac{1}{8}\cos 2t\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{16} + \frac{1}{4}.$$

(3) $\int_{-1}^{1} \frac{1}{1+2^{\frac{1}{x}}} dx$

解 $f(x)=\frac{1}{1+2^{\frac{1}{x}}}$, $\because$ $f(0^-)=\lim\limits_{x\to 0^-}\frac{1}{1+2^{\frac{1}{x}}}=1,$

$f(0^+)=\lim\limits_{x\to 0^+}\frac{1}{1+2^{\frac{1}{x}}}=0,$

$\therefore$ $x=0$是 $f(x)$的第一类跳跃间断点 .

故此积分是定积分.

$$\int_{-1}^{1}\frac{1}{1+2^{\frac{1}{x}}}\,\mathrm{d}x$$

$$\boxed{\int_{-a}^{a}f(x)\mathrm{d}x=\int_{0}^{a}[f(x)+f(-x)]\mathrm{d}x}$$

$$=\int_{0}^{1}\left(\frac{1}{1+2^{\frac{1}{x}}}+\frac{1}{1+2^{-\frac{1}{x}}}\right)\mathrm{d}x$$

$$=\int_{0}^{1}\left(\frac{1}{1+2^{\frac{1}{x}}}+\frac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}\right)\mathrm{d}x=\int_{0}^{1}\mathrm{d}x=1.$$

$$\boxed{\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx}$$

(4) 求 $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x+\cos^n x}\,\mathrm{d}\,x.$

解 令 $I=\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x+\cos^n x}\,\mathrm{d}\,x = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x+(\sqrt{1-\sin^2 x})^2}\,\mathrm{d}\,x$

$J=\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x+\sin^n x}\,\mathrm{d}\,x,$ 则 $I=J$

而 $I+J=\int_0^{\frac{\pi}{2}} \frac{\cancel{\sin^n x+\cos^n x}}{\cancel{\sin^n x+\cos^n x}}\,\mathrm{d}\,x=\int_0^{\frac{\pi}{2}}\mathrm{d}\,x=\frac{\pi}{2}$

$\therefore\ 2I=\frac{\pi}{2},\qquad I=\frac{\pi}{4}.$

(5) $\int_{-\frac{1}{2}}^{\frac{1}{2}}[\frac{\sin x}{x^8+1}+\sqrt{\ln^2(1-x)}]\,\mathrm{d}x.$

解 原式 $=0+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\ln(1-x)\right|\mathrm{d}x$

$$=\int_{-\frac{1}{2}}^{0}\ln(1-x)\,\mathrm{d}x-\int_{0}^{\frac{1}{2}}\ln(1-x)\,\mathrm{d}x$$

$$=\frac{3}{2}\ln\frac{3}{2}+\ln\frac{1}{2}.$$

(6) $\int_{-2}^{2}\min\{\frac{1}{|x|},x^2\}\mathrm{d}\,x.$

解 $\because \min\{\frac{1}{|x|},x^2\}=\begin{cases} x^2, & |x|\le 1 \\ \frac{1}{|x|}, & |x|>1 \end{cases}$ 是偶函数,

$$原式=2\int_{0}^{2}\min\{\frac{1}{|x|},x^2\}\mathrm{d}\,x$$

$$=2\int_{0}^{1}x^2dx+2\int_{1}^{2}\frac{1}{x}\mathrm{d}\,x=\frac{2}{3}+2\ln 2.$$

例4 已知 $\int_0^{\pi} \frac{\cos x}{(x+2)^2} \mathrm{d}x = A$, 求 $I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{x+1} \mathrm{d}x.$

解 $I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{x+1} \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{x+1} \mathrm{d}x$

$$\overset{令 t=2x}{=\!=\!=} \frac{1}{2}\int_0^{\pi} \frac{\sin t}{\frac{t}{2}+1} \cdot \frac{1}{2} \mathrm{d}t = \frac{1}{2}\int_0^{\pi} \frac{\sin t}{t+2} \mathrm{d}t$$

$$= -\frac{1}{2}\int_0^{\pi} \frac{1}{t+2} \mathrm{d}(\cos t)$$

$$I=-\frac{1}{2}\int_0^{\pi}\frac{1}{t+2}\mathrm{d}(\cos t)$$

$$=-\frac{1}{2}\left[\frac{\cos t}{t+2}\bigg|_0^{\pi}-\int_0^{\pi}\frac{-1}{(t+2)^2}\cdot(\cos t)\,\mathrm{d}t\right]$$

$$=-\frac{1}{2}\left[\left(\frac{-1}{\pi+2}-\frac{1}{2}\right)\bigg|+\int_0^{\pi}\frac{\cos t}{(t+2)^2}\mathrm{d}t\right]$$

$$=\frac{1}{2(\pi+2)}+\frac{1}{4}-\frac{1}{2}\int_0^{\pi}\frac{\cos x}{(x+2)^2}\mathrm{d}x$$

$$=\frac{1}{2(\pi+2)}+\frac{1}{4}-\frac{1}{2}A.$$

类似题 已知 $\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$，求 $\int_0^{+\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x.$

解 $$\int_0^{+\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x=\lim_{b\to+\infty}\int_0^{b}\frac{\sin^2 x}{x^2}\mathrm{d}x$$ 定积分

$$\because\quad \int_0^{b}\frac{\sin^2 x}{x^2}\mathrm{d}x=\lim_{t\to0^+}\int_t^{b}\frac{\sin^2 x}{x^2}\mathrm{d}x$$

$$=-\lim_{t\to0^+}\int_t^{b}\sin^2 x\,\mathrm{d}\left(\frac{1}{x}\right)$$

$$=-\lim_{t\to0^+}\left[\frac{1}{x}\cdot\sin^2 x\Big|_t^b-\int_t^{b}\frac{2\sin x\cos x}{x}\mathrm{d}x\right]$$

$$= -\lim_{t\to 0^+}[(\frac{1}{b}\cdot\sin^2 b - \frac{\sin t}{t}\cdot\sin t) - \int_t^b \frac{\sin 2x}{x}\mathrm{d}x]$$

$$= -\frac{1}{b}\cdot\sin^2 b + \int_0^b \frac{\sin 2x}{x}\mathrm{d}x$$

$$\therefore \int_0^{+\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x = \lim_{b\to+\infty}[-\frac{1}{b}\cdot\sin^2 b + \int_0^b \frac{\sin 2x}{x}\mathrm{d}x]$$

$$= \int_0^{+\infty}\frac{\sin 2x}{x}\mathrm{d}x \overset{令 u=2x}{=\!=\!=} \int_0^{+\infty}\frac{\sin u}{u}\mathrm{d}t$$

$$= \int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}.$$

例5 设 $f(x)=\dfrac{x-1}{x(x-2)}$，求 $\displaystyle\int_1^3 \frac{f'(x)}{1+f^2(x)}\,\mathrm{d}x$.

解 令 $\varphi(x)=\dfrac{f'(x)}{1+f^2(x)}$，则

$x=2$ 是 $\varphi(x)$ 的可去间断点 . 事实上，

$$\lim_{x\to2}\varphi(x)=\lim_{x\to2}\frac{x(x-2)-(x-1)(2x-2)}{x^2(x-2)^2+(x-1)^2}=-2$$

$\therefore$ $\varphi(x)$ 在 $[1,2]$ 上无原函数 .

故对 $\displaystyle\int_1^3\varphi(x)\,\mathrm{d}x$ 不能直接用牛顿 – 莱布尼茨公式.

$$\text{原式} = \int_1^2 \frac{f'(x)}{1+f^2(x)}\,\mathrm{d}x + \int_2^3 \frac{f'(x)}{1+f^2(x)}\,\mathrm{d}x$$

$$= \lim_{t\to 2^-}\int_1^t \frac{1}{1+f^2(x)}\,\mathrm{d}f(x) + \lim_{t\to 2^+}\int_t^3 \frac{1}{1+f^2(x)}\,\mathrm{d}f(x)$$

$$= \lim_{t\to 2^-} \arctan f(x)\Big|_1^t + \lim_{t\to 2^+} \arctan f(x)\Big|_t^3$$

$$= \lim_{t\to 2^-}[\arctan f(t) - \arctan f(1)]$$

$$+ \lim_{t\to 2^+}[\arctan f(3) - \arctan f(t)]$$

$\because \quad f(x)=\dfrac{x-1}{x(x-2)}, \quad \therefore \quad f(1)=0, \quad f(3)=\dfrac{2}{3},$

又 $\because \quad \lim\limits_{t\to 2^-} f(t)=\lim\limits_{t\to 2^-}\dfrac{t-1}{t(t-2)}=-\infty$

$\therefore \quad \lim\limits_{t\to 2^-}\arctan f(t)=-\dfrac{\pi}{2},$

同理 $\quad \lim\limits_{t\to 2^+}\arctan f(t) = \lim\limits_{t\to 2^+}\arctan\dfrac{t-1}{t(t-2)}=\dfrac{\pi}{2}$

从而 原式 $=\arctan\dfrac{2}{3}-\pi.$

例6 设 $f(x)=3x-\sqrt{1-x^2}\int_0^1 f^2(x)\mathrm{d}x$, 求 $f(x)$.

解 令 $a=\boxed{\int_0^1 f^2(x)\mathrm{d}x}$, 则 $f(x)=3x-a\sqrt{1-x^2}$

定积分是一个数

$$f^2(x)=(3x-a\sqrt{1-x^2})^2$$

$$=9x^2-6ax\sqrt{1-x^2}+a^2(1-x^2)$$

等式两边积分:

$$a=\int_0^1 f^2(x)\mathrm{d}x=\int_0^1[9x^2-6ax\sqrt{1-x^2}+a^2(1-x^2)]\mathrm{d}x$$

$$=3-2a+\frac{2}{3}a^2,$$

即 $2a^2-9a+9=0.$

即 $2a^2-9a+9=0,$

$$(2a-3)(a-3)=0$$

解得 $a=\frac{3}{2},\ a=3.$

$\therefore\quad f(x)=3x-\frac{3}{2}\sqrt{1-x^2}$

及 $f(x)=3x-3\sqrt{1-x^2}.$

例7 (1) 设 $f(\sin^2 x)=\dfrac{x}{\sin x}$，求 $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}} f(x)\,\mathrm{d}x$.

解　令 $u=\sin^2 x$，　则

$$\sin x=\pm\sqrt{u},\quad x=\arcsin(\pm\sqrt{u})$$

$$f(u)=\frac{\arcsin\sqrt{u}}{\sqrt{u}},$$

$$\therefore\quad f(x)=\frac{\arcsin\sqrt{x}}{\sqrt{x}},$$

于是 $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}} f(x)\,\mathrm{d}x = \int \frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x$

$$= -2\int \arcsin\sqrt{x}\ \mathrm{d}(\sqrt{1-x})$$

$$= -2\left(\sqrt{1-x}\arcsin\sqrt{x} - \int \cancel{\sqrt{1-x}} \cdot \frac{1}{\cancel{\sqrt{1-x}}}\mathrm{d}\sqrt{x}\right)$$

$$= -2\sqrt{1-x}\arcsin\sqrt{x} + 2\sqrt{x} + C.$$

(2) 设$f(x)=\int_1^{\sqrt{x}} e^{-t^2}\,dt$，求$\int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx$.

解 $f'(x)=e^{-x}\cdot\frac{1}{2\sqrt{x}}$.

分析 e^{-t^2}无初等函数形式的原函数，故无法直接求出$f(x)$，所以采用分部积分法.

$$\int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx$$

$$=2\int_0^1 f(x)\,d\sqrt{x}=2\left[\sqrt{x}f(x)\Big|_0^1-\int_0^1\sqrt{x}f'(x)\,dx\right]$$

$$=-\int_0^1 e^{-x}\,dx=\frac{1}{e}-1.$$

$$\boxed{f(1)=\int_1^1 e^{-t^2}\,dt=1}$$

例8 计算下列积分：

(1) $$\int_0^{+\infty}\frac{x^{\frac{n}{2}}}{1+x^{n+2}}\,\mathrm{d}x\quad(n>-2)$$

解 原式 $=\displaystyle\int_0^{+\infty}\frac{x^{\frac{n}{2}}}{1+(x^{\frac{n}{2}+1})^2}\,\mathrm{d}x$

$$=\frac{1}{\frac{n}{2}+1}\int_0^{+\infty}\frac{1}{1+(x^{\frac{n}{2}+1})^2}\,\mathrm{d}(x^{\frac{n}{2}+1})$$

$$=\frac{2}{n+2}\arctan x^{\frac{n}{2}+1}\Big|_0^{+\infty}=\frac{2}{n+2}\cdot\frac{\pi}{2}=\frac{\pi}{n+2}.$$

(2) $\int_0^1 \frac{x^{\frac{n}{2}}}{\sqrt{x(1-x)}}\,\mathrm{d}x$ （n为正奇数）

解 $\because \lim\limits_{x\to 1-0} f(x)=+\infty$, 故此积分是广义积分.

原式 $\overset{\sqrt{x}=\sin t}{\underset{x=\sin^2 t}{=}} \int_0^{\frac{\pi}{2}} \frac{(\sin t)^n}{\cancel{(\sin t)}\cancel{\sqrt{1-\sin^2 t}}} \cdot 2\cancel{\sin t}\cancel{\cos t}\,\mathrm{d}t$

$$= 2\int_0^{\frac{\pi}{2}} \sin^n t\,\mathrm{d}t = 2\frac{(n-1)!!}{n!!}.$$

(3) $\int \sin^{n-1} x \sin(n+1)x \,\mathrm{d}x$

解 原式 $= \int (\sin^{n-1} x)(\sin nx \cos x + \sin x \cos nx)\,\mathrm{d}x$

$$= \int \sin nx \cdot \sin^{n-1} x \cos x \,\mathrm{d}x + \int \sin^{n} x \cos nx \,\mathrm{d}x$$

$$= \frac{1}{n}\int \sin nx \,\mathrm{d}(\sin^{n} x) + \int \sin^{n} x \cos nx \,\mathrm{d}x$$

$$= \frac{1}{n}\sin nx \sin^{n} x - \frac{1}{n}\int \cancel{\sin^{n} x \cdot n \cos nx \,\mathrm{d}x} + \int \cancel{\sin^{n} x \cos nx} \,\mathrm{d}x$$

$$= \frac{1}{n}\sin nx \sin^{n} x + C.$$

(4) $I_n = \int_0^{\pi} \frac{\sin(2n-1)x}{\sin x} \mathrm{d}\, x \quad (n \ge 1)$ 定积分

解 $I_1 = \int_0^{\pi} \frac{\sin x}{\sin x} \mathrm{d}\, x = \int_0^{\pi} \mathrm{d}\, x = \pi$

$$I_n = \int_0^{\pi} \frac{\sin 2nx \cos x - \sin x \cos 2nx}{\sin x} \mathrm{d}\, x$$

$$= \int_0^{\pi} \frac{\sin 2nx \cos x}{\sin x} \mathrm{d}\, x - \int_0^{\pi} \cos 2nx \,\mathrm{d}\, x$$

$$= \frac{1}{2} \int_0^{\pi} \frac{\sin(2n+1)x + \sin(2n-1)x}{\sin x} \mathrm{d}\, x - \frac{1}{2n} \sin 2nx \Big|_0^{\pi}$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin(2n+1)x + \sin(2n-1)x}{\sin x}\,\mathrm{d}x - \frac{1}{2n}\sin 2nx\Big|_0^{\pi}$$

$$= \frac{1}{2}(I_{n+1} + I_n) - 0$$

$$\therefore \quad I_{n+1} = I_n \quad (n \geq 1)$$

于是 $I_n = I_1 = \pi$.

例9 设$f(x)$是连续函数，

(1) 利用定义证明函数 $F(x)=\int_0^x f(t)\mathrm{d}t$可导，且

$$F'(x)=f(x);$$

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证 由导数定义，$\forall x\in R$,

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\int_0^{x+h}f(t)\mathrm{d}t-\int_0^x f(t)\mathrm{d}t}{h}$$

$f(x)$连续，由积分中值定理

$$=\lim_{h\to 0}\frac{\int_x^{x+h}f(t)\mathrm{d}t}{h}=\lim_{h\to 0}\frac{f(\xi)\cdot h}{h}=\lim_{h\to 0}f(\xi)=f(x)$$

(ξ介于x与$x+h$之间)

(2) 当 $f(x)$ 是以2为周期的周期函数时， 证明:

$$G(x)=2\int_0^x f(t)\,\mathrm{d}t-x\int_0^2 f(t)\,\mathrm{d}t$$ 也是以 2为周期的周期函数 .

需证: $\forall x\in \boldsymbol{R}$

$$G(x+2)=G(x)$$

证(方法1) 因为 $f(x)$ 连续，所以

$G(x)$可导. 令 $H(x)=G(x+2)-G(x)$,

则 $H'(x)=G'(x+2)-G'(x)$

$$=[2f(x+2)-\cancel{\int_0^2 f(t)\,\mathrm{d}t}]-[2f(x)-\cancel{\int_0^2 f(t)\,\mathrm{d}t}]$$

$$=2[f(x+2)-f(x)]=0$$

$\therefore \quad H(x) \equiv C \quad$ (常数)

又 $\because \quad H(0) = G(2) - G(0) = 0$

$$G(0) = 0,$$
$$G(2) = 0$$

$\therefore \quad C = 0 \qquad \therefore \quad H(x) \equiv 0$

从而 $\quad \forall x \in R,\ \ G(x+2) = G(x).$

即 $G(x)$是以2为周期的周期函数 .

$$H(x) = G(x+2) - G(x)$$
$$G(x) = 2\int_0^x f(t)\,\mathrm{d}t - x\int_0^2 f(t)\,\mathrm{d}t$$

(方法2) $$G(x+2)=2\int_0^{x+2} f(t)\mathrm{d}t-(x+2)\int_0^2 f(t)\mathrm{d}t$$

$$=2[\int_0^2 f(t)\mathrm{d}t+\int_2^{x+2} f(t)\mathrm{d}t]-(x+2)\int_0^2 f(t)\mathrm{d}t$$

$$=2\int_2^{x+2} f(t)\mathrm{d}t-x\int_0^2 f(t)\mathrm{d}t \qquad (\text{令 } t=u+2)$$

$$=2\int_0^{x} f(u+2)\mathrm{d}u-x\int_0^2 f(t)\mathrm{d}t$$

$$=2\int_0^{x} f(u)\mathrm{d}u-x\int_0^2 f(t)\mathrm{d}t=G(x).$$

$$\boxed{G(x)=2\int_0^{x} f(t)\mathrm{d}t-x\int_0^2 f(t)\mathrm{d}t}$$

例10 设 $f(a)=\int_0^{\pi}\ln(1-2a\cos x+a^2)dx$，证明：

(1) $f(a)$是偶函数；

(2) $f(a^2)=2f(a)$.

证 (1) $f(-a)=\int_0^{\pi}\ln(1+2a\cos x+a^2)dx$

$$\overset{\text{令 } x=\pi-t}{\underset{t=\pi-x}{=\!=\!=}}\int_{\pi}^{0}\ln(1-2a\cos t+a^2)(-dt)$$

$$=\int_0^{\pi}\ln(1-2a\cos t+a^2)dt=f(a).$$

$\therefore\quad f(a)$是偶函数 .

证 (2) $2f(a)=f(a)+f(-a)$

$$=\int_0^{\pi}\ln(1-2a\cos x+a^2)dx$$

$$+\int_0^{\pi}\ln(1+2a\cos x+a^2)dx$$

$$=\int_0^{\pi}\ln(1-2a\cos x+a^2)(1+2a\cos x+a^2)dx$$

$$=\int_0^{\pi}\ln[(1+a^2)^2-4a^2\cos^2 x]dx$$

$$=\int_0^{\pi}\ln[1+2a^2(1-2\cos^2 x)+a^4]dx$$

$$= \int_0^{\pi} \ln[1 + 2a^2(1 - 2\cos^2 x) + a^4]dx$$

$$= \int_0^{\pi} \ln(1 - 2a^2 \cos 2x + a^4)dx$$

$$\underline{\underline{u = 2x}} \int_0^{2\pi} \ln(1 - 2a^2 \cos u + a^4) \cdot \frac{1}{2} du$$

$$= \frac{1}{2}[\int_0^{\pi} \ln(1 - 2a^2 \cos u + a^4)du$$

$$+ \underline{\int_{\pi}^{2\pi} \ln(1 - 2a^2 \cos u + a^4)du]}$$

$$\because \quad \int_{\pi}^{2\pi} \ln(1-2a^2\cos u + a^4)du$$

$$\overset{u=2\pi-t}{\underset{t=2\pi-u}{=\!=\!=}} \int_{\pi}^{0} \ln(1-2a^2\cos t + a^4)(-dt)$$

$$= \int_{0}^{\pi} \ln(1-2a^2\cos t + a^4)dt$$

$$\therefore \quad 2f(a) = \frac{1}{2}[\int_{0}^{\pi} \ln(1-2a^2\cos u + a^4)du$$

$$+ \int_{0}^{\pi} \ln(1-2a^2\cos t + a^4)dt]$$

$$= \int_{0}^{\pi} \ln(1-2a^2\cos u + a^4)du = f(a^2).$$

例11 设$f(x)$连续，常数 $a>0$，证明：

$$\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{\mathrm{d}x}{x}=\int_1^a f\left(x+\frac{a^2}{x}\right)\frac{\mathrm{d}x}{x}.$$

分析 显然要用换元法.

$$x=\varphi(t)=?$$

原则：先看被积函数，再看限.

令 $t=x^2\ (x=\sqrt{t})$，则

$$\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{\mathrm{d}x}{x}=\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{\mathrm{d}x^2}{2x^2}$$

$$\xlongequal{令\ t=x^2}\frac{1}{2}\int_1^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}$$

右端 $=\int_1^a f\left(x+\dfrac{a^2}{x}\right)\dfrac{\mathrm{d}x}{x}$

$$=\frac{1}{2}\left[\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}+\underline{\int_a^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}}\right]$$

需证: $\displaystyle\int_a^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}\overset{?}{=}\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}$

需证: $\int_a^{a^2} f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t} \overset{?}{=} \int_1^a f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t}$

问: 能否作变换 $u=\frac{t}{a}$? 否

$$\int_a^{a^2} f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t} = \int_1^a f(au+\frac{a}{u})\frac{\mathrm{d}u}{u}$$

被积函数未达到要求!

要求: $t+\frac{a^2}{t}=u+\frac{a^2}{u}$, 即 $(t-u)+a^2\frac{u-t}{ut}=0$

即 $(t-u)+a^2\dfrac{u-t}{ut}=0$

亦即 $(t-u)(1-\dfrac{a^2}{tu})=0$

$\therefore\quad 1-\dfrac{a^2}{tu}=0,\quad \boxed{u=\dfrac{a^2}{t}}$

证 $\displaystyle\int_1^a f(x^2+\frac{a^2}{x^2})\frac{\mathrm{d}x}{x}=\int_1^a f(x^2+\frac{a^2}{x^2})\frac{\mathrm{d}x^2}{2x^2}$

$\xlongequal{\text{令 } t=x^2} \displaystyle\frac{1}{2}\int_1^{a^2} f(t+\frac{a^2}{t})\frac{\mathrm{d}t}{t}$

$$=\frac{1}{2}\left[\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}+\underline{\int_a^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}}\right]$$

$$\int_a^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t} \xlongequal{\text{令}u=\frac{a^2}{t}} \int_a^1 f\left(\frac{a^2}{u}+u\right)\frac{u}{a^2}\cdot\left(-\frac{a^2}{u^2}\right)\mathrm{d}u$$

$$=\int_1^a f\left(u+\frac{a^2}{u}\right)\frac{\mathrm{d}u}{u}=\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{\mathrm{d}t}{t}$$

代入上式，得

$$\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{\mathrm{d}x}{x}=\int_1^a f\left(x+\frac{a^2}{x}\right)\frac{\mathrm{d}x}{x}.$$

例12 求 $I(x)=\int_{-1}^{1}|t-x|e^{t}\,dt$在$[-1,1]$上的最大值.

解 $I(x)=\int_{-1}^{x}|t-x|e^{t}\,dt+\int_{x}^{1}|t-x|e^{t}\,dt$

$=\int_{-1}^{x}(x-t)e^{t}\,dt+\int_{x}^{1}(t-x)e^{t}\,dt$

$=2e^{x}-(e+e^{-1})x-2e^{-1}$

$I'(x)=2e^{x}-(e+e^{-1})$

令 $I'(x)=0$, 得唯一驻点：$x=\ln\dfrac{e+e^{-1}}{2}=\ln ch1$

$\because \quad I''(x) = 2\mathrm{e}^{x} > 0$

$\therefore \quad x = \ln \mathrm{ch}\, 1$是$I(x)$的极小值点，从而是最小值点.

又$\because \quad I(-1) = \mathrm{e} + \mathrm{e}^{-1} > I(1) = \mathrm{e} - 3\mathrm{e}^{-1}$

$$\therefore \quad \max_{x\in[-1,1]} I(x) = I(-1) = \mathrm{e} + \mathrm{e}^{-1}.$$

例13 求极限

$$I=\lim_{n\to\infty}\left(\frac{\sin\frac{\pi}{n}}{n+1}+\frac{\sin\frac{2\pi}{n}}{n+\frac{1}{2}}+\cdots+\frac{\sin\pi}{n+\frac{1}{n}}\right).$$

解 $$I=\lim_{n\to\infty}\sum_{i=1}^{n}\boxed{\frac{\sin\frac{i}{n}\pi}{n+\frac{1}{i}}}$$

$$\frac{\sin\pi\frac{i}{n}}{n+\frac{1}{i}}\overset{?}{=}f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$$

不是

$$\frac{\sin\pi\frac{i}{n}}{n+1}<\frac{\sin\pi\frac{i}{n}}{n+\frac{1}{i}}<\frac{\sin\pi\frac{i}{n}}{n}\qquad(i=1,2,\cdots,n)$$

$$\frac{\sin \pi \frac{i}{n}}{n+1} < \frac{\sin \pi \frac{i}{n}}{n+\frac{1}{i}} < \frac{\sin \pi \frac{i}{n}}{n} \quad (i=1,2,\cdots,n)$$

$$\because \quad \lim_{n\to\infty} \sum_{i=1}^{n} \sin \pi \frac{i}{n} \cdot \frac{1}{n}$$

$$= \int_0^1 \sin \pi x \, \mathrm{d}x = -\frac{1}{\pi} \cos \pi x \Big|_0^1 = \frac{2}{\pi}.$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin \pi \frac{i}{n}}{n+1} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \sum_{i=1}^{n} \sin \pi \frac{i}{n} \cdot \frac{1}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sin\pi\frac{i}{n}}{n+1}=\lim_{n\to\infty}\frac{n}{n+1}\cdot\sum_{i=1}^{n}\sin\pi\frac{i}{n}\cdot\frac{1}{n}$$

$$=\lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\sin\pi\frac{i}{n}\cdot\frac{1}{n}$$

$$=1\cdot\int_0^1\sin\pi x\,\mathrm{d}x=\frac{2}{\pi}.$$

由夹逼准则，得

$$\therefore\quad I=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sin\frac{i}{n}\pi}{n+\frac{1}{i}}=\frac{2}{\pi}.$$

类似题 (1) $\lim\limits_{n\to\infty}\sin\dfrac{\pi}{n}\sum\limits_{k=1}^{n}\cos^2\dfrac{k\pi}{n}=$ $\underline{\dfrac{\pi}{2}}$.

解 原式$=\lim\limits_{n\to\infty}\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\dfrac{\pi}{n}\sum\limits_{k=1}^{n}\cos^2\dfrac{k\pi}{n}$

$$=1\times\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\cos^2\frac{k\pi}{n}=\pi\lim_{n\to\infty}\sum_{k=1}^{n}\cos^2\pi\frac{k}{n}\cdot\frac{1}{n}$$

$$=\pi\int_0^1\cos^2\pi x\,\mathrm{d}x\ \underline{\underline{t=\pi x}}\ \int_0^{\pi}\cos^2 t\,\mathrm{d}t$$

$$=\int_0^{\pi}\frac{1+\cos 2t}{2}\mathrm{d}t=\frac{\pi}{2}.$$

(2) 求极限 $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}$.

解 令 $x_n=\dfrac{\sqrt[n]{n!}}{n}$，则

这是一个瑕积分，瑕点为：$x=0$.

$$\because\quad \ln x_n=\ln\sqrt[n]{\frac{n!}{n^n}}=\frac{1}{n}\ln\frac{n!}{n^n}$$

$$=\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+\cdots+\ln\frac{n}{n}\right)=\sum_{i=1}^{n}\left(\ln\frac{i}{n}\right)\cdot\frac{1}{n}$$

$$\therefore\quad \lim_{n\to\infty}\ln x_n=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\ln\frac{i}{n}\right)\cdot\frac{1}{n}=\boxed{\int_0^1\ln x\,dx}$$

$$\int_0^1 \ln x \,\mathrm{d}x = \lim_{t\to 0^+}\int_t^1 \ln x \,\mathrm{d}x = \lim_{t\to 0^+}\left(x\ln x\Big|_t^1 - \int_t^1 \mathrm{d}x\right)$$

$$= \lim_{t\to 0^+}(-t\ln t - 1 + t) = -1$$

又 $\because$ $\displaystyle \lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t}{\left(\frac{1}{t}\right)} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{\left(-\frac{1}{t^2}\right)}$

$$= \lim_{t\to 0^+}(-t) = 0$$

$\therefore$ $\displaystyle \lim_{n\to\infty}\ln x_n = \int_0^1 \ln x \,\mathrm{d}x = -1$, 从而

原式 $\displaystyle = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \mathrm{e}^{\ln x_n} = \mathrm{e}^{-1}$.

(3) 求 $I=\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+n}\right).$

解 $I=\lim\limits_{n\to\infty}\frac{1}{n}+\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{n+i}$

$$=0+\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\Delta x_i$$

$$=\int_0^1\frac{1}{1+x}\,\mathrm{d}\,x=\ln(1+x)\Big|_0^1=\ln 2.$$

$f\left(\frac{i}{n}\right)=\frac{1}{1+\frac{i}{n}}$,

$f(x)=\frac{1}{1+x}\in C[0,1]$

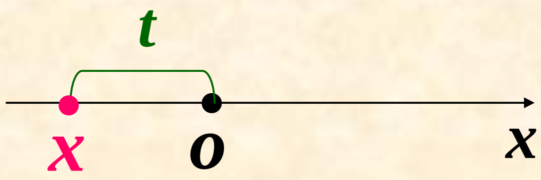
$\xi_i=\frac{i}{n}\quad(i=1,2,\cdots,n)$

$\Delta x_i=\frac{1}{n},\quad x\in[0,1]$

例14 设 $f(x)=\begin{cases} 1, & x>0 \\ 0, & x=0, \\ -1, & x<0 \end{cases}$ $F(x)=\int_0^x f(t)\mathrm{d}t$

讨论$F(x)$的连续性及可导性.

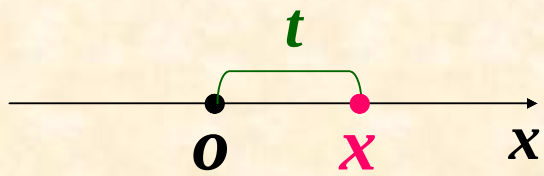
解 当$x<0$时,



$$F(x)=\int_0^x f(t)\mathrm{d}t=\int_0^x(-1)\mathrm{d}t=-x$$

当$x=0$时, $F(0)=0$

当$x>0$时,



$$F(x)=\int_0^x f(t)\mathrm{d}t=\int_0^x 1\mathrm{d}t=x$$

$\therefore \quad F(x) = |x|$

$\therefore \quad F(x)$在R上连续，在 $x = 0$处不可导，

在$x \neq 0$处可导.

类似题:

1.设 $f(x)=\begin{cases}\sin x, & 0\le x<\pi \\ 2, & \pi\le x\le 2\pi\end{cases}$, $F(x)=\int_0^x f(t)\,\mathrm{d}t.$

讨论 $F(x)$ 在 $x=\pi$ 处的连续性及可导性 **2013考研**

解 $F(x)=\int_0^x f(t)\,\mathrm{d}t$

$$=\begin{cases}\int_0^x \sin t\,\mathrm{d}t, & 0\le x<\pi \\ \int_0^\pi \sin t\,\mathrm{d}t+\int_\pi^x 2\,\mathrm{d}t, & \pi\le x\le 2\pi\end{cases}$$

$$=\begin{cases}1-\cos x, & 0\le x<\pi \\ 2+2x-2\pi, & \pi\le x\le 2\pi\end{cases}$$

$$F(x)=\begin{cases}1-\cos x, & 0\le x<\pi\\ 2x-2\pi+2, & \pi\le x\le 2\pi\end{cases}$$

(1) 连续性

$$\because \quad F(\pi^-)=\lim_{x\to\pi^-}(1-\cos x)=1-\cos\pi=2$$

$$=F(\pi^+)=\lim_{x\to\pi^+}(2x-2\pi+2)=2$$

$$=F(\pi)$$

$\therefore F(x)$在$x=\pi$处连续.

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(2) 可导性

$$\because\ F'_-(\pi) = \lim_{x\to\pi^-} \frac{F(x) - F(\pi)}{x - \pi}$$

$$= \lim_{x\to\pi^-} \frac{(1 - \cos x) - 2}{x - \pi} = \lim_{x\to\pi^-} \frac{\sin x}{1} = 0$$

$$F'_+(\pi) = \lim_{x\to\pi^+} \frac{F(x) - F(\pi)}{x - \pi}$$

$$= \lim_{x\to\pi^-} \frac{(2x - 2\pi + 2) - 2}{x - \pi} = 2$$

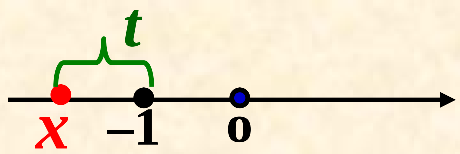
$$F'_-(\pi) \neq F'_+(\pi)$$

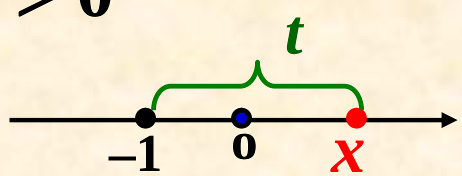
$\therefore F(x)$在$x = \pi$处不可导.

2. 设 $f(x)=|x|$, 求 $\int_{-1}^{x} f(t)\,\mathrm{d}t,\ x\in(-\infty,+\infty)$.

解

$$\int_{-1}^{x} f(t)dt=\begin{cases}\int_{-1}^{x}(-t)dt, & x\le 0\\ \int_{-1}^{0}(-t)dt+\int_{0}^{x} tdt, & x>0\end{cases}$$





$$=\begin{cases}\dfrac{1}{2}-\dfrac{x^2}{2}, & x\le 0\\ \dfrac{1}{2}+\dfrac{x^2}{2}, & x>0\end{cases}$$

3\. 设 $f(x)=x,\ g(x)=\begin{cases}\sin x, & 0\le x\le \dfrac{\pi}{2}\\ 0, & x>\dfrac{\pi}{2}\end{cases}$,

求 $\int_0^x f(t)g(x-t)\,\mathrm{d}t$,在$[0,+\infty)$上的表达式.

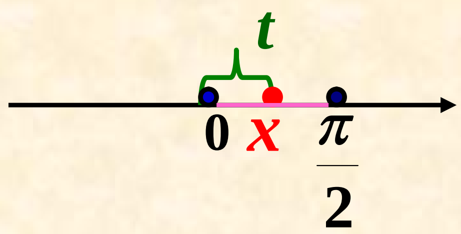
解 $\int_0^x f(t)g(x-t)\,\mathrm{d}t \xlongequal{u=x-t} \int_x^0 f(x-u)g(u)(-\mathrm{d}u)$

$=\int_0^x f(x-u)g(u)\,\mathrm{d}u$

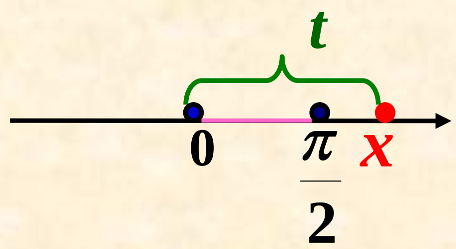
$=\int_0^x (x-u)g(u)\,\mathrm{d}u$

$$= \int_0^x (x-u)g(u)\,\mathrm{d}u$$

$$= \begin{cases} \int_0^x (x-u)\sin u\,\mathrm{d}u, & 0 \le x \le \dfrac{\pi}{2} \\ \int_0^{\frac{\pi}{2}} (x-u)\sin u\,\mathrm{d}u + \int_{\frac{\pi}{2}}^x (x-u)\cdot 0\,\mathrm{d}u, & x > \dfrac{\pi}{2} \end{cases}$$



$$= \begin{cases} x - \sin x, & 0 \le x \le \dfrac{\pi}{2} \\ x - 1, & x > \dfrac{\pi}{2} \end{cases}$$



4. 设 $f(x)=\begin{cases}2x, & 0<x<1\\ 0, & \text{其他}\end{cases}$，$g(x)=\begin{cases}e^{-x}, & x>0\\ 0, & x\le 0\end{cases}$，

求 $h(t)=\int_{-\infty}^{+\infty} f(x)g(t-x)\,\mathrm{d}x$ 的表达式.

解 $$h(t)=\int_{-\infty}^{0} \underbrace{\boxed{f(x)}}_{0} g(t-x)\,\mathrm{d}x+\int_{0}^{1} f(x)g(t-x)\,\mathrm{d}x$$

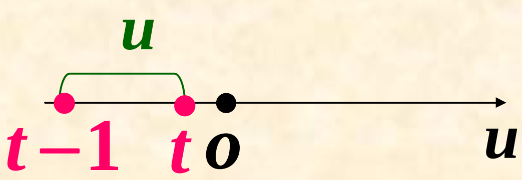
$$+\int_{1}^{+\infty} \underbrace{\boxed{f(x)}}_{0} g(t-x)\,\mathrm{d}x$$

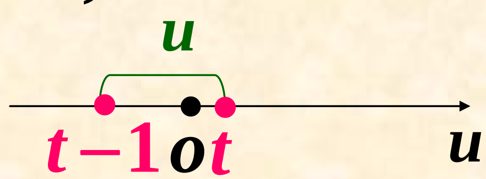
$$=\int_{0}^{1} f(x)g(t-x)\,\mathrm{d}x=\int_{0}^{1} 2xg(t-x)\,\mathrm{d}x$$

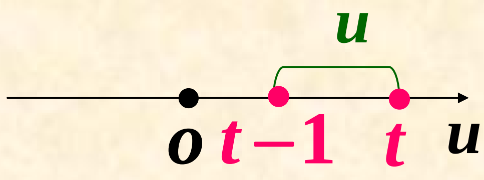
$$h(t)=\int_0^1 2xg(t-x)\,\mathrm{d}x \qquad g(x)=\begin{cases} e^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$$

$$\overset{u=t-x}{=\!=\!=}\ 2\int_t^{t-1}(t-u)g(u)(-\mathrm{d}u)=2\int_{t-1}^{t}(t-u)g(u)\,\mathrm{d}u$$

$$=\begin{cases} 0, & \text{当}\,t\le 0\text{时} \\ 2\int_{t-1}^{0}(t-u)\underbrace{\boxed{g(u)}}_{0}\mathrm{d}u+2\int_0^t (t-u)\,e^{-u}\,\mathrm{d}u, & \text{当}\,0<t\le 1\text{时} \\ 2\int_{t-1}^{t}(t-u)e^{-u}\,\mathrm{d}u, & \text{当}\,t>1\text{时} \end{cases}$$







$$h(t)=\begin{cases} 0, & t\le 0 \\ 2(e^{-t}+t-1), & 0<t\le 1 \\ 2e^{-t}, & t>1 \end{cases}$$