第四节

多元复合函数的求导法则

- 一、主要内容
- 二、典型例题
- 三、同步练习
- 四、同步练习解答

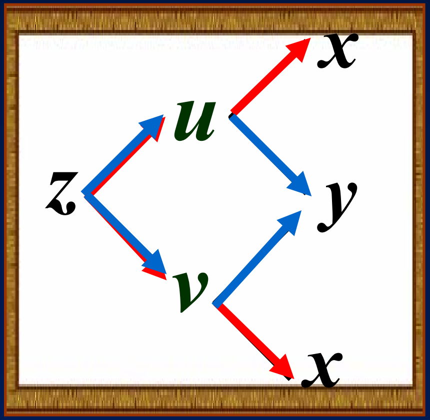
一、主要内容

（一）多元复合函数求导的链式法则

定理8.5　设函数 $u=\varphi(x,y)$ 和 $v=\psi(x,y)$ 在点 (x,y) 具有对 x 及 y 的偏导数，$z=f(u,v)$ 在对应点 (u,v) 处偏导数连续，则复合函数

$$z=f[\varphi(x,y),\psi(x,y)]$$

在点 (x,y) 处可导，且



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

若引入记号:

$$f_1' = \frac{\partial f(u,v)}{\partial u}, \quad f_2' = \frac{\partial f(u,v)}{\partial v},$$

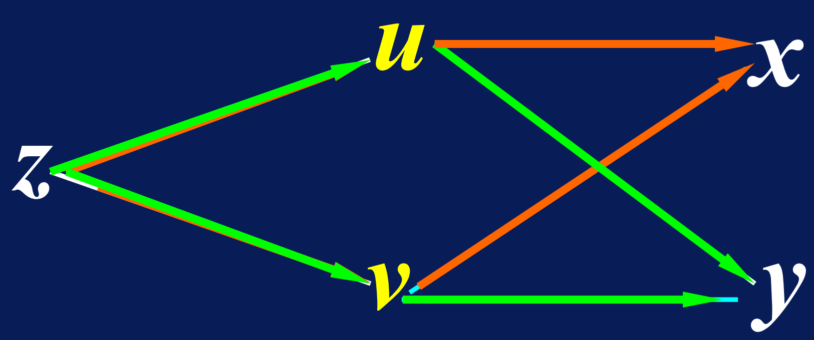
$$\varphi_1' = \frac{\partial \varphi(x,y)}{\partial x}, \quad \psi_1' = \frac{\partial \psi(x,y)}{\partial x}, \quad \varphi_2' = \frac{\partial \varphi(x,y)}{\partial y}, \quad \psi_2' = \frac{\partial \psi(x,y)}{\partial y}$$

则

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1'\varphi_1' + f_2'\psi_1'$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = f_1'\varphi_2' + f_2'\psi_2'$$

注 1° 复合关系图(结构图)



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

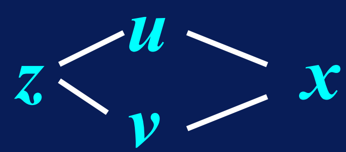
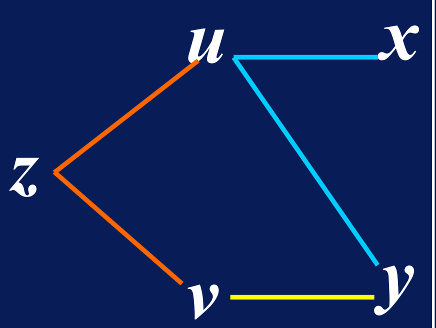
口诀：“项数 = 通向该自变量的路径数”.

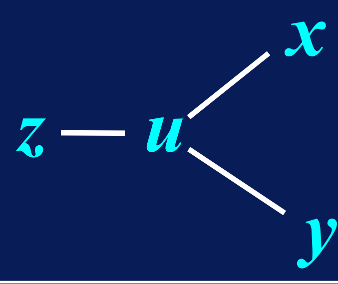
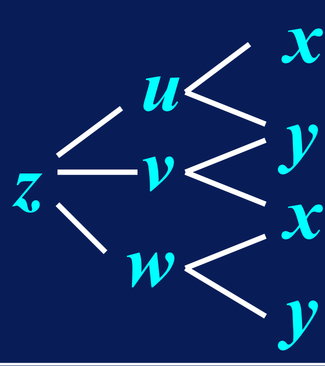
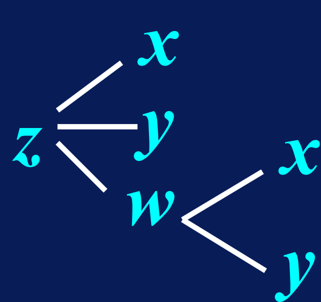
“连线相乘，分线相加”;

“单路全导, 叉路偏导”

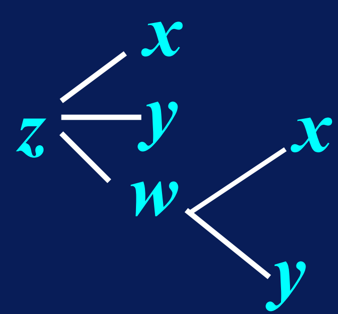
2° 其他情形

全导数

函数关系	结构图	求导公式
$z=f(u,v)$ $u=\varphi(x)$ $v=\psi(x)$		$\boxed{\frac{\mathrm{d}z}{\mathrm{d}x}}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}$
$z=f(u,v)$ $u=\varphi(x,y)$ $v=\psi(y)$		$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}$ $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}y}$

函数关系	关系图	求导公式
$z = f(u)$ $u = \varphi(x, y)$		$\dfrac{\partial z}{\partial x} = \dfrac{\mathrm{d} z}{\mathrm{d} u}\dfrac{\partial u}{\partial x}, \quad \dfrac{\partial z}{\partial y} = \dfrac{\mathrm{d} z}{\mathrm{d} u}\dfrac{\partial u}{\partial y}$
$z = f(u, v, w)$ $u = u(x, y)$ $v = v(x, y)$ $w = w(x, y)$		$\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial x}$ $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} + \dfrac{\partial z}{\partial w}\dfrac{\partial w}{\partial y}$
★ $z = f(x, y, w)$ $w = \varphi(x, y)$		$\dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial x}$ $\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial y}$

★ $z=f(x,y,w),\ w=\varphi(x,y)$

即 

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

$$=\boxed{\frac{\partial f}{\partial x}}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x},$$

把复合函数
$z=f[x,y,\varphi(x,y)]$
中的y看作不变，而
对x的偏导数

两者的区别

把 $z=f(x,y,w)$中
的y及 w看作不变
而对 x的偏导数

3° 若将定理条件: $f(u,v)$ 在点(u,v)偏导数连续减弱为偏导数存在, 则定理结论不一定成立.

如: $$z=f(u,v)=\begin{cases}\dfrac{u^2v}{u^2+v^2}, & u^2+v^2\neq 0\\ 0, & u^2+v^2=0\end{cases}$$ $u=t$, $v=t$

可复合为 $$z=f(t,t)=\frac{t^2t}{t^2+t^2}=\frac{t}{2}$$

虽然 $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)}=0$, $\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)}=0$ 但 $\dfrac{\partial z}{\partial u},\dfrac{\partial z}{\partial v}$在(0,0)不连续

$$\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0}=\frac{1}{2}\neq\left.\left(\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}\right)\right|_{(0,0)}=0\cdot 1+0\cdot 1=0$$

（二）一阶全微分形式不变性

设 $z=f(u,v)$有连续的偏导数，则

当 u,v 是自变量时，有

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

当 u,v 是中间变量时，若 $u=\varphi(x,y),v=\psi(x,y)$

均有连续的偏导数，则

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x})\mathrm{d}x+(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y})\mathrm{d}y$$

$$\mathrm{d}z = \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$ —— 一阶 全微分形式不变性

一阶全微分形式不变性的实质:

无论 u,v 是自变量还是中间变量，函数的一阶全微分表达形式都一样，均为

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

二、典型例题

1. 中间变量均为多元函数的复合函数求导

例1 设 $z = e^u \sin v$, $u = xy$, $v = x + y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解 (方法1)

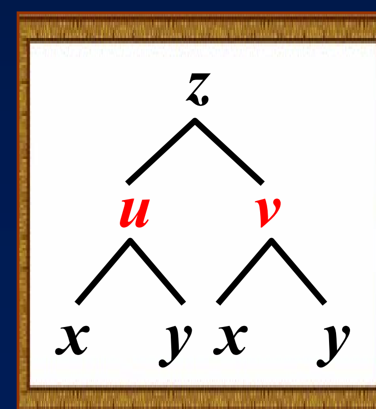
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1 = e^u (y \sin v + \cos v),$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^{xy} (x \sin(x+y) + \cos(x+y)).$$



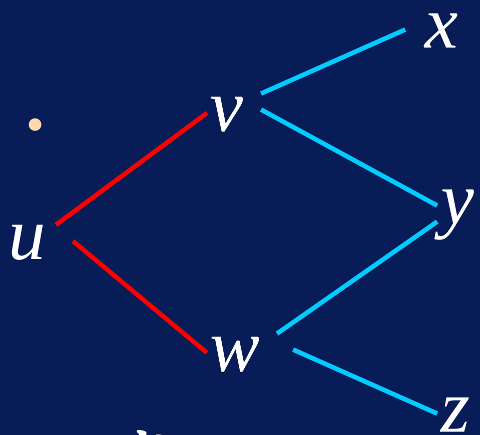
(方法2) $z = e^u \sin v = e^{xy}\sin(x+y)$

$$\frac{\partial z}{\partial x} = e^{xy} y \cdot \sin(x+y) + e^{xy} \cdot \cos(x+y) \cdot 1$$

$$= e^{xy}[y\sin(x+y) + \cos(x+y)]$$

(对x 求偏导数时，暂视 y 为常数)

例2 设 $u = f(\frac{x}{y}, \frac{y}{z})$，其中 f 有一阶连续偏导数，求函数 u 的一阶偏导数 .



解 设 $v = \frac{x}{y}, w = \frac{y}{z}$，则函数

由 $u = f(v, w), v = \frac{x}{y}, w = \frac{y}{z}$ 复合而成 .

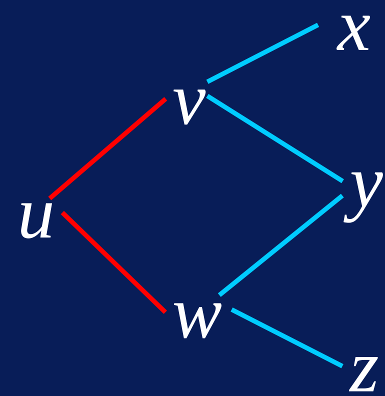
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial f}{\partial v} \cdot \frac{1}{y},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y} = \frac{\partial f}{\partial v} \cdot (-\frac{x}{y^2}) + \frac{\partial f}{\partial w} \cdot \frac{1}{z}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial z}=\frac{\partial f}{\partial w}\cdot\left(-\frac{y}{z^2}\right)=-\frac{y}{z^2}\cdot\frac{\partial f}{\partial w}.$$

若使用记号: $\frac{\partial f(v,w)}{\partial v}=f_1'$, $\frac{\partial f(v,w)}{\partial w}=f_2'$

则上述结果可表示为:



$$\boxed{\begin{aligned}&u=f(v,w),\\&v=\frac{x}{y},w=\frac{y}{z}\end{aligned}}$$

$$\frac{\partial u}{\partial x}=\frac{1}{y}f_1',$$

$$\frac{\partial u}{\partial y}=-\frac{x}{y^2}f_1'+\frac{1}{z}f_2',\quad \frac{\partial u}{\partial z}=-\frac{y}{z^2}f_2'.$$

例3 设 $z=f[xy+\varphi(y)]$，其中 f,φ 可微，

求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解 令 $u=xy+\varphi(y)$

$$\begin{tikzcd}[row sep=small] & & x \\ z \arrow[r, dash] & u \arrow[ur, dash] \arrow[dr, dash] & \\ & & y \end{tikzcd}$$

$$\frac{\partial z}{\partial x}=\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial x}=f'(u)\cdot y=y\,f'[xy+\varphi(y)]$$

$$\frac{\partial z}{\partial y}=\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial y}=f'(u)\cdot[x+\varphi'(y)]$$

$$=[x+\varphi'(y)]\,f'[xy+\varphi(y)]$$

2. 中间变量均为一元函数的复合函数求导

例4 设 $y=[f(x)]^{\varphi(x)}$，其中 $f(x)>0$，求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

解 令 $u=f(x),\ v=\varphi(x)$,

则 $y=[f(x)]^{\varphi(x)}$ 可看作由 $y=u^{v}$,

$u=f(x), v=\varphi(x)$ 复合而成 .

$$
\begin{tikzcd}[column sep=small, row sep=tiny]
 & u \arrow[dr, dash] & \\
y \arrow[ur, dash] \arrow[dr, dash] & & x \\
 & v \arrow[ur, dash] &
\end{tikzcd}
$$

所以
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\partial y}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial y}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$=vu^{v-1}f'(x)+u^{v}(\ln u)\varphi'(x)$$

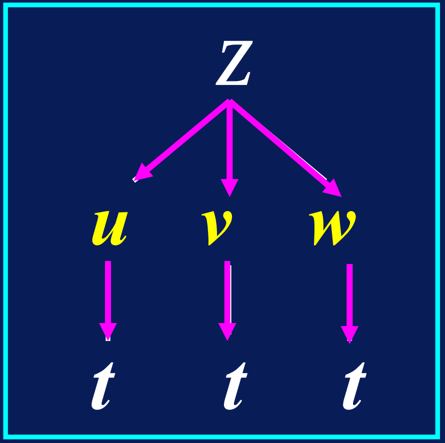
$$=[f(x)]^{\varphi(x)}\left[\frac{\varphi(x)}{f(x)}f'(x)+\varphi'(x)\ln f(x)\right].$$

推广：假设下面所涉及到的函数都可微．

当中间变量多于两个时，**例如:**

$$z = f(u, v, w),$$

$$u = \varphi(t), v = \psi(t), w = \omega(t)$$



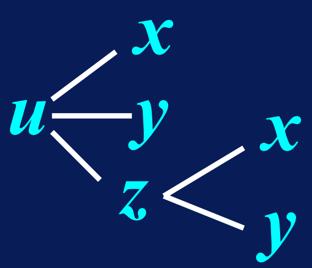
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

$$= f_1'\varphi' + f_2'\psi' + f_3'\omega'$$

3.中间变量既有一元函数，又有多元函数的复合函数求导

例5 设 $u=e^{x^2+y^2+z^2}$, $z=x^2\sin y$, 求 $\dfrac{\partial u}{\partial x}$ 及 $\dfrac{\partial u}{\partial y}$.

解 (方法1) 令 $u=f(x,y,z)=e^{x^2+y^2+z^2}$



$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

$$=e^{x^2+y^2+z^2}\cdot 2x+e^{x^2+y^2+z^2}\cdot 2z\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}.$$

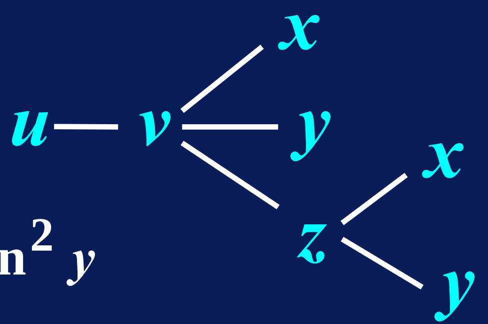
$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=e^{x^2+y^2+z^2}\cdot 2y+e^{x^2+y^2+z^2}\cdot 2z\cdot x^2\cos y$$

$$=(2y+x^4\sin 2y)e^{x^2+y^2+x^4\sin^2 y}.$$

(方法2) 令 $v=x^2+y^2+z^2$，则 $u=e^v$,, $z=x^2\sin y$.

$$\frac{\partial u}{\partial x}=\frac{\mathrm{d}u}{\mathrm{d}v}\cdot\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=e^v\cdot\left(2x+2z\cdot\frac{\partial z}{\partial x}\right)$$

$$=e^v\cdot(2x+2z\cdot 2x\sin y)$$



$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

注 对具体函数，用方法2较简单.

(方法3) 由一阶全微分的形式不 变性，

$$\mathrm{d}u = \mathrm{e}^{x^2+y^2+z^2}\mathrm{d}(x^2+y^2+z^2)$$

$$= \mathrm{e}^{x^2+y^2+z^2}(2x\mathrm{d}x + 2y\mathrm{d}y + 2z\mathrm{d}z)$$

$$= \mathrm{e}^{x^2+y^2+z^2}(2x\mathrm{d}x + 2y\mathrm{d}y + 2z\underline{\mathrm{d}(x^2\sin y)})$$

$$= \mathrm{e}^{x^2+y^2+z^2}[2x\mathrm{d}x + 2y\mathrm{d}y + 2z(2x\sin y\mathrm{d}x + x^2\cos y\mathrm{d}y)]$$

$$= \mathrm{e}^{x^2+y^2+z^2}[2x(1+2z\sin y)\mathrm{d}x + (2y+2x^2 z\cos y)\mathrm{d}y]$$

$$= \mathrm{e}^{x^2+y^2+x^4\sin^2 y}[2x(1+2x^2\sin^2 y)\mathrm{d}x + (2y+2x^4\sin y\cos y)\mathrm{d}y]$$

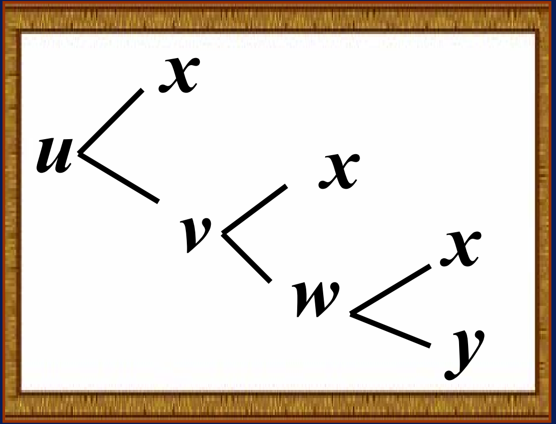
$$\mathrm{d}u = \mathrm{e}^{x^2+y^2+x^4\sin^2 y}[2x(1+2x^2\sin^2 y)\mathrm{d}x + (2y+x^4\sin 2y)\mathrm{d}y]$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x(1+2x^2\sin^2 y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y},$$

$$\frac{\partial u}{\partial y} = (2y+x^4\sin 2y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}.$$

例6 设 $u = xf(x, \frac{y}{x})$, f的二阶偏导数存在.

求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ 及 $\frac{\partial^2 u}{\partial x \partial y}$.



解 令 $w = \frac{y}{x}$, $v = f(x, w)$, $u = xv$.

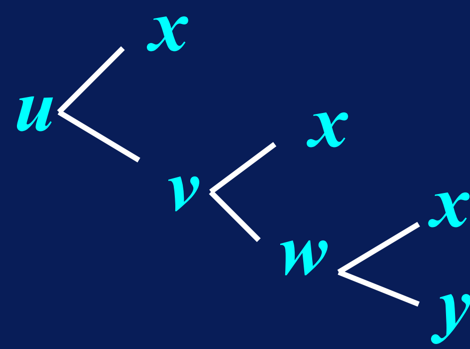
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial v} \cdot \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial w} \cdot \frac{\partial w}{\partial x}\right) = v + x\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}\right)$$

$$= f + x[f_1' + f_2' \cdot (-\frac{y}{x^2})] = f + xf_1' - \frac{y}{x} f_2'.$$

可记 $f = f(x, \frac{y}{x})$, $f_1' = \frac{\partial f}{\partial x}$, $f_2' = \frac{\partial f}{\partial w}$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial w}\cdot\frac{\partial w}{\partial y}$$

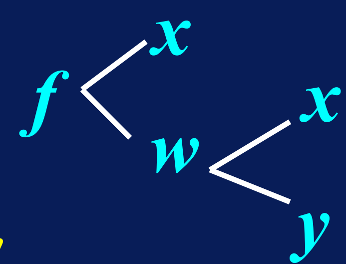
$$=x\cdot f_2'\cdot\frac{1}{x}=f_2'$$



$$u=xv$$
$$v=f(x,w)$$
$$w=\frac{y}{x}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(f+x f_1'-\frac{y}{x}f_2'\right)$$

$$=\cancel{f_2'\cdot\frac{1}{x}}+x\cdot f_{12}''\cdot\frac{1}{x}$$



$$-\left(\cancel{\frac{1}{x}f_2'}+\frac{y}{x}f_{22}''\cdot\frac{1}{x}\right)=f_{12}''-\frac{y}{x^2}f_{22}''.$$

例7 已知 $e^{-xy}-2z+e^{z}=0$，求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

解 $\because \quad \mathrm{d}(e^{-xy}-2z+e^{z})=0,$

$\therefore \quad e^{-xy}\,\mathrm{d}(-xy)-2\,\mathrm{d}z+e^{z}\,\mathrm{d}z=0,$

$$(e^{z}-2)\,\mathrm{d}z=e^{-xy}(x\,\mathrm{d}y+y\,\mathrm{d}x)$$

$$\mathrm{d}z=\frac{ye^{-xy}}{(e^{z}-2)}\mathrm{d}x+\frac{xe^{-xy}}{(e^{z}-2)}\mathrm{d}y$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

三、同步练习

1. 设 $z=\arctan\dfrac{x}{y}, x=u+v, y=u-v$, 求 $\dfrac{\partial z}{\partial v}$.

2. 设 $u=f(\dfrac{x}{y},\dfrac{y}{z})$ 其中f可微,求u的一阶偏导数.

3. 设 $z=f(u,x,y),\ u=xe^{y}$ 求 $\dfrac{\partial^2 z}{\partial x\partial y},\dfrac{\partial^2 z}{\partial x^2}$.

4. 设 $z=f(u,v), u=xy, v=e^{x}$,

求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

5. 已知 $f(x,y)\Big|_{y=x^2}=1$， $f_1'(x,y)\Big|_{y=x^2}=2x$，求 $f_2'(x,y)\Big|_{y=x^2}$.

6. 设 $z=f(xy,x^2+y^2)$, $y=\varphi(x)$, f 可微，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

7. 设 $z=uv+\sin t$, $u=e^t$, $v=\cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

8. 设$z=f(x+\varphi(y))$， 其中f具有二阶连续偏导数，试证 ：$\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial^2 z}{\partial x^2}$.

9. 设 $z=f(u)$, 方程 $u=\varphi(u)+\int_y^x p(t)dt$ 确定 u 是 x,y 的函数，其中 $f(u)$, $\varphi(u)$ 可微, $p(t),\varphi'(u)$ 连续, 且 $\varphi'(u)\neq 1$, 求 $p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}$.

10. 设 $z=\frac{1}{x}f(xy)+y\varphi(x+y)$, f,φ 具有连续导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

11. 设 $z=f(u,v)$, $u=xy$, $v=e^x$，求 $\frac{\partial^2 z}{\partial x\partial y}$.

12. 设 $u = f(x, y, z),\ y = g(x, t),\ t = h(x, z)$ 均可微，求 $\dfrac{\partial u}{\partial x}$ 及 $\dfrac{\partial u}{\partial z}$.

13. 设函数 $z = f(x, y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1) = 1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 3,$$

$\varphi(x) = f(x, f(x, x))$，求 $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. (2001考研)

14. 设$u = x - 2\sqrt{y}, v = x + 2\sqrt{y}, (y > 0)$,变换

方程: $\dfrac{\partial^2 z}{\partial x^2} - y\dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{2}\dfrac{\partial z}{\partial y}$为$u$, v的方程

(其中所涉及的函数 z的二阶偏导数假定都连 续).

15. 设$u = f(x, y)$二阶偏导数连续,求下列表达式在

极坐标系下的形式

$$(1)\ (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2,\quad (2)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

四、同步练习解答

1. 设 $z=\arctan\frac{x}{y}$, $x=u+v$, $y=u-v$, 求 $\frac{\partial z}{\partial v}$.

解

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}\cdot 1+\frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2})\cdot(-1)$$

$$=\frac{y+x}{x^2+y^2}=\frac{u}{u^2+v^2}$$

2. 设 $u = f(\frac{x}{y}, \frac{y}{z})$ 其中f可微,求u的一阶偏导数.

解

$$\frac{\partial u}{\partial x} = f_1' \cdot \frac{1}{y} = \frac{1}{y} f_1',$$

$$\frac{\partial u}{\partial y} = f_1' \cdot (-\frac{x}{y^2}) + f_2' \cdot \frac{1}{z} = -\frac{x}{y^2} f_1' + \frac{1}{z} f_2',$$

$$\frac{\partial u}{\partial z} = f_2' \cdot (-\frac{y}{z^2}) = -\frac{y}{z^2} f_2'.$$

3. 设 $z=f(u,x,y)$, $u=x\mathrm{e}^y$ 求 $\dfrac{\partial^2 z}{\partial x\partial y}, \dfrac{\partial^2 z}{\partial x^2}$.

其中f具有连续的二阶偏导数.

解

$$\frac{\partial z}{\partial x}=f_1'\cdot \mathrm{e}^y+f_2'$$

$$\frac{\partial^2 z}{\partial x\partial y}=\mathrm{e}^y f_1'+\mathrm{e}^y\cdot(f_{11}''\cdot x\mathrm{e}^y+f_{13}'')$$

$$+x\mathrm{e}^y f_{21}''+f_{23}''$$

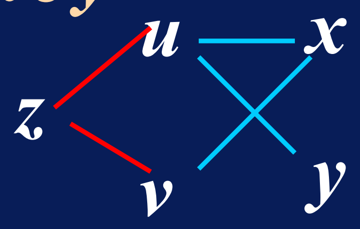
$$\frac{\partial^2 z}{\partial x^2}=\mathrm{e}^y(f_{11}''\cdot \mathrm{e}^y+f_{12}'')+f_{21}''\cdot \mathrm{e}^y+f_{22}''$$

$$=\mathrm{e}^{2y}f_{11}''+2\mathrm{e}^y f_{12}''+f_{22}''.$$

4. 设 $z = f(u, v), u = xy, v = e^x$,

求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

解



$$z_x = f_1' \cdot y + f_2' \cdot e^x = y f_1' + e^x f_2',$$

$$z_{xy} = f_1' + y f_{11}'' \cdot x + e^x f_{21}'' \cdot x$$

$$= f_1' + xy f_{11}'' + x e^x f_{21}''.$$

5. 已知 $f(x,y)\Big|_{y=x^2}=1$， $f_1'(x,y)\Big|_{y=x^2}=2x$, 求 $f_2'(x,y)\Big|_{y=x^2}$.

解 由$f(x,x^2)=1$两边对 x 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

6. 设 $z = u^2 \ln v$, 而 $u = \dfrac{x}{y}$, $v = 3x - 2y$, 求 $\dfrac{\partial z}{\partial y}$.

解 (方法1) 把 u, v代入，得到复合函数

$$z = \frac{x^2}{y^2}\ln(3x - 2y),$$

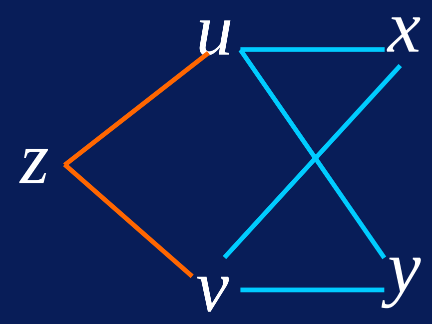
再利用多元函数求偏导 数的方法求 $\dfrac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial y} = -\frac{2x^2}{y^3}\ln(3x - 2y) + \frac{x^2}{y^2}\cdot\frac{-2}{3x - 2y}$$

$$= -\frac{2x^2}{y^3}\ln(3x - 2y) - \frac{2x^2}{y^2}\cdot\frac{1}{3x - 2y}.$$

(方法2) 利用多元复合函数的求导法则:

$$z = u^2 \ln v$$
$$u = \frac{x}{y},\ v = 3x - 2y$$



画出关系

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

写出公式

$$= 2u\ln v\cdot\left(-\frac{x}{y^2}\right) + u^2\cdot\frac{1}{v}(-2)$$

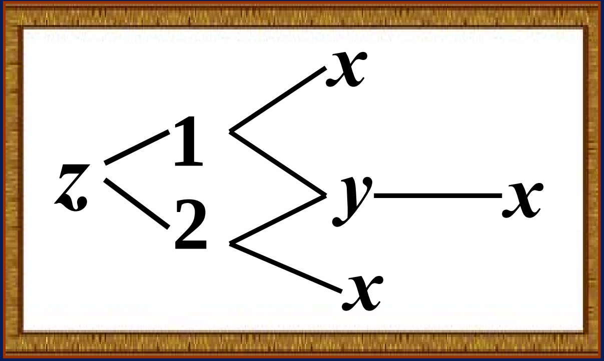
求出各偏导数

$$= -\frac{2x^2}{y^3}\ln(3x-2y) - \frac{2x^2}{y^2(3x-2y)}.$$

将x，y代入

6. 设 $z=f(xy,x^2+y^2),\ y=\varphi(x)$, f 可微,

求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.



解 $$\frac{\mathrm{d}z}{\mathrm{d}x}=f_1'\cdot\left(y+x\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$+f_2'\cdot\left(2x+2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

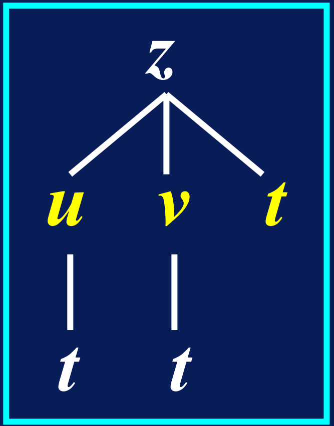
$$=[y+x\varphi'(x)]f_1'+2[x+\varphi(x)\cdot\varphi'(x)]f_2'.$$

7. 设 $z = uv + \sin t,\ u = e^t,\ v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

$$= v e^t - u\sin t + \cos t$$

$$= e^t(\cos t - \sin t) + \cos t$$



8. 设 $z = f(x + \varphi(y))$， 其中 f 具有

二阶连续偏导数，试证 ：$\dfrac{\partial z}{\partial x} \cdot \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial^2 z}{\partial x^2}$.

证 令 $u = x + \varphi(y)$, 则 $z = f(u), u = x + \varphi(y)$

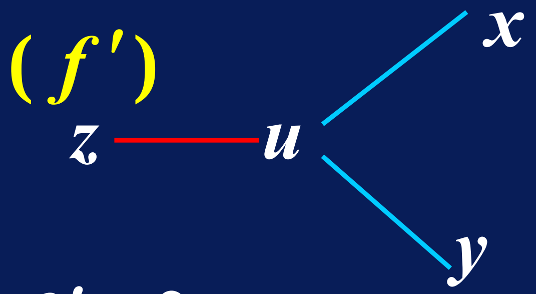
故

$$\begin{tikzcd}[arrows=dash] & & x \\ z \arrow[r] & u \arrow[ur] \arrow[dr] & \\ & & y \end{tikzcd}$$

于是

$$\frac{\partial z}{\partial x} = \frac{\mathrm{d} z}{\mathrm{d} u} \cdot \frac{\partial u}{\partial x} = f',$$

$$\frac{\partial z}{\partial y}=\frac{\mathrm{d}\,z}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial y}=f'\cdot\varphi',$$



$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}(f')=\frac{\mathrm{d}\,f'}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial y}=f''\cdot\varphi',$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(f')=\frac{\mathrm{d}\,f'}{\mathrm{d}\,u}\cdot\frac{\partial u}{\partial x}=f'',$$

故 $\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial^2 z}{\partial x\partial y}=f'\cdot f''\cdot\varphi'=\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial^2 z}{\partial x^2}.$

9. 设 $z=f(u)$, 方程 $u=\varphi(u)+\int_y^x p(t)\,dt$

确定 u 是 x,y 的函数, 其中 $f(u)$, $\varphi(u)$ 可微, $p(t),\varphi'(u)$ 连续, 且 $\varphi'(u)\neq 1$, 求 $p(y)\dfrac{\partial z}{\partial x}+p(x)\dfrac{\partial z}{\partial y}$.

解 $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$, $\dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\left\{\begin{aligned}\frac{\partial u}{\partial x}&=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}&=\frac{-p(y)}{1-\varphi'(u)}\end{aligned}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[\,p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\,\right]=0$$

10. 设 $z=\frac{1}{x}f(xy)+y\varphi(x+y)$, f,φ具有连续导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解 $\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}[\frac{1}{x}f(xy)]+\frac{\partial}{\partial x}[y\varphi(x+y)]$

$$=[(-\frac{1}{x^2})f(xy)+\frac{1}{x}f'(xy)\cdot y]+y\varphi'(x+y)\cdot 1$$

$$=-\frac{1}{x^2}f(xy)+\frac{y}{x}f'(xy)+y\varphi'(x+y)$$

$$\frac{\partial z}{\partial x}=-\frac{1}{x^2}f(xy)+\frac{y}{x}f'(xy)+y\varphi'(x+y)$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$=\cancel{\left(-\frac{1}{x^2}\right)f'(xy)\cdot x}+\cancel{\frac{1}{x}f'(xy)}+\frac{y}{x}f''(xy)x$$

$$+[\varphi'(x+y)+y\varphi''(x+y)]$$

$$=yf''(xy)+\varphi'(x+y)+y\varphi''(x+y)$$

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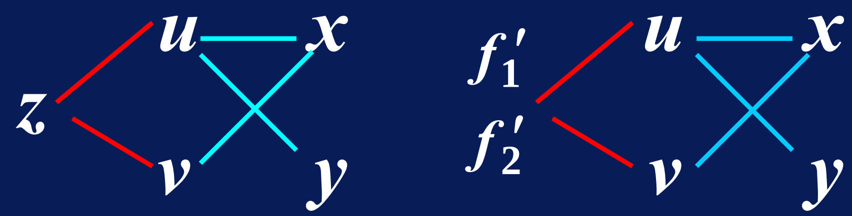
11. 设 $z = f(u,v), u = xy, v = e^x$，求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

解



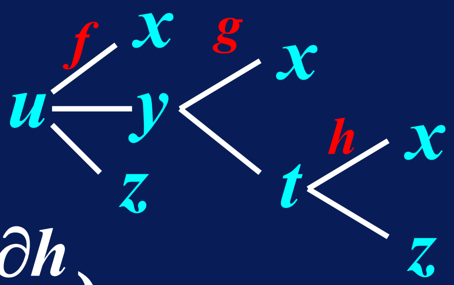
$$z_x = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1'\cdot y + f_2'\cdot e^x,$$

$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_1'\cdot 1 + y\cdot f_{11}''\cdot x + e^x\cdot f_{21}''\cdot x$$

$$= f_{11}''\cdot xy + f_1' + f_{21}''\cdot xe^x.$$

12. 设 $u = f(x, y, z),\ y = g(x, t), t = h(x, z)$ 均可微，求 $\dfrac{\partial u}{\partial x}$ 及 $\dfrac{\partial u}{\partial z}$.

解 $$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}$$



$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t}\frac{\partial h}{\partial x}\right)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial t}\frac{\partial h}{\partial x}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial t}\frac{\partial h}{\partial z}$$

13. 设函数 $z=f(x,y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$\varphi(x)=f(x,f(x,x))$，求 $\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. (2001考研)

解 由题设 $\varphi(1)=f(1,f(1,1))=f(1,1)=1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}=3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$=3\varphi^2(1)\Big[f_1'(x,f(x,x))+f_2'(x,f(x,x))\big(f_1'(x,x)+f_2'(x,x)\big)\Big]\Big|_{x=1}$$

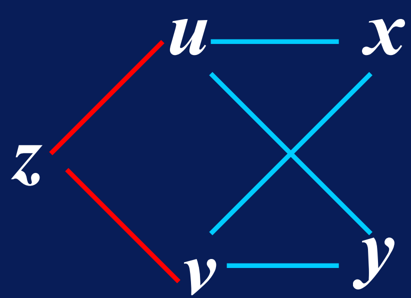
$$=3\cdot\big[2+3\cdot(2+3)\big]=51$$

14. 设$u = x - 2\sqrt{y}, v = x + 2\sqrt{y}, (y > 0)$,变换

方程: $\dfrac{\partial^2 z}{\partial x^2} - y\dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{2}\dfrac{\partial z}{\partial y}$为$u$，$v$的方程

(其中所涉及的函数 z的二阶偏导数假定都连 续).

解 z, u, v, x, y 的关系为



于是

$$z_x = z_u \cdot u_x + z_v v_x = z_u + z_v$$

$$z_{xx} = z_{uu} \cdot u_x + z_{uv} \cdot v_x + z_{vu} \cdot u_x + z_{vv} \cdot v_x$$

$$= z_{uu} + 2z_{uv} + z_{vv}$$

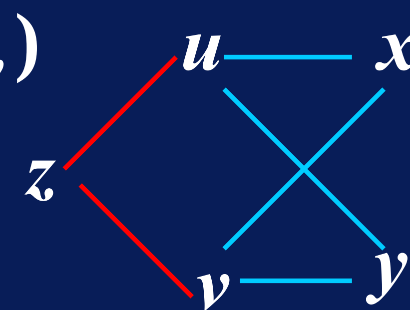
$$z_y = z_u \cdot u_y + z_v \cdot v_y = -\frac{1}{\sqrt{y}} z_u + \frac{1}{\sqrt{y}} z_v \qquad \boxed{\begin{aligned} u &= x - 2\sqrt{y} \\ v &= x + 2\sqrt{y} \end{aligned}}$$

$$= \frac{1}{\sqrt{y}}(-z_u + z_v)$$

$$z_{yy} = -\frac{1}{2\sqrt{y^3}}(-z_u + z_v) + \frac{1}{\sqrt{y}}\left[-z_{uu} \cdot \left(-\frac{1}{\sqrt{y}}\right) - z_{uv} \cdot \frac{1}{\sqrt{y}}\right.$$

$$\left. + z_{vu} \cdot \left(-\frac{1}{\sqrt{y}}\right) + z_{vv} \cdot \frac{1}{\sqrt{y}}\right]$$

$$= -\frac{1}{2\sqrt{y^3}}(-z_u + z_v) + \frac{1}{y}(z_{uu} - 2z_{uv} + z_{vv})$$



将z_{xx},z_{yy},z_y代入式：

$$\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}=\frac{1}{2}\frac{\partial z}{\partial y}$$

可得

$$4z_{uv}+\frac{1}{2\sqrt{y}}(-z_u+z_v)=\frac{1}{2\sqrt{y}}(-z_u+z_v)$$

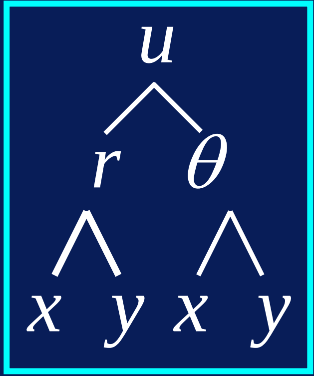
化简得 $$z_{uv}=0.$$

这是一个二阶双曲型偏微分方程的标准形式.

15. 设 $u=f(x,y)$ 二阶偏导数连续，求下列表达式在极坐标系下的形式 (1) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$, (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

解 已知 $x=r\cos\theta$, $y=r\sin\theta$, 则

$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}$$



(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\boxed{\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{-y}{x^2+y^2}}$$

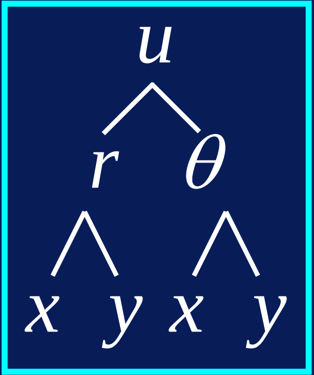
$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\boxed{\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}}$$

$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$



$$\therefore\quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial x}$$

$$r \quad \theta$$

$$x \quad y \quad x \quad y$$

(2) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$+\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\left(-\frac{\sin\theta}{r}\right)$$

注意利用已有公式

$$=\left(\frac{\partial^2 u}{\partial r^2}\cos\theta-\frac{\partial^2 u}{\partial \theta\partial r}\frac{\sin\theta}{r}+\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r^2}\right)\cos\theta$$

$$+\left(\frac{\partial^2 u}{\partial r\partial \theta}\cos\theta+\frac{\partial u}{\partial r}(-\sin\theta)-\frac{\partial^2 u}{\partial \theta^2}\frac{\sin\theta}{r}-\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$-\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$