第二节　多元函数的偏导数

习题　8-2

1．求下列函数的偏导数:

(1) $z = ax^2y + axy^2$;　　(2) $z = \tan^2(x^2 + y^2)$;

(3) $z = \dfrac{x}{y} + \dfrac{y}{x}$;　　(4) $z = \arctan\dfrac{x}{y^2}$;

(5) $z = \ln(x + \sqrt{x^2 - y^2})$;　　(6) $z = x\mathrm{e}^{-y} + y\mathrm{e}^{-x}$;

(7) $u = \ln(x + 2^{yz})$;　　(8) $z = (1 + xy)^y$.

解　(1) $z_x = ay \cdot 2x + ay^2 = 2axy + ay^2$,

$$z_y = ax^2 + ax \cdot 2y = ax^2 + 2axy.$$

(2) $z_x = 2\tan(x^2 + y^2) \cdot \sec^2(x^2 + y^2) \cdot 2x$

$$= 4x\tan(x^2 + y^2)\sec^2(x^2 + y^2),$$

由所给出函数关于自变量 x, y 的对称性, 所以有

$$z_y = 4y\tan(x^2 + y^2)\sec^2(x^2 + y^2).$$

(3) $z_x = \dfrac{1}{y} + y \cdot (-\dfrac{1}{x^2}) = \dfrac{1}{y} - \dfrac{y}{x^2}$,

$$z_y = x \cdot (-\frac{1}{y^2}) + \frac{1}{x} = \frac{1}{x} - \frac{x}{y^2}.$$

(4) $z_x = \dfrac{1}{1 + (\dfrac{x}{y^2})^2} \cdot \dfrac{1}{y^2} = \dfrac{y^2}{x^2 + y^4}$,

$$z_y = \frac{1}{1 + (\dfrac{x}{y^2})^2} \cdot x \cdot (-2)\frac{1}{y^3} = \frac{-2xy}{x^2 + y^4}.$$

(5) $z_x = \dfrac{1}{x + \sqrt{x^2 - y^2}} \cdot [1 + \dfrac{1}{2}(x^2 - y^2)^{-\frac{1}{2}} \cdot 2x]$

$$=\frac{1}{x+\sqrt{x^2-y^2}}\cdot(1+\frac{x}{\sqrt{x^2-y^2}})$$

$$=\frac{1}{x+\sqrt{x^2-y^2}}\cdot\frac{\sqrt{x^2-y^2}+x}{\sqrt{x^2-y^2}}=\frac{1}{\sqrt{x^2-y^2}},$$

$$z_y=\frac{1}{x+\sqrt{x^2-y^2}}\cdot\frac{1}{2}(x^2-y^2)^{-\frac{1}{2}}\cdot(-2y)=-\frac{y}{(x+\sqrt{x^2-y^2})\sqrt{x^2-y^2}}$$

$$=-\frac{y}{x\sqrt{x^2-y^2}+x^2-y^2}.$$

(6) $z_x=\mathrm{e}^{-y}+y\mathrm{e}^{-x}\cdot(-1)=\mathrm{e}^{-y}-y\mathrm{e}^{-x}$,

$z_y=x\mathrm{e}^{-y}\cdot(-1)+\mathrm{e}^{-x}=-x\mathrm{e}^{-y}+\mathrm{e}^{-x}$.

(7) $u_x=\dfrac{1}{x+2^{yz}}$,

$u_y=\dfrac{1}{x+2^{yz}}\cdot 2^{yz}\cdot\ln 2\cdot z=\dfrac{z2^{yz}\ln 2}{x+2^{yz}}$,

由所给函数关于自变量 y,z 的对称性，所以有

$$u_z=\frac{y2^{yz}\ln 2}{x+2^{yz}}.$$

注意 常见的错误是遗漏了步骤: $\dfrac{\partial}{\partial y}(yz)=z$，而得到错误结果, $u_y=\dfrac{2^{yz}\ln 2}{x+2^{yz}}$.

(8) 求 z_y 时，用幂函数的导数公式，得

$$z_x=y(1+xy)^{y-1}\cdot y=y^2(1+xy)^{y-1}.$$

求 z_y 时，把 x 暂时看作常数，这时 z 是关于 y 的幂指函数，所以

$$z_y=\frac{\partial}{\partial y}[\mathrm{e}^{y\ln(1+xy)}]=\mathrm{e}^{y\ln(1+xy)}[\ln(1+xy)+\frac{xy}{1+xy}]$$

$$=(1+xy)^y[\ln(1+xy)+\frac{xy}{1+xy}].$$

2. 求下列函数在指定点处的一阶偏导数:

(1) $f(x,y)=x+(y-1)\arcsin\sqrt{\dfrac{x}{y}}$ 在点 $(x,1)$ 对 x 的偏导数 $f_x(x,1)$;

(2) $f(x,y)=x^2\mathrm{e}^y+(x-1)\arctan\dfrac{y}{x}$在点$(1,0)$的两个偏导数$f_x(1,0)$与$f_y(1,0)$.

解 (1) **法 1** 因$f(x,1)=x$，所以

$$f_x(x,1)=1.$$

法 2 $f_x(x,y)=1+\dfrac{y-1}{\sqrt{1-\dfrac{x}{y}}}\cdot\dfrac{1}{2}\sqrt{\dfrac{y}{x}}\cdot\dfrac{1}{y}$，于是

$$f_x(x,1)=f_x(x,y)\big|_{y=1}=1.$$

(2) **法 1** 因$f(x,0)=x^2$，所以$f_x(x,0)=2x$，故

$$f_x(1,0)=2,$$

因$f(1,y)=\mathrm{e}^y$，所以$f_y(1,y)=\mathrm{e}^y$，故

$$f_y(1,0)=1.$$

法 2 $f_x(x,y)=2x\mathrm{e}^y+\arctan\dfrac{y}{x}+(x-1)\cdot\dfrac{1}{1+(\dfrac{y}{x})^2}\cdot y(-\dfrac{1}{x^2})$,

$$f_y(x,y)=x^2\mathrm{e}^y+(x-1)\cdot\frac{1}{1+(\dfrac{y}{x})^2}\cdot\frac{1}{x},$$

于是

$$f_x(1,0)=2,\ f_y(1,0)=1.$$

注意 计算偏导数$f_x(x_0,y_0)$时，可以利用本题的解法 1，将$y=y_0$先代入$f(x,y)$中，再对x求导. 显然，本题如果用解法 2，先求$f_x(x,y)$，$f_y(x,y)$，后代入$x=x_0,y=y_0$的值，则要麻烦多了.

3. 求曲线$\begin{cases} z=\dfrac{1}{4}(x^2+y^2), \\ y=2, \end{cases}$在点$M_0(2,2,2)$处的切线关于$x$轴的倾角.

解 根据偏导数的几何意义，$f_x(2,2)$就是曲线在点$M_0(2,2,2)$处的切线关于x

轴的斜率, 而

$$f_x(2,2)=\frac{1}{2}x\big|_{x=2}=1\text{, 即斜率 } k=\tan\alpha=1\text{,}$$

于是倾角 $\alpha=\frac{\pi}{4}$.

4. 设 $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0 & , x^2+y^2=0,\end{cases}$

试证函数 $f(x,y)$ 在点 $(0,0)$ 处连续且偏导数存在, 并求出 $f_x(0,0)$ 及 $f_y(0,0)$ 的值.

解 因为函数 $f(x,y)$ 在点 $(0,0)$ 的邻域内有定义, 且

$$\lim_{\substack{x\to 0\\ y\to 0}}f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}=\lim_{\rho\to 0}\rho^2\sin\frac{1}{\rho}=0=f(0,0)\text{,}$$

所以 $f(x,y)$ 在点 $(0,0)$ 处连续. 又因为

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(\Delta x)^2\sin\frac{1}{\sqrt{(\Delta x)^2}}}{\Delta x}=\lim_{\Delta x\to 0}\Delta x\sin\frac{1}{\sqrt{(\Delta x)^2}}=0\text{,}$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{(\Delta y)^2\sin\frac{1}{\sqrt{(\Delta y)^2}}}{\Delta y}=\lim_{\Delta y\to 0}\Delta y\sin\frac{1}{\sqrt{(\Delta y)^2}}=0\text{.}$$

所以 $f(x,y)$ 在点 $(0,0)$ 处偏导数存在, 且 $f_x(0,0)=0$, $f_y(0,0)=0$.

注意 如同一元函数一样, 分段函数在分界点处的偏导数应按定义来求.

5. 求下列函数的所有二阶偏导数:

(1) $z=\cos^2(ax-by)$; (2) $z=\mathrm{e}^{-\alpha x}\sin\beta y$;

(3) $z=x\mathrm{e}^{-xy}$; (4) $z=y^x$.

解 (1) $z_x=2\cos(ax-by)\cdot(-1)\sin(ax-by)\cdot a=-a\sin 2(ax-by)$,

$$z_y=2\cos(ax-by)\cdot(-1)\sin(ax-by)\cdot(-b)=b\sin 2(ax-by)\text{,}$$

$$z_{xx}=-a\cos 2(ax-by)\cdot 2a=-2a^2\cos 2(ax-by)\text{,}$$

$$z_{xy}=z_{yx}=-a\cos 2(ax-by)\cdot(-2b)=2ab\cos 2(ax-by)\text{,}$$

$$z_{yy}=b\cos 2(ax-by)\cdot(-2b)=-2b^2\cos 2(ax-by)\,.$$

(2) $z_x=\mathrm{e}^{-\alpha x}(-\alpha)\sin\beta y=-\alpha\mathrm{e}^{-\alpha x}\sin\beta y$,

$z_y=\mathrm{e}^{-\alpha x}\cos\beta y\cdot\beta=\beta\mathrm{e}^{-\alpha x}\cos\beta y$,

$z_{xx}=\alpha^2\mathrm{e}^{-\alpha x}\sin\beta y,\qquad z_{xy}=z_{yx}=-\alpha\beta\mathrm{e}^{-\alpha x}\cos\beta y$,

$z_{yy}=-\beta^2\mathrm{e}^{-\alpha x}\cos\beta y$.

(3) $z_x=\mathrm{e}^{-xy}+x\mathrm{e}^{-xy}\cdot(-y)=(1-xy)\mathrm{e}^{-xy}$, $\qquad z_y=-x^2\mathrm{e}^{-xy}$,

$z_{xx}=-y\cdot\mathrm{e}^{-xy}+(1-xy)\cdot\mathrm{e}^{-xy}\cdot(-y)=(-2y+xy^2)\mathrm{e}^{-xy}$,

$z_{xy}=z_{yx}=-2x\cdot\mathrm{e}^{-xy}-x^2\cdot\mathrm{e}^{-xy}\cdot(-y)=(-2x+x^2y)\mathrm{e}^{-xy}$,

$z_{yy}=-x^2\mathrm{e}^{-xy}\cdot(-x)=x^3\mathrm{e}^{-xy}$.

(4) $z_x=y^x\ln y$, $\qquad z_y=xy^{x-1}$, $\qquad z_{xx}=y^x\ln^2 y$,

$z_{xy}=z_{yx}=y^{x-1}+xy^{x-1}\ln y=y^{x-1}(1+x\ln y)$, $\qquad z_{yy}=x(x-1)y^{x-2}$.

6. 求下列函数的指定的高阶偏导数:

(1) $z=x\ln(xy)$, z_{xxy}, z_{xyy};

(2) $u=x^a y^b z^c$, $\dfrac{\partial^6 u}{\partial x\partial y^2\partial z^3}$;

(3) $f(x,y,z)=xy^2+yz^2+zx^2$, $f_{xz}(1,0,2)$ 及 $f_{yz}(0,-1,0)$.

解 (1) $z_x=\ln(xy)+x\cdot\dfrac{y}{xy}=\ln(xy)+1$,

$z_{xx}=\dfrac{y}{xy}=\dfrac{1}{x},\qquad z_{xxy}=0$,

$z_{xy}=\dfrac{x}{xy}=\dfrac{1}{y},\qquad z_{xyy}=-\dfrac{1}{y^2}$.

(2) $\dfrac{\partial u}{\partial x}=ax^{a-1}y^bz^c$, $\qquad \dfrac{\partial^2 u}{\partial x\partial y}=abx^{a-1}y^{b-1}z^c$,

$$\frac{\partial^3 u}{\partial x \partial y^2} = ab(b-1)x^{a-1}y^{b-2}z^c \quad , \quad \frac{\partial^4 u}{\partial x \partial y^2 \partial z} = abc(b-1)x^{a-1}y^{b-2}z^{c-1} \quad ,$$

$$\frac{\partial^5 u}{\partial x \partial y^2 \partial z^2} = abc(b-1)(c-1)x^{a-1}y^{b-2}z^{c-2} ,$$

$$\frac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = abc(b-1)(c-1)(c-2)x^{a-1}y^{b-2}z^{c-3} .$$

(3) 因为 $f_x = y^2 + 2xz$, $f_y = 2xy + z^2$, $f_{xz} = 2x$, $f_{yz} = 2z$,

所以

$$f_{xz}(1,0,2) = 2 \, , \, f_{yz}(0,-1,0) = 0 \, .$$

7. 验证函数 $z = \mathrm{e}^{-kn^2t}\sin nx$ 满足热传导方程

$$z_t = kz_{xx} \, .$$

证 因为 $z_t = -kn^2\mathrm{e}^{-kn^2t}\sin nx$, $z_x = n\mathrm{e}^{-kn^2t}\cos nx$,

$$z_{xx} = -n^2\mathrm{e}^{-kn^2t}\sin nx \, .$$

所以

$$z_t = k(-n^2\mathrm{e}^{-kn^2t}\sin nx) = kz_{xx} \, .$$

8. 验证函数 $u = \sin(x-at) + \ln(x+at)$ 满足波动方程

$$u_{tt} = a^2 u_{xx} \, .$$

证 因为 $u_t = -a\cos(x-at) + \dfrac{a}{x+at}$,

$$u_{tt} = -a \cdot (-1)\sin(x-at) \cdot (-a) + (-1)\frac{a^2}{(x+at)^2}$$

$$= -a^2 \cdot [\sin(x-at) + \frac{1}{(x+at)^2}] ,$$

$$u_x = \cos(x-at) + \frac{1}{x+at} , \quad u_{xx} = -\sin(x-at) - \frac{1}{(x+at)^2} \, .$$

所以

$$u_{tt} = a^2[-\sin(x-at) - \frac{1}{(x+at)^2}] = a^2 u_{xx} .$$

9. 验证函数 $u = \arctan\frac{x}{y}$ 满足拉普拉斯方程

$$u_{xx} + u_{yy} = 0 .$$

证 因为 $$u_x = \frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y} = \frac{y}{x^2+y^2} ,$$

$$u_{xx} = y\cdot(-1)\frac{1}{(x^2+y^2)^2}\cdot(2x) = -\frac{2xy}{(x^2+y^2)^2} ,$$

$$u_y = \frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2}) = -\frac{x}{x^2+y^2} ,$$

$$u_{yy} = x\cdot\frac{1}{(x^2+y^2)^2}\cdot 2y = \frac{2xy}{(x^2+y^2)^2} .$$

所以

$$u_{xx} + u_{yy} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0 .$$