*第八节　二元函数的泰勒公式

*习题　8-8

1.　将函数 $f(x,y)=2x^2-xy-y^2-6x-3y+5$ 在点 $(1,-2)$ 展成泰勒公式.

解　$f(1,-2)=5$，　$f_x(1,-2)=(4x-y-6)\big|_{(1,-2)}=0$，

$f_y(1,-2)=(-x-2y-3)\big|_{(1,-2)}=0$，

$f_{xx}(1,-2)=4$，　$f_{xy}(1,-2)=-1$，　$f_{yy}(1,-2)=-2$，

函数为 2 次多项式, 3 阶及 3 阶以上的各偏导数均为零.

又因为 $h=x-1,\ k=y+2$，

将以上各项代入泰勒公式，得

$$f(x,y)=f(1,-2)+(x-1)f_x(1,-2)+(y+2)f_y(1,-2)$$

$$+\frac{1}{2!}[(x-1)^2f_{xx}(1,-2)+2(x-1)(y+2)f_{xy}(1,-2)+(y+2)^2f_{yy}(1,-2)]$$

$$=5+\frac{1}{2}[4(x-1)^2-2(x-1)(y+2)-2(y+2)^2]$$

$$=5+2(x-1)^2-(x-1)(y+2)-(y+2)^2.$$

2.　将函数 $f(x,y)=\mathrm{e}^{x+y}$ 在点 $(1,-1)$ 展成泰勒公式.

解　$f(1,-1)=1,\ f_x(1,-1)=\mathrm{e}^{x+y}\big|_{(1,-1)}=1,\ f_y(1,-1)=\mathrm{e}^{x+y}\big|_{(1,-1)}=1,$

$\cdots,\ f^{(n)}_{x^m y^{n-m}}\ (1,-1)=\mathrm{e}^{x+y}\big|_{(1,-1)}=1\quad(m=0,1,\cdots,n),\cdots$

又因为 $h=x-1,\ k=y+1$,

将以上各项代入泰勒公式，得

$$\mathrm{e}^{x+y}=1+(x-1)\cdot1+(y+1)\cdot1+\frac{1}{2!}[(x-1)^2\cdot1+2(x-1)(y+1)\cdot1+(y+1)^2\cdot1]$$

$$+\frac{1}{3!}[(x-1)^3\cdot1+3(x-1)^2(y+1)\cdot1+3(x-1)(y+1)^2\cdot1+(y+1)^3\cdot1]$$

$$+\ldots+\frac{1}{n!}[(x-1)+(y+1)]^n+\cdots$$

$$=\sum_{k=0}^{n}\frac{[(x-1)+(y+1)]^k}{k!}=\sum_{k=0}^{\infty}[\sum_{m=0}^{k}\frac{1}{k!}\frac{k!}{m!(k-m)!}(x-1)^m(y+1)^{k-m}]$$

$$=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{m!(k-m)!}(x-1)^m(y+1)^{k-m}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(x-1)^m(y+1)^n}{m!n!}.$$

3. 求函数 $f(x,y)=\mathrm{e}^x\ln(1+y)$ 的三阶麦克劳林公式.

解 $f_x(x,y)=\mathrm{e}^x\ln(1+y)$， $f_y(x,y)=\dfrac{\mathrm{e}^x}{1+y}$，

$$f_{xx}(x,y)=\mathrm{e}^x\ln(1+y),\quad f_{xy}(x,y)=\frac{\mathrm{e}^x}{1+y},\quad f_{yy}(x,y)=-\frac{\mathrm{e}^x}{(1+y)^2},$$

$$f_{xxx}(x,y)=\mathrm{e}^x\ln(1+y),\quad f_{yyy}(x,y)=\frac{2\mathrm{e}^x}{(1+y)^3},$$

于是

$$(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(0,0)=hf_x(0,0)+kf_y(0,0)=k,$$

$$(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2f(0,0)=h^2f_{xx}f(0,0)+2hkf_{xy}(0,0)+k^2f_{yy}(0,0)$$

$$=2hk-k^2,$$

$$(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^3f(0,0)=h^3f_{xxx}(0,0)+3h^2kf_{xxy}(0,0)+3hk^2f_{xyy}(0,0)+k^3f_{yyy}(0,0)$$

$$=3h^2k-3hk^2+2k^3.$$

又因为

$$f(0,0)=0,h=x,k=y,$$

将以上各项代入三阶麦克劳林公式，得

$$\mathrm{e}^x\ln(1+y)=y+\frac{1}{2!}(2xy-y^2)+\frac{1}{3!}(3x^2y-3xy^2+2y^3)+R_3,$$

其中 $R_3=\dfrac{1}{4!}[(h\dfrac{\partial}{\partial x}+k\dfrac{\partial}{\partial y})^4f(\theta h,\theta k)]_{h=x,k=y}$

$$=\frac{\mathrm{e}^{\theta x}}{24}[x^4\ln(1+\theta y)+\frac{4x^3y}{1+\theta y}-\frac{6x^2y^2}{(1+\theta y)^2}+\frac{8xy^3}{(1+\theta y)^3}-\frac{6y^4}{(1+\theta y)^4}],\quad(0<\theta<1).$$