第三节

定积分的换元积分法与分部积分法

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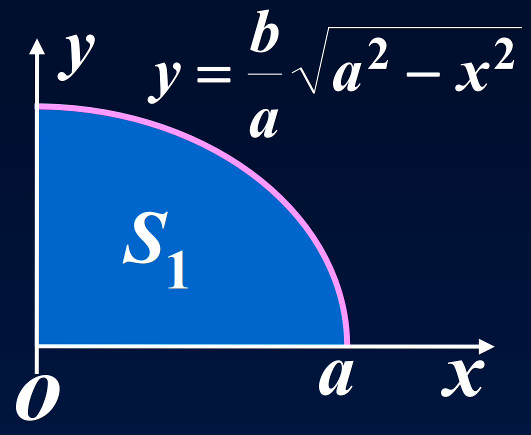
一、主要内容

(一) 定积分的换元积分法



引例 求椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 围成平面图形的面积 S.

解(方法1) $S=4S_1$

$$=4\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,\mathrm{d}x$$

令$x=a\sin t$ $t\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$=4\frac{b}{a}\,\frac{a^2}{2}\left[\arcsin\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right]_0^a$$

$$=\pi ab$$ 运算繁!

(方法2)　$S = 4S_1 = 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x$

令 $x = a\sin t$

$= 4\frac{b}{a}\int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t$　(形式上)

$= 4ab\left[\frac{1}{2}(t + \frac{\sin 2t}{2})\right]_0^{\frac{\pi}{2}}$

$= 2ab\,[\frac{\pi}{2} - 0]$

$= \pi ab$　这是巧合吗?

问题　定积分是否有换元积分法?　有.

定理5.4 设 $f(x)\in C[a,b]$, 单值函数 $x=\varphi(t)$ 满足:

1) $\varphi(\alpha)=a,\ \varphi(\beta)=b;$

2) 在 $[\alpha,\beta]$ 或 $[\beta,\alpha]$ 上 $\varphi(t)$ 具有连续导数

且 $a\le\varphi(t)\le b,$ 则

$$\boxed{\int_a^b f(x)\,\mathrm{d}x=\int_\alpha^\beta f\,[\varphi(t)]\,\varphi'(t)\,\mathrm{d}\,t}$$

—— 换元公式

注 1° α 不一定小于 β, 当 $\alpha > \beta$ 时,换元公式仍成立；

2° 换元要换限， 下限 $a \leftrightarrow$ 下限 α,

上限 $b \leftrightarrow$ 上限 β;

3° 换元公式双向使用：

令 $x = \varphi(t)$

$$\int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)\,\mathrm{d}t = \int_{a}^{b} f(x)\,\mathrm{d}x$$

配元不换限

或配元 $\int_{\alpha}^{\beta} f[\varphi(t)]\underline{\varphi'(t)\,\mathrm{d}t} = \int_{\alpha}^{\beta} f[\varphi(t)]\,\mathrm{d}\varphi(t)$

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4° 定积分与不定积分换元法的异同

(1) 相同处

换元目的: 改变被积函数，以简化计算.

(2) 不同处

换元要换限，变量不代回 .

(二) 定积分的分部积分法

定理5.5 设 $u(x), v(x)$ 在 $[a,b]$ 上导数连续，则

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,\mathrm{d}x$$

即 $$\boxed{\int_a^b u\,\mathrm{d}v = uv\Big|_a^b - \int_a^b v\,\mathrm{d}u}$$

—— 分部积分公式

二、典型例题

例1 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x \mathrm{d}x.$

解(方法1) 令 $t = \cos x$, $\mathrm{d}t = -\sin x \mathrm{d}x$,

$$x = \frac{\pi}{2} \quad : \; t = 0,$$

$$x = 0 \quad : \; t = 1,$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x \mathrm{d}x = -\int_1^0 t^5 \mathrm{d}t = \left.\frac{t^6}{6}\right|_0^1 = \frac{1}{6}.$$

换元要换限！

(方法2) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x \, \mathrm{d}x$

$$= -\int_0^{\frac{\pi}{2}} \cos^5 x \, \mathrm{d}(\cos x)$$

$$= -\frac{1}{6}\cos^6 x \Big|_0^{\frac{\pi}{2}} = \frac{1}{6}.$$

注　不明显写出新变量t, 积分限不动!

例2 求 $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,\mathrm{d}x.$

解 $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,\mathrm{d}x \xlongequal[\substack{x=0:\ t=0 \\ x=\frac{1}{2}:\ t=\frac{\pi}{6}}]{\text{令}x=\sin t} \int_0^{\frac{\pi}{6}} \cos t \cdot \cos t\,\mathrm{d}t$

$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1+\cos 2t)\,\mathrm{d}t$

$= \frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12}+\frac{\sqrt{3}}{8}.$

例3 计算$\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,\mathrm{d}x.$

解 $\because f(x)=\sqrt{\sin^3 x-\sin^5 x}=|\cos x|(\sin x)^{\frac{3}{2}}$

$$\therefore \int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,\mathrm{d}x=\int_0^{\pi}|\cos x|(\sin x)^{\frac{3}{2}}\,\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\cos x\,(\sin x)^{\frac{3}{2}}\,\mathrm{d}x-\int_{\frac{\pi}{2}}^{\pi}\cos x(\sin x)^{\frac{3}{2}}\,\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}\,\mathrm{d}\sin x-\int_{\frac{\pi}{2}}^{\pi}(\sin x)^{\frac{3}{2}}\,\mathrm{d}\sin x$$

$$=\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}-\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{4}{5}.$$

例4 设$f(x)$在$[-a,a]$上连续，则

(1) $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} [f(x) + f(-x)]\,\mathrm{d}x$

(2) $\int_{-a}^{a} f(x)\,\mathrm{d}x = \begin{cases} 0, & \text{当} f(x)\text{为奇函数时} \\ 2\int_{0}^{a} f(x)\,\mathrm{d}x, & \text{当} f(x)\text{为偶函数时} \end{cases}$

证 (1) $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x,$

在$\int_{-a}^{0} f(x)\,\mathrm{d}x$中，令$x = -t,$

$$\int_{-a}^{0} f(x)\,\mathrm{d}x \xlongequal{x=-t} \int_{a}^{0} f(-t)(-\mathrm{d}t) = \int_{0}^{a} f(-x)\,\mathrm{d}x$$

$$\therefore \quad \int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} [f(x) + f(-x)]\,\mathrm{d}x$$

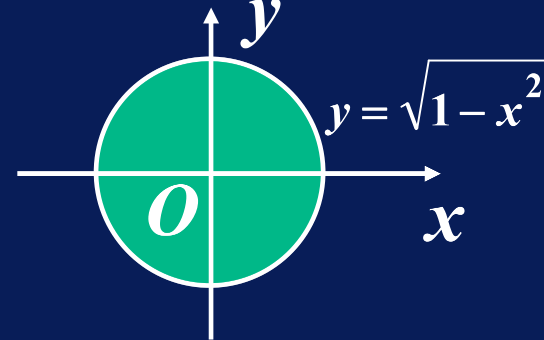
(2) 若 $f(x)$ 为偶函数，则 $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} [f(x) + f(x)]\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$$

若 $f(x)$ 为奇函数，则 $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} [f(x) - f(x)]\,\mathrm{d}x = 0.$$

例5 计算下列定积分:



(1) $\displaystyle\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}\,\mathrm{d}x.$

解
$$原式=\int_{-1}^{1}\underbrace{\boxed{\frac{2x^2}{1+\sqrt{1-x^2}}}}_{偶函数}\mathrm{d}x+\int_{-1}^{1}\underbrace{\boxed{\frac{x\cos x}{1+\sqrt{1-x^2}}}}_{奇函数}\mathrm{d}x$$

$$=4\int_0^1\frac{x^2}{1+\sqrt{1-x^2}}\mathrm{d}x+0=4\int_0^1\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}\mathrm{d}x$$

$$=4\int_0^1(1-\sqrt{1-x^2})\mathrm{d}x=4-\underbrace{\boxed{4\int_0^1\sqrt{1-x^2}\,\mathrm{d}x}}_{单位圆的面积}=4-\pi.$$

(2) $\int_0^2 x\sqrt{2x-x^2}\,\mathrm{d}x$

解 原式 $=\int_0^2 x\sqrt{1-(x-1)^2}\,\mathrm{d}x$

$$\underline{\underline{t=x-1}}\ \int_{-1}^{1}(t+1)\sqrt{1-t^2}\,\mathrm{d}t$$

$$=\int_{-1}^{1}\underbrace{\boxed{t\sqrt{1-t^2}}}_{\text{奇函数}}\,\mathrm{d}t+\int_{-1}^{1}\underbrace{\boxed{\sqrt{1-t^2}}}_{\text{偶函数}}\,\mathrm{d}t$$

$$=0+\frac{\pi\cdot 1^2}{2}=\frac{\pi}{2}.$$

(3) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 x}{1+\mathrm{e}^{-x}} \mathrm{d}x$

解 原式 $=\int_{0}^{\frac{\pi}{4}}[f(x)+f(-x)]\mathrm{d}x$

$$=\int_{0}^{\frac{\pi}{4}}\left[\frac{\cos^2 x}{1+\mathrm{e}^{-x}}+\frac{\cos^2(-x)}{1+\mathrm{e}^{x}}\right]\mathrm{d}x$$

$$=\int_{0}^{\frac{\pi}{4}}(\cos^2 x)\left[\frac{\mathrm{e}^{x}}{1+\mathrm{e}^{x}}+\frac{1}{1+\mathrm{e}^{x}}\right]\mathrm{d}x$$

$$=\int_{0}^{\frac{\pi}{4}}\cos^2 x\,\mathrm{d}x=\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)\Big|_{0}^{\frac{\pi}{4}}=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right).$$

例6 若$f(x)$在$[0, 1]$上连续，证明:

(1) $\int_0^{\pi} f(\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x$;

(2) $\int_0^{\pi} x f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$;

(3) $\int_0^{\pi} f(\sin x)\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$.

$x = 0 \;:\; t = \frac{\pi}{2}$,

$x = \frac{\pi}{2} \;:\; t = 0$.

证 (1) 设 $x = \frac{\pi}{2} - t$, $\mathrm{d}x = -\mathrm{d}t$,

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2} - t\right)\right]\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x.$$

$$(2)\ \int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x.$$

证(2) 设 $x = \pi - t$, $\mathrm{d}x = -\mathrm{d}t$,

$x = 0$: $t = \pi$,

$x = \pi$: $t = 0$,

$$\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = -\int_{\pi}^{0} (\pi - t) f[\sin(\pi - t)]\,\mathrm{d}t$$

$$= \int_0^{\pi} (\pi - t) f(\sin t)\,\mathrm{d}t = \pi\int_0^{\pi} f(\sin t)\,\mathrm{d}t - \int_0^{\pi} xf(\sin x)\,\mathrm{d}x,$$

$$\therefore \int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x.$$

$$(3)\ \int_0^{\pi} f(\sin x)\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$$

证(3) $\int_0^{\pi} f(\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,\mathrm{d}x$

$$\because\ \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,\mathrm{d}x \xlongequal[t=\pi-x]{x=\pi-t} \int_{\frac{\pi}{2}}^{0} f[\sin(\pi-t)](-\mathrm{d}t)$$

$$= \int_0^{\frac{\pi}{2}} f(\sin t)\,\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$$

$$\therefore\ \int_0^{\pi} f(\sin x)\,\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$$

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例7 计算 $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x.$

解(方法1) $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x=\int_0^{\pi}x\cdot\frac{\sin x}{2-\sin^2 x}\mathrm{d}x$

$$=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x=-\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos^2 x}\mathrm{d}(\cos x)$$

$$=-\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$=-\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)=\frac{\pi^2}{4}.$$

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(方法2) $$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x \xlongequal[t=x-\frac{\pi}{2}]{x=\frac{\pi}{2}+t} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\frac{\pi}{2}+t)\cos t}{1+\sin^2 t}\,\mathrm{d}t$$

$$=\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\boxed{\frac{\cos t}{1+\sin^2 t}}}_{\text{偶函数}}\mathrm{d}t+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\boxed{\frac{t\cos t}{1+\sin^2 t}}}_{\text{奇函数}}\mathrm{d}t$$

$$=\frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{\cos t}{1+\sin^2 t}\,\mathrm{d}t$$

$$=\pi\arctan(\sin x)\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2}{4}.$$

例8 设 $f(x)$是以 T为周期的连续函数，则

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$$

其中 a为任意实数.

证(方法1) $\int_a^{a+T} f(x)\,\mathrm{d}x$

$$= \int_a^0 f(x)\,\mathrm{d}x + \int_0^T f(x)\,\mathrm{d}x + \int_T^{a+T} f(x)\,\mathrm{d}x$$

$$\because \int_T^{a+T} f(x)\,\mathrm{d}x \;\underline{\underline{x=T+t}}\; \int_0^a f(T+t)\,\mathrm{d}t = -\int_a^0 f(x)\,\mathrm{d}x$$

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$$\int_a^{a+T} f(x)\,\mathrm{d}x$$

$$=\cancel{\int_a^0 f(x)\,\mathrm{d}x}+\int_0^T f(x)\,\mathrm{d}x-\cancel{\int_a^0 f(x)\,\mathrm{d}x}$$

$$\therefore \quad \int_a^{a+T} f(x)\,\mathrm{d}x=\int_0^T f(x)\,\mathrm{d}x$$

(方法2) 令 $F(a)=\int_a^{a+T} f(x)\,\mathrm{d}x$

$\because f(x)$连续， $\therefore F(a)$可导

$$F'(a)=f(a+T)(a+T)'-f(a)\cdot 1$$

$$=f(a)-f(a)=0$$

$\therefore F(a)\equiv C$(常数)， 令$a=0$,

由$F(0)=\int_0^T f(x)\,\mathrm{d}x$, 得 $C=\int_0^T f(x)\,\mathrm{d}x$

命题得证.

例9 计算 $\int_0^{100\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x.$

解 $f(x)=\sqrt{1-\cos 2x}$

$$f(x+\pi)=f(x),\quad T=\pi$$

$$\int_0^{100\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x=\int_0^{\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x+\int_{\pi}^{2\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x$$

$$+\cdots+\int_{99\pi}^{100\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x$$

$$=100\int_0^{\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x=100\int_0^{\pi}\sqrt{2}\sin x\,\mathrm{d}x=200\sqrt{2}.$$

例10 设 $f(x)=\begin{cases}\dfrac{1}{1+x},\text{当}x\geq 0\text{时},\\ \dfrac{1}{1+\mathrm{e}^x},\text{当}x<0\text{时},\end{cases}$ 求 $\int_0^2 f(x-1)\,\mathrm{d}x$.

解 $\int_0^2 f(x-1)\,\mathrm{d}x \overset{t=x-1}{=\!=\!=} \int_{-1}^1 f(t)\,\mathrm{d}t$

$$=\int_{-1}^0 f(t)\,\mathrm{d}t+\int_0^1 f(t)\,\mathrm{d}t=\int_{-1}^0 \frac{1}{1+\mathrm{e}^t}\,\mathrm{d}t+\int_0^1 \frac{1}{1+t}\,\mathrm{d}t$$

$$=\int_{-1}^0 \left(1-\frac{\mathrm{e}^t}{1+\mathrm{e}^t}\right)\mathrm{d}t+\ln(1+t)\Big|_0^1$$

$$=1-\ln(1+\mathrm{e}^t)\Big|_{-1}^0+\ln 2=\ln(1+\mathrm{e}).$$

例11 已知$f(x)$连续，$\int_0^x tf(x-t)\,\mathrm{d}t = 1-\cos x$，求$\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x$的值．

解 令$u=x-t$，则有$t=x-u,\ \mathrm{d}t=-\mathrm{d}u$，

且当$t=0$时，$u=x$; 当$t=x$时，$u=0$. 从而

$$\int_0^x t\,f(x-t)\,\mathrm{d}t = \int_x^0 (x-u)f(u)(-\mathrm{d}u)$$

$$= \int_0^x (x-u)f(u)\,\mathrm{d}u$$

$$= x\int_0^x f(u)\,\mathrm{d}u - \int_0^x uf(u)\,\mathrm{d}u,$$

于是有 $x\int_0^x f(u)\,\mathrm{d}u - \int_0^x uf(u)\,\mathrm{d}u = 1-\cos x,$

两边对 x 求导，得

$$\int_0^x f(u)\,\mathrm{d}u + \cancel{xf(x)} - \cancel{xf(x)} = \sin x,$$

即 $$\int_0^x f(u)\,\mathrm{d}u = \sin x;$$

在上式中令 $x=\frac{\pi}{2}$，得 $\int_0^{\frac{\pi}{2}} f(u)\,\mathrm{d}u = 1,$

即 $$\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x = 1.$$

例12 计算 $\int_0^{\frac{\pi}{4}} \frac{x\,\mathrm{d}x}{1+\cos 2x}$.

解 $\because 1+\cos 2x = 2\cos^2 x,$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{x}{1+\cos 2x}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{x}{2\cos^2 x}\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{4}} x\cdot\sec^2 x\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} x\,\mathrm{d}(\tan x) = \frac{1}{2}\left\{[x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,\mathrm{d}x\right\}$$

$$= \frac{\pi}{8} - \frac{1}{2}[\ln\sec x]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{\ln 2}{4}.$$

例13 计算 $\displaystyle\int_0^1 \frac{\ln(1+x)}{(2+x)^2}\mathrm{d}x.$

解 $$\int_0^1 \frac{\ln(1+x)}{(2+x)^2}\mathrm{d}x = -\int_0^1 \ln(1+x)\mathrm{d}\left(\frac{1}{2+x}\right)$$

$$= -\left[\frac{\ln(1+x)}{2+x}\right]_0^1 + \int_0^1 \frac{1}{2+x}\mathrm{d}\ln(1+x)$$

$$= -\frac{\ln 2}{3} + \int_0^1 \boxed{\frac{1}{2+x}\cdot\frac{1}{1+x}}\mathrm{d}x \longrightarrow \left(\frac{1}{1+x} - \frac{1}{2+x}\right)$$

$$= -\frac{\ln 2}{3} + [\ln(1+x) - \ln(2+x)]_0^1$$

$$= \frac{5}{3}\ln 2 - \ln 3.$$

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例14 证明定积分公式(*Wallis* 公式)

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x$$

$$= \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & n \text{ 为正偶数} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \cdots \dfrac{4}{5} \cdot \dfrac{2}{3}, & n \text{ 为大于1的正奇数} \end{cases}$$

证 当$n = 0$时， $I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \dfrac{\pi}{2}$

当 $n=1$ 时，$I_1 = \int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}x = -\cos x \Big|_0^{\frac{\pi}{2}} = 1$

当 $n \geq 2$ 时，$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \underbrace{\sin^{n-1} x}_{u} \cdot \underbrace{\sin x \,\mathrm{d}x}_{\mathrm{d}v}$

$$= -\int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \mathrm{d}(\cos x)$$

$$= -[\underbrace{\sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}}}_{0} - \int_0^{\frac{\pi}{2}} \cos x \cdot \mathrm{d}(\sin^{n-1} x)]$$

$$= \int_0^{\frac{\pi}{2}} \cos x \cdot (n-1)\sin^{n-2} x \cdot \cos x \,\mathrm{d}x$$

$$I_n = \int_0^{\frac{\pi}{2}} \cos x \cdot (n-1)\sin^{n-2} x \cdot \cos x \, \mathrm{d}x$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} \, \mathrm{d}x$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x) \, \mathrm{d}x$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, \mathrm{d}x - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore \boxed{I_n = \frac{n-1}{n} I_{n-2}, \quad n \geq 2.}$$ 积分I_n关于下标的递推公式

$$\boxed{I_n = \frac{n-1}{n} I_{n-2} \quad (n \ge 2)}$$

$$\therefore \quad I_n = \frac{n-}{n} \cdot \frac{n-3}{n-2} I_{n-4}$$

$\cdots\cdots$，直到下标减到0或1为止

$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \frac{2m-5}{2m-4} \cdots\cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$= \frac{(2m-1)!!}{(2m)!!} I_0 = \frac{(2m-1)!!}{(2m)!!} \cdot \frac{\pi}{2}$$

$$\boxed{I_0 = \frac{\pi}{2}}$$

$(m = 1, 2, \cdots)$

$$\boxed{I_n = \frac{n-1}{n} I_{n-2} \quad (n \ge 2)}$$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} I_2 \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

$$= \frac{(2m)!!}{(2m+1)!!} I_1 = \frac{(2m)!!}{(2m+1)!!} \cdot 1 \quad (m = 1, 2, \cdots)$$

综上所述: $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}\, x = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}\, x$

$$= \begin{cases} \dfrac{(n-1)!!}{n!!} \cdot \dfrac{\pi}{2}, & n\text{为正偶数;} \\[2ex] \dfrac{(n-1)!!}{n!!} \cdot 1, & n\text{为大于1的正奇数.} \end{cases}$$

例15 求 $I_n=\int_0^{\pi} x\sin^n x\,\mathrm{d}x \quad (n\ge 2)$

解 $I_n=\int_0^{\pi} x\sin^n x\,\mathrm{d}x \ (n\ge 2)$

$$=\frac{\pi}{2}\int_0^{\pi}\sin^n x\,\mathrm{d}x$$

$$\boxed{\int_0^{\pi} x f(\sin x)\,\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x}$$

$$=\frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x$$

$$\boxed{\int_0^{\pi} f(\sin x)\,\mathrm{d}x=2\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x}$$

$$=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi^2}{2}, & n\text{为偶数 } (n\ge 2);\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}\cdot\pi, & n\text{为奇数 } (n\ge 3).\end{cases}$$

例16 计算 $\int_0^3 \arcsin\sqrt{\dfrac{x}{x+1}}\,\mathrm{d}x.$

解 令 $t=\arcsin\sqrt{\dfrac{x}{x+1}}$, 则 $\sin^2 t=\dfrac{x}{x+1}$

$$\cos^2 t=1-\sin^2 t=1-\frac{x}{x+1}=\frac{1}{x+1},\ \tan^2 t=x$$

$$\therefore \int_0^3 \arcsin\sqrt{\frac{x}{x+1}}\,\mathrm{d}x=\int_0^{\frac{\pi}{3}} t\,\mathrm{d}(\tan^2 t)$$

$$=t\tan^2 t\Big|_0^{\frac{\pi}{3}}-\int_0^{\frac{\pi}{3}}\tan^2 t\,\mathrm{d}t=\frac{\pi}{3}\cdot 3-\int_0^{\frac{\pi}{3}}(\sec^2 t-1)\,\mathrm{d}t$$

$$=\pi-\tan t\Big|_0^{\frac{\pi}{3}}+\frac{\pi}{3}=\frac{4\pi}{3}-\sqrt{3}.$$

例17 若$f''(x)$在$[a,b]$上连续，证明:

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}[f(a)+f(b)] + \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x$$

证 $\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x = \int_a^b (x-a)(x-b)\,\mathrm{d}f'(x)$

$$= \underbrace{(x-a)(x-b)f'(x)\Big|_a^b}_{0} - \int_a^b (2x-a-b)f'(x)\,\mathrm{d}x$$

$$= -\int_a^b (2x-a-b)\,\mathrm{d}f(x)$$

$$\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x = -\int_a^b (2x-a-b)\,\mathrm{d}\,f(x)$$

$$= -\left[(2x-a-b)f(x)\Big|_a^b - \int_a^b 2f(x)\,\mathrm{d}x\right]$$

$$= -\left[(b-a)f(b)-(a-b)f(a)-2\int_a^b f(x)\,\mathrm{d}x\right]$$

$$= -(b-a)[f(b)+f(a)]+2\int_a^b f(x)\,\mathrm{d}x$$

$$\therefore \quad \int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}[f(a)+f(b)]$$

$$+\frac{1}{2}\int_a^b f''(x)(x-a)(x-b)\,\mathrm{d}x$$

例18 设 $f'(\mathrm{e}^x)=1+x, f(1)=0$, 求 $f(x)$及 $\int_2^3 f(x-1)\,\mathrm{d}x$.

解 令 $u=\mathrm{e}^x$, 即 $x=\ln u$

则 $f'(u)=1+\ln u$

$\therefore$ $f'(x)=1+\ln x$

$$f(x)-f(1)=\int_1^x f'(t)\,\mathrm{d}t=\int_1^x (1+\ln t)\,\mathrm{d}t$$

$\because$ $f(1)=0$, $\therefore f(x)=\int_1^x (1+\ln t)\,\mathrm{d}t$

$$\therefore f(x)=\int_1^x (1+\ln t)\,\mathrm{d}t$$

$$=t(1+\ln t)\Big|_1^x-\int_1^x t\cdot\frac{1}{t}\,\mathrm{d}t$$

$$=[x(1+\ln x)-1]-(x-1)=x\ln x$$

$$\int_2^3 f(x-1)\,\mathrm{d}x \underline{\underline{u=x-1}} \int_1^2 f(u)\,\mathrm{d}u=\int_1^2 u\ln u\,\mathrm{d}u$$

$$=\frac{1}{2}\int_1^2 \ln u\,\mathrm{d}u^2=\frac{1}{2}\left(u^2\ln u\Big|_1^2-\int_1^2 u\,\mathrm{d}u\right)$$

$$=\frac{1}{2}\left(4\ln 2-\frac{1}{2}u^2\Big|_1^2\right)=2\ln2-\frac{3}{4}.$$

例19 设 $f(x)=\int_1^{x^2}\frac{\sin t}{t}\,\mathrm{d}t$, 求 $\int_0^1 xf(x)\,\mathrm{d}x$.

分析 因为 $\frac{\sin t}{t}$ 没有初等函数形式的原函数，

无法直接求出 $f(x)$，所以采用分部积分法.

解

$$\int_0^1 xf(x)\,\mathrm{d}x=\frac{1}{2}\int_0^1 f(x)\,\mathrm{d}(x^2)$$

$$=\frac{1}{2}\left[x^2 f(x)\right]_0^1-\frac{1}{2}\int_0^1 x^2\,\mathrm{d}f(x)$$

$$=\frac{1}{2}f(1)-\frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x$$

$$\because\ f(x)=\int_1^{x^2}\frac{\sin t}{t}\,\mathrm{d}t,\qquad f(1)=\int_1^{1}\frac{\sin t}{t}\,\mathrm{d}t=0,$$

$$f'(x)=\frac{\sin x^2}{x^2}\cdot 2x=\frac{2\sin x^2}{x},$$

$$\therefore\int_0^1 xf(x)\,\mathrm{d}x=\frac{1}{2}f(1)-\frac{1}{2}\int_0^1 x^2 f'(x)\,\mathrm{d}x$$

$$=-\frac{1}{2}\int_0^1 2x\sin x^2\,\mathrm{d}x=-\frac{1}{2}\int_0^1\sin x^2\,\mathrm{d}x^2$$

$$=\frac{1}{2}\left[\cos x^2\right]_0^1=\frac{1}{2}(\cos 1-1).$$

例20(综合题) 设 $F(x)=\int_0^x (x^2-t^2)f(t)\,\mathrm{d}t$, $f(x)$可导,

且$f(0)=0$. 证明:

(1) 若$f(x)$为偶函数，则 $F(x)$为奇函数;

(2) 若$f(x)>0(x>0)$,则$F(x)$在$[0,+\infty)$上单调增加;

(3) 当$x\to 0$时， $F'(x)$与x^3是等价无穷小，求 $f'(0)$.

证 (1) $f(-x)=f(x)$

$$\because\ F(-x)=\int_0^{-x}[(-x)^2-t^2]f(t)\,\mathrm{d}t$$

$$=\int_0^{-x}(x^2-t^2)f(t)\,\mathrm{d}t$$

$$F(x)=\int_0^{-x}(x^2-t^2)f(t)\,\mathrm{d}t$$

$$\overset{令u=-t}{=\!=\!=}\int_0^{x}[x^2-(-u)^2]f(-u)(-\mathrm{d}u)$$

$$=-\int_0^{x}(x^2-u)^2 f(u)\,\mathrm{d}u=-F(x)$$

$\therefore\quad F(x)$是奇函数.

(2) $\quad F(x)=\int_0^{x}(x^2-t^2)f(t)\,\mathrm{d}t$

$$=x^2\cdot\int_0^{x}f(t)\,\mathrm{d}t-\int_0^{x}t^2f(t)\,\mathrm{d}t$$

$$F'(x)=2x\int_0^{x}f(t)\,\mathrm{d}t+\cancel{x^2f(x)}-\cancel{x^2f(x)}$$

$$F'(x) = 2x\int_0^x f(t)\,\mathrm{d}t$$

$\because \quad f(x) > 0\ (x > 0)$

$\therefore$ 当 $x > 0$ 时，$\int_0^x f(t)\,\mathrm{d}t > 0$

从而当 $x > 0$ 时，$F'(x) > 0$

又 $\because F(x)$ 在 $[0, +\infty)$ 上连续

$\therefore F(x)$ 在 $[0, +\infty)$ 上单调增加.

$$(3)\quad 1=\lim_{x\to 0}\frac{F'(x)}{x^3}=\lim_{x\to 0}\frac{2x\int_0^x f(t)\,\mathrm{d}t}{x^3}$$

$$=2\lim_{x\to 0}\frac{\int_0^x f(t)\,\mathrm{d}t}{x^2}\qquad \left(\frac{0}{0}\right)$$

$$=2\lim_{x\to 0}\frac{f(x)}{2x}=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\qquad (f(0)=0)$$

$$=f'(0)$$

$$\therefore\quad f'(0)=1.$$

三、同步练习

1. 计算$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\mathrm{d}x$.

2. 设$f(x)$在$[a,b]$上连续，证明:

$$\int_0^{\frac{\pi}{2}} f(|\cos x|)\mathrm{d}x = \frac{1}{4}\int_0^{2\pi} f(|\cos x|)\mathrm{d}x.$$

3. 计算 $I=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\mathrm{d}x$.

4. 设 $f(x)=\begin{cases} 1+x^2, x<0, \\ \mathrm{e}^{-x}, x\geq 0. \end{cases}$ 求$\int_1^3 f(x-2)\mathrm{d}x$.

5. 设 $f(x)$可导，且 $f(0)=0$,

$$F(x)=\int_0^x t^{n-1} f(x^n-t^n)\,\mathrm{d}t \quad (n\in N),$$

求 $\lim\limits_{x\to 0}\dfrac{F(x)}{x^{2n}}$.

6. 计算 $\int_1^4 \dfrac{\ln x}{\sqrt{x}}\,\mathrm{d}x$.

7. 计算 $\int_0^{\sqrt{3}} \arctan x\,\mathrm{d}x$.

8. 计算 $\int_0^1 (1-x^2)^2\sqrt{1-x^2}\,\mathrm{d}x$.

9. 计算$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^3+\sin^2 x)\cos^2 x\,\mathrm{d}x$.

10. 计算 $\int_{100-\frac{\pi}{2}}^{100+\frac{\pi}{2}} \tan^2 x\sin^2 2x\,\mathrm{d}x$.

11. 计算$\int_0^1(\arcsin x)^2\,\mathrm{d}x$.

12. 设 $f''(x)$在$[0,1]$连续, 且$f(0)=1, f(2)=3,$ $f'(2)=5,$ 求$\int_0^1 x\,f''(2x)\,\mathrm{d}x$.

13. 设$f(x)=\int_0^{\sqrt{x}}\mathrm{e}^{-t^2}\,\mathrm{d}t,\quad f(1)=0,$ 求$\int_0^1\frac{1}{\sqrt{x}}f(x)\,\mathrm{d}x$.

四、同步练习解答

1\. 计算$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\mathrm{d}x$.

解 令$\sqrt{2x+1}=t$,则$x=\frac{t^2-1}{2}$,$\mathrm{d}x=t\,\mathrm{d}t$,且当$x=0$时,$t=1$;当$x=4$时,于是

$$\int_0^4 \frac{x+2}{\sqrt{2x+1}}\mathrm{d}x=\int_1^3 \frac{\frac{t^2-1}{2}+2}{t}\cdot t\,\mathrm{d}t$$

$$=\frac{1}{2}\int_1^3 \left(t^2+3\right)\mathrm{d}t=\frac{1}{2}\left(\frac{t^3}{3}\Big|_1^3+6\right)=\frac{22}{3}.$$

2. 设 $f(x)$ 在 $[\,a\,,b\,]$ 上连续，证明:

$$\int_0^{\frac{\pi}{2}} f(|\cos x|)\,\mathrm{d}\,x=\frac{1}{4}\int_0^{2\pi} f(|\cos x|)\,\mathrm{d}\,x.$$

证 $\int_0^{2\pi} f(|\cos x|)\,\mathrm{d}\,x$

偶函数

$$\overset{x=\pi-t}{\underset{t=\pi-x}{=\!=\!=}}\int_{\pi}^{-\pi} f[|\cos(\pi-t)|](-\mathrm{d}\,t)=\int_{-\pi}^{\pi}\boxed{f(|\cos t|)}\,\mathrm{d}\,t$$

$$=2\boxed{\int_0^{\pi} f(|\cos t|)\,\mathrm{d}\,t}\ \overset{t=\frac{\pi}{2}-u}{\underset{u=\frac{\pi}{2}-t}{=\!=\!=\!=}}\ 2\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} f\left[\left|\cos\left(\frac{\pi}{2}-u\right)\right|\right](-\mathrm{d}\,u)$$

偶函数

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \boxed{f(|\sin u|)}\,\mathrm{d}u$$

$$= 4\int_{0}^{\frac{\pi}{2}} f(|\sin u|)\,\mathrm{d}u = 4\int_{0}^{\frac{\pi}{2}} f(|\cos u|)\,\mathrm{d}u$$

即 $$\int_{0}^{2\pi} f(|\cos x|)\,\mathrm{d}x = 4\int_{0}^{\frac{\pi}{2}} f(|\cos x|)\,\mathrm{d}x$$

$$\therefore \quad \int_{0}^{\frac{\pi}{2}} f(|\cos x|)\,\mathrm{d}x = \frac{1}{4}\int_{0}^{2\pi} f(|\cos x|)\,\mathrm{d}x.$$

3. 计算 $I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}\mathrm{d}x.$

解 $\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-\sin^2 x}}{\sin x+\sqrt{1-\sin^2 x}}\mathrm{d}x$

$=\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-\cos^2 x}}{\cos x+\sqrt{1-\cos^2 x}}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x,$

$\therefore\quad 2I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}\mathrm{d}x+\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}\mathrm{d}x=\int_0^{\frac{\pi}{2}}\mathrm{d}x=\frac{\pi}{2},$

故 $I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}\mathrm{d}x=\frac{\pi}{4}.$

4. 设 $f(x)=\begin{cases}1+x^2, x<0,\\ e^{-x}, x\geq 0.\end{cases}$ 求 $\int_1^3 f(x-2)\,\mathrm{d}x$.

分析 先用换元法公式把被积函数化成 $f(t)$, 再把 $f(t)$ 的表达式代入积分式, 要注意 $f(t)$ 是以 $t=0$ 为分段点的分段函数。

解 令 $x-2=t$, 则 $\mathrm{d}x=\mathrm{d}t$, 且当 $x=1$ 时, $t=-1$; 当 $x=3$ 时, $t=1$, 于是

$$\int_{1}^{3} f(x-2)\,\mathrm{d}x = \int_{-1}^{1} f(t)\,\mathrm{d}t$$

$$= \int_{-1}^{0} \left(1+t^2\right)\mathrm{d}t + \int_{0}^{1} \mathrm{e}^{-t}\,\mathrm{d}t$$

$$= 1 + \left.\frac{t^3}{3}\right|_{-1}^{0} - \left.\mathrm{e}^{-t}\right|_{0}^{1} = \frac{7}{3} - \frac{1}{\mathrm{e}}.$$

$$f(x) = \begin{cases} 1+x^2, x<0, \\ \mathrm{e}^{-x}, x \ge 0. \end{cases}$$

5. 设 $f(x)$可导，且 $f(0)=0$,

$$F(x)=\int_0^x t^{n-1}f(x^n-t^n)\,\mathrm{d}t \quad (n\in N),$$

求 $\lim\limits_{x\to 0}\dfrac{F(x)}{x^{2n}}$.

解 $F(x)=\int_0^x t^{n-1}f(x^n-t^n)\,\mathrm{d}t=\dfrac{1}{n}\int_0^x f(x^n-t^n)\,\mathrm{d}(t^n)$

$$=-\frac{1}{n}\int_0^x f(x^n-t^n)\,\mathrm{d}(x^n-t^n)$$

$$\overset{令u=x^n-t^n}{=\!=\!=\!=}-\frac{1}{n}\int_{x^n}^0 f(u)\,\mathrm{d}u=\frac{1}{n}\int_0^{x^n} f(u)\,\mathrm{d}u$$

$$\therefore \quad F(x)=\frac{1}{n}\int_0^{x^n} f(u)\,\mathrm{d}u$$

$$F'(x)=\frac{1}{n}f(x^n)\cdot nx^{n-1}=f(x^n)\cdot x^{n-1}$$

$$\lim_{x\to0}\frac{F(x)}{x^{2n}}=\lim_{x\to0}\frac{F'(x)}{2nx^{2n-1}}=\frac{1}{2n}\lim_{x\to0}\frac{f(x^n)\cdot x^{n-1}}{x^{2n-1}}$$

$$=\frac{1}{2n}\lim_{x\to0}\frac{f(x^n)}{x^n}=\frac{1}{2n}\lim_{x\to0}\frac{f(x^n)-f(0)}{x^n}\qquad (f(0)=0)$$

$$=\frac{1}{2n}f'(0).$$

6. 计算 $\int_1^4 \frac{\ln x}{\sqrt{x}} \mathrm{d}x$.

解 根据定积分的分部积分 公式，得

$$\int_1^4 \frac{\ln x}{\sqrt{x}} \mathrm{d}x = 2\int_1^4 \ln x \,\mathrm{d}\sqrt{x} = 2\sqrt{x}\ln x\Big|_1^4 - 2\int_1^4 \sqrt{x}\,\mathrm{d}\ln x$$

$$= 4\ln 4 - 2\int_1^4 \sqrt{x}\cdot\frac{1}{x}\mathrm{d}x$$

$$= 4\ln 4 - 2\int_1^4 \frac{1}{\sqrt{x}}\mathrm{d}x$$

$$= 4\ln 4 - 4\sqrt{x}\Big|_1^4 = 4(\ln 4 - 1).$$

7. 计算$\int_0^{\sqrt{3}} \arctan x \, \mathrm{d} x$.

解 根据定积分的分部积分 公式， 得

$$\int_0^{\sqrt{3}} \arctan x \, \mathrm{d} x = x \arctan x \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} x \, \mathrm{d} \arctan x$$

$$= \sqrt{3} \arctan \sqrt{3} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} \mathrm{d} x$$

$$= \frac{\sqrt{3}}{3}\pi - \frac{1}{2}\ln(1+x^2)\Big|_0^{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}\pi - \ln 2.$$

8. 计算 $\int_0^1 (1-x^2)^2\sqrt{1-x^2}\,\mathrm{d}x.$

解 令 $x=\sin t$，则 $\mathrm{d}x=\cos t\,\mathrm{d}t$，且当 $x=0$ 时，$t=0$；

当 $x=1$ 时，$t=\dfrac{\pi}{2}$. 于是有

$$\int_0^1 (1-x^2)^2\sqrt{1-x^2}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}(1-\sin^2 t)^2\sqrt{1-\sin^2 t}\cos t\,\mathrm{d}t$$

$$=\int_0^{\frac{\pi}{2}}\cos^6 t\,\mathrm{d}t=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=\frac{5}{32}\pi.$$

9. 计算 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^3+\sin^2 x)\cos^2 x\,\mathrm{d}x.$

解 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^3+\sin^2 x)\cos^2 x\,\mathrm{d}x$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^3\cos^2 x\,\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\cos^2 x\,\mathrm{d}x$$

$$=0+2\int_0^{\frac{\pi}{2}}\sin^2 x(1-\sin^2 x)\,\mathrm{d}x$$

$$=2\left(\int_0^{\frac{\pi}{2}}\sin^2 x\,\mathrm{d}x-\int_0^{\frac{\pi}{2}}\sin^4 x\,\mathrm{d}x\right)$$

$$=2\left(\frac{1}{2}\cdot\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)=\frac{\pi}{8}.$$

10. 计算 $\int_{100-\frac{\pi}{2}}^{100+\frac{\pi}{2}} \tan^2 x \sin^2 2x \,\mathrm{d}x$.

解 $\tan^2 x \cdot \sin^2 2x = \dfrac{\sin^2 x}{\cos^2 x} \cdot 4\sin^2 x \cdot \cos^2 x = 4\sin^4 x$

其周期为π，故

$$\int_{100-\frac{\pi}{2}}^{100+\frac{\pi}{2}} \tan^2 x \sin^2 2x \,\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\sin^4 x \,\mathrm{d}x$$

$$= 8\int_0^{\frac{\pi}{2}} \sin^4 x \,\mathrm{d}x$$

$$= 8 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{2}\pi.$$

11. 计算$\int_0^1(\arcsin x)^2\,\mathrm{d}x$.

解 先用换元积分法，再用分部积分法 .

令$\arcsin x=t$，则$x=\sin t$，$\mathrm{d}x=\cos t\,\mathrm{d}t$，

$x=0:\ t=0;\qquad x=1:\ t=\dfrac{\pi}{2}$. 于是有

$$\int_0^1(\arcsin x)^2\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}t^2\cos t\,\mathrm{d}t=\int_0^{\frac{\pi}{2}}t^2\,\mathrm{d}\sin t$$

$$=t^2\sin t\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}2t\sin t\,\mathrm{d}t=\frac{\pi^2}{4}+2t\cos t\Big|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\cos t\,\mathrm{d}t$$

$$=\frac{\pi}{4}-2\sin t\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{4}-2.$$

12. 设 $f''(x)$ 在 $[0,1]$ 连续, 且 $f(0)=1, f(2)=3,$

$f'(2)=5$, 求 $\int_0^1 x\, f''(2x)\,\mathrm{d}x$.

解 $\int_0^1 x\, \underline{f''(2x)\,\mathrm{d}x} = \frac{1}{2}\int_0^1 x\, \mathrm{d} f'(2x)$ (分部积分)

$$= \frac{1}{2}\left[\, x f'(2x)\Big|_0^1 - \int_0^1 f'(2x)\,\mathrm{d}x \,\right]$$

$$= \frac{5}{2} - \frac{1}{4} f(2x)\Big|_0^1 = 2$$

13. 设$f(x)=\int_0^{\sqrt{x}} e^{-t^2}\,dt$, $f(1)=0$,

求$\int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx$.

分析 e^{-t^2}无初等函数形式的原函数，故无法直接求出$f(x)$，所以采用分部积分法.

解 $f'(x)=e^{-x}\cdot\frac{1}{2\sqrt{x}}$.

$$\int_0^1 \frac{1}{\sqrt{x}} f(x)\,dx = 2\int_0^1 f(x)\,d\sqrt{x}$$

$$= 2\sqrt{x}f(x)\Big|_0^1 - 2\int_0^1 \sqrt{x} f'(x)\,dx = -\int_0^1 e^{-x}\,dx$$

$$= \frac{1}{e} - 1.$$