

第二节 多元函数的偏导数

习题 8-2

1. 求下列函数的偏导数:

$$(1) \quad z = ax^2y + axy^2; \quad (2) \quad z = \tan^2(x^2 + y^2);$$

$$(3) \quad z = \frac{x}{y} + \frac{y}{x}; \quad (4) \quad z = \arctan \frac{x}{y^2};$$

$$(5) \quad z = \ln(x + \sqrt{x^2 - y^2}); \quad (6) \quad z = xe^{-y} + ye^{-x};$$

$$(7) \quad u = \ln(x + 2^{yz}); \quad (8) \quad z = (1 + xy)^y.$$

解 (1) $z_x = ay \cdot 2x + ay^2 = 2axy + ay^2,$

$$z_y = ax^2 + ax \cdot 2y = ax^2 + 2axy.$$

$$(2) \quad z_x = 2 \tan(x^2 + y^2) \cdot \sec^2(x^2 + y^2) \cdot 2x$$

$$= 4x \tan(x^2 + y^2) \sec^2(x^2 + y^2),$$

由所给出函数关于自变量 x, y 的对称性, 所以有

$$z_y = 4y \tan(x^2 + y^2) \sec^2(x^2 + y^2).$$

$$(3) \quad z_x = \frac{1}{y} + y \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{y} - \frac{y}{x^2},$$

$$z_y = x \cdot \left(-\frac{1}{y^2}\right) + \frac{1}{x} = \frac{1}{x} - \frac{x}{y^2}.$$

$$(4) \quad z_x = \frac{1}{1 + \left(\frac{x}{y^2}\right)^2} \cdot \frac{1}{y^2} = \frac{y^2}{x^2 + y^4},$$

$$z_y = \frac{1}{1 + \left(\frac{x}{y^2}\right)^2} \cdot x \cdot (-2) \frac{1}{y^3} = \frac{-2xy}{x^2 + y^4}.$$

$$(5) \quad z_x = \frac{1}{x + \sqrt{x^2 - y^2}} \cdot \left[1 + \frac{1}{2}(x^2 - y^2)^{-\frac{1}{2}} \cdot 2x\right]$$

$$\begin{aligned}
&= \frac{1}{x + \sqrt{x^2 - y^2}} \cdot \left(1 + \frac{x}{\sqrt{x^2 - y^2}}\right) \\
&= \frac{1}{x + \sqrt{x^2 - y^2}} \cdot \frac{\sqrt{x^2 - y^2} + x}{\sqrt{x^2 - y^2}} = \frac{1}{\sqrt{x^2 - y^2}}, \\
z_y &= \frac{1}{x + \sqrt{x^2 - y^2}} \cdot \frac{1}{2} (x^2 - y^2)^{-\frac{1}{2}} \cdot (-2y) = -\frac{y}{(x + \sqrt{x^2 - y^2})\sqrt{x^2 - y^2}} \\
&= -\frac{y}{x\sqrt{x^2 - y^2} + x^2 - y^2}.
\end{aligned}$$

$$(6) \quad z_x = e^{-y} + ye^{-x} \cdot (-1) = e^{-y} - ye^{-x},$$

$$z_y = xe^{-y} \cdot (-1) + e^{-x} = -xe^{-y} + e^{-x}.$$

$$(7) \quad u_x = \frac{1}{x + 2^{yz}},$$

$$u_y = \frac{1}{x + 2^{yz}} \cdot 2^{yz} \cdot \ln 2 \cdot z = \frac{z 2^{yz} \ln 2}{x + 2^{yz}},$$

由所给函数关于自变量 y, z 的对称性, 所以有

$$u_z = \frac{y 2^{yz} \ln 2}{x + 2^{yz}}.$$

注意 常见的错误是遗漏了步骤: $\frac{\partial}{\partial y}(yz) = z$, 而得到错误结果, $u_y = \frac{2^{yz} \ln 2}{x + 2^{yz}}.$

(8) 求 z_y 时, 用幂函数的导数公式, 得

$$z_x = y(1 + xy)^{y-1} \cdot y = y^2(1 + xy)^{y-1}.$$

求 z_y 时, 把 x 暂时看作常数, 这时 z 是关于 y 的幂指函数, 所以

$$\begin{aligned}
z_y &= \frac{\partial}{\partial y} [e^{y \ln(1+xy)}] = e^{y \ln(1+xy)} [\ln(1+xy) + \frac{xy}{1+xy}] \\
&= (1+xy)^y [\ln(1+xy) + \frac{xy}{1+xy}].
\end{aligned}$$

2. 求下列函数在指定点处的一阶偏导数:

$$(1) \quad f(x, y) = x + (y-1) \arcsin \sqrt{\frac{x}{y}} \text{ 在点 } (x, 1) \text{ 对 } x \text{ 的偏导数 } f_x(x, 1);$$

(2) $f(x, y) = x^2 e^y + (x-1) \arctan \frac{y}{x}$ 在点 $(1, 0)$ 的两个偏导数 $f_x(1, 0)$ 与 $f_y(1, 0)$.

解 (1) 法 1 因 $f(x, 1) = x$, 所以

$$f_x(x, 1) = 1.$$

法 2 $f_x(x, y) = 1 + \frac{y-1}{\sqrt{1-\frac{x}{y}}} \cdot \frac{1}{2} \sqrt{\frac{y}{x}} \cdot \frac{1}{y}$, 于是

$$f_x(x, 1) = f_x(x, y) \Big|_{y=1} = 1.$$

(2) 法 1 因 $f(x, 0) = x^2$, 所以 $f_x(x, 0) = 2x$, 故

$$f_x(1, 0) = 2,$$

因 $f(1, y) = e^y$, 所以 $f_y(1, y) = e^y$, 故

$$f_y(1, 0) = 1.$$

法 2 $f_x(x, y) = 2xe^y + \arctan \frac{y}{x} + (x-1) \cdot \frac{1}{1+(\frac{y}{x})^2} \cdot y(-\frac{1}{x^2})$,

$$f_y(x, y) = x^2 e^y + (x-1) \cdot \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{1}{x},$$

于是

$$f_x(1, 0) = 2, f_y(1, 0) = 1.$$

注意 计算偏导数 $f_x(x_0, y_0)$ 时, 可以利用本题的解法 1, 将 $y = y_0$ 先代入 $f(x, y)$ 中, 再对 x 求导. 显然, 本题如果用解法 2, 先求 $f_x(x, y)$, $f_y(x, y)$, 后代入 $x = x_0, y = y_0$ 的值, 则要麻烦多了.

3. 求曲线 $\begin{cases} z = \frac{1}{4}(x^2 + y^2), \\ y = 2, \end{cases}$ 在点 $M_0(2, 2, 2)$ 处的切线关于 x 轴的倾角.

解 根据偏导数的几何意义, $f_x(2, 2)$ 就是曲线在点 $M_0(2, 2, 2)$ 处的切线关于 x

轴的斜率, 而

$$f_x(2,2) = \frac{1}{2}x|_{x=2} = 1, \text{ 即斜率 } k = \tan \alpha = 1,$$

于是倾角 $\alpha = \frac{\pi}{4}$.

$$4. \text{ 设 } f(x,y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$$

试证函数 $f(x,y)$ 在点 $(0,0)$ 处连续且偏导数存在, 并求出 $f_x(0,0)$ 及 $f_y(0,0)$ 的值.

解 因为函数 $f(x,y)$ 在点 $(0,0)$ 的邻域内由定义, 且

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x,y) = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}} = \lim_{\rho \rightarrow 0} \rho^2 \sin \frac{1}{\rho} = 0 = f(0,0),$$

所以 $f(x,y)$ 在点 $(0,0)$ 处连续. 又因为

$$f_x(0,0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 \sin \frac{1}{\sqrt{(\Delta x)^2}}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \Delta x \sin \frac{1}{\sqrt{(\Delta x)^2}} = 0,$$

$$f_y(0,0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{(\Delta y)^2 \sin \frac{1}{\sqrt{(\Delta y)^2}}}{\Delta y} = \lim_{\Delta y \rightarrow 0} \Delta y \sin \frac{1}{\sqrt{(\Delta y)^2}} = 0.$$

所以 $f(x,y)$ 在点 $(0,0)$ 处偏导数存在, 且 $f_x(0,0) = 0, f_y(0,0) = 0$.

注意 如同一元函数一样, 分段函数在分界点处的偏导数应按定义来求.

5. 求下列函数的所有二阶偏导数:

$$(1) \quad z = \cos^2(ax - by);$$

$$(2) \quad z = e^{-\alpha x} \sin \beta y;$$

$$(3) \quad z = xe^{-xy};$$

$$(4) \quad z = y^x.$$

解 (1) $z_x = 2 \cos(ax - by) \cdot (-1) \sin(ax - by) \cdot a = -a \sin 2(ax - by),$

$$z_y = 2 \cos(ax - by) \cdot (-1) \sin(ax - by) \cdot (-b) = b \sin 2(ax - by),$$

$$z_{xx} = -a \cos 2(ax - by) \cdot 2a = -2a^2 \cos 2(ax - by),$$

$$z_{xy} = z_{yx} = -a \cos 2(ax - by) \cdot (-2b) = 2ab \cos 2(ax - by),$$

$$z_{yy} = b \cos 2(ax - by) \cdot (-2b) = -2b^2 \cos 2(ax - by).$$

$$(2) \quad z_x = e^{-\alpha x}(-\alpha) \sin \beta y = -\alpha e^{-\alpha x} \sin \beta y,$$

$$z_y = e^{-\alpha x} \cos \beta y \cdot \beta = \beta e^{-\alpha x} \cos \beta y,$$

$$z_{xx} = \alpha^2 e^{-\alpha x} \sin \beta y, \quad z_{xy} = z_{yx} = -\alpha \beta e^{-\alpha x} \cos \beta y,$$

$$z_{yy} = -\beta^2 e^{-\alpha x} \cos \beta y.$$

$$(3) \quad z_x = e^{-xy} + x e^{-xy} \cdot (-y) = (1 - xy) e^{-xy}, \quad z_y = -x^2 e^{-xy},$$

$$z_{xx} = -y \cdot e^{-xy} + (1 - xy) \cdot e^{-xy} \cdot (-y) = (-2y + xy^2) e^{-xy},$$

$$z_{xy} = z_{yx} = -2x \cdot e^{-xy} - x^2 \cdot e^{-xy} \cdot (-y) = (-2x + x^2 y) e^{-xy},$$

$$z_{yy} = -x^2 e^{-xy} \cdot (-x) = x^3 e^{-xy}.$$

$$(4) \quad z_x = y^x \ln y, \quad z_y = xy^{x-1}, \quad z_{xx} = y^x \ln^2 y,$$

$$z_{xy} = z_{yx} = y^{x-1} + xy^{x-1} \ln y = y^{x-1}(1 + x \ln y), \quad z_{yy} = x(x-1)y^{x-2}.$$

6. 求下列函数的指定的高阶偏导数:

$$(1) \quad z = x \ln(xy), \quad z_{xxy}, \quad z_{xyy};$$

$$(2) \quad u = x^a y^b z^c, \quad \frac{\partial^6 u}{\partial x \partial y^2 \partial z^3};$$

$$(3) \quad f(x, y, z) = xy^2 + yz^2 + zx^2, \quad f_{xz}(1, 0, 2) \text{ 及 } f_{yz}(0, -1, 0).$$

解 (1) $z_x = \ln(xy) + x \cdot \frac{y}{xy} = \ln(xy) + 1,$

$$z_{xx} = \frac{y}{xy} = \frac{1}{x}, \quad z_{xxy} = 0,$$

$$z_{xy} = \frac{x}{xy} = \frac{1}{y}, \quad z_{xyy} = -\frac{1}{y^2}.$$

$$(2) \quad \frac{\partial u}{\partial x} = ax^{a-1} y^b z^c, \quad \frac{\partial^2 u}{\partial x \partial y} = abx^{a-1} y^{b-1} z^c,$$

$$\frac{\partial^3 u}{\partial x \partial y^2} = ab(b-1)x^{a-1}y^{b-2}z^c, \quad \frac{\partial^4 u}{\partial x \partial y^2 \partial z} = abc(b-1)x^{a-1}y^{b-2}z^{c-1},$$

$$\frac{\partial^5 u}{\partial x \partial y^2 \partial z^2} = abc(b-1)(c-1)x^{a-1}y^{b-2}z^{c-2},$$

$$\frac{\partial^6 u}{\partial x \partial y^2 \partial z^3} = abc(b-1)(c-1)(c-2)x^{a-1}y^{b-2}z^{c-3}.$$

(3) 因为 $f_x = y^2 + 2xz$, $f_y = 2xy + z^2$, $f_{xz} = 2x$, $f_{yz} = 2z$,

所以

$$f_{xz}(1, 0, 2) = 2, f_{yz}(0, -1, 0) = 0.$$

7. 验证函数 $z = e^{-kn^2 t} \sin nx$ 满足热传导方程

$$z_t = kz_{xx}.$$

证 因为 $z_t = -kn^2 e^{-kn^2 t} \sin nx$, $z_x = ne^{-kn^2 t} \cos nx$,

$$z_{xx} = -n^2 e^{-kn^2 t} \sin nx.$$

所以

$$z_t = k(-n^2 e^{-kn^2 t} \sin nx) = kz_{xx}.$$

8. 验证函数 $u = \sin(x - at) + \ln(x + at)$ 满足波动方程

$$u_{tt} = a^2 u_{xx}.$$

证 因为 $u_t = -a \cos(x - at) + \frac{a}{x + at}$,

$$u_{tt} = -a \cdot (-1) \sin(x - at) \cdot (-a) + (-1) \frac{a^2}{(x + at)^2}$$

$$= -a^2 \cdot [\sin(x - at) + \frac{1}{(x + at)^2}],$$

$$u_x = \cos(x - at) + \frac{1}{x + at}, \quad u_{xx} = -\sin(x - at) - \frac{1}{(x + at)^2}.$$

所以

$$u_{tt} = a^2 \left[-\sin(x - at) - \frac{1}{(x + at)^2} \right] = a^2 u_{xx} .$$

9. 验证函数 $u = \arctan \frac{x}{y}$ 满足拉普拉斯方程

$$u_{xx} + u_{yy} = 0 .$$

证 因为
$$u_x = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2} ,$$

$$u_{xx} = y \cdot (-1) \frac{1}{(x^2 + y^2)^2} \cdot (2x) = -\frac{2xy}{(x^2 + y^2)^2} ,$$

$$u_y = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2} ,$$

$$u_{yy} = x \cdot \frac{1}{(x^2 + y^2)^2} \cdot 2y = \frac{2xy}{(x^2 + y^2)^2} .$$

所以

$$u_{xx} + u_{yy} = -\frac{2xy}{(x^2 + y^2)^2} + \frac{2xy}{(x^2 + y^2)^2} = 0 .$$