第八章

多元函数微分学

一、主要内容

1. 基本概念

(1) 多元函数极限

(2) 连续

(3) 偏导数

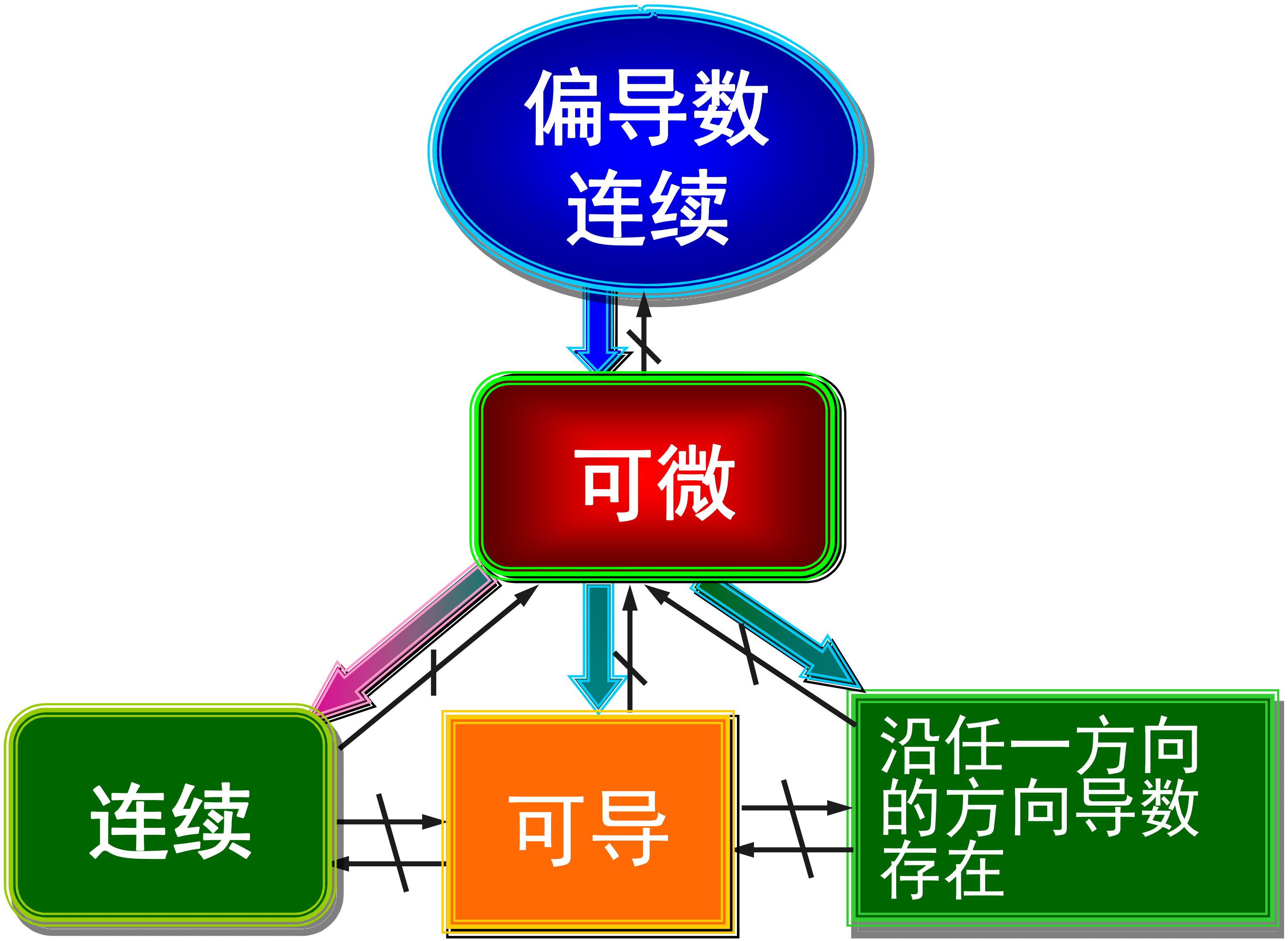
(4) 全微分

(5) 方向导数

(6) 梯度

(7) 极值

2. 概念之间的关系　　对于二元函数



3. 多元函数微分法

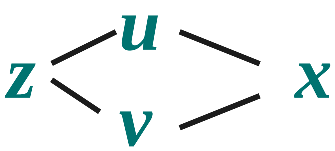
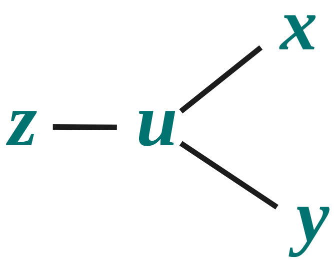
(1) 复合函数求导法

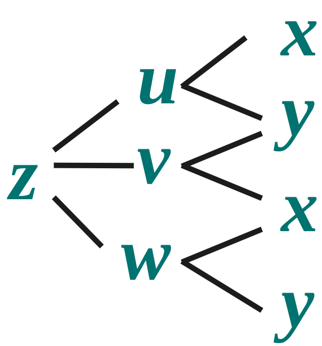
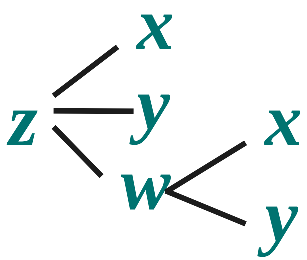
(2) 隐函数(组)求导法

(3) 全微分形式不变性

(1) 复合函数求导法

全导数

函数关系	结构图	求导公式
$z = f(u,v)$ $u = \varphi(x)$ $v = \psi(x)$		$\boxed{\dfrac{dz}{dx}} = \dfrac{\partial z}{\partial u}\cdot\dfrac{du}{dx} + \dfrac{\partial z}{\partial v}\cdot\dfrac{dv}{dx}$
$z = f(u)$ $u = \varphi(x,y)$		$\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x}$ $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}\dfrac{\partial u}{\partial y}$

函数关系	关系图	求导公式
$z = f(u,v,w)$ $u = u(x,y)$ $v = v(x,y)$ $w = w(x,y)$		$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$ $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$
$z = f(x,y,w)$ $w = \varphi(x,y)$ ★		$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$ $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$

(2) 隐函数(组)求导法

1) $F(x,y)=0$

隐函数存在定理 1 设函数$F(x,y)$在点$P(x_0,y_0)$的某一邻域内具有连续的偏导数，且

$$F(x_0,y_0)=0,\ F_y(x_0,y_0)\neq 0,$$

则方程$F(x,y)=0$在某 U(P)内恒能唯一确定一个单值连续且具有连续导数的函数$y=f(x)$，它满足条件：$y_0=f(x_0)$，

隐函数的求导公式

并有
$$\frac{dy}{dx}=-\frac{F_x}{F_y}. \qquad (2.1)$$

2) $F(x,y,z)=0$

隐函数存在定理 2 设函数$F(x,y,z)$在点$P(x_0,y_0,z_0)$的某一邻域内有连续的偏导数且 $F(x_0,y_0,z_0)=0$， $F_z(x_0,y_0,z_0)\neq 0$，则方程$F(x,y,z)=0$在点$P(x_0,y_0,z_0)$的某一邻域内恒能唯一确定一个单值连续且具有连续偏导数的函数$z=f(x,y)$，它满足条件 $z_0=f(x_0,y_0)$，

并有
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}. \qquad (2.2)$$

3) 方程组的情形 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$

隐函数存在定理 3 设 $F(x,y,u,v)$, $G(x,y,u,v)$ 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内有对各个变量的连续偏导数，又 $F(x_0,y_0,u_0,v_0)=0$,

$$G(x_0,y_0,u_0,v_0)=0\ ,$$

且偏导数所组成的函数行列式（或称雅可比式）

$$J=\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_P=\left.\begin{vmatrix} \dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v} \\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v} \end{vmatrix}\right|_P \neq 0$$

例 目 回 上 下 停

则方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$

在点$P(x_0,y_0,u_0,v_0)$的某一邻域内恒能唯一确定一组单值连续且具有连续偏导数的函数$u=u(x,y)$，$v=v(x,y)$，它们满足条件：$u_0=u(x_0,y_0),\ v_0=v(x_0,y_0)$，并有

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}, \qquad (2.3)$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)} = -\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} = -\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix},$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)} = -\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}.$$

注 情形3的特例:

$$\begin{cases} F(x,u,v)=0 \\ G(x,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x) \\ v=v(x) \end{cases}$$

$$\begin{cases} F(x,u(x),v(x))\equiv 0 \\ G(x,u(x),v(x))\equiv 0 \end{cases}$$

$$\begin{cases} F_x+F_u\cdot\dfrac{du}{dx}+F_v\cdot\dfrac{dv}{dx}=0 \\ G_x+G_u\cdot\dfrac{du}{dx}+G_v\cdot\dfrac{dv}{dx}=0 \end{cases}$$

$$\begin{cases} F_u \cdot \boxed{\dfrac{du}{dx}} + F_v \cdot \dfrac{dv}{dx} = -F_x \\ G_u \cdot \boxed{\dfrac{du}{dx}} + G_v \cdot \dfrac{dv}{dx} = -G_x \end{cases}$$

$$\frac{du}{dx} = \frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{1}{J}\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}$$

$$= -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}, \qquad \frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}.$$

(3) 全微分形式不变性

设函数$z = f(u,v)$具有连续偏导数，则有全微分

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv;$$

当$u = \phi(x,y)$、$v = \psi(x,y)$时，有

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

全微分形式不变形的实质：

无论z是自变量u、v的函数还是中间变量u、v的函数，它的全微分形式是一样的.

4. 应用

(1) 方向导数、梯度、散度 与旋度

(2) 切线、法平面

(3) 切平面、法线

(4) 极值 $\begin{cases} \text{无条件极值} \\ \text{条件极值} \end{cases}$

(5) 最值

(1) 空间曲线的切线与法平面

1) 曲线方程为参数方程的情形

$$\Gamma: \quad x=\varphi(t),\ y=\psi(t),\ z=\omega(t).$$

曲线 Γ 的切向量:

$$\vec{T}=(\varphi'(t_0),\psi'(t_0),\omega'(t_0))$$

切线方程为 $$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{\omega'(t_0)}.$$

法平面方程为

$$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+\omega'(t_0)(z-z_0)=0.$$

2) 曲线方程为一般方程的情形

$$\Gamma:\begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases} \Longrightarrow \begin{cases}x=x\\ y=\varphi(x)\\ z=\psi(x)\end{cases}$$

$$\left(\text{若 } J=\frac{\partial(F,G)}{\partial(y,z)}\neq 0\ \right)$$

求曲线 Γ 的**切向量**:

(方法1) 由 $\begin{cases}F(x,\varphi(x),\psi(x))\equiv 0\\ G(x,\varphi(x),\psi(x))\equiv 0\end{cases}$, 有

$$\begin{cases}F_x+F_y\cdot\varphi'(x)+F_z\cdot\psi'(x)=0\\ G_x+G_y\cdot\varphi'(x)+G_z\cdot\psi'(x)=0\end{cases}$$

$$\begin{cases} F_x + F_y \cdot \varphi'(x) + F_z \cdot \psi'(x) = 0 \\ G_x + G_y \cdot \varphi'(x) + G_z \cdot \psi'(x) = 0 \end{cases}$$

可求得曲线在$M(x_0, y_0, z_0)$处的切向量:

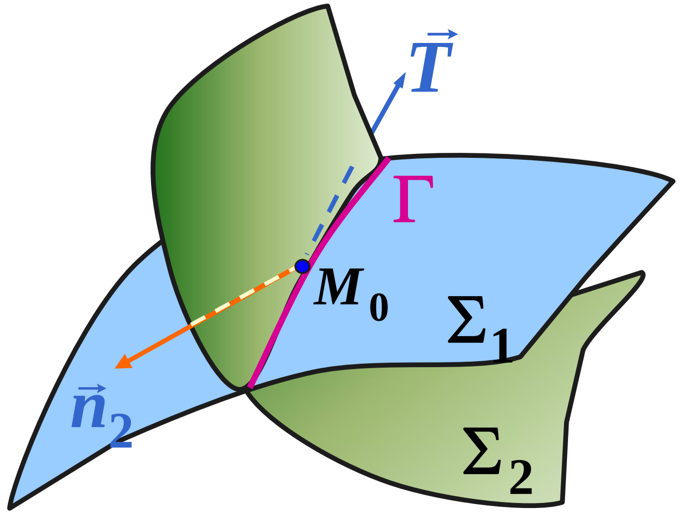
$$\vec{T} = \{1, \varphi'(x_0), \psi'(x_0)\}$$

切向量求法之一

$$= \left\{ 1, -\frac{1}{J}\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}, -\frac{1}{J}\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix} \right\}_M$$

$$= \frac{1}{J}\left\{ J, \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}, \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \right\}_M, \text{其中 } J = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}.$$

(方法2) $\Gamma: \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$



$\because \quad \vec{T} \perp \vec{n}_1, \quad \vec{T} \perp \vec{n}_2$

$$\vec{n}_1 = (F_x(x_0,y_0,z_0), F_y(x_0,y_0,z_0), F_z(x_0,y_0,z_0))$$

$$\vec{n}_2 = (G_x(x_0,y_0,z_0), G_y(x_0,y_0,z_0), G_z(x_0,y_0,z_0))$$

$\therefore$ 曲线Γ在点M_0处的切向量：

切向量求法之二

$$\vec{T} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}_{M_0} \qquad (\vec{n}_1 \not\parallel \vec{n}_2)$$

(2) 曲面的切平面与法线

$$\pi: \quad F(x, y, z) = 0.$$

切平面方程为:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0)$$

$$+ F_z(x_0, y_0, z_0)(z - z_0) = 0$$

法线方程为:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

(3) 多元函数的极值

1) 无条件极值

定理 1（必要条件）

设函数$z = f(x, y)$在点(x_0, y_0)具有偏导数，且在点(x_0, y_0)处有极值，则它在该点的偏导数必然为零：

$$f_x(x_0, y_0) = 0, \qquad f_y(x_0, y_0) = 0.$$

定义 一阶偏导数同时为零的点，均称为多元函数的驻点.

注意 驻点 $\xleftarrow{\quad\quad}$ 极值点

驻点 $\cancel{\xrightarrow{\quad\quad}}$ 极值点

定理 2（充分条件）

设函数$z = f(x,y)$在点(x_0,y_0)的某邻域内有一阶及二阶连续偏导数，

又 $f_x(x_0,y_0)=0$，$f_y(x_0,y_0)=0$，

令 $f_{xx}(x_0,y_0)=A$，$f_{xy}(x_0,y_0)=B$，

$f_{yy}(x_0,y_0)=C$，$\Delta = AC - B^2$，

则有

Δ	$f(x_0,y_0)$	
>0	$A>0$, 极小值	是极值
	$A<0$, 极大值	
<0	非极值	
$=0$	不定(需用其他方法确定)	

求函数$z = f(x, y)$极值的一般步骤：

1° 求极值可疑点：驻点、偏导数不存在的点；

2° 判断

(1) 利用极值的充分判定法　　，

(2) 若充分条件不满足，则 利用极值的定义 .

2) 条件极值：对自变量有附加条件的极值.

拉格朗日乘数法

要找函数$z = f(x, y)$在条件$\varphi(x, y) = 0$下的可能极值点，

先构造函数$F(x, y) = f(x, y) + \lambda\varphi(x, y)$，

其中λ为某一常数，可由

$$\begin{cases} f_x(x, y) + \lambda\varphi_x(x, y) = 0, \\ f_y(x, y) + \lambda\varphi_y(x, y) = 0, \\ \varphi(x, y) = 0. \end{cases}$$

解出x, y, λ，其中x, y就是可能的极值点的坐标.

二、典型例题

1. 偏导数与全微分

(1) 基本概念

例1 已知 $f(x,y)=\mathrm{e}^{\sqrt{x^2+y^4}}$，求 $f'_x(0,0), f'_y(0,0)$.

解 $f(x,0)=\mathrm{e}^{|x|}$，$f(0,y)=\mathrm{e}^{y^2}$，

$$\lim_{x\to 0^-}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0^-}\frac{\mathrm{e}^{|x|}-1}{x}$$

$$=\lim_{x\to 0^-}\frac{\mathrm{e}^{-x}-1}{x}=-1$$

$$\lim_{x\to 0^+}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0^+}\frac{e^{|x|}-1}{x}=\lim_{x\to 0^+}\frac{e^{x}-1}{x}=1$$

$$\because\quad \lim_{x\to 0^+}\frac{f(x,0)-f(0,0)}{x}\neq\lim_{x\to 0^-}\frac{f(x,0)-f(0,0)}{x}$$

$\therefore\quad f'_x(0,0)$不存在.

$$f'_y(0,0)=\left.\frac{\mathrm{d}\,f(0,y)}{\mathrm{d}\,y}\right|_{y=0}=\left.2y\mathrm{e}^{y^2}\right|_{y=0}=0.$$

注　1° 何时必须用偏导数定义求偏导数?

当 $x=x_0$ 是 $f(x,y_0)$的分段点时，求 $f_x(x_0,y_0)$ 须用偏导数定义.

2° 在某些情形下，用偏导数定义求偏导数较简单.

如: $f(x,y)=\dfrac{(\sin xy)(\cos\sqrt{y+2})-(y-1)\cos x}{1+\sin x+\sin(y-1)}$,

则 $f_y(0,1)=$ ____-1____.

解　此题用求导法很繁，而用偏导数定义

$$f_y(0,1)=\lim_{y\to1}\frac{f(0,y)-f(0,1)}{y-1}=\lim_{y\to1}\frac{-\dfrac{\cancel{(y-1)}}{1+\sin(y-1)}-0}{\cancel{y-1}}=-1.$$

目 回 上 下 停

例2 设连续函数 $z = f(x,y)$ 满足:

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$$\lim_{\substack{x\to 0\\ y\to 1}} \frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}} = 0$$

则 $\mathrm{d}z\big|_{(0,1)} = \underline{\ 2\mathrm{d}x-\mathrm{d}y\ }$.

解 依题设条件，知

$$\lim_{\substack{x\to 0\\ y\to 1}} [f(x,y)-2x+y-2] = 0$$

$$f(0,1) = \lim_{\substack{x\to 0\\ y\to 1}} f(x,y) = 1 \qquad (f(x,y)\text{连续})$$

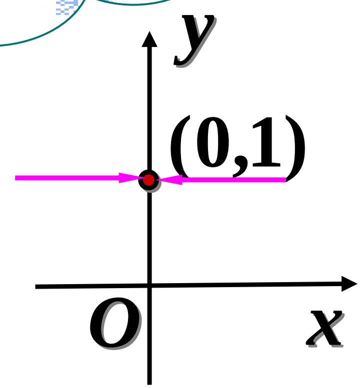
由 $\lim\limits_{x\to 0}\dfrac{f(x,1)-2x-1}{|x|}=0$ (沿 $y=1$)

无穷小与有界函数的乘积仍为无穷小

知 $\lim\limits_{x\to 0}\left[\dfrac{f(x,1)-1}{x}-2\right]$

$=\lim\limits_{x\to 0}\left[\dfrac{f(x,1)-2x-1}{|x|}\right]\cdot\dfrac{|x|}{x}$

$=0$

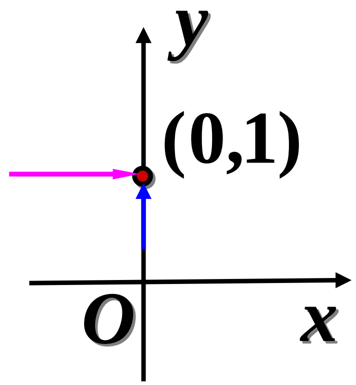


从而 $f_x'(0,1)=2$

$$\lim_{\substack{x\to 0\\ y\to 1}}\frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}}=0$$

由 $\lim\limits_{y\to 1}\dfrac{f(0,y)+y-2}{|y-1|}=0$ (沿 $x=0$)

知 $\lim\limits_{y\to 1}\left[\dfrac{f(0,y)-1}{y-1}+1\right]$



$$=\lim_{y\to 1}\left[\frac{f(0,y)+y-2}{|y-1|}\right]\cdot\frac{|y-1|}{y-1}$$

$=0,$

从而 $f_y'(0,1)=-1$

$$\boxed{\lim_{\substack{x\to 0\\ y\to 1}}\frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}}=0}$$

$$\because \quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta z-[f_x'(0,1)\Delta x+f_y'(0,1)\Delta y]}{\rho}$$

$$=\lim_{\substack{x\to 0\\ y\to 1}} \frac{[f(x,y)-f(0,1)]-[2x+(-1)(y-1)]}{\sqrt{x^2+(y-1)^2}}$$

$$=\lim_{\substack{x\to 0\\ y\to 1}} \frac{f(x,y)-2x+y-2}{\sqrt{x^2+(y-1)^2}}=0$$

$\therefore \quad z=f(x,y)$在$(0,1)$处可微，且

$$\mathrm{d}z\big|_{(01)}=f_x'(0,1)\,\mathrm{d}x+f_y'(0,1)\,\mathrm{d}y=2\,\mathrm{d}x-\mathrm{d}y$$

类似题 二元函数 $f(x,y)$ 在点(0,0)处可微的一个充分条件是(C).

概念之间的关系

(A) $\lim\limits_{(x,y)\to(0,0)}[f(x,y)-f(0,0)]=0$

(B) $\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x}=0$ 且 $\lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y}=0.$

(C) $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=0$

(D) $\lim\limits_{x\to 0}[f_x(x,0)-f_x(0,0)]=0$ 且

$\lim\limits_{y\to 0}[f_y(0,y)-f_y(0,0)]=0.$

(2) 求导法

题型1 已知函数或偏导数的一些特殊关系，求偏导数或微分.

例3 设 $z = g(xy) + yf(2x - y, \sin y)$，其中$g$二阶可导，$f$ 具有二阶连续偏导数，求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

解 $$\frac{\partial z}{\partial x} = g'(xy) \cdot y + y \cdot f_1' \cdot 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\, g'(xy) \cdot y + 2y \cdot f_1'(2x - y, \sin y)\right]$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\, g'(xy)\cdot y + 2y\cdot\, f_1'(2x - y, \sin y)\right]$$

$$= g'(xy) + y\frac{\partial}{\partial y}[g'(xy)] + 2\left(1\cdot\, f_1' + y\frac{\partial f_1'}{\partial y}\right)$$

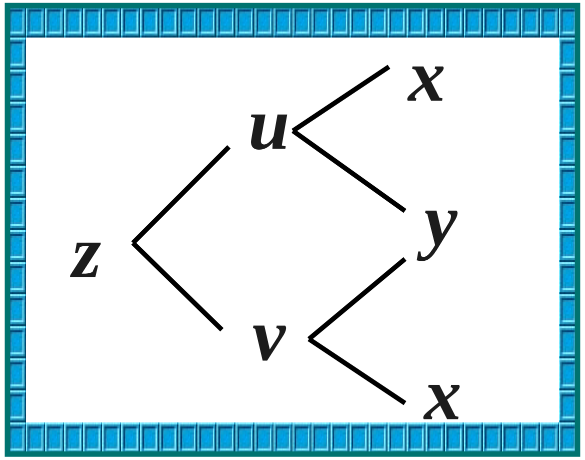
$$= g'(xy) + yg''(xy)\cdot x$$

$$+ 2\{\, f_1' + y[\, f_{11}''\cdot(-1) + \, f_{12}''\cdot\cos y\,]\}$$

$$= g' + xyg'' + 2\, f_1' - 2yf_{11}'' + 2y(\cos y)\, f_{12}''.$$

例4 设 $z = f(x, y)$有二阶连续偏导数，令

$u = xy,\ v = \dfrac{x}{y},$ 试将方程:



$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

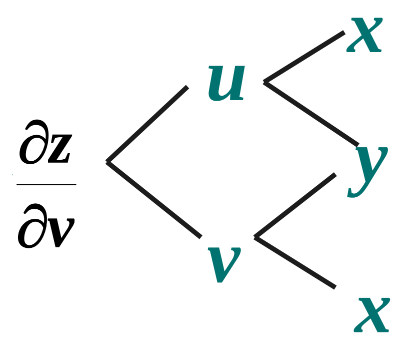
化为关于 u, v的方程.

解 $z = f(x, y) = f(\sqrt{uv}, \sqrt{\dfrac{u}{v}})$

$= F(u, v)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot y + \frac{\partial z}{\partial v} \cdot \frac{1}{y}$$

$$\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}$$



$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right)$$

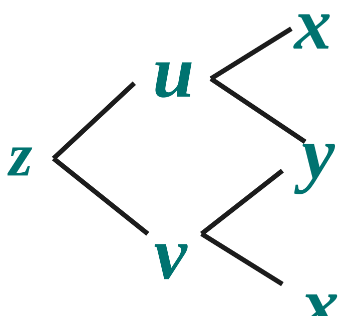
$$= y\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = y\cdot\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$+\frac{1}{y}\cdot\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$\boxed{\frac{\partial u}{\partial x} = y, \quad \frac{\partial v}{\partial x} = \frac{1}{y}}$$

$$= y\cdot\left(\frac{\partial^2 z}{\partial u^2}\cdot y + \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{1}{y}\right) + \frac{1}{y}\cdot\left(\frac{\partial^2 z}{\partial v\partial u}\cdot y + \frac{\partial^2 z}{\partial v^2}\cdot\frac{1}{y}\right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}$$

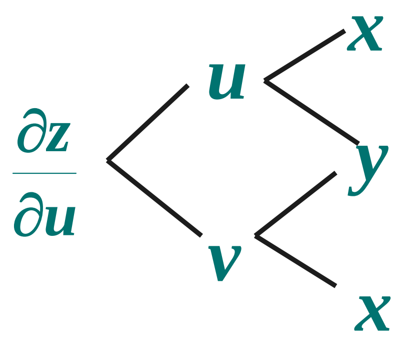
$$u = xy,\quad v = \frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x,\quad \frac{\partial v}{\partial y} = -\frac{x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right)$$

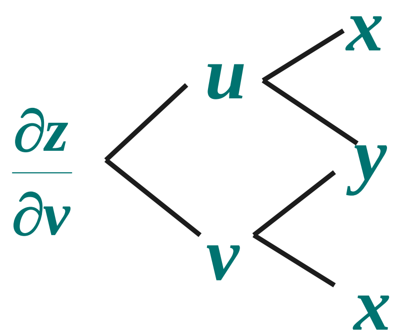
$$= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial y}\left(\frac{1}{y^2}\frac{\partial z}{\partial v}\right)$$

$$= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial y}\left(\frac{1}{y^2}\frac{\partial z}{\partial v}\right)$$



$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial y}\right]$$

$$- x\left[-\frac{2}{y^3}\frac{\partial z}{\partial v} + \frac{1}{y^2}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\right]$$



$$= x\left[\frac{\partial^2 z}{\partial u^2}\cdot x + \frac{\partial^2 z}{\partial u \partial v}\cdot\left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$- \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 z}{\partial u^2} - \frac{x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$-\frac{x}{y^2}[\frac{\partial^2 z}{\partial v\partial u}\cdot x + \frac{\partial^2 z}{\partial v^2}\cdot(-\frac{x}{y^2})]$$

$\because$ z有二阶连续偏导数

$\therefore$ $z_{uv} = z_{vu}$

$$= x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 4x^2\frac{\partial^2 z}{\partial u\partial v} - \frac{2x}{y}\frac{\partial z}{\partial v}$$

$$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}$$

$$=4x^2\frac{\partial^2 z}{\partial u\partial v}-\frac{2x}{y}\frac{\partial z}{\partial v}$$

$$=4uv\frac{\partial^2 z}{\partial u\partial v}-2v\frac{\partial z}{\partial v}$$

$$u=xy,\quad v=\frac{x}{y}$$

$$x^2=uv,$$

$\therefore$ 当$v\neq 0$时，原方程可化为：

$$2u\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial z}{\partial v}=0.$$

题型2　已知函数偏导数，求函数.

例5　设 $f(u)$在$(0,+\infty)$内具有二阶导数，且

$z=f(\sqrt{x^2+y^2})$ 满足等式

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

复合函数求(偏)导、微分方程求解

(1) 验证 $f''(u)+\dfrac{f'(u)}{u}=0$;

(2) 若 $f(1)=0$，$f'(1)=1$，求$f(u)$的表达式.

证 (1) $\dfrac{\partial z}{\partial x} = f'(u)\cdot\dfrac{\partial u}{\partial x}$

$$= f'(u)\cdot\frac{x}{\sqrt{x^2+y^2}}$$

$$z = f(u)$$

$$u\sqrt{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[f'(u)\frac{x}{\sqrt{x^2+y^2}}\right]$$

$$= \frac{\partial f'(u)}{\partial x}\cdot\frac{x}{\sqrt{x^2+y^2}} + f'(u)\cdot\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= f''(u)\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + f'(u)\cdot\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(u)\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + f'(u)\cdot\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

由x, y的轮换对称性，知

$$\frac{\partial^2 z}{\partial y^2} = f''(u)\cdot\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + f'(u)\cdot\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

于是 $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f''(u) + f'(u)\cdot\frac{1}{(x^2+y^2)^{\frac{1}{2}}} = 0$$

即 $$f''(u) + \frac{1}{u} f'(u) = 0$$

(2) 若 $f(1)=0$, $f'(1)=1$, 求$f(u)$的表达式.

解 $f''(u)+\frac{1}{u}f'(u)=0$ 属于$y''=f(x,y')$型,

令 $p=f'(u)$, 则 $\frac{\mathrm{d}p}{\mathrm{d}u}+\frac{1}{u}p=0$,

$\int\frac{\mathrm{d}p}{p}=\int-\frac{1}{u}\mathrm{d}u$, $\ln p=-\ln u+\ln C_1$

$f'(u)=p=\frac{C_1}{u}$. 由 $f'(1)=1$, 得 $C_1=1$

$\therefore$ $f'(u)=\frac{1}{u}$, $f(u)=\ln u+C_2$,

由 $f(1)=0$, 得 $C_2=0$. $\therefore$ $f(u)=\ln u$.

题型3　利用公式求全微分.

例6　设 $f(u)$可微，且 $f'(0)=\frac{1}{2}$，则

$z=f(4x^2-y^2)$在点(1,2)处的全微分

$\mathrm{d}z\big|_{(1,2)}=\underline{4\mathrm{d}x-2\mathrm{d}y}$.

解　$\mathrm{d}z\big|_{(1,2)}=\left(\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y\right)\Big|_{(1,2)}$

$$\overset{或}{=}\mathrm{d}f(4x^2-y^2)\big|_{(1,2)}$$

目　回　上　下　停

$$= f'(4x^2 - y^2)\Big|_{(1,2)} \cdot \mathrm{d}(4x^2 - y^2)\Big|_{(1,2)}$$

$$= f'(0) \cdot (8x\,\mathrm{d}x - 2y\,\mathrm{d}y)\Big|_{(1,2)}$$

$$= \frac{1}{2} \cdot (8\,\mathrm{d}x - 4\,\mathrm{d}y)$$

$$= 4\mathrm{d}x - 2\mathrm{d}y.$$

题型4　隐函数求导.

例7　设 $z=z(x,y)$ 由方程:

$$x-az=f(y-bz) \qquad (1)$$

$(a,b$为非零常数$)$所确定，证明:

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=1,\ \text{且求}\ \frac{\partial^2 z}{\partial x^2}.$$

证法1　**(公式法)**

$$\text{令}\ F(x,y,z)=f(y-bz)-x+az$$

$$\text{则}\ (1)\Leftrightarrow F(x,y,z)=0$$

$$F_x=-1,\ F_y=f',\quad F_z=f'\cdot(-b)+a,$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(-1)}{-bf' + a}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{f'}{-bf' + a}$$

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \frac{a}{-bf' + a} + \frac{b\cdot(-f')}{-bf' + a} = 1$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{a - bf'}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{a - bf'}\right)$$

$$= \frac{-1}{(a - bf')^2} \cdot [-bf'' \cdot \frac{\partial}{\partial x}(y - bz)]$$

$$= \frac{-1}{(a - bf')^2} \cdot [-bf'' \cdot (-b\frac{\partial z}{\partial x})]$$

$$= -\frac{b^2 f''}{(a - bf')^3}.$$

证法2 (复合函数链导法)

$$x - az(x,y) \equiv f[y - bz(x,y)]$$

$$\text{“}\frac{\partial}{\partial x}\text{”}: \quad 1 - a\frac{\partial z}{\partial x} = f' \cdot (-b\frac{\partial z}{\partial x})$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{a - bf'}$$

类似地，可求得 $\dfrac{\partial z}{\partial x} = \dfrac{-f'}{a - bf'}$.

证法3 (全微分形式不变性)

$$x - az = f(y - bz)$$

$$\because \quad d(x - az) = df(y - bz)$$

$$dx - adz = f' \cdot d(y - bz)$$

$$= f' \cdot (dy - bdz)$$

$$(a - bf')dz = dx - f'dy$$

$$dz = \boxed{\frac{1}{a - bf'}}dx \boxed{-\frac{f'}{a - bf'}}dy$$

$$\frac{\partial z}{\partial x} \qquad \frac{\partial z}{\partial y}$$

类似题

设 $z=z(x,y)$是由方程

$$x^2+y^2-z=\varphi(x+y+z)$$

所确定的函数，其中 φ具有二阶导数，且 $\varphi'\neq-1.$

(1) 求 $\mathrm{d}z$;

(2) 记 $u(x,y)=\dfrac{1}{x-y}(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y})$, 求 $\dfrac{\partial u}{\partial x}$.

2. 方向导数、梯度、散度与旋度

$$u = u(x, y, z)$$

$$\vec{A} = (P, Q, R)$$

梯度

$$grad\,u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}$$

散度

$$div\vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\parallel$$

旋度

$$\boxed{rot\vec{A} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}}$$

例8 设 $u(x,y,z)=1+\frac{x^2}{6}+\frac{y^2}{12}+\frac{z^2}{18}$,

单位向量 $\vec{n}=\frac{1}{\sqrt{3}}(1,1,1)$, 则 $\left.\frac{\partial u}{\partial n}\right|_{(1,2,3)}=$ $\underline{\quad\frac{1}{\sqrt{3}}\quad}$.

解 $\frac{\partial u}{\partial x}=\frac{x}{3}$, $\frac{\partial u}{\partial y}=\frac{y}{6}$, $\frac{\partial u}{\partial z}=\frac{z}{9}$.

$$\operatorname{grad}u\big|_{(1,2,3)}=\left.\left(\frac{x}{3},\frac{y}{6},\frac{z}{9}\right)\right|_{(1,2,3)}=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

$$\therefore\quad \left.\frac{\partial u}{\partial n}\right|_{(1,2,3)}=\operatorname{grad}u\cdot\vec{n}\big|_{(1,2,3)}=\frac{1}{\sqrt{3}}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)=\frac{1}{\sqrt{3}}.$$

3. 几何应用

例9 曲线 $\begin{cases} x^2+y^2+z^2=6 \\ x^2+y^2-z^2=4 \end{cases}$ 在点(2,1,1)处的切线与y轴的夹角余弦是 $\underline{\pm\dfrac{2}{\sqrt{5}}}$.

解 切向量：

$$\vec{T}=\pm\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2x & 2y & 2z \\ 2x & 2y & -2z \end{vmatrix}_{(2,1,1)}=\pm 8(-1,2,0)$$

$$\vec{e}_{\vec{T}}=\pm\left(-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}},0\right)$$

例10 曲面 $z=x^2+y^2$ 与平面 $2x+4y+z=0$ 平行的切平面的方程是 $\underline{2x+4y+z+5=0}$.

解 曲面 $z=x^2+y^2$ 的法向量：

$$\vec{n}=(2x,2y,-1)$$

$F(x,y,z)=0$ 的法向量：$\vec{n}=(F_x,F_y,F_z)$

所给平面的法向量：$\vec{n}_1=(2,4,1)$

依题设，知 $\vec{n}\,//\,\vec{n}_1$， 且 $z=x^2+y^2$

$$\therefore \quad \frac{2x}{2}=\frac{2y}{4}=\frac{-1}{1} \quad 且\ z=x^2+y^2$$

$\therefore$ 切点为 $(-1,-2,5)$，切平面方程为：

$$2(x+1)+4(y+2)+1\cdot(z-5)=0$$

例11 过直线 L: $\begin{cases} 10x + 2y - 2z = 27 \\ x + y - z = 0 \end{cases}$,

作曲面Σ: $3x^2 + y^2 - z^2 = 27$的切平面，求此切平面.

解 Σ上点$P(x_0, y_0, z_0)$处的法向量:

$$\vec{n} = (F_x, F_y, F_z)_P = (6x, 2y, -2z)_P$$

$$= 2(3x_0, y_0, -z_0)$$

$\therefore$ Σ上点P处的切平面:

$$3x_0(x - x_0) + y_0(y - y_0) - z_0(z - z_0) = 0$$

$\because$ $P \in \Sigma$, $3x_0^2 + y_0^2 - z_0^2 = 27$

$\therefore$ Σ上点P处的切平面：

$$3x_0x + y_0y - z_0z = 27 \qquad (1)$$

另一方面，过直线L的平面束：

$$(10x + 2y - 2z - 27) + \lambda(x + y - z) = 0$$

即 $(10 + \lambda)x + (2 + \lambda)y - (2 + \lambda)z = 27 \qquad (2)$

由于平面(1)是(2)中的某个平面，故

$$\begin{cases} 10 + \lambda = 3x_0 \\ 2 + \lambda = y_0 \\ -(2 + \lambda) = -z_0 \end{cases}，即 \begin{cases} x_0 = \dfrac{\lambda + 10}{3} \\ y_0 = 2 + \lambda \\ z_0 = 2 + \lambda \end{cases},$$

代入Σ的方程，得

$$3\left(\frac{\lambda+10}{3}\right)^2+\cancel{(2+\lambda)^2}-\cancel{(2+\lambda)^2}=27$$

$\lambda_1=-1,\ \lambda_2=-19$

代入(2)，得所求切平面：

$$9x+y-z-27=0$$

及　$9x+17y-17z+27=0.$

例12 设Σ为椭球面 $\dfrac{x^2}{2}+\dfrac{y^2}{2}+z^2=1$的上半部分，

点$P(x,y,z)\in\Sigma,\pi$ 为Σ在点P处的切平面，

$\rho(x,y,z)$为原点到平面 π 的距离，求

$$I=\iint_{\Sigma}\frac{z}{\rho(x,y,z)}dS.$$

解 设(X,Y,Z)为切平面 π上任意一点，则 π的方程：

$$\frac{xX}{2}+\frac{yY}{2}+zZ=1$$

$$\therefore\quad \rho=\frac{|0-1|}{\sqrt{\dfrac{x^2}{4}+\dfrac{y^2}{4}+z^2}}=\left(\frac{x^2}{4}+\frac{y^2}{4}+z^2\right)^{-\frac{1}{2}}$$

由 $z=\sqrt{1-(\frac{x^2}{2}+\frac{y^2}{2})}$, 得

$$dS=\sqrt{1+z_x^2+z_y^2}\,dxdy=\frac{\sqrt{4-x^2-y^2}}{2z}dxdy$$

$$\therefore\quad I=\iint\limits_{\Sigma}\frac{z}{\rho(x,y,z)}dS$$

$$=\iint\limits_{D_{xy}}\sqrt{\frac{x^2}{4}+\frac{y^2}{4}+(1-\frac{x^2}{2}-\frac{y^2}{2})}\cdot\frac{\sqrt{4-x^2-y^2}}{2}dxdy$$

$$=\frac{1}{4}\iint\limits_{D_{xy}}(4-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{4}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{2}}(4-\rho^2)\rho\,\mathrm{d}\rho$$

$$=\frac{3\pi}{2}.$$

例13 选择题

设 $f(x,y)$在点$(0,0)$的某邻域内有定义，且 $f_x(0,0)=3,\ f_y(0,0)=1$,则(C).

$A.\quad \mathrm{d}z\big|_{(0,0)}=3\,\mathrm{d}x+\mathrm{d}y$

$B.$曲面$z=f(x,y)$在点$(0,0,f(0,0))$处的法向量为$(3,1,1)$

$C.$曲线$\begin{cases} z=f(x,y) \\ y=0 \end{cases}$在点$(0,0,f(0,0))$处的切向量为$(1,0,3)$

$D.$曲线$\begin{cases} z=f(x,y) \\ y=0 \end{cases}$在点$(0,0,f(0,0))$处的切向量为$(3,0,1)$

解　1°　可导 $\nRightarrow$ 可微，　$\therefore A$不可取.

2°　$\Sigma: F(x,y,z) = f(x,y) - z = 0,$

法向量：$\vec{n} = (F_x, F_y, F\,)_{(0,0,f(0,0))}$

$$= (f_x, f_y, -1)_{(0,0,f(0,0))} \qquad \therefore \quad B\text{错}$$

$$= (3,1,-1)$$

3° 曲线 $\begin{cases} z = f(x,y) \\ y = 0 \end{cases} \Leftrightarrow \begin{cases} x = x \\ y = 0 \\ z = f(x,0) \end{cases} \qquad \therefore \quad$ 选C.

切向量：$\vec{T} = (\varphi'(x), \psi'(x), \omega'(x))_{(0,0,f(0,0))}$

$$= (1,\ 0,\ f_x(x,0))_{(0,0,f(0,0))} = (1,0,3)$$

4. 极值、最值

题型1　显函数的极值

例14 求 $z = x^3 + y^3 - 3axy$ (a为常数)的极值.

解　1° 求驻点

$$\begin{cases} z_x = 3x^2 - 3ay = 0 & ① \\ z_y = 3y^2 - 3ax = 0 & ② \end{cases}$$

当 $a=0$ 时，有唯一驻点：(0,0)

当 $a \neq 0$ 时，

①－②：

$$(x^2 - y^2) + a(x - y) = 0$$

$$(x - y)(x + y + a) = 0$$

$$\because\ x + y + a \neq 0$$

否则　$x + y + a = 0$

$$z_x = 3[x^2 + a(x + a)]$$

$$= 3(x^2 + ax + a^2) > 0$$

$\therefore \quad x = y$ 代入①，

得 $\quad x^2 - ax = 0,$

$$x = 0, \quad x = a$$

$$\begin{cases} z_x = 3x^2 - 3ay = 0 & ① \\ z_y = 3y^2 - 3ax = 0 & ② \end{cases}$$

有驻点： $(0,0)$, (a,a)

2° 判断 $\quad z_x = 3x^2 - 3ay\ ,\ z_y = 3y^2 - 3ax$

$$A = z_{xx} = 6x,$$

$$B = z_{xy} = -3a,$$

$$C = z_{yy} = 6y,$$

$$\Delta = AC - B^2 = 36xy - 9a^2$$

(1) 当$a \neq 0$时，

<table>
<tr><td>驻点</td><td>$(0,0)$</td><td colspan="2">(a,a)</td></tr>
<tr><td>Δ</td><td>$-9a^2 < 0$</td><td colspan="2">$27a^2 > 0$</td></tr>
<tr><td rowspan="2">A</td><td rowspan="2"></td><td colspan="2">$6a$</td></tr>
<tr><td>$+$
$(a > 0)$</td><td>$-$
$(a < 0)$</td></tr>
<tr><td>$z(x,y)$</td><td>非极值</td><td>极小值</td><td>极大值</td></tr>
</table>

即当$a \neq 0$时，$z = x^3 + y^3 - 3axy$ 在$(0,0)$不取得极值.

当$a > 0$时，$z = x^3 + y^3 - 3axy$ 在(a,a)取得极小值：$z(a,a) = -a^3$;

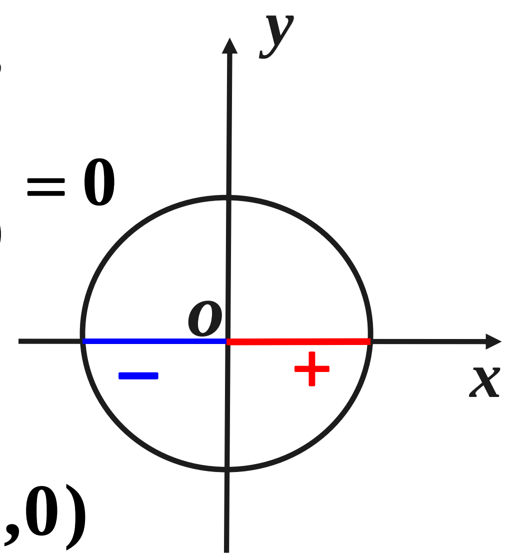
当$a<0$时，$z=x^3+y^3-3axy$ 在(a,a)取

得极大值：$z(a,a)=-a^3$.

(2) 当$a=0$ 时，在唯一驻点(0,0)处，

$$\Delta = AC-B^2=(36xy-9a^2)\Big|_{(0,0)}=0$$

充分判别法失效！



此时，$z=x^3+y^3$，$z(0,0)=0$

当 $x>0$时，$z(x,0)=x^3>0=z(0,0)$

当 $x<0$时，$z(x,0)=x^3<0=z(0,0)$

$\therefore$ (0,0)不是$z=x^3+y^3$的极值点 .

当$a=0$ 时，$z=x^3+y^3-3axy$ 无极值.

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类似题

2012考研

求函数 $f(x,y) = x\mathrm{e}^{-\frac{x^2+y^2}{2}}$ 的极值.

解　1° 求驻点
$$\begin{cases} f_x = (1-x^2)\mathrm{e}^{-\frac{x^2+y^2}{2}} = 0 \\ f_y = -xy\,\mathrm{e}^{-\frac{x^2+y^2}{2}} = 0 \end{cases}$$

驻点：　$(1,0)$, $(-1,0)$

2° 判断
$$A = f_{xx} = x(x^2-3)\mathrm{e}^{-\frac{x^2+y^2}{2}},$$
$$B = f_{xy} = y(x^2-1)\mathrm{e}^{-\frac{x^2+y^2}{2}},$$
$$C = f_{yy} = x(y^2-1)\mathrm{e}^{-\frac{x^2+y^2}{2}},$$

$$\Delta = AC - B^2$$

驻点	$(1,0)$	$(-1,0)$
Δ	$2e^{-1} > 0$	$2e^{-1} > 0$
A	$-2e^{-\frac{1}{2}} < 0$	$2e^{-\frac{1}{2}} > 0$
$f(x,y)$	极大值 $e^{-\frac{1}{2}}$	极小值 $-e^{-\frac{1}{2}}$

$$f(x,y) = x e^{-\frac{x^2+y^2}{2}},$$

$$A = f_{xx} = x(x^2-3)e^{-\frac{x^2+y^2}{2}},$$

$$B = f_{xy} = y(x^2-1)e^{-\frac{x^2+y^2}{2}},$$

$$C = f_{yy} = x(y^2-1)e^{-\frac{x^2+y^2}{2}},$$

例15 若 $f(x,y)=2x^2+ax+xy^2+2y$在点$(1,-1)$处取得极值，则常数 $a=$ $\underline{\quad -5 \quad}$.

解 由可导函数取得极值的 必要条件，得

$$\begin{cases} f_x(1,-1)=(4x+a+y^2)_{(1,-1)} \\ \qquad =5+a=0 \\ f_y(1,-1)=(2xy+2)_{(1,-1)}=0 \end{cases}$$

$$\therefore \quad a=-5$$

题型2　隐函数的极值

例16　求由方程 $x^2+y^2+z^2-2x+2y-4z-10=0$ 确定的函数 $z=f(x,y)$ 的极值.

隐函数求极值问题

解　将方程两边分别对 x, y 求偏导

$$\begin{cases} 2x+2z\cdot z_x-2-4z_x=0 \\ 2y+2z\cdot z_y+2-4z_y=0 \end{cases}$$

$$\begin{cases} z_x=\dfrac{1-x}{z-2} \\ z_y=\dfrac{-1-y}{z-2} \end{cases} \qquad (z\neq 2)$$

令 $z_x = 0, z_y = 0,$ 得 $x = 1, y = -1,$

即驻点为 $P(1, -1)$,

将上方程组再分别对 x, y 求偏导数,

$$A = z_{xx}|_P = \frac{1}{2-z},$$

$$B = z_{xy}|_P = 0,$$

$$C = z_{yy}|_P = \frac{1}{2-z},$$

故 $\Delta = AC - B^2 = \dfrac{1}{(2-z)^2} > 0 \quad (z \neq 2)$,

函数在P有极值.

将$P(1,-1)$代入原方程， 有$z_1 = -2$, $z_2 = 6$,

当$z_1 = -2$时，$A = z_{xx}\big|_{(1,-1,-2)} = \dfrac{1}{2-z}\Big|_{z=-2} = \dfrac{1}{4} > 0$

所以$z = f(1,-1) = -2$为极小值；

当$z_2 = 6$时，$A = -\dfrac{1}{4} < 0$,

所以$z = f(1,-1) = 6$为极大值.

题型3　条件极值

例17　求函数 $u = x^2 + y^2 + z^2$ 在约束条件：

$$z = x^2 + y^2 \text{和} x + y + z = 4$$

下的最大值与最小值．

解　作拉格朗日函数：

$$F(x,y,z,\lambda,\mu) = x^2 + y^2 + z^2 + \lambda(x^2 + y^2 - z) + \mu(x + y + z - 4)$$

令 $$\begin{cases} F_x = 2x + 2\lambda x + \mu = 0 \\ F_y = 2y + 2\lambda y + \mu = 0 \\ F_z = 2z - \lambda + \mu = 0 \\ F_\lambda = x^2 + y^2 - z = 0 \\ F_\mu = x + y + z - 4 = 0 \end{cases}$$

解方程组得

$$(x_1, y_1, z_1) = (1,1,2), (x_2, y_2, z_2) = (-2,-2,8)$$

故所求最大值：$u_{\max} = u(-2,-2,8) = 72;$

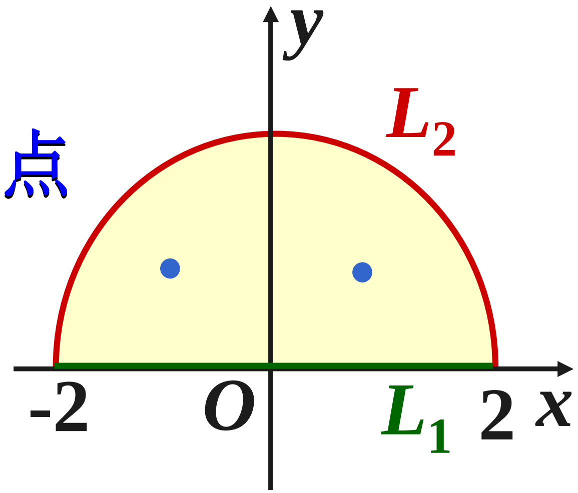
最小值：$u_{\min} = u(1,1,2) = 6.$

题型4　$f(x, y)$在有界闭区域上的最值.

例18　求函数 $f(x, y) = x^2 + 2y^2 - x^2y^2$ 在区域

$$D = \{(x, y) \mid x^2 + y^2 \le 4, y \ge 0\}$$

上的最大值和最小值.



解　1° 先求 $f(x, y)$在D内的驻点

由
$$\begin{cases} f_x = 2x - 2xy^2 = 0, \\ f_y = 4y - 2x^2y = 0. \end{cases}$$

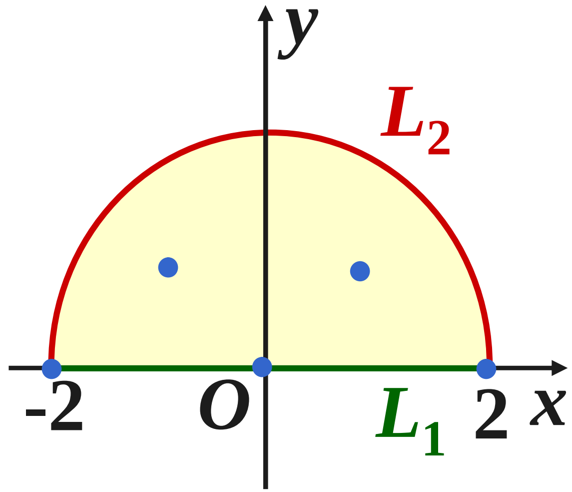
得D内驻点为：$(-\sqrt{2},1), (\sqrt{2},1)$,

且 $f(\pm\sqrt{2},1) = 2.$

2° 再求 $f(x,y)$在D边界上的最值

在边界L_1：$y=0\ (-2\le x\le 2)$上，记

$$g(x)=f(x,0)=x^2$$



在L_1上，$f(x,y)$ 的最大值

为 $g(\pm 2)=f(\pm 2,0)=4$，

最小值为

$$g(0)=f(0,0)=0.$$

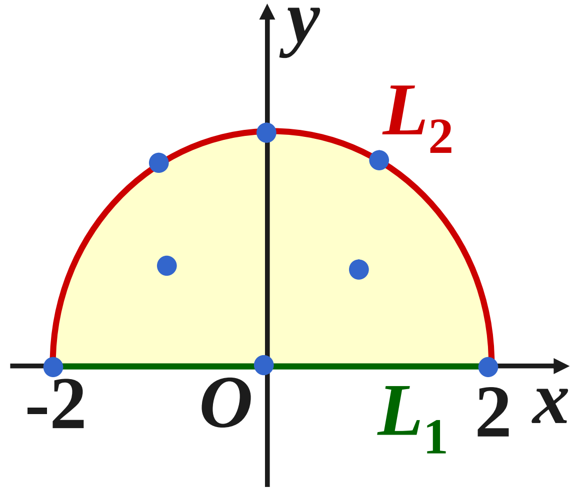
在边界L_2：$x^2+y^2=4(y\ge 0)$上，

构造函数

$$F(x,y,\lambda)=x^2+2y^2-x^2y^2+\lambda(x^2+y^2-4),$$

$$F(x,y,\lambda)=x^2+2y^2-x^2y^2+\lambda(x^2+y^2-4),$$

令 $\begin{cases} F_x = 2x-2xy^2+2\lambda x=0 \\ F_y = 4y-2x^2y+2\lambda y=0 \\ F_\lambda = x^2+y^2-4=0 \end{cases}$



解得极值可疑点: $\begin{cases} x=\pm\sqrt{\dfrac{5}{2}}, \\ y=\sqrt{\dfrac{3}{2}} \end{cases}$ $\begin{cases} x=0, \\ y=2. \end{cases}$

$f(\pm\sqrt{\dfrac{5}{2}},\sqrt{\dfrac{3}{2}})=\dfrac{7}{4},\quad f(0,2)=8$

综上, $f(x,y)$ 在D上的最大值为8，最小值为0.

题型5　二重极限与极值

例19 已知函数 $f(x,y)$在点(0,0)的某邻域内连续，且

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)-xy}{(x^2+y^2)^2}=1,$$ 则(A).

(A) 点(0,0)不是 $f(x,y)$的极值点；

(B) 点(0,0)是 $f(x,y)$的极大值点；

(C) 点(0,0)是 $f(x,y)$的极小值点；

(D) 根据所给条件无法判断　点(0,0)是否是 $f(x,y)$的极值点 .

解 由 $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{f(x,y)-xy}{(x^2+y^2)^2}=1,$

及 $f(x,y)$ 在 $(0,0)$ 处连续，得

$$\lim_{\substack{x\to 0\\ y\to 0}}[f(x,y)-xy]=0,$$

从而 $f(0,0)=\lim\limits_{\substack{x\to 0\\ y\to 0}}f(x,y)=0,$

$$\frac{f(x,y)-xy}{(x^2+y^2)^2}=1+\alpha，\text{其中}\lim_{\substack{x\to 0\\ y\to 0}}\alpha=0.$$

$$f(x,y)=xy+(1+\alpha)(x^2+y^2)^2,$$

其中 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \alpha = 0.$

$\because \quad f(x,x)=x^2+(1+\alpha)\cdot 4x^4$

$$=x^2+o(x^2)$$

$$f(x,-x)=-x^2+o(x^2)$$

$\therefore \quad \exists\delta>0,$ 当$x\in \mathring{U}(0,\delta)$时，有

$$f(x,x)=x^2+o(x^2)>0=f(0,0)$$

$$f(x,-x)=-x^2+o(x^2)<0=f(0,0)$$

题型6　应用题

例20 已知曲线 $C:\begin{cases} x^2+y^2-2z^2=0 \\ x+y+3z=5 \end{cases}$，

求C上距离xOy面最远的点和最近的点．

解 点(x,y,z)到xOy面的距离为：

$$d=|z|$$

故求C上距离xOy面最远点和最近点等价于：

求 $H=d^2=z^2$ 在条件 $x^2+y^2-2z^2=0$ 与

$x+y+3z=5$下的最大值点和最小值点.

令 $F(x,y,z,\lambda,\mu)=z^2+\lambda(x^2+y^2-2z^2)$
$+\mu(x+y+3z-5)$

由 $\begin{cases} F_x=2\lambda x+\mu=0 \\ F_y=2\lambda y+\mu=0 \\ F_z=2z-4\lambda z+3\mu=0 \\ F_\lambda=x^2+y^2-2z^2=0 \\ F_\mu=x+y+3z-5=0 \end{cases}$，解得 $x=y$

从而 $\begin{cases} 2x^2-2z^2=0 \\ 2x+3z=5 \end{cases}$，解得 $\begin{cases} x=-5 \\ y=-5 \\ z=5 \end{cases}$ 或 $\begin{cases} x=1 \\ y=1 \\ z=1 \end{cases}$

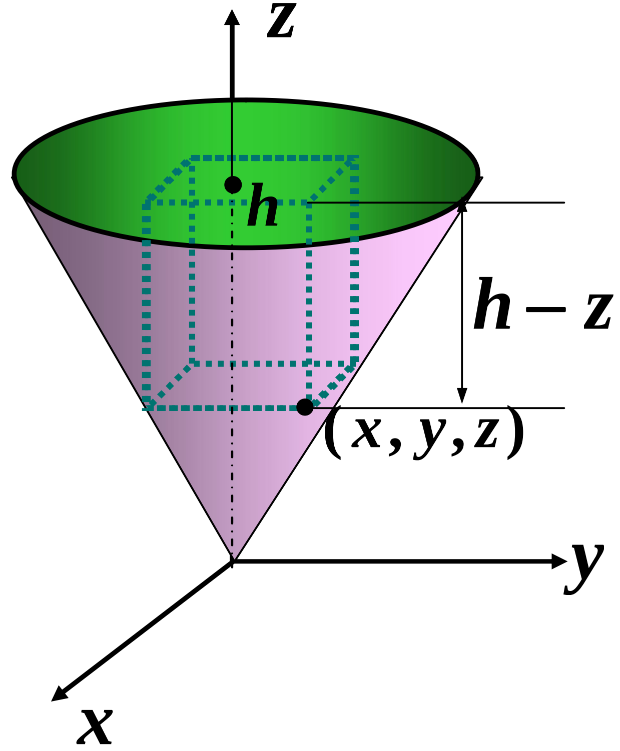
根据几何意义，曲线C上一定存在距离xOy面最远的点和最近的点，故所求点依次为

$$(-5-5,5)\ \text{和}\ (1,1,1).$$

例21 试求在圆锥面 $Rz = h\sqrt{x^2 + y^2}$ 和平面 $z = h$ 所围锥体内作出的底面平 行于 xOy 面的最大长方体体积 $(R > 0, h > 0)$.

解 设长方体位于第一卦限内的一个顶点的坐标为 (x, y, z), 则长方体的长，宽，高分别为 $2x,\ 2y,\ h - z$.

故长方体的体积：



$$V=2x\cdot 2y\cdot(h-z)=4xy\ (h-z),\ \begin{pmatrix}0<x,y<R\\0<z<h\end{pmatrix}$$

目标函数

约束条件： $h\sqrt{x^2+y^2}-Rz=0.$

$$F(x,y,z,\lambda)=xy(h-z)+\lambda(h\sqrt{x^2+y^2}-Rz),$$

$$\begin{cases}F_x=y(h-z)+\lambda\dfrac{hx}{\sqrt{x^2+y^2}}=0, & ①\\[2ex] F_y=x(h-z)+\lambda\dfrac{hy}{\sqrt{x^2+y^2}}=0, & ②\\[2ex] F_z=-xy-\lambda R=0, & ③\\[1ex] F_\lambda=h\sqrt{x^2+y^2}-Rz=0. & ④\end{cases}$$

这种解法具有一般性

①$\cdot y$ - ②$\cdot x$, 得 $y = x$,

代入④得 $z = \dfrac{\sqrt{2}h}{R}x$, 代入③得 $\lambda = -\dfrac{x^2}{R}$.

进一步可解得 $x = y = \dfrac{\sqrt{2}}{3}R, z = \dfrac{2}{3}h.$

由实际问题存在最大值, 及可疑的极值点唯一,有

$$V_{\max} = 4xy(h-z) = 4\cdot\left(\frac{\sqrt{2}}{3}R\right)^2\cdot\frac{1}{3}h = \frac{8}{27}R^2h.$$

例22 在第一卦限内作椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 的切平面，使切平面与三个坐标面所围成的四面体体积最小，求切点坐标.

解 设$P(x_0,y_0,z_0)$为椭球面上一点,

令 $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1,$

则 $F_x|_P=\frac{2x_0}{a^2}$， $F_y|_P=\frac{2y_0}{b^2}$， $F_z|_P=\frac{2z_0}{c^2}$

过$P(x_0,y_0,z_0)$的切平面方程为

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)+\frac{z_0}{c^2}(z-z_0)=0,$$

化简为 $\dfrac{x\cdot x_0}{a^2}+\dfrac{y\cdot y_0}{b^2}+\dfrac{z\cdot z_0}{c^2}=1,$

该切平面在三个轴上的截距各为

$$x=\frac{a^2}{x_0},\quad y=\frac{b^2}{y_0},\quad z=\frac{c^2}{z_0},$$

所围四面体的体积 $V=\dfrac{1}{6}xyz=\dfrac{a^2b^2c^2}{6x_0y_0z_0},$

在条件$\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2}=1$下求 V 的最小值,

$$V=\frac{a^2b^2c^2}{6x_0y_0z_0}\text{最小}$$

$\Leftrightarrow \ln V$最小

$\Leftrightarrow u$最大

令 $u=\ln x_0+\ln y_0+\ln z_0$,

$$G(x_0,y_0,z_0)$$

$$=\ln x_0+\ln y_0+\ln z_0+\lambda\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}-1\right),$$

由 $$\begin{cases} G_{x_0}=0, \quad G_{y_0}=0, \quad G_{z_0}=0 \\ \dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{y_0^2}{c^2}-1=0 \end{cases},$$

即 $\begin{cases} \dfrac{1}{x_0} + \dfrac{2\lambda x_0}{a^2} = 0 \\ \dfrac{1}{y_0} + \dfrac{2\lambda y_0}{b^2} = 0 \\ \dfrac{1}{z_0} + \dfrac{2\lambda z_0}{c^2} = 0 \\ \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} + \dfrac{z_0^2}{c^2} - 1 = 0 \end{cases}$ 可得 $\begin{cases} x_0 = \dfrac{a}{\sqrt{3}} \\ y_0 = \dfrac{b}{\sqrt{3}}, \\ z_0 = \dfrac{c}{\sqrt{3}} \end{cases}$

当切点坐标为

$(\dfrac{a}{\sqrt{3}}, \dfrac{b}{\sqrt{3}}, \dfrac{c}{\sqrt{3}})$时,

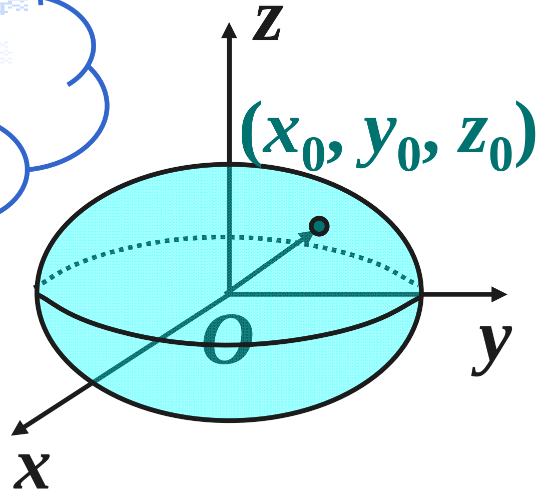
四面体的体积最小$V_{\min} = \dfrac{\sqrt{3}}{2}abc$.

例23 已知力场 $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$，问质点从原点沿直线移到曲面 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ 的第一卦限部分上的哪一点作功最大？并求出最大功.

第二类曲线积分、条件极值

解 目标函数：

功 $W = \int_L yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z$

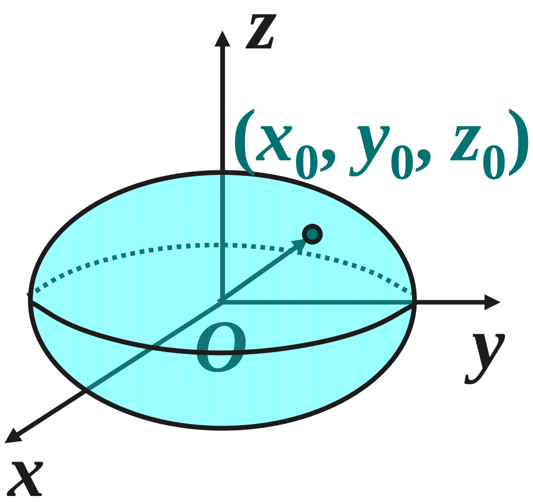


先求W的表达式:

(方法1) 直接法

直线段的参数方程:

$$\begin{cases} x = x_0 t \\ y = y_0 t \\ z = z_0 t \end{cases} \qquad t: 0 \mapsto 1$$



$$\therefore \quad W = \int_L yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z = \int_0^1 (x_0 y_0 z_0) \cdot 3t^2\,\mathrm{d}t$$

$$= (x_0 y_0 z_0)\, t^3 \Big|_0^1 = x_0 y_0 z_0$$

(方法2) 原函数法

$$\because \quad yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\mathrm{d}z = \mathrm{d}(xyz)$$

$$\therefore \quad W = \int_L yz\,\mathrm{d}x + zx\,\mathrm{d}y + xy\,\mathrm{d}z$$

$$= xyz\Big|_{(0,0,0)}^{(x_0,y_0,z_0)} = x_0 y_0 z_0$$

再求$W = xyz$ 在条件: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$下的极值.

作 $F(x,y,z) = xyz + \lambda(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1)$

$$(x \ge 0, y \ge 0, z \ge 0)$$

解 $\begin{cases} F_x = yz + \dfrac{2x}{a^2}\lambda = 0 \\ F_y = xz + \dfrac{2y}{b^2}\lambda = 0 \\ F_z = xy + \dfrac{2z}{c^2}\lambda = 0 \\ F_\lambda = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0 \end{cases}$ 得 $x = \dfrac{a}{\sqrt{3}}, y = \dfrac{b}{\sqrt{3}}, z = \dfrac{c}{\sqrt{3}}$

由于实际问题最大值一定存在，所以

$$W_{\max} = \frac{\sqrt{3}}{9}abc.$$

题型7　不等式证明

例24 当 $x>0, y>0, z>0$ 时，求函数

$$u=\ln x+2\ln y+3\ln z$$

在球面 $x^2+y^2+z^2=6r^2$ 上的最大值，并证明：

对于任何正数 a,b,c，有

$$ab^2c^3\le 108\left(\frac{a+b+c}{6}\right)^6.$$

解(1) 作

$$F(x,y,z)=\ln x+2\ln y+\ln z+\lambda(x^2+y^2+z^2-6r^2)$$

$$\begin{cases} F_x=\dfrac{1}{x}+2x\lambda=0 & (1) \\ F_y=\dfrac{2}{y}+2y\lambda=0 & (2) \\ F_z=\dfrac{3}{z}+2z\lambda=0 & (3) \\ F_\lambda=x^2+y^2+z^2-6r^2=0 & (4) \end{cases}$$

$(1)\times y-(2)\times x:\quad y^2=2x^2$

$\because\ x>0, y>0 \quad \therefore\ y=\sqrt{2}x$

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$(2)\times z-(3)\times y$: $z=\sqrt{3}x$

$$y=\sqrt{2}x$$

代入(4)，得 $x=r$.

$$\boxed{u=\ln x+2\ln y+3\ln z}$$

故有惟一极值可疑点：$(r,\sqrt{2}r,\sqrt{3}r)$

$\because\quad \lim\limits_{x\to 0^+} u=-\infty,\quad \lim\limits_{y\to 0^+} u=-\infty,\quad \lim\limits_{z\to 0^+} u=-\infty$

而 u 在$x>0,\ y>0,\ z>0$内可微

$\therefore$ u 在条件 $x^2+y^2+z^2=6r^2(x>0,\ y>0,\ z>0)$

下的最大值为：

$$M=u(r,\sqrt{2}r,\sqrt{3}r)=\ln(6\sqrt{3}r^6)$$

证(2) 由(1)，知 $\boxed{u = \ln x + 2\ln y + 3\ln z}$

当 $x^2 + y^2 + z^2 = 6r^2 (x > 0, y > 0, z > 0)$时，有

$$u(x, y, z) \le M = u(r, \sqrt{2}r, \sqrt{3}r)$$

即 $\ln xy^2z^3 \le \ln(6\sqrt{3}r^6)$

亦即 $xy^2z^3 \le 6\sqrt{3}r^6 = 6\sqrt{3}\cdot(\dfrac{x^2 + y^2 + z^2}{6})^3$

两边平方，得

$$x^2y^4z^6 \le (6\sqrt{3})^2\cdot(\frac{x^2 + y^2 + z^2}{6})^6$$

即 $x^2\cdot(y^2)^2(z^2)^3 \le 108\cdot(\dfrac{x^2 + y^2 + z^2}{6})^6$

$\therefore \quad \forall\, a>0, b>0, c>0$，令

$$x=\sqrt{a},\ y=\sqrt{b},\ z=\sqrt{c}$$

代入

$$x^2\cdot(y^2)^2(z^2)^3 \le 108\cdot\left(\frac{x^2+y^2+z^2}{6}\right)^6$$

便得

$$ab^2c^3 \le 108\left(\frac{a+b+c}{6}\right)^6.$$

类似题

利用条件极值的方法证明：对于任何正数 a, b, c 下列不等式成立：

$$abc^3 \le 27\left(\frac{a+b+c}{5}\right)^5.$$

分析 $\forall\, a>0, b>0, c>0,$ 令 $d=a+b+c$

若能证明： 在条件

$$x+y+z=d \quad (x,y,z>0)$$

下，有 $f(x,y,z)=xyz^3 \le f\left(\frac{d}{5},\frac{d}{5},\frac{3d}{5}\right)=27\left(\frac{d}{5}\right)^5$

则由于 $a+b+c=d$，特别地，应有

$$f(a,b,c)=abc^3$$

$$\le f(\frac{d}{5},\frac{d}{5},\frac{3d}{5})=27(\frac{d}{5})^5=27(\frac{a+b+c}{5})^5.$$

从而原不等式成立.

证 $\forall\, a>0, b>0, c>0$, 令

$$d=a+b+c$$

考虑 $f(x,y,z)=xyz^3$

在条件: $x+y+z=d \quad (x,y,z>0)$

下的极值问题.

作 $F(x,y,z)=xyz^3+\lambda(x+y+z-d)$

由 $\begin{cases} F_x=yz^3+\lambda=0 \\ F_y=xz^3+\lambda=0 \\ F_z=3xyz^2+\lambda=0 \\ x+y+z=d \end{cases}$，解得

唯一极值可疑点：$x=y=\dfrac{d}{5},\ z=\dfrac{3d}{5}$.

$$f\left(\frac{d}{5},\frac{d}{5},\frac{3d}{5}\right)=\frac{d}{5}\cdot\frac{d}{5}\cdot\left(\frac{3d}{5}\right)^3=27\left(\frac{d}{5}\right)^5$$

$\because \quad f(x,y,z) = xyz^3$在有界闭集$D$:

$$x + y + z = d \quad (x, y, z \geq 0)$$

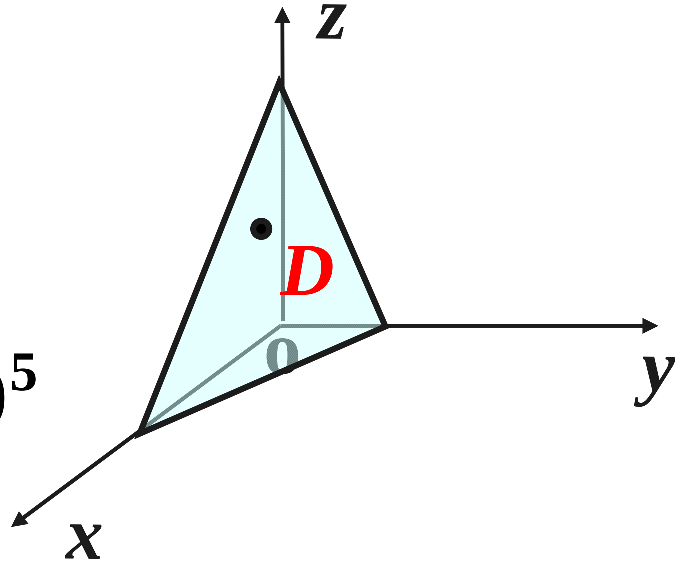
上连续，

$\therefore \quad f(x,y,z)$必在$D$上取得最大值 .

又$\because$ 在D的边界上，$f = 0$

$$\therefore \quad M = \max_{(x,y,z)\in D} f(x,y,z)$$

$$= f\left(\frac{d}{5}, \frac{d}{5}, \frac{3d}{5}\right) = 27\left(\frac{d}{5}\right)^5$$



$\because \quad (a,b,c) \in D$

$\therefore \quad f(a,b,c) \le M = 27\left(\frac{d}{5}\right)^5$

即 $\quad abc^3 \le 27\left(\frac{a+b+c}{5}\right)^5.$

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