

## 第七节 方向导数与梯度

### 习题 8-7

1. 求下列函数在指定点  $M_0$  处沿指定方向  $l$  的方向导数:

(1)  $z = \cos(x+y)$ ,  $M_0(0, \frac{\pi}{2})$ ,  $l = (3, -4)$ ;

(2)  $u = xyz$ ,  $M_0(1, 1, 1)$ ,  $l = (1, 1, 1)$ .

解 (1) 由方向  $l = (3, -4)$  可求出与  $l$  同向的单位向量为

$$e_l = (\frac{3}{5}, -\frac{4}{5}),$$

因为函数可微分, 且

$$\left. \frac{\partial z}{\partial x} \right|_{(0, \frac{\pi}{2})} = -\sin(x+y) \Big|_{(0, \frac{\pi}{2})} = -1, \quad \left. \frac{\partial z}{\partial y} \right|_{(0, \frac{\pi}{2})} = -\sin(x+y) \Big|_{(0, \frac{\pi}{2})} = -1,$$

故所求方向导数为

$$\left. \frac{\partial z}{\partial l} \right|_{(0, \frac{\pi}{2})} = (-1) \cdot \frac{3}{5} + (-1) \cdot (-\frac{4}{5}) = \frac{1}{5}.$$

(2) 函数  $u = xyz$  在平面上处处可微, 则

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma,$$

因为  $\frac{\partial u}{\partial x} = yz$ ,  $\frac{\partial u}{\partial y} = xz$ ,  $\frac{\partial u}{\partial z} = xy$ , 所以在点  $(1, 1, 1)$  处,  $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 1$ .

由  $l = (1, 1, 1)$  有  $|l| = \sqrt{3}$ , 于是

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{1}{\sqrt{3}},$$

故所求方向导数为

$$\left. \frac{\partial u}{\partial l} \right|_{(1, 1, 1)} = 1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

2. 求函数  $z = \ln(x+y)$  在抛物线  $y^2 = 4x$  上点  $(1, 2)$  处, 沿着这抛物线在该点处与  $x$  轴正向夹角为锐角的切线方向的方向导数.

解 先求切线斜率: 在  $y^2 = 4x$  两边分别求导, 得  $2y \frac{dy}{dx} = 4$ ,

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于是  $\frac{dy}{dx} = \frac{2}{y}$ , 斜率  $k = \frac{dy}{dx}\big|_{(1,2)} = 1$ .

切线方向为  $\boldsymbol{l} = (1, 1)$ , 与  $\boldsymbol{l}$  同向的单位向量为  $\boldsymbol{e}_l = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , 又因为

$$\frac{\partial z}{\partial x}\big|_{(1,2)} = \frac{1}{x+y}\big|_{(1,2)} = \frac{1}{3}, \quad \frac{\partial z}{\partial y}\big|_{(1,2)} = \frac{1}{x+y}\big|_{(1,2)} = \frac{1}{3},$$

所以

$$\frac{\partial z}{\partial l}\big|_{(1,2)} = \frac{1}{3} \cdot \frac{\sqrt{2}}{2} + \frac{1}{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{3}.$$

3. 设  $f(x, y)$  具有一阶连续的偏导数, 已给四点  $A(1, 3)$ ,  $B(3, 3)$ ,  $C(1, 7)$ ,  $D(6, 15)$ , 若  $f(x, y)$  在点  $A$  处沿  $\overline{AB}$  方向的方向导数等于 3, 而沿  $\overline{AC}$  方向的方向导数等于 26, 求  $f(x, y)$  在点  $A$  处沿  $\overline{AD}$  方向的方向导数.

**解** 根据题意可求得方向  $\overline{AB} = (2, 0)$ , 与  $\overline{AB}$  同向的单位向量为  $\boldsymbol{e}_{\overline{AB}} = (1, 0)$ , 则有

$$\frac{\partial f(x, y)}{\partial \overline{AB}}\big|_{(1,3)} = f'_x(1, 3) \cdot 1 + f'_y(1, 3) \cdot 0 = f'_x(1, 3) = 3,$$

又因为方向  $\overline{AC} = (0, 4)$ , 与  $\overline{AC}$  同向的单位向量为  $\boldsymbol{e}_{\overline{AC}} = (0, 1)$ ,

则有

$$\frac{\partial f(x, y)}{\partial \overline{AC}}\big|_{(1,3)} = f'_x(1, 3) \cdot 0 + f'_y(1, 3) \cdot 1 = f'_y(1, 3) = 26,$$

而方向  $\overline{AD} = (5, 12)$ , 与  $\overline{AD}$  同向的单位向量为

$$\boldsymbol{e}_{\overline{AD}} = \left(\frac{5}{\sqrt{5^2 + 12^2}}, \frac{12}{\sqrt{5^2 + 12^2}}\right) = \left(\frac{5}{13}, \frac{12}{13}\right),$$

所以

$$\frac{\partial f(x, y)}{\partial \overline{AD}}\big|_{(1,3)} = f'_x(1, 3) \cdot \frac{5}{13} + f'_y(1, 3) \cdot \frac{12}{13} = 3 \cdot \frac{5}{13} + 26 \cdot \frac{12}{13} = \frac{327}{13}.$$

4. 设  $z = f(x, y) = \sqrt[3]{xy}$ , 证明函数  $f$  在原点  $O(0, 0)$  连续, 且  $f_x(0, 0)$  与  $f_y(0, 0)$  都存在, 但  $f$  在原点沿着向量  $\boldsymbol{l} = (a, b)$  方向的方向导数不存在(其中  $a, b$  为任意非零常数).

**证** 函数  $z = f(x, y) = \sqrt[3]{xy}$  在点  $(0, 0)$  的邻域有定义, 且

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} z = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \sqrt[3]{xy} = 0 = f(0,0),$$

故函数  $f$  在原点  $O(0,0)$  处连续. 又

$$f_x(0,0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0-0}{\Delta x} = 0,$$

同理

$$f_y(0,0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0-0}{\Delta y} = 0,$$

所以  $f_x(0,0)$  与  $f_y(0,0)$  都存在.

而函数  $f$  在原点  $O(0,0)$  沿方向  $\boldsymbol{l}$  的方向导数为

$$\left. \frac{\partial f}{\partial l} \right|_{(0,0)} = \lim_{\rho \rightarrow 0} \frac{f(0 + \Delta x, 0 + \Delta y) - f(0,0)}{\rho} = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\sqrt[3]{\Delta x \Delta y}}{\sqrt{\Delta x^2 + \Delta y^2}},$$

让点  $(\Delta x, \Delta y)$  沿直线  $\Delta y = \Delta x$  趋于点  $(0,0)$ , 即  $\Delta y = \Delta x \rightarrow 0$ , 得

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y = \Delta x \rightarrow 0}} \frac{\sqrt[3]{\Delta x \Delta y}}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^{\frac{2}{3}}}{\sqrt{2} \cdot \Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{2}(\Delta x)^{\frac{1}{3}}} \text{ 不存在.}$$

即  $f$  在  $(0,0)$  点沿方向  $\boldsymbol{l}$  的方向导数不存在.

**注意** 方向导数是沿任意指定方向的变化率, 偏导数是沿坐标轴方向的变化率, 故可将方向导数看作偏导数的推广. 函数在某点处的偏导数都存在, 并不意味着函数在该点处沿任一方向  $\boldsymbol{l}$  的方向导数也存在, 但是如果函数在该点处是可微的, 则函数在该点沿任一方向  $\boldsymbol{l}$  的方向导数都存在.

5. 求函数  $u = x^2 + y^2 + z^2$  在曲线  $x = t, y = t^2, z = t^3$  上点  $(1,1,1)$  处, 沿曲线在该点的切线正方向 (对应于  $t$  增大的方向) 的方向导数.

**解** 先求曲线在给定点的切线方向. 因为

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 3t^2,$$

所以曲线在点  $(1,1,1)$  处的切线的方向向量为  $\boldsymbol{T} = (1, 2, 3)$ , 与  $\boldsymbol{T}$  同向的单位向量为

$$\boldsymbol{e}_T = \left( \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right),$$

又因为

$$\left. \frac{\partial u}{\partial x} \right|_{(1,1,1)} = \left. \frac{\partial u}{\partial y} \right|_{(1,1,1)} = \left. \frac{\partial u}{\partial z} \right|_{(1,1,1)} = 2,$$

所以

$$\left. \frac{\partial u}{\partial T} \right|_{(1,1,1)} = 2 \cdot \frac{1}{\sqrt{14}} + 2 \cdot \frac{2}{\sqrt{14}} + 2 \cdot \frac{3}{\sqrt{14}} = \frac{6}{\sqrt{14}}.$$

6. 求函数  $u = x + y + z$  在球面  $x^2 + y^2 + z^2 = 1$  上点  $(x_0, y_0, z_0)$  处, 沿球面在该点的外法线方向的方向导数.

解 设  $F(x, y, z) = x^2 + y^2 + z^2 - 1$ , 则

$$F_x = 2x, F_y = 2y, F_z = 2z,$$

于是球面在  $(x_0, y_0, z_0)$  处的外法线方向向量可取为

$$\boldsymbol{l} = (F_x, F_y, F_z) \Big|_{(x_0, y_0, z_0)} = (2x_0, 2y_0, 2z_0),$$

$\boldsymbol{l}$  的方向余弦为

$$\cos \alpha = \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}, \cos \beta = \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}},$$

$$\cos \gamma = \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}},$$

又因为

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 1,$$

所以

$$\begin{aligned} \frac{\partial u}{\partial l} \Big|_{(x_0, y_0, z_0)} &= \left( \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma \right) \Big|_{(x_0, y_0, z_0)} \\ &= 1 \cdot \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} + 1 \cdot \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} + 1 \cdot \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} \\ &= \frac{x_0 + y_0 + z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}. \end{aligned}$$

注意到点  $(x_0, y_0, z_0)$  在球面  $x^2 + y^2 + z^2 = 1$  上, 有  $x_0^2 + y_0^2 + z_0^2 = 1$ , 故

$$\frac{\partial u}{\partial l} \Big|_{(x_0, y_0, z_0)} = x_0 + y_0 + z_0.$$

7. 求函数  $u = x^3 + y^3 + z^3 - 3xyz$  的梯度, 并问在何点处其梯度:

(1) 垂直于  $z$  轴; (2) 平行于  $z$  轴; (3) 等于零向量.

解 因为  $\frac{\partial u}{\partial x} = 3x^2 - 3yz$ ,  $\frac{\partial u}{\partial y} = 3y^2 - 3xz$ ,  $\frac{\partial u}{\partial z} = 3z^2 - 3xy$ ,

所以

$$\operatorname{grad} u = (3x^2 - 3yz, 3y^2 - 3xz, 3z^2 - 3xy).$$

取  $z$  轴的方向向量为  $\boldsymbol{l} = (0, 0, 1)$ ,

(1) 由于梯度垂直于  $z$  轴, 则

$$\boldsymbol{l} \cdot \operatorname{grad} u = 0, \text{ 即 } (0, 0, 1) \cdot (3x^2 - 3yz, 3y^2 - 3xz, 3z^2 - 3xy) = 0,$$

于是有

$$3z^2 - 3xy = 0, \text{ 即 } z^2 = xy,$$

所以曲面  $z^2 = xy$  上的点梯度垂直于  $z$  轴.

(2) 由于梯度平行于  $z$  轴, 则

$$\boldsymbol{l} // \operatorname{grad} u, \text{ 可得 } \frac{3x^2 - 3yz}{0} = \frac{3y^2 - 3xz}{0} = \frac{3z^2 - 3xy}{1}, \text{ 于是有}$$

$$\begin{cases} 3x^2 - 3yz = 0, \\ 3y^2 - 3xz = 0, \end{cases} \quad \text{即} \quad \begin{cases} x^2 = yz, \\ y^2 = xz, \end{cases}$$

所以曲线  $\begin{cases} x^2 = yz, \\ y^2 = xz, \end{cases}$  上的点梯度平行于  $z$  轴.

(3) 由  $\operatorname{grad} u = (3x^2 - 3yz, 3y^2 - 3xz, 3z^2 - 3xy) = 0$ , 有

$$3x^2 - 3yz = 3y^2 - 3xz = 3z^2 - 3xy = 0,$$

即

$$x = y = z,$$

所以直线  $x = y = z$  上的点梯度等于零向量.

8. 已知  $u = x^2 + y^2 + z^2 - xy + yz$ , 点  $P_0 = (1, 1, 1)$ . 求  $u$  在点  $P_0$  处的方向导数  $\frac{\partial u}{\partial l}$  的最大、最小值, 并指出相应的方向  $\boldsymbol{l}$ , 再指出沿什么方向, 其方向导数为零.

解  $\frac{\partial u}{\partial x} = 2x - y, \frac{\partial u}{\partial y} = 2y - x + z, \frac{\partial u}{\partial z} = 2z + y$ , 于是

$$\left. \frac{\partial u}{\partial x} \right|_{(1,1,1)} = 1, \quad \left. \frac{\partial u}{\partial y} \right|_{(1,1,1)} = 2, \quad \left. \frac{\partial u}{\partial z} \right|_{(1,1,1)} = 3,$$

所以

$$\operatorname{grad} u_{(1,1,1)} = (1, 2, 3).$$

因为函数  $u = x^2 + y^2 + z^2 - xy + yz$  在点  $P_0(1, 1, 1)$  处可微分,

$$\mathbf{e}_l = (\cos \alpha, \cos \beta, \cos \gamma),$$

是与方向  $\mathbf{l}$  同向的单位向量, 则

$$\begin{aligned} \frac{\partial u}{\partial l} \Big|_{(1,1,1)} &= \frac{\partial u}{\partial x} \Big|_{(1,1,1)} \cos \alpha + \frac{\partial u}{\partial y} \Big|_{(1,1,1)} \cos \beta + \frac{\partial u}{\partial z} \Big|_{(1,1,1)} \cos \gamma \\ &= 1 \cdot \cos \alpha + 2 \cdot \cos \beta + 3 \cdot \cos \gamma \\ &= \operatorname{grad} u(1,1,1) \cdot \mathbf{e}_l = |\operatorname{grad} u(1,1,1)| \cos \theta, \end{aligned}$$

其中  $\theta = (\operatorname{grad} u(1,1,1), \mathbf{e}_l)$ .

由此可知

当向量  $\mathbf{e}_l$  与  $\operatorname{grad} u(1,1,1)$  的夹角  $\theta = 0$ , 即沿梯度方向

$$\mathbf{l} = \operatorname{grad} u(1,1,1) = (1, 2, 3),$$

方向导数最大, 这个最大值为  $|\operatorname{grad} u(1,1,1)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ ;

当向量  $\mathbf{e}_l$  与  $\operatorname{grad} u(1,1,1)$  的夹角  $\theta = \pi$ , 即沿方向

$$-\mathbf{l} = -\operatorname{grad} u(1,1,1) = (-1, -2, -3),$$

方向导数最小, 这个最小值为  $-|\operatorname{grad} u(1,1,1)| = -\sqrt{14}$ ;

当向量  $\mathbf{e}_l$  与  $\operatorname{grad} u(1,1,1)$  的夹角  $\theta = \frac{\pi}{2}$ , 即垂直于  $\mathbf{l} = (1, 2, 3)$  的方向, 方向导数为零.

9. 设一金属球体内各点处的温度与该点离球心的距离成反比, 证明: 球体内任意(异于球心的)一点处沿着指向球心的方向温度上升得最快.

证 设  $p(x, y, z)$  为球体内任意一点,  $p_0(x_0, y_0, z_0)$  为球心坐标,  $T$  为球体内该点的温度, 则

$$T = \frac{k}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \quad (k \text{ 为常数}),$$

$$\frac{\partial T}{\partial x} = \frac{-k(x-x_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}},$$

$$\frac{\partial T}{\partial y} = \frac{-k(y-y_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}},$$

$$\frac{\partial T}{\partial z} = \frac{-k(z-z_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}},$$

温度  $T$  在  $p$  点处的梯度方向, 就是温度上升得最快的方向,

$$\begin{aligned} & \text{grad} T \Big|_p \\ &= \left( \frac{\partial T}{\partial x} \Big|_p, \frac{\partial T}{\partial y} \Big|_p, \frac{\partial T}{\partial z} \Big|_p \right) \\ &= \left( \frac{-k(x-x_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}}, \frac{-k(y-y_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}}, \right. \\ & \quad \left. \frac{-k(z-z_0)}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}} \right) \\ &= \frac{-k}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{3}{2}}} (x-x_0, y-y_0, z-z_0), \end{aligned}$$

即球体内任意 (异于球心的) 点  $p(x, y, z)$  处沿着指向球心  $p_0(x_0, y_0, z_0)$  的方向温度上升得最快.

10. 设  $u(x, y), v(x, y)$  都具有一阶连续偏导数, 证明:

(1)  $\text{grad}(u+v) = \text{grad} u + \text{grad} v$ ;

(2)  $\text{grad}(uv) = v \text{grad} u + u \text{grad} v$ ;

(3)  $\text{grad}\left(\frac{u}{v}\right) = \frac{v \text{grad} u - u \text{grad} v}{v^2}$ ;

(4)  $\text{grad} f(u) = f'(u) \text{grad} u$  (设  $f'(u)$  连续).

证 (1)  $\text{grad}(u+v) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \right)$   
 $= \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) + \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right)$   
 $= \text{grad} u + \text{grad} v.$

(2)  $\text{grad}(uv) = \left( \frac{\partial}{\partial x}(uv), \frac{\partial}{\partial y}(uv), \frac{\partial}{\partial z}(uv) \right)$   
 $= \left( \frac{\partial u}{\partial x} v + u \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} v + u \frac{\partial v}{\partial y}, \frac{\partial u}{\partial z} v + u \frac{\partial v}{\partial z} \right)$   
 $= v \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) + u \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right)$   
 $= v \text{grad} u + u \text{grad} v$

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$$\begin{aligned}
(3) \quad \text{grad}\left(\frac{u}{v}\right) &= \left(\frac{\partial}{\partial x}\left(\frac{u}{v}\right), \frac{\partial}{\partial y}\left(\frac{u}{v}\right), \frac{\partial}{\partial z}\left(\frac{u}{v}\right)\right) \\
&= \left(\frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}, \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}, \frac{v\frac{\partial u}{\partial z} - u\frac{\partial v}{\partial z}}{v^2}\right) \\
&= \frac{v\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) - u\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right)}{v^2} \\
&= \frac{v\text{grad}u - u\text{grad}v}{v^2}.
\end{aligned}$$

$$\begin{aligned}
(4) \quad \text{grad}f(u) &= \left(\frac{\partial}{\partial x}f(u), \frac{\partial}{\partial y}f(u), \frac{\partial}{\partial z}f(u)\right) \\
&= \left(f'(u)\frac{\partial u}{\partial x}, f'(u)\frac{\partial u}{\partial y}, f'(u)\frac{\partial u}{\partial z}\right) \\
&= f'(u)\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = f'(u)\text{grad}u.
\end{aligned}$$