第五节

隐函数的微分法

- 一、主要内容
- 二、典型例题
- 三、同步练习
- 四、同步练习解答

一、主要内容

（一）由一个方程确定的隐函数的微分法

1. $F(x, y) = 0$

问题的提出：$F(x, y) = 0 \xrightarrow{?} y = f(x)$

例如，方程 $x^2 + \sqrt{y} + C = 0$

当 $C < 0$ 时，能确定隐函数；

当 $C > 0$ 时，不能确定隐函数；

问题1. 在何种条件下，能确定一个隐函数？

在方程（或方程组）能确定隐函数时，即

$$F(x,y)=0 \longrightarrow y=f(x)$$

$$F(x,f(x))\equiv 0,\quad x\in I$$

问题2. 在何种条件下，$f'(x)$ 存在？

求导方法？ 求导公式？

$$\frac{\mathrm{d}y}{\mathrm{d}x}=?$$

定理8.7 设函数 $F(x,y)$在点(x_0,y_0)的某邻域内满足

① 具有连续偏导数;

② $F(x_0\,,\,y_0)=0$;

③ $F_y(x_0\,,\,y_0)\neq 0$

注意公式里的负号

则方程$F(x,y)=0$在点(x_0,y_0)的某邻域内能唯一确定一个函数$y=f(x)$，满足条件 $y_0=f(x_0)$，并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

—— 隐函数求导公式

若$F(x,y)$的二阶偏导数也都连续，则还有

二阶导数：

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = = -\frac{F_{xx}F_y{}^2 - 2F_{xy}F_xF_y + F_{yy}F_x{}^2}{F_y^3}$$

求二阶导数时，要注意y是x的函数！

2. $F(x,y,z)=0$

定理8.8 若 $F(x,y,z)$ 满足:

① 在点 (x_0,y_0,z_0) 的某邻域内具有连续偏导数,

② $F(x_0,y_0,z_0)=0$

③ $F_z(x_0,y_0,z_0)\neq 0$

则方程 $F(x,y,z)=0$ 在点 (x_0,y_0,z_0) 的某一邻域内可唯一确定一个函数 $z=f(x,y)$ 满足 $z_0=f(x_0,y_0)$,

并有连续偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

注意公式里的负号

注 在公式 $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ 中，

F_x：将 $F(x,y,z)$中的y,z暂视为常数，

对x求偏导数;

F_z：将 $F(x,y,z)$中的x,y暂视为常数，

对z求偏导数;

(二) 由方程组确定的隐函数微分法

以两个方程确定两个隐函数的情况为例，即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

由 函数F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为函数F、G 的雅可比(Jacobi)行列式.

定理8.9 设函数 $F(x,y,u,v), G(x,y,u,v)$ 满足:

① 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$;

③ $J\Big|_P=\dfrac{\partial(F,G)}{\partial(u,v)}\Big|_P\neq 0$

则方程组 $F(x,y,u,v)=0$, $G(x,y,u,v)=0$

在点(x_0,y_0,u_0,v_0) 的某一邻域内能唯一确定

一对满足条件 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$,

具有连续偏导数的函数

$$u = u(x, y),\ v = v(x, y),$$

且有

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(\underline{x}, v)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(\underline{y}, v)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{x})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{y})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}$$

注　情形二的特例：若方程组

$$\begin{cases}F(x,\ y,z)=0,\\ G(x,y,z)=0\end{cases}$$

满足定理8.9的条件,则

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)},$$

$$\frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,x)}.$$

函数个数=方程个数;

自变量个数=方程组所含变量个数–方程个数

二、典型例题

例1 已知 $\ln\sqrt{x^2+y^2}=\arctan\dfrac{y}{x}$，求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ 及 $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

解(方法1) 公式法

令 $$F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\frac{y}{x}$$

$$=\frac{1}{2}\ln(x^2+y^2)-\arctan\frac{y}{x},$$

则 $$F_x(x,y)=\frac{1}{2}\cdot\frac{2x}{x^2+y^2}-\frac{-\dfrac{y}{x^2}}{1+(\dfrac{y}{x})^2}=\frac{x+y}{x^2+y^2},$$

西北工业大学 NORTHWESTERN POLYTECHNICAL UNIVERSITY

$$F_x(x,y)=\frac{x+y}{x^2+y^2},\quad F_y(x,y)=\frac{y-x}{x^2+y^2},$$

$$\therefore\quad \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=-\frac{x+y}{y-x}.$$

x

y —— x

求二阶导数时，要注意y是x的函数！

$$\therefore\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{F_x}{F_y}\right)=-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+y}{y-x}\right)$$

$$=-\frac{\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)(y-x)-(x+y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}-1\right)}{(y-x)^2}=\frac{2(x^2+y^2)}{(x-y)^3}.$$

(方法2) 复合函数求导法

$$\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$$

即 $$\frac{1}{2}\ln(x^2+y^2)=\arctan\frac{y}{x}$$

两端同时对 x 求导， 得

用此法求导数时，要注意 y 是 x 的函数!

$$\frac{1}{2}\cdot\frac{2x+2yy'}{x^2+y^2}=\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{y'x-y}{x^2}$$

$$x+yy'=xy'-y,\qquad \therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x+y}{x-y}.$$

(方法3) 全微分法

$$\frac{1}{2}\ln(x^2+y^2)=\arctan\frac{y}{x}$$

一阶全微分形式不变性,

两端同时取全微分,得

$$\frac{1}{2}\cdot\frac{1}{x^2+y^2}\mathrm{d}(x^2+y^2)=\frac{1}{1+(\frac{y}{x})^2}\mathrm{d}(\frac{y}{x})$$

$$\frac{1}{2}\cdot\frac{2x\mathrm{d}x+2y\mathrm{d}y}{x^2+y^2}=\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{x\mathrm{d}y-y\mathrm{d}x}{x^2}$$

解得 $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x+y}{x-y}$.

例2 设 $z = z(x, y)$ 由方程:

$$F(x+\frac{z}{y}, y+\frac{z}{x}) = 0 \qquad (1)$$

所确定，证明: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy$

证 方程（1）两边同时取全微分得

$$\mathrm{d}F(x+\frac{z}{y}, y+\frac{z}{x})$$

$$= F_1' \cdot \mathrm{d}(x+\frac{z}{y}) + F_2' \cdot \mathrm{d}(y+\frac{z}{x})$$

$$= F_1' \cdot [\mathrm{d}x + \mathrm{d}(\frac{z}{y})] + F_2' \cdot [\mathrm{d}y + \mathrm{d}(\frac{z}{x})]$$

$$= F_1' \cdot [\mathrm{d}x + \mathrm{d}(\frac{z}{y})] + F_2' \cdot [\mathrm{d}y + \mathrm{d}(\frac{z}{x})]$$

$$= F_1' \cdot (\mathrm{d}x + \frac{y\,\mathrm{d}z - z\,\mathrm{d}y}{y^2}) + F_2' \cdot (\mathrm{d}y + \frac{x\,\mathrm{d}z - z\,\mathrm{d}x}{x^2})$$

$$= (\frac{F_1'}{y} + \frac{F_2'}{x})\,\mathrm{d}z + (F_1' - \frac{z}{x^2}F_2')\,\mathrm{d}x + (F_2' - \frac{z}{y^2}F_1')\,\mathrm{d}y = 0$$

$$\mathrm{d}z = \underbrace{\boxed{\frac{(\frac{z}{x^2}F_2' - F_1')}{\frac{F_1'}{y} + \frac{F_2'}{x}}}}_{\frac{\partial z}{\partial x}}\mathrm{d}x + \underbrace{\boxed{\frac{(\frac{z}{y^2}F_1' - F_2')}{\frac{F_1'}{y} + \frac{F_2'}{x}}}}_{\frac{\partial z}{\partial y}}\mathrm{d}y$$

故 $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\cdot\frac{\frac{z}{x^2}F_2'-F_1'}{\frac{F_1'}{y}+\frac{F_2'}{x}}+y\cdot\frac{\frac{z}{y^2}F_1'-F_2'}{\frac{F_1'}{y}+\frac{F_2'}{x}}$$

$$=\frac{z\cancel{\left(\frac{F_1'}{y}+\frac{F_2'}{x}\right)}-xy\cancel{\left(\frac{F_1'}{y}+\frac{F_2'}{x}\right)}}{\cancel{\frac{F_1'}{y}+\frac{F_2'}{x}}}$$

$$=z-xy.$$

例3 设 $xu - yv = 0$， $yu + xv = 1$，

求 $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$和$\dfrac{\partial v}{\partial y}$.

解（方法1）直接套公式

（方法2）复合函数求导法

将所给方程的两边对 x 求偏导数，并移项

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}, \quad J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2,$$

在$J\neq 0$的条件下，解此方程组得

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-u & -y\\ -v & x\end{vmatrix}}{\begin{vmatrix}x & -y\\ y & x\end{vmatrix}}=-\frac{xu+yv}{x^2+y^2},\quad \frac{\partial v}{\partial x}=\frac{\begin{vmatrix}x & -u\\ y & -v\end{vmatrix}}{\begin{vmatrix}x & -y\\ y & x\end{vmatrix}}=\frac{yu-xv}{x^2+y^2},$$

将所给方程的两边对 y 求偏导数，并解方程组得

$$\frac{\partial u}{\partial y}=\frac{xv-yu}{x^2+y^2},\qquad \frac{\partial v}{\partial y}=-\frac{xu+yv}{x^2+y^2}.$$

例4 设 $\begin{cases} z = x^2 + y^2, \\ x^2 + 2y^2 + 3z^2 = 20, \end{cases}$ 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}, \dfrac{\mathrm{d}z}{\mathrm{d}x}, \dfrac{\mathrm{d}^2 z}{\mathrm{d}x^2}$.

分析

> 函数个数=方程个数;
>
> 自变量个数=方程组所含变量个数−方程个数

本题目方程组中包含两个方程， 故有两个函数.

由题目知 y、z 是函数，x是自变量，故 y, z 均是由

方程组确定的自变量x的一元函数.

解 对方程组中每一个方程 的两端同时关于 x 求导数，得

$$\begin{cases} z = x^2 + y^2, \\ x^2 + 2y^2 + 3z^2 = 20, \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}x} = 2x + 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}, \\ 2x + 4y\dfrac{\mathrm{d}y}{\mathrm{d}x} + 6z\dfrac{\mathrm{d}z}{\mathrm{d}x} = 0. \end{cases}$$

解得 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x(1+6z)}{y(1+3z)}$, $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{x}{1+3z}$.

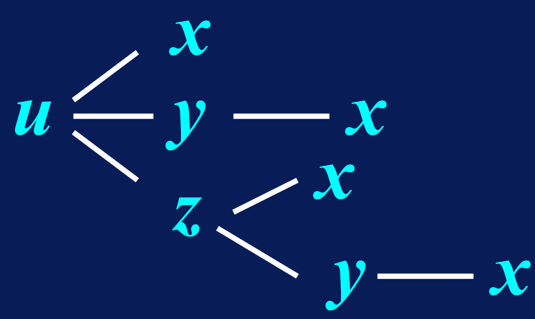
$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1+3z}\right) = \frac{(1+3z) - x\cdot 3\dfrac{\mathrm{d}z}{\mathrm{d}x}}{(1+3z)^2} = \frac{(1+3z)^2 - 3x^2}{(1+3z^2)^3}.$$

求二阶导数时，要注意y是x的函数！

例5 设 $u = f(x, y, z), \varphi(x^2, e^y, z) = 0, y = \sin x,$

$(f, \varphi$ 具有一阶连续偏导数 $)$, 且 $\dfrac{\partial \varphi}{\partial z} \neq 0$, 求 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

解 $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial z} \boxed{\frac{\mathrm{d}z}{\mathrm{d}x}}$$



$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x,$$

由 $\varphi(x^2, e^y, z) = 0$，两边对 x 求导数，得

$$\varphi_1' \cdot 2x + \varphi_2' \cdot e^y \frac{\mathrm{d}y}{\mathrm{d}x} + \varphi_3' \frac{\mathrm{d}z}{\mathrm{d}x} = 0$$

于是可得,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1}{\varphi_3'}(2x\varphi_1' + e^{\sin x} \cdot \cos x \cdot \varphi_2')$$

故 $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}$$

$$= \frac{\partial f}{\partial x} + (\cos x)\frac{\partial f}{\partial y} - \frac{1}{\varphi_3'}(2x\varphi_1' + \mathrm{e}^{\sin x} \cdot \cos x \cdot \varphi_2')\frac{\partial f}{\partial z}.$$

例6 设 $y=f(x,t)$，其中 $t=t(x,y)$ 由 $F(x,y,t)=0$ 所确定，f,F 有一阶连续的偏导数，求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

解（方法1）由方程组确定的隐函数求导法

$$\begin{cases} y=f(x,t) \\ F(x,y,t)=0 \end{cases} \longrightarrow \begin{cases} y=y(x) \\ t=t(x) \end{cases} \longrightarrow \begin{cases} y(x)\equiv f[x,t(x)] \\ F[x,y(x),t(x)]\equiv 0 \end{cases},$$

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x}=f_x+f_t\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x} \\ F_x+F_y\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}+F_t\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}=0 \end{cases}$$

y,t 都是 x 的函数

即 $\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} - f_t \cdot \dfrac{\mathrm{d}t}{\mathrm{d}x} = f_x \\ F_y \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x} + F_t \cdot \dfrac{\mathrm{d}t}{\mathrm{d}x} = -F_x \end{cases}$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\begin{vmatrix} f_x & -f_t \\ -F_x & F_t \end{vmatrix}}{\begin{vmatrix} 1 & -f_t \\ F_y & F_t \end{vmatrix}} = \frac{f_x F_t - f_t F_x}{F_t + f_t F_y}.$$

（方法2）全微分法

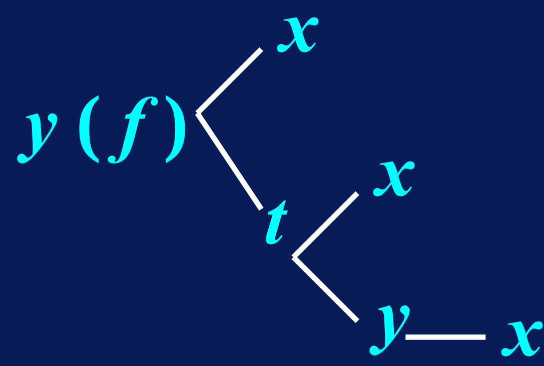
由 $\begin{cases} y = f(x,t) \\ F(x,y,t) = 0 \end{cases}$，得 $\begin{cases} \mathrm{d}\,y = \mathrm{d}\,f(x,t) \\ \mathrm{d}[F(x,y,t)] = 0 \end{cases}$

$$\begin{cases} \mathrm{d}\,y = f_x\,\mathrm{d}\,x + f_t\,\mathrm{d}\,t \\ F_x\,\mathrm{d}\,x + F_y\,\mathrm{d}\,y + F_t\,\mathrm{d}\,t = 0 \end{cases}$$

$$\begin{cases} \mathrm{d}\,y - f_t\,\mathrm{d}t = f_x\,\mathrm{d}\,x \\ F_y\,\mathrm{d}\,y + F_t\,\mathrm{d}\,t = -F_x\,\mathrm{d}\,x \end{cases} \quad \therefore \ \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{f_x F_t - f_t F_x}{F_t + f_t F_y}.$$

西北工業大學 NORTHWESTERN POLYTECHNICAL UNIVERSITY

（方法3）复合函数求导法



$$y = f(x,t),$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f_x + f_t \cdot \left(\frac{\partial t}{\partial x} + \frac{\partial t}{\partial y} \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x}\right)$$

$\because t = t(x,y)$ 由 $F(x,y,t) = 0$ 所确定

$$\therefore \frac{\partial t}{\partial x} = -\frac{F_x}{F_t}, \quad \frac{\partial t}{\partial y} = -\frac{F_y}{F_t}$$

故 $$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = f_x + f_t \cdot \left(-\frac{F_x}{F_t} - \frac{F_y}{F_t} \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,x}\right)$$

故 $$\frac{\mathrm{d}y}{\mathrm{d}x}=f_x+f_t\cdot\left(-\frac{F_x}{F_t}-\frac{F_y}{F_t}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$\left(1+\frac{f_tF_y}{F_t}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=f_x-\frac{f_tF_x}{F_t}$$

$$\therefore\quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{f_xF_t-f_tF_x}{F_t+f_tF_y}.$$

三、同步练习

1. 设 $z = f(x+y+z, xyz)$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial x}{\partial z}, \dfrac{\partial x}{\partial y}$.

2. 设函数 $x = x(u,v), y = y(u,v)$ 在点 (u,v) 的某一邻域内有连续的偏导数，且 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$

1) 证明函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 (u,v) 对应的点 (x,y) 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u = u(x,y), v = v(x,y)$.

2) 求 $u = u(x,y), v = v(x,y)$ 对 x, y 的偏导数.

西北工業大學 NORTHWESTERN POLYTECHNICAL UNIVERSITY

3. 验证方程 $\sin y + e^x - xy - 1 = 0$

在 (0,0)点某邻域可确定一个单值可导隐函数

$y = f(x)$, 并求 $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}$, $\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$

4. 设 $x^2 + y^2 + z^2 - 4z = 0$, 求 $\dfrac{\partial^2 z}{\partial x^2}$.

5. 设 $\dfrac{x}{z} - \ln\dfrac{z}{y} = 0$, 求 $\dfrac{\partial^2 z}{\partial x \partial y}$.

6. 设$F(u, v)$具有连续偏导数, 已知方程$F(\dfrac{x}{z}, \dfrac{y}{z}) = 0$,

求 $\mathrm{d}z$.

7\. 设 $\begin{cases} u = f(ux, v+y), \\ v = g(u-x, v^2 y), \end{cases}$ 其中 f, g 具有一阶连续偏导数，求 u_y.

8\. 设 $y = y(x), z = z(x)$ 是由方程 $z = x f(x+y)$ 和 $F(x, y, z) = 0$ 所确定的函数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. (99考研)

9\. 设 $u = f(x, y, z)$ 有连续的一阶偏导数，又函数 $y = y(x)$ 及 $z = z(x)$ 分别由下列两式确定：

$e^{xy} - xy = 2$, $e^x = \int_0^{x-z} \dfrac{\sin t}{t} \mathrm{d}t$, 求 $\dfrac{\mathrm{d}u}{\mathrm{d}x}$. (2001考研)

四、同步练习解答

1. 设 $z = f(x+y+z, xyz)$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial x}{\partial z}, \dfrac{\partial x}{\partial y}$.

解（方法1）

- $$\frac{\partial z}{\partial x} = f_1' \cdot \left(1 + \frac{\partial z}{\partial x}\right) + f_2' \cdot \left(yz + xy\frac{\partial z}{\partial x}\right)$$

$$\Longrightarrow \frac{\partial z}{\partial x} = \frac{f_1' + yzf_2'}{1 - f_1' - xyf_2'}$$

- $$1 = f_1' \cdot \left(\frac{\partial x}{\partial z} + 1\right) + f_2' \cdot \left(yz\frac{\partial x}{\partial z} + xy\right)$$

$$\Longrightarrow \frac{\partial x}{\partial z} = \frac{1 - f_1' - xyf_2'}{f_1' + yzf_2'}$$

- $0 = f_1' \cdot \left(\frac{\partial x}{\partial y} + 1 \right) + f_2' \cdot \left(yz \frac{\partial x}{\partial y} + xz \right)$

$$\Longrightarrow \frac{\partial x}{\partial y} = -\frac{f_1' + xz f_2'}{f_1' + yz f_2'}$$

（方法2）全微分法

$$z = f(x + y + z, xyz)$$

$$\mathrm{d}z = f_1' \cdot (\mathrm{d}x + \mathrm{d}y + \mathrm{d}z) + f_2' \cdot (yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z)$$

解出 $\mathrm{d}x$：

$$\mathrm{d}x = \frac{-(f_1' + xz f_2')\mathrm{d}y + (1 - f_1' - xy f_2')\mathrm{d}z}{f_1' + yz f_2'}$$

$$\mathrm{d}x=\frac{-(f_1'+xzf_2')\mathrm{d}y+(1-f_1'-xyf_2')\mathrm{d}z}{f_1'+yzf_2'}$$

$\mathrm{d}\,y,\ \mathrm{d}\,z$ 的系数分别是 $\dfrac{\partial x}{\partial y}$, $\dfrac{\partial x}{\partial z}$.

问题 如何用全微分法求 $\dfrac{\partial z}{\partial x}$?

将$\mathrm{d}\,z$进行整理，其中 $\mathrm{d}\,x$的系数就是 $\dfrac{\partial z}{\partial x}$.

2. 设函数 $x=x(u,v),\ y=y(u,v)$ 在点 (u,v) 的某一邻域内有连续的偏导数, 且 $\dfrac{\partial(x,y)}{\partial(u,v)}\neq 0$

1) 证明函数组 $\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$ 在与点 (u,v) 对应的点 (x,y) 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u=u(x,y),\ v=v(x,y)$.

2) 求 $u=u(x,y),\ v=v(x,y)$ 对 x,y 的偏导数.

解 1) 令 $F(x,y,u,v)\equiv x-x(u,v)=0$

$$G(x,y,u,v)\equiv y-y(u,v)=0$$

则有 $J=\dfrac{\partial(F,G)}{\partial(u,v)}=\dfrac{\partial(x,y)}{\partial(u,v)}\neq 0,$

由定理 3 可知结论 1) 成立.

2) 求反函数的偏导数.

$$\begin{cases} x\equiv x(u(x,y),v(x,y)) \\ y\equiv y(u(x,y),v(x,y)) \end{cases} \qquad ①$$

①式两边对 x 求偏导数, 得

$$\begin{cases} 1=\dfrac{\partial x}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial x}{\partial v}\cdot\dfrac{\partial v}{\partial x} \\ 0=\dfrac{\partial y}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial y}{\partial v}\cdot\dfrac{\partial v}{\partial x} \end{cases} \qquad ②$$

注意$J\neq 0$，从方程组②解得

$$\frac{\partial u}{\partial x}=\frac{1}{J}\begin{vmatrix}1 & \dfrac{\partial x}{\partial v}\\ 0 & \dfrac{\partial y}{\partial v}\end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v},\quad \frac{\partial v}{\partial x}=\frac{1}{J}\begin{vmatrix}\dfrac{\partial x}{\partial u} & 1\\ \dfrac{\partial y}{\partial u} & 0\end{vmatrix}=-\frac{1}{J}\frac{\partial y}{\partial u}$$

同理，①式两边对 y 求偏导数，可得

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial x}{\partial v},\qquad \frac{\partial v}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

本题的应用：计算极坐标变换 $x = r\cos\theta$, $y = r\sin\theta$

的逆变换的导数.

$$\frac{\partial r}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r}$$

由于 $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

所以 $\frac{\partial r}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \theta} = \frac{1}{r} r\cos\theta = \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r} = -\frac{1}{r}\sin\theta = -\frac{y}{x^2+y^2}$$

同样有 $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$ $\qquad$ $\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$

3. 验证方程 $\sin y+e^{x}-xy-1=0$ 在 $(0,0)$点某邻域可确定一个单值可导隐函数 $y=f(x)$, 并求 $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}$, $\left.\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}$

解 令 $F(x,y)=\sin y+e^{x}-xy-1$, 则

① $F_x=e^{x}-y,\ F_y=\cos y-x$ 连续 ,

② $F(0,0)=0$,

③ $F_y(0,0)=1\neq 0$

由 定理1 可知, 在 $x=0$ 的某邻域内方程存在单值可导的隐函数 $y=f(x)$, 且

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -\left.\frac{F_x}{F_y}\right|_{x=0} = -\left.\frac{e^x - y}{\cos y - x}\right|_{x=0,\,y=0} = -1$$

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x - y}{\cos y - x}\right)\Bigg|_{x=0,\ y=0,\ y'=-1}$$

$$= -\frac{(e^x - y')(\cos y - x) - (e^x - y)(-\sin y \cdot y' - 1)}{(\cos y - x)^2}\Bigg|_{\substack{x=0\\ y=0\\ y'=-1}}$$

$$= -3$$

西北工业大学 NORTHWESTERN POLYTECHNICAL UNIVERSITY

导数的另一求法 — 复合函数求导法

$$\sin y + e^x - xy - 1 = 0,$$

$\downarrow$ 两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

$$\boxed{\begin{aligned} y'\Big|_{x=0} &= -\frac{e^x - y}{\cos y - x}\Bigg|_{(0,0)} \\ &= -1 \end{aligned}}$$

$\downarrow$ 两边再对 x 求导

$$-\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - xy'' = 0$$

$\downarrow$ 令 $x = 0$，注意此时 $y = 0$，$y' = -1$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\Bigg|_{x=0} = -3$$

4. 设 $x^2+y^2+z^2-4z=0$，求 $\dfrac{\partial^2 z}{\partial x^2}$.

解（方法1）复合函数求导法

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

注意本方法中，始终将 z 看作 x 与 y 的函数

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

（方法2）公式法

设 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $F_x=2x,\ F_z=2z-4$

$$\therefore\quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x}{2z-4}=\frac{x}{2-z}$$

用公式法求 F_x 时，先不将 z 看作 x 与 y 的函数！应暂视 y,z 为常数

对 $\dfrac{\partial z}{\partial x}=\dfrac{x}{2-z}$ 两端关于 x 求偏导数，得

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{(2-z)-x\left(-\frac{\partial z}{\partial x}\right)}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

求二阶导数时，要视z是x,y的函数！

5. 设 $\dfrac{x}{z}-\ln\dfrac{z}{y}=0$, 求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

解 方程两端取全微分：$\dfrac{z\mathrm{d}x-x\mathrm{d}z}{z^2}-\dfrac{y}{z}\cdot\dfrac{y\mathrm{d}z-z\mathrm{d}y}{y^2}=0$,

解得 $\mathrm{d}z=\dfrac{y^2z\mathrm{d}x+yz^2\mathrm{d}y}{y^2(x+z)}$, $\quad\therefore\quad \dfrac{\partial z}{\partial x}=\dfrac{z}{x+z}$,

$$\frac{\partial z}{\partial y}=\frac{z^2}{y(x+z)}.$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{z}{x+z}\right)=\frac{\dfrac{\partial z}{\partial y}(x+z)-z\cdot\dfrac{\partial z}{\partial y}}{(x+z)^2}=\frac{\partial z}{\partial y}\cdot\frac{x}{(x+z)^2}.$$

$$=\frac{xz^2}{y(x+z)^3}.$$

6. 设$F(u,v)$具有连续偏导数, 已知方程$F(\frac{x}{z},\frac{y}{z})=0$, 求 $\mathrm{d}z$.

解（方法1）先求偏导数　设 $z=f(x,y)$ 是由方程 $F(\frac{x}{z},\frac{y}{z})=0$ 确定的隐函数, 则

$$\frac{\partial z}{\partial x}=-\frac{F_1'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_1'}{xF_1'+yF_2'}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_2'}{xF_1'+yF_2'}$$

故　$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

（方法2）全微分法　对方程两边求全微分:

$$F(\frac{x}{z},\frac{y}{z})=0$$

$$F_1'\cdot \mathrm{d}(\frac{x}{z})+F_2'\cdot \mathrm{d}(\frac{y}{z})=0$$

$$F_1'\cdot(\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2})+F_2'\cdot(\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2})=0$$

$$-\frac{xF_1'+yF_2'}{z^2}\mathrm{d}z+\frac{F_1'\mathrm{d}x+F_2'\mathrm{d}y}{z}=0$$

$$\mathrm{d}z=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

7. 设$\begin{cases} u = f(ux, v+y), \\ v = g(u-x, v^2 y), \end{cases}$ 其中f, g具有一阶连续偏导数,求u_y.

解（方法1）复合函数求导法

对每一个方程关于 y求偏导数,

得 $$\begin{cases} u_y = f_1' \cdot x\, u_y + f_2' \cdot (v_y + 1), \\ v_y = g_1' \cdot u_y + g_2'(2yv\, v_y + v^2). \end{cases}$$

解此关于u_y, v_y的二元一次方程组

得 $$u_y = \frac{v^2 f_2' g_2' + f_2'(1-2yvg_2')}{f_2' g_1' - (1-xf_1')(1-2yvg_2')},$$

$$v_y = \frac{f_2' g_1' + v^2 g_2'(1-xf_1')}{(1-xf_1')(1-2yvg_2') - f_2' g_1'}.$$

(方法2) 全微分法 $\begin{cases} u = f(ux, v+y), \\ v = g(u-x, v^2 y), \end{cases}$

对每一个方程两端同时 取全微分

得 $$\begin{cases} \mathrm{d}u = f_1'(u\,\mathrm{d}x + x\,\mathrm{d}u) + f_2'(\mathrm{d}v + \mathrm{d}y), \\ \mathrm{d}v = g_1'(\mathrm{d}u - \mathrm{d}x) + g_2'(v^2\,\mathrm{d}y + 2vy\,\mathrm{d}v). \end{cases}$$

解此关于 $\mathrm{d}u, \mathrm{d}v$ 的二元一次方程组，得

$$\mathrm{d}u = \frac{[uf_1'(1-2yvg_2') - f_2'g_1]\mathrm{d}x + [v^2 f_2'g_2' + f_2'(1-2yvg_2')]\mathrm{d}y}{f_2'g_1' - (1-xf_1')(1-2yvg_2')}$$

$$\mathrm{d}v = \frac{[g_1'(1-xf_1') - uf_1'g_1']\mathrm{d}x + [v^2 g_2'(xf_1'-1) - f_2'g_1']\mathrm{d}y}{(1-xf_1')(1-2yvg_2') - f_2'g_1'}$$

由此可得 u_y，v_y，同时可得 u_x，v_x.

8. 设 $y=y(x), z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. (99考研)

解 （方法1）分别在各方程两端对 x 求导，得

$$\begin{cases} z'=f+x\cdot f'\cdot(1+y') \\ F_x+F_y\cdot y'+F_z\cdot z'=0 \end{cases} \Longrightarrow \begin{cases} -xf'\cdot y'+\underline{z'}=f+xf' \\ F_y\cdot y'+F_z\cdot \underline{z'}=-F_x \end{cases}$$

$$\therefore\quad \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\begin{vmatrix} -xf' & f+xf' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -xf' & 1 \\ F_y & F_z \end{vmatrix}}=\frac{(f+xf')F_y-xf'\cdot F_x}{F_y+xf'\cdot F_z}$$

$$(F_y+xf'\cdot F_z\neq 0)$$

（方法2）全微分法

$$z = x f(x+y),\ F(x,y,z)=0$$

对各方程两边分别求全微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + x f' \cdot (\mathrm{d}x + \mathrm{d}y) \\ F_1'\mathrm{d}x + F_2'\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

化简得

$$\begin{cases} (f + x f')\,\mathrm{d}x + x f'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1'\,\mathrm{d}x + F_2'\,\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

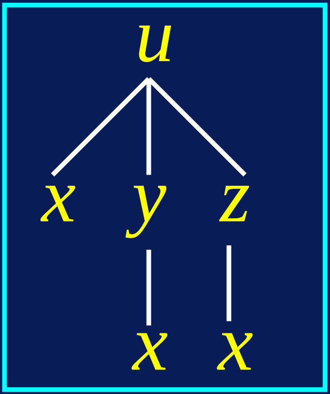
消去$\mathrm{d}y$可得$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

9. 设 $u = f(x,y,z)$ 有连续的一阶偏导数，又函数 $y = y(x)$ 及 $z = z(x)$ 分别由下列两式确定：

$e^{xy} - xy = 2$，$e^{x} = \int_0^{x-z} \frac{\sin t}{t}\,\mathrm{d}t$，求 $\frac{\mathrm{d}u}{\mathrm{d}x}$．(2001考研)

解 每个方程两边都对 x 求导，得

$$\begin{cases} e^{xy}(y + xy') - (y + xy') = 0 \\ e^{x} = \dfrac{\sin(x-z)}{x-z}(1 - z') \end{cases}$$



解得 $y' = -\dfrac{y}{x}$， $z' = 1 - \dfrac{e^{x}(x-z)}{\sin(x-z)}$

因此 $$\frac{\mathrm{d}u}{\mathrm{d}x} = f_1' + f_2' \cdot y' + f_3' \cdot z' = f_1' - \frac{y}{x} f_2' + \Big[1 - \frac{e^{x}(x-z)}{\sin(x-z)}\Big] f_3'$$