第六节　极限的存在准则与两个重要极限

习题　1-6

1. 计算下列极限:

(1) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}(\beta\neq 0)$;　　(2) $\lim\limits_{x\to 0^+}\sqrt{x}\cot\sqrt{x}$;

(3) $\lim\limits_{n\to\infty}3^n\sin\dfrac{\pi}{3^n}$;　　(4) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x\sin x}$;

(5) $\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{1-\cos x}}$;　　(6) $\lim\limits_{x\to\infty}\dfrac{\sin x-x}{2x+\cos x}$.

解　(1)　若 $\alpha\neq 0$, $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}=\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\sin\beta x}\cdot\cos\beta x=\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\alpha x}\cdot\dfrac{\beta x}{\sin\beta x}\cdot\dfrac{\alpha x}{\beta x}=\dfrac{\alpha}{\beta}$;

若 $\alpha=0$, 易知 $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}=0=\dfrac{\alpha}{\beta}$.

(2) $$\lim_{x\to 0^+}\sqrt{x}\cot\sqrt{x}=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sin\sqrt{x}}\cdot\cos\sqrt{x}=\lim_{x\to 0^+}\frac{\sqrt{x}}{\sin\sqrt{x}}\cdot\lim_{x\to 0^+}\cos\sqrt{x}=1.$$

(3) $$\lim_{n\to\infty}3^n\sin\frac{\pi}{3^n}=\lim_{n\to\infty}\frac{\sin\frac{\pi}{3^n}}{\frac{\pi}{3^n}}\cdot\pi=\pi.$$

(4) $$\lim_{x\to 0}\frac{1-\cos 2x}{x\sin x}=\lim_{x\to 0}\frac{2\sin^2 x}{x\sin x}=2.$$

(5) $$\lim_{x\to 0^+}\frac{x}{\sqrt{1-\cos x}}=\lim_{x\to 0^+}\frac{x}{\sqrt{2}\sin\frac{x}{2}}=\lim_{x\to 0^+}\sqrt{2}\cdot\frac{\frac{x}{2}}{\sin\frac{x}{2}}=\sqrt{2}.$$

(6) $$\lim_{x\to\infty}\frac{\sin x-x}{2x+\cos x}=\lim_{x\to\infty}\frac{\frac{\sin x}{x}-1}{2+\frac{\cos x}{x}}=-\frac{1}{2}.$$

2. 计算下列极限：

(1) $\lim\limits_{x\to 0}(1+ax)^{\frac{b}{x}}(a,b>0)$;　　(2) $\lim\limits_{x\to\infty}(\dfrac{x-1}{x+1})^x$;

(3) $\lim\limits_{x\to 0}\sqrt[x]{1-2x}$; (4) $\lim\limits_{x\to\frac{\pi}{2}}(1+\cos x)^{2\sec x}$;

(5) $\lim\limits_{n\to\infty}(1-\dfrac{1}{n})^{kn}$ (k 为正整数); (6) $\lim\limits_{n\to\infty}(\dfrac{n+1}{n-1})^{n}$.

解 (1) $\lim\limits_{x\to 0}(1+ax)^{\frac{b}{x}}=\lim\limits_{x\to 0}(1+ax)^{\frac{1}{ax}\cdot ab}=\mathrm{e}^{ab}$.

(2) $\lim\limits_{x\to\infty}(\dfrac{x-1}{x+1})^{x}=\lim\limits_{x\to\infty}(1-\dfrac{2}{x+1})^{(-\frac{x+1}{2})\cdot(-\frac{2x}{x+1})}=\mathrm{e}^{-2}$.

(3) $\lim\limits_{x\to 0}\sqrt[x]{1-2x}=\lim\limits_{x\to 0}(1-2x)^{(-\frac{1}{2x})(-2)}=\mathrm{e}^{-2}$.

(4) $\lim\limits_{x\to\frac{\pi}{2}}(1+\cos x)^{2\sec x}=\lim\limits_{x\to\frac{\pi}{2}}(1+\cos x)^{\frac{2}{\cos x}}=\mathrm{e}^{2}$.

(5) $\lim\limits_{n\to\infty}(1-\dfrac{1}{n})^{kn}=\lim\limits_{n\to\infty}[(1-\dfrac{1}{n})^{-n}]^{-k}=\mathrm{e}^{-k}$.

(6) $\lim\limits_{n\to\infty}(\dfrac{n+1}{n-1})^{n}=\lim\limits_{n\to\infty}(1+\dfrac{2}{n-1})^{\frac{n-1}{2}\cdot\frac{2n}{n-1}}=\mathrm{e}^{2}$.

3. 利用夹逼准则证明下列极限:

(1) $\lim\limits_{n\to\infty}(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}})=1$;

(2) $\lim\limits_{n\to\infty}(\dfrac{1}{n^2+1}+\dfrac{2}{n^2+2}+\cdots+\dfrac{n}{n^2+n})=\dfrac{1}{2}$;

(3) $\lim\limits_{n\to\infty}(\sin\dfrac{\pi}{\sqrt{n^2+1}}+\sin\dfrac{\pi}{\sqrt{n^2+2}}+\cdots+\sin\dfrac{\pi}{\sqrt{n^2+n}})=\pi$;

(4) $\lim\limits_{x\to 0}\sqrt[n]{1+x}=1$.

证 (1) 因为

$$\frac{n}{\sqrt{n^2+n}}<\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}<\frac{n}{\sqrt{n^2+1}},$$

又 $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\frac{1}{n}}}=1$; $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+1}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\frac{1}{n^2}}}=1$;

所以 $\lim\limits_{n\to\infty}(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}})=1$.

(2) 因为 $\dfrac{\frac{1}{2}n(n+1)}{n^2+n}=\dfrac{1}{n^2+n}+\dfrac{2}{n^2+n}+\cdots+\dfrac{n}{n^2+n}<\dfrac{1}{n^2+1}+\dfrac{2}{n^2+2}+\cdots+\dfrac{n}{n^2+n}$

$$<\frac{1}{n^2+1}+\frac{2}{n^2+1}+\cdots+\frac{n}{n^2+1}=\frac{\frac{1}{2}n(n+1)}{n^2+1},$$

又 $\lim\limits_{n\to\infty}\dfrac{\frac{1}{2}n(n+1)}{n^2+n}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{2}n(n+1)}{n^2+1}=\dfrac{1}{2}$，故

$$\lim_{n\to\infty}\left(\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n}\right)=\frac{1}{2}.$$

(3) 因为

$$n\sin\frac{\pi}{\sqrt{n^2+n}}\le\sin\frac{\pi}{\sqrt{n^2+1}}+\sin\frac{\pi}{\sqrt{n^2+2}}+\cdots+\sin\frac{\pi}{\sqrt{n^2+n}}\le n\sin\frac{\pi}{\sqrt{n^2+1}},$$

又 $\lim\limits_{n\to\infty}n\sin\dfrac{\pi}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\dfrac{\sin\frac{\pi}{\sqrt{n^2+n}}}{\frac{\pi}{\sqrt{n^2+n}}}\cdot\dfrac{n\pi}{\sqrt{n^2+n}}=\pi$，同理 $\lim\limits_{n\to\infty}n\sin\dfrac{\pi}{\sqrt{n^2+1}}=\pi$，故

$$\lim_{n\to\infty}\left(\sin\frac{\pi}{\sqrt{n^2+1}}+\sin\frac{\pi}{\sqrt{n^2+2}}+\cdots+\sin\frac{\pi}{\sqrt{n^2+n}}\right)=\pi.$$

(4) 当 $x>0$ 时，$1<\sqrt[n]{1+x}<1+x$，故 $\lim\limits_{x\to0^+}\sqrt[n]{1+x}=1$；

当 $-1<x<0$ 时，$1+x<\sqrt[n]{1+x}<1$，故 $\lim\limits_{x\to0^-}\sqrt[n]{1+x}=1$.

故
$$\lim_{x\to0}\sqrt[n]{1+x}=1.$$

4. 利用单调有界准则证明下面数列存在极限，并求其极限值:

(1) $a_1=\sqrt{2},a_2=\sqrt{2\sqrt{2}},\cdots,a_n=\sqrt{2\sqrt{2\cdots\sqrt{2}}}$ （n 次复合）;

(2) $x_1=1,x_2=1+\dfrac{x_1}{x_1+1},\cdots,x_n=1+\dfrac{x_{n-1}}{x_{n-1}+1}$.

证 (1) 易知 $a_{n+1}=\sqrt{2a_n}\,(n=1,2,\cdots)$，下证此数列单调有界:

当 $n=1$ 时，$a_1=\sqrt{2}<2$，假设 $n=k$ 时，$a_k<2$，则当 $n=k+1$ 时，$a_{k+1}=\sqrt{2a_k}<2$，即 $a_n<2(n=1,2,\cdots)$，即此数列有界;

因为 $a_{n+1}-a_n=\sqrt{2a_n}-a_n=\dfrac{2a_n-a_n^2}{\sqrt{2a_n}+a_n}=\dfrac{-a_n(a_n-2)}{\sqrt{2a_n}+a_n}$，由 $a_n<2$，故 $a_{n+1}-a_n>0$，

即 $a_{n+1}>a_n$.

综上，$\lim\limits_{n\to\infty}a_n$ 存在，令 $\lim\limits_{n\to\infty}a_n=A$.

又 $a_{n+1}=\sqrt{2a_n}$，故 $a_{n+1}^2=2a_n$，因此 $\lim\limits_{n\to\infty}a_{n+1}^2=2\lim\limits_{n\to\infty}a_n$，即 $A^2=2A$，

解得 $A_1=2$，$A_2=0$ (舍去)，故 $\lim\limits_{n\to\infty}a_n=2$.

(2) 易知 $x_n>0$，先证此数列单调有界:

当 $n=1$ 时，$x_1=1\le 2$，当 $n>1$ 时，$x_n=1+\dfrac{x_{n-1}}{x_{n-1}+1}\le 2$，即 $x_n\le 2(n=1,2,\cdots)$，即此数列有界;

又 $x_2-x_1=\dfrac{1}{2}>0$，故

$$\begin{aligned}x_{n+1}-x_n&=(1+\frac{x_n}{x_n+1})-(1+\frac{x_{n-1}}{x_{n-1}+1})=\frac{x_n-x_{n-1}}{(x_n+1)(x_{n-1}+1)}\\&=\cdots=\frac{x_2-x_1}{(x_n+1)(x_{n-1}+1)\cdots(x_2+1)(x_1+1)}>0,\end{aligned}$$

即 $x_{n+1}>x_n$.

综上，$\lim\limits_{n\to\infty}x_n$ 存在，令 $\lim\limits_{n\to\infty}x_n=A$.

又 $x_n=1+\dfrac{x_{n-1}}{x_{n-1}+1}$，因此 $\lim\limits_{n\to\infty}x_n=1+\dfrac{\lim\limits_{n\to\infty}x_{n-1}}{\lim\limits_{n\to\infty}x_{n-1}+1}$，即 $A=1+\dfrac{A}{A+1}$，

解得 $A_1=\dfrac{1+\sqrt{5}}{2}$，$A_1=\dfrac{1-\sqrt{5}}{2}$ (舍去)，故 $\lim\limits_{n\to\infty}x_n=\dfrac{1+\sqrt{5}}{2}$.

5. 记 $(2n-1)!!=1\cdot3\cdot5\cdot7\cdots(2n-1)$，$(2n)!!=2\cdot4\cdot6\cdot8\cdots(2n)$.

设 $x_n=\dfrac{(2n-1)!!}{(2n)!!}$，试证明 $\dfrac{1}{\sqrt{4n}}<x_n<\dfrac{1}{\sqrt{2n+1}}$，并求极限 $\lim\limits_{n\to\infty}x_n$.

证 易知 $x_{n+1}=\dfrac{(2(n+1)-1)!!}{(2(n+1))!!}=\dfrac{2n+1}{2n+2}\cdot x_n$，

当 $n=1$ 时，$\dfrac{1}{\sqrt{4}}<x_1=\dfrac{1}{2}<\dfrac{1}{\sqrt{2+1}}$，假设 $n=k$ 时，$\dfrac{1}{\sqrt{4k}}<x_k<\dfrac{1}{\sqrt{2k+1}}$，则当 $n=k+1$ 时，

$$\frac{1}{\sqrt{4(k+1)}} < \frac{2k+1}{2k+2} \cdot \frac{1}{\sqrt{4k}} < x_{k+1} = \frac{2k+1}{2k+2} \cdot x_k < \frac{2k+1}{2k+2} \cdot \frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2(k+1)+1}},$$

故 $\frac{1}{\sqrt{4n}} < x_n < \frac{1}{\sqrt{2n+1}}$. 由 $\lim\limits_{n\to\infty} \frac{1}{\sqrt{4n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$, 故 $\lim\limits_{n\to\infty} x_n = 0$.