第二节 不定积分的换元积分法

习题 4-2

1. 在下列各式等号右端的空白处填入适当的系数，使等式成立(例如: $\mathrm{d}x=\frac{1}{3}\mathrm{d}(3x+8)$):

(1) $\mathrm{d}x=\quad\mathrm{d}(-6x+7)$; (2) $\frac{\mathrm{d}x}{\sqrt{x}}=\quad\mathrm{d}(\sqrt{x}+6)$;

(3) $\frac{1}{x^2}\mathrm{d}x=\quad\mathrm{d}(\frac{1}{x})$; (4) $\sqrt{x}\mathrm{d}x=\quad\mathrm{d}(x^{\frac{3}{2}}-5)$;

(5) $x\mathrm{d}x=\quad\mathrm{d}(3x^2+4)$; (6) $x^2\mathrm{d}x=\quad\mathrm{d}(x^3+7)$;

(7) $x^3\mathrm{d}x=\quad\mathrm{d}(5x^4-6)$; (8) $\sin 2x\mathrm{d}x=\quad\mathrm{d}(\cos 2x)$;

(9) $\cos\frac{7}{3}x\mathrm{d}x=\quad\mathrm{d}(5-\sin\frac{7}{3}x)$; (10) $\csc^2 x\mathrm{d}x=\quad\mathrm{d}(\cot x)$;

(11) $\frac{\mathrm{d}x}{x}=\quad\mathrm{d}(5-\ln|x|)$; (12) $\mathrm{e}^{6x}\mathrm{d}x=\quad\mathrm{d}(\mathrm{e}^{6x}-6)$;

(13) $\mathrm{e}^{-\frac{x}{5}}\mathrm{d}x=\quad\mathrm{d}(2+\mathrm{e}^{-\frac{x}{5}})$; (14) $\frac{\mathrm{d}x}{\sqrt{1-x^2}}=\quad\mathrm{d}(2-\arcsin x)$;

(15) $\frac{x}{\sqrt{1-x^2}}\mathrm{d}x=\quad\mathrm{d}(\sqrt{1-x^2})$; (16) $\frac{2x}{\sqrt{1+x^2}}\mathrm{d}x=\quad\mathrm{d}(\sqrt{1+x^2})$;

(17) $\frac{\mathrm{d}x}{1+4x^2}=\quad\mathrm{d}(\arctan 2x)$; (18) $\frac{\mathrm{d}x}{\sqrt{1-4x^2}}=\quad\mathrm{d}(\arccos 2x+3)$;

(19) $\csc^2 9x\mathrm{d}x=\quad\mathrm{d}(\cot 9x)$; (20) $\sec^2 3x\mathrm{d}x=\quad\mathrm{d}(\tan 3x)$.

解 (1) $\mathrm{d}x=-\frac{1}{6}\mathrm{d}(-6x+7)$; (2) $\frac{\mathrm{d}x}{\sqrt{x}}=2\mathrm{d}(\sqrt{x}+6)$;

(3) $\frac{1}{x^2}\mathrm{d}x=-1\mathrm{d}(\frac{1}{x})$; (4) $\sqrt{x}\mathrm{d}x=\frac{2}{3}\mathrm{d}(x^{\frac{3}{2}}-5)$;

(5) $x\mathrm{d}x=\frac{1}{6}\mathrm{d}(3x^2+4)$; (6) $x^2\mathrm{d}x=\frac{1}{3}\mathrm{d}(x^3+7)$;

(7) $x^3\mathrm{d}x=\frac{1}{20}\mathrm{d}(5x^4-6)$; (8) $\sin 2x\mathrm{d}x=-\frac{1}{2}\mathrm{d}(\cos 2x)$;

(9) $\cos\frac{7}{3}x\mathrm{d}x=-\frac{3}{7}\mathrm{d}(5-\sin\frac{7}{3}x)$; (10) $\csc^2 x\mathrm{d}x=-1\mathrm{d}(\cot x)$;

(11) $\frac{\mathrm{d}x}{x}=-1\mathrm{d}(5-\ln|x|)$; (12) $\mathrm{e}^{6x}\mathrm{d}x=\frac{1}{6}\mathrm{d}(\mathrm{e}^{6x}-6)$;

(13) $\mathrm{e}^{-\frac{x}{5}}\mathrm{d}x=-5\mathrm{d}(2+\mathrm{e}^{-\frac{x}{5}})$; (14) $\frac{\mathrm{d}x}{\sqrt{1-x^2}}=-1\mathrm{d}(2-\arcsin x)$;

(15) $\frac{x}{\sqrt{1-x^2}}\mathrm{d}x=-1\mathrm{d}(\sqrt{1-x^2})$; (16) $\frac{2x}{\sqrt{1+x^2}}\mathrm{d}x=2\mathrm{d}(\sqrt{1+x^2})$;

(17) $\frac{\mathrm{d}x}{1+4x^2}=\frac{1}{2}\mathrm{d}(\arctan 2x)$; (18) $\frac{\mathrm{d}x}{\sqrt{1-4x^2}}=-\frac{1}{2}\mathrm{d}(\arccos 2x+3)$;

(19) $\csc^2 9x\mathrm{d}x=-\frac{1}{9}\mathrm{d}(\cot 9x)$; (20) $\sec^2 3x\mathrm{d}x=\frac{1}{3}\mathrm{d}(\tan 3x)$.

2. 求下列不定积分:

(1) $\int\sqrt{7+4x}\mathrm{d}x$; (2) $\int\frac{1}{\sqrt[3]{2-3x}}\mathrm{d}x$;

(3) $\int\frac{\mathrm{d}x}{5x-2}$; (4) $\int\mathrm{e}^{1-3x}\mathrm{d}x$;

(5) $\int\mathrm{sh}(5x-1)\mathrm{d}x$; (6) $\int\mathrm{ch}(1-\frac{1}{2}x)\mathrm{d}x$;

(7) $\int\frac{1}{\sin^2 8x}\mathrm{d}x$; (8) $\int\tan(3x-5)\mathrm{d}x$;

(9) $\int\frac{x\mathrm{d}x}{\sqrt{2x^2+3}}$; (10) $\int\frac{x}{1+x^2}\mathrm{d}x$;

(11) $\int\frac{x^2}{(4+x^3)^2}\mathrm{d}x$; (12) $\int\frac{x^2}{\sqrt[3]{x^3+1}}\mathrm{d}x$;

(13) $\int 2x\mathrm{e}^{-x^2}\mathrm{d}x$; (14) $\int\mathrm{e}^{\mathrm{e}^x+x}\mathrm{d}x$;

(15) $\int\mathrm{e}^x\sin\mathrm{e}^x\mathrm{d}x$; (16) $\int\frac{\mathrm{e}^{2x}-1}{\mathrm{e}^x}\mathrm{d}x$;

(17) $\int\frac{\ln(\ln x)}{x\ln x}\mathrm{d}x$; (18) $\int\frac{\sqrt{1+\ln x}}{x}\mathrm{d}x$;

(19) $\int \frac{\sin\frac{1}{x}}{x^2}\mathrm{d}x$;　(20) $\int (1-\frac{1}{x^2})\sin(x+\frac{1}{x})\mathrm{d}x$;

(21) $\int \frac{\sec^2\frac{1}{x}}{x^2}\mathrm{d}x$;　(22) $\int \frac{a^{\frac{1}{x}}}{x^2}\mathrm{d}x$;

(23) $\int 10^{-3x+2}\mathrm{d}x$;　(24) $\int \frac{(\arctan x)^2}{1+x^2}\mathrm{d}x$;

(25) $\int \sqrt{\frac{\arcsin x}{1-x^2}}\mathrm{d}x$;　(26) $\int \frac{\mathrm{d}x}{\cos^2 x\sqrt{1+\tan x}}$;

(27) $\int \frac{\mathrm{d}x}{(\arcsin x)^2\sqrt{1-x^2}}$;　(28) $\int \cos^4 x\mathrm{d}x$;

(29) $\int \cos 3x\cos 4x\mathrm{d}x$;　(30) $\int \frac{\mathrm{d}x}{1+(2x-3)^2}$;

(31) $\int \frac{\cos x-\sin x}{\cos x+\sin x}\mathrm{d}x$;　(32) $\int \frac{2x+2}{x^2+2x+9}\mathrm{d}x$;

(33) $\int \frac{\ln\tan x}{\sin x\cos x}\mathrm{d}x$;　(34) $\int \frac{x^2}{1+x^2}\mathrm{d}x$;

(35) $\int \frac{\mathrm{d}x}{9+25x^2}$;　(36) $\int \frac{\sec^2 x}{2+\tan^2 x}\mathrm{d}x$;

(37) $\int \frac{\mathrm{d}x}{x(1+\ln x)}$;　(38) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x$;

(39) $\int \frac{x^2}{\sqrt{4-x^2}}\mathrm{d}x$;　(40) $\int \frac{\mathrm{d}x}{x\sqrt{9-x^2}}$;

(41) $\int \frac{\sqrt{a^2-x^2}}{x^2}\mathrm{d}x\ \ (a>0)$;　(42) $\int t\sqrt{25-t^2}\mathrm{d}t$;

(43) $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$;　(44) $\int \frac{\mathrm{d}x}{x^2\sqrt{a^2+x^2}}\ \ (a>0)$;

(45) $\int \frac{\sqrt{x^2-2}}{x}\mathrm{d}x$;　(46) $\int \frac{2x-1}{\sqrt{9x^2-4}}\mathrm{d}x$;

(47) $\int \frac{x}{\sqrt{x^2+2x+2}}\mathrm{d}x$;　(48) $\int \frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}\mathrm{d}x$.

解 (1) $\int\sqrt{7+4x}\mathrm{d}x=\frac{1}{4}\int(7+4x)^{\frac{1}{2}}\mathrm{d}(7+4x)=\frac{1}{6}(7+4x)^{\frac{3}{2}}+C$.

(2) $\int\frac{1}{\sqrt[3]{2-3x}}\mathrm{d}x=-\frac{1}{3}\int(2-3x)^{-\frac{1}{3}}\mathrm{d}(2-3x)=-\frac{1}{2}(2-3x)^{\frac{2}{3}}+C$.

(3) $\int\frac{\mathrm{d}x}{5x-2}=\frac{1}{5}\int\frac{\mathrm{d}(5x-2)}{5x-2}=\frac{1}{5}\ln|5x-2|+C$.

(4) $\int\mathrm{e}^{1-3x}\mathrm{d}x=-\frac{1}{3}\int\mathrm{e}^{1-3x}\mathrm{d}(1-3x)=-\frac{1}{3}\mathrm{e}^{1-3x}+C$.

(5) $\int\mathrm{sh}(5x-1)\mathrm{d}x=\frac{1}{5}\int\mathrm{sh}(5x-1)\mathrm{d}(5x-1)=\frac{1}{5}\mathrm{ch}(5x-1)+C$.

(6) $\int\mathrm{ch}(1-\frac{1}{2}x)\mathrm{d}x=-2\int\mathrm{ch}(1-\frac{1}{2}x)\mathrm{d}(1-\frac{1}{2}x)=-2\mathrm{sh}(1-\frac{1}{2}x)+C$.

(7) $\int\frac{1}{\sin^2 8x}\mathrm{d}x=\int\csc^2 8x\mathrm{d}x=\frac{1}{8}\int\csc^2 8x\mathrm{d}(8x)=-\frac{1}{8}\cot 8x+C$.

(8) $\int\tan(3x-5)\mathrm{d}x=\int\frac{\sin(3x-5)}{\cos(3x-5)}\mathrm{d}x=-\frac{1}{3}\int\frac{1}{\cos(3x-5)}\mathrm{d}\cos(3x-5)$

$$=-\frac{1}{3}\ln|\cos(3x-5)|+C.$$

(9) $\int\frac{x\mathrm{d}x}{\sqrt{2x^2+3}}=\frac{1}{4}\int\frac{\mathrm{d}(2x^2+3)}{\sqrt{2x^2+3}}=\frac{1}{2}\sqrt{2x^2+3}+C$.

(10) $\int\frac{x}{1+x^2}\mathrm{d}x=\frac{1}{2}\int\frac{1}{1+x^2}\mathrm{d}(1+x^2)=\frac{1}{2}\ln(1+x^2)+C$.

(11) $\int\frac{x^2}{(4+x^3)^2}\mathrm{d}x=\frac{1}{3}\int(4+x^3)^{-2}\mathrm{d}(4+x^3)=-\frac{1}{3(4+x^3)}+C$.

(12) $\int\frac{x^2}{\sqrt[3]{x^3+1}}\mathrm{d}x=\frac{1}{3}\int(x^3+1)^{-\frac{1}{3}}\mathrm{d}(x^3+1)=\frac{1}{2}(x^3+1)^{\frac{2}{3}}+C$.

(13) $\int 2x\mathrm{e}^{-x^2}\mathrm{d}x=-\int\mathrm{e}^{-x^2}\mathrm{d}(-x^2)=-\mathrm{e}^{-x^2}+C$.

(14) $\int\mathrm{e}^{\mathrm{e}^x+x}\mathrm{d}x=\int\mathrm{e}^{\mathrm{e}^x}\cdot\mathrm{e}^x\mathrm{d}x=\int\mathrm{e}^{\mathrm{e}^x}\mathrm{d}\mathrm{e}^x=\mathrm{e}^{\mathrm{e}^x}+C$.

(15) $\int\mathrm{e}^x\sin\mathrm{e}^x dx=\int\sin\mathrm{e}^x\mathrm{d}\mathrm{e}^x=-\cos\mathrm{e}^x+C$.

(16) $\int\frac{\mathrm{e}^{2x}-1}{\mathrm{e}^x}\mathrm{d}x=\int(\mathrm{e}^x-\mathrm{e}^{-x})\mathrm{d}x=\int\mathrm{e}^x\mathrm{d}x-\int\mathrm{e}^{-x}\mathrm{d}x=\mathrm{e}^x+\int\mathrm{e}^{-x}\mathrm{d}(-x)$

$$=e^x+e^{-x}+C.$$

(17) $\int\frac{\ln(\ln x)}{x\ln x}dx=\int\frac{\ln(\ln x)}{\ln x}d\ln x=\int\ln(\ln x)d\ln(\ln x)$

$$=\frac{1}{2}(\ln\ln x)^2+C.$$

(18) $\int\frac{\sqrt{1+\ln x}}{x}dx=\int\sqrt{1+\ln x}d(1+\ln x)=\frac{2}{3}(1+\ln x)^{\frac{3}{2}}+C.$

(19) $\int\frac{\sin\frac{1}{x}}{x^2}dx=-\int\sin\frac{1}{x}d\frac{1}{x}=\cos\frac{1}{x}+C.$

(20) $\int(1-\frac{1}{x^2})\sin(x+\frac{1}{x})dx=\int\sin(x+\frac{1}{x})d(x+\frac{1}{x})=-\cos(x+\frac{1}{x})+C.$

(21) $\int\frac{\sec^2\frac{1}{x}}{x^2}dx=-\int\sec^2\frac{1}{x}d\frac{1}{x}=-\tan\frac{1}{x}+C.$

(22) $\int\frac{a^{\frac{1}{x}}}{x^2}dx=-\int a^{\frac{1}{x}}d\frac{1}{x}=-\frac{a^{\frac{1}{x}}}{\ln a}+C.$

(23) $\int 10^{-3x+2}dx=-\frac{1}{3}\int 10^{-3x+2}d(-3x+2)=-\frac{10^{-3x+2}}{3\ln 10}+C.$

(24) $\int\frac{(\arctan x)^2}{1+x^2}dx=\int(\arctan x)^2 d\arctan x=\frac{1}{3}(\arctan x)^3+C.$

(25) $\int\sqrt{\frac{\arcsin x}{1-x^2}}dx=\int\sqrt{\arcsin x}d\arcsin x=\frac{2}{3}(\arcsin x)^{\frac{3}{2}}+C.$

(26) $\int\frac{dx}{\cos^2 x\sqrt{1+\tan x}}=\int(1+\tan x)^{-\frac{1}{2}}d(1+\tan x)=2\sqrt{1+\tan x}+C.$

(27) $\int\frac{dx}{(\arcsin x)^2\sqrt{1-x^2}}=\int\frac{d\arcsin x}{(\arcsin x)^2}=-\frac{1}{\arcsin x}+C.$

(28) $\int\cos^4 xdx=\int(\cos^2 x)^2dx=\frac{1}{4}\int(1+\cos 2x)^2dx$

$$=\frac{1}{4}\int(1+2\cos 2x+\cos^2 2x)dx$$

$$=\frac{1}{4}(x+\sin 2x+\int\frac{1+\cos 4x}{2}dx)$$

$$=\frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{8}(x+\frac{1}{4}\sin 4x)+C$$

$$=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.$$

(29) $\int\cos 3x\cos 4x\mathrm{d}x=\frac{1}{2}\int(\cos 7x+\cos x)\mathrm{d}x$

$$=\frac{1}{2}[\sin x+\frac{1}{7}\int\cos 7x\mathrm{d}(7x)]$$

$$=\frac{1}{2}\sin x+\frac{1}{14}\sin 7x+C.$$

(30) $\int\frac{\mathrm{d}x}{1+(2x-3)^2}=\frac{1}{2}\int\frac{\mathrm{d}(2x-3)}{1+(2x-3)^2}=\frac{1}{2}\arctan(2x-3)+C.$

(31) $\int\frac{\cos x-\sin x}{\cos x+\sin x}\mathrm{d}x=\int\frac{1}{\cos x+\sin x}\mathrm{d}(\cos x+\sin x)$

$$=\ln\left|\cos x+\sin x\right|+C.$$

(32) $\int\frac{2x+2}{x^2+2x+9}\mathrm{d}x=\int\frac{1}{x^2+2x+9}\mathrm{d}(x^2+2x+9)=\ln(x^2+2x+9)+C.$

(33) $\int\frac{\ln\tan x}{\sin x\cos x}\mathrm{d}x=\int\frac{\ln\tan x}{\tan x\cdot\cos^2 x}\mathrm{d}x=\int\frac{\ln\tan x}{\tan x}\mathrm{d}\tan x$

$$=\int\ln\tan x\mathrm{d}\ln\tan x=\frac{1}{2}(\ln\tan x)^2+C.$$

注意　被积函数较复杂是积分时的困难之处，注意到$\sin x\cos x=\tan x\cos^2 x$且$(\tan x)'=\frac{1}{\cos^2 x}$，所以可凑成微分$\frac{1}{\cos^2 x}\mathrm{d}x=\mathrm{d}\tan x$.

(34) $\int\frac{x^2}{1+x^2}\mathrm{d}x=\int\frac{x^2+1-1}{1+x^2}\mathrm{d}x=\int(1-\frac{1}{1+x^2})\mathrm{d}x=x-\arctan x+C.$

(35) $\int\frac{\mathrm{d}x}{9+25x^2}=\frac{1}{9}\int\frac{\mathrm{d}x}{1+(\frac{5}{3}x)^2}=\frac{1}{15}\int\frac{1}{1+(\frac{5}{3}x)^2}\mathrm{d}(\frac{5}{3}x)$

$$=\frac{1}{15}\arctan(\frac{5}{3}x)+C.$$

(36) $\int\frac{\sec^2 x}{2+\tan^2 x}\mathrm{d}x=\int\frac{1}{2+\tan^2 x}\mathrm{d}\tan x=\frac{\sqrt{2}}{2}\int\frac{1}{1+(\frac{\tan x}{\sqrt{2}})^2}\mathrm{d}\frac{\tan x}{\sqrt{2}}$

$$=\frac{\sqrt{2}}{2}\arctan\frac{\tan x}{\sqrt{2}}+C.$$

(37) $\int \frac{\mathrm{d}x}{x(1+\ln x)} = \int \frac{\mathrm{d}(1+\ln x)}{1+\ln x} = \ln|1+\ln x| + C$.

(38) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x = 2\int \frac{\arctan\sqrt{x}}{1+x}\mathrm{d}\sqrt{x} = 2\int \arctan\sqrt{x}\mathrm{d}(\arctan\sqrt{x})$

$$= (\arctan\sqrt{x})^2 + C .$$

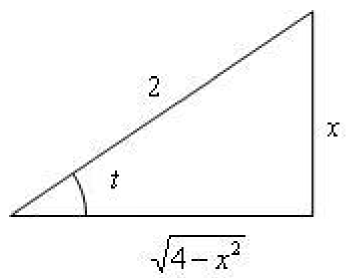
(39) 令 $x = 2\sin t \ (-\frac{\pi}{2} < x < \frac{\pi}{2})$, $\mathrm{d}x = 2\cos t\mathrm{d}t$.

$$\int \frac{x^2}{\sqrt{4-x^2}}\mathrm{d}x = \int \frac{4\sin^2 t}{2\cos t} 2\cos t\mathrm{d}t = 2\int (1-\cos 2t)\mathrm{d}t$$

$$= 2(t - \frac{1}{2}\sin 2t) + C = 2t - \sin t\cos t + C$$

$$= 2\arcsin\frac{x}{2} - \frac{x}{4}\sqrt{4-x^2} + C .$$

注意 将原函数还原为 x 的函数时，碰到较复杂的形式如 $\frac{1}{2}\sin 2t$ ，就易出错，这时应: ① 将 2 倍角的形式化为单角形式; ② 利用直角三角形.



(40) 令 $x = 3\sin t \ (-\frac{\pi}{2} < t < \frac{\pi}{2})$, $\mathrm{d}x = 3\cos t \ \mathrm{d}t$.

$$\int \frac{\mathrm{d}x}{x\sqrt{9-x^2}} = \int \frac{1}{3\sin t \cdot 3\cos t} \cdot 3\cos t\mathrm{d}t = \frac{1}{3}\int \csc t\mathrm{d}t$$

$$= \frac{1}{3}\int \frac{\csc^2 t + \csc t\cot t}{\csc t + \cot t}\mathrm{d}t = -\frac{1}{3}\int \frac{\mathrm{d}(\csc t + \cot t)}{\csc t + \cot t}$$

$$= -\frac{1}{3}\ln|\csc t + \cot t| + C = -\frac{1}{3}\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + C$$

$$=\frac{1}{3}\ln\left|\frac{x}{3+\sqrt{9-x^2}}\right|+C.$$

(41) 令 $x=a\sin t\ \ (-\frac{\pi}{2}<t<\frac{\pi}{2})$, $\mathrm{d}x=a\cos t\mathrm{d}t$.

$$\int\frac{\sqrt{a^2-x^2}}{x^2}\mathrm{d}x=\int\frac{a\cos t}{a^2\sin^2 t}\cdot a\cos t\mathrm{d}t=\int\cot^2 t\mathrm{d}t$$

$$=\int(\csc^2 t-1)\mathrm{d}t=-\cot t-t+C$$

$$=-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C.$$

(42) $\int t\sqrt{25-t^2}\mathrm{d}t=-\frac{1}{2}\int(25-t^2)^{\frac{1}{2}}\mathrm{d}(25-t^2)=-\frac{1}{3}(25-t^2)^{\frac{3}{2}}+C$.

(43) 令 $x=\frac{3}{2}\tan t$, $\mathrm{d}x=\frac{3}{2}\sec^2 t\mathrm{d}t$.

$$\int\frac{\mathrm{d}x}{\sqrt{4x^2+9}}=\int\frac{1}{3\sec t}\frac{3}{2}\sec^2 t\mathrm{d}t=\frac{1}{2}\int\sec t\mathrm{d}t$$

$$=\frac{1}{2}\int\frac{\sec^2 t+\sec t\cdot\tan t}{\sec t+\tan t}\mathrm{d}t$$

$$=\frac{1}{2}\int\frac{1}{\sec t+\tan t}\mathrm{d}(\sec t+\tan t)=\frac{1}{2}\ln\left|\sec t+\tan t\right|$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}}{3}+\frac{2x}{3}\right|+C_1=\frac{1}{2}\ln\left|\sqrt{4x^2+9}+2x\right|+C.$$

(44) 令 $x=a\tan t,\ \mathrm{d}x=a\sec^2 t\mathrm{d}t$.

$$\int\frac{\mathrm{d}x}{x^2\sqrt{a^2+x^2}}=\int\frac{a\sec^2 t}{a^2\tan^2 t\cdot a\sec t}\mathrm{d}t=\frac{1}{a^2}\int\frac{\cos t}{\sin^2 t}\mathrm{d}t$$

$$=\frac{1}{a^2}\int\frac{1}{\sin^2 t}\mathrm{d}\sin t=-\frac{1}{a^2\sin t}+C$$

$$=-\frac{\sqrt{a^2+x^2}}{a^2x}+C.$$

(45) 令 $x=\sqrt{2}\sec t\ \ (0<t<\pi),\ \mathrm{d}x=\sqrt{2}\sec t\tan t\mathrm{d}t$.

$$\int\frac{\sqrt{x^2-2}}{x}\mathrm{d}x=\int\frac{\sqrt{2}\tan t}{\sqrt{2}\sec t}\cdot\sqrt{2}\sec t\tan t\mathrm{d}t=\sqrt{2}\int\tan^2 t\mathrm{d}t$$

$$=\sqrt{2}\int(\sec^2 t-1)\mathrm{d}t=\sqrt{2}(\tan t-t)+C$$

$$=\sqrt{2}\left(\frac{\sqrt{x^2-2}}{\sqrt{2}}-\arccos\frac{\sqrt{2}}{|x|}\right)+C$$

$$=\sqrt{x^2-2}-\sqrt{2}\arccos\frac{\sqrt{2}}{|x|}+C\ .$$

(46)　令 $x=\frac{2}{3}\sec t\ \ (0<t<\pi),\ \mathrm{d}x=\frac{2}{3}\sec t\tan t\mathrm{d}t$.

$$\int\frac{2x-1}{\sqrt{9x^2-4}}\mathrm{d}x=\int\frac{\frac{4}{3}\sec t-1}{2\tan t}\cdot\frac{2}{3}\sec t\tan t\mathrm{d}t=\frac{4}{9}\int\sec^2 t\mathrm{d}t-\frac{1}{3}\int\sec t\mathrm{d}t$$

$$=\frac{4}{9}\tan t-\frac{1}{3}\int\frac{\sec^2 t+\sec t\tan t}{\sec t+\tan t}\mathrm{d}t$$

$$=\frac{4}{9}\tan t-\frac{1}{3}\int\frac{\mathrm{d}(\sec t+\tan t)}{\sec t+\tan t}$$

$$=\frac{4}{9}\tan t-\frac{1}{3}\ln|\sec t+\tan t|+C_1$$

$$=\frac{2}{9}\sqrt{9x^2-4}-\frac{1}{3}\ln\left|\frac{3x}{2}+\frac{\sqrt{9x^2-4}}{2}\right|+C_1$$

$$=\frac{2}{9}\sqrt{9x^2-4}-\frac{1}{3}\ln\left|3x+\sqrt{9x^2-4}\right|+C\ .$$

(47)　令 $x+1=\tan t,\ \mathrm{d}x=\sec^2 t\mathrm{d}t$.

$$\int\frac{x}{\sqrt{x^2+2x+2}}\mathrm{d}x=\int\frac{x}{\sqrt{(x+1)^2+1}}\mathrm{d}x=\int\frac{\tan t-1}{\sec t}\cdot\sec^2 t\mathrm{d}t$$

$$=\int\tan t\sec t\mathrm{d}t-\int\sec t\mathrm{d}t=\sec t-\int\frac{\sec t(\sec t+\tan t)}{\sec t+\tan t}\mathrm{d}t$$

$$=\sec t-\int\frac{1}{\sec t+\tan t}\mathrm{d}(\sec t+\tan t)$$

$$=\sec t-\ln|\sec t+\tan t|+C$$

$$=\sqrt{(x+1)^2+1}-\ln\left|\sqrt{(x+1)^2+1}+x+1\right|+C$$

$$=\sqrt{x^2+2x+2}-\ln\left|\sqrt{x^2+2x+2}+x+1\right|+C.$$

(48) $\displaystyle\int\frac{e^x-1}{e^x+1}dx=\int\frac{e^x}{e^x+1}dx-\int\frac{1}{e^x+1}dx=\int\frac{e^x}{e^x+1}dx-\int\frac{1+e^x-e^x}{e^x+1}dx$

$$=2\int\frac{e^x}{e^x+1}dx-\int 1dx=2\int\frac{1}{e^x+1}d(e^x+1)-x$$

$$=2\ln(e^x+1)-x+C.$$

注意 ① 被积函数直接凑微分比较困难，不妨将其拆为两项，然后再积分.

② 计算$\displaystyle\int\frac{1}{1+e^x}dx$时，除了上述解法，常用的解法还有以下两种:

$$\int\frac{1}{1+e^x}dx=\int\frac{e^x}{e^x(1+e^x)}dx=\int(\frac{1}{e^x}-\frac{1}{1+e^x})e^xdx$$

$$=\int\frac{1}{e^x}d(e^x)+\int\frac{1}{1+e^x}d(1+e^x)$$

$$=\ln e^x-\ln(1+e^x)+C=x-\ln(1+e^x)+C.$$

及

$$\int\frac{1}{1+e^x}dx=\int\frac{e^{-x}}{e^{-x}(1+e^x)}dx=\int\frac{e^{-x}}{1+e^{-x}}dx=-\int\frac{1}{1+e^{-x}}d(1+e^{-x})$$

$$=-\ln(1+e^{-x})+C.$$