第三节　可利用变量代换法求解的一阶微分方程

习题　12-3

1. 求下列微分方程的通解:

(1) $xy'-y=xe^{\frac{y}{x}}$;　(2) $x\dfrac{dy}{dx}=y(\ln|y|-\ln|x|)$;

(3) $xy'-y=\sqrt{x^2+y^2}$;　(4) $(x^2+y^2)dx-2xydy=0$;

(5) $2x^3y'=y(2x^2-y^2)$;　(6) $(x+y\cos\dfrac{y}{x})dx-x\cos\dfrac{y}{x}dy=0$;

(7) $(x^3+y^3)dx-3xy^2dy=0$;　(8) $xy'=y+x\tan\dfrac{y}{x}$.

解 (1) 方程化为

$$y'-\frac{y}{x}=e^{\frac{y}{x}},$$

令 $\dfrac{y}{x}=u$, $y'=u+x\dfrac{du}{dx}$, 方程化为

$$u+x\frac{du}{dx}-u=e^{u},$$

$$e^{-u}du=\frac{dx}{x}.$$

解此方程, 得

$$-e^{-u}=\ln|x|+\ln|C|,$$

$$e^{-\frac{y}{x}}+\ln|Cx|=0,$$

(2) 方程化为

$$\frac{dy}{dx}=\frac{y}{x}\ln\left|\frac{y}{x}\right|,$$

令 $\dfrac{y}{x}=u$, $y'=u+x\dfrac{du}{dx}$, 方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=u\ln|u|,$$

$$\frac{\mathrm{d}u}{u(\ln|u|-1)}=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$\ln|\ln|u|-1|=\ln|x|+\ln|C|,$$

$$y=x\mathrm{e}^{Cx+1}.$$

(3)　方程化为

$$y'-\frac{y}{x}=\sqrt{1+(\frac{y}{x})^2},$$

令 $\frac{y}{x}=u$，$y'=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$x\frac{\mathrm{d}u}{\mathrm{d}x}=\sqrt{1+u^2},$$

$$\frac{\mathrm{d}u}{\sqrt{1+u^2}}=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$\ln(u+\sqrt{1+u^2})=\ln x+\ln C,$$

$$y+\sqrt{x^2+y^2}=Cx^2.$$

(4)　方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(\frac{x}{y}+\frac{y}{x}),$$

令 $\frac{y}{x}=u$，$y'=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2u}+\frac{u}{2},$$

$$\frac{2u\mathrm{d}u}{1-u^2}=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$-\ln|1-u^2|+\ln|C|=\ln|x|,$$

$$y^2=x(x-C).$$

(5)　方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\frac{y}{x}[2-(\frac{y}{x})^2],$$

令 $\frac{y}{x}=u$，$y'=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{u}{2}(2-u^2),$$

$$-\frac{2\mathrm{d}u}{u^3}=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$u^{-2}=\ln|x|+C,$$

$$x^2=y^2(\ln|x|+C).$$

(6) 方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sec\frac{y}{x}+\frac{y}{x},$$

令 $\frac{y}{x}=u$，$y'=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\sec u+u,$$

$$\cos u\mathrm{d}u=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$\sin u=\ln|x|+C,$$

$$\sin\frac{y}{x}-\ln|x|=C.$$

(7) 方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3}[(\frac{x}{y})^2+\frac{y}{x}],$$

令 $\frac{y}{x}=u$，$y'=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{3}(\frac{1}{u^2}+u),$$

$$\frac{3u^2\mathrm{d}u}{1-2u^3}=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$-\frac{1}{2}\ln\left|1-2u^3\right|=\ln|x|+C,$$

$$x^3 - 2y^3 = Cx.$$

(8) 方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \tan\frac{y}{x},$$

令 $\frac{y}{x} = u$, $y' = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$, 方程化为

$$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = u + \tan u,$$

$$\cot u\mathrm{d}u = \frac{\mathrm{d}x}{x},$$

解此方程, 得

$$\ln|\sin u| = \ln|x| + \ln|C|,$$

$$\sin\frac{y}{x} = Cx.$$

2. 求下列微分方程满足所给初始条件的特解:

(1) $xy' = y + \frac{x^2}{y}$, $y(1) = 2$;

(2) $(y^2 - 3x^2)\mathrm{d}y + 2xy\mathrm{d}x = 0$, $y(0) = 1$;

(3) $(x^2 + 2xy - y^2)\mathrm{d}x + (y^2 + 2xy - x^2)\mathrm{d}y = 0$, $y(1) = 1$;

(4) $y - xy' = 2(x + yy')$, $y(1) = 1$.

解 (1) 方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{x}{y},$$

令 $\frac{y}{x} = u$, $y' = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$, 方程化为

$$u + x\frac{\mathrm{d}u}{\mathrm{d}x} = u + \frac{1}{u},$$

$$u\mathrm{d}u = \frac{\mathrm{d}x}{x},$$

解此方程, 得

$$\frac{u^2}{2} = \ln|x| + C,$$

$$y^2 = 2x^2(\ln|x| + C),$$

将初始条件代入得 $C=2$，特解为 $y^2=2x^2(\ln|x|+2)$.

(2) 方程化为

$$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{3}{2}\frac{x}{y}-\frac{1}{2}\frac{y}{x},$$

令 $\frac{x}{y}=u$, $\frac{\mathrm{d}x}{\mathrm{d}y}=u+y\frac{\mathrm{d}u}{\mathrm{d}y}$，方程化为

$$u+y\frac{\mathrm{d}u}{\mathrm{d}y}=\frac{3}{2}u-\frac{1}{2u},$$

$$\frac{2u}{(u^2-1)}\mathrm{d}u=\frac{\mathrm{d}y}{y},$$

解此方程，得

$$\ln\left|u^2-1\right|=\ln|y|+\ln|C|,$$

$$x^2-y^2=Cy^3,$$

将初始条件代入得 $C=-1$，特解为 $y^3=y^2-x^2$.

(3) 方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(\frac{y}{x})^2-2\frac{y}{x}-1}{(\frac{y}{x})^2+2\frac{y}{x}-1},$$

令 $\frac{y}{x}=u$, $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{u^2-2u-1}{u^2+2u-1},$$

$$-\frac{u^2+2u-1}{u^3+u^2+u+1)}\mathrm{d}u=(\frac{1}{u+1}-\frac{2u}{u^2+1})\mathrm{d}u=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$\ln\left|\frac{u+1}{u^2+1}\right|=\ln|x|+\ln|C|$$

$$\frac{x+y}{x^2+y^2}=C,$$

将初始条件代入得 $C=1$，特解为 $x+y=x^2+y^2$.

(4) 方程化为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{y}{x}-2}{1+2\frac{y}{x}},$$

令 $\frac{y}{x}=u$，$\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$，方程化为

$$u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{u-2}{1+2u},$$

$$-\frac{1+2u}{2(1+u^2)}\mathrm{d}u=\frac{\mathrm{d}x}{x},$$

解此方程，得

$$-\frac{1}{2}[\arctan u+\ln(1+u^2)]=\ln|x|+C,$$

$$\arctan\frac{y}{x}+\ln(x^2+y^2)=C,$$

将初始条件代入得 $C=\frac{\pi}{4}+\ln 2=\ln(2\mathrm{e}^{\frac{\pi}{4}})$，特解为 $\arctan\frac{y}{x}=\ln\frac{2\mathrm{e}^{\frac{\pi}{4}}}{x^2+y^2}$.

3. 求下列微分方程的通解:

(1) $y'+y=(\cos x-\sin x)y^2$;　　(2) $y'-\frac{4y}{x}=x\sqrt{y}$;

(3) $y'-y+2xy^{-1}=0$;　　(4) $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}=a\ln|x|\cdot y^2$;

(5) $(y-x^2)\mathrm{d}y+2xy\mathrm{d}x=0$.

解 (1) 方程化为

$$y^{-2}y'+y^{-1}=(\cos x-\sin x),$$

令 $z=y^{-1}$，方程化为

$$z'-z=\sin x-\cos x,$$

求解此线性方程，得

$$z=\mathrm{e}^{\int\mathrm{d}x}[\int(\sin x-\cos x)\mathrm{e}^{-\int\mathrm{d}x}\mathrm{d}x+C]=\mathrm{e}^x(-\mathrm{e}^{-x}\sin x+C),$$

$$\frac{1}{y}=-\sin x+C\mathrm{e}^x.$$

(2) 方程化为

$$y^{-\frac{1}{2}}y'-\frac{4}{x}y^{\frac{1}{2}}=x,$$

令 $z=y^{\frac{1}{2}}$，方程化为

$$z'-\frac{2}{x}z=\frac{x}{2},$$

求解此线性方程，得

$$z=\mathrm{e}^{\int\frac{2}{x}\mathrm{d}x}[\int\frac{x}{2}\mathrm{e}^{-\int\frac{2}{x}\mathrm{d}x}\mathrm{d}x+C]=x^2(\frac{1}{2}\ln|x|+C),$$

$$y=x^4(C+\frac{1}{2}\ln|x|)^2.$$

(3)　方程化为

$$yy'-y^2=-2x,$$

令 $z=y^2$，方程化为

$$z'-2z=-4x,$$

求解此线性方程，得

$$z=\mathrm{e}^{\int 2\mathrm{d}x}[\int -4x\mathrm{e}^{-\int 2\mathrm{d}x}\mathrm{d}x+C]=\mathrm{e}^{2x}[(2x+1)\mathrm{e}^{-2x}+C]=2x+1+C\mathrm{e}^{2x},$$

$$y^2=C\mathrm{e}^{2x}+2x+1.$$

(4)　方程化为

$$y^{-2}y'+\frac{1}{x}y^{-1}=a\ln|x|,$$

令 $z=y^{-1}$，方程化为

$$z'-\frac{1}{x}z=-a\ln|x|,$$

求解此线性方程，得

$$z=\mathrm{e}^{\int\frac{1}{x}\mathrm{d}x}[\int -a\ln|x|\mathrm{e}^{-\int\frac{1}{x}\mathrm{d}x}\mathrm{d}x+C]=x(-\frac{a}{2}\ln^2|x|+C)$$

$$xy\left[C-\frac{a}{2}(\ln|x|)^2\right]=1.$$

(5)　方程化为

$$2x\frac{\mathrm{d}x}{\mathrm{d}y}-\frac{1}{y}x^2=-1,$$

方程进一步化为

$$\frac{\mathrm{d}x^2}{\mathrm{d}y}-\frac{1}{y}x^2=-1,$$

求解此线性方程，得

$$x^2=\mathrm{e}^{\int\frac{1}{y}\mathrm{d}y}\left(\int-\mathrm{e}^{-\int\frac{1}{y}\mathrm{d}y}\mathrm{d}y+C\right)=y(-\ln|y|+C),$$

$$x^2=y(C-\ln|y|).$$

4. 用适当的变换，求下列微分方程的通解:

(1) $xy'+y=y\ln|xy|$，　　(2) $(x+y)^2y'=a^2$，

(3) $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2}{x}+\tan\frac{y^2}{x}$，　　(4) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(x-y)^4}+1$.

解 (1) 方程化为

$$(xy)'=y\ln|xy|,$$

令 $u=xy$，方程化为

$$u'=\frac{u}{x}\ln|u|,$$

$$\frac{\mathrm{d}u}{u\ln|u|}=\frac{\mathrm{d}x}{x},$$

求解此微分方程，得

$$\ln|\ln|u||=\ln|x|+\ln|C|,$$

$$xy=\mathrm{e}^{Cx}.$$

(2) 令 $u=x+y$，方程化为

$$u^2(u'-1)=a^2,$$

$$\frac{u^2\mathrm{d}u}{a^2+u^2}=\mathrm{d}x,$$

求解此微分方程，得

$$u-a\arctan\frac{u}{a}=x+C,$$

$$y=a\arctan\frac{x+y}{a}+C.$$

(3) 令 $u=\frac{y^2}{x}$，方程化为

$$u + xu' = u + \tan u ,$$

$$\cot u \mathrm{d}u = \frac{\mathrm{d}x}{x},$$

求解此微分方程，得

$$\ln|\sin u| = \ln|x| + \ln|C| ,$$

$$\sin\frac{y^2}{x} = Cx .$$

(4)　令 $u = x - y$，方程化为

$$1 - u' = \frac{1}{u^4} + 1 ,$$

$$u^4 \mathrm{d}u = -\mathrm{d}x ,$$

求解此微分方程，得

$$\frac{u^5}{5} = -x + C ,$$

$$(x - y)^5 = -5x + C .$$