第四节　有理函数的积分与积分表的使用

习题　4-4

求下列不定式积分:

1. $\int \frac{\mathrm{d}x}{3x^2-2x+2}$.

解　$\int \frac{\mathrm{d}x}{3x^2-2x+2} = \int \frac{\mathrm{d}x}{3(x-\frac{1}{3})^2+\frac{5}{3}} = \frac{1}{3}\int \frac{\mathrm{d}x}{(x-\frac{1}{3})^2+\frac{5}{9}}$

$$= \frac{1}{3}\int \frac{\mathrm{d}(x-\frac{1}{3})}{(x-\frac{1}{3})^2+(\frac{\sqrt{5}}{3})^2} = \frac{1}{3}\cdot\frac{3}{\sqrt{5}}\arctan\frac{x-\frac{1}{3}}{\frac{\sqrt{5}}{3}}+C$$

$$= \frac{1}{\sqrt{5}}\arctan\frac{3x-1}{\sqrt{5}}+C.$$

2. $\int \frac{x^5+x^4-8}{x^3-x}\mathrm{d}x$.

解　$\int \frac{x^5+x^4-8}{x^3-x}\mathrm{d}x = \int \frac{x^2(x^3-x)+x(x^3-x)+(x^3-x)+(x^2+x)-8}{x^3-x}\mathrm{d}x$

$$= \int (x^2+x+1+\frac{1}{x-1}-\frac{8}{x^3-x})\mathrm{d}x$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|-8\int \frac{\mathrm{d}x}{x^3-x}.$$

因为
$$\frac{1}{x^3-x} = -\frac{1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)},$$

故
$$\int \frac{\mathrm{d}x}{x^3-x} = -\ln|x|+\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+C_1,$$

所以
$$\int \frac{x^5+x^4-8}{x^3-x}\mathrm{d}x = \frac{1}{3}x^3+\frac{1}{2}x^2+x+8\ln|x|-3\ln|x-1|-4\ln|x+1|+C.$$

注意　被积函数是有理假分式，应先将其化为多项式与有理真分式之和, 并将有理真分式分解为部分分式, 然后积分.

3. $\int \frac{\mathrm{d}x}{(x^2+a^2)^2}$　$(a>0)$.

解　令 $x = a\tan t$, $\mathrm{d}x = a\sec^2 t\mathrm{d}t$.

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^2} = \int \frac{a\sec^2 t\mathrm{d}t}{(a^2\tan^2 t+a^2)^2} = \int \frac{a\sec^2 t\mathrm{d}t}{a^4\sec^4 t} = \frac{1}{a^3}\int \cos^2 t\mathrm{d}t$$

$$= \frac{1}{a^3}\int \frac{1+\cos 2t}{2}\mathrm{d}t = \frac{1}{2a^3}(t+\frac{1}{2}\sin 2t)+C$$

$$= \frac{1}{2a^3}(t+\sin t\cos t)+C$$

$$= \frac{1}{2a^3}(\arctan\frac{x}{a}+\frac{x}{\sqrt{x^2+a^2}}\frac{a}{\sqrt{x^2+a^2}})+C$$

$$= \frac{1}{2a^3}(\arctan\frac{x}{a}+\frac{ax}{x^2+a^2})+C.$$

注意　此题若利用拆项的方法求解，设

$\frac{1}{(x^2+a^2)^2} = \frac{Ax+B}{(x^2+a^2)^2}+\frac{Cx+D}{x^2+1}$，再求得定常数，将很繁琐.

4. $\int \frac{\mathrm{d}x}{x^2(1-x)}$.

解　因为 $\frac{1}{x^2(1-x)} = \frac{1}{x}+\frac{1}{x^2}+\frac{1}{1-x}$，所以

$$\int \frac{\mathrm{d}x}{x^2(1-x)} = \int (\frac{1}{x}+\frac{1}{x^2}+\frac{1}{1-x})\mathrm{d}x = \ln|x|-\frac{1}{x}-\ln|1-x|+C$$

$$= -\frac{1}{x}-\ln\left|\frac{1-x}{x}\right|+C.$$

5. $\int \frac{x+5}{x^2-2x-1}\mathrm{d}x$.

解　$$\int \frac{x+5}{x^2-2x-1}\mathrm{d}x = \frac{1}{2}\int \frac{2x-2}{x^2-2x-1}\mathrm{d}x+6\int \frac{1}{x^2-2x-1}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{\mathrm{d}(x^2-2x-1)}{x^2-2x-1}+6\int \frac{\mathrm{d}(x-1)}{(x-1)^2-(\sqrt{2})^2}$$

$$= \frac{1}{2}\ln|x^2-2x-1|+3\ln\left|\frac{x-1-\sqrt{2}}{x-1+\sqrt{2}}\right|+C.$$

6. $\int \frac{x}{x^3-1}\mathrm{d}x$.

解　$$\int \frac{x}{x^3-1}\mathrm{d}x = \int \frac{x-1+1}{(x-1)(x^2+x+1)}\mathrm{d}x$$

$$= \int \frac{1}{x^2+x+1}\mathrm{d}x+\int \frac{1}{(x-1)(x^2+x+1)}\mathrm{d}x$$

$$=\int\frac{1}{x^2+x+1}\mathrm{d}x+\int[\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}]\mathrm{d}x$$

$$=\int\frac{1}{x^2+x+1}\mathrm{d}x+\frac{1}{3}\int\frac{1}{x-1}\mathrm{d}x-\frac{1}{6}\int\frac{(2x+1)+3}{x^2+x+1}\mathrm{d}x$$

$$=\int\frac{1}{x^2+x+1}\mathrm{d}x+\frac{1}{3}\int\frac{1}{x-1}\mathrm{d}x-\frac{1}{6}\int\frac{2x+1}{x^2+x+1}\mathrm{d}x$$

$$-\frac{1}{2}\int\frac{1}{x^2+x+1}\mathrm{d}x$$

$$=\frac{1}{2}\int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\mathrm{d}x+\frac{1}{3}\ln|x-1|-\frac{1}{6}\int\frac{\mathrm{d}(x^2+x+1)}{x^2+x+1}$$

$$=\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)+C.$$

7. $\int\tan^4 x\mathrm{d}x$.

解 $\int\tan^4 x\mathrm{d}x=\int(\sec^2 x-1)\tan^2 x\mathrm{d}x=\int\sec^2 x\tan^2 x\mathrm{d}x-\int\tan^2 x\mathrm{d}x$

$$=\int\tan^2 x\mathrm{d}\tan x-\int(\sec^2 x-1)\mathrm{d}x$$

$$=\frac{1}{3}\tan^3 x-\tan x+x+C.$$

8. $\int\frac{\mathrm{d}x}{1+\cos x}$.

解 令 $\tan\frac{x}{2}=u$，则 $\cos x=\frac{1-u^2}{1+u^2}$，$\mathrm{d}x=\frac{2}{1+u^2}\mathrm{d}u$.

$$\int\frac{\mathrm{d}x}{1+\cos x}=\int\frac{1}{1+\frac{1-u^2}{1+u^2}}\cdot\frac{2\mathrm{d}u}{1+u^2}=\int\frac{2\mathrm{d}u}{(1+u^2)+(1-u^2)}$$

$$=\int\mathrm{d}u=u+C=\tan\frac{x}{2}+C.$$

9. $\int\frac{\mathrm{d}x}{\sin x+\cos x}$.

解 令 $\tan\frac{x}{2}=u$，则 $\sin x=\frac{2u}{1+u^2},\cos x=\frac{1-u^2}{1+u^2}$，$\mathrm{d}x=\frac{2}{1+u^2}\mathrm{d}u$.

$$\int\frac{\mathrm{d}x}{\sin x+\cos x}=\int\frac{1}{\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2\mathrm{d}u}{1+u^2}=2\int\frac{\mathrm{d}u}{1+2u-u^2}$$

$$=-2\int\frac{\mathrm{d}u}{(u-1)^2-2}=-2\int\frac{\mathrm{d}(u-1)}{(u-1)^2-(\sqrt{2})^2}$$

$$=-2\cdot\frac{1}{2\sqrt{2}}\ln\left|\frac{u-1-\sqrt{2}}{u-1+\sqrt{2}}\right|+C=\frac{\sqrt{2}}{2}\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|+C$$

$$=\frac{\sqrt{2}}{2}\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|+C\,.$$

10. $\displaystyle\int\frac{\mathrm{d}x}{1+\sin x+\cos x}$.

解　令 $\tan\frac{x}{2}=u$，则 $\sin x=\frac{2u}{1+u^2}$，$\cos x=\frac{1-u^2}{1+u^2}$，$\mathrm{d}x=\frac{2}{1+u^2}\mathrm{d}u$.

$$\int\frac{\mathrm{d}x}{1+\sin x+\cos x}=\int\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2\mathrm{d}u}{1+u^2}=\int\frac{\mathrm{d}u}{1+u}=\ln|1+u|+C$$

$$=\ln\left|1+\tan\frac{x}{2}\right|+C\,.$$

11. $\displaystyle\int\frac{\mathrm{d}x}{2+5\cos x}$.

解　令 $\tan\frac{x}{2}=u$，则 $\cos x=\frac{1-u^2}{1+u^2}$，$\mathrm{d}x=\frac{2}{1+u^2}\mathrm{d}u$.

$$\int\frac{\mathrm{d}x}{2+5\cos x}=\int\frac{1}{2+5\frac{1-u^2}{1+u^2}}\cdot\frac{2\mathrm{d}u}{1+u^2}=2\int\frac{\mathrm{d}u}{7-3u^2}$$

$$=\frac{2}{\sqrt{3}}\int\frac{\mathrm{d}(\sqrt{3}u)}{(\sqrt{7})^2-(\sqrt{3}u)^2}=\frac{2}{\sqrt{3}}\cdot\frac{1}{2\sqrt{7}}\ln\left|\frac{\sqrt{3}u+\sqrt{7}}{\sqrt{3}u-\sqrt{7}}\right|+C$$

$$=\frac{1}{\sqrt{21}}\ln\left|\frac{\sqrt{3}\tan\frac{x}{2}+\sqrt{7}}{\sqrt{3}\tan\frac{x}{2}-\sqrt{7}}\right|+C\,.$$

12. $\displaystyle\int\frac{1-\tan x}{1+\tan x}\mathrm{d}x$.

解　$\displaystyle\int\frac{1-\tan x}{1+\tan x}\mathrm{d}x=\int\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\mathrm{d}x=\int\frac{\cos x-\sin x}{\cos x+\sin x}\mathrm{d}x$

$$=\int\frac{\mathrm{d}(\cos x+\sin x)}{\cos x+\sin x}=\ln|\sin x+\cos x|+C\,.$$

注意　对三角函数有理式的积分，尽可能利用三角恒等变形，拆项积分，在万

不得已时才用半角代换.

13. $\int \frac{\sqrt{x-1}}{x}\mathrm{d}x$.

解 令$\sqrt{x-1}=t$，则$x=1+t^2$，$\mathrm{d}x=2t\mathrm{d}t$.

$$\int \frac{\sqrt{x-1}}{x}\mathrm{d}x=\int \frac{t}{t^2+1}\cdot 2t\mathrm{d}t=2\int \frac{t^2}{t^2+1}\mathrm{d}t=2\int (1-\frac{1}{t^2+1})\mathrm{d}t$$

$$=2(t-\arctan t)+C=2(\sqrt{x-1}-\arctan\sqrt{x-1})+C.$$

14. $\int \frac{\mathrm{d}x}{1+\sqrt[3]{x+2}}$.

解 令$\sqrt[3]{x+2}=t$，则$x=t^3-2$，$\mathrm{d}x=3t^2\mathrm{d}t$.

$$\int \frac{\mathrm{d}x}{1+\sqrt[3]{x+2}}=\int \frac{3t^2}{t+1}\mathrm{d}t=3\int \frac{t^2-1+1}{t+1}\mathrm{d}t=3\int (t-1+\frac{1}{t+1})\mathrm{d}t$$

$$=3(\frac{t^2}{2}-t+\ln|1+t|)+C$$

$$=\frac{3}{2}\sqrt[3]{(x+2)^2}-3\sqrt[3]{x+2}+3\ln\left|1+\sqrt[3]{x+2}\right|+C.$$

15. $\int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt[3]{x})}$.

解 令$x=t^6$，$\mathrm{d}x=6t^5\mathrm{d}t$.

$$\int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt[3]{x})}=\int \frac{6t^5\mathrm{d}t}{t^3(1+t^2)}=6\int \frac{t^2\mathrm{d}t}{1+t^2}=6\int (1-\frac{1}{1+t^2})\mathrm{d}t$$

$$=6(t-\arctan t)+C=6(\sqrt[6]{x}-\arctan\sqrt[6]{x})+C.$$

注意 被积函数中既含有$\sqrt[3]{x}$，又有$\sqrt{x}$，要使其有理化, 必须令$x=t^6$.

16. $\int \frac{\mathrm{d}x}{\sqrt{2x+1}-\sqrt[4]{2x+1}}$.

解 令$\sqrt[4]{2x+1}=t$, $x=\frac{t^4-1}{2}$, $\mathrm{d}x=2t^3\mathrm{d}t$.

$$\int \frac{\mathrm{d}x}{\sqrt{2x+1}-\sqrt[4]{2x+1}}=\int \frac{2t^3\mathrm{d}t}{t^2-t}=2\int \frac{t^2-1+1}{t-1}\mathrm{d}t=2\int (t+1+\frac{1}{t-1})\mathrm{d}t$$

$$=2(\frac{1}{2}t^2+t+\ln|t-1|)+C=t^2+2t+2\ln|t-1|+C$$

$$=\sqrt{2x+1}+2\sqrt[4]{2x+1}+2\ln\left|\sqrt[4]{2x+1}-1\right|+C.$$

17. $\int\frac{\sqrt{1+x}}{1+\sqrt{1+x}}\mathrm{d}x$.

解　令 $\sqrt{1+x}=t,\ x=t^2-1,\ \mathrm{d}x=2t\mathrm{d}t$.

$$\int\frac{\sqrt{1+x}}{1+\sqrt{1+x}}\mathrm{d}x=\int\frac{t}{1+t}\cdot 2t\mathrm{d}t=2\int\frac{t^2-1+1}{1+t}\mathrm{d}t=2\int(t-1+\frac{1}{1+t})\mathrm{d}t$$

$$=2(\frac{t^2}{2}-t+\ln|1+t|)+C_1$$

$$=x+1-2\sqrt{1+x}+2\ln(\sqrt{1+x}+1)+C_1$$

$$=x-2\sqrt{1+x}+2\ln(\sqrt{1+x}+1)+C.$$

18. $\int\frac{x}{\sqrt{1+x-x^2}}\mathrm{d}x$.

解　令 $x-\frac{1}{2}=\frac{\sqrt{5}}{2}\sin t,\ x=\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t,\ \mathrm{d}x=\frac{\sqrt{5}}{2}\cos t\mathrm{d}t$.

$$\int\frac{x}{\sqrt{1+x-x^2}}\mathrm{d}x=\int\frac{x}{\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}}\mathrm{d}x=\int\frac{\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t}{\frac{\sqrt{5}}{2}\cos t}\cdot\frac{\sqrt{5}}{2}\cos t\mathrm{d}t$$

$$=\int(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t)\mathrm{d}t=\frac{1}{2}t-\frac{\sqrt{5}}{2}\cos t+C$$

$$=\frac{1}{2}\arcsin\frac{2x-1}{\sqrt{5}}-\frac{\sqrt{5}}{2}\sqrt{1-\frac{(2x-1)^2}{5}}+C$$

$$=\frac{1}{2}\arcsin\frac{2x-1}{\sqrt{5}}-\sqrt{1+x-x^2}+C.$$

19. $\int\sqrt{\frac{1-x}{1+x}}\mathrm{d}x$.

解　法 1　$\int\sqrt{\frac{1-x}{1+x}}\mathrm{d}x=\int\frac{1-x}{\sqrt{1-x^2}}\mathrm{d}x=\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x-\int\frac{x}{\sqrt{1-x^2}}\mathrm{d}x$

$$=\arcsin x+\frac{1}{2}\int\frac{\mathrm{d}(1-x^2)}{\sqrt{1-x^2}}=\arcsin x+\sqrt{1-x^2}+C.$$

注意　若令$\sqrt{1-x}$或$\sqrt{1+x}$为t将不易积分.

法 2　令$x=\sin t\ \ (-\frac{\pi}{2}<t<\frac{\pi}{2}),\ \mathrm{d}x=\cos t\mathrm{d}t$.

$$\int\sqrt{\frac{1-x}{1+x}}\mathrm{d}x=\int\frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x=\int\frac{\cos^2 t}{1+\sin t}\mathrm{d}t=\int\frac{1-\sin^2 t}{1+\sin t}\mathrm{d}t=\int(1-\sin t)\mathrm{d}t$$

$$=t+\cos t+C=\arcsin x+\sqrt{1-x^2}+C.$$

20. $\int\frac{\mathrm{d}x}{\sqrt{x(1+x)}}$.

解　法 1　$$\int\frac{\mathrm{d}x}{\sqrt{x(1+x)}}=\int\frac{\mathrm{d}x}{\sqrt{x}\cdot\sqrt{1+x}}=\int\frac{2\mathrm{d}\sqrt{x}}{\sqrt{1+(\sqrt{x})^2}}$$

$$=2\ln(\sqrt{x}+\sqrt{1+x})+C_1=\ln\left|2x+1+2\sqrt{x^2+x}\right|+C_1$$

$$=\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C.$$

法 2　令$x+\frac{1}{2}=\frac{1}{2}\sec t,\ x=-\frac{1}{2}+\frac{1}{2}\sec t,\ \mathrm{d}x=\frac{1}{2}\sec t\tan t\mathrm{d}t$.

$$\int\frac{\mathrm{d}x}{\sqrt{x(1+x)}}=\int\frac{\mathrm{d}x}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}}=\int\frac{\sec t\cdot\tan t}{\tan t}\mathrm{d}t=\int\sec t\mathrm{d}t$$

$$=\ln\left|\sec t+\tan t\right|=\ln\left|2x+1+\sqrt{(2x+1)^2-1}\right|+C_1$$

$$=\ln\left|2x+1+2\sqrt{x^2+x}\right|+C_1=\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C.$$