第四节　多元复合函数的求导法则

习题　8-4

1. 求下列函数的全导数:

(1) $z=\dfrac{v}{u}$， $u=\ln x, v=\mathrm{e}^{x}$;

(2) $z=\arcsin(x-y)$， $x=3t, y=t^{3}$;

(3) $u=xy+yz$， $y=\mathrm{e}^{x}, z=\sin x$;

(4) $u=\mathrm{e}^{2x}(y+z)$， $x=2t$， $y=\sin t, z=2\cos t$.

解 (1) $\dfrac{\mathrm{d}z}{\mathrm{d}x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\mathrm{d}v}{\mathrm{d}x}=-\dfrac{v}{u^{2}}\cdot\dfrac{1}{x}+\dfrac{1}{u}\cdot\mathrm{e}^{x}$

$$=-\frac{\mathrm{e}^{x}}{\ln^{2}x}\cdot\frac{1}{x}+\frac{1}{\ln x}\cdot\mathrm{e}^{x}=\frac{\mathrm{e}^{x}}{\ln x}(1-\frac{1}{x\ln x}).$$

(2) $\dfrac{\mathrm{d}z}{\mathrm{d}t}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{1}{\sqrt{1-(x-y)^{2}}}\cdot 3+\dfrac{(-1)}{\sqrt{1-(x-y)^{2}}}\cdot 3t^{2}$

$$=\frac{3}{\sqrt{1-(3t-t^{3})^{2}}}-\frac{3t^{2}}{\sqrt{1-(3t-t^{3})^{2}}}=\frac{3(1-t^{2})}{\sqrt{1-(3t-t^{3})^{2}}}.$$

(3) $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\mathrm{d}z}{\mathrm{d}x}=y+(x+z)\cdot\mathrm{e}^{x}+y\cdot\cos x$

$$=\mathrm{e}^{x}+(x+\sin x)\mathrm{e}^{x}+\mathrm{e}^{x}\cos x=\mathrm{e}^{x}(1+x+\sin x+\cos x).$$

(4) $\dfrac{\mathrm{d}u}{\mathrm{d}t}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\mathrm{d}y}{\mathrm{d}t}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\mathrm{d}z}{\mathrm{d}t}$

$$=2\mathrm{e}^{2x}(y+z)\cdot 2+\mathrm{e}^{2x}\cdot\cos t+\mathrm{e}^{2x}\cdot(-2\sin t)$$

$$=4\mathrm{e}^{4t}(\sin t+2\cos t)+\mathrm{e}^{4t}\cos t-2\mathrm{e}^{4t}\sin t$$

$$=\mathrm{e}^{4t}(2\sin t+9\cos t).$$

注意　① 第(3)小题是三个中间变量 x, y, z， 一个自变量 x 的情形，其中 x 既是自变量，又是中间变量，此题常错误地解答为:

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial u}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}x}=\mathrm{e}^{x}(x+\sin x+\cos x).$$

该解法的错误在于复合关系不清楚，因而丢掉一项.

② 此类题目也可以将中间变量代入到原函数中，将其化为一元函数，再用一元函数求导法则求出全导数. 例如，第(1) 小题也可用如下求法：先写出 $z=\dfrac{\mathrm{e}^{x}}{\ln x}$，然后求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

2. 求下列函数的一阶偏导数:

(1) $z=u\mathrm{e}^{\frac{u}{v}}$， $u=x^2+y^2, v=xy$;

(2) $z=x^2\ln y$， $x=\dfrac{s}{t}, y=3s-2t$;

(3) $z=x\arctan(xy)$， $x=t^2, y=s\mathrm{e}^{t}$.

解 (1) $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=(\mathrm{e}^{\frac{u}{v}}+u\mathrm{e}^{\frac{u}{v}}\cdot\frac{1}{v})\cdot 2x+u\mathrm{e}^{\frac{u}{v}}\cdot(-\frac{u}{v^2})\cdot y$$

$$=(\mathrm{e}^{\frac{x^2+y^2}{xy}}+\frac{x^2+y^2}{xy}\mathrm{e}^{\frac{x^2+y^2}{xy}})\cdot 2x-\mathrm{e}^{\frac{x^2+y^2}{xy}}\frac{(x^2+y^2)^2}{x^2y^2}\cdot y$$

$$=(2x+\frac{2x^2+2y^2}{y}-\frac{x^4+2x^2y^2+y^4}{x^2y})\mathrm{e}^{\frac{x^2+y^2}{xy}}$$

$$=\frac{1}{x^2y}(x^4-y^4+2x^3y)\mathrm{e}^{\frac{x^2+y^2}{xy}},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\quad=(\mathrm{e}^{\frac{u}{v}}+\frac{u}{v}\mathrm{e}^{\frac{u}{v}})\cdot 2y-\frac{u^2}{v^2}\mathrm{e}^{\frac{u}{v}}\cdot x$$

$$=(\mathrm{e}^{\frac{x^2+y^2}{xy}}+\frac{x^2+y^2}{xy}\mathrm{e}^{\frac{x^2+y^2}{xy}})\cdot 2y-\mathrm{e}^{\frac{x^2+y^2}{xy}}\cdot\frac{(x^2+y^2)^2}{x^2y^2}\cdot x$$

$$=(2y+\frac{2x^2+2y^2}{x}-\frac{x^4+2x^2y^2+y^4}{xy^2})\mathrm{e}^{\frac{x^2+y^2}{xy}}$$

$$=\frac{1}{xy^2}(y^4-x^4+2xy^3)\mathrm{e}^{\frac{x^2+y^2}{xy}}.$$

(2) $\dfrac{\partial z}{\partial s}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial s}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial s}=2x\ln y\cdot\dfrac{1}{t}+\dfrac{x^2}{y}\cdot 3$

$$=\frac{2s}{t}\ln(3s-2t)\cdot\frac{1}{t}+3\frac{s^2}{(3s-2t)t^2}$$

$$=\frac{2s}{t^2}\ln(3s-2t)+\frac{3s^2}{(3s-2t)t^2},$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}=2x\ln y\cdot(-\frac{s}{t^2})+\frac{x^2}{y}\cdot(-2)$$

$$=-\frac{2s^2}{t^3}\ln(3s-2t)-\frac{2s^2}{(3s-2t)t^2}.$$

(3) $\dfrac{\partial z}{\partial s}=\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial s}=x\dfrac{x}{1+(xy)^2}\cdot \mathrm{e}^t=\dfrac{t^4\mathrm{e}^t}{1+t^4s^2\mathrm{e}^{2t}}$,

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=[\arctan(xy)+x\cdot\frac{y}{1+(xy)^2}]\cdot 2t+\frac{x^2}{1+(xy)^2}\cdot s\mathrm{e}^t$$

$$=2t[\arctan(t^2s\mathrm{e}^t)+\frac{t^2s\mathrm{e}^t}{1+t^4s^2\mathrm{e}^{2t}}]+\frac{t^4s\mathrm{e}^t}{1+t^4s^2\mathrm{e}^{2t}}.$$

注意 常见错误是最后结果中仍有中间变量出现.

3. 设 f 具有一阶连续偏导数, 求下列复合函数的偏导数:

(1) $z=f(x^2-y^2,\mathrm{e}^{xy})$;　　(2) $z=f(x,x+y,x-y)$;

(3) $z=xy+\dfrac{y}{x}f(xy)$;　　(4) $u=f(x,xy,xyz)$.

解 (1) 将中间变量 x^2-y^2, e^{xy} 依次编号为1, 2，则

$$\frac{\partial z}{\partial x}=f_1'\cdot\frac{\partial}{\partial x}(x^2-y^2)+f_2'\cdot\frac{\partial}{\partial x}(\mathrm{e}^{xy})=2xf_1'+y\mathrm{e}^{xy}f_2',$$

$$\frac{\partial z}{\partial y}=f_1'\cdot\frac{\partial}{\partial y}(x^2-y^2)+f_2'\cdot\frac{\partial}{\partial y}(\mathrm{e}^{xy})=-2yf_1'+x\mathrm{e}^{xy}f_2'.$$

(2) 将中间变量 x, $x+y$, $x-y$ 依次编号为1, 2, 3，则

$$\frac{\partial z}{\partial x} = f_1'\cdot 1 + f_2'\cdot 1 + f_3'\cdot 1 = f_1' + f_2' + f_3',$$

$$\frac{\partial z}{\partial y} = f_2'\cdot 1 + f_3'\cdot(-1) = f_2' - f_3'.$$

(3) $\frac{\partial z}{\partial x} = y - \frac{y}{x^2} f(xy) + \frac{y}{x} f'(xy)\cdot y = y - \frac{y}{x^2} f(xy) + \frac{y^2}{x} f'(xy)$,

$$\frac{\partial z}{\partial y} = x + \frac{1}{x} f(xy) + \frac{y}{x} f'(xy)\cdot x = x + \frac{1}{x} f(xy) + yf'(xy).$$

(4) 将中间变量 x, xy, xyz 依次编号为1, 2, 3，则

$$\frac{\partial u}{\partial x} = f_1'\cdot 1 + f_2'\cdot y + f_3'\cdot yz = f_1' + yf_2' + yzf_3'$$

$$\frac{\partial u}{\partial y} = f_2'\cdot x + f_3'\cdot xz = xf_2' + xzf_3',$$

$$\frac{\partial u}{\partial z} = f_3'\cdot xy = xyf_3'.$$

4. 设 f 具有二阶连续偏导数, 求下列函数的指定的偏导数:

(1) $z = f(ax, by)$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$;

(2) $u = f(x^2 + y^2 + z^2)$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^3 z}{\partial x \partial y \partial z}$;

(3) $u = f(xy^2, yz^2)$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial y \partial z}$;

(4) $z = f(x\ln x, 2x - y)$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$.

解 (1) 将中间变量 ax, by 依次编号为1, 2，则

$$\frac{\partial z}{\partial x} = f_1'\cdot a = af_1', \qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(af_1') = a\frac{\partial}{\partial x}(f_1') = a^2 f_{11}'',$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(af_1') = a\frac{\partial}{\partial y}(f_1') = abf_{12}''.$$

(2) $\dfrac{\partial u}{\partial x}= f'\cdot 2x = 2xf'$,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(2xf')=2f'+2x\cdot f''\cdot 2x=2f'+4x^2f'',$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}(2xf')=2xf''\cdot 2y=4xyf'',$$

$$\frac{\partial^3 u}{\partial x\partial y\partial z}=\frac{\partial}{\partial z}(4xyf'')=4xyf'''\cdot 2z=8xyzf'''.$$

(3) 将中间变量 xy^2，yz^2 依次编号为1, 2，则

$$\frac{\partial u}{\partial y}=f_1'\cdot\frac{\partial}{\partial y}(xy^2)+f_2'\cdot\frac{\partial}{\partial y}(yz^2)=f_1'\cdot 2xy+f_2'\cdot z^2$$

$$=2xyf_1'+z^2f_2',$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(2xyf_1'+z^2f_2')=\frac{\partial}{\partial y}(2xy)\cdot f_1'+2xy\frac{\partial}{\partial y}(f_1')+z^2\frac{\partial}{\partial y}(f_2')$$

$$=2xf_1'+2xy[f_{11}''\frac{\partial}{\partial y}(xy^2)+f_{12}''\frac{\partial}{\partial y}(yz^2)]$$

$$+z^2[f_{21}''\frac{\partial}{\partial y}(xy^2)+f_{22}''\frac{\partial}{\partial y}(yz^2)]$$

$$=2xf_1'+2xy(f_{11}''\cdot 2xy+f_{12}''\cdot z^2)+z^2(f_{21}''\cdot 2xy+f_{22}''\cdot z^2)$$

$$=2xf_1'+4x^2y^2f_{11}''+4xyz^2f_{12}''+z^4f_{22}'',$$

$$\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial}{\partial z}(2xyf_1'+z^2f_2')=2xy\frac{\partial}{\partial z}(f_1')+\frac{\partial}{\partial z}(z^2)f_2'+z^2\frac{\partial}{\partial z}(f_2')$$

$$=2xy\cdot f_{12}''\cdot\frac{\partial}{\partial z}(yz^2)+2zf_2'+z^2\cdot f_{22}''\cdot\frac{\partial}{\partial z}(yz^2)$$

$$=2xy\cdot f_{12}''\cdot 2yz+2zf_2'+z^2\cdot f_{22}''\cdot 2yz$$

$$=4xy^2zf_{12}''+2yz^2f_{22}''+2zf_2'.$$

注意 二阶偏导数常错求为:

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(2xyf_1'+z^2 f_2')=\frac{\partial}{\partial y}(2xy)\cdot f_1'+2xy\frac{\partial}{\partial y}(f_1')+z^2\frac{\partial}{\partial y}(f_2')$$

$$=2xf_1'+2xyf_{11}''\cdot y^2+z^2 f_{22}''\cdot z^2=2xf_1'+2xy^3 f_{11}''+z^4 f_{22}''.$$

在求多元复合函数的二阶偏导数时，要特别记住抽象的多元复合函数的偏导函数与原来的函数具有相同的复合结构．此题产生错误的原因是没有认识到 f_1'， f_2' 与 f 有一样的复合结构，当它们继续对自变量(x 或 y)求偏导数时，必须再次运用复合函数的求导法则，这是一个很容易出错的问题，应特别注意.

(4)　将中间变量 $x\ln x$，$2x-y$ 依次编号为1，2，则

$$\frac{\partial z}{\partial x}=f_1'\frac{\partial}{\partial x}(x\ln x)+f_2'\frac{\partial}{\partial x}(2x-y)=f_1'(1+\ln x)+f_2'\cdot 2$$

$$=(1+\ln x)f_1'+2f_2',$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}[(1+\ln x)f_1'+2f_2']=\frac{\partial}{\partial x}(1+\ln x)f_1'+(1+\ln x)\frac{\partial}{\partial x}(f')+2\frac{\partial}{\partial x}(f_2')$$

$$=\frac{1}{x}f_1'+(1+\ln x)[f_{11}''\frac{\partial}{\partial x}(x\ln x)+f_{12}''\frac{\partial}{\partial x}(2x-y)]$$

$$+2[f_{21}''\frac{\partial}{\partial x}(x\ln x)+f_{22}''\frac{\partial}{\partial x}(2x-y)]$$

$$=\frac{1}{x}f_1'+(1+\ln x)[f_{11}''(1+\ln x)+f_{12}''\cdot 2]+2[f_{21}''(1+\ln x)+f_{22}''\cdot 2]$$

$$=\frac{1}{x}f_1'+(1+\ln x)^2 f_{11}''+4(1+\ln x)f_{12}''+4f_{22}'',$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}[(1+\ln x)f_1'+2f_2']=(1+\ln x)\frac{\partial}{\partial y}(f_1')+2\frac{\partial}{\partial y}(f_2')$$

$$=(1+\ln x)f_{12}''\frac{\partial}{\partial y}(2x-y)+2f_{22}''\frac{\partial}{\partial y}(2x-y)$$

$$=(1+\ln x)f_{12}''\cdot(-1)+2f_{22}''\cdot(-1)=-(1+\ln x)f_{12}''-2f_{22}''.$$

5.　证明函数 $u=\varphi(x-ct)+\psi(x+ct)$ 满足弦振动方程:

$$c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}.$$

证　因为

$$\frac{\partial u}{\partial x}=\varphi'(x-ct)\cdot 1+\psi'(x+ct)\cdot 1=\varphi'(x-ct)+\psi'(x+ct)\,,$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial x}[\varphi'(x-ct)+\psi'(x+ct)]=\varphi''(x-ct)\cdot 1+\psi''(x+ct)\cdot 1\\&=\varphi''(x-ct)+\psi''(x+ct)\,,\end{aligned}$$

$$\frac{\partial u}{\partial t}=\varphi'(x-ct)\cdot(-c)+\psi'(x+ct)\cdot c=-c\varphi'(x-ct)+c\psi'(x+ct)\,,$$

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}&=\frac{\partial}{\partial t}[-c\varphi'(x-ct)+c\psi'(x+ct)]\\&=-c\varphi''(x-ct)\cdot(-c)+c\psi''(x+ct)\cdot c\\&=c^2[\varphi''(x-ct)+\psi''(x+ct)]\,,\end{aligned}$$

所以

$$c^2\frac{\partial^2 u}{\partial x^2}=c^2[\varphi''(x-ct)+\psi''(x+ct)]=\frac{\partial^2 u}{\partial t^2}\,.$$

6. 若 $f(u,v)$ 的二阶偏导数连续，且满足拉普拉斯方程:

$$\Delta f=\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}=0\,.$$

证明 函数 $z=f(x^2-y^2,2xy)$ 也满足拉普拉斯方程:

$$\Delta z=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0\,.$$

证 令 $u=x^2-y^2$， $v=2xy$，则 $z=f(u,v)$.

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}\cdot 2x+\frac{\partial f}{\partial v}\cdot 2y=2x\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v}\,,$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=\frac{\partial}{\partial x}(2x\frac{\partial f}{\partial u}+2y\frac{\partial f}{\partial v})\\&=2\frac{\partial f}{\partial u}+2x(\frac{\partial^2 f}{\partial u^2}\cdot 2x+\frac{\partial^2 f}{\partial u\partial v}\cdot 2y)+2y(\frac{\partial^2 f}{\partial v\partial u}\cdot 2x+\frac{\partial^2 f}{\partial v^2}\cdot 2y)\\&=2\frac{\partial f}{\partial u}+4x^2\frac{\partial^2 f}{\partial u^2}+8xy\frac{\partial^2 f}{\partial u\partial v}+4y^2\frac{\partial^2 f}{\partial v^2}\,,\end{aligned}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial f}{\partial u}\cdot(-2y)+\frac{\partial f}{\partial v}\cdot 2x=-2y\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v}\,,$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(-2y\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v})$$

$$=-2\frac{\partial f}{\partial u}+(-2y)[\frac{\partial^2 f}{\partial u^2}\cdot(-2y)+\frac{\partial^2 f}{\partial u\partial v}\cdot 2x]$$

$$+2x[\frac{\partial^2 f}{\partial v\partial u}\cdot(-2y)+\frac{\partial^2 f}{\partial v^2}\cdot 2x]$$

$$=-2\frac{\partial f}{\partial u}+4y^2\frac{\partial^2 f}{\partial u^2}-8xy\frac{\partial^2 f}{\partial u\partial v}+4x^2\frac{\partial^2 f}{\partial v^2},$$

所以

$$\Delta z=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4x^2(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2})+4y^2(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2})$$

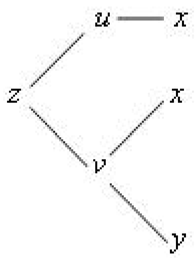
$$=(4x^2+4y^2)(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2})=(4x^2+4y^2)\cdot\Delta f=0.$$

7. 作自变量变换: $u=x,\ v=xy$，求方程

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0$$

的解.

解 z, x, y, u, v 的复合关系如下:



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}\cdot 1+\frac{\partial z}{\partial v}\cdot y=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial z}{\partial v}\cdot x=x\frac{\partial z}{\partial v},$$

将 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$ 的表示式代入方程 $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0$，得

$$x(\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v})-y(x\frac{\partial z}{\partial v})=0 \text{ 即 } x\frac{\partial z}{\partial u}=0,$$

由变换 $u=x$，可得 $u\frac{\partial z}{\partial u}=0$，于是有

$$\frac{\partial z}{\partial u}=0,$$

所以此方程的解为

$$z=f(v),$$

其中 f 为任意可微函数.

故方程 $x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}=0$ 的解为

$$z=f(x,y),$$

其中 f 为任意可微函数.

8. 用一阶全微分形式不变性求下列复合函数的全微分:

(1) $z=f(t),\ t=x+y$; (2) $z=\sin(2x+\mathrm{e}^y)$.

解 (1) $\mathrm{d}z=\mathrm{d}f(t)=f'(t)\mathrm{d}t=f'(x+y)\mathrm{d}(x+y)=f'(x+y)(\mathrm{d}x+\mathrm{d}y)$.

(2) $\mathrm{d}z=\mathrm{d}\sin(2x+\mathrm{e}^y)=\cos(2x+\mathrm{e}^y)\mathrm{d}(2x+\mathrm{e}^y)$

$$=\cos(2x+\mathrm{e}^y)[\mathrm{d}(2x)+\mathrm{d}\mathrm{e}^y]=\cos(2x+\mathrm{e}^y)(2\mathrm{d}x+\mathrm{e}^y\mathrm{d}y).$$