第二节

多元函数的偏导数

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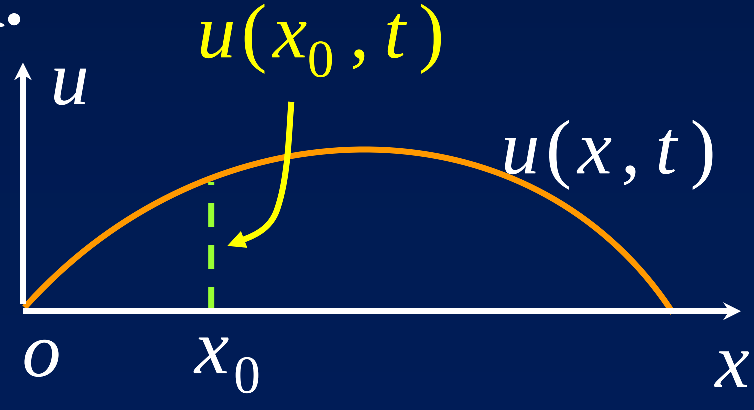
一、主要内容

（一）偏导数的概念

1.引例 弦线的振动问题.

研究弦在点 x_0 处的振动速度与加速度，就是将振幅 $u(x,t)$ 中的 x 固定于x_0处，求 $u(x_0,t)$ 关于 t 的一阶导数与二阶导数.



2. 定义8.6 设函数 $z=f(x,y)$在点$P_0(x_0,y_0)$的某邻域$U(P_0)$内有定义．若当固定 y在 y_0，$z=f(x,y_0)$ 在 $x=x_0$处的导数存在，即

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

存在，则称此极限为 $z=f(x,y)$在点(x_0,y_0)处 对x 的**偏导数**，记为

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)};\ \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)};\ z_x\big|_{(x_0,y_0)};\ f_x(x_0,y_0).$$

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注 $f_x(x_0, y_0)$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x, y_0)\Big|_{x=x_0}$$

同样可定义 函数 $f(x, y)$ 在点 (x_0, y_0) 对 y 的偏导数

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

$$= \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y=y_0}$$

记为 $\left.\dfrac{\partial z}{\partial y}\right|_{(x_0, y_0)}$; $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0, y_0)}$; $z_y\big|_{(x_0, y_0)}$; $f_y(x_0, y_0)$.

注　1° 偏导函数

若函数 $z=f(x,y)$ 在域 D 内每一点 (x,y) 处对 x 的（或 y）偏导数都存在，称该偏导数为 $z=f(x,y)$ 对自变量 x (或 y) 的**偏导函数**，也简称为**偏导数**，记为

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ z_x,\ f_x(x,y),\ f_1'(x,y)$$

$$\left(\ \frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y},\ z_y,\ f_y(x,y),\ f_2'(x,y)\ \right)$$

由此可知： $f_x(x_0,y_0)=f_x(x,y)\Big|_{(x_0,y_0)}$

$$f_y(x_0,y_0)=f_y(x,y)\Big|_{(x_0,y_0)}$$

2° 偏导数的概念可以推广到二元以上函数

例如: 三元函数 $u=f(x,y,z)$ 在点 (x,y,z) 处对 x 的偏导数定义为

$$f_x(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}$$

$$f_y(x,y,z)=?$$

$$f_z(x,y,z)=?$$

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3° 可(偏)导

若 $z=f(x,y)$在点(x_0,y_0)处的两个偏导数

$f_x(x_0,y_0), f_y(x_0,y_0)$均存在，则称 $f(x,y)$

在点(x_0,y_0)处可(偏)导.

4° 偏导数 $\dfrac{\partial z}{\partial x}$是一个整体记号，不能看作分子

与分母的商！ $\dfrac{\partial z}{\partial x} \neq \partial z \big/ \partial x$

5° 若 $f(x, y_0)$为分段函数，分段点为 x_0，则求 $f_x(x_0, y_0)$时，须用偏导数定义 .

3.多元函数在一点连续与偏导数存在的关系

对于一元函数：可导 $\underset{\not\Longleftarrow}{\Longrightarrow}$ 连续

对于多元函数：可（偏）导 $\underset{\not\Longleftarrow}{\not\Longrightarrow}$ 连续

（二）偏导数的计算

由偏导数的定义可知，偏导数的计算可归结为一元函数的导数计算.

$$f_x(x_0,y_0)=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

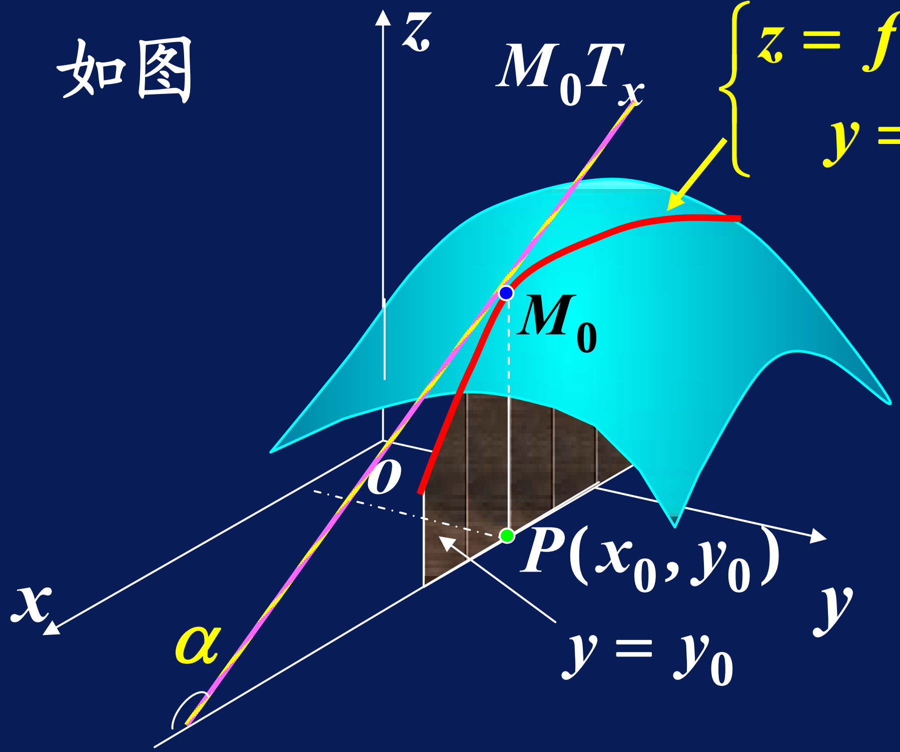
$$f_y(x_0,y_0)=\frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

求某个具体的点处的偏导数时方便

（三）偏导数的几何意义

设 $M_0(x_0, y_0, f(x_0, y_0))$ 为曲面 $z = f(x, y)$ 上一点,

如图



$$\begin{cases} z = f(x, y) \\ y = y_0 \end{cases} \Longleftrightarrow \begin{cases} z = f(x, y_0) \\ y = y_0 \end{cases}$$

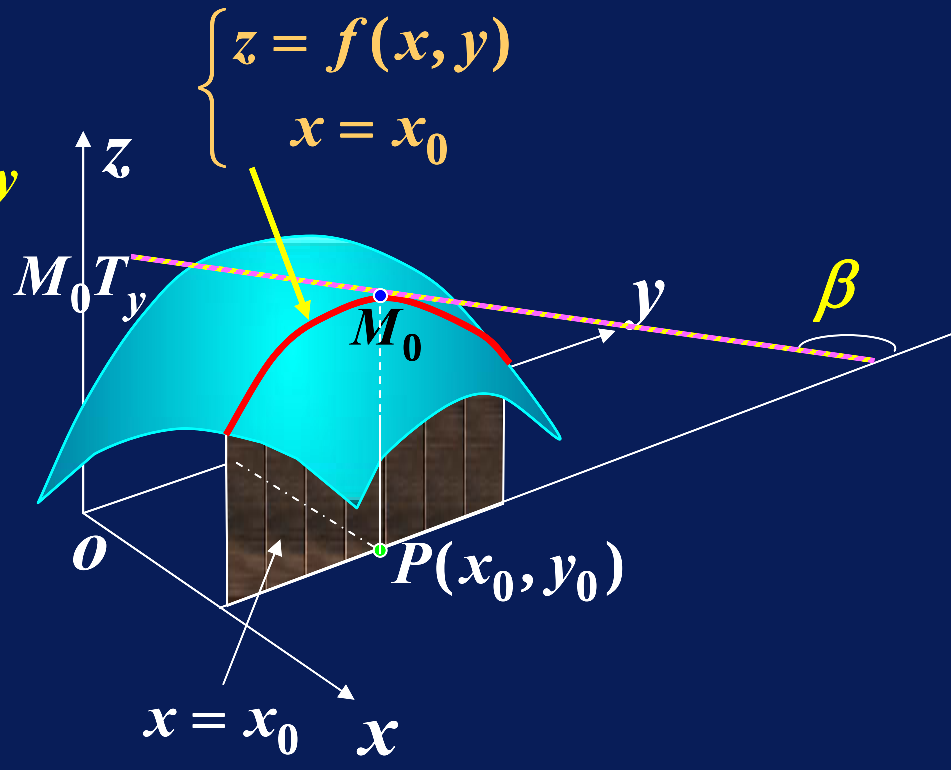
$$\tan\alpha = f_x(x_0, y_0) = \left.\frac{\mathrm{d} f(x, y_0)}{\mathrm{d}x}\right|_{x=x_0}$$

$$\tan\beta = f_y(x_0, y_0) = \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y = y_0}$$

是曲线 $\begin{cases} z = f(x, y) \\ x = x_0 \end{cases}$

在点 M_0 处的切线 M_0T_y

对 y 轴的斜率.



（四）高阶偏导数

设 $z=f(x,y)$在域 D 内存在连续的偏导数

$$\frac{\partial z}{\partial x}=f_x(x,y), \qquad \frac{\partial z}{\partial y}=f_y(x,y)$$

若这两个偏导数仍存在偏导数，则称它们是

$z=f(x,y)$的二阶偏导数．按求导顺序不同，

有下列四个二阶偏导数:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x \partial y}=f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y \partial x}=f_{yx}(x,y);\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

类似可以定义更高阶的偏导数.

例如，$z=f(x,y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial^3 z}{\partial x^3}$$

$z=f(x,y)$ 关于 x 的 $n-1$ 阶偏导数，再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right)=\frac{\partial^n z}{\partial x^{n-1}\partial y}$$

问题: 具备怎样的条件,混合偏导数 相等?

定理 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续, 则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

本定理对 n 元函数的高阶混合偏导数也成立.

例如, 对三元函数 $u = f(x, y, z)$, 当三阶混合偏导数在点 (x, y, z) 连续时, 有

$$f_{xyz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z)$$
$$= f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{zyx}(x,y,z)$$

二、典型例题

例1 证明：函数 $z=\sqrt{x^2+y^2}$ 在 $(0,0)$ 点连续，

但两个偏导数均不存在 .

证 $\forall \varepsilon>0$, 取 $\delta\le\varepsilon$,

则当 $\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+y^2}<\delta$ 时，

便有 $\left|\sqrt{x^2+y^2}-\sqrt{0^2+0^2}\right|$

$$=\sqrt{x^2+y^2}<\delta\le\varepsilon,$$

故函数在点 $(0,0)$ 处连续 .

$f(x,0)=\sqrt{x^2}=|x|$为分段函数，分段点：　$x=0$

故求 $f_x(0,0)$时，须用偏导数定义　.

但 $$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}$$

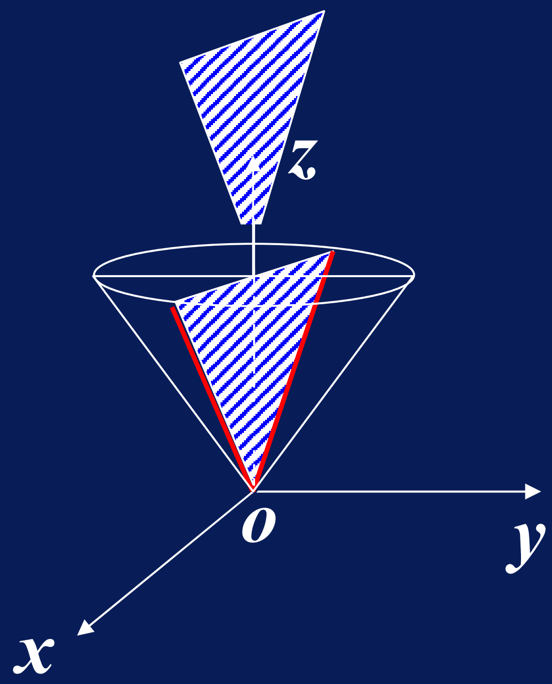
$$=\lim_{x\to 0}\frac{\sqrt{x^2}-0}{x}=\lim_{x\to 0}\frac{|x|}{x}$$

此极限不存在，

故函数在 (0,0)点处关于

自变量x的偏导数不存在．

同理,关于自变量y的偏导数也不存在．



注 对于二元函数:

可偏导 $\not\longleftarrow$ 连续

例2 设 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$,

求 $f_x(0,0)$, $f_y(0,0)$, 并讨论$f(x,y)$在$(0,0)$处的连续性.

解(方法1) $$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}$$

$$=\lim_{x\to 0}\frac{\dfrac{x\cdot 0}{\sqrt{x^2+0}}-0}{x}=0$$

同理可求得 $f_y(0,0)=0.$

$$\because \lim_{\substack{x\to 0\\ y=kx\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y=kx\to 0}}\frac{xy}{x^2+y^2}$$

$$=\lim_{x\to 0}\frac{x\cdot kx}{x^2+(kx)^2}=\frac{k}{1+k^2}$$

其值随 k 的不同而变化,

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2}\text{不存在.}$$

从而 $f(x,y)$ 在点$(0,0)$并不连续!

(方法2) $z = f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0\quad, & x^2+y^2 = 0 \end{cases}$

$\because \quad f(x,0) \equiv 0, \quad f(0,y) \equiv 0,$

$$\therefore \quad f_x(0,0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x,0)\Big|_{x=0} = 0$$

$$f_y(0,0) = \frac{\mathrm{d}}{\mathrm{d}y} f(0,y)\Big|_{y=0} = 0$$

注 对于二元函数: 可偏导 $\not\longrightarrow$ 连续

例3 求 $z = x^2 + 3xy + y^2$ 在点(1 , 2) 处的偏导数.

解(方法1) 先求后代

$$\frac{\partial z}{\partial x} = 2x + 3y, \quad \frac{\partial z}{\partial y} = 3x + 2y$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2\cdot 1 + 3\cdot 2 = 8,$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 3\cdot 1 + 2\cdot 2 = 7$$

先代后求

(方法2)

$$z\big|_{y=2}=x^2+6x+4$$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,\,2)}=\left.\frac{\mathrm{d}z(x,2)}{\mathrm{d}x}\right|_{x=1}=\frac{\mathrm{d}}{\mathrm{d}x}(x^2+6x+4)\big|_{x=1}$$

$$=(2x+6)\big|_{x=1}=8$$

$$z\big|_{x=1}=1+3y+y^2$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,\,2)}=\left.\frac{\mathrm{d}z(1,y)}{\mathrm{d}y}\right|_{y=2}=\frac{\mathrm{d}}{\mathrm{d}y}(1+3y+y^2)\big|_{y=2}$$

$$=(3+2y)\big|_{y=2}=7$$

例4 求曲线 $\begin{cases} z = \sqrt{1+x^2+y^2}, \\ x = 1, \end{cases}$

在点 $(1,1,\sqrt{3})$ 处的切线与 y 轴正向的夹角 β

解 根据偏导数的几何意义，有

$$\tan\beta = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}} = \left.\frac{2y}{2\sqrt{1+x^2+y^2}}\right|_{\substack{x=1\\y=1}}$$

$$= \frac{1}{\sqrt{3}} \qquad \text{故} \quad \beta = \frac{\pi}{6}$$

例5 求函数 $z=e^{x+2y}$ 的二阶偏导数及 $\dfrac{\partial^3 z}{\partial y\partial x^2}$.

解 $\dfrac{\partial z}{\partial x}=e^{x+2y}$ $\quad\dfrac{\partial z}{\partial y}=2e^{x+2y}$ $\quad\dfrac{\partial^2 z}{\partial x^2}=e^{x+2y}$

$\dfrac{\partial^2 z}{\partial x\partial y}=2e^{x+2y}$ $\quad\dfrac{\partial^2 z}{\partial y\partial x}=2e^{x+2y}$ $\quad\dfrac{\partial^2 z}{\partial y^2}=4e^{x+2y}$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=2e^{x+2y}$$

注 此处 $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$,但这一结论并不总是成立.

问题：二阶混合偏导数一定都相等吗？不一定！

例如：
$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0\,, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y}=\lim_{y\to 0}\frac{-y}{y}=-1 \\ f_{yx}(0,0)&=\lim_{x\to 0}\frac{f_y(x,0)-f_y(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1 \end{aligned}\right\} \boxed{\text{二者不等}}$$

三、同步练习

1. 设$z = f(x, y) = \sqrt{|xy|}$, 求$f_x(0, 0)$, $f_y(0, 0)$.

2. 设$z = \arcsin\dfrac{x}{\sqrt{x^2 + y^2}}$, 求$\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

3. 设$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0. \end{cases}$

求$f_x(x, y)$及$f_y(x, y)$.

4. 求下列函数的一阶和二阶偏导数

(1) $z=\ln(x+y^2)$;　　(2) $z=x^y$.

5. 求下列函数的偏导数

(1) $F(x,y)=\int_{3y}^{2x}\frac{\sin t}{t}\,\mathrm{d}t$, 求 F_x, F_y;

(2) $F(x,y)=x\int_{y}^{x}\mathrm{e}^{-3t^2}\,\mathrm{d}t$, 求 $\frac{\partial^2 F}{\partial x\partial y}$.

6. 验证函数 $u(x,y)=\ln\sqrt{x^2+y^2}$ 满足二维拉普拉斯方程 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

四、同步练习解答

1. 设$z = f(x, y) = \sqrt{|xy|}$, 求$f_x(0, 0)$, $f_y(0, 0)$.

解 $$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x}$$

$$= \lim_{x\to 0}\frac{\sqrt{|x\cdot 0|} - 0}{x} = 0$$

同理可求得 $f_y(0,0) = 0.$

2. 设 $z=\arcsin\dfrac{x}{\sqrt{x^2+y^2}}$，求 $\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)_x' \qquad (y\neq 0)$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}} \qquad (\sqrt{y^2}=|y|)$$

$$=\frac{|y|}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_y$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{(-xy)}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\operatorname{sgn} y \qquad (y\neq 0)$$

3. 设$f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$

求$f_x(x,y)$及$f_y(x,y)$.

解: 当 $x^2+y^2\neq 0$ 时,

$$f_x(x,y)=\frac{\partial}{\partial x}\left(\frac{x^2y}{x^2+y^2}\right)=\frac{2xy(x^2+y^2)-x^2y\cdot 2x}{(x^2+y^2)^2}$$

$$=\frac{2xy^3}{(x^2+y^2)^2}$$

$$f_y(x,y)=\frac{\partial}{\partial y}\left(\frac{x^2 y}{x^2+y^2}\right)=\frac{x^2(x^2+y^2)-x^2 y\cdot 2y}{(x^2+y^2)^2}$$

$$=\frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$$

在（0,0）点，注意到 $f(x,0)=\begin{cases}0, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}=0$

故有 $f_x(0,0)=\dfrac{\mathrm{d}}{\mathrm{d}x}f(x,0)\Big|_{x=0}=0$

同理 $f_y(0,0)=\dfrac{\mathrm{d}}{\mathrm{d}y}f(0,y)\Big|_{y=0}=0.$

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$$\therefore f_x(x,y)=\begin{cases}\dfrac{2xy^3}{(x^2+y^2)^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

$$f_y(x,y)=\begin{cases}\dfrac{x^2(x^2-y^2)}{(x^2+y^2)^2}, & (x^2+y^2)\neq 0,\\ 0, & (x^2+y^2)\neq 0.\end{cases}$$

4. 求下列函数的一阶和二阶偏导数

(1) $z=\ln(x+y^2)$;

解 $\dfrac{\partial z}{\partial x}=\dfrac{1}{x+y^2}$, $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{-1}{(x+y^2)^2}$,

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{-2y}{(x+y^2)^2},$$

$$\frac{\partial z}{\partial y}=\frac{2y}{x+y^2},\quad \frac{\partial^2 z}{\partial y^2}=\frac{2\cdot(x+y^2)-2y\cdot 2y}{(x+y^2)^2}=\frac{2(x-y^2)}{(x+y^2)^2}.$$

(2) $z = x^y$.

$$\frac{\partial z}{\partial x} = yx^{y-1},$$

$$\frac{\partial^2 z}{\partial x^2} = y(y-1)x^{y-.2},$$

$$\frac{\partial^2 z}{\partial x \partial y} = x^{y-1} + yx^{y-1}\ln x,$$

$$\frac{\partial z}{\partial y} = x^y \ln x,$$

$$\frac{\partial^2 z}{\partial y^2} = x^y \ln^2 x.$$

5. 求下列函数的偏导数

(1) $F(x,y)=\int_{3y}^{2x}\frac{\sin t}{t}\mathrm{d}t$,求$F_x, F_y$;

(2) $F(x,y)=x\int_{y}^{x}\mathrm{e}^{-3t^2}\mathrm{d}t$,求$\frac{\partial^2 F}{\partial x\partial y}$.

解 (1) $F_x=2\cdot\frac{\sin 2x}{2x}=\frac{\sin 2x}{x}$.

$$F_y=-\frac{\sin 3y}{3y}\cdot 3=-\frac{\sin 3y}{y}.$$

(2) $F_x=\int_{y}^{x}\mathrm{e}^{-3t^2}\mathrm{d}t+x\cdot\mathrm{e}^{-3x^2}$

$$F_{xy}=-\mathrm{e}^{-3y^2}.$$

6. 验证函数$u(x,y)=\ln\sqrt{x^2+y^2}$满足二维拉普拉斯方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

解 $\because \quad u=\ln\sqrt{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2),$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$$