第五节　极限的运算法则

习题　1-5

1. 下列运算是否正确，为什么?

(1) $\lim\limits_{n\to\infty}(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+n})$

$=\lim\limits_{n\to\infty}\frac{1}{n}+\lim\limits_{n\to\infty}\frac{1}{n+1}+\cdots+\lim\limits_{n\to\infty}\frac{1}{n+n}$

$=0+0+\cdots+0=0$;

(2) $\lim\limits_{x\to+\infty}(\sqrt{x+1}-\sqrt{x-1})=\lim\limits_{x\to+\infty}\sqrt{x+1}-\lim\limits_{x\to+\infty}\sqrt{x-1}=\infty-\infty=0$;

(3) $\lim\limits_{x\to0}x\sin\frac{1}{x}=\lim\limits_{x\to0}x\cdot\lim\limits_{x\to0}\sin\frac{1}{x}=0$.

解　(1)　不正确，因为只有有限个数列和的极限(且这有限个数列的极限都存在)才等于它们极限的和.

(2)　不正确，因为只有当两函数极限都存在时，才有两函数差的极限等于它们极限的差.

(3)　不正确，因为 $\lim\limits_{x\to0}\sin\frac{1}{x}$ 不存在.

2. 计算下列各极限:

(1) $\lim\limits_{n\to\infty}\frac{1+2+3+\cdots+(n-1)}{n^2}$;　　(2) $\lim\limits_{n\to\infty}(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})$;

(3) $\lim\limits_{n\to\infty}\frac{5n^2+2n+3}{n^3-n+3}$;　　(4) $\lim\limits_{n\to\infty}(\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{n(n+1)})$;

(5) $\lim\limits_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-2n})$;　　(6) $\lim\limits_{n\to\infty}\frac{(n+1)(2n+1)(3n+1)}{3n^3}$.

解　(1)　$\lim\limits_{n\to\infty}\frac{1+2+3+\cdots+(n-1)}{n^2}=\lim\limits_{n\to\infty}\frac{\frac{n(n-1)}{2}}{n^2}=\frac{1}{2}$.

(2) $\lim\limits_{n\to\infty}(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})=\lim\limits_{n\to\infty}\frac{1\cdot(1-(\frac{1}{2})^{n+1})}{1-\frac{1}{2}}=\lim\limits_{n\to\infty}2(1-(\frac{1}{2})^{n+1})=2$.

(3) $\lim\limits_{n\to\infty}\dfrac{5n^2+2n+3}{n^3-n+3}=\lim\limits_{n\to\infty}\dfrac{\frac{5}{n}+\frac{2}{n^2}+\frac{3}{n^3}}{1-\frac{1}{n^2}+\frac{3}{n^3}}=\dfrac{\lim\limits_{n\to\infty}(\frac{5}{n}+\frac{2}{n^2}+\frac{3}{n^3})}{\lim\limits_{n\to\infty}(1-\frac{1}{n^2}+\frac{3}{n^3})}=0$.

(4) $\lim\limits_{n\to\infty}(\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)})=\lim\limits_{n\to\infty}(1-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\cdots+\dfrac{1}{n}-\dfrac{1}{n+1})$

$$=\lim_{n\to\infty}(1-\frac{1}{n+1})=1.$$

(5) $\lim\limits_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-2n})=\lim\limits_{n\to\infty}\dfrac{1+2n}{\sqrt{n^2+1}+\sqrt{n^2-2n}}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}+2}{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{2}{n}}}$

$=1$.

(6) $\lim\limits_{n\to\infty}\dfrac{(n+1)(2n+1)(3n+1)}{3n^3}=\lim\limits_{n\to\infty}\dfrac{(1+\frac{1}{n})(2+\frac{1}{n})(3+\frac{1}{n})}{3}=2$.

3. 计算下列各极限:

(1) $\lim\limits_{x\to 0}(1-\dfrac{2}{x-3})$;

(2) $\lim\limits_{x\to\infty}\dfrac{3x^2-7x+1}{5x^2+2x-3}$;

(3) $\lim\limits_{x\to 0}\dfrac{x^2}{1-\sqrt{1+x^2}}$;

(4) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$;

(5) $\lim\limits_{x\to\infty}(1+\dfrac{1}{x})(2-\dfrac{1}{x^2})$;

(6) $\lim\limits_{x\to 1}\dfrac{x^2-1}{2x^2-x-1}$.

解 (1) $\lim\limits_{x\to 0}(1-\dfrac{2}{x-3})=1-\lim\limits_{x\to 0}\dfrac{2}{x-3}=1-\dfrac{2}{-3}=\dfrac{5}{3}$.

(2) $\lim\limits_{x\to\infty}\dfrac{3x^2-7x+1}{5x^2+2x-3}=\lim\limits_{x\to\infty}\dfrac{3-\frac{7}{x}+\frac{1}{x^2}}{5+\frac{2}{x}-\frac{3}{x^2}}=\dfrac{\lim\limits_{x\to\infty}(3-\frac{7}{x}+\frac{1}{x^2})}{\lim\limits_{x\to\infty}(5+\frac{2}{x}-\frac{3}{x^2})}=\dfrac{3}{5}$.

(3) $\lim\limits_{x\to 0}\dfrac{x^2}{1-\sqrt{1+x^2}}=-\lim\limits_{x\to 0}(1+\sqrt{1+x^2})=-2$.

(4) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}=\lim\limits_{x\to 1}(x^2+x+1)=3$.

(5) $\lim\limits_{x\to\infty}(1+\dfrac{1}{x})(2-\dfrac{1}{x^2})=(1+\lim\limits_{x\to\infty}\dfrac{1}{x})(2-\lim\limits_{x\to\infty}\dfrac{1}{x^2})=2$.

(6) $\lim\limits_{x\to 1}\dfrac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to 1}\dfrac{x+1}{2x+1}=\dfrac{2}{3}$.

4. 计算下列各极限:

(1) $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$;　　　　(2) $\lim\limits_{n\to\infty}(1-\dfrac{1}{2^2})(1-\dfrac{1}{3^2})\cdots(1-\dfrac{1}{n^2})$.

解 (1) $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}=\lim\limits_{h\to 0}(2x+h)=2x$.

(2) $\lim\limits_{n\to\infty}(1-\dfrac{1}{2^2})(1-\dfrac{1}{3^2})\cdots(1-\dfrac{1}{n^2})=\lim\limits_{n\to\infty}(1-\dfrac{1}{2})(1+\dfrac{1}{2})(1-\dfrac{1}{3})(1+\dfrac{1}{3})\cdots(1-\dfrac{1}{n})(1+\dfrac{1}{n})$

$$=\lim_{n\to\infty}\frac{1}{2}(1+\frac{1}{n})=\frac{1}{2}.$$

5. 表述并证明 $x\to\infty$ 时函数极限的四则运算法则.

解 若 $\lim\limits_{x\to\infty}f(x)=A$，$\lim\limits_{x\to\infty}g(x)=B$，则

(1) $\lim\limits_{x\to\infty}[f(x)\pm g(x)]=A\pm B=\lim\limits_{x\to\infty}f(x)\pm\lim\limits_{x\to\infty}g(x)$;

(2) $\lim\limits_{x\to\infty}[f(x)\cdot g(x)]=AB=\lim\limits_{x\to\infty}f(x)\cdot\lim\limits_{x\to\infty}g(x)$;

(3) 若 $B\neq 0$，则有 $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\dfrac{A}{B}=\dfrac{\lim\limits_{x\to\infty}f(x)}{\lim\limits_{x\to\infty}g(x)}$.

证明如下:

(1) 仅证明和的形式.

由 $\lim\limits_{x\to\infty}f(x)=A$，$\lim\limits_{x\to\infty}g(x)=B$ 知，$\forall\varepsilon>0$，$\exists X_1>0, X_2>0$，当 $|x|>X_1$ 时，有 $|f(x)-A|<\dfrac{\varepsilon}{2}$；当 $|x|>X_2$ 时，有 $|g(x)-B|<\dfrac{\varepsilon}{2}$.

取 $X=\max\{X_1,X_2\}$，则当 $|x|>X$ 时，$|[f(x)+g(x)]-(A+B)|\leq|f(x)-A|+|g(x)-B|<\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$，因此 $\lim\limits_{x\to\infty}[f(x)+g(x)]=A+B=\lim\limits_{x\to\infty}f(x)+\lim\limits_{x\to\infty}g(x)$.

(2) 由于 $|f(x)\cdot g(x)-AB|=|f(x)\cdot g(x)-Bf(x)+Bf(x)-AB|$

$$\leq|f(x)|\cdot|g(x)-B|+|B|\cdot|f(x)-A|.$$

由 $\lim\limits_{x\to\infty}f(x)=A$ 及函数极限的局部有界性得，$\forall\varepsilon>0$，$\exists X_1>0$ 及 $M>0$，当 $|x|>X_1$ 时，有 $|f(x)-A|<\dfrac{\varepsilon}{2C}$，且 $|f(x)|\leq M$，其中 $C=\max\{M,|B|\}$.

又 $\lim\limits_{x\to\infty}g(x)=B$，故 $\exists X_2>0$，当 $|x|>X_2$ 时，有 $|g(x)-B|<\dfrac{\varepsilon}{2C}$.

取 $X=\max\{X_1,X_2\}$，则当$|x|>X$ 时，有

$$|f(x)\cdot g(x)-AB|\le|f(x)|\cdot|g(x)-B|+|B|\cdot|f(x)-A|$$

$$<M\cdot\frac{\varepsilon}{2C}+|B|\cdot\frac{\varepsilon}{2C}\le C\cdot\frac{\varepsilon}{2C}+C\cdot\frac{\varepsilon}{2C}=\varepsilon\,.$$

因此 $\lim\limits_{x\to\infty}[f(x)\cdot g(x)]=AB=\lim\limits_{x\to\infty}f(x)\cdot\lim\limits_{x\to\infty}g(x)$.

(3) 因为 $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to\infty}[f(x)\cdot\dfrac{1}{g(x)}]$，故由(2)，只需证当 $B\ne 0$ 时，有

$$\lim_{x\to\infty}\frac{1}{g(x)}=\frac{1}{B}=\frac{1}{\lim\limits_{x\to\infty}g(x)}\,.$$

$$\left|\frac{1}{g(x)}-\frac{1}{B}\right|=\left|\frac{B-g(x)}{g(x)\cdot B}\right|=\frac{1}{|B|}\cdot\frac{1}{|g(x)|}\cdot|g(x)-B|\,.$$

由 $\lim\limits_{x\to\infty}g(x)=B$ 及函数极限的局部有界性知，$\forall\varepsilon>0$，$\exists X>0$ 及 $M>0$，当 $|x|>X$ 时，有$|g(x)-B|<\dfrac{|B|}{M}\varepsilon$，且 $\dfrac{1}{|g(x)|}\le M$，所以，

$$\left|\frac{1}{g(x)}-\frac{1}{B}\right|=\frac{1}{|B|}\cdot\frac{1}{|g(x)|}\cdot|g(x)-B|<\frac{1}{|B|}\cdot M\cdot\frac{|B|}{M}\varepsilon=\varepsilon\,.$$

即 $\lim\limits_{x\to\infty}\dfrac{1}{g(x)}=\dfrac{1}{B}=\dfrac{1}{\lim\limits_{x\to\infty}g(x)}$.