第二节　微积分基本定理

习题　5-2

1. 求下列函数 $y=y(x)$ 的导数 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

(1) $y=\int_0^t \sin t^2\mathrm{d}t$;　　(2) $y=\int_{x^2}^{x^3}\dfrac{1}{\sqrt{1+t^4}}\mathrm{d}t$;

(3) $y=\int_x^{x^2} t^2\mathrm{e}^{-t}\mathrm{d}t$;　　(4) $y=\int_0^{x^2}\dfrac{\sin t^2}{1+\mathrm{e}^t}\mathrm{d}t$;

(5) $y=\int_{\cos x}^{\sin x}\mathrm{e}^{t^2}\mathrm{d}t$;　　(6) $\int_0^y \mathrm{e}^t\mathrm{d}t+\int_0^x \cos t\mathrm{d}t=0$;

(7) $\begin{cases} x=\int_0^t \sin u\mathrm{d}u, \\ y=\int_0^t \cos u\mathrm{d}u; \end{cases}$　　(8) $\begin{cases} x=\int_0^{t^2} \cos u^2\mathrm{d}u, \\ y=\sin t^4; \end{cases}$

(9) $\int_0^y \mathrm{e}^t\mathrm{d}t+\int_0^{xy}\cos t\mathrm{d}t=0$.

解　(1) $y'=\sin x^2$.

(2) $$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{x^3}\frac{\mathrm{d}t}{\sqrt{1+t^4}}=\frac{1}{\sqrt{1+x^{12}}}\cdot 3x^2-\frac{1}{\sqrt{1+x^8}}\cdot 2x$$

$$=\frac{3x^2}{\sqrt{1+x^{12}}}-\frac{2x}{\sqrt{1+x^8}}.$$

(3) $y'=x^4\mathrm{e}^{-x^2}\cdot 2x-x^2\mathrm{e}^{-x}=x^2(2x^3\mathrm{e}^{-x^2}-\mathrm{e}^{-x})$.

(4) $y'=\dfrac{\sin x^4}{1+\mathrm{e}^{x^2}}\cdot 2x=\dfrac{2x\sin x^4}{1+\mathrm{e}^{x^2}}$.

(5) $y'=\mathrm{e}^{\sin^2 x}\cdot\cos x-\mathrm{e}^{\cos^2 x}\cdot(-\sin x)=\mathrm{e}^{\sin^2 x}\cos x+\mathrm{e}^{\cos^2 x}\sin x$.

(6) 方程两边对 x 求导得

$$\mathrm{e}^y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+\cos x=0,$$

所以 y 对 x 的导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\cos x}{\mathrm{e}^y}.$$

(7) $\because x_t'=\sin t,\ y_t'=\cos t,$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y_t'}{x_t'}=\frac{\cos t}{\sin t}=\cot t.$$

(8) $\because \dfrac{\mathrm{d}x}{\mathrm{d}t}=\cos t^4\cdot 2t=2t\cos t^4,$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\cos t^4\cdot 4t^3=4t^3\cos t^4,$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4t^3\cos t^4}{2t\cos t^4}=2t^2.$$

(9) 方程两边对 x 求导得

$$\mathrm{e}^y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+\cos(xy)\cdot(y+x\cdot\frac{\mathrm{d}y}{\mathrm{d}x})=0,$$

所以 y 对 x 的导数为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y\cdot\cos(xy)}{\mathrm{e}^y+x\cos(xy)}.$$

2. 求下列极限:

(1) $\lim\limits_{x\to 0}\dfrac{\int_0^x\ln(1+t)\mathrm{d}t}{x^2}$;　　(2) $\lim\limits_{x\to 0}\dfrac{\int_0^x\cos^2 t\mathrm{d}t}{x}$;

(3) $\lim\limits_{x\to 0}\dfrac{(\int_0^x\mathrm{e}^{t^2}\mathrm{d}t)^2}{\int_0^x t\mathrm{e}^{2t^2}\mathrm{d}t}$;　　(4) $\lim\limits_{x\to\infty}\dfrac{1}{x}\int_0^x(1+t^2)\mathrm{e}^{t^2-x^2}\mathrm{d}t$.

解 (1) $\lim\limits_{x\to 0}\dfrac{\int_0^x\ln(1+t)\mathrm{d}t}{x^2}=\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{2x}=\lim\limits_{x\to 0}\dfrac{x}{2x}=\dfrac{1}{2}$.

(2) $\lim\limits_{x\to 0}\dfrac{\int_0^x\cos^2 t\mathrm{d}t}{x}=\lim\limits_{x\to 0}\dfrac{\cos^2 x}{1}=1$.

(3) $\lim\limits_{x\to 0}\dfrac{(\int_0^x\mathrm{e}^{t^2}\mathrm{d}t)^2}{\int_0^x t\mathrm{e}^{2t^2}\mathrm{d}t}=\lim\limits_{x\to 0}\dfrac{2\int_0^x\mathrm{e}^{t^2}\mathrm{d}t\cdot\mathrm{e}^{x^2}}{x\mathrm{e}^{2x^2}}=\lim\limits_{x\to 0}\dfrac{2\int_0^x\mathrm{e}^{t^2}\mathrm{d}t}{x\mathrm{e}^{x^2}}$

$$=2\lim_{x\to 0}\frac{\mathrm{e}^{x^2}}{(1+2x^2)\mathrm{e}^{x^2}}=2\lim_{x\to 0}\frac{1}{1+2x^2}=2.$$

(4) $\lim\limits_{x\to\infty}\dfrac{1}{x}\displaystyle\int_0^x(1+t^2)\mathrm{e}^{t^2-x^2}\,\mathrm{d}t=\lim\limits_{x\to\infty}\dfrac{\mathrm{e}^{-x^2}\displaystyle\int_0^x(1+t^2)\mathrm{e}^{t^2}\,\mathrm{d}t}{x}=\lim\limits_{x\to\infty}\dfrac{\displaystyle\int_0^x(1+t^2)\mathrm{e}^{t^2}\,\mathrm{d}t}{x\mathrm{e}^{x^2}}$

$$=\lim_{x\to\infty}\frac{(1+x^2)\mathrm{e}^{x^2}}{(1+2x^2)\mathrm{e}^{x^2}}=\lim_{x\to\infty}\frac{1+x^2}{1+2x^2}$$

$$=\lim_{x\to\infty}\frac{\dfrac{1}{x^2}+1}{\dfrac{1}{x^2}+2}=\frac{1}{2}.$$

3. 设 $f(x)=\sqrt{1-x^2}$，找 $\xi\in(-1,1)$，使 $\int_{-1}^{1}f(x)\mathrm{d}x=2f(\xi)$.

解 $\int_{-1}^{1}f(x)\mathrm{d}x$ 表示曲线 $f(x)=\sqrt{1-x^2}$ 与 x 轴在 $[-1,1]$ 内所围的面积，显然是圆 $x^2+y^2=1$ 的面积的 $\dfrac{1}{2}$，因此

$$\int_{-1}^{1}f(x)\mathrm{d}x=\frac{\pi}{2}.$$

又由积分中值定理，存在 $\xi\in(-1,1)$，使得

$$\int_{-1}^{1}f(x)\mathrm{d}x=2f(\xi)=2\sqrt{1-\xi^2},$$

从而有

$$2\sqrt{1-\xi^2}=\frac{\pi}{2},$$

故 $\xi=\pm\sqrt{1-\dfrac{\pi^2}{16}}$.

4. 设 $f(x)=\int_0^x t\mathrm{e}^{-t^2}\mathrm{d}t$，求 $f(x)$ 的极值点与拐点.

解 $f'(x)=x\mathrm{e}^{-x^2}$，$f''(x)=\mathrm{e}^{-x^2}-2x^2\mathrm{e}^{-x^2}=\mathrm{e}^{-x^2}(1-2x^2)$.

令 $f'(x)=0$ 得 $x=0$. 当 $x<0$ 时，$f'(x)<0$; 当 $x>0$ 时，$f'(x)>0$，故 $x=0$ 为 $f(x)$ 的唯一极值点且为极小值点.

令 $f''(x)=0$ 得 $x=\pm\dfrac{\sqrt{2}}{2}$，故 $f(x)$ 图形的拐点为 $(\dfrac{\sqrt{2}}{2},-\dfrac{1}{2}\mathrm{e}^{-\frac{1}{2}})$，$(-\dfrac{\sqrt{2}}{2},-\dfrac{1}{2}\mathrm{e}^{-\frac{1}{2}})$.

5. 设 $f(x)$ 连续，且 $\int_0^x f(t)\mathrm{d}t=x^2(1+x)$，求 $f(2)$.

解 对方程$\int_0^x f(t)\mathrm{d}t = x^2(1+x)$两边关于$x$求导，有

$$f(x) = 2x(1+x) + x^2 = 3x^2 + 2x.$$

令$x=2$，则$f(2)=16$.

6. 计算下列各定积分:

(1) $\int_1^2 (x+\frac{1}{x})^2 \mathrm{d}x$; (2) $\int_4^9 \sqrt{x}(1+\sqrt{x})\mathrm{d}x$;

(3) $\int_1^{\sqrt{3}} \frac{1+2x^2}{x^2(1+x^2)}\mathrm{d}x$; (4) $\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}} \frac{|\ln x|}{x}\mathrm{d}x$;

(5) $\int_0^1 \frac{x}{\sqrt{1+x^2}}\mathrm{d}x$; (6) $\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin\frac{1}{y}}{y^2}\mathrm{d}y$;

(7) $\int_{-1}^0 \frac{3x^4+3x^2+1}{1+x^2}\mathrm{d}x$; (8) $\int_0^{\frac{\pi}{4}} \tan^3\theta \mathrm{d}\theta$;

(9) $\int_{-(\mathrm{e}+1)}^{-2} \frac{1}{1+x}\mathrm{d}x$; (10) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^3 x - \cos^5 x}\mathrm{d}x$;

(11) $\int_0^{\frac{\pi}{2}} |\sin x - \cos x|\mathrm{d}x$; (12) $\int_0^1 \frac{1}{x^2-x+1}\mathrm{d}x$;

(13) $\int_0^{\pi} \sqrt{1+\cos 2x}\mathrm{d}x$; (14) $\int_0^2 |1-x|\mathrm{d}x$;

(15) $\int_1^{\mathrm{e}} \frac{1}{x^2(1+x^2)}\mathrm{d}x$.

解 (1) $\int_1^2 (x+\frac{1}{x})^2 \mathrm{d}x = \int_1^2 (x^2+2+\frac{1}{x^2})\mathrm{d}x = (\frac{1}{3}x^3+2x-\frac{1}{x})\Big|_1^2 = \frac{29}{6}$.

(2) $\int_4^9 \sqrt{x}(1+\sqrt{x})\mathrm{d}x = \int_4^9 (\sqrt{x}+x)\mathrm{d}x = (\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2)\Big|_4^9 = 45\frac{1}{6}$.

(3) $\int_1^{\sqrt{3}} \frac{1+2x^2}{x^2(1+x^2)}\mathrm{d}x = \int_1^{\sqrt{3}} \frac{1+x^2+x^2}{x^2(1+x^2)}\mathrm{d}x = \int_1^{\sqrt{3}} (\frac{1}{x^2}+\frac{1}{1+x^2})\mathrm{d}x$

$$= (-\frac{1}{x}+\arctan x)\Big|_1^{\sqrt{3}} = 1-\frac{\sqrt{3}}{3}+\frac{\pi}{12}.$$

(4) $\int_{\frac{1}{e}}^{e}\frac{|\ln x|}{x}dx=\int_{\frac{1}{e}}^{1}\frac{-\ln x}{x}dx+\int_{1}^{e}\frac{\ln x}{x}dx=-\int_{\frac{1}{e}}^{1}\ln x d\ln x+\int_{1}^{e}\ln x d\ln x$

$$=-\frac{1}{2}\ln^2 x\Big|_{\frac{1}{e}}^{1}+\frac{1}{2}\ln^2 x\Big|_{1}^{e}=\frac{1}{2}+\frac{1}{2}=1.$$

注意 常见错误是$\int_{\frac{1}{e}}^{e}\frac{|\ln x|}{x}dx=\frac{1}{2}\ln^2 x\Big|_{\frac{1}{e}}^{e}$，产生错误的原因是忽略了$\ln x$的取值范围．事实上当$\frac{1}{e}\le x<1$时，$\ln x<0$，因此取掉绝对值号前面要加负号，$1<x\le e$时，$\ln x>0$.

(5) $\int_0^1\frac{x}{\sqrt{1+x^2}}dx=\int_0^1 d\sqrt{1+x^2}=\sqrt{1+x^2}\Big|_0^1=\sqrt{2}-1$.

(6) $\int_{\frac{1}{\pi}}^{\frac{2}{\pi}}\frac{\sin\frac{1}{y}}{y^2}dy=\int_{\frac{1}{\pi}}^{\frac{2}{\pi}}d\cos\frac{1}{y}=\cos\frac{1}{y}\Big|_{\frac{1}{\pi}}^{\frac{2}{\pi}}=1$.

(7) $\int_{-1}^{0}\frac{3x^4+3x^2+1}{1+x^2}dx=\int_{-1}^{0}(3x^2+\frac{1}{1+x^2})dx=(x^3+\arctan x)\Big|_{-1}^{0}=1+\frac{\pi}{4}$.

(8) $\int_0^{\frac{\pi}{4}}\mathrm{tg}^3\theta d\theta=\int_0^{\frac{\pi}{4}}\frac{\sin^3\theta}{\cos^3\theta}d\theta=-\int_0^{\frac{\pi}{4}}\frac{\sin^2\theta}{\cos^3\theta}d\cos\theta=-\int_0^{\frac{\pi}{4}}\frac{1-\cos^2\theta}{\cos^3\theta}d\cos\theta$

$$=(\ln|\cos\theta|+\frac{1}{2\cos^2\theta})\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}(1-\ln 2).$$

(9) $\int_{-(e+1)}^{-2}\frac{1}{1+x}dx=\ln|1+x|\Big|_{-e-1}^{-2}=\ln 1-\ln e=-1$.

(10) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos^3 x-\cos^5 x}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos^3 x(1-\cos^2 x)}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos^3 x\cdot\sin^2 x}dx$

$$=2\int_0^{\frac{\pi}{2}}(\cos x)^{\frac{3}{2}}\sin x dx=-2\int_0^{\frac{\pi}{2}}(\cos x)^{\frac{3}{2}}d\cos x$$

$$=-2\cdot\frac{2}{5}(\cos x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}=\frac{4}{5}.$$

(11) $\int_0^{\frac{\pi}{2}}|\sin x-\cos x|dx=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x-\cos x)dx$

$$=(\sin x+\cos x)\Big|_0^{\frac{\pi}{4}}-(\sin x+\cos x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\sqrt{2}-1-1+\sqrt{2}$$

$$=2(\sqrt{2}-1).$$

(12) $$\int_0^1\frac{\mathrm{d}x}{x^2-x+1}=\int_0^1\frac{\mathrm{d}x}{(x-\frac{1}{2})^2+\frac{3}{4}}=\frac{4}{3}\int_0^1\frac{\mathrm{d}x}{1+\frac{4}{3}(x-\frac{1}{2})^2}$$

$$=\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int_0^1\frac{1}{1+[\frac{2}{\sqrt{3}}(x-\frac{1}{2})]^2}\mathrm{d}[\frac{2}{\sqrt{3}}(x-\frac{1}{2})]$$

$$=\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2(x-\frac{1}{2})}{\sqrt{3}}\Bigg|_0^1=\frac{2\sqrt{3}}{9}\pi.$$

(13) $$\int_0^{\pi}\sqrt{1+\cos 2x}\,\mathrm{d}x=\int_0^{\pi}\sqrt{2\cos^2 x}\,\mathrm{d}x=\sqrt{2}\int_0^{\pi}|\cos x|\mathrm{d}x$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}}\cos x\mathrm{d}x+\sqrt{2}\int_{\frac{\pi}{2}}^{\pi}(-\cos x)\mathrm{d}x$$

$$=\sqrt{2}(\sin x\Big|_0^{\frac{\pi}{2}}-\sin x\Big|_{\frac{\pi}{2}}^{\pi})=2\sqrt{2}.$$

(14) $$\int_0^2|1-x|\mathrm{d}x=\int_0^1(1-x)\mathrm{d}x+\int_1^2(x-1)\mathrm{d}x=(x-\frac{1}{2}x^2)\Big|_0^1+(\frac{1}{2}x^2-x)\Big|_1^2=1.$$

(15) $$\int_1^{\mathrm{e}}\frac{1}{x^2(1+x^2)}\mathrm{d}x=\int_1^{\mathrm{e}}(\frac{1}{x^2}-\frac{1}{1+x^2})\mathrm{d}x$$

$$=(-\frac{1}{x}-\operatorname{arctg}x)\Big|_1^{\mathrm{e}}$$

$$=1-\frac{1}{\mathrm{e}}-\operatorname{arctg}\mathrm{e}-\frac{\pi}{4}.$$

7. 已知 $f(x)=\begin{cases}\tan^2 x, & 0\le x\le\frac{\pi}{4},\\ \sin x\cos^3 x, & \frac{\pi}{4}<x\le\frac{\pi}{2}.\end{cases}$ 计算 $\int_0^{\frac{\pi}{2}}f(x)\mathrm{d}x$.

解 $$\int_0^{\frac{\pi}{2}}f(x)\mathrm{d}x=\int_0^{\frac{\pi}{4}}\tan^2 x\mathrm{d}x+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin x\cos^3 x\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{4}}(\sec^2 x-1)\mathrm{d}x-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^3 x\mathrm{d}\cos x=(\tan x-x)\Big|_0^{\frac{\pi}{4}}-\frac{1}{4}\cos^4 x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=1-\frac{\pi}{4}+\frac{1}{16}=\frac{17}{16}-\frac{\pi}{4}.$$

8. 设 m 、 n 为正整数，证明下列各式:

(1) $\int_{-\pi}^{\pi}\sin mx\mathrm{d}x=0$;　　(2) $\int_{-\pi}^{\pi}\cos mx\mathrm{d}x=0$;

(3) $\int_{-\pi}^{\pi}\sin mx\cos nx\mathrm{d}x=0$;　　(4) $\int_{-\pi}^{\pi}\sin mx\sin nx\mathrm{d}x=0(m\neq n)$;

(5) $\int_{-\pi}^{\pi}\cos mx\cos nx\mathrm{d}x=0(m\neq n)$;　　(6) $\int_{-\pi}^{\pi}\sin^2 mx\mathrm{d}x=\pi$;

(7) $\int_{-\pi}^{\pi}\cos^2 mx\mathrm{d}x=\pi$.

证 (1) $\int_{-\pi}^{\pi}\sin mx\mathrm{d}x=-\frac{1}{m}\cos mx\Big|_{-\pi}^{\pi}$

$$=-\frac{1}{m}(\cos m\pi-\cos m\pi)=0.$$

(2) $\int_{-\pi}^{\pi}\cos mx\mathrm{d}x=\frac{1}{m}\sin mx\Big|_{-\pi}^{\pi}$

$$=\frac{1}{m}(\sin m\pi+\sin m\pi)=0.$$

(3) 因为 $\sin mx\cos nx$ 是奇函数，故

$$\int_{-\pi}^{\pi}\sin mx\cos nx\mathrm{d}x=0.$$

(4) $\int_{-\pi}^{\pi}\sin mx\sin nx\mathrm{d}x=-\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x-\cos(m-n)x]\mathrm{d}x=0(m\neq n)$

$$=-\frac{1}{2}\Big[\frac{\sin(m+n)x}{m+n}-\frac{\sin(m-n)x}{m-n}\Big]\Big|_{-\pi}^{\pi}=0.$$

(5) $\int_{-\pi}^{\pi}\cos mx\cos nx\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x+\cos(m-n)x]\mathrm{d}x$

$$=\frac{1}{2}\Big[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\Big]\Big|_{-\pi}^{\pi}=0.$$

(6) $\int_{-\pi}^{\pi}\sin^2 mx\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2mx)\mathrm{d}x$

$$=\frac{1}{2}\Big(x\Big|_{-\pi}^{\pi}\Big)-\frac{1}{4m}\Big(\sin 2mx\Big|_{-\pi}^{\pi}\Big)=\pi.$$

(7) $\int_{-\pi}^{\pi}\cos^2 mx\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 2mx)\mathrm{d}x$

$$=\frac{1}{2}(x\big|_{-\pi}^{\pi})+\frac{1}{4m}(\sin 2mx\big|_{-\pi}^{\pi})=\pi\ .$$

9. 设 $f(x)=\begin{cases}x^2, & x\in[0,1),\\ \sin x\cos^3 x, & x\in[1,2].\end{cases}$ 求 $\Phi(x)=\int_0^x f(t)\mathrm{d}t$ 在 $[0,2]$ 上的表达式，并讨论 $\Phi(x)$ 在 $(0,2)$ 内的连续性.

解 当 $x\in[0,1)$ 时,

$$\Phi(x)=\int_0^x f(t)\mathrm{d}t=\int_0^x t^2\mathrm{d}t=(\frac{1}{3}t^3)\bigg|_0^x=\frac{1}{3}x^3-\frac{1}{3}x^0=\frac{x^3}{3};$$

当 $x\in[1,2]$ 时,

$$\Phi(x)=\int_0^x f(t)\mathrm{d}t=\int_0^1 f(t)\mathrm{d}t+\int_1^x f(t)\mathrm{d}t$$

$$=\int_0^1 t^2\mathrm{d}t+\int_1^x t\mathrm{d}t=(\frac{1}{3}t^3)\bigg|_0^1+(\frac{1}{2}t^2)\bigg|_1^x=\frac{x^2}{2}-\frac{1}{6};$$

故
$$\Phi(x)=\begin{cases}\dfrac{x^3}{3}, & x\in[0,1),\\ \dfrac{x^2}{2}-\dfrac{1}{6}, & x\in[1,2].\end{cases}$$

因为
$$\lim_{x\to1^-}\Phi(x)=\lim_{x\to1^-}\frac{x^3}{3}=\frac{1}{3}=\Phi(1),$$
$$\lim_{x\to1^+}\Phi(x)=\lim_{x\to1^+}(\frac{x^2}{2}-\frac{1}{6})=\frac{1}{3}=\Phi(1),$$

故 $\Phi(x)$ 在 $x=1$ 处连续. 显然在 $(0,1),(1,2)$ 内 $\Phi(x)$ 为初等函数，故连续.

综上有 $\Phi(x)$ 在 $(0,2)$ 内连续.

10. 设 $f(x)$ 在 $[a,b]$ 上连续，在 (a,b) 内可导，且 $f'(x)\le 0$，$F(x)=\frac{1}{x-a}\int_a^x f(t)\mathrm{d}t$. 证明: 在 (a,b) 内 $F'(x)\le 0$.

证 由条件知 $f(x)$ 在 $[a,b]$ 上单调递减，对 $F(x)=\frac{1}{x-a}\int_a^x f(t)\mathrm{d}t$ 两边关于 x 求导，得

$$F'(x)=(\frac{1}{x-a})'\cdot\int_a^x f(t)\mathrm{d}t+\frac{1}{x-a}\cdot(\int_a^x f(t)\mathrm{d}t)'$$
$$=-\frac{1}{(x-a)^2}\int_a^x f(t)\mathrm{d}t+\frac{f(x)}{x-a}$$

$$=\frac{f(x)}{x-a}-\frac{1}{(x-a)^2}[(x-a)\cdot f(\xi)] \qquad (a\le\xi\le x)$$

$$=\frac{f(x)}{x-a}-\frac{f(\xi)}{x-a}=\frac{1}{x-a}[f(x)-f(\xi)],$$

且 $x\neq a$.

由 $a<x\le b$ 得 $x-a>0$ 从而 $\frac{1}{x-a}>0$ ，又由于 $f(x)$ 在 $[a,b]$ 上单调递减，并且 $\xi\le x$ ，所以有

$$f(\xi)\ge f(x)，即\ f(x)-f(\xi)\le 0,$$

故

$$F'(x)=\frac{1}{x-a}[f(x)-f(\xi)]\le 0.$$