第三节　任意项级数的审敛法

习题　11-3

1. 判定下列级数是否收敛，如果收敛，是条件收敛还是绝对收敛?

(1) $1-\dfrac{1}{3^2}+\dfrac{1}{5^2}-\dfrac{1}{7^2}+\cdots$;

(2) $\dfrac{1}{\ln 2}-\dfrac{1}{\ln 3}+\dfrac{1}{\ln 4}-\dfrac{1}{\ln 5}+\cdots$;

(3) $\sum\limits_{n=2}^{\infty}\dfrac{\sqrt{n}\cos n\pi}{n-1}$;

(4) $\sum\limits_{n=1}^{\infty}(-1)^{\frac{n(n-1)}{2}}\dfrac{n^{10}}{2^n}$;

(5) $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{2^{n^2}}{n!}$;

(6) $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n-\ln n}$;

(7) $\dfrac{1}{\pi^2}\sin\dfrac{\pi}{2}-\dfrac{1}{\pi^3}\sin\dfrac{\pi}{3}+\dfrac{1}{\pi^4}\sin\dfrac{\pi}{4}-\cdots$;

(8) $\dfrac{1}{\sqrt{2}-1}-\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{3}-1}-\dfrac{1}{\sqrt{3}+1}+\cdots+\dfrac{1}{\sqrt{n}-1}-\dfrac{1}{\sqrt{n}+1}+\cdots$;

(9) $\dfrac{1}{a+b}-\dfrac{1}{2a+b}+\dfrac{1}{3a+b}-\dfrac{1}{4a+b}+\cdots(a>0, b>0)$;

(10) $\sum\limits_{n=2}^{\infty}(-1)^{n-1}\dfrac{1\cdot 3\cdot\cdots\cdot(2n-3)}{2\cdot 4\cdot\cdots\cdot(2n)}$;

(11) $\sum\limits_{n=1}^{\infty}(-1)^{n}\dfrac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}$;

(12) $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\sqrt{n}+(-1)^n}$;

(13) $\sum\limits_{n=1}^{\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})$.

解　(1)　设 $u_n=(-1)^{n-1}\dfrac{1}{(2n-1)^2}$，则 $|u_n|=\dfrac{1}{(2n-1)^2}\le\dfrac{1}{4(n-1)^2}(n\ge 2)$，而 $\sum\limits_{n=2}^{\infty}\dfrac{1}{4(n-1)^2}$ 收敛，所以原级数绝对收敛.

(2)　设 $u_n=(-1)^{n-1}\dfrac{1}{\ln(n+1)}$，$a_n=\dfrac{1}{\ln(n+1)}$，显然 $a_n\ge a_{n+1}$，且 $\lim\limits_{n\to\infty}a_n=0$，故原级数收敛，但又因为 $\lim\limits_{n\to\infty}\dfrac{|u_n|}{\frac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{n}{\ln(n+1)}=\infty$，所以原级数条件收敛.

(3) 设 $u_n=\dfrac{\sqrt{n}\cos n\pi}{n-1}$, $a_n=\dfrac{\sqrt{n}}{(n-1)}$, 再设 $f(x)=\dfrac{\sqrt{x}}{x-1}$, 则

$$f'(x)=\frac{-x-1}{2\sqrt{x}(x-1)^2}<0\,(x>0),$$

故 $a_n\ge a_{n+1}$, 而 $\lim\limits_{n\to\infty}a_n=0$ 显然成立, 所以原级数收敛, 但又因为 $\lim\limits_{n\to\infty}\dfrac{|u_n|}{\dfrac{1}{\sqrt{n}}}=1$, 因此原级数条件收敛.

(4) 设 $u_n=(-1)^{\frac{n(n-1)}{2}}\dfrac{n^{10}}{2^n}$, 而 $\lim\limits_{n\to\infty}\dfrac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\dfrac{(n+1)^{10}}{2^{n+1}}\dfrac{2^n}{n^{10}}=\dfrac{1}{2}<1$, 所以原级数绝对收敛.

(5) 设 $u_n=(-1)^{n+1}\dfrac{2^{n^2}}{n!}$, 而 $\lim\limits_{n\to\infty}\dfrac{|u_{n+1}|}{|u_n|}=\lim\limits_{n\to\infty}\dfrac{2^{(n+1)^2}}{(n+1)!}\dfrac{n!}{2^{n^2}}=\lim\limits_{n\to\infty}\dfrac{2^{2n+1}}{n+1}=\infty$, 所以原级数发散.

(6) 设 $u_n=\dfrac{(-1)^n}{n-\ln n}$, $a_n=\dfrac{1}{n-\ln n}$, 显然 $\lim\limits_{n\to\infty}a_n=0$. 再设 $f(x)=\dfrac{1}{x-\ln x}$, 则 $f'(x)=\dfrac{1-x}{x(x-\ln x)^2}<0\,(x>1)$, 故 $a_n\ge a_{n+1}(n\ge 2)$, 所以原级数收敛, 但又因为 $\lim\limits_{n\to\infty}\dfrac{|u_n|}{\dfrac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{n}{n-\ln n}=1$, 因此原级数条件收敛.

(7) 设 $u_n=(-1)^n\dfrac{1}{\pi^n}\sin\dfrac{\pi}{n}(n=2,3,\cdots)$, 则 $|u_n|\le\dfrac{1}{\pi^n}$, 而 $\sum\limits_{n=2}^{\infty}\dfrac{1}{\pi^n}$ 收敛, 所以原级数绝对收敛.

(8) 设级数 $(\dfrac{1}{\sqrt{2}-1}-\dfrac{1}{\sqrt{2}+1})+(\dfrac{1}{\sqrt{3}-1}-\dfrac{1}{\sqrt{3}+1})+\cdots+(\dfrac{1}{\sqrt{n}-1}-\dfrac{1}{\sqrt{n}+1})+\cdots$, 其的一般项为 $u_n=\dfrac{1}{\sqrt{n}-1}-\dfrac{1}{\sqrt{n}+1}(n=2,3,\cdots)$, 故 $\lim\limits_{n\to\infty}\dfrac{|u_n|}{\dfrac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{2n}{(\sqrt{n}-1)(\sqrt{n}+1)}=2$,

所以 $(\dfrac{1}{\sqrt{2}-1}-\dfrac{1}{\sqrt{2}+1})+(\dfrac{1}{\sqrt{3}-1}-\dfrac{1}{\sqrt{3}+1})+\cdots+(\dfrac{1}{\sqrt{n}-1}-\dfrac{1}{\sqrt{n}+1})+\cdots$ 发散, 从而原级数发散.

(9) 设 $u_n=(-1)^{n-1}\frac{1}{na+b}$，$a_n=\frac{1}{na+b}$，显然 $a_n\ge a_{n+1}$且$\lim\limits_{n\to\infty}a_n=0$，所以原级数收敛，而$\sum\limits_{n=1}^{\infty}\frac{1}{na+b}$发散，因此原级数条件收敛.

(10) 设 $u_n=(-1)^{n-1}\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{2\cdot 4\cdot\cdots\cdot(2n)}$，$a_n=\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{2\cdot 4\cdot\cdots\cdot(2n)}$．由 $\frac{a}{b}<\frac{a+1}{b+1}(0<a<b)$ 得

$$a_n<\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\cdots\times\frac{2n-4}{2n-3}\times\frac{2n-2}{2n-1}\times\frac{1}{2n}\times\frac{2n}{2n}=\frac{1}{a_n}\frac{1}{4n^2(2n-1)},$$

所以$0<a_n<\frac{1}{2n\sqrt{2n-1}}(n\ge 2)$，而$\sum\limits_{n=2}^{\infty}\frac{1}{2n\sqrt{2n-1}}$收敛，因此原级数绝对收敛.

(11) 设$u_n=(-1)^n\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}$，$a_n=\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}$，因为

$$a_n=1\times\frac{3}{2}\times\frac{5}{4}\cdots\frac{2n-1}{2n-2}\times\frac{1}{2n}>\frac{1}{2n}(n\ge 2),$$

而$\sum\limits_{n=1}^{\infty}\frac{1}{2n}$发散，所以$\sum\limits_{n=1}^{\infty}|u_n|$发散．又因$a_n\ge a_{n+1}$，再由$\frac{a}{b}<\frac{a+1}{b+1}(0<a<b)$得

$$a_n=\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\cdots\frac{2n-1}{2n}<\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\cdots\frac{2n}{2n+1}=\frac{1}{a_n}\frac{1}{2n+1},$$

所以$0<a_n<\frac{1}{\sqrt{2n+1}}\to 0(n\to\infty)$，故由莱布尼兹判别法知原级数收敛，从而条件收敛.

(12) $u_n=\frac{(-1)^n}{\sqrt{n}+(-1)^n}=\frac{(-1)^n\sqrt{n}}{n-1}-\frac{1}{n-1}$，显然$\sum\limits_{n=2}^{\infty}\frac{1}{n-1}$发散且易由莱布尼兹判别法知$\sum\limits_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{n-1}$收敛，所以原级数发散.

(13) $u_n=(-1)^n(\sqrt{n+1}-\sqrt{n})=\frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$，$a_n=\frac{1}{\sqrt{n+1}+\sqrt{n}}$，而$\lim\limits_{n\to\infty}\frac{|u_n|}{\frac{1}{\sqrt{n}}}=1$，且$a_n\ge a_{n+1},\lim\limits_{n\to\infty}a_n=0$，所以原级数条件收敛.

2. 设$a_n<c_n<b_n$，且级数$\sum\limits_{n=1}^{\infty}a_n$、$\sum\limits_{n=1}^{\infty}b_n$均收敛，证明$\sum\limits_{n=1}^{\infty}c_n$收敛.

证　由题知$0 \le b_n - c_n \le b_n - a_n$且$\sum_{n=1}^{\infty}(b_n - a_n)$收敛，从而$\sum_{n=1}^{\infty}(b_n - c_n)$收敛，所以$\sum_{n=1}^{\infty}c_n$收敛.