第三节　不定积分的分部积分法

习题　4-3

求下列不定式积分:

1. $\int x\sin x\mathrm{d}x$.

解　$\int x\sin x\mathrm{d}x=-\int x\mathrm{d}\cos x=-x\cos x+\int\cos x\mathrm{d}x$

$$=-x\cos x+\sin x+C.$$

2. $\int\ln x\mathrm{d}x$.

解　$\int\ln x\mathrm{d}x=x\ln x-\int x\mathrm{d}\ln x=x\ln x-\int x\cdot\frac{1}{x}\mathrm{d}x=x(\ln x-1)+C$.

3. $\int(\frac{1}{x}+\ln x)\mathrm{e}^x\mathrm{d}x$.

解　$\int(\frac{1}{x}+\ln x)\mathrm{e}^x\mathrm{d}x=\int\frac{1}{x}\mathrm{e}^x\mathrm{d}x+\int\mathrm{e}^x\ln x\mathrm{d}x=\int\mathrm{e}^x\mathrm{d}\ln x+\int\mathrm{e}^x\ln x\mathrm{d}x$

$$=\mathrm{e}^x\ln x-\int\ln x\mathrm{d}(\mathrm{e}^x)+\int\mathrm{e}^x\ln x\mathrm{d}x+C$$

$$=\mathrm{e}^x\ln x-\int\mathrm{e}^x\ln x\mathrm{d}x+\int\mathrm{e}^x\ln x\mathrm{d}x+C$$

$$=\mathrm{e}^x\ln x+C.$$

4. $\int\ln(x+\sqrt{x^2+1})\mathrm{d}x$.

解　$\int\ln(x+\sqrt{x^2+1})\mathrm{d}x=x\ln(x+\sqrt{x^2+1})-\int x\mathrm{d}\ln(x+\sqrt{x^2+1})$

$$=x\ln(x+\sqrt{x^2+1})-\int x\frac{1}{x+\sqrt{x^2+1}}(1+\frac{1}{2}\frac{1}{\sqrt{x^2+1}}\cdot 2x)\mathrm{d}x$$

$$=x\ln(x+\sqrt{x^2+1})-\int\frac{x}{\sqrt{x^2+1}}\mathrm{d}x$$

$$=x\ln(x+\sqrt{x^2+1})-\frac{1}{2}\int(x^2+1)^{-\frac{1}{2}}\mathrm{d}(x^2+1)$$

$$= x\ln(x+\sqrt{x^2+1})-\sqrt{x^2+1}+C.$$

5. $\int \frac{x}{\sin^2 x}\mathrm{d}x$.

解 $$\int \frac{x}{\sin^2 x}\mathrm{d}x=\int x\csc^2 x\mathrm{d}x=-\int x\mathrm{d}\cot x=-x\cot x+\int \cot x\mathrm{d}x$$

$$=-x\cot x+\int \frac{\cos x}{\sin x}\mathrm{d}x=-x\cot x+\int \frac{\mathrm{d}\sin x}{\sin x}$$

$$=-x\cot x+\ln\left|\sin x\right|+C.$$

6. $\int x^2\arctan x\mathrm{d}x$.

解 $$\int x^2\arctan x\mathrm{d}x=\frac{1}{3}\int \arctan x\mathrm{d}(x^3)=\frac{1}{3}(x^3\arctan x-\int x^3\mathrm{d}\arctan x)$$

$$=\frac{1}{3}x^3\arctan x-\frac{1}{3}\int \frac{x^3}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{3}x^3\arctan x-\frac{1}{6}\int (1-\frac{1}{1+x^2})\mathrm{d}(x^2)$$

$$=\frac{1}{3}x^3\arctan x-\frac{x^2}{6}+\frac{1}{6}\int \frac{\mathrm{d}(1+x^2)}{1+x^2}$$

$$=\frac{1}{3}x^3\arctan x-\frac{x^2}{6}+\frac{1}{6}\ln(1+x^2)+C.$$

7. $\int x^2\mathrm{e}^{3x}\mathrm{d}x$.

解 $$\int x^2\mathrm{e}^{3x}\mathrm{d}x=\frac{1}{3}\int x^2\mathrm{d}(\mathrm{e}^{3x})=\frac{1}{3}[x^2\mathrm{e}^{3x}-\int \mathrm{e}^{3x}\mathrm{d}(x^2)]$$

$$=\frac{1}{3}(x^2\mathrm{e}^{3x}-2\int x\mathrm{e}^{3x}\mathrm{d}x)=\frac{1}{3}x^2\mathrm{e}^{3x}-\frac{2}{9}\int x\mathrm{d}(\mathrm{e}^{3x})$$

$$=\frac{1}{3}x^2\mathrm{e}^{3x}-\frac{2}{9}(x\mathrm{e}^{3x}-\int \mathrm{e}^{3x}\mathrm{d}x)$$

$$=\frac{1}{3}x^2\mathrm{e}^{3x}-\frac{2}{9}x\mathrm{e}^{3x}+\frac{2}{27}\mathrm{e}^{3x}+C=\frac{\mathrm{e}^{3x}}{3}(x^2-\frac{2}{3}x+\frac{2}{9})+C.$$

8. $\int \sin(\ln x)\mathrm{d}x$.

解 $$\int \sin(\ln x)\mathrm{d}x=x\sin(\ln x)-\int x\mathrm{d}\sin(\ln x)$$

$$=x\sin(\ln x)-\int x\cdot\cos(\ln x)\cdot\frac{1}{x}\mathrm{d}x$$

$$=x\sin(\ln x)-\int \cos(\ln x)\mathrm{d}x$$

$$= x\sin(\ln x) - x\cos(\ln x) + \int x \mathrm{d}\cos(\ln x)$$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\mathrm{d}x .$$

移项，解出

$$\int \sin(\ln x)\mathrm{d}x = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C .$$

9. $\int \arctan x\mathrm{d}x$.

解 $\int \arctan x\mathrm{d}x = x\arctan x - \int x\mathrm{d}\arctan x = x\arctan x - \int \frac{x}{1+x^2}\mathrm{d}x$

$$= x\arctan x - \frac{1}{2}\int \frac{\mathrm{d}(1+x^2)}{1+x^2}$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C .$$

10. $\int \mathrm{arc}\cot \mathrm{d}x$.

解 $\int \mathrm{arc}\cot \mathrm{d}x = x\,\mathrm{arc}\cot x - \int x\mathrm{d}\,\mathrm{arc}\cot x = x\,\mathrm{arc}\cot x + \int \frac{x}{1+x^2}\mathrm{d}x$

$$= x\,\mathrm{arc}\cot x + \frac{1}{2}\int \frac{\mathrm{d}(1+x^2)}{1+x^2}$$

$$= x\,\mathrm{arc}\cot x + \frac{1}{2}\ln(1+x^2) + C .$$

11. $\int \arccos x\mathrm{d}x$.

解 $\int \arccos x\mathrm{d}x = x\arccos x - \int x\mathrm{d}\arccos x$

$$= x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$$

$$= x\arccos x - \frac{1}{2}\int \frac{\mathrm{d}(1-x^2)}{\sqrt{1-x^2}}$$

$$= x\arccos x - \sqrt{1-x^2} + C .$$

12. $\int \mathrm{e}^{3x}\cos 2x\mathrm{d}x$.

解 $\int \mathrm{e}^{3x}\cos 2x\mathrm{d}x = \frac{1}{3}\int \cos 2x\mathrm{d}(\mathrm{e}^{3x}) = \frac{1}{3}(\mathrm{e}^{3x}\cos 2x - \int \mathrm{e}^{3x}\mathrm{d}\cos 2x)$

$$=\frac{1}{3}e^{3x}\cos 2x+\frac{2}{3}\int e^{3x}\sin 2x dx$$

$$=\frac{1}{3}e^{3x}\cos 2x+\frac{2}{9}\int \sin 2x d(e^{3x})$$

$$=\frac{1}{3}e^{3x}\cos 2x+\frac{2}{9}(e^{3x}\sin 2x-\int e^{3x}d\sin 2x)$$

$$=\frac{1}{3}e^{3x}\cos 2x+\frac{2}{9}e^{3x}\sin 2x-\frac{4}{9}\int e^{3x}\cos 2x dx.$$

移项，解出

$$\int e^{3x}\cos 2x dx=\frac{e^{3x}}{13}(3\cos 2x+2\sin 2x)+C.$$

注意　产生循环现象，从而求出积分，是分部积分法的主要作用之一．应当注意的是：在反复使用分部积分法的过程中，每次所选 u 均应是同一类函数，如本题两次都选三角函数作为 u，否则不仅不会产生循环现象，反而会一来一往地恢复原状，毫无所得．

13. $\int(\arcsin x)^2 dx$.

解 $\int(\arcsin x)^2 dx=x\arcsin^2 x-\int x d(\arcsin x)^2$

$$=x\arcsin^2 x-2\int x\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=x\arcsin^2 x+2\int \arcsin x d\sqrt{1-x^2}$$

$$=x\arcsin^2 x+2(\sqrt{1-x^2}\arcsin x-\int\sqrt{1-x^2}d\arcsin x)$$

$$=x\arcsin^2 x+2\sqrt{1-x^2}\arcsin x-2\int\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}dx$$

$$=x\arcsin^2 x+2\sqrt{1-x^2}\arcsin x-2x+C.$$

14. $\int\frac{\ln x}{\sqrt{x}}dx$.

解 $\int\frac{\ln x}{\sqrt{x}}dx=2\int\ln x d(x^{\frac{1}{2}})=2(x^{\frac{1}{2}}\ln x-\int x^{\frac{1}{2}}d\ln x)$

$$=2\sqrt{x}\ln x-2\int x^{-\frac{1}{2}}dx=2\sqrt{x}\ln x-4\sqrt{x}+C.$$

15. $\int\frac{\ln(\ln x)}{x}\mathrm{d}x$.

解　令 $\ln x=t,\ x=\mathrm{e}^t,\ \mathrm{d}x=\mathrm{e}^t\mathrm{d}t$.

$$\int\frac{\ln(\ln x)}{x}\mathrm{d}x=\int\frac{\ln t}{\mathrm{e}^t}\cdot\mathrm{e}^t\mathrm{d}t=\int\ln t\mathrm{d}t=t\ln t-\int t\mathrm{d}\ln t$$

$$=t\ln t-\int t\cdot\frac{1}{t}\mathrm{d}t=t\ln t-t+C=\ln x\ln(\ln x)-\ln x+C\ .$$

16. $\int\frac{\ln\sin x}{\cos^2 x}\mathrm{d}x$.

解　$\int\frac{\ln\sin x}{\cos^2 x}\mathrm{d}x=\int\ln\sin x\mathrm{d}\tan x=\tan x\ln\sin x-\int\tan x\mathrm{d}\ln\sin x$

$$=\tan x\ln\sin x-\int\tan\frac{1}{\sin x}\cdot\cos x\mathrm{d}x$$

$$=\tan x\ln\sin x-x+C\ .$$

17. $\int\frac{x\cos x}{\sin^3 x}\mathrm{d}x$.

解　$\int\frac{x\cos x}{\sin^3 x}\mathrm{d}x=\int\frac{x\cos x}{\sin x\cdot\sin^2 x}\mathrm{d}x=-\int x\cot x\mathrm{d}\cot x$

$$=-(x\cot^2 x-\int\cot x\mathrm{d}x\cot x)$$

$$=-x\cot^2 x+\int(\cot^2 x-\frac{x\cos x}{\sin^3 x})\mathrm{d}x$$

$$=-x\cot^2 x+\int(\csc^2 x-1)\mathrm{d}x-\int\frac{x\cos x}{\sin^3 x}\mathrm{d}x$$

$$=-x\cot^2 x-\cot x-x-\int\frac{x\cos x}{\sin^3 x}\mathrm{d}x\ .$$

移项，解出

$$\int\frac{x\cos x}{\sin^3 x}\mathrm{d}x=-\frac{1}{2}(x\cot^2 x+\cot x+x)+C\ .$$

18. $\int\frac{\arcsin x\mathrm{e}^{\arcsin x}}{\sqrt{1-x^2}}\mathrm{d}x$.

解　令 $\arcsin x=t$.

$$\int\frac{\arcsin x\mathrm{e}^{\arcsin x}}{\sqrt{1-x^2}}\mathrm{d}x=\int\arcsin x\mathrm{e}^{\arcsin x}\mathrm{d}\arcsin x$$

$$=\int t\mathrm{e}^t\mathrm{d}t=\int t\mathrm{d}(\mathrm{e}^t)=t\mathrm{e}^t-\int\mathrm{e}^t\mathrm{d}t$$

$$=te^t-e^t+C=\arcsin x\cdot e^{\arcsin x}-e^{\arcsin x}+C$$

$$=e^{\arcsin x}(\arcsin x-1)+C\ .$$

19. $\int\frac{x\arcsin x}{\sqrt{1-x^2}}dx$.

解　令 $\arcsin x=t,\ x=\sin t,\ dx=\cos t dt$.

$$\int\frac{x\arcsin x}{\sqrt{1-x^2}}dx=\int\frac{t\sin t}{\cos t}\cos t dt=\int t\sin t dt=-\int t d\cos t$$

$$=-t\cos t+\int\cos t dt=-\sqrt{1-x^2}\arcsin x+x+C\ .$$

注意　视被积函数的情况交替使用换元积分法与分部积分法是计算不定积分经常会遇到的情况，一般有先分部后凑微分，先换元后分部或先分部后换元，本例中是先换元后分部.

20. $\int\frac{\arctan e^x}{e^x}dx$.

解　法 1　$\int\frac{\arctan e^x}{e^x}dx=-\int\arctan e^x d(e^{-x})$

$$=-e^{-x}\arctan e^x+\int e^{-x}d\arctan e^x$$

$$=-e^{-x}\arctan e^x+\int\frac{e^{-x}e^x}{1+e^{2x}}dx$$

$$=-e^{-x}\arctan e^x+\int\frac{1+e^{2x}-e^{2x}}{1+e^{2x}}dx$$

$$=-e^{-x}\arctan e^x+x-\frac{1}{2}\int\frac{1}{1+e^{2x}}d(1+e^{2x})$$

$$=-e^{-x}\arctan e^x+x-\frac{1}{2}\ln(1+e^{2x})+C\ .$$

法 2　令 $e^x=t,\ x=\ln t,\ dx=\frac{dt}{t}$.

$$\int\frac{\arctan e^x}{e^x}dx=\int\frac{1}{t^2}\arctan t dt=-\frac{1}{t}\arctan t+\int\frac{1}{t(1+t^2)}dt$$

$$=-\frac{1}{t}\arctan t+\int(\frac{1}{t}-\frac{t}{1+t^2})dt$$

$$=-\frac{1}{t}\arctan t+\ln t-\frac{1}{2}\ln(1+t^2)+C$$

$$= -\mathrm{e}^{-x}\arctan \mathrm{e}^{x} + x - \frac{1}{2}\ln(1+\mathrm{e}^{2x}) + C .$$

21. $\int \cos^2\sqrt{x}\mathrm{d}x$.

解　令 $\sqrt{x} = t,\ x = t^2,\ \mathrm{d}x = 2t\mathrm{d}t$.

$$\int \cos^2\sqrt{x}\mathrm{d}x = 2\int t\cos^2 t\mathrm{d}t = 2\int t\cos t\mathrm{d}\sin t$$

$$= 2t\sin t\cos t - 2\int \sin t\mathrm{d}(t\cos t)$$

$$= t\sin 2t - 2\int \sin t(\cos t - t\sin t)\mathrm{d}t$$

$$= t\sin 2t - \int \sin 2t\mathrm{d}t + 2\int t(1-\cos^2 t)\mathrm{d}t$$

$$= t\sin 2t + \frac{1}{2}\cos 2t + t^2 - 2\int t\cos^2 t\mathrm{d}t$$

移项，解出

$$\int \cos^2\sqrt{x}\mathrm{d}x = 2\int t\cos^2 t\mathrm{d}t = \frac{1}{2}(t\sin 2t + \frac{1}{2}\cos 2t + t^2) + C$$

$$= \frac{1}{2}\sqrt{x}\sin 2\sqrt{x} + \frac{1}{4}\cos 2\sqrt{x} + \frac{x}{2} + C .$$

22. $\int \frac{x\arctan x}{\sqrt{1+x^2}}\mathrm{d}x$.

解　令 $\arctan x = t,\ x = \tan t,\ \mathrm{d}x = \sec^2 t\mathrm{d}t$.

$$\int \frac{x\arctan x}{\sqrt{1+x^2}}\mathrm{d}x = \int \frac{t\cdot\tan t}{\sqrt{1+\tan^2 t}}\sec^2 t\mathrm{d}t = \int t\cdot\sec t\cdot\tan t\mathrm{d}t = \int t\mathrm{d}\sec t$$

$$= t\sec t - \int \sec t\mathrm{d}t = t\sec t - \int \frac{\sec t(\sec t + \tan t)}{\sec t + \tan t}\mathrm{d}t$$

$$= t\sec t - \int \frac{\mathrm{d}(\sec t + \tan t)}{\sec t + \tan t} = t\sec t - \ln\left|\sec t + \tan t\right| + C$$

$$= \sqrt{1+x^2}\arctan x - \ln(x + \sqrt{1+x^2}) + C .$$

23. $\int xf''(x)\mathrm{d}x$.

解　$\int xf''(x)\mathrm{d}x = \int x\mathrm{d}f'(x) = xf'(x) - \int f'(x)\mathrm{d}x = xf'(x) - f(x) + C$.

24. $\int e^{\sqrt[3]{x}}dx$.

解　令 $\sqrt[3]{x}=t,\ x=t^3,\ dx=3t^2dt$.

$$\int e^{\sqrt[3]{x}}dx=3\int t^2e^tdt=3\int t^2d(e^t)=3[t^2e^t-\int e^td(t^2)]=3(t^2e^t-2\int te^tdt)$$

$$=3[t^2e^t-2\int td(e^t)]=3(t^2e^t-2te^t+2\int e^tdt)$$

$$=3(t^2e^t-2te^t+2e^t)+C=3(x^{\frac{2}{3}}-2x^{\frac{1}{3}}+2)e^{\sqrt[3]{x}}+C.$$

25. $\int\frac{\ln x}{(1-x)^2}dx$.

解　$$\int\frac{\ln x}{(1-x)^2}dx=-\int\frac{\ln x}{(1-x)^2}d(1-x)=\int\ln xd\frac{1}{1-x}$$

$$=\frac{\ln x}{1-x}-\int\frac{1}{1-x}d\ln x=\frac{\ln x}{1-x}-\int\frac{1}{x(1-x)}dx$$

$$=\frac{\ln x}{1-x}-\int(\frac{1}{1-x}+\frac{1}{x})dx=\frac{\ln x}{1-x}+\ln|1-x|-\ln x+C$$

$$=\frac{\ln x}{1-x}+\ln\frac{|1-x|}{x}+C.$$

26. $\int e^x\sin^2 xdx$.

解　$$\int e^x\sin^2 xdx=\int e^x\frac{1-\cos 2x}{2}dx=\frac{1}{2}\int e^xdx-\frac{1}{2}\int e^x\cos 2xdx$$

$$=\frac{e^x}{2}-\frac{1}{4}\int e^xd\sin 2x=\frac{e^x}{2}-\frac{1}{4}e^x\sin 2x+\frac{1}{4}\int e^x\sin 2xdx$$

$$=\frac{e^x}{2}-\frac{1}{4}e^x\sin 2x-\frac{1}{8}\int e^xd\cos 2x$$

$$=\frac{e^x}{2}-\frac{1}{4}e^x\sin 2x-\frac{1}{8}e^x\cos 2x+\frac{1}{8}\int e^x\cos 2xdx.$$

移项，解出

$$\int e^x\cos 2xdx=\frac{2}{5}e^x(\sin 2x+\frac{1}{2}\cos 2x)+C_1,$$ 代入原式得

$$\int e^x\sin^2 xdx=\frac{e^x}{2}-\frac{e^x}{5}(\sin 2x+\frac{1}{2}\cos 2x)+C$$

$$=e^x(\frac{1}{2}-\frac{1}{10}\cos 2x-\frac{1}{5}\sin 2x)+C.$$

注意　利用半角公式将 $\sin^2 x$ 化为 $\frac{1-\cos 2x}{2}$，便得到属于分部积分法解决的类

型$\int e^x \cos 2x dx$.