

## 第五节 隐函数的微分法

### 习题 8-5

1. 求下列方程所确定的隐函数  $y = y(x)$  的一阶导数:

$$(1) \quad xy - \ln y = a; \quad (2) \quad \ln \sqrt{x^2 + y^2} = \arctan \frac{y}{x}.$$

解 (1) 法 1 用隐函数求导公式求  $\frac{dy}{dx}$ .

令  $F(x, y) = xy - \ln y - a$ , 则  $F_x = y$ ,  $F_y = x - \frac{1}{y}$ , 所以

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y}{x - \frac{1}{y}} = \frac{y^2}{1 - xy}.$$

法 2 用隐函数求导法则求  $\frac{dy}{dx}$ .

等式两边对  $x$  求导, 注意  $y$  是  $x$  的函数, 有

$$y + x \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx} = 0,$$

所以

$$\frac{dy}{dx} = \frac{-y}{x - \frac{1}{y}} = \frac{y^2}{1 - xy}.$$

(2) 法 1 用隐函数求导公式求  $\frac{dy}{dx}$ .

令  $F(x, y) = \ln \sqrt{x^2 + y^2} - \arctan \frac{y}{x}$ , 则

$$F_x = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2x}{2\sqrt{x^2 + y^2}} - \frac{1}{1 + (\frac{y}{x})^2} \cdot (-\frac{y}{x^2}) = \frac{x + y}{x^2 + y^2},$$

$$F_y = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2y}{2\sqrt{x^2 + y^2}} - \frac{1}{1 + (\frac{y}{x})^2} \cdot \frac{1}{x} = \frac{y - x}{x^2 + y^2},$$

所以

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{x + y}{x^2 + y^2}}{\frac{y - x}{x^2 + y^2}} = \frac{x + y}{x - y}.$$

法 2 用隐函数求导法则求  $\frac{dy}{dx}$ .

等式两边对  $x$  求导, 注意  $y$  是  $x$  的函数, 有

$$\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{x+y \frac{dy}{dx}}{\sqrt{x^2+y^2}} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{x \frac{dy}{dx} - y}{x^2},$$

化简得

$$x+y \frac{dy}{dx} = x \frac{dy}{dx} - y,$$

所以

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

2. 求下列方程所确定的隐函数  $z = z(x, y)$  的一价偏导数:

$$(1) \quad e^{-(x+y+z)} = x+y+z; \quad (2) \quad \frac{x}{z} = \ln \frac{z}{y}.$$

解 (1) 法 1 用隐函数求导公式求  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ .

令  $F(x, y, z) = e^{-(x+y+z)} - x - y - z$ , 则

$$F_x = e^{-(x+y+z)} \cdot (-1) - 1 = -e^{-(x+y+z)} - 1,$$

由对称性可得

$$F_y = F_z = -e^{-(x+y+z)} - 1,$$

所以

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -1, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -1.$$

法 2 直接用复合函数求偏导数的方法求  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ .

等式两边对  $x$  求偏导, 注意  $z$  是  $x$  和  $y$  的二元函数, 则有

$$e^{-(x+y+z)} \cdot (1 + \frac{\partial z}{\partial x}) = 1 + \frac{\partial z}{\partial x},$$

所以

$$\frac{\partial z}{\partial x} = \frac{1 - e^{-(x+y+z)}}{e^{-(x+y+z)} - 1} = -1,$$

同理, 由对称性可得

$$\frac{\partial z}{\partial y} = -1.$$

法3 用全微分形式不变性求  $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ .

将  $x, y, z$  均看作自变量, 方程两边同时取全微分, 得

$$de^{-(x+y+z)} = d(x+y+z),$$

即

$$e^{-(x+y+z)} d[-(x+y+z)] = d(x+y+z),$$

$$-e^{-(x+y+z)} (dx + dy + dz) = dx + dy + dz,$$

这时, 再将  $z$  看作  $x, y, z$  的函数, 解出  $z$  的全微分  $dz$ , 有

$$dz = \frac{1 + e^{-(x+y+z)}}{-e^{-(x+y+z)} - 1} dx + \frac{1 + e^{-(x+y+z)}}{-e^{-(x+y+z)} - 1} dy = -dx - dy,$$

所以

$$\frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1.$$

(2) 法1 公式法.

令  $F(x, y, z) = \frac{x}{z} - \ln \frac{z}{y}$ , 则

$$F_x = \frac{1}{z}, \quad F_y = -\frac{y}{z} \cdot \left(-\frac{z}{y^2}\right) = \frac{1}{y},$$

$$F_z = -\frac{x}{z^2} - \frac{y}{z} \cdot \frac{1}{y} = -\frac{x+z}{z^2},$$

所以

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1}{z} \cdot \frac{z^2}{x+z} = \frac{z}{x+z},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{1}{y} \cdot \frac{-z^2}{x+z} = \frac{z^2}{y(x+z)}.$$

法2 复合函数求导法.

等式两边对  $x$  求偏导, 得

$$\frac{z-x\frac{\partial z}{\partial x}}{z^2} = \frac{y}{z} \cdot \frac{1}{y} \frac{\partial z}{\partial x}, \quad \text{即} \quad z-x\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial x},$$

所以

$$\frac{\partial z}{\partial x} = \frac{z}{x+z}.$$

等式两边对  $y$  求偏导, 则有

$$-\frac{x}{z^2} \frac{\partial z}{\partial y} = \frac{y}{z} \frac{y \frac{\partial z}{\partial y} - z}{y^2}, \text{ 即 } -x \frac{\partial z}{\partial y} = z \frac{\partial z}{\partial y} - \frac{z^2}{y},$$

所以

$$\frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}.$$

法 3 全微分法.

方程两边同时取全微分, 得

$$d \frac{x}{z} = d \ln \frac{z}{y},$$

即

$$\begin{aligned} \frac{zdx - xdz}{z^2} &= \frac{y}{z} \cdot \frac{ydz - zdy}{y^2}, \\ \frac{1}{z} dx - \frac{x}{z^2} dz &= \frac{1}{z} dz - \frac{1}{y} dy, \end{aligned}$$

解出

$$dz = \frac{\frac{1}{z}}{\frac{x}{z^2} + \frac{1}{z}} dx + \frac{\frac{1}{y}}{\frac{x}{z^2} + \frac{1}{z}} dy = \frac{z}{x+z} dx + \frac{z^2}{y(x+z)} dy,$$

所以

$$\frac{\partial z}{\partial x} = \frac{z}{x+z}, \quad \frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}.$$

**注意** 对于隐函数求导, 一般有三种方法: 公式法, 复合函数求导法, 全微分法, 在不同的情况下, 各有方便之处, 但必须注意: 在等式两边关于  $x$  或  $y$  求偏导数时, 应当将  $z$  看作是  $x, y$  的函数, 而对  $F(x, y, z)$  关于  $x$  和  $y$  求偏导数时, 将  $z$  看作是常数.

3. 设  $f$  可微, 且方程  $y+z=xf(y^2-z^2)$  确定了  $z=z(x, y)$ , 计算  $x \frac{\partial z}{\partial x} + z \frac{\partial z}{\partial y}$ .

**解** 令  $F(x, y, z) = y+z-xf(y^2-z^2)$ , 则

$$F_x = -f(y^2-z^2),$$

$$F_y = 1 - xf'(y^2-z^2) \cdot 2y = 1 - 2xyf'(y^2-z^2),$$

$$F_z = 1 - xf'(y^2 - z^2) \cdot (-2z) = 1 + 2xzf'(y^2 - z^2),$$

于是

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-f(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} = \frac{f(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{1 - 2xyf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)},$$

所以

$$\begin{aligned} x \frac{\partial z}{\partial x} + z \frac{\partial z}{\partial y} &= \frac{xf(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} - \frac{z - 2xyzf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} \\ &= \frac{[xf(y^2 - z^2) - z] + 2xyzf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} \\ &= \frac{y + 2xyzf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} = y. \end{aligned}$$

4. 设方程  $f(ax - cz, ay - bz) = 0$  确定  $z = z(x, y)$ , 证明:

$$c \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = a.$$

证 因为  $f_x = f'_1 \cdot a = af'_1$ ,  $f_y = f'_2 \cdot a = af'_2$ ,

$$f_z = f'_1 \cdot (-c) + f'_2 \cdot (-b) = -cf'_1 - bf'_2,$$

于是

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{f_x}{f_z} = -\frac{af'_1}{-cf'_1 - bf'_2} = \frac{af'_1}{cf'_1 + bf'_2}, \\ \frac{\partial z}{\partial y} &= -\frac{f_y}{f_z} = -\frac{af'_2}{-cf'_1 - bf'_2} = \frac{af'_2}{cf'_1 + bf'_2}, \end{aligned}$$

所以

$$c \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = \frac{acf'_1}{cf'_1 + bf'_2} + \frac{abf'_2}{cf'_1 + bf'_2} = a.$$

5. 设  $x = x(y, z)$ ,  $y = y(z, x)$ ,  $z = z(x, y)$  都是由方程  $F(x, y, z) = 0$  所确定的函数, 且都具有连续偏导数, 证明:

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1.$$

证 由隐函数求导公式, 得

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}, \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}, \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

所以

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = \left(-\frac{F_y}{F_x}\right) \cdot \left(-\frac{F_z}{F_y}\right) \cdot \left(-\frac{F_x}{F_z}\right) = -1.$$

6. 求下列方程所确定的隐函数的指定偏导数:

$$(1) \quad e^z - xyz = 0, \frac{\partial^2 z}{\partial x^2}; \quad (2) \quad xy + yz + zx = 1, \frac{\partial^2 z}{\partial x \partial y};$$

$$(3) \quad z + \ln z - \int_y^x e^{-t^2} dt = 0, \frac{\partial^2 z}{\partial x \partial y}.$$

解 (1) 令  $F(x, y, z) = e^z - xyz$ , 则

$$F_x = -yz, \quad F_z = e^z - xy,$$

所以

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = \frac{yz}{e^z - xy}, \\ \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{yz}{e^z - xy} \right) = \frac{y \frac{\partial z}{\partial x} (e^z - xy) - yz(e^z \frac{\partial z}{\partial x} - y)}{(e^z - xy)^2}, \end{aligned}$$

把  $\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$  代入上式, 并化简, 有

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2 z e^z - y^2 z^2 e^z - 2xy^3 z}{(e^z - xy)^3},$$

又因为  $e^z = xyz$ , 所以

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy^3 z^2 - xy^3 z^3 - 2xy^3 z}{(xyz - xy)^3} = \frac{z(z^2 - 2z + 2)}{x^2(1 - z)^2}.$$

(2) 令  $F(x, y, z) = xy + yz + zx - 1$ , 则

$$F_x = y + z, \quad F_y = x + z, \quad F_z = x + y,$$

于是

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y+z}{x+y}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{x+y},$$

所以

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{y+z}{x+y} \right) = -\frac{(1+\frac{\partial z}{\partial y})(x+y) - (y+z) \cdot 1}{(x+y)^2} \\ &= -\frac{(1-\frac{x+z}{x+y})(x+y) - (y+z)}{(x+y)^2} = \frac{2z}{(x+y)^2}. \end{aligned}$$

(3) 原等式两边对  $x$  求偏导, 得

$$\frac{\partial z}{\partial x} + \frac{1}{z} \cdot \frac{\partial z}{\partial x} - e^{-x^2} = 0,$$

于是

$$\frac{\partial z}{\partial x} = \frac{e^{-x^2}}{1+\frac{1}{z}} = \frac{ze^{-x^2}}{z+1}.$$

原等式两边对  $y$  求偏导, 得

$$\frac{\partial z}{\partial y} + \frac{1}{z} \cdot \frac{\partial z}{\partial y} + e^{-y^2} = 0,$$

于是

$$\frac{\partial z}{\partial y} = \frac{-e^{-y^2}}{1+\frac{1}{z}} = -\frac{ze^{-y^2}}{z+1}.$$

所以

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left( \frac{ze^{-x^2}}{z+1} \right) = \frac{e^{-x^2} \frac{\partial z}{\partial y} (z+1) - ze^{-x^2} \cdot \frac{\partial z}{\partial y}}{(z+1)^2} \\ &= \frac{e^{-x^2} \left( -\frac{ze^{-y^2}}{z+1} \right) (z+1) + ze^{-x^2} \frac{ze^{-y^2}}{z+1}}{(z+1)^2} = \frac{-z}{(z+1)^3} e^{-(x^2+y^2)}. \end{aligned}$$

7. 求由方程  $f(x-y, y-z, z-x)=0$  所确定的函数  $z=z(x, y)$  的全微分  $dz$ .

解 用全微分形式不变性求  $dz$ . 方程两边同时取全微分, 得

$$f'_1 \cdot d(x-y) + f'_2 \cdot d(y-z) + f'_3 \cdot d(z-x) = 0,$$

$$f'_1(dx-dy) + f'_2(dy-dz) + f'_3(dz-dx) = 0,$$

化简得

$$(f'_3 - f'_2)dz = (f'_3 - f'_1)dx + (f'_1 - f'_2)dy,$$

所以

$$dz = \frac{(f'_3 - f'_1)dx + (f'_1 - f'_2)dy}{f'_3 - f'_2}.$$

8. 求由下列方程组所确定的隐函数的导数或偏导数:

$$(1) \begin{cases} x + y + z = 2, \\ x^2 + y^2 = \frac{1}{2}z^2, \end{cases} \text{ 求 } \frac{dx}{dz}, \frac{dy}{dz};$$

$$(2) \text{ 设 } u(x, y) = e^{3x-y}, \quad x^2 + y = t^2, \quad x - y = t + 2, \quad \text{求 } \frac{du}{dt} \Big|_{t=0};$$

$$(3) \begin{cases} xu - yv = 0, \\ yu + xv = 1, \end{cases} \text{ 求 } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y};$$

$$(4) \begin{cases} xy^2 - uv = 1, \\ x^2 + y^2 - u + v = 0, \end{cases} \text{ 求 } \frac{\partial u}{\partial x} \Big|_{\substack{x=1 \\ y=1}}, \frac{\partial v}{\partial x} \Big|_{\substack{x=1 \\ y=1}}.$$

**解** (1) 此方程组可确定两个一元隐函数  $x = x(z)$  和  $y = y(z)$ , 方程两边对  $z$  求导并移项, 得

$$\begin{cases} \frac{dx}{dz} + \frac{dy}{dz} = -1, \\ 2x \frac{dx}{dz} + 2y \frac{dy}{dz} = z, \end{cases}$$

在  $D = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = 2y - 2x \neq 0$  的条件下, 解方程组求得

$$\frac{dx}{dz} = \frac{\begin{vmatrix} -1 & 1 \\ z & 2y \end{vmatrix}}{D} = \frac{-2y - z}{2y - 2x} = \frac{z + 2y}{2(x - y)},$$

$$\frac{dy}{dz} = \frac{\begin{vmatrix} 1 & -1 \\ 2x & z \end{vmatrix}}{D} = \frac{z + 2x}{2y - 2x} = \frac{z + 2x}{2(y - x)}.$$

$$(2) \text{ 方程组 } \begin{cases} x^2 + y = t^2, \\ x - y = t + 2, \end{cases} \text{ 可确定两个一元隐函数 } x = x(t) \text{ 和 } y = y(t), \text{ 方程两边}$$

对  $t$  求导, 得



$$\begin{cases} 2x \frac{dx}{dt} + \frac{dy}{dt} = 2t, \\ \frac{dx}{dt} - \frac{dy}{dt} = 1, \end{cases}$$

在  $D = \begin{vmatrix} 2x & 1 \\ 1 & -1 \end{vmatrix} = -2x - 1 \neq 0$  的条件下, 解方程组求得

$$\begin{aligned} \frac{dx}{dt} &= \frac{\begin{vmatrix} 2t & 1 \\ 1 & 1 \end{vmatrix}}{D} = \frac{2t-1}{-2x-1} = \frac{1-2t}{1+2x}, \\ \frac{dy}{dt} &= \frac{\begin{vmatrix} 2x & 2t \\ 1 & 1 \end{vmatrix}}{D} = \frac{2x-2t}{-2x-1} = \frac{2t-2x}{1+2x}, \end{aligned}$$

由  $u(x, y) = e^{3x-y}$  可得

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = e^{3x-y} \cdot 3 \cdot \frac{1-2t}{1+2x} + e^{3x-y} \cdot (-1) \cdot \frac{2t-2x}{1+2x} \\ &= \frac{3-8t+2x}{1+2x} e^{3x-y}. \end{aligned}$$

当  $t=0$  时, 有  $\begin{cases} x^2 + y = 0, \\ x - y = 2, \end{cases}$  可解得

$$\begin{cases} x=1, \\ y=-1, \end{cases} \text{ 或 } \begin{cases} x=-2, \\ y=-4. \end{cases}$$

所以

$$\left. \frac{du}{dt} \right|_{\substack{t=0 \\ x=1 \\ y=-1}} = \frac{5}{3} e^4 \quad \text{或} \quad \left. \frac{du}{dt} \right|_{\substack{t=0 \\ x=-2 \\ y=-4}} = \frac{1}{3} e^{-2}.$$

(3) 法 1 将所给方程的两边对  $x$  求导, 得

$$\begin{cases} u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x} = 0, \\ y \frac{\partial u}{\partial x} + v + x \frac{\partial v}{\partial x} = 0, \end{cases}$$

移项, 得

$$\begin{cases} x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x} = -u, \\ y \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} = -v. \end{cases}$$

在  $J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$  的条件下, 解方程组求得

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{J} = \frac{-xu - yv}{x^2 + y^2} = -\frac{xu + yv}{x^2 + y^2},$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{J} = \frac{-xv + yu}{x^2 + y^2} = \frac{yu - xv}{x^2 + y^2}.$$

将所给方程的两边对  $y$  求导, 用同样的方法在  $J = x^2 + y^2 \neq 0$  的条件下可得

$$\frac{\partial u}{\partial y} = \frac{xv - yu}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = -\frac{xu + yv}{x^2 + y^2}.$$

法 2 对方程组求全微分可得

$$\begin{cases} udx + xdu - vdy - ydv = 0, \\ udy + ydu + vdx + xdv = 0, \end{cases}$$

移项, 得

$$\begin{cases} xdu - ydv = -udx + vdy, \\ ydu + xdv = -vdx - udy. \end{cases}$$

在  $D = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$  的条件下, 解方程组求得

$$du = \frac{\begin{vmatrix} -udx + vdy & -y \\ -vdx - udy & x \end{vmatrix}}{x^2 + y^2} = \frac{-(xu + yv)dx + (xv - yu)dy}{x^2 + y^2},$$

$$dv = \frac{\begin{vmatrix} x & -udx + vdy \\ y & -vdx - udy \end{vmatrix}}{x^2 + y^2} = \frac{(yu - xv)dx - (xu + yv)dy}{x^2 + y^2},$$

所以

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\frac{xu + yv}{x^2 + y^2}, & \frac{\partial u}{\partial y} &= \frac{xv - yu}{x^2 + y^2}, \\ \frac{\partial v}{\partial x} &= \frac{yu - xv}{x^2 + y^2}, & \frac{\partial v}{\partial y} &= -\frac{xu + yv}{x^2 + y^2}.\end{aligned}$$

(4) 将所给方程的两边对  $x$  求导, 得

$$\begin{cases} y^2 - v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} = 0, \\ 2x - \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0, \end{cases}$$

移项, 得

$$\begin{cases} v \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x} = y^2, \\ \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 2x. \end{cases}$$

在  $J = \begin{vmatrix} v & u \\ 1 & -1 \end{vmatrix} = -v - u = -(u + v) \neq 0$  的条件下, 解方程组求得

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} y^2 & u \\ 2x & -1 \end{vmatrix}}{J} = \frac{-y^2 - 2xu}{-(u + v)} = \frac{y^2 + 2xu}{u + v},$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} v & y^2 \\ 1 & 2x \end{vmatrix}}{J} = \frac{2xv - y^2}{-(u + v)} = \frac{y^2 - 2xv}{u + v}.$$

当  $x=1, y=1$  时,  $\begin{cases} uv=0, \\ u-v=2, \end{cases}$  解得

$$\begin{cases} u=0, \\ v=-2, \end{cases} \text{ 或 } \begin{cases} u=2, \\ v=0, \end{cases}$$

所以

$$\left. \frac{\partial u}{\partial x} \right|_{\substack{x=1 \\ y=1}} = -\frac{1}{2}, \quad \left. \frac{\partial v}{\partial x} \right|_{\substack{x=1 \\ y=1}} = -\frac{5}{2}$$

或

$$\left. \frac{\partial u}{\partial x} \right|_{\substack{x=1 \\ y=1}} = \frac{5}{2}, \quad \left. \frac{\partial v}{\partial x} \right|_{\substack{x=1 \\ y=1}} = \frac{1}{2}.$$

9. 设  $u+v=x+y$ ,  $\frac{\sin u}{\sin v}=\frac{x}{y}$ , 求  $du, dv$ .

解 对方程组  $\begin{cases} u+v=x+y, \\ \frac{\sin u}{\sin v}=\frac{x}{y}, \end{cases}$  求全微分可得

$$\begin{cases} du+dv=dx+dy, \\ \frac{\sin v \cdot \cos u \cdot du - \sin u \cdot \cos v \cdot dv}{\sin^2 v} = \frac{ydx - xdy}{y^2}, \end{cases}$$

移项整理, 得

$$\begin{cases} du+dv=dx+dy, \\ \cos u \sin v du - \sin u \cos v dv = \frac{\sin^2 v}{y^2}(ydx - xdy). \end{cases}$$

在  $D = \begin{vmatrix} 1 & 1 \\ \cos u \sin v & -\sin u \cos v \end{vmatrix} = -\sin u \cos v - \cos u \sin v \neq 0$  的条件下, 解方程组

求得

$$\begin{aligned} du &= \frac{\begin{vmatrix} dx+dy & 1 \\ \frac{\sin^2 v}{y^2}(ydx - xdy) & -\sin u \cos v \end{vmatrix}}{D} \\ &= \frac{-\sin u \cos v dx - \sin u \cos v dy - \frac{\sin^2 v}{y} dx + \frac{x \sin^2 v}{y^2} dy}{-\sin u \cos v - \cos u \sin v} \\ &= \frac{(y^2 \sin u \cos v + y \sin^2 v)dx + (y^2 \sin u \cos v - x \sin^2 v)dy}{y^2 \sin u \cos v + y^2 \cos u \sin v}, \end{aligned}$$

由题设  $\frac{\sin u}{\sin v}=\frac{x}{y}$ , 得  $y \sin u = x \sin v$  代入上式, 化简得

$$du = \frac{(\sin v + x \cos v)dx - (\sin u - x \cos v)dy}{x \cos v + y \cos u}.$$

同理可解得

$$dv = \frac{(y \cos u - \sin v)dx + (\sin u + y \cos u)dy}{x \cos v + y \cos u}.$$

10. 设  $x=u+v$ ,  $y=u^2+v^2$ ,  $z=u^3+v^3$  确定了  $z$  是  $x, y$  的函数, 求

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}.$$

解 对方程组  $\begin{cases} u+v=x, \\ u^2+v^2=y, \end{cases}$  求全微分可得

$$\begin{cases} du+dv=dx, \\ 2udu+2v dv=dy, \end{cases}$$

在  $D = \begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix} = 2v - 2u \neq 0$  的条件下, 解方程组求得

$$\begin{aligned} du &= \frac{\begin{vmatrix} dx & 1 \\ dy & 2v \end{vmatrix}}{D} = \frac{2vdx - dy}{2v - 2u}, \\ dv &= \frac{\begin{vmatrix} 1 & dx \\ 2u & dy \end{vmatrix}}{D} = \frac{-2udx + dy}{2v - 2u}. \end{aligned}$$

对方程  $z = u^3 + v^3$  两边求全微分可得

$$\begin{aligned} dz &= 3u^2 du + 3v^2 dv = 3u^2 \cdot \frac{2vdx - dy}{2v - 2u} + 3v^2 \cdot \frac{-2udx + dy}{2v - 2u} \\ &= -3uv dx + \frac{3}{2}(u+v)dy, \end{aligned}$$

所以

$$\frac{\partial z}{\partial x} = -3uv, \quad \frac{\partial z}{\partial y} = \frac{3}{2}(u+v).$$

11. 设  $u = xyz^2$ , 而  $z = z(x, y)$  是由方程  $x^2 + y^2 + z^2 = 3xyz$  所确定的隐函数,

求  $\frac{\partial u}{\partial x} \Big|_{(1,1,1)}$ .

解 先由方程  $x^2 + y^2 + z^2 = 3xyz$  求  $\frac{\partial z}{\partial x}$ .

令  $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$ , 则

$$F_x = 2x - 3yz, \quad F_z = 2z - 3xy,$$

于是

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy}.$$

由题设  $u = xy^2z^3$  可得

$$\begin{aligned}\frac{\partial u}{\partial x} &= 1 \cdot y^2 z^3 + xy^2 \cdot 3z^2 \cdot \frac{\partial z}{\partial x} = y^2 z^3 + 3xy^2 z^2 \cdot \left(-\frac{2x-3yz}{2z-3xy}\right), \\ \frac{\partial u}{\partial x} \Big|_{(1,1,1)} &= 1 + 3 \cdot \left(-\frac{2-3}{2-3}\right) = -2.\end{aligned}$$