第六节　傅里叶级数

习题　11-6

1. 将下列以 2π 为周期的函数(已给出函数在一个周期内的表达式)展开成傅里叶级数:

(1)　$f(x)=2x+1\ (-\pi<x\le\pi)$;　　（2）　$f(x)=\mathrm{e}^x+1\ (-\pi<x\le\pi)$;

(3)　$f(x)=\begin{cases} bx, & -\pi<x\le 0, \\ ax, & 0<x\le\pi, \end{cases}$　(常数$a,b:a>b>0$).

解　(1)　$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}(2x+1)\mathrm{d}x=\dfrac{1}{\pi}\int_{-\pi}^{\pi}2x\mathrm{d}x+\dfrac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{d}x=2$,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}2x\cdot\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\mathrm{d}x$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\mathrm{d}x=0,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(2x+1)\cdot\sin nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}2x\cdot\sin nx\mathrm{d}x+\frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\mathrm{d}x$$

$$=\frac{4}{\pi}\int_{0}^{\pi}x\cdot\sin nx\mathrm{d}x=\frac{4}{\pi}\{-[\frac{x}{n}\cos nx]_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\mathrm{d}x\}=\frac{4}{n}(-1)^{n+1}.$$

$f(x)$ 满足收敛定理条件，在 $(-\pi,\pi)$ 内连续，而在 $x=\pm\pi$ 处不连续，故级数在 $x\in(-\infty,+\infty)$ 且 $x\ne(2k+1)\pi\ (k=0,\pm1,\pm2,\cdots)$ 时收敛于 $f(x)$，即

$$f(x)=1+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx,\quad x\in(-\infty,+\infty)\text{ 且 }x\ne(2k+1)\pi(k=0,\pm1,\pm2,\cdots).$$

(2)　$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}(\mathrm{e}^x+1)\mathrm{d}x=\dfrac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{e}^x\mathrm{d}x+\dfrac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{d}x=2+\dfrac{1}{\pi}[\mathrm{e}^{\pi}-\mathrm{e}^{-\pi}]$,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\mathrm{e}^x+1)\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{e}^x\cdot\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\mathrm{d}x$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{e}^x\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\frac{(-1)^n(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})}{1+n^2},$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(\mathrm{e}^x+1)\cdot\sin nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{e}^x\cdot\sin nx\mathrm{d}x+\frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\mathrm{d}x$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{e}^x\cdot\sin nx\mathrm{d}x=-\frac{n}{\pi}\frac{(-1)^n(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})}{1+n^2}.$$

$f(x)$ 满足收敛定理条件，在 $(-\pi,\pi)$ 内连续，而在 $x=\pm\pi$ 处不连续，故级数在

$x\in(-\infty,+\infty)$，且 $x\neq(2k+1)\pi\ \ (k=0,\pm1,\pm2,\cdots)$ 时收敛于 $f(x)$，即

$$f(x)=1+\frac{1}{2\pi}(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})\ +(\mathrm{e}^{\pi}-\mathrm{e}^{-\pi})\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx-n\sin nx),$$

$x\in(-\infty,+\infty)$，且 $x\neq(2k+1)\pi\ \ (k=0,\pm1,\pm2,\cdots)$.

(3) $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{0}bx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}ax\mathrm{d}x=\frac{\pi}{2}(a-b)$,

$$a_n=\frac{b}{\pi}\int_{-\pi}^{0}x\cdot\cos nx\mathrm{d}x+\frac{a}{\pi}\int_{0}^{\pi}x\cdot\cos nx\mathrm{d}x$$

$$=\frac{b}{\pi}[\frac{x}{n}\sin nx+\frac{1}{n^2}\cdot\cos nx]_{-\pi}^{0}+\frac{a}{\pi}[\frac{x}{n}\sin nx+\frac{1}{n^2}\cdot\cos nx]_{0}^{\pi}$$

$$=\frac{b-a}{\pi n^2}(1-\cos n\pi)=\frac{b-a}{\pi n^2}[1-(-1)^n],$$

$$b_n=\frac{b}{\pi}\int_{-\pi}^{0}x\cdot\sin nx\mathrm{d}x+\frac{a}{\pi}\int_{0}^{\pi}x\cdot\sin nx\mathrm{d}x$$

$$=\frac{b}{\pi}[-\frac{x}{n}\cos nx+\frac{1}{n^2}\cdot\sin nx]_{-\pi}^{0}+\frac{a}{\pi}[-\frac{x}{n}\cos nx+\frac{1}{n^2}\cdot\sin nx]_{0}^{\pi}=\frac{(-1)^{n-1}(a+b)}{n}.$$

$f(x)$ 满足收敛定理条件，在 $(-\pi,\pi)$ 内连续，而在 $x=\pm\pi$ 处不连续，故级数在 $x\in(-\infty,+\infty)$，且 $x\neq(2k+1)\pi\ \ (k=0,\pm1,\pm2,\cdots)$ 时收敛于 $f(x)$，即

$$f(x)=\frac{\pi}{4}(a-b)+\sum_{n=1}^{\infty}\{\frac{[1-(-1)^n](b-a)}{n^2\pi}\cos nx+\frac{(-1)^{n-1}(a+b)}{n}\sin nx\},$$

$x\in(-\infty,+\infty)$，且 $x\neq(2k+1)\pi\ \ (k=0,\pm1,\pm2,\cdots)$.

2. 设下列函数 $f(x)$ 是周期为 2π 的周期函数，它们在 $(-\pi,\pi]$ 上的表达式分别为:

(1) $f(x)=\begin{cases}1+\dfrac{2x}{\pi}, & -\pi<x\leq 0,\\ 1-\dfrac{2x}{\pi}, & 0<x\leq\pi.\end{cases}$

(2) $f(x)=\begin{cases}-\dfrac{\pi}{2}, & -\pi<x\leq-\dfrac{\pi}{2},\\ x, & -\dfrac{\pi}{2}<x\leq\dfrac{\pi}{2},\\ \dfrac{\pi}{2}, & \dfrac{\pi}{2}<x\leq\pi.\end{cases}$

试将 $f(x)$ 展开成傅里叶级数.

解 (1) $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}(1+\frac{2x}{\pi})\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}(1-\frac{2x}{\pi})\mathrm{d}x=0$,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}(1+\frac{2x}{\pi})\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}(1-\frac{2x}{\pi})\cos nx\mathrm{d}x$$

$$=\frac{4}{\pi n^2}(1-(-1)^n)=\begin{cases}0, & n=2,4,6,\cdots,\\ \dfrac{8}{\pi n^2}, & n=1,3,5,\cdots.\end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\sin nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}(1+\frac{2x}{\pi})\sin nx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}(1-\frac{2x}{\pi})\sin nx\mathrm{d}x=0\,.$$

$f(x)$ 满足收敛定理条件，在 $(-\infty,+\infty)$ 内连续，故级数在 $x\in(-\infty,+\infty)$ 内收敛于 $f(x)$，从而

$$f(x)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2},\quad x\in(-\infty,+\infty).$$

(2) $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}}(-\frac{\pi}{2})\mathrm{d}x+\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\mathrm{d}x+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\mathrm{d}x=0$,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\mathrm{d}x$$

$$=\frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}}(-\frac{\pi}{2}\cos nx)\mathrm{d}x+\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\cos nx\mathrm{d}x=0\,,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\sin nx\mathrm{d}x$$

$$=\frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}}(-\frac{\pi}{2}\sin nx)\mathrm{d}x+\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\sin nx\mathrm{d}x+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{\pi}{2}\sin nx\mathrm{d}x$$

$$=\int_{\frac{\pi}{2}}^{\pi}\sin nx\mathrm{d}x+\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}x\sin nx\mathrm{d}x=\frac{(-1)^{n+1}}{n}+\frac{2}{n^2\pi}\sin\frac{n\pi}{2}\,.$$

$f(x)$ 满足收敛定理条件，在 $(-\pi,\pi)$ 内连续，而在 $x=\pm\pi$ 处不连续，故级数在 $x\in(-\infty,+\infty)$，且 $x\neq(2k+1)\pi\ \ (k=0,\pm1,\pm2,\cdots)$ 时收敛于 $f(x)$，从而

$$f(x)=\sum_{n=1}^{\infty}\{\frac{(-1)^{n+1}}{n}+\frac{2}{n^2\pi}\sin\frac{n\pi}{2}\}\sin nx,\quad x\in(-\infty,+\infty)\text{ 且 }x\neq(2k+1)\pi\ \ (k=0,\pm1,\pm2,\cdots)\,.$$

3. 设函数 $f(x)$ 以 2π 为周期，证明 $f(x)$ 的傅里叶系数为:

$$a_n=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\cos nx\mathrm{d}x\quad(n=0,1,2,\cdots)\,,$$

$$b_n=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\sin nx\mathrm{d}x\quad(n=1,2,\cdots)\,.$$

证 $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cdot\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cdot\cos nx\mathrm{d}x$.

因为令 $x+2\pi=t$ 有

$$\int_{-\pi}^{0}f(x)\cdot\cos nx\mathrm{d}x=\int_{\pi}^{2\pi}f(t-2\pi)\cdot\cos n(t-2\pi)\mathrm{d}t=\int_{\pi}^{2\pi}f(t)\cdot\cos nt\mathrm{d}t,$$

所以 $$a_n=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\cos nx\mathrm{d}x\quad(n=0,1,2,\cdots).$$

同理 $$b_n=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\sin nx\mathrm{d}x\quad(n=1,2,\cdots).$$

4. 设函数 $f(x)$ 以 2π 为周期, 证明

(1) 如果 $f(x-\pi)=-f(x)$, 则 $f(x)$ 的傅里叶系数 $a_0=0, a_{2k}=0$, $b_{2k}=0\ (k=1,2,\cdots)$;

(2) 如果 $f(x-\pi)=f(x)$, 则 $f(x)$ 的傅里叶系数 $a_{2k+1}=0$, $b_{2k+1}=0\ (k=0,1,2,\cdots)$.

证 (1) $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cdot\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cdot\cos nx\mathrm{d}x$,
因为令 $x-\pi=y$ 有

$$\int_{0}^{\pi}f(x)\cdot\cos nx\mathrm{d}x=\int_{0}^{\pi}-f(x-\pi)\cdot\cos nx\mathrm{d}x=-\int_{-\pi}^{0}f(y)\cdot\cos n(y+\pi)\mathrm{d}y$$

$$=-\int_{-\pi}^{0}f(y)\cdot(-1)^n\cdot\cos ny\mathrm{d}y=(-1)^{n+1}\int_{-\pi}^{0}f(x)\cdot\cos nx\mathrm{d}x,$$

从而 $a_n=\frac{1}{\pi}[1+(-1)^{n+1}]\int_{-\pi}^{0}f(x)\cdot\cos nx\mathrm{d}x$, 于是得 $a_0=0, a_{2k}=0,\ (k=1,2,\cdots)$.

又 $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\sin nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cdot\sin nx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cdot\sin nx\mathrm{d}x$, 因为令 $x-\pi=y$ 有

$$\int_{0}^{\pi}f(x)\cdot\sin nx\mathrm{d}x=\int_{0}^{\pi}-f(x-\pi)\cdot\sin nx\mathrm{d}x$$

$$=-\int_{-\pi}^{0}f(y)\cdot\sin n(y+\pi)\mathrm{d}y=-\int_{-\pi}^{0}f(y)\cdot(-1)^n\cdot\sin ny\mathrm{d}y=(-1)^{n+1}\int_{-\pi}^{0}f(x)\cdot\sin nx\mathrm{d}x,$$

所以得 $b_{2k}=0,\ (k=1,2,\cdots)$.

(2) $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\mathrm{d}x=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cdot\cos nx\mathrm{d}x+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cdot\cos nx\mathrm{d}x$, 因为令 $x-\pi=y$ 有

$$\int_0^{\pi} f(x)\cdot\cos nx\mathrm{d}x=\int_0^{\pi} f(x-\pi)\cdot\cos nx\mathrm{d}x=\int_{-\pi}^{0} f(y)\cdot\cos n(y+\pi)\mathrm{d}y$$

$$=\int_{-\pi}^{0} f(y)\cdot(-1)^n\cdot\cos ny\mathrm{d}y=(-1)^n\int_{-\pi}^{0} f(x)\cdot\cos nx\mathrm{d}x,$$

从而 $a_n=\frac{1}{\pi}[1+(-1)^n]\int_{-\pi}^{0} f(x)\cdot\cos nx\mathrm{d}x$，于是得 $a_{2k+1}=0,\ (k=0,1,2,\cdots)$.

又 $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nxdx=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\cdot\sin nx\mathrm{d}x+\frac{1}{\pi}\int_0^{\pi} f(x)\cdot\sin nx\mathrm{d}x$，因为令 $x-\pi=y$ 有

$$\int_0^{\pi} f(x)\cdot\sin nx\mathrm{d}x=\int_0^{\pi} f(x-\pi)\cdot\sin nx\mathrm{d}x=\int_{-\pi}^{0} f(y)\cdot\sin n(y+\pi)\mathrm{d}y$$

$$=\int_{-\pi}^{0} f(y)\cdot(-1)^n\cdot\sin ny\mathrm{d}y=(-1)^n\int_{-\pi}^{0} f(x)\cdot\sin nx\mathrm{d}x,$$

所以得 $b_{2k+1}=0,\ (k=0,1,2,\cdots)$.

5. 证明下列等式(m,n 均为自然数):

(1) $\int_{-\pi}^{\pi}\cos nx\cos mx\mathrm{d}x=0\quad(m\neq n)$；　(2) $\int_{-\pi}^{\pi}\cos nx\sin mx\mathrm{d}x=0$；

(3) $\int_{-\pi}^{\pi}\sin^2 nx\mathrm{d}x=\pi$.

证 (1) $\int_{-\pi}^{\pi}\cos nx\cos mx\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\cos(mx+nx)+\cos(mx-nx)]\mathrm{d}x$

$$=[\frac{\sin(m+n)x}{m+n}]_0^{\pi}+[\frac{\sin(m-n)x}{m-n}]_0^{\pi}=0.$$

(2) $\int_{-\pi}^{\pi}\cos nx\sin mx\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}[\sin(mx+nx)+\sin(mx-nx)]\mathrm{d}x=0.$

(3) $\int_{-\pi}^{\pi}\sin^2 nx\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2nx)\mathrm{d}x=\frac{1}{2}\int_{-\pi}^{\pi}\mathrm{d}x-\frac{1}{2}\int_{-\pi}^{\pi}\cos 2nx\mathrm{d}x$

$$=[\frac{1}{2}x]_{-\pi}^{\pi}-[\frac{\sin 2nx}{4n}]_{-\pi}^{\pi}=\pi.$$