第五节　隐函数的微分法

习题　8-5

1.　求下列方程所确定的隐函数 $y=y(x)$ 的一阶导数:

(1)　$xy-\ln y=a$;　　　　(2)　$\ln\sqrt{x^2+y^2}=\arctan\dfrac{y}{x}$.

解　(1)　**法 1**　用隐函数求导公式求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

令 $F(x,y)=xy-\ln y-a$, 则 $F_x=y,\ F_y=x-\dfrac{1}{y}$, 所以

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=-\frac{y}{x-\dfrac{1}{y}}=\frac{y^2}{1-xy}.$$

法 2　用隐函数求导法则求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

等式两边对 x 求导, 注意 y 是 x 的函数, 有

$$y+x\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

所以

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-y}{x-\dfrac{1}{y}}=\frac{y^2}{1-xy}.$$

(2)　**法 1**　用隐函数求导公式求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

令 $F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\dfrac{y}{x}$, 则

$$F_x=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x}{2\sqrt{x^2+y^2}}-\frac{1}{1+(\dfrac{y}{x})^2}\cdot(-\frac{y}{x^2})=\frac{x+y}{x^2+y^2},$$

$$F_y=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2y}{2\sqrt{x^2+y^2}}-\frac{1}{1+(\dfrac{y}{x})^2}\cdot\frac{1}{x}=\frac{y-x}{x^2+y^2},$$

所以

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=-\frac{\dfrac{x+y}{x^2+y^2}}{\dfrac{y-x}{x^2+y^2}}=\frac{x+y}{x-y}.$$

法 2　用隐函数求导法则求 $\frac{dy}{dx}$.

等式两边对 x 求导，注意 y 是 x 的函数，有

$$\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{x+y\frac{dy}{dx}}{\sqrt{x^2+y^2}}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{x\frac{dy}{dx}-y}{x^2},$$

化简得

$$x+y\frac{dy}{dx}=x\frac{dy}{dx}-y,$$

所以

$$\frac{dy}{dx}=\frac{x+y}{x-y}.$$

2.　求下列方程所确定的隐函数 $z=z(x,y)$ 的一阶偏导数:

(1)　$e^{-(x+y+z)}=x+y+z$;　　　　(2)　$\frac{x}{z}=\ln\frac{z}{y}$.

解　(1)　**法 1**　用隐函数求导公式求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

令 $F(x,y,z)=e^{-(x+y+z)}-x-y-z$，则

$$F_x=e^{-(x+y+z)}\cdot(-1)-1=-e^{-(x+y+z)}-1,$$

由对称性可得

$$F_y=F_z=-e^{-(x+y+z)}-1,$$

所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-1,\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-1.$$

法 2　直接用复合函数求偏导数的方法求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

等式两边对 x 求偏导，注意 z 是 x 和 y 的二元函数，则有

$$e^{-(x+y+z)}\cdot(1+\frac{\partial z}{\partial x})=1+\frac{\partial z}{\partial x},$$

所以

$$\frac{\partial z}{\partial x}=\frac{1-e^{-(x+y+z)}}{e^{-(x+y+z)}-1}=-1,$$

同理，由对称性可得

$$\frac{\partial z}{\partial y}=-1 .$$

法 3 用全微分形式不变性求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

将 x, y, z 均看作自变量，方程两边同时取全微分，得

$$\mathrm{d}e^{-(x+y+z)}=\mathrm{d}(x+y+z) ,$$

即

$$e^{-(x+y+z)}\mathrm{d}[-(x+y+z)]=\mathrm{d}(x+y+z) ,$$

$$-e^{-(x+y+z)}(\mathrm{d}x+\mathrm{d}y+\mathrm{d}z)=\mathrm{d}x+\mathrm{d}y+\mathrm{d}z ,$$

这时，再将 z 看作 x, y, z 的函数，解出 z 的全微分 $\mathrm{d}z$，有

$$\mathrm{d}z=\frac{1+e^{-(x+y+z+)}}{-e^{-(x+y+z)}-1}\mathrm{d}x+\frac{1+e^{-(x+y+z)}}{-e^{-(x+y+z)}-1}\mathrm{d}y=-\mathrm{d}x-\mathrm{d}y ,$$

所以

$$\frac{\partial z}{\partial x}=-1,\ \frac{\partial z}{\partial y}=-1 .$$

(2) **法 1** 公式法.

令 $F(x,y,z)=\frac{x}{z}-\ln\frac{z}{y}$，则

$$F_x=\frac{1}{z}, \qquad F_y=-\frac{y}{z}\cdot(-\frac{z}{y^2})=\frac{1}{y} ,$$

$$F_z=-\frac{x}{z^2}-\frac{y}{z}\cdot\frac{1}{y}=-\frac{x+z}{z^2} ,$$

所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{1}{z}\frac{z^2}{x+z}=\frac{z}{x+z} ,$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{1}{y}\frac{-z^2}{x+z}=\frac{z^2}{y(x+z)} .$$

法 2 复合函数求导法.

等式两边对 x 求偏导，得

$$\frac{z-x\frac{\partial z}{\partial x}}{z^2}=\frac{y}{z}\cdot\frac{1}{y}\frac{\partial z}{\partial x} ,\ \text{即}\ z-x\frac{\partial z}{\partial x}=z\frac{\partial z}{\partial x} ,$$

所以

$$\frac{\partial z}{\partial x}=\frac{z}{x+z}.$$

等式两边对 y 求偏导，则有

$$-\frac{x}{z^2}\frac{\partial z}{\partial y}=\frac{y}{z}\frac{y\frac{\partial z}{\partial y}-z}{y^2}，即 -x\frac{\partial z}{\partial y}=z\frac{\partial z}{\partial y}-\frac{z^2}{y},$$

所以

$$\frac{\partial z}{\partial y}=\frac{z^2}{y(x+z)}.$$

法 3　全微分法.

方程两边同时取全微分，得

$$\mathrm{d}\frac{x}{z}=\mathrm{d}\ln\frac{z}{y},$$

即

$$\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2}=\frac{y}{z}\cdot\frac{y\mathrm{d}z-z\mathrm{d}y}{y^2},$$

$$\frac{1}{z}\mathrm{d}x-\frac{x}{z^2}\mathrm{d}z=\frac{1}{z}\mathrm{d}z-\frac{1}{y}\mathrm{d}y,$$

解出

$$\mathrm{d}z=\frac{\frac{1}{z}}{\frac{x}{z^2}+\frac{1}{z}}\mathrm{d}x+\frac{\frac{1}{y}}{\frac{x}{z^2}+\frac{1}{z}}\mathrm{d}y=\frac{z}{x+z}\mathrm{d}x+\frac{z^2}{y(x+z)}\mathrm{d}y,$$

所以

$$\frac{\partial z}{\partial x}=\frac{z}{x+z},\quad \frac{\partial z}{\partial y}=\frac{z^2}{y(x+z)}.$$

注意　对于隐函数求导，一般有三种方法：公式法，复合函数求导法，全微分法，在不同的情况下，各有方便之处，但必须注意：在等式两边关于 x 或 y 求偏导数时，应当将 z 看作是 x, y 的函数，而对 $F(x,y,z)$ 关于 x 和 y 求偏导数时，将 z 看作是常数.

3.　设 f 可微，且方程 $y+z=xf(y^2-z^2)$ 确定了 $z=z(x,y)$，计算 $x\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}$.

解　令 $F(x,y,z)=y+z-xf(y^2-z^2)$，则

$$F_x=-f(y^2-z^2),$$

$$F_y=1-xf'(y^2-z^2)\cdot 2y=1-2xyf'(y^2-z^2),$$

$$F_z = 1 - xf'(y^2 - z^2) \cdot (-2z) = 1 + 2xzf'(y^2 - z^2)\,,$$

于是

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-f(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} = \frac{f(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)}\,,$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{1 - 2xyf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)}\,,$$

所以

$$x\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = \frac{xf(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} - \frac{z - 2xyzf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)}$$

$$= \frac{[xf(y^2 - z^2) - z] + 2xyzf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)}$$

$$= \frac{y + 2xyzf'(y^2 - z^2)}{1 + 2xzf'(y^2 - z^2)} = y\,.$$

4. 设方程 $f(ax - cz, ay - bz) = 0$ 确定 $z = z(x, y)$，证明:

$$c\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = a\,.$$

证 因为 $f_x = f_1' \cdot a = af_1'$, $f_y = f_2' \cdot a = af_2'$,

$$f_z = f_1' \cdot (-c) + f_2' \cdot (-b) = -cf_1' - bf_2'\,,$$

于是

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{af_1'}{-cf_1' - bf_2'} = \frac{af_1'}{cf_1' + bf_2'}\,,$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{af_2'}{-cf_1' - bf_2'} = \frac{af_2'}{cf_1' + bf_2'}\,,$$

所以

$$c\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \frac{acf_1'}{cf_1' + bf_2'} + \frac{abf_2'}{cf_1' + bf_2'} = a\,.$$

5. 设 $x = x(y, z)$，$y = y(z, x)$，$z = z(x, y)$ 都是由方程 $F(x, y, z) = 0$ 所确定的函数，且都具有连续偏导数，证明:

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1\,.$$

证　由隐函数求导公式，得

$$\frac{\partial x}{\partial y}=-\frac{F_y}{F_x}\,,\quad \frac{\partial y}{\partial z}=-\frac{F_z}{F_y}\,,\quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}\,,$$

所以

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=(-\frac{F_y}{F_x})\cdot(-\frac{F_z}{F_y})\cdot(-\frac{F_x}{F_z})=-1\,.$$

6. 求下列方程所确定的隐函数的指定偏导数:

(1)　$\mathrm{e}^z-xyz=0$，$\dfrac{\partial^2 z}{\partial x^2}$；　　　　(2)　$xy+yz+zx=1$，$\dfrac{\partial^2 z}{\partial x\partial y}$；

(3)　$z+\ln z-\int_y^x \mathrm{e}^{-t^2}\mathrm{d}t=0$，$\dfrac{\partial^2 z}{\partial x\partial y}$.

解　(1)　令 $F(x,y,z)=\mathrm{e}^z-xyz$，则

$$F_x=-yz\,,\quad F_z=\mathrm{e}^z-xy\,,$$

所以

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{yz}{\mathrm{e}^z-xy}\,,$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})=\frac{\partial}{\partial x}(\frac{yz}{\mathrm{e}^z-xy})=\frac{y\frac{\partial z}{\partial x}(\mathrm{e}^z-xy)-yz(\mathrm{e}^z\frac{\partial z}{\partial x}-y)}{(\mathrm{e}^z-xy)^2}\,,$$

把 $\dfrac{\partial z}{\partial x}=\dfrac{yz}{\mathrm{e}^z-xy}$ 代入上式，并化简，有

$$\frac{\partial^2 z}{\partial x^2}=\frac{2y^2z\mathrm{e}^z-y^2z^2\mathrm{e}^z-2xy^3z}{(\mathrm{e}^z-xy)^3}\,,$$

又因为 $\mathrm{e}^z=xyz$，所以

$$\frac{\partial^2 z}{\partial x^2}=\frac{2xy^3z^2-xy^3z^3-2xy^3z}{(xyz-xy)^3}=\frac{z(z^2-2z+2)}{x^2(1-z)^2}\,.$$

(2)　令 $F(x,y,z)=xy+yz+zx-1$，则

$$F_x=y+z\,,\quad F_y=x+z\,,\quad F_z=x+y\,,$$

于是

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{y+z}{x+y}, \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{x+z}{x+y},$$

所以

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})=\frac{\partial}{\partial y}(-\frac{y+z}{x+y})=-\frac{(1+\frac{\partial z}{\partial y})(x+y)-(y+z)\cdot 1}{(x+y)^2}$$

$$=-\frac{(1-\frac{x+z}{x+y})(x+y)-(y+z)}{(x+y)^2}=\frac{2z}{(x+y)^2}.$$

(3) 原等式两边对 x 求偏导，得

$$\frac{\partial z}{\partial x}+\frac{1}{z}\cdot\frac{\partial z}{\partial x}-\mathrm{e}^{-x^2}=0,$$

于是

$$\frac{\partial z}{\partial x}=\frac{\mathrm{e}^{-x^2}}{1+\frac{1}{z}}=\frac{z\mathrm{e}^{-x^2}}{z+1}.$$

原等式两边对 y 求偏导，得

$$\frac{\partial z}{\partial y}+\frac{1}{z}\cdot\frac{\partial z}{\partial y}+\mathrm{e}^{-y^2}=0,$$

于是

$$\frac{\partial z}{\partial y}=\frac{-\mathrm{e}^{-y^2}}{1+\frac{1}{z}}=-\frac{z\mathrm{e}^{-y^2}}{z+1}.$$

所以

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(\frac{\partial z}{\partial x})=\frac{\partial}{\partial y}(\frac{z\mathrm{e}^{-x^2}}{z+1})=\frac{\mathrm{e}^{-x^2}\frac{\partial z}{\partial y}(z+1)-z\mathrm{e}^{-x^2}\cdot\frac{\partial z}{\partial y}}{(z+1)^2}$$

$$=\frac{\mathrm{e}^{-x^2}(-\frac{z\mathrm{e}^{-y^2}}{z+1})(z+1)+z\mathrm{e}^{-x^2}\frac{z\mathrm{e}^{-y^2}}{z+1}}{(z+1)^2}=\frac{-z}{(z+1)^3}\mathrm{e}^{-(x^2+y^2)}.$$

7. 求由方程 $f(x-y,y-z,z-x)=0$ 所确定的函数 $z=z(x,y)$ 的全微分 $\mathrm{d}z$.

解 用全微分形式不变性求 $\mathrm{d}z$. 方程两边同时取全微分，得

$$f_1'\cdot\mathrm{d}(x-y)+f_2'\cdot\mathrm{d}(y-z)+f_3'\cdot\mathrm{d}(z-x)=0,$$

$$f_1'(\mathrm{d}x-\mathrm{d}y)+f_2'(\mathrm{d}y-\mathrm{d}z)+f_3'(\mathrm{d}z-\mathrm{d}x)=0,$$

化简得

$$(f_3'-f_2')\mathrm{d}z=(f_3'-f_1')\mathrm{d}x+(f_1'-f_2')\mathrm{d}y,$$

所以

$$\mathrm{d}z=\frac{(f_3'-f_1')\mathrm{d}x+(f_1'-f_2')\mathrm{d}y}{f_3'-f_2'}.$$

8. 求由下列方程组所确定的隐函数的导数或偏导数:

(1) $\begin{cases}x+y+z=2,\\ x^2+y^2=\dfrac{1}{2}z^2,\end{cases}$ 求 $\dfrac{\mathrm{d}x}{\mathrm{d}z}$, $\dfrac{\mathrm{d}y}{\mathrm{d}z}$;

(2) 设 $u(x,y)=\mathrm{e}^{3x-y}$, $x^2+y=t^2$, $x-y=t+2$, 求 $\dfrac{\mathrm{d}u}{\mathrm{d}t}\Big|_{t=0}$;

(3) $\begin{cases}xu-yv=0,\\ yu+xv=1,\end{cases}$ 求 $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$, $\dfrac{\partial v}{\partial y}$;

(4) $\begin{cases}xy^2-uv=1,\\ x^2+y^2-u+v=0,\end{cases}$ 求 $\dfrac{\partial u}{\partial x}\Big|_{\substack{x=1\\y=1}}$, $\dfrac{\partial v}{\partial x}\Big|_{\substack{x=1\\y=1}}$.

解 (1) 此方程组可确定两个一元隐函数 $x=x(z)$ 和 $y=y(z)$, 方程两边对 z 求导并移项, 得

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}z}+\dfrac{\mathrm{d}y}{\mathrm{d}z}=-1,\\ 2x\dfrac{\mathrm{d}x}{\mathrm{d}z}+2y\dfrac{\mathrm{d}y}{\mathrm{d}z}=z,\end{cases}$$

在 $D=\begin{vmatrix}1 & 1\\ 2x & 2y\end{vmatrix}=2y-2x\neq 0$ 的条件下, 解方程组求得

$$\frac{\mathrm{d}x}{\mathrm{d}z}=\frac{\begin{vmatrix}-1 & 1\\ z & 2y\end{vmatrix}}{D}=\frac{-2y-z}{2y-2x}=\frac{z+2y}{2(x-y)},$$

$$\frac{\mathrm{d}y}{\mathrm{d}z}=\frac{\begin{vmatrix}1 & -1\\ 2x & z\end{vmatrix}}{D}=\frac{z+2x}{2y-2x}=\frac{z+2x}{2(y-x)}.$$

(2) 方程组 $\begin{cases}x^2+y=t^2,\\ x-y=t+2,\end{cases}$ 可确定两个一元隐函数 $x=x(t)$ 和 $y=y(t)$, 方程两边对 t 求导, 得

$$\begin{cases} 2x\dfrac{\mathrm{d}x}{\mathrm{d}t}+\dfrac{\mathrm{d}y}{\mathrm{d}t}=2t, \\ \dfrac{\mathrm{d}x}{\mathrm{d}t}-\dfrac{\mathrm{d}y}{\mathrm{d}t}=1\ , \end{cases}$$

在 $D=\begin{vmatrix} 2x & 1 \\ 1 & -1 \end{vmatrix}=-2x-1\neq 0$ 的条件下，解方程组求得

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\begin{vmatrix} 2t & 1 \\ 1 & 1 \end{vmatrix}}{D}=\frac{2t-1}{-2x-1}=\frac{1-2t}{1+2x}\ ,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\begin{vmatrix} 2x & 2t \\ 1 & 1 \end{vmatrix}}{D}=\frac{2x-2t}{-2x-1}=\frac{2t-2x}{1+2x}\ ,$$

由 $u(x,y)=\mathrm{e}^{3x-y}$ 可得

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t}&=\frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{3x-y}\cdot 3\cdot\frac{1-2t}{1+2x}+\mathrm{e}^{3x-y}\cdot(-1)\cdot\frac{2t-2x}{1+2x} \\ &=\frac{3-8t+2x}{1+2x}\mathrm{e}^{3x-y}\ . \end{aligned}$$

当 $t=0$ 时，有 $\begin{cases} x^2+y=0, \\ x-y=2, \end{cases}$ 可解得

$$\begin{cases} x=1, \\ y=-1, \end{cases} \text{或} \begin{cases} x=-2, \\ y=-4. \end{cases}$$

所以

$$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{\substack{t=0\\x=1\\y=-1}}=\frac{5}{3}\mathrm{e}^4 \quad \text{或} \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{\substack{t=0\\x=-2\\y=-4}}=\frac{1}{3}\mathrm{e}^{-2}\ .$$

(3)　**法 1**　将所给方程的两边对 x 求导，得

$$\begin{cases} u+x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=0, \\ y\dfrac{\partial u}{\partial x}+v+x\dfrac{\partial v}{\partial x}=0, \end{cases}$$

移项，得

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u, \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v. \end{cases}$$

在 $J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$ 的条件下，解方程组求得

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{J} = \frac{-xu - yv}{x^2 + y^2} = -\frac{xu + yv}{x^2 + y^2},$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{J} = \frac{-xv + yu}{x^2 + y^2} = \frac{yu - xv}{x^2 + y^2}.$$

将所给方程的两边对 y 求导，用同样的方法在 $J = x^2 + y^2 \neq 0$ 的条件下可得

$$\frac{\partial u}{\partial y} = \frac{xv - yu}{x^2 + y^2}, \qquad \frac{\partial v}{\partial y} = -\frac{xu + yv}{x^2 + y^2}.$$

法 2　对方程组求全微分可得

$$\begin{cases} u\mathrm{d}x + x\mathrm{d}u - v\mathrm{d}y - y\mathrm{d}v = 0, \\ u\mathrm{d}y + y\mathrm{d}u + v\mathrm{d}x + x\mathrm{d}v = 0, \end{cases}$$

移项，得

$$\begin{cases} x\mathrm{d}u - y\mathrm{d}v = -u\mathrm{d}x + v\mathrm{d}y, \\ y\mathrm{d}u + x\mathrm{d}v = -v\mathrm{d}x - u\mathrm{d}y. \end{cases}$$

在 $D = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$ 的条件下，解方程组求得

$$\mathrm{d}u = \frac{\begin{vmatrix} -u\mathrm{d}x + v\mathrm{d}y & -y \\ -v\mathrm{d}x - u\mathrm{d}y & x \end{vmatrix}}{x^2 + y^2} = \frac{-(xu + yv)\mathrm{d}x + (xv - yu)\mathrm{d}y}{x^2 + y^2},$$

$$\mathrm{d}v = \frac{\begin{vmatrix} x & -u\mathrm{d}x + v\mathrm{d}y \\ y & -v\mathrm{d}x - u\mathrm{d}y \end{vmatrix}}{x^2 + y^2} = \frac{(yu - xv)\mathrm{d}x - (xu + yv)\mathrm{d}y}{x^2 + y^2},$$

所以

$$\frac{\partial u}{\partial x}=-\frac{xu+yv}{x^2+y^2}, \qquad \frac{\partial u}{\partial y}=\frac{xv-yu}{x^2+y^2},$$

$$\frac{\partial v}{\partial x}=\frac{yu-xv}{x^2+y^2}, \qquad \frac{\partial v}{\partial y}=-\frac{xu+yv}{x^2+y^2}.$$

(4) 将所给方程的两边对 x 求导，得

$$\begin{cases} y^2-v\dfrac{\partial u}{\partial x}-u\dfrac{\partial v}{\partial x}=0, \\ 2x-\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial x}=0, \end{cases}$$

移项，得

$$\begin{cases} v\dfrac{\partial u}{\partial x}+u\dfrac{\partial v}{\partial x}=y^2, \\ \dfrac{\partial u}{\partial x}-\dfrac{\partial v}{\partial x}=2x. \end{cases}$$

在 $J=\begin{vmatrix} v & u \\ 1 & -1 \end{vmatrix}=-v-u=-(u+v)\neq 0$ 的条件下，解方程组求得

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix} y^2 & u \\ 2x & -1 \end{vmatrix}}{J}=\frac{-y^2-2xu}{-(u+v)}=\frac{y^2+2xu}{u+v},$$

$$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix} v & y^2 \\ 1 & 2x \end{vmatrix}}{J}=\frac{2xv-y^2}{-(u+v)}=\frac{y^2-2xv}{u+v}.$$

当 $x=1,\ y=1$ 时，$\begin{cases} uv=0, \\ u-v=2, \end{cases}$ 解得

$$\begin{cases} u=0, \\ v=-2, \end{cases} \text{或} \begin{cases} u=2, \\ v=0, \end{cases}$$

所以

$$\left.\frac{\partial u}{\partial x}\right|_{\substack{x=1\\y=1}}=-\frac{1}{2}, \qquad \left.\frac{\partial v}{\partial x}\right|_{\substack{x=1\\y=1}}=-\frac{5}{2}$$

或

$$\left.\frac{\partial u}{\partial x}\right|_{\substack{x=1\\y=1}}=\frac{5}{2}, \qquad \left.\frac{\partial v}{\partial x}\right|_{\substack{x=1\\y=1}}=\frac{1}{2}.$$

9. 设 $u+v=x+y$， $\dfrac{\sin u}{\sin v}=\dfrac{x}{y}$，求 $\mathrm{d}u$, $\mathrm{d}v$.

解 对方程组 $\begin{cases} u+v=x+y, \\ \dfrac{\sin u}{\sin v}=\dfrac{x}{y}, \end{cases}$ 求全微分可得

$$\begin{cases} \mathrm{d}u+\mathrm{d}v=\mathrm{d}x+\mathrm{d}y \ , \\ \dfrac{\sin v\cdot\cos u\cdot\mathrm{d}u-\sin u\cdot\cos v\cdot\mathrm{d}v}{\sin^2 v}=\dfrac{y\mathrm{d}x-x\mathrm{d}y}{y^2}, \end{cases}$$

移项整理，得

$$\begin{cases} \mathrm{d}u+\mathrm{d}v=\mathrm{d}x+\mathrm{d}y, \\ \cos u\sin v\mathrm{d}u-\sin u\cos v\mathrm{d}v=\dfrac{\sin^2 v}{y^2}(y\mathrm{d}x-x\mathrm{d}y). \end{cases}$$

在 $D=\begin{vmatrix} 1 & 1 \\ \cos u\sin v & -\sin u\cos v \end{vmatrix}=-\sin u\cos v-\cos u\sin v\neq 0$ 的条件下，解方程组

求得

$$\mathrm{d}u=\frac{\begin{vmatrix} \mathrm{d}x+\mathrm{d}y & 1 \\ \dfrac{\sin^2 v}{y^2}(y\mathrm{d}x-x\mathrm{d}y) & -\sin u\cos v \end{vmatrix}}{D}$$

$$=\frac{-\sin u\cos v\mathrm{d}x-\sin u\cos v\mathrm{d}y-\dfrac{\sin^2 v}{y}\mathrm{d}x+\dfrac{x\sin^2 v}{y^2}\mathrm{d}y}{-\sin u\cos v-\cos u\sin v}$$

$$=\frac{(y^2\sin u\cos v+y\sin^2 v)\mathrm{d}x+(y^2\sin u\cos v-x\sin^2 v)\mathrm{d}y}{y^2\sin u\cos v+y^2\cos u\sin v},$$

由题设 $\dfrac{\sin u}{\sin v}=\dfrac{x}{y}$，得 $y\sin u=x\sin v$ 代入上式，化简得

$$\mathrm{d}u=\frac{(\sin v+x\cos v)\mathrm{d}x-(\sin u-x\cos v)\mathrm{d}y}{x\cos v+y\cos u}.$$

同理可解得

$$\mathrm{d}v=\frac{(y\cos u-\sin v)\mathrm{d}x+(\sin u+y\cos u)\mathrm{d}y}{x\cos v+y\cos u}.$$

10. 设 $x=u+v$, $y=u^2+v^2$， $z=u^3+v^3$ 确定了 z 是 x,y 的函数，求

$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解 对方程组$\begin{cases} u+v=x, \\ u^2+v^2=y, \end{cases}$求全微分可得

$$\begin{cases} \mathrm{d}u+\mathrm{d}v=\mathrm{d}x, \\ 2u\mathrm{d}u+2v\mathrm{d}v=\mathrm{d}y, \end{cases}$$

在$D=\begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix}=2v-2u\neq 0$的条件下，解方程组求得

$$\mathrm{d}u=\frac{\begin{vmatrix} \mathrm{d}x & 1 \\ \mathrm{d}y & 2v \end{vmatrix}}{D}=\frac{2v\mathrm{d}x-\mathrm{d}y}{2v-2u},$$

$$\mathrm{d}v=\frac{\begin{vmatrix} 1 & \mathrm{d}x \\ 2u & \mathrm{d}y \end{vmatrix}}{D}=\frac{-2u\mathrm{d}x+\mathrm{d}y}{2v-2u}.$$

对方程$z=u^3+v^3$两边求全微分可得

$$\mathrm{d}z=3u^2\mathrm{d}u+3v^2\mathrm{d}v=3u^2\cdot\frac{2v\mathrm{d}x-\mathrm{d}y}{2v-2u}+3v^2\cdot\frac{-2u\mathrm{d}x+\mathrm{d}y}{2v-2u}$$

$$=-3uv\mathrm{d}x+\frac{3}{2}(u+v)\mathrm{d}y,$$

所以

$$\frac{\partial z}{\partial x}=-3uv, \qquad \frac{\partial z}{\partial y}=\frac{3}{2}(u+v).$$

11. 设$u=xy^2z^3$，而$z=z(x,y)$是由方程$x^2+y^2+z^2=3xyz$所确定的隐函数，求$\left.\dfrac{\partial u}{\partial x}\right|_{(1,1,1)}$.

解 先由方程$x^2+y^2+z^2=3xyz$求$\dfrac{\partial z}{\partial x}$.

令$F(x,y,z)=x^2+y^2+z^2-3xyz$，则

$$F_x=2x-3yz, \quad F_z=2z-3xy,$$

于是

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x-3yz}{2z-3xy}.$$

由题设$u=xy^2z^3$可得

$$\frac{\partial u}{\partial x}=1\cdot y^2z^3+xy^2\cdot 3z^2\cdot\frac{\partial z}{\partial x}=y^2z^3+3xy^2z^2\cdot(-\frac{2x-3yz}{2z-3xy}),$$

$$\frac{\partial u}{\partial x}\Big|_{(1,1,1)}=1+3\cdot(-\frac{2-3}{2-3})=-2.$$