第二节　洛必达法则

习题　3-2

1. 用洛必达法则求下列极限:

(1) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\tan x}$;　　(2) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\ln\sin x}{(\pi-2x)^2}$;

(3) $\lim\limits_{x\to 0}x^2e^{\frac{1}{x^2}}$;　　(4) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$;

(5) $\lim\limits_{x\to 0^+}\dfrac{\ln\sin ax}{\ln\sin bx}\,(a>0,b>0)$;　　(6) $\lim\limits_{x\to 1}(\dfrac{2}{x^2-1}-\dfrac{1}{x-1})$;

(7) $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}$;　　(8) $\lim\limits_{x\to 0}(\dfrac{1}{x}\cot x-\dfrac{1}{x^2})$;

(9) $\lim\limits_{x\to\frac{\pi}{2}^-}(\cos x)^{\frac{\pi}{2}-x}$;　　(10) $\lim\limits_{x\to+\infty}(\dfrac{2}{\pi}\arctan x)^x$;

(11) $\lim\limits_{x\to 0}(\dfrac{\sin x}{x})^{\frac{1}{x^2}}$;　　(12) $\lim\limits_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$;

(13) $\lim\limits_{x\to\infty}(1+\dfrac{3}{x})^{2x}$;　　(14) $\lim\limits_{x\to+\infty}(x+e^x)^{\frac{1}{x}}$.

解　(1) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}}{\tan x}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}}{\sec^2 x}=\lim\limits_{x\to 0}[\cos^2 x(e^x+e^{-x})]=2.$

(2) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\ln\sin x}{(\pi-2x)^2}\overset{\frac{0}{0}}{=}\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\frac{1}{\sin x}\cos x}{2(\pi-2x)(-2)}=-\dfrac{1}{4}\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\pi-2x}\overset{\frac{0}{0}}{=}-\dfrac{1}{4}\lim\limits_{x\to\frac{\pi}{2}}\dfrac{-\sin x}{-2}=-\dfrac{1}{8}.$

(3) $\lim\limits_{x\to 0}x^2e^{\frac{1}{x^2}}=\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x^2}}}{\frac{1}{x^2}}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x^2}}(-2\frac{1}{x^3})}{-2\frac{1}{x^3}}=\lim\limits_{x\to 0}e^{\frac{1}{x^2}}=+\infty.$

(4) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{6x}=\dfrac{1}{6}.$

(5) $\lim\limits_{x\to 0^+}\dfrac{\ln\sin ax}{\ln\sin bx}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to 0^+}\dfrac{\dfrac{1}{\sin ax}\cos ax\cdot a}{\dfrac{1}{\sin bx}\cos bx\cdot b}=1.$

(6) $\lim\limits_{x\to 1}(\dfrac{2}{x^2-1}-\dfrac{1}{x-1})=\lim\limits_{x\to 1}\dfrac{1-x}{x^2-1}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 1}\dfrac{-1}{2x}=-\dfrac{1}{2}.$

(7) $\because\ \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=\mathrm{e}^{\lim\limits_{x\to 0}\frac{1}{x}\ln(1+x)}=\mathrm{e},$

$\therefore\ \lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-\mathrm{e}}{x}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}[\frac{1}{x}\ln(1+x)]'}{1}$

$=\lim\limits_{x\to 0}\{(1+x)^{\frac{1}{x}}[\dfrac{1}{x(1+x)}+\ln(1+x)(-\dfrac{1}{x^2})]\}$

$=\mathrm{e}\lim\limits_{x\to 0}\dfrac{x-(1+x)\ln(1+x)}{x^2(1+x)}\qquad(\dfrac{0}{0})$

$=\mathrm{e}\lim\limits_{x\to 0}\dfrac{1-1-\ln(1+x)}{2x(1+x)+x^2}=-\dfrac{1}{2}\mathrm{e}.$

(8) $\lim\limits_{x\to 0}(\dfrac{1}{x}\cot x-\dfrac{1}{x^2})=\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^2\tan x}=\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^3}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{1-\sec^2 x}{3x^2}$

$=\lim\limits_{x\to 0}\dfrac{\cos^2 x-1}{3x^2\cos^2 x}=-\lim\limits_{x\to 0}\dfrac{\sin^2 x}{3x^2}=-\dfrac{1}{3}.$

(9) 令 $y=(\cos x)^{\frac{\pi}{2}-x}$，则 $\ln y=(\dfrac{\pi}{2}-x)\ln\cos x.$

$\because\ \lim\limits_{x\to\frac{\pi}{2}^-}\ln y=\lim\limits_{x\to\frac{\pi}{2}^-}(\dfrac{\pi}{2}-x)\ln\cos x=\lim\limits_{x\to\frac{\pi}{2}^-}\dfrac{\ln\cos x}{\dfrac{1}{\frac{\pi}{2}-x}}\overset{\frac{\infty}{\infty}}{=}\lim\limits_{x\to\frac{\pi}{2}^-}\dfrac{\dfrac{-\sin x}{\cos x}}{\dfrac{1}{(\frac{\pi}{2}-x)^2}}$

$=-\lim\limits_{x\to\frac{\pi}{2}^-}\dfrac{(\frac{\pi}{2}-x)^2}{\cos x}\overset{\frac{0}{0}}{=}-\lim\limits_{x\to\frac{\pi}{2}^-}\dfrac{2(\frac{\pi}{2}-x)(-1)}{-\sin x}=0,$

$$\therefore \lim_{x\to\frac{\pi}{2}^-}(\cos x)^{\frac{\pi}{2}-x} = \lim_{x\to\frac{\pi}{2}^-} y = e^{\lim\limits_{x\to\frac{\pi}{2}^-}\ln y} = e^0 = 1.$$

(10) 令 $y=(\frac{2}{\pi}\arctan x)^x$，则 $\ln y = x\ln(\frac{2}{\pi}\arctan x)$.

$$\because \lim_{x\to+\infty}\ln y = \lim_{x\to+\infty} x\ln(\frac{2}{\pi}\arctan x) = \lim_{x\to+\infty}\frac{\ln(\frac{2}{\pi}\arctan x)}{\frac{1}{x}} \qquad (\frac{0}{0})$$

$$= \lim_{x\to+\infty}\frac{\frac{1}{\frac{2}{\pi}\arctan x}\frac{2}{\pi}\frac{1}{1+x^2}}{-\frac{1}{x^2}} = -\lim_{x\to+\infty}\frac{x^2}{(1+x^2)\arctan x} = -\frac{2}{\pi},$$

$$\therefore \lim_{x\to+\infty}(\frac{2}{\pi}\arctan x)^x = \lim_{x\to+\infty} y = e^{\lim\limits_{x\to+\infty}\ln y} = e^{-\frac{2}{\pi}}.$$

(11) 令 $y=(\frac{\sin x}{x})^{\frac{1}{x^2}}$，则 $\ln y = \frac{1}{x^2}\ln(\frac{\sin x}{x})$.

$$\because \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(\frac{\sin x}{x})}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\frac{x}{\sin x}\frac{\cos x\cdot x-\sin x}{x^2}}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\cos x\cdot x-\sin x}{x^3} \qquad (\frac{0}{0})$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\cos x + x(-\sin x)-\cos x}{3x^2} = \frac{1}{2}\lim_{x\to 0}\frac{-x\sin x}{3x^2} = -\frac{1}{6},$$

$$\therefore \lim_{x\to 0}(\frac{\sin x}{x})^{\frac{1}{x^2}} = \lim_{x\to 0} y = e^{-\frac{1}{6}}.$$

(12) 令 $y=(\cot x)^{\frac{1}{\ln x}}$，则 $\ln y = \frac{1}{\ln x}\ln(\cot x)$.

$$\because \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(\cot x)}{\ln x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to 0^+}\frac{\frac{1}{\cot x}(-\csc^2 x)}{\frac{1}{x}} = -\lim_{x\to 0^+}\frac{x}{\sin x\cos x} = -1,$$

$$\therefore \lim_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}} = \lim_{x\to 0^+} y = e^{-1}.$$

(13) 令 $y=(1+\frac{3}{x})^{2x}$，则 $\ln y = 2x\ln(1+\frac{3}{x})$.

$$\because \lim_{x\to\infty}\ln y = 2\lim_{x\to\infty}\frac{\ln(1+\frac{3}{x})}{\frac{1}{x}} \overset{\frac{0}{0}}{=} 2\lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}(-\frac{3}{x^2})}{-\frac{1}{x^2}} = 6\lim_{x\to\infty}\frac{x}{x+3} = 6,$$

$$\therefore \lim_{x\to\infty}(1+\frac{3}{x})^{2x} = \lim_{x\to\infty} y = \mathrm{e}^6.$$

(14) 令 $y=(x+\mathrm{e}^x)^{\frac{1}{x}}$，则 $\ln y = \frac{1}{x}\ln(x+\mathrm{e}^x)$.

$$\because \lim_{x\to+\infty}\ln y = \lim_{x\to+\infty}\frac{\ln(x+\mathrm{e}^x)}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to+\infty}\frac{1+\mathrm{e}^x}{x+\mathrm{e}^x} = 1,$$

$$\therefore \lim_{x\to+\infty}(x+\mathrm{e}^x)^{\frac{1}{x}} = \lim_{x\to+\infty} y = \mathrm{e}.$$

2. 验证极限 $\lim\limits_{x\to 0}\frac{\sin^2 x\sin\frac{1}{x}}{x}$ 存在，但不能用洛必达法则求出.

解 由于 $\lim\limits_{x\to 0}\frac{\sin^2 x}{x}=0$，函数 $\sin\frac{1}{x}$ 为有界函数，根据无穷小的性质知，极限 $\lim\limits_{x\to 0}\frac{\sin^2 x\sin\frac{1}{x}}{x}$ 存在且为零.

若用洛必达法则，则有

$$\lim_{x\to 0}\frac{\sin^2 x\sin\frac{1}{x}}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{2\sin x\cos x\sin\frac{1}{x}+\sin^2 x\cos\frac{1}{x}(-\frac{1}{x^2})}{1}.$$

由于 $\lim\limits_{x\to 0}2\sin x\cos x\sin\frac{1}{x}=0$，而 $\lim\limits_{x\to 0}[\sin^2 x\cos\frac{1}{x}(-\frac{1}{x^2})]=\lim\limits_{x\to 0}(-\frac{\sin^2 x}{x^2}\cos\frac{1}{x})$ 不存在，因此极限 $\lim\limits_{x\to 0}\frac{\sin^2 x\sin\frac{1}{x}}{x}$ 是不能用洛必达法则求出的.

3. 设 $f(x)=\begin{cases}\frac{g(x)}{x}, & x\neq 0,\\ 0, & x=0,\end{cases}$ 以及 $g(0)=g'(0)=0$，$g''(0)=2$，求 $f'(0)$.

解 $f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{\frac{g(x)}{x}-0}{x-0}=\lim\limits_{x\to 0}\frac{g(x)}{x^2}\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\frac{g'(x)}{2x}$

$$=\frac{1}{2}\lim_{x\to 0}\frac{g'(x)-g'(0)}{x-0}=\frac{1}{2}g''(0)=1.$$

注意 易犯的错误是

$$f'(0)=\lim_{x\to 0}\frac{g(x)}{x^2}=\lim_{x\to 0}\frac{g'(x)}{2x}=\lim_{x\to 0}\frac{g''(x)}{2}=\frac{1}{2}g''(0)=1.$$

错误有二: (1) 洛必达法则的再次使用. 实际上, 第二次已不满足洛必达法则的使用条件.

(2) 利用了函数 $g''(x)$ 在点 $x=0$ 处的连续性, 而这是未知的.