第三节　定积分的换元积分法与分部积分法

习题　5-3

1. 计算下列定积分:

(1) $\int_0^{\frac{\pi}{2}}\cos^5 x\sin x\mathrm{d}x$;　(2) $\int_{-2}^{-\sqrt{2}}\frac{\mathrm{d}x}{\sqrt{x^2-1}}$;

(3) $\int_1^4\frac{1}{1+\sqrt{x}}\mathrm{d}x$;　(4) $\int_4^9\frac{\sqrt{x}}{\sqrt{x}-1}\mathrm{d}x$;

(5) $\int_{-1}^1\frac{x}{\sqrt{5-4x}}\mathrm{d}x$;　(6) $\int_{-3}^{-1}\frac{\mathrm{d}x}{x^2+4x+5}$;

(7) $\int_1^2\frac{\mathrm{e}^{\frac{1}{x}}}{x^2}\mathrm{d}x$;　(8) $\int_0^1\frac{\mathrm{d}x}{\mathrm{e}^x+\mathrm{e}^{-x}}$;

(9) $\int_0^3\frac{x}{\sqrt{x+1}}\mathrm{d}x$;　(10) $\int_{-1}^1\frac{\mathrm{d}x}{(1+x^2)^2}$;

(11) $\int_0^\pi(1-\sin^3\theta)\mathrm{d}\theta$;　(12) $\int_{-2}^0\frac{1}{x^2+2x+2}\mathrm{d}x$;

(13) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\cos 2x\mathrm{d}x$;　(14) $\int_1^{\sqrt{3}}\frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}$;

(15) $\int_1^{\mathrm{e}^2}\frac{\mathrm{d}x}{x\sqrt{1+\ln x}}$;　(16) $\int_{-2}^1\frac{\mathrm{d}x}{(11+5x)^3}$.

解　(1) $\int_0^{\frac{\pi}{2}}\cos^5 x\sin x\mathrm{d}x=-\int_0^{\frac{\pi}{2}}\cos^5 x\mathrm{d}\cos x=-\frac{1}{6}\cos^6 x\Big|_0^{\frac{\pi}{2}}=\frac{1}{6}$.

(2) 由 $\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$ 知

$$\int_{-2}^{-\sqrt{2}}\frac{\mathrm{d}x}{\sqrt{x^2-1}}=\ln\left|x+\sqrt{x^2-1}\right|\Big|_{-2}^{\sqrt{2}}=\ln\left|-\sqrt{2}+1\right|-\ln\left|-2+\sqrt{3}\right|$$

$$=\ln(\sqrt{2}-1)-\ln(2-\sqrt{3}).$$

(3) 令 $\sqrt{x}=t$, $\mathrm{d}x=2t\mathrm{d}t$, $x=1$, $t=1$; $x=4$, $t=2$,

$$\int_1^4 \frac{1}{1+\sqrt{x}}\mathrm{d}x=\int_1^2 \frac{2t}{1+t}\mathrm{d}t=2(\int_1^2 \mathrm{d}t-\int_1^2 \frac{\mathrm{d}t}{1+t})$$

$$=2[t\Big|_1^2-\ln(1+t)\Big|_1^2]$$

$$=2(1-\ln\frac{3}{2})=2+2\ln\frac{2}{3}.$$

(4) $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}\mathrm{d}x=\int_4^9 \frac{\sqrt{x}-1+1}{\sqrt{x}-1}\mathrm{d}x=\int_4^9 \mathrm{d}x+\int_4^9 \frac{1}{\sqrt{x}-1}\mathrm{d}x$

$$=5+\int_4^9 \frac{1}{\sqrt{x}-1}\mathrm{d}x.$$

令 $t=\sqrt{x}$，则 $\mathrm{d}x=2t\mathrm{d}t$, $x=4$, $t=2$; $x=9$, $t=3$, 于是

$$\int_4^9 \frac{1}{\sqrt{x}-1}\mathrm{d}x=2\int_2^3 \frac{t}{t-1}\mathrm{d}t=2\int_2^3 (1+\frac{1}{t-1})\mathrm{d}t$$

$$=2t\Big|_2^3+2\int_2^3 \frac{1}{t-1}\mathrm{d}(t-1)=2+2\ln|t-1|\Big|_2^3=2+2\ln 2,$$

所以
$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}\mathrm{d}x=7+2\ln 2.$$

(5) 令 $\sqrt{5-4x}=u$，则 $x=\frac{5}{4}-\frac{u^2}{4}$, $\mathrm{d}x=-\frac{1}{2}u\mathrm{d}u$, $x=-1$, $u=3$; $x=1$, $u=1$,

$$\int_{-1}^1 \frac{x}{\sqrt{5-4x}}\mathrm{d}x=\int_3^1 \frac{\frac{1}{4}(5-u^2)(-\frac{1}{2}u)}{u}\mathrm{d}u$$

$$=\int_1^3 \frac{1}{8}\cdot(5-u^2)\mathrm{d}u=\frac{5}{8}u\Big|_1^3-\frac{1}{8}\frac{u^3}{3}\Bigg|_1^3=\frac{1}{6}.$$

(6) $\int_{-3}^{-1} \frac{\mathrm{d}x}{x^2+4x+5}=\int_{-3}^{-1} \frac{\mathrm{d}x}{(x+2)^2+1}=\int_{-3}^{-1} \frac{\mathrm{d}(x+2)}{(x+2)^2+1}$

$$=\arctan(x+2)\Big|_{-3}^{-1}=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}.$$

(7) $\int_1^2 \frac{\mathrm{e}^{\frac{1}{x}}}{x^2}\mathrm{d}x=-\int_1^2 \mathrm{d}\mathrm{e}^{\frac{1}{x}}=-\mathrm{e}^{\frac{1}{x}}\Bigg|_1^2=\mathrm{e}-\sqrt{\mathrm{e}}.$

(8) $\int_0^1 \frac{\mathrm{d}x}{\mathrm{e}^x+\mathrm{e}^{-x}}=\int_0^1 \frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1}\mathrm{d}x=\int_0^1 \frac{\mathrm{d}\mathrm{e}^x}{\mathrm{e}^{2x}+1}=\arctan\mathrm{e}^x\Big|_0^1=\arctan\mathrm{e}-\frac{\pi}{4}$.

(9) 令$t=\sqrt{x+1}$，则 $\mathrm{d}x=2t\mathrm{d}t$，$x=0$，$t=1$；$x=3$，$t=2$，于是

$$\int_0^3 \frac{x}{\sqrt{x+1}}\mathrm{d}x=\int_1^2 \frac{t^2-1}{t}\cdot 2t\mathrm{d}t=2\int_1^2 (t^2-1)\mathrm{d}t$$

$$=2(\frac{1}{3}t^3-t)\Big|_1^2=\frac{8}{3}.$$

(10) 因为$\frac{1}{(1+x^2)^2}$是偶函数，故

$$\int_{-1}^1 \frac{\mathrm{d}x}{(1+x^2)^2}=2\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}.$$

令 $x=\tan\theta$，则$\mathrm{d}x=\sec^2\theta\mathrm{d}\theta$，$x=0$，$\theta=0$；$x=1$，$\theta=\frac{\pi}{4}$，于是

$$\int_{-1}^1 \frac{\mathrm{d}x}{(1+x^2)^2}=2\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}=2\int_0^{\frac{\pi}{4}} \frac{1}{\sec^4\theta}\cdot\sec^2\theta\mathrm{d}\theta$$

$$=2\int_0^{\frac{\pi}{4}}\cos^2\theta\mathrm{d}\theta=\int_0^{\frac{\pi}{4}}(\cos 2\theta+1)\mathrm{d}\theta=(\frac{1}{2}\sin 2\theta+\theta)\Big|_0^{\frac{\pi}{4}}=\frac{\pi}{4}+\frac{1}{2}.$$

(11) $\int_0^{\pi}(1-\sin^3\theta)\mathrm{d}\theta=\int_0^{\pi}\mathrm{d}\theta+\int_0^{\pi}\sin^2\theta\mathrm{d}\cos\theta=\theta\Big|_0^{\pi}+\int_0^{\pi}(1-\cos^2\theta)\mathrm{d}\cos\theta$

$$=\pi+(\cos\theta-\frac{1}{3}\cos^3\theta)\Big|_0^{\pi}=\pi-\frac{4}{3}.$$

(12) $\int_{-2}^0 \frac{1}{x^2+2x+2}\mathrm{d}x=\int_{-2}^0 \frac{\mathrm{d}x}{1+(x+1)^2}=\arctan(x+1)\Big|_{-2}^0$

$$=\arctan 1-\arctan(-1)=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}.$$

(13) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\cos 2x\mathrm{d}x=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos 3x+\cos x)\mathrm{d}x$

$$=\frac{1}{2}(\frac{1}{3}\sin 3x+\sin x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}(-\frac{2}{3}+\frac{6}{3})=\frac{2}{3}.$$

(14) 令 $x=\tan\theta$，则 $\mathrm{d}x=\sec^2\theta\mathrm{d}\theta$， $x=1,\ \theta=\dfrac{\pi}{4};\ x=\sqrt{3},\ \theta=\dfrac{\pi}{3}$,

$$\int_1^{\sqrt{3}}\frac{\mathrm{d}x}{x^2\sqrt{1+x^2}}=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2\theta}{\tan^2\theta\sec\theta}\mathrm{d}\theta=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\sin\theta)^{-2}\mathrm{d}\sin\theta=\sqrt{2}-\frac{2}{3}\sqrt{3}.$$

(15) $$\int_1^{\mathrm{e}^2}\frac{\mathrm{d}x}{x\sqrt{1+\ln x}}=\int_1^{\mathrm{e}^2}(1+\ln x)^{-\frac{1}{2}}\mathrm{d}\ln x$$

$$=\int_1^{\mathrm{e}^2}(1+\ln x)^{-\frac{1}{2}}\mathrm{d}(1+\ln x)=2\sqrt{1+\ln x}\Big|_1^{\mathrm{e}^2}$$

$$=2\sqrt{1+\ln\mathrm{e}^2}-2\sqrt{1+\ln 1}=2\sqrt{3}-2.$$

(16) $$\int_{-2}^{1}\frac{\mathrm{d}x}{(11+5x)^3}=\frac{1}{5}\int_{-2}^{1}(11+5x)^{-3}\mathrm{d}(11+5x)$$

$$=\frac{1}{5}\cdot[(-\frac{1}{2})(11+5x)^{-2}]\Big|_{-2}^{1}=\frac{51}{512}.$$

2. $f(x)=\begin{cases}x^2\mathrm{e}^{-x^2}, & x\geq 0,\\ \dfrac{1}{1+\cos x}, & -1<x<0.\end{cases}$ 计算 $\int_1^4 f(x-2)\mathrm{d}x$.

解 令 $t=x-2$，则 $\mathrm{d}x=\mathrm{d}t,\ x=1,\ t=-1;\ x=4,\ t=2$, 于是

$$\int_1^4 f(x-2)\mathrm{d}x=\int_{-1}^{2}f(t)\mathrm{d}t$$

$$=\int_{-1}^{0}\frac{1}{1+\cos t}\mathrm{d}t+\int_0^2 t\mathrm{e}^{-t^2}\mathrm{d}t=\int_{-1}^{0}\frac{1}{2\cos^2\dfrac{t}{2}}\mathrm{d}t-\frac{1}{2}\int_0^2\mathrm{d}\mathrm{e}^{-t^2}$$

$$=\frac{1}{2}\int_{-1}^{0}\sec^2\frac{t}{2}\mathrm{d}t-\frac{1}{2}\mathrm{e}^{-t^2}\Big|_0^2=\mathrm{tg}\frac{t}{2}\Big|_{-1}^{0}-\frac{1}{2}\mathrm{e}^{-4}+\frac{1}{2}$$

$$=\mathrm{tg}\frac{1}{2}-\frac{1}{2}\mathrm{e}^{-4}+\frac{1}{2}.$$

3. 利用函数的奇偶性计算下列定积分:

(1) $\int_{-\pi}^{\pi}x^6\sin x\mathrm{d}x$; (2) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\dfrac{x^3}{1+\cos x}\mathrm{d}x$;

(3) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x\mathrm{d}x$; (4) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^8 x\mathrm{d}x$;

(5) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$; (6) $\int_{-\pi}^{\pi} x\sin x\mathrm{d}x$.

解 (1) 因 $x^6\sin x$ 为奇函数，故

$$\int_{-\pi}^{\pi} x^6\sin x\mathrm{d}x = 0.$$

(2) 因 $\frac{x^3}{1+\cos x}$ 为奇函数，故

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{x^3}{1+\cos x}\mathrm{d}x = 0.$$

(3) 因 $\cos^5 x$ 为偶函数，故

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \cos^5 x\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \sin^5 x\mathrm{d}x$$

$$= 2\cdot\frac{2\cdot 4}{1\cdot 3\cdot 5} = \frac{16}{15}.$$

(4) 因 $\sin^8 x$ 为偶函数，故

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^8 x\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \sin^8 x\mathrm{d}x = 2\cdot\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2} = \frac{35}{128}\pi.$$

(5) 因 $\frac{x\arcsin x}{\sqrt{1-x^2}}$ 为偶函数，故

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x = 2\int_0^{\frac{1}{2}} \frac{x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x = -2\int_0^{\frac{1}{2}} \arcsin x\mathrm{d}\sqrt{1-x^2}$$

$$= -2\sqrt{1-x^2}\arcsin x\Big|_0^{\frac{1}{2}} + 2\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}\mathrm{d}x$$

$$= -\frac{\sqrt{3}}{6}\pi + 2x\Big|_0^{\frac{1}{2}} = -\frac{\sqrt{3}}{6}\pi + 1.$$

(6) 因 $x\sin x$ 为偶函数，故

$$\int_{-\pi}^{\pi} x\sin x\mathrm{d}x = 2\int_0^{\pi} x\sin x\mathrm{d}x = -2\int_0^{\pi} x\mathrm{d}\cos x$$

$$= -x\cos x\Big|_0^{\pi} + 2\int_0^{\pi}\cos x\mathrm{d}x = 2\pi + 2\sin x\Big|_0^{\pi}$$

$$= 2\pi .$$

4. 证明: $\int_{-a}^{a}\varphi(x^2)\mathrm{d}x = 2\int_0^{a}\varphi(x^2)\mathrm{d}x$，其中 $\varphi(u)$ 为连续函数.

证 因为被积函数 $\varphi(x^2)$ 是 x 的偶函数，又因为积分区间 $[-a,a]$ 关于原点对称，所以有 $\int_{-a}^{a}\varphi(x^2)\mathrm{d}x = 2\int_0^{a}\varphi(x^2)\mathrm{d}x$.

5. 设 $f(x)$ 在 $[a,b]$ 上连续，证明

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x.$$

证 令 $a+b-x=t$，则 $x=a+b-t$，$\mathrm{d}x=-\mathrm{d}t$，$x=a$，$t=b$；$x=b$，$t=a$，于是

$$\int_a^b f(a+b-x)\mathrm{d}x = \int_b^a f(t)(-\mathrm{d}t) = \int_a^b f(t)\mathrm{d}t.$$

由于 $\int_a^b f(t)\mathrm{d}t = \int_a^b f(x)\mathrm{d}x$，故 $\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x.$

6. 设 $f(x)$ 在 $[-b,b]$ 上连续，证明

$$\int_{-b}^{b} f(x)\mathrm{d}x = \int_{-b}^{b} f(-x)\mathrm{d}x.$$

证 令 $-x=t$ 则 $x=-t$，$\mathrm{d}x=-\mathrm{d}t$，$x=-b$，$t=b$；$x=b$，$t=-b$，于是

$$\int_{-b}^{b} f(-x)\mathrm{d}x = \int_{b}^{-b} f(t)(-\mathrm{d}t) = \int_{-b}^{b} f(t)\mathrm{d}t,$$

而 $\int_{-b}^{b} f(t)\mathrm{d}t = \int_{-b}^{b} f(x)\mathrm{d}x$ （如 $\int_{-b}^{b}\sin t\mathrm{d}t = \int_{-b}^{b}\sin x\mathrm{d}x$），故

$$\int_{-b}^{b} f(x)\mathrm{d}x = \int_{-b}^{b} f(-x)\mathrm{d}x.$$

7. 证明: $\int_x^1 \frac{\mathrm{d}x}{1+x^2} = \int_1^{\frac{1}{x}} \frac{\mathrm{d}x}{1+x^2}(x>0)$.

证 令 $x=\frac{1}{u}$，则 $\mathrm{d}x = -\frac{1}{u^2}\mathrm{d}u$，$x=x$，$u=\frac{1}{x}$；$x=1$，$u=1$，于是

$$\int_x^1 \frac{\mathrm{d}x}{1+x^2} = \int_{\frac{1}{x}}^1 \frac{-\frac{1}{u^2}}{1+\frac{1}{u^2}}\mathrm{d}u = \int_1^{\frac{1}{x}} \frac{1}{u^2+1}\mathrm{d}u = \int_1^{\frac{1}{x}} \frac{1}{x^2+1}\mathrm{d}x,$$

故 $$\int_x^1 \frac{\mathrm{d}x}{1+x^2} = \int_1^{\frac{1}{x}} \frac{\mathrm{d}x}{1+x^2} (x>0).$$

8. 证明: $\int_0^1 x^m (1-x)^n \mathrm{d}x = \int_0^1 x^n (1-x)^m \mathrm{d}x$.

证 令$1-x=t$则$x=1-t$, $\mathrm{d}x=-\mathrm{d}t$, $x=0$, $t=1$; $x=1$, $t=0$, 于是

$$\int_0^1 x^m (1-x)^n \mathrm{d}x = \int_1^0 (1-t)^m t^n (-\mathrm{d}t) = \int_0^1 t^n (1-t)^m \mathrm{d}t$$

$$= \int_0^1 x^n (1-x)^m \mathrm{d}x.$$

9. 证明: $\int_0^{\pi} \sin^n x \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x$.

证 $$\int_0^{\pi} \sin^n x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \sin^n x \mathrm{d}x,$$

而 $$\int_{\frac{\pi}{2}}^{\pi} \sin^n x \mathrm{d}x \xlongequal{令x=\pi-t} \int_{\frac{\pi}{2}}^{0} \sin^n (\pi-t)(-\mathrm{d}t)$$

$$= \int_0^{\frac{\pi}{2}} \sin^n t \mathrm{d}t = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x,$$

故 $$\int_0^{\pi} \sin^n x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x + \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = 2\int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x.$$

10. 若$f(t)$是连续函数且为奇函数, 证明$\int_0^x f(t)\mathrm{d}t$是偶函数; 若$f(t)$是连续函数且为偶函数, 证明$\int_0^x f(t)\mathrm{d}t$是奇函数.

证 令$F(x) = \int_0^x f(t)\mathrm{d}t$, 则$F(-x) = \int_0^{-x} f(t)\mathrm{d}t$.

令$t=-u$则$\mathrm{d}t=-\mathrm{d}u$, $t=0$, $u=0$; $t=-x$, $u=x$, 于是

$$F(-x) = \int_0^{-x} f(-u)(-\mathrm{d}u) = -\int_0^x f(-u)\mathrm{d}u.$$

若$f(t)$是连续奇函数, 则$f(-u) = -f(u)$, 因而

$$F(-x) = -\int_0^x f(-u)\mathrm{d}u = \int_0^x f(u)\mathrm{d}u$$

$$= \int_0^x f(t)\mathrm{d}t = F(x),$$

故$\int_0^x f(t)\mathrm{d}t$是偶函数.

类似地, 若$f(t)$是连续偶函数, 则$f(-u) = f(u)$, 因而

$$F(-x)=-\int_0^x f(u)\mathrm{d}u=-\int_0^x f(t)\mathrm{d}t=-F(x),$$

故 $\int_0^x f(t)\mathrm{d}t$ 是奇函数.

11. 计算下列定积分:

(1) $\int_0^{\frac{1}{2}}\arcsin x\mathrm{d}x$; (2) $\int_0^1 \mathrm{e}^{\sqrt{x}}\mathrm{d}x$;

(3) $\int_{\frac{1}{\sqrt{2}}}^1 \frac{\sqrt{1-x^2}}{x^2}\mathrm{d}x$; (4) $\int_0^1 x\mathrm{e}^{2x}\mathrm{d}x$;

(5) $\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}|\ln x|\mathrm{d}x$; (6) $\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x$;

(7) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{x}{\sin^2 x}\mathrm{d}x$; (8) $\int_1^4 \frac{\ln x}{\sqrt{x}}\mathrm{d}x$;

(9) $\int_0^1 x\mathrm{e}^{-x}\mathrm{d}x$; (10) $\int_0^{\frac{\pi^2}{4}}\cos\sqrt{x}\mathrm{d}x$;

(11) $\int_0^{\pi}(x\sin x)^2\mathrm{d}x$; (12) $\int_0^1 (1-x^2)^{\frac{m}{2}}\mathrm{d}x$ (m 为自然数);

(13) $J_m=\int_0^{\pi}x\sin^m x\mathrm{d}x$ (m 为自然数).

解 (1) $\int_0^{\frac{1}{2}}\arcsin x\mathrm{d}x = x\arcsin x\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{x}{\sqrt{1-x^2}}\mathrm{d}x=\frac{\pi}{12}+\int_0^{\frac{1}{2}}\mathrm{d}\sqrt{1-x^2}$

$$=\frac{\pi}{12}+\sqrt{1-x^2}\Big|_0^{\frac{1}{2}}=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1.$$

(2) 令 $t=\sqrt{x}$, 则 $\mathrm{d}x=2t\mathrm{d}t$, $x=0$, $t=0$; $x=1$, $t=1$, 于是

$$\int_0^1 \mathrm{e}^{\sqrt{x}}\mathrm{d}x=2\int_0^1 t\mathrm{e}^t\mathrm{d}t=2\int_0^1 t\mathrm{d}\mathrm{e}^t=2t\mathrm{e}^t\Big|_0^1-2\int_0^1\mathrm{e}^t\mathrm{d}t$$

$$=2\mathrm{e}-2\mathrm{e}^t\Big|_0^1=2\mathrm{e}-2\mathrm{e}+2=2.$$

(3) $\int_{\frac{1}{\sqrt{2}}}^1 \frac{\sqrt{1-x^2}}{x^2}\mathrm{d}x=-\int_{\frac{1}{\sqrt{2}}}^1\sqrt{1-x^2}\,\mathrm{d}\frac{1}{x}=-\frac{\sqrt{1-x^2}}{x}\Bigg|_{\frac{1}{\sqrt{2}}}^1-\int_{\frac{1}{\sqrt{2}}}^1\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$$=1-\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x=1-\arcsin x\Big|_{\frac{1}{\sqrt{2}}}^{1}$$

$$=1-\frac{\pi}{2}+\frac{\pi}{4}=1-\frac{\pi}{4}.$$

(4) $\int_0^1 x\mathrm{e}^{2x}\mathrm{d}x=\frac{1}{2}\int_0^1 x\mathrm{d}\mathrm{e}^{2x}=\frac{1}{2}x\mathrm{e}^{2x}\Big|_0^1-\frac{1}{2}\int_0^1\mathrm{e}^{2x}\mathrm{d}x$

$$=\frac{\mathrm{e}^2}{2}-\frac{1}{4}\mathrm{e}^{2x}\Big|_0^1=\frac{\mathrm{e}^2}{2}-\frac{\mathrm{e}^2}{4}+\frac{1}{4}=\frac{1}{4}(\mathrm{e}^2+1).$$

(5) 由于在$[\frac{1}{\mathrm{e}},1]$上，$\ln x<0$知$|\ln x|=-\ln x$；在$[1,\mathrm{e}]$上，$\ln x>0$知$|\ln x|=\ln x$，故

$$\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}|\ln x|\mathrm{d}x=-\int_{\frac{1}{\mathrm{e}}}^{1}\ln x\mathrm{d}x+\int_1^{\mathrm{e}}\ln x\mathrm{d}x,$$

而 $$\int\ln x\mathrm{d}x=x\ln x-\int x\cdot\frac{1}{x}\mathrm{d}x=x\ln x-x+C,$$

于是 $$\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}}|\ln x|\mathrm{d}x=-[x\ln x-x]\Big|_{\frac{1}{\mathrm{e}}}^{1}+[x\ln x-x]\Big|_1^{\mathrm{e}}$$

$$=-[(\ln 1-1)-(\frac{1}{\mathrm{e}}\ln\frac{1}{\mathrm{e}}-\frac{1}{\mathrm{e}})]+[(\mathrm{e}\ln\mathrm{e}-\mathrm{e})-(\ln 1-1)]=2-\frac{2}{\mathrm{e}}.$$

(6) $\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x=\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\mathrm{d}\sin x=\mathrm{e}^{2x}\sin x\Big|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\sin x\mathrm{e}^{2x}\mathrm{d}x$

$$=\mathrm{e}^{\pi}+2\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\mathrm{d}\cos x$$

$$=\mathrm{e}^{\pi}+2\mathrm{e}^{2x}\cos x\Big|_0^{\frac{\pi}{2}}-4\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x$$

$$=\mathrm{e}^{\pi}-2-4\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x,$$

故 $$5\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x=\mathrm{e}^{\pi}-2,$$

即 $$\int_0^{\frac{\pi}{2}}\mathrm{e}^{2x}\cos x\mathrm{d}x=\frac{\mathrm{e}^{\pi}-2}{5}.$$

(7) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{x}{\sin^2 x}\mathrm{d}x=-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}x\mathrm{d}(\cot x)=-(x\cot x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\cot x\mathrm{d}x$

$$=(\frac{1}{4}-\frac{\sqrt{3}}{9})\pi+\ln\sin x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=(\frac{1}{4}-\frac{\sqrt{3}}{9})\pi+\frac{1}{2}\ln\frac{3}{2}.$$

(8) $\int_1^4\frac{\ln x}{\sqrt{x}}\mathrm{d}x=2\int_1^4\ln x\mathrm{d}\sqrt{x}=2[\sqrt{x}\ln x\Big|_1^4-\int_1^4\sqrt{x}\mathrm{d}\ln x]$

$$=8\ln 2-2\int_1^4 x^{-\frac{1}{2}}\mathrm{d}x$$

$$=8\ln 2-2\cdot 2\cdot x^{\frac{1}{2}}\Big|_1^4=8\ln 2-4.$$

(9) $\int_0^1 x\mathrm{e}^{-x}\mathrm{d}x=-\int_0^1 x\mathrm{d}\mathrm{e}^{-x}=-[x\mathrm{e}^{-x}\Big|_0^1-\int_0^1\mathrm{e}^{-x}\mathrm{d}x]$

$$=-[(x\mathrm{e}^{-x}+\mathrm{e}^{-x})\Big|_0^1]=1-\frac{2}{\mathrm{e}}.$$

(10) 令$t=\sqrt{x}$，则$\mathrm{d}x=2t\mathrm{d}t$，$x=0$，$t=0$；$x=\frac{\pi^2}{4}$，$t=\frac{\pi}{2}$，于是

$$\int_0^{\frac{\pi^2}{4}}\cos\sqrt{x}\mathrm{d}x=2\int_0^{\frac{\pi}{2}}t\cos t\mathrm{d}t=2\int_0^{\frac{\pi}{2}}t\mathrm{d}\sin t$$

$$=2t\sin t\Big|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\sin t\mathrm{d}t=\pi+2\cos t\Big|_0^{\frac{\pi}{2}}=\pi-2.$$

(11) $\int_0^{\pi}(x\sin x)^2\mathrm{d}x=\int_0^{\pi}x^2\frac{1-\cos 2x}{2}\mathrm{d}x$

$$=\frac{1}{2}\int_0^{\pi}x^2\mathrm{d}x-\frac{1}{2}\int_0^{\pi}x^2\cos 2x\mathrm{d}x$$

$$=\frac{x^3}{6}\Big|_0^{\pi}-\frac{1}{4}\int_0^{\pi}x^2\mathrm{d}\sin 2x$$

$$=\frac{\pi^3}{6}-\frac{1}{4}[x^2\sin 2x\Big|_0^{\pi}-\int_0^{\pi}\sin 2x\cdot 2x\mathrm{d}x]$$

$$=\frac{\pi^3}{6}-\frac{1}{4}\int_0^{\pi}x\mathrm{d}\cos 2x=\frac{\pi^3}{6}-\frac{1}{4}[x\cos 2x\Big|_0^{\pi}-\int_0^{\pi}\cos 2x\mathrm{d}x]$$

$$=\frac{\pi^3}{6}-\frac{1}{4}[\pi-\frac{\sin 2x}{2}\Big|_0^{\pi}]=\frac{\pi^3}{6}-\frac{\pi}{4}.$$

(12) 令$x=\sin t$，则$\mathrm{d}x=\cos t\mathrm{d}t$，$x=0$，$t=0$；$x=1$，$t=\frac{\pi}{2}$，于是

$$\int_0^1 (1-x^2)^{\frac{m}{2}} dx = \int_0^{\frac{\pi}{2}} (\cos^2 t)^{\frac{m}{2}} \cos t dt = \int_0^{\frac{\pi}{2}} \cos^{m+1} t dt$$

$$= \begin{cases} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot m}{2\cdot 4\cdot 6\cdot\cdots\cdot(m+1)}\cdot\dfrac{\pi}{2}, & \text{当}m\text{为奇数时}, \\ \dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot m}{1\cdot 3\cdot 5\cdot\cdots\cdot(m+1)}, & \text{当}m\text{为偶数时}. \end{cases}$$

(13) 令 $x=\pi-t$，则 $dx=-dt$，$x=0$，$t=\pi$；$x=\pi$，$t=0$，于是

$$J_m = \int_\pi^0 (\pi-t)\sin^m(\pi-t)(-dt) = \int_0^\pi (\pi-t)\sin^m(\pi-t)dt$$

$$= \pi\int_0^\pi \sin^m t dt - \int_0^\pi t\sin^m t dt = \pi\int_0^\pi \sin^m x dx - \int_0^\pi x\sin^m x dx$$

所以有 $$J_m = \frac{\pi}{2}\int_0^\pi \sin^m x dx = \frac{\pi}{2}\cdot 2\cdot\int_0^{\frac{\pi}{2}} \sin^m x dx = \pi\int_0^{\frac{\pi}{2}} \sin^m x dx$$

$$= \begin{cases} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(m-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot m}\cdot\dfrac{\pi^2}{2}, & \text{当}m\text{为偶数时}, \\ \dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(m-1)}{1\cdot 3\cdot 5\cdot\cdots\cdot m}\pi, & \text{当}m\text{为大于1的奇数时}, \quad (J_1=\pi). \end{cases}$$

12. $f(x)$ 在 $(-\infty,+\infty)$ 上连续，且 $F(x)=\int_0^x (2t-x)f(t)dt$.

证明: (1) 若 $f(x)$ 是偶函数，则 $F(x)$ 也是偶函数;

(2) 若 $f(x)$ 单调递减，则 $F(x)$ 也单调递减.

证 (1) $F(-x)=\int_0^{-x}(2t+x)f(t)dt$.

令 $u=-t$，则 $du=-dt$，$t=0$，$u=0$；$t=-x$，$u=x$，从而

$$F(-x)=\int_0^x(-2u+x)f(-u)(-du)$$

$$=\int_0^x(2u-x)f(-u)du$$

$$=\int_0^x(2u-x)f(u)du=F(x), \qquad (\because f(-u)=f(u)).$$

(2) 由于 $f(x)$ 为连续函数，所以 $F(x)$ 为可导函数，且

$$F(x)=\int_0^x 2tf(t)dt - x\int_0^x f(t)dt.$$

对 $F(x)$ 两边关于 x 求导，有

$$F'(x)=2xf(x)-\int_0^x f(t)dt - xf(x) = xf(x)-\int_0^x f(t)dt$$

$$=\int_0^x f(x)dt - \int_0^x f(t)dt = \int_0^x [f(x)-f(t)]dt.$$

当 $0<t<x$ 时，由于 $f(x)$ 单调递减，可知有 $f(t)>f(x)$，即 $f(x)-f(t)<0$，故 $F'(x)<0$.

当 $x<t<0$ 时，有 $f(x)-f(t)>0$，同样有 $F'(x)<0$.

因此在 $(-\infty,+\infty)$ 内总有 $F'(x)<0$，故当 $f(x)$ 单调递减时，$F(x)$ 也必是单调递减函数.