
# 边界元方法基础

# Fundamentals of boundary element method

## Chapter 4: Numerical Methods

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# Outline


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2 Boundary discretization

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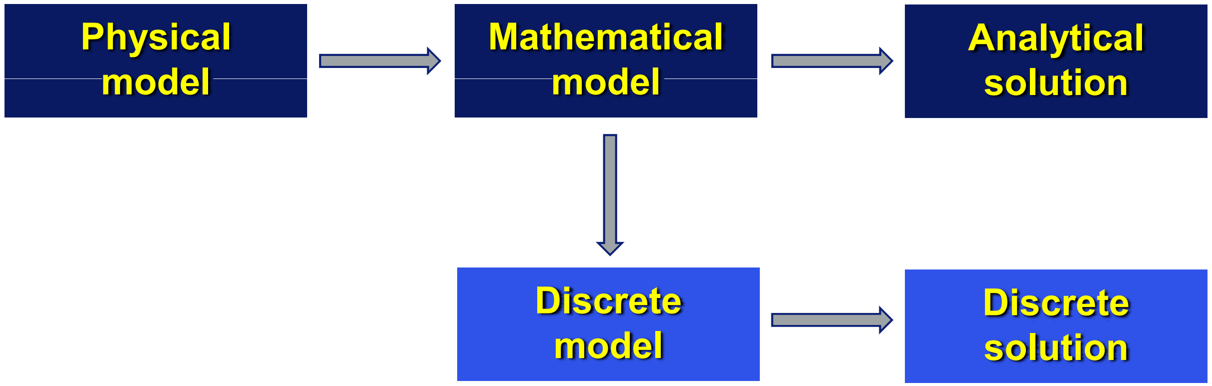
# Overview

# Overview



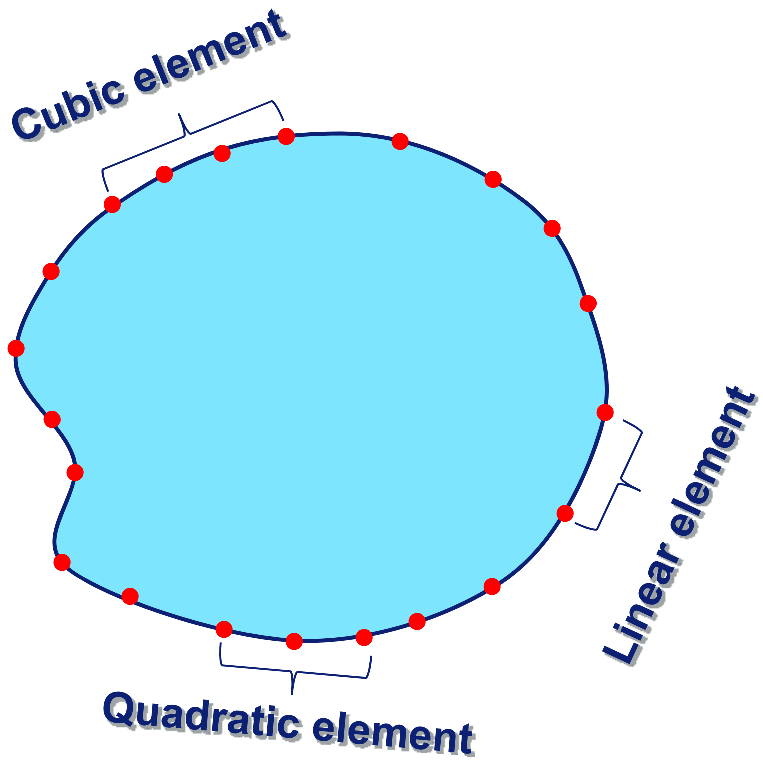

**Numerical methods**

- Discretization: boundary, integral equation
- integration
- Singularity
- linear equations

# Boundary discretization

## Boundary discretization

- **Discretize the boundary into a number of curvilinear elements**
  - Linear element – two nodes
  - Quadratic element – three nodes
  - Cubic element – four nodes




- **Constant element**
  - Parameters are constant on each element
  - Denotes by the central point

# Linear element

- **The geometry of each element can be represented by interpolation between the nodal points**

$$\begin{cases} x = \sum_{i=1}^{n} x_i N_i(\xi) \\ y = \sum_{i=1}^{n} y_i N_i(\xi) \end{cases}$$

$x_i, y_i$ - the coordinates at nodal points

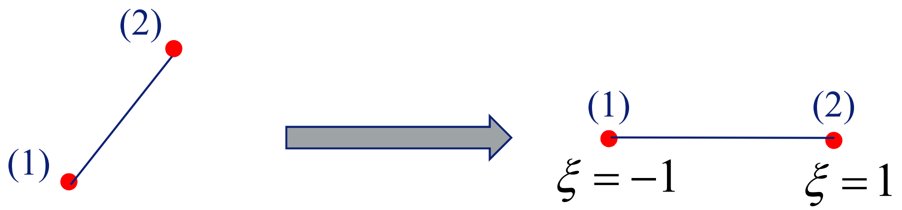
$N_i(\xi)$ - the shape functions defined on a master element with a local coordinate $-1 \le \xi \le 1$

$n$ - the number of nodes on the element

- **For linear element**

$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$




Real element

Master element

# Segment of the boundary

- A typical infinitesimal segment of the boundary can be evaluated by

$$\mathrm{d}S = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)^2}\,\mathrm{d}\xi = J\mathrm{d}\xi$$

where $J$ is the Jacobian, and

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = \sum_{i=1}^{n} x_i \frac{\mathrm{d}N_i(\xi)}{\mathrm{d}\xi}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = \sum_{i=1}^{n} y_i \frac{\mathrm{d}N_i(\xi)}{\mathrm{d}\xi}$$

- For linear element

$$\left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\xi} &= \frac{x_2 - x_1}{2} \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= \frac{y_2 - y_1}{2} \end{aligned}\right\} \Rightarrow J = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

# Numerical integration

# Numerical integration

- **Gaussian quadrature**
  - A **quadrature rule** is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration.
  - An *n*-point **Gaussian quadrature rule**

$$y = \int_{-1}^{1} f(x)\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i)$$

- **Example: 2-point Gaussian quadrature**
  - An integral over $[a, b]$ must be changed into an integral over $[-1, 1]$ before applying the Gaussian quadrature rule

$$y = \int_{a}^{b} f(x)\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f(\frac{b-a}{2}x + \frac{b+a}{2})\mathrm{d}x$$

  - Applying the Gaussian quadrature rule then results in the following approximation

$$y = \int_{a}^{b} f(x)\mathrm{d}x \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f(\frac{b-a}{2}x_i + \frac{b+a}{2})$$

# Gaussian integration points and the weights

| Number of points, $n$ | Points, $x_i$ | Approximately, $x_i$ | Weights, $w_i$ | Approximately, $w_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 2 |
| 2 | $\pm\frac{1}{\sqrt{3}}$ | ±0.57735 | 1 | 1 |
| 3 | 0 | 0 | $\frac{8}{9}$ | 0.888889 |
| | $\pm\sqrt{\frac{3}{5}}$ | ±0.774597 | $\frac{5}{9}$ | 0.555556 |
| 4 | $\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$ | ±0.339981 | $\frac{18+\sqrt{30}}{36}$ | 0.652145 |
| | $\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$ | ±0.861136 | $\frac{18-\sqrt{30}}{36}$ | 0.347855 |
| 5 | 0 | 0 | $\frac{128}{225}$ | 0.568889 |
| | $\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$ | ±0.538469 | $\frac{322+13\sqrt{70}}{900}$ | 0.478629 |
| | $\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$ | ±0.90618 | $\frac{322-13\sqrt{70}}{900}$ | 0.236927 |

# Singularity treatments

# Singularity

$$p(\mathbf{x}) = -\int_S p(\mathbf{y}) \frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \mathrm{d}S$$

- **For 3D problems**

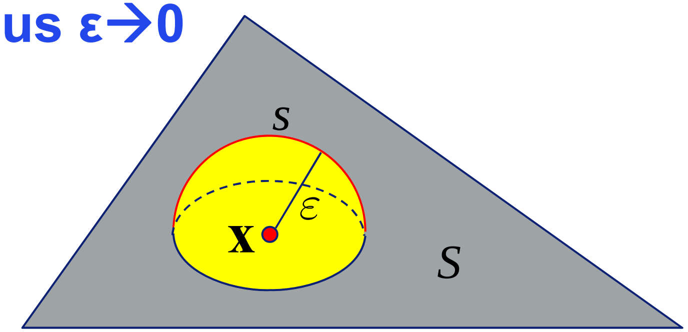
$$G_0(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r} e^{ikr} \Rightarrow \frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} = -\frac{e^{ikr}(ikr-1)}{4\pi r^2} \frac{\mathbf{r}\cdot\mathbf{n}(\mathbf{y})}{r}$$

- **If $\mathbf{y} \to \mathbf{x}$ , $G_0$ is singular.** $r \to 0$
  - Weakly singular $f(r) \to O(\ln r)$
  - Singular $f(r) \to O(r^{-1})$
  - Strongly singular $f(r) \to O(r^{-2})$
  - Hypersingular $f(r) \to O(r^{-n}), n \geq 3$
- **When the field point is located on the surface, the kernel of the integral in the right-hand side is strongly singular.**

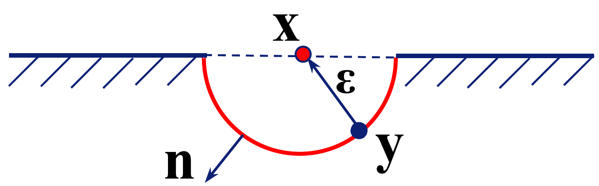
$$\frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \to O\left(r^{-2}\right), \quad r \to 0$$

# Integral equation for surface fields

- The surface is assumed to be augment by an hemisphere of radius ε→0




$$e^{ik\varepsilon} \to 1, \frac{\boldsymbol{\varepsilon}\cdot\mathbf{n}(\mathbf{y})}{\varepsilon} \to -1$$




$$p(\mathbf{x}) = -\int_{S-s} p(\mathbf{y})\frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\mathrm{d}S - \int_{s} p(\mathbf{y})\frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\mathrm{d}S$$

- For surface s, ε→0

$$\int_s \frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\mathrm{d}S = -\frac{e^{ik\varepsilon}(ik\varepsilon - 1)}{4\pi\varepsilon^2}\frac{\boldsymbol{\varepsilon}\cdot\mathbf{n}(\mathbf{y})}{\varepsilon}s$$

$$\to \frac{ik\varepsilon - 1}{4\pi\varepsilon^2}2\pi\varepsilon^2 = \frac{ik\varepsilon - 1}{2} \to -\frac{1}{2}$$

- The integral equations becomes

$$\boxed{\frac{1}{2}p(\mathbf{x}) = -\int_{S\setminus\mathbf{x}} p(\mathbf{y})\frac{\partial G_0(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})}\mathrm{d}S}$$

# Discretization of the integral equation

# Field on scattering surface

- For a unit monopole point source, the integral solution is

$$G(\mathbf{x},\mathbf{y}) = G_0(\mathbf{x},\mathbf{y}) - \int_{\mathrm{S}} G(\mathbf{z},\mathbf{y}) \left[ \frac{\partial G_0(\mathbf{x},\mathbf{z})}{\partial \mathbf{n}(\mathbf{z})} - \beta G_0(\mathbf{x},\mathbf{z}) \right] \mathrm{d}S(\mathbf{z})$$

- For constant elements, the solution can be written as

$$G(\mathbf{x},\mathbf{y}) = G_0(\mathbf{x},\mathbf{y}) - \sum_{n=1}^{M} G(\mathbf{z}_n,\mathbf{y}) \int_{\mathrm{S}} \left[ \frac{\partial G_0(\mathbf{x},\mathbf{z}_n)}{\partial \mathbf{n}(\mathbf{z}_n)} - \beta G_0(\mathbf{x},\mathbf{z}_n) \right] \mathrm{d}S(\mathbf{z}_n)$$

- For computing the $G(\mathbf{z}_n,\mathbf{y})$ , the observer is assumed to be located on the scattering surface, so that the integral equation is

$$G(\mathbf{z}_m,\mathbf{y}) = G_0(\mathbf{z}_m,\mathbf{y}) - \sum_{n=1}^{M} G(\mathbf{z}_n,\mathbf{y}) \int_{\mathrm{S}} \left[ \frac{\partial G_0(\mathbf{z}_m,\mathbf{z}_n)}{\partial \mathbf{n}(\mathbf{z}_n)} - \beta G_0(\mathbf{z}_m,\mathbf{z}_n) \right] \mathrm{d}S(\mathbf{z}_n)$$

# Discretization form of the integral equation

$$G(\mathbf{z}_m,\mathbf{y})+\sum_{n=1}^{M}G(\mathbf{z}_n,\mathbf{y})\int_{\mathrm{S}}\left[\frac{\partial G_0(\mathbf{z}_m,\mathbf{z}_n)}{\partial \mathbf{n}(\mathbf{z}_n)}-\beta G_0(\mathbf{z}_m,\mathbf{z}_n)\right]\mathrm{d}S(\mathbf{z}_n)=G_0(\mathbf{z}_m,\mathbf{y})$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MM} \end{bmatrix}\begin{bmatrix} G(\mathbf{z}_1,\mathbf{y}) \\ G(\mathbf{z}_2,\mathbf{y}) \\ \vdots \\ G(\mathbf{z}_M,\mathbf{y}) \end{bmatrix}=\begin{bmatrix} G_0(\mathbf{z}_1,\mathbf{y}) \\ G_0(\mathbf{z}_2,\mathbf{y}) \\ \vdots \\ G_0(\mathbf{z}_M,\mathbf{y}) \end{bmatrix}$$

or $\mathbf{AG}=\mathbf{G}_0$

$$I_1=\int_{\mathrm{S}_n}\beta G_0(\mathbf{z}_m,\mathbf{z}_n)\mathrm{d}S(\mathbf{z}_n)$$

$$I_2=\int_{\mathrm{S}_n}\frac{\partial G_0(\mathbf{z}_m,\mathbf{z}_n)}{\partial \mathbf{n}(\mathbf{z}_n)}\mathrm{d}S(\mathbf{z}_n)$$

$$A_{mn}=\begin{cases}\frac{1}{2}-I_1, & m=n,\\ I_2-I_1, & m\neq n.\end{cases}$$

# Linear system solver

# Iteration method

$$G(\mathbf{z}_m,\mathbf{y}) = G_0(\mathbf{z}_m,\mathbf{y}) - \sum_{n=1}^{M} G(\mathbf{z}_n,\mathbf{y}) \int_{\mathrm{S}} \left[ \frac{\partial G_0(\mathbf{z}_m,\mathbf{z}_n)}{\partial \mathbf{n}(\mathbf{z}_n)} - \beta G_0(\mathbf{z}_m,\mathbf{z}_n) \right] \mathrm{d}S(\mathbf{z}_n)$$

- **Let** $G^{(0)}(\mathbf{z},\mathbf{y}) = G_0(\mathbf{z},\mathbf{y})$
- **Use** $G^{(i-1)}(\mathbf{z},\mathbf{y})$ **to compute** $G^{(i)}(\mathbf{z},\mathbf{y})$**.**
- **Define**

$$err = \max\left\{ \mathrm{abs}\left( G^{(i)}(\mathbf{z}_m,\mathbf{y}) - G^{(i-1)}(\mathbf{z}_m,\mathbf{y}) \right) \Big/ \mathrm{abs}\left( G^{(i)}(\mathbf{z}_m,\mathbf{y}) \right) \right\}$$

**if** $err$ **is less than a given small value** $\varepsilon$**, for example,** $\varepsilon = 1\%$ **, the solution is converged.**

To be continued.

Thank you!

Email me if you have any questions:

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