Review of Lecture #15/第15次课复习

Chapter 10 Compressible Flow through Nozzles, Diffusers, and Wind Tunnels

第十章 通过喷管、扩压器和风洞的可压缩流

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National Key Laboratory of Science and Technology on Aerodynamic Design and Research

路线图

Governing equations for quasi-one-dimensional flow
(准一维流动控制方程的推导)

Nozzle flows
(喷管流动)

Diffusers
(扩压器)

Supersonic wind tunnels
(超声速风洞)

10.3节 喷管流动问题的补充：

10.3 节有关喷管内各种流动的讨论，是基于给定了管道形状，即我们事先给定了$A=A(x)$的前提下的。当我们把所得结果看做是每一截面处的平均值时，本章给出的准一维流动理论能很好地估算管内的流动。

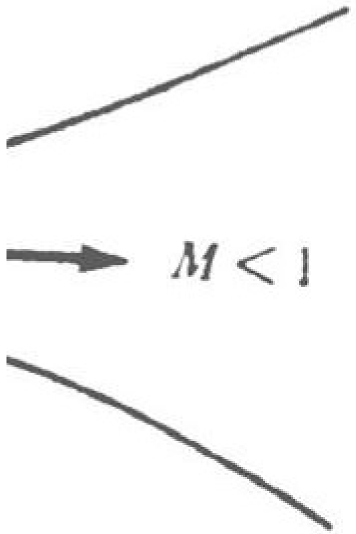
但是，这一理论不能告诉我们如何去设计喷管的形状，在现实中，如果喷管壁没有得到适当的设计，就可能出现斜激波，而得不到我们需要的等熵超声速流。如果要设计喷管壁的外形，必须考虑三维真实流动。本书第十三章有采用特征线法设计超声速喷管扩张段的内容，感兴趣的同学可以自学。

10.4 Diffusers(扩压器)

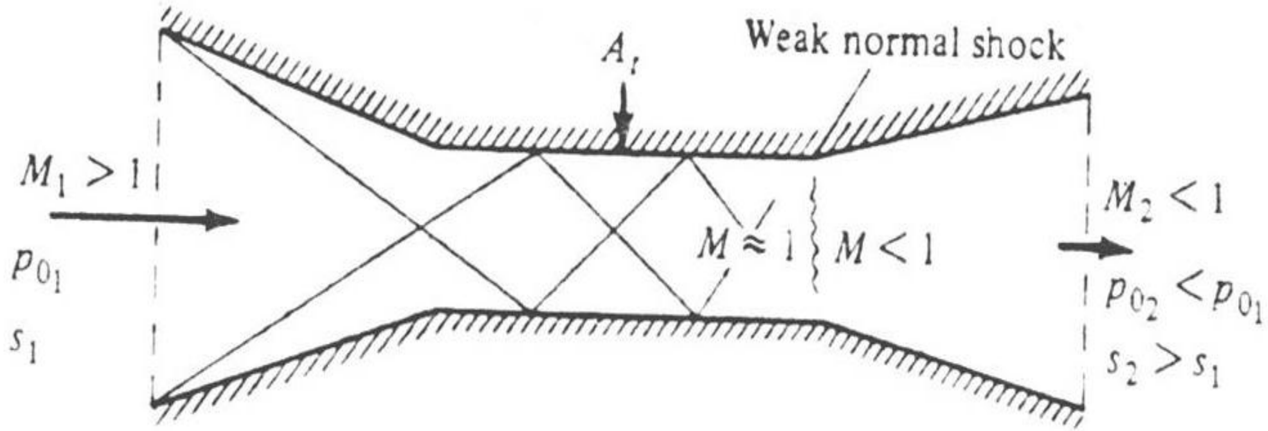
扩压器的作用：减速、增压。

扩压器的定义：

扩压器是这样的一段管道，它的作用是使气流以尽可能小的总压损失通过管道并在其出口降低速度。



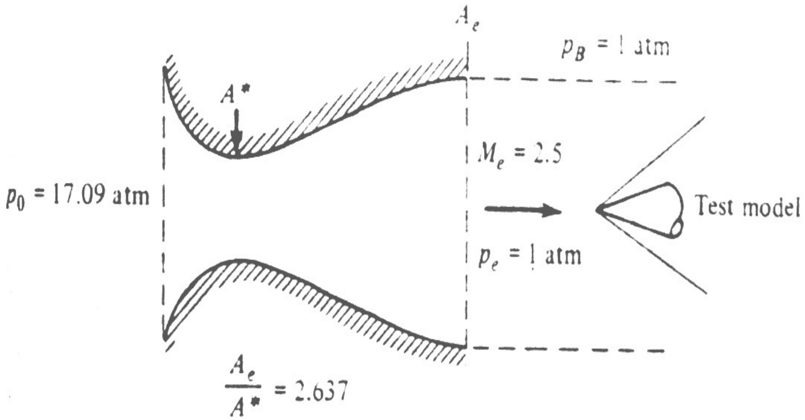
亚声速扩压器



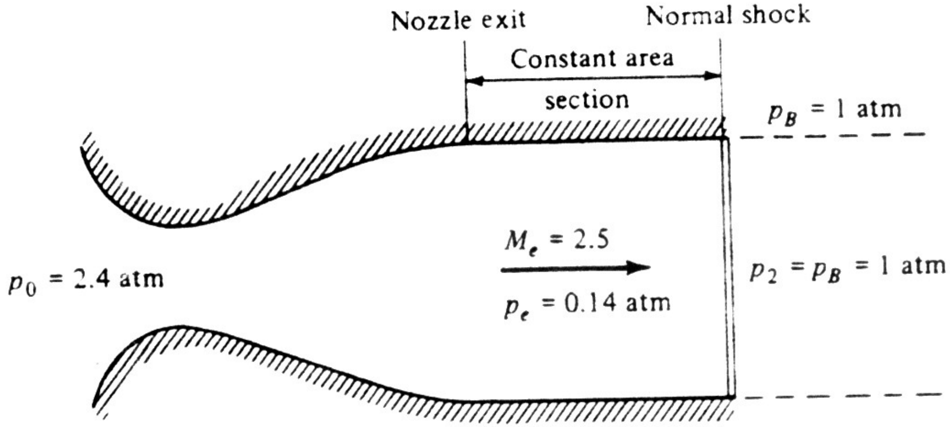
(b) Actual supersonic diffuser

超声速扩压器

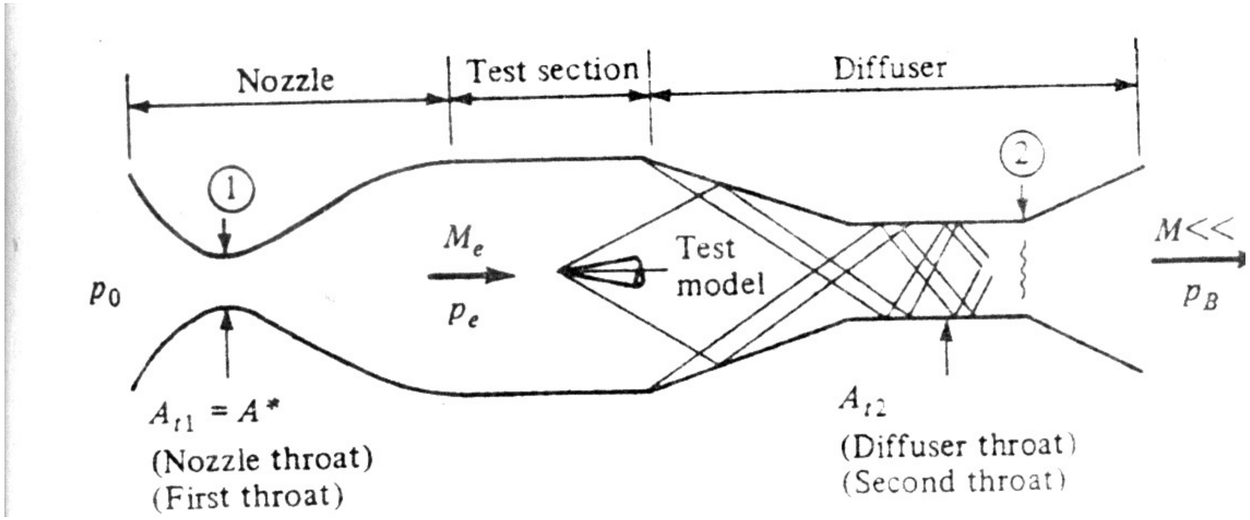
在实验室进行一个超声速飞行器的模型试验的三种实现方法：



（1）效率低、耗能大



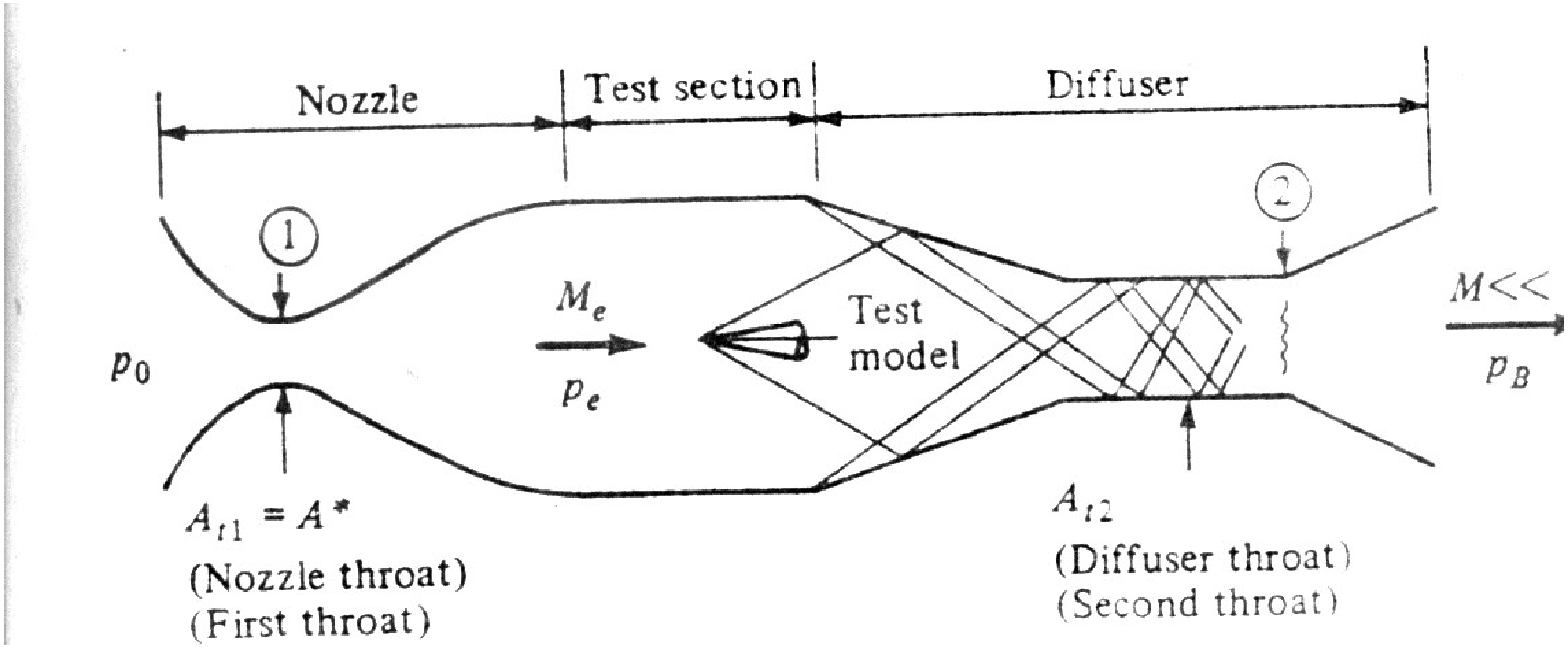
（2）虽然效率提高，耗能减小，但不能实用



（3）实用的超声速风洞布局，效率提高，耗能减小

- 超声速风洞第二喉道的讨论：

喷管段　　试验段　　扩压段



喷管喉道

第一喉道

扩压器喉道

第二喉道

第二喉道面积与第一喉道面积比

$$\boxed{\frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}}} \qquad \textbf{(10.39)}$$

结论：

$$\boxed{A_{t,2} > A_{t,1}}$$

10.6 Viscous Flow: Shock-Vave/Boundary Layer Interaction inside Nozzels（黏性流：喷管内的激波/边界层干扰）

准一维流无粘假设

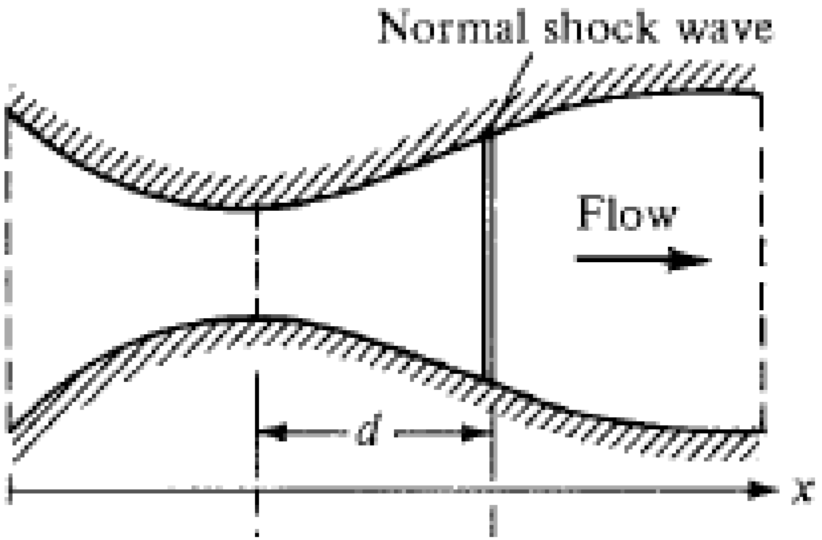


Figure 10.15 Supersonic nozzle flow with a normal shock inside the nozzle.

真实管内流动：激波/边界层干扰

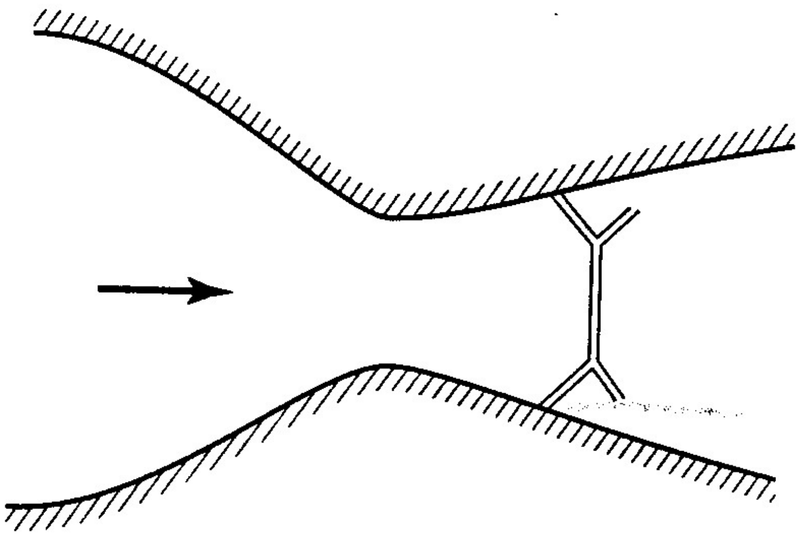
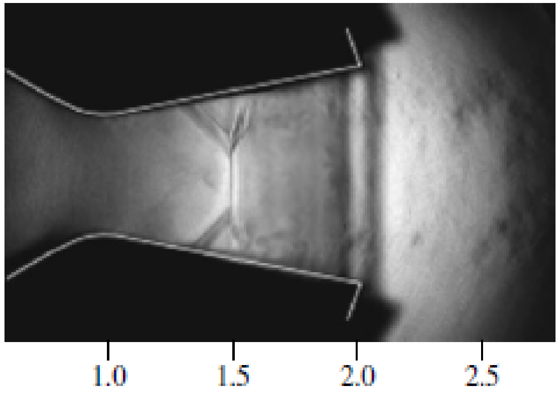


Figure 10.21 Sketch of an overexpanded nozzle flow with flow separation. (*Source: Craig Hunter, NASA.*).

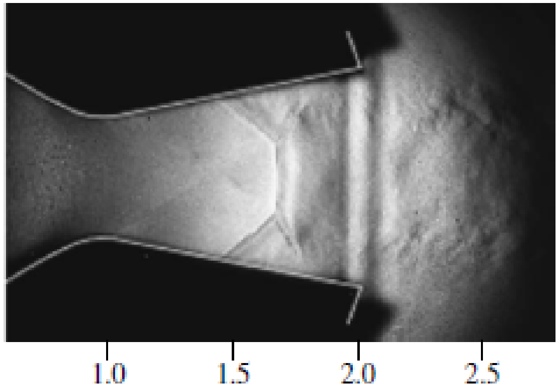
纹影照像（过膨胀喷管内的激波/边界层干扰）

作者：Craig A. Hunter*
单位：*NASA Langley Research Center, Hampton, Virginia 23681*

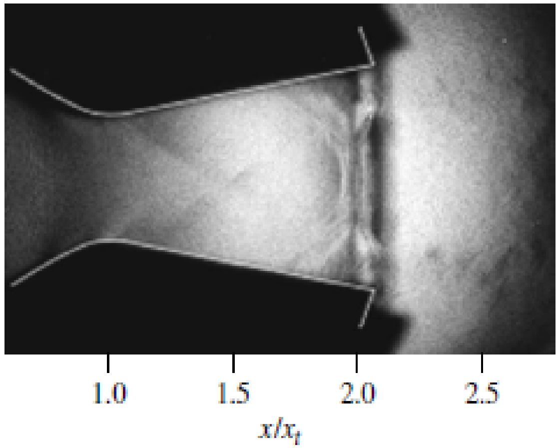
标题：
Experimental Investigation of Separated Nozzle Flow
JOURNAL OF PROPULSION AND POWER Vol. 20, No. 3, May–June 2004



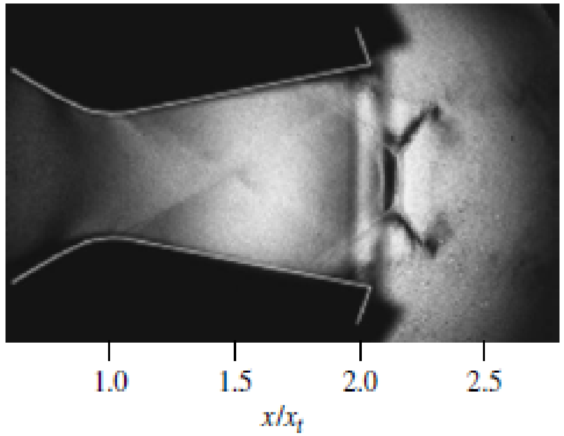
Exit-to-reservoir ratio=0.5



Exit-to-reservoir ratio=0.417



Exit-to-reservoir ratio=0.333



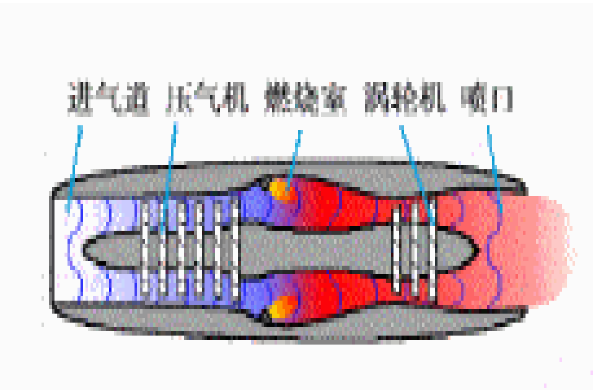
Exit-to-reservoir ratio=0.294

Figure 10.22 Schlieren photographs of the shock-wave/boundary-layer interaction inside an overexpanded nozzle flow. Exit-to-reservoir pressure ratio is (*a*) 0.5, (*b*) 0.417, (*c*) 0.333, (*d*) 0.294. (*Source: Craig Hunter, NASA.*)

结束语

本章学习的基本概念和基本关系式，其应用并不局限于我们在本章讨论的喷管、扩压器和超声速风洞。

我们讨论的准一维流动可以应用到涉及管道流的任何内流。例如，喷气式发动机的进气道，战斗机的推力矢量喷管，遵循的都是本章的气动原理。同样，火箭发动机实际上就是一个精心设计的、以得到最优推力为目标的超声速喷管。



涡喷发动机原理

战斗机推力矢量喷管

Apollo F-1火箭发动机喷管
（美国堪萨斯宇宙与太空中心博物馆）

投票 最多可选1项

设置

请对上一次课内容的掌握情况进行投票

- A 完全掌握了这部分知识内容
- B 掌握了大部分
- C 掌握了一小部分
- D 完全不懂

提交

Review of Lecture #15 ended!

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Lecture #16/第16次课

Chapter 11 Subsonic Compressible Flow Over Airfoils: Linear Theory

(绕翼型的可压缩亚声速流:线化理论)

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2019年11月27日 Wednesday

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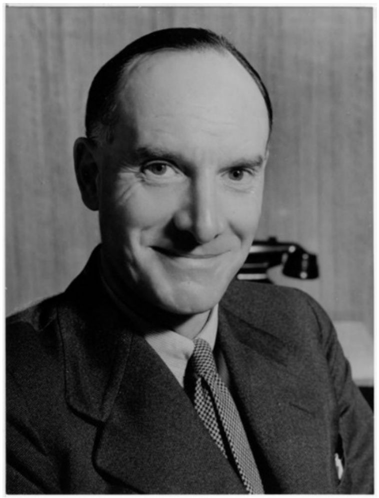
National Key Laboratory of Science and Technology on Aerodynamic Design and Research

历史故事：高速飞行的探路者DH-108



设计师DeHavilland爵士

DH-108（“吸血鬼”F.MK1的机身）

DH-108（TG283，TG306）

DH-108（VM120）

布局特点
- 大后掠：
  TG283-43度
  TG306-45度
- 无平尾
  俯仰和滚转控制由机翼后缘襟翼外侧的升降舵实现

11.1 Introduction/引言

The purpose of this chapter is to examine the properties of two-dimensional airfoils at Mach number above 0.3 but below 1, where we can no longer assume incompressible flow.

本章的目的是研究马赫数大于0.3小于1时二维翼型的流动特性，这时不可压假设不再成立。

Road Map for Chap.11.

Velocity potential equation

↓

Linearized velocity potential equation

- Prandtl-Glauert Compressibilty correction
- Improved compressibilty Correction

↓

Critical Mach number

↓

Drag-Divergence Mach number: Sound Barrier

↓

The area rule for transonic flow

↓

Supercritical airfoils

十一章路线图

速度势方程

↓

线化速度势方程 → 普朗特－葛劳特压缩性修正；改进的压缩性修正

↓

临界马赫数

↓

阻力发散马赫数：声障

↓

跨声速面积律

↓

超临界翼型

11.2 The Velocity Potential Equation (速度势方程)

For two-dimensional , steady, irrotational , isentropic flow, a velocity potential $\phi = \phi(x, y)$ can be defined such that :

$$V = \nabla \phi \tag{11.1}$$

or in terms of the Cartesian velocity components:

$$u = \frac{\partial \phi}{\partial x} \tag{11.2a}$$

$$v = \frac{\partial \phi}{\partial y} \tag{11.2b}$$

问题： 如何由无粘可压缩流动控制方程推导出速度势方程？

Question: How to derive the equation for ϕ according to the governing equations of a steady, inviscid and compressible flow ?

"速度势方程"的英文：

velocity potential equation

Let us proceed to obtain an equation for ϕ which represents a combination of the continuity, momentum, and energy equations. Such an equation would be very useful, because it would be simply one governing equation in terms of one unknown, namely ϕ.

现在我们开始推导一个用 ϕ 表示的方程, 这个方程代表了连续方程、动量方程及能量方程的结合。这样的方程（速度势方程）将非常有用，因为它把原有关于多个未知数的控制方程简化为了关于一个未知数 ϕ 的控制方程。

速度势方程推导

The continuity equation for steady, two-dimensional flow is :

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{11.3}$$

or

$$\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\frac{\partial v}{\partial y}=0 \tag{11.4}$$

Substituting $u=\frac{\partial \phi}{\partial x}, v=\frac{\partial \phi}{\partial y}$ into it, we get

$$\rho\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)+\frac{\partial \phi}{\partial x}\frac{\partial \rho}{\partial x}+\frac{\partial \phi}{\partial y}\frac{\partial \rho}{\partial y}=0 \tag{11.5}$$

速度势方程推导 （续）

To eliminate ρ from above equation, we consider the momentum equation :

$$dp = -\rho V dV \qquad (3.12)$$

$$dp = -\rho V dV = -\frac{\rho}{2} dV^2 = -\frac{\rho}{2} d(u^2 + v^2)$$

$$dp = -\frac{\rho}{2} d\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] \qquad (11.6)$$

Since the flow we are considering is isentropic, therefore

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \qquad (11.7)$$

速度势方程推导 （续）

Hence
$$d\rho = -\frac{\rho}{2a^2} d\left[(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 \right] \qquad (11.9)$$

$$\frac{\partial \rho}{\partial x} = -\frac{\rho}{2a^2} \frac{\partial}{\partial x}\left[(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 \right] \qquad (11.10)$$

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{2a^2} \frac{\partial}{\partial y}\left[(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 \right] \qquad (11.11)$$

Substitute them into

$$\rho(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}) + \frac{\partial \phi}{\partial x}\frac{\partial \rho}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \rho}{\partial y} = 0$$

速度势方程推导 （续）

We get the *velocity potential equation*:

$$\boxed{\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} -\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0} \tag{11.12}$$

In this equation , the speed of sound is also the function of ϕ :

$$a^2=a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right] \tag{11.13}$$

$$a_0^2=a_\infty^2+\frac{\gamma-1}{2}V_\infty^2=const.$$

$$\boxed{\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} -\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0} \qquad \textbf{(11.12)}$$

Eq. 11.12 represents a combination of continuity, momentum, energy equations. In principle, it can be solved to obtain Φ for the flow field around any two-dimensional shape, subject of course to the usual boundary conditions at infinity and along the body surface.

边界条件

- **The infinite boundary condition (远场边界条件) is**

$$u = \frac{\partial \phi}{\partial x} = V_\infty$$

$$v = \frac{\partial \phi}{\partial y} = 0$$

- **The wall boundary condition (物面边界条件) is**

$$\frac{\partial \phi}{\partial n} = 0$$

Once ϕ is known, all the other flow variables are directly obtained as follows:

1. Calculate u and v: $u = \frac{\partial \phi}{\partial x}$ $v = \frac{\partial \phi}{\partial y}$

2. Calculate a:

$$a = \sqrt{a_0^{\ 2} - \frac{\gamma - 1}{2}\left[(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 \right]}$$

3.Calculate *M*: $M = \frac{V}{a} = \frac{\sqrt{u^2 + v^2}}{a}$

4. Calculate T, p, ρ :

$$T = T_0 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1}$$

$$p = p_0 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}}$$

$$\rho = \rho_0 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma - 1}}$$

单选题 1分

设置

将下列内容按定常、无粘、可压缩无旋流动的速度势方程求解方法的正确步骤排列顺序。

①根据定常、绝热、无粘流动能量方程计算流场中每一点的声速；②求解定常、无粘、可压缩无旋流动速度势方程，得到速度势，对速度势求梯度得到每一点的速度；③利用等熵流关系式，以及自由来流的总温、总压、总密度，得到每一点的压强；④计算流场中每一点的马赫数；

A ①②④③

B ②④③①

C ③②①④

D ②①④③

提交

11.3 The Linearized Velocity Potential Equation 线化速度势方程

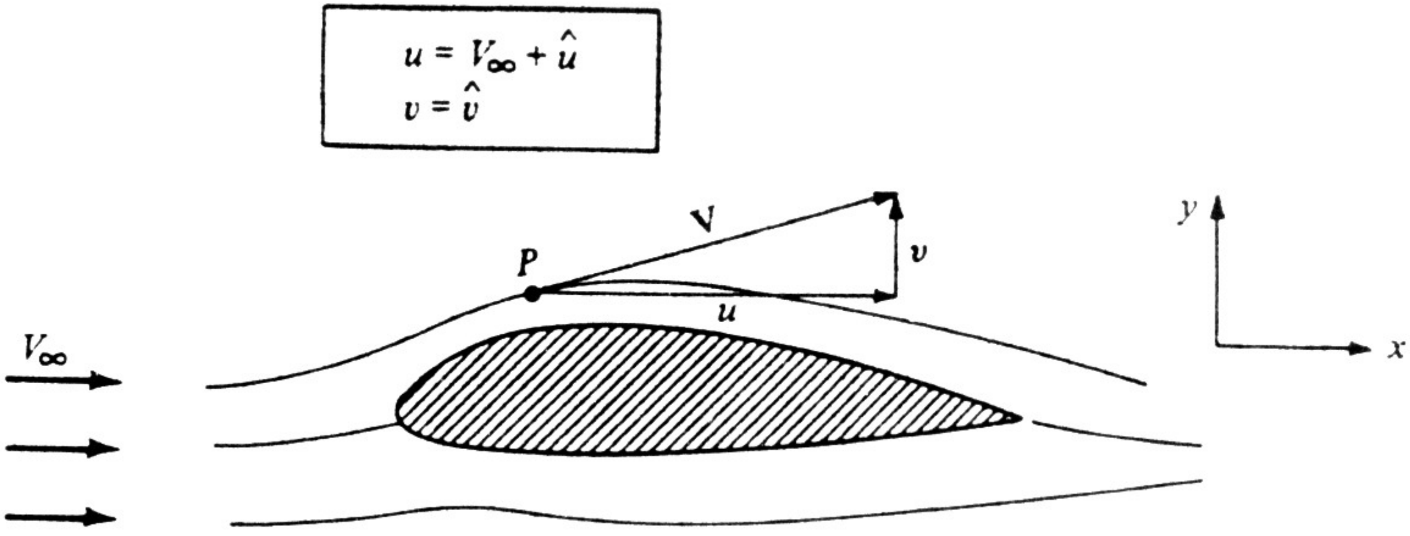


FIGURE 11.2
Uniform flow and perturbed flow.

If we define $u = V_\infty + \hat{u}$, $v = \hat{v}$, we call $\hat{u}$ and $\hat{v}$ the perturbation velocities (扰动速度).

扰动速度势与扰动速度势方程

因此，我们可定义一个扰动速度势 $\hat{\phi}$

有 $\phi = V_\infty x + \hat{\phi}$

代入方程（11.12), 两边同时乘以 a^2 ：

$$\left[a^2 - (V_\infty + \frac{\partial \hat{\phi}}{\partial x})^2\right]\frac{\partial^2 \hat{\phi}}{\partial x^2} + \left[a^2 - (\frac{\partial \hat{\phi}}{\partial y})^2\right]\frac{\partial^2 \hat{\phi}}{\partial y^2} - 2(V_\infty + \frac{\partial \hat{\phi}}{\partial x})(\frac{\partial \hat{\phi}}{\partial y})\frac{\partial^2 \hat{\phi}}{\partial x \partial y} = 0 \tag{11.14}$$

This equation is called the ***perturbation velocity potential equation***(该方程称为**扰动速度势方程**).

我们将方程（11.14) 重新写成扰动速度的形式

$$[a^2-(V_\infty+\hat{u})^2]\frac{\partial\hat{u}}{\partial x}+(a^2-\hat{v}^2)\frac{\partial\hat{v}}{\partial y}-2(V_\infty+\hat{u})\hat{v}\frac{\partial\hat{u}}{\partial y}=0 \qquad (11.14a)$$

能量方程如下

$$\frac{a_\infty^2}{\gamma-1}+\frac{V_\infty^2}{2}=\frac{a^2}{\gamma-1}+\frac{(V_\infty+\hat{u})^2+\hat{v}^2}{2} \qquad (11.15)$$

或

$$a^2=a_\infty^2-\frac{\gamma-1}{2}(2V_\infty\hat{u}+\hat{u}^2+\hat{v}^2) \qquad (11.15a)$$

将方程(11.15a) 带入 (11.14a), 我们得到：

$$(1-M_\infty^2)\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}=M_\infty^2[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}]\frac{\partial\hat{u}}{\partial x}$$

$$+M_\infty^2[(\gamma-1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}]\frac{\partial\hat{v}}{\partial y}$$

$$+M_\infty^2[\frac{\hat{v}}{V_\infty}(1+\frac{\hat{u}}{V_\infty})(\frac{\partial\hat{u}}{\partial y}+\frac{\partial\hat{v}}{\partial x})] \tag{11.16}$$

Eq.(11.16) is still exact for inviscid , irrotational, isentropic flow, whenever the perturbation is small or large.

该方程对于无粘、无旋等熵流动，无论扰动大还是小，仍然精确成立。

小扰动假设（small perturbation)

在图11.2中，假设物体为细长体，且攻角为小攻角，则物体引起的扰动比较小

$$\frac{\hat{u}}{V_\infty},\frac{\hat{v}}{V_\infty}\langle\langle 1, \qquad \frac{\hat{u}^2}{V_\infty^2},\frac{\hat{v}^2}{V_\infty^2}\langle\langle\langle 1$$

且：扰动速度 $\hat{u}$ 和 $\hat{v}$ 的导数也是小量

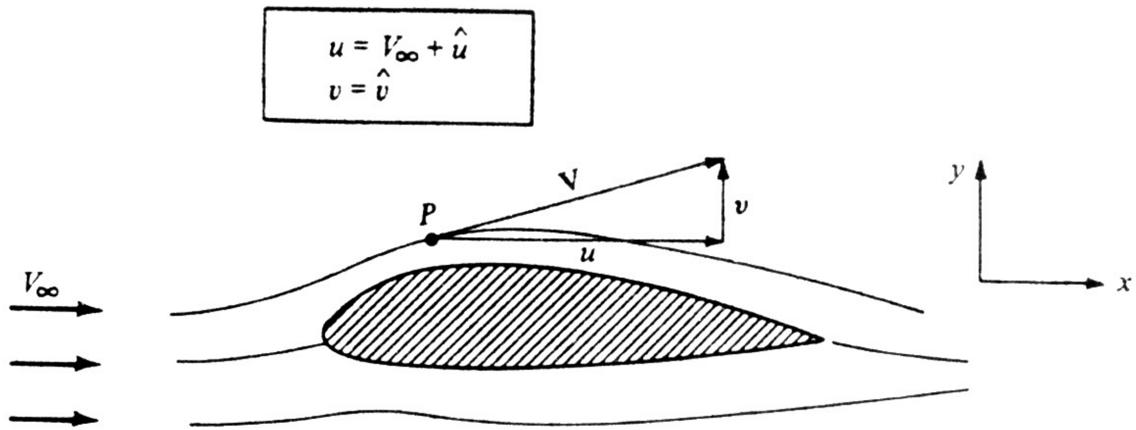


FIGURE 11.2
Uniform flow and perturbed flow.

小扰动线化速度势方程 （续）

$$(1-M_\infty^2)\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}=M_\infty^2[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}]\frac{\partial\hat{u}}{\partial x}$$

$$+M_\infty^2[(\gamma-1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}]\frac{\partial\hat{v}}{\partial y} \qquad (11.16)$$

$$+M_\infty^2[\frac{\hat{v}}{V_\infty}(1+\frac{\hat{u}}{V_\infty})(\frac{\partial\hat{u}}{\partial y}+\frac{\partial\hat{v}}{\partial x})]$$

1. If $0\le M_\infty\le 0.8$ **or** $M_\infty\ge 1.2$ **, then we have**

$$M_\infty^2[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}]\frac{\partial\hat{u}}{\partial x}<<<(1-M_\infty^2)\frac{\partial\hat{u}}{\partial x}$$

小扰动线化速度势方程

$$(1-M_\infty^2)\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}=\mathbf{0}$$

$$+M_\infty^2[(\gamma-1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}]\frac{\partial\hat{v}}{\partial y} \quad (11.16)$$

$$+M_\infty^2[\frac{\hat{v}}{V_\infty}(1+\frac{\hat{u}}{V_\infty})(\frac{\partial\hat{u}}{\partial y}+\frac{\partial\hat{v}}{\partial x})]$$

1. If $0\le M_\infty\le 0.8$ **or** $M_\infty\ge 1.2$ **, then we have**

$$M_\infty^2[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}]\frac{\partial\hat{u}}{\partial x}<<<(1-M_\infty^2)\frac{\partial\hat{u}}{\partial x}$$

小扰动线化速度势方程 （续）

$$(1-M_\infty^2)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}= \mathbf{0}$$

$$+\ \mathbf{0} \qquad \text{⌊.16)}$$

$$+M_\infty^2[\frac{\hat{v}}{V_\infty}(1+\frac{\hat{u}}{V_\infty})(\frac{\partial \hat{u}}{\partial y}+\frac{\partial \hat{v}}{\partial x})]$$

2. For $M_\infty < 5$, approximately we have

$$M_\infty^2[(\gamma-1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}]\frac{\partial \hat{v}}{\partial y} <<< \frac{\partial \hat{v}}{\partial y}$$

小扰动线化速度势方程 （续）

$$(1-M_\infty^2)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}= \mathbf{0}$$

$$+\ \mathbf{0} \qquad (11.16)$$

$$+\ \mathbf{0}$$

3. For $M_\infty < 5$, approximately we have

$$M_\infty^2\left[\frac{\hat{v}}{V_\infty}(1+\frac{\hat{u}}{V_\infty})(\frac{\partial \hat{u}}{\partial y}+\frac{\partial \hat{v}}{\partial x})\right]\approx 0$$

线化小扰动速度势方程

$$(1-M_\infty^2)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=0 \longrightarrow \boxed{(1-M_\infty^2)\frac{\partial^2 \hat{\phi}}{\partial x^2}+\frac{\partial^2 \hat{\phi}}{\partial y^2}=0} \qquad (11.18)$$

Read P 719

Examine Eq.(11.18). It is a linear partial differential equation, and therefore is inherently simpler than its parent equation. However, we have paid a price for this simplification. It is no longer exact. It is only an approximation to the physics of the flow. It is not valid for transonic flows $0.8 \le M_\infty \le 1.2$ and hypersonic flows $M_\infty > 5$.

该方程是一个线性偏微分方程，相比原方程来说，大大简化了。但是，由于进行了线化，我们付出了精度代价，该方程只是对满足小扰动条件的真实物理问题的近似。对于跨声速流动和高超声速流动，该方程不成立。

线化小扰动速势方程的适用条件

Due to the assumptions made in obtaining Eq. (11.18), it is reasonably valid for the following situations（小扰动速势方程的适用条件）:

1、*Small* perturbation, i.e., thin bodies at small angles of attack (小扰动，即小攻角下的薄物体）

2、*Subsonic* and *supersonic* Mach numbers (亚声速或超声速）

$$0 \le M_\infty \le 0.8 \qquad 1.2 \le M_\infty < 5$$

讨论问题：绕翼型流动是否处处满足小扰动假设?

具有钝前缘和后缘角的翼型在前、后缘处不满足小扰动假设。

补充：跨声速小扰动速度势方程的推导

用扰动速度表示的精确成立的方程为：

$$(1-M_\infty^2)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=M_\infty^2[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}]\frac{\partial \hat{u}}{\partial x}$$
$$+M_\infty^2[(\gamma-1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}]\frac{\partial \hat{v}}{\partial y}$$
$$+M_\infty^2[\frac{\hat{v}}{V_\infty}(1+\frac{\hat{u}}{V_\infty})(\frac{\partial \hat{u}}{\partial y}+\frac{\partial \hat{v}}{\partial x})] \qquad (11.16)$$

跨声速时，由于 $(1-M_\infty^2)$ 量级较小，右端项中 $M_\infty^2(\gamma+1)\frac{\hat{u}}{V_\infty}\frac{\partial \hat{u}}{\partial x}$ 不能被忽略了。

因此，对应的跨声速小扰动速度势方程是:

$$(1-M_\infty^2)\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty}\right]\frac{\partial \hat{u}}{\partial x}$$

or

$$\{1-M_\infty^2[1+(\gamma+1)\frac{\hat{u}}{V_\infty}]\}\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}=0$$

$$[1-M_\infty^2(1+\frac{\gamma+1}{V_\infty}\frac{\partial \hat{\phi}}{\partial x})]\frac{\partial^2 \hat{\phi}}{\partial x^2}+\frac{\partial^2 \hat{\phi}}{\partial y^2}=0$$

注意：对于跨声速流，即使是小扰动方程也是非线性的，因此很难用理论方法求解。

钱学森大师的第一个博士生罗时均先生是我国最早实现跨声速小扰动速度势方程数值求解的。

压强系数 （Pressure coefficient）

压强系数的精确表达式为/Exact expression for pressure coefficient

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)} \qquad (11.22)$$

引入小扰动线化后的压强系数/Linearized pressure coefficient

$$\boxed{C_p = -\frac{2\hat{u}}{V_\infty}} \qquad (11.32)$$

小扰动线化压强系数 推导：

$$C_p \equiv \frac{p - p_\infty}{q_\infty} \qquad \textbf{(11.19)}$$

Since $$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2$$

$$= \frac{\gamma}{2} p_\infty \frac{1}{\gamma p_\infty / \rho_\infty} V_\infty^2 \qquad \textbf{(11.20)}$$

$$q_\infty = \frac{\gamma}{2} p_\infty \frac{V_\infty^2}{a_\infty^2} = \frac{\gamma}{2} p_\infty M_\infty^2 \qquad \textbf{(11.21)}$$

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)} \qquad \textbf{(11.22)}$$

小扰动线化压强系数 推导（续）：

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)} \qquad (11.22)$$

$$T + \frac{V^2}{2c_p} = T_\infty + \frac{V_\infty^2}{2c_p} \qquad (11.23)$$

$$c_p = \frac{\gamma R}{\gamma - 1} \qquad T - T_\infty = \frac{V^2 - V_\infty^2}{2\gamma R/(\gamma - 1)} \qquad (11.24)$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{\gamma R T_\infty} = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2} \qquad (11.25)$$

小扰动线化压强系数 推导（续）：

$$V^2 = (V_\infty + \hat{u})^2 + \hat{v}^2$$

$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2) \qquad \textbf{(11.26)}$$

$$\frac{p}{p_\infty} = (\frac{T}{T_\infty})^{\frac{\gamma}{\gamma-1}}$$

$$= \left[1 - \frac{\gamma - 1}{2a_\infty^2}(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2)\right]^{\frac{\gamma}{\gamma-1}}$$

小扰动线化压强系数 推导（续）：

$$\frac{p}{p_\infty}=\left[1-\frac{\gamma-1}{2}M_\infty^2\left(\frac{2\hat{u}}{V_\infty}+\frac{\hat{u}^2+\hat{v}^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}} \qquad \textbf{(11.27)}$$

(11.27)仍然是一个精确表达式。引入小扰动假设：

$$\frac{\hat{u}}{V_\infty}\ll 1, \qquad \frac{\hat{u}^2}{V_\infty^2}\lll 1, \qquad \frac{\hat{v}^2}{V_\infty^2}\lll 1$$

$$\frac{p}{p_\infty}=1-\frac{\gamma}{2}M_\infty^2\left(\frac{2\hat{u}}{V_\infty}+\frac{\hat{u}^2+\hat{v}^2}{V_\infty^2}\right)+\cdots \qquad \textbf{(11.30)}$$

小扰动线化压强系数 推导（续）：

$$C_p = -\frac{2\hat{u}}{V_\infty} - \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2} \qquad \textbf{(11.31)}$$

$$\boxed{C_p = -\frac{2\hat{u}}{V_\infty}} \qquad \textbf{(11.32)}$$

式（**11.32)**是压强系数的线化形式，只适用于小扰动情况；公式（**11.32)** 说明压强系数只依赖于x方向的扰动速度。

线化方程的边界条件/Boundary conditions for linearized equation

远场边界条件 $\hat{\phi} = \text{constant};\quad or\ \hat{u} = \hat{v} = 0$

物面边界条件 $$\tan\theta = \frac{v}{u} = \frac{\hat{v}}{V_\infty + \hat{u}} \tag{11.33}$$

上式中 θ 为物面切向与自由来流的夹角。

公式(11.33)为精确的流动与物面相切条件。在小扰动假设下，它变为

$$\tan\theta = \frac{\hat{v}}{V_\infty} \qquad \boxed{\frac{\partial\hat{\phi}}{\partial y} = V_\infty \tan\theta} \tag{11.34}$$

小结：本节推导的三个重要公式

$$\boxed{(1-M_\infty^2)\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}=0} \qquad \textbf{(11.18)}$$

$$\boxed{C_p=-\frac{2\hat{u}}{V_\infty}} \qquad \textbf{(11.32)}$$

$$\boxed{\frac{\partial\hat{\phi}}{\partial y}=V_\infty\tan\theta} \qquad \textbf{(11.34)}$$

方程（**11.18)**为亚声速或超声速小扰动速度势方程；方程（**11.32)**是线性化的压强系数表达式；方程（**11.34)**是物面流动相切条件的近似表达式。三个公式具有一致的精度。

主观题 2分

设置

小扰动线化速度势方程的适用条件是什么？

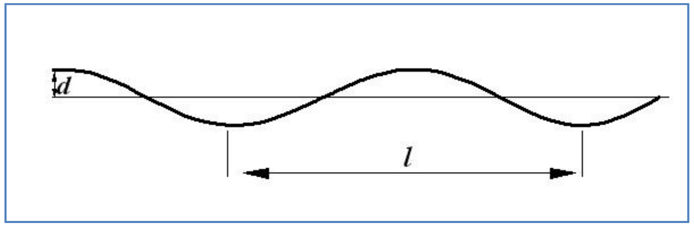
正常使用主观题需2.0以上版本雨课堂

作答

补充例题：亚声速小扰动流过波形壁的解

$$\beta^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

$$\boxed{y_s = d\cos(2\pi x / l)}$$



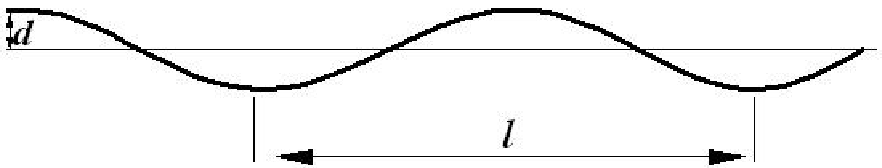
物面边界条件：
$$\left(\frac{\partial \hat{\phi}}{\partial y}\right)_{y=0} = V_\infty \frac{dy_s}{dx}$$

无穷远边界条件：
$$\frac{\partial \hat{\phi}}{\partial x} = 0$$

$$\frac{\partial \hat{\phi}}{\partial y} = 0$$

波形壁表面方程：

$$y_s = d\cos(2\pi x / l)$$



求解椭圆型线性偏微分方程的分离变量法：

$$\hat{\phi} = F(x)G(y)$$

$$\frac{\partial^2 \hat{\phi}}{\partial x^2} = F''(x)G(y)$$

$$\frac{\partial^2 \hat{\phi}}{\partial y^2} = F(x)G''(y)$$

代入方程： $$\beta^2\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}=0$$

得： $$\beta^2F''(x)G(y)+F(x)G''(y)=0$$

$$\frac{F''(x)}{F(x)}=-\frac{1}{\beta^2}\frac{G''(y)}{G(y)}=-k^2$$

$$F(x)=c_1\cos kx+c_2\sin kx$$

$$G(y)=c_3e^{\beta ky}+c_4e^{-\beta ky}$$

$$\hat{\phi} = (c_1 \cos kx + c_2 \sin kx)(c_3 e^{\beta ky} + c_4 e^{-\beta ky})$$

由无穷远边界条件 $\dfrac{\partial \hat{\phi}}{\partial y} = 0$ 可知：$c_3 = 0$

由物面边界条件 $\left.\dfrac{\partial \hat{\phi}}{\partial y}\right|_{y=0} = V_\infty \dfrac{dy_s}{dx}$ 可得：

$$-c_1 c_4 \beta k \cos kx - c_2 c_4 \beta k \sin kx = -V_\infty \frac{2\pi d}{l} \sin \frac{2\pi x}{l}$$

即 $c_1 c_4 = 0$ $\quad k = \dfrac{2\pi}{l}$

$$c_2 c_4 \beta k = V_\infty \frac{2\pi d}{l} \quad \Rightarrow \quad c_2 c_4 = V_\infty \frac{d}{\beta}$$

$$\hat{\phi} = \frac{V_\infty d}{\beta} e^{-\frac{2\pi\beta y}{l}} \sin\frac{2\pi x}{l}$$

$$\phi = V_\infty x + \frac{V_\infty d}{\sqrt{1-M_\infty{}^2}} e^{-\frac{2\pi\sqrt{1-M_\infty{}^2}\,y}{l}} \sin\frac{2\pi x}{l}$$

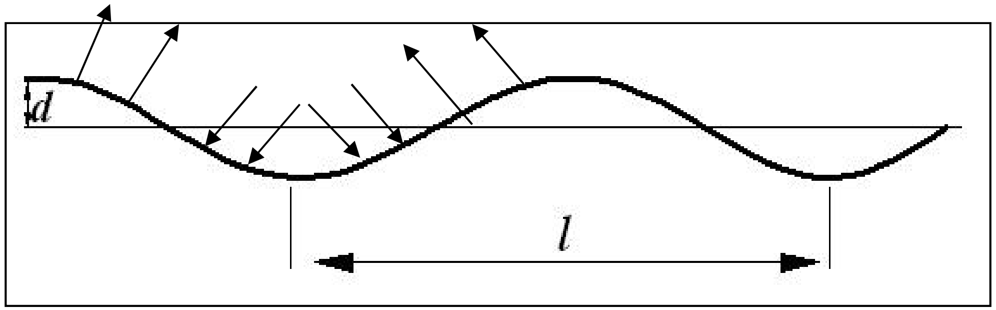
$$\widehat{u} = \frac{\partial \hat{\phi}}{\partial x} = \frac{2\pi V_\infty d}{\beta l} e^{-\frac{2\pi\sqrt{1-M_\infty^{\ 2}}\, y}{l}} \cos\frac{2\pi x}{l}$$

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial y} = -\frac{2\pi V_\infty d}{l} e^{-\frac{2\pi\sqrt{1-M_\infty^{\ 2}}\, y}{l}} \cos\frac{2\pi x}{l}$$

$$C_p = -\frac{2}{V_\infty}\frac{\partial \hat{\phi}}{\partial x} = -\frac{4\pi d}{\beta l} e^{-\frac{2\pi\sqrt{1-M_\infty{}^2}\,y}{l}} \cos\frac{2\pi x}{l}$$

$$(C_p)_{y=0} = -\frac{2}{V_\infty}\frac{\partial \hat{\phi}}{\partial x} = -\frac{4\pi d}{\beta l}\cos\frac{2\pi x}{l}$$

$$c_d = 0$$



作业：**Problem 11.1r**

更正：考虑直角坐标系下的亚声速可压缩流的速度势为：

$$\phi(x)=V_\infty x+\frac{V_\infty l}{10\sqrt{1-M_\infty{}^2}}e^{-2\pi\sqrt{1-M_\infty{}^2}\frac{y}{l}}\sin\frac{2\pi x}{l}$$

自由来流的 $V_\infty=700\,ft/s,\ \ p_\infty=1\text{atm},T_\infty=519^o R$

特征长度 $l=1\,ft$

计算点（**$(x,y)=($0.25ft,0.25ft$)$**处的***M*，*p*，*T*。**）

Lecture #16 ended！

衷心感谢同学们的支持！

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