第四章　平面问题的极坐标解答

要点：（1）极坐标中平面问题的基本方程：

——— 平衡方程、几何方程、物理方程、相容方程、边界条件。

（2）极坐标中平面问题的求解方法及应用

应用：圆盘、圆环、厚壁圆筒、楔形体、半无限平面体等的应力与变形分析。

主　要　内　容

§ 4-1 极坐标中的平衡微分方程

1. 极坐标中的微元体

体力：f_r, f_θ

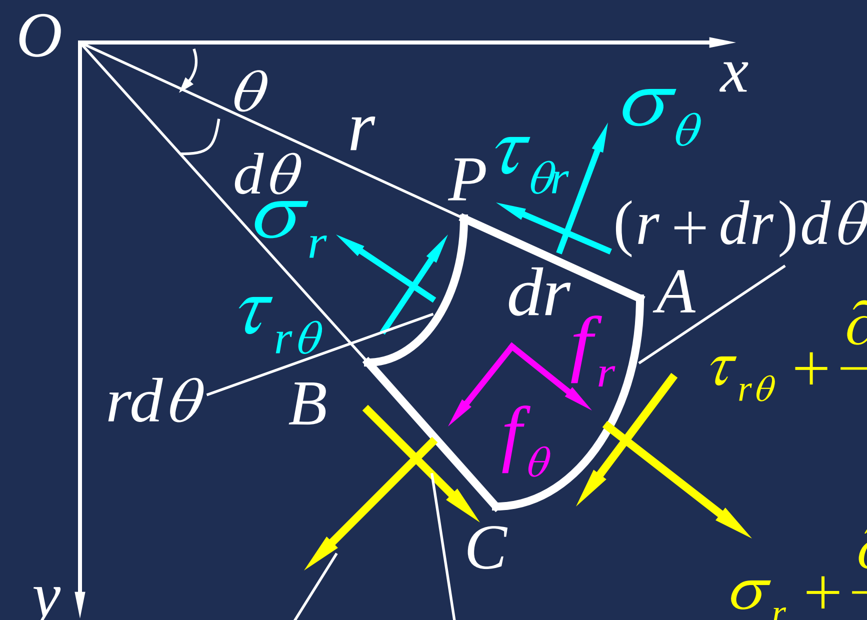
应力：

PA面 $\sigma_\theta, \tau_{\theta r}$

PB面 $\sigma_r, \tau_{r\theta}$

BC面 $\begin{cases} \sigma_\theta + \dfrac{\partial \sigma_\theta}{\partial \theta} d\theta \\ \tau_{\theta r} + \dfrac{\partial \tau_{\theta r}}{\partial \theta} d\theta \end{cases}$

AC面 $\begin{cases} \sigma_r + \dfrac{\partial \sigma_r}{\partial r} dr \\ \tau_{r\theta} + \dfrac{\partial \tau_{r\theta}}{\partial r} dr \end{cases}$



$$\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr$$

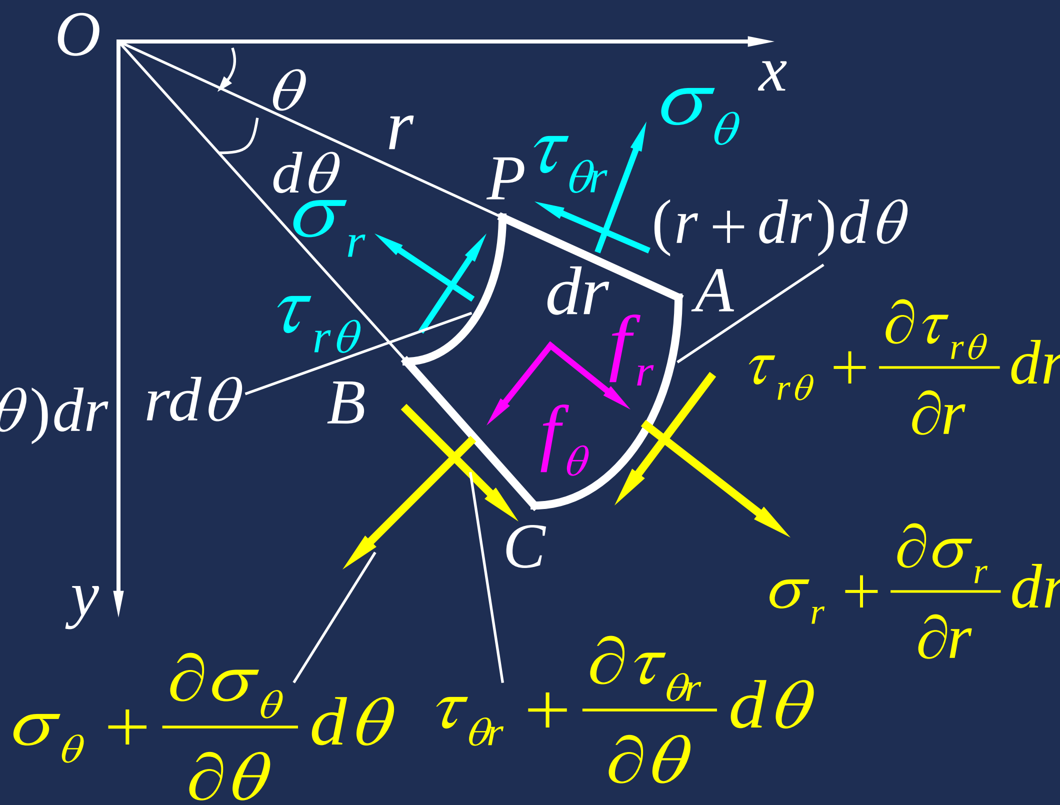
$$\sigma_r + \frac{\partial \sigma_r}{\partial r} dr$$

$$\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta \qquad \tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta$$

应力正向规定：

- 正应力 —— 拉为正，压为负；
- 剪应力 —— r、θ的正面上，与坐标方向一致时为正；
  r、θ的负面上，与坐标方向相反时为正。

2. 平衡微分方程



考虑微元体平衡（取厚度为1）：

$$\sum F_r = 0,$$

$$-\sigma_r r d\theta - \cancel{\tau_{\theta r}} dr + (\cancel{\tau_{\theta r}} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta) dr$$

$$+ (\sigma_r + \frac{\partial \sigma_r}{\partial r} dr)(r + dr) d\theta$$

$$-\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right) dr \frac{d\theta}{2}$$

$$-\sigma_\theta dr \frac{d\theta}{2} + f_r r dr d\theta = 0$$

（高阶小量，舍去）

将上式化开：

$$-\cancel{\sigma_r r d\theta} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta dr + \cancel{\sigma_r r d\theta} + \frac{\partial \sigma_r}{\partial r} r dr d\theta + \sigma_r dr d\theta + \frac{\partial \sigma_r}{\partial r} \cancel{(dr)^2 d\theta}$$

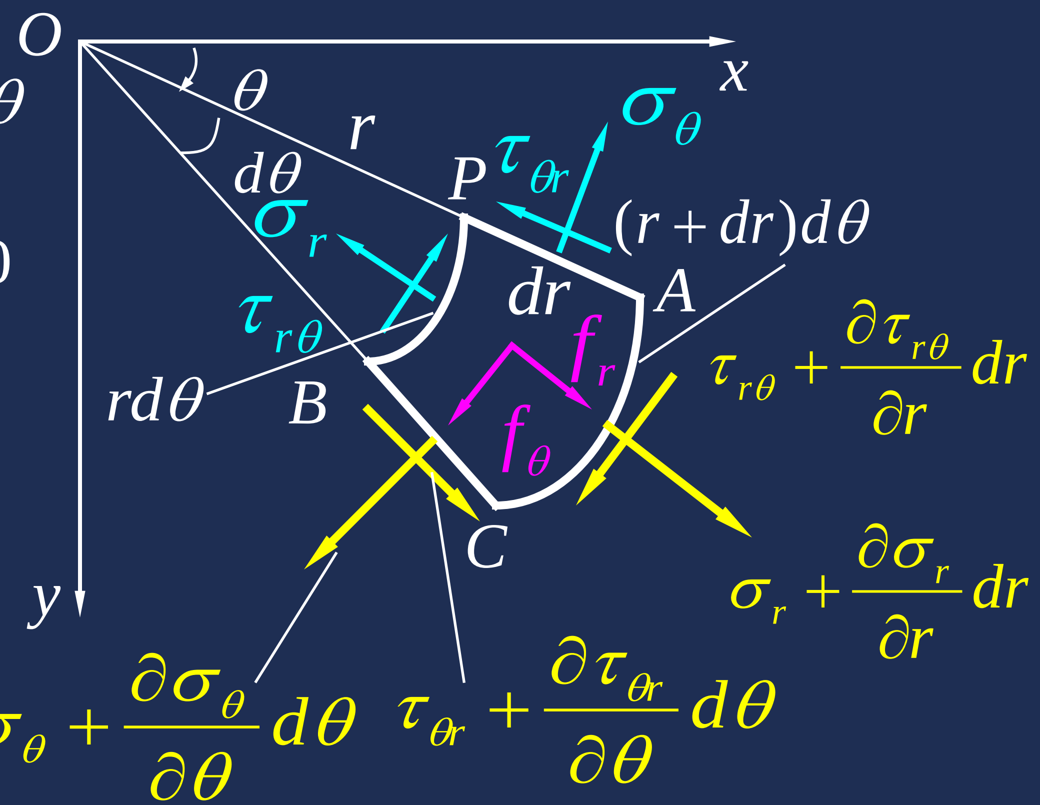
$$-\sigma_\theta dr \frac{d\theta}{2} - \left(\frac{\partial \sigma_\theta}{\partial \theta} \cancel{d\theta}\right) dr \frac{d\theta}{2} - \sigma_\theta dr \frac{d\theta}{2} + f_r r dr d\theta = 0$$

$$+\frac{\partial \sigma_r}{\partial r} r dr d\theta + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta dr + \sigma_r dr d\theta$$

$$-\sigma_\theta dr d\theta + f_r r dr d\theta = 0$$

两边同除以 $rdrd\theta$：

$$\boxed{\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0}$$



$$\sum F_\theta = 0,$$

$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right) dr - \sigma_\theta dr + \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} dr\right)(r+dr)d\theta - \tau_{r\theta} r d\theta$$

$$+\left(\tau_{\theta r} + \frac{\partial \tau_{\theta r}}{\partial \theta} d\theta\right) dr \frac{d\theta}{2} + \tau_{\theta r} dr \frac{d\theta}{2} + f_\theta r dr d\theta = 0$$

两边同除以 $rdrd\theta$，并略去高阶小量：

$$\boxed{\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0}$$

$$\sum M = 0, \quad \tau_{r\theta} = \tau_{\theta r}$$

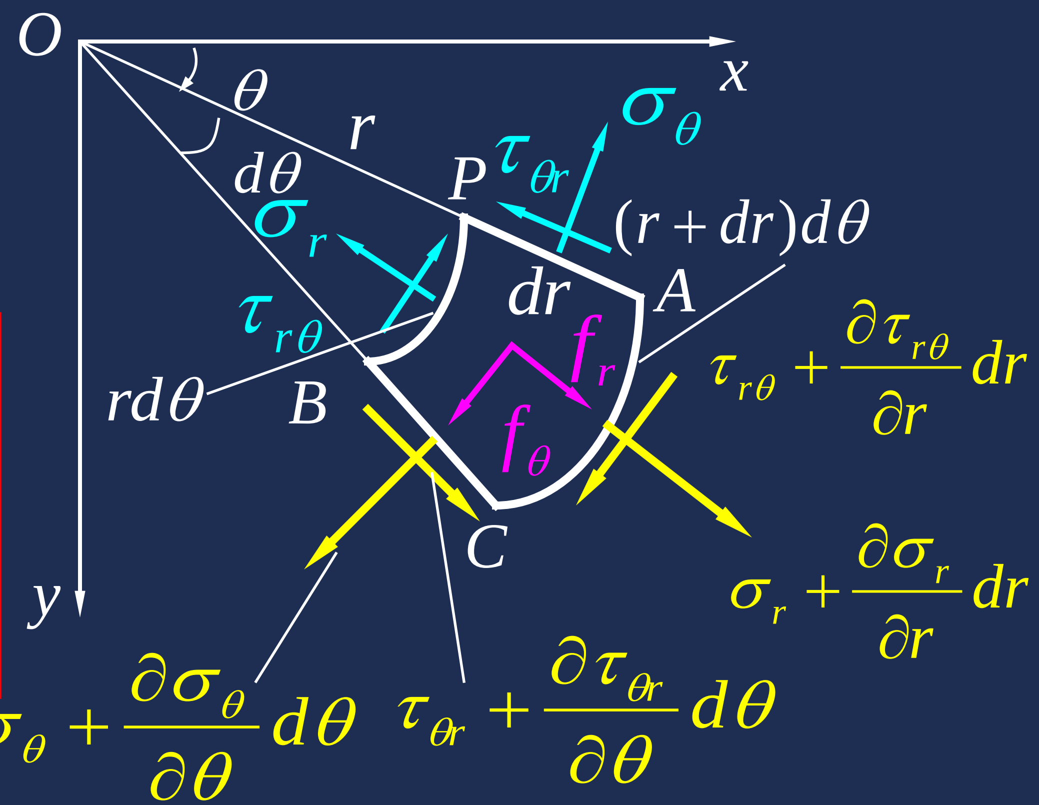
—— 剪应力互等定理

于是，极坐标下的平衡方程为：

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0$$

（4—1）

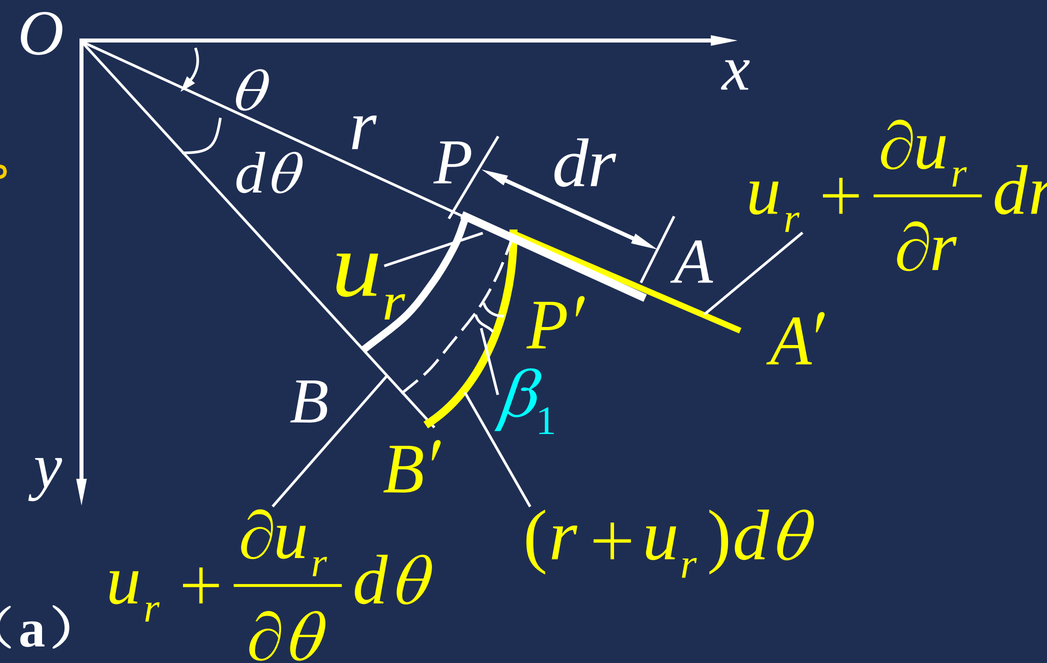


方程（4—1）中包含三个未知量，而只有二个方程，是一次超静定问题，需考虑变形协调条件才能求解。

§ 4-2 极坐标中的几何方程与物理方程

1. 几何方程

(1) 只有径向变形，无环向变形。



径向线段PA的相对伸长：

$$\varepsilon_{r1} = \frac{P'A' - PA}{PA} = \frac{AA' - PP'}{PA}$$

$$= \frac{u_r + \frac{\partial u_r}{\partial r} dr - u_r}{dr} = \frac{\partial u_r}{\partial r} \quad \text{(a)}$$

径向线段PA的转角： $\alpha_1 = 0$ （b）

线段PB的相对伸长： $\varepsilon_{\theta 1} = \frac{P'B' - PB}{PB} = \frac{(r+u_r)d\theta - rd\theta}{rd\theta} = \frac{u_r}{r}$ （c）

环向线段PB的转角：

$$\tan \beta_1 \approx \beta_1 = \frac{BB' - PP'}{PB} = \frac{(u_r + \frac{\partial u_r}{\partial \theta} d\theta) - u_r}{rd\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} \quad \text{(d)}$$

径向线段PA的相对伸长：

$$\boxed{\varepsilon_{r1} = \frac{\partial u_r}{\partial r}} \qquad \text{(a)}$$

径向线段PA的转角：

$$\alpha_1 = 0 \qquad \text{(b)}$$

环向线段PB的相对伸长：

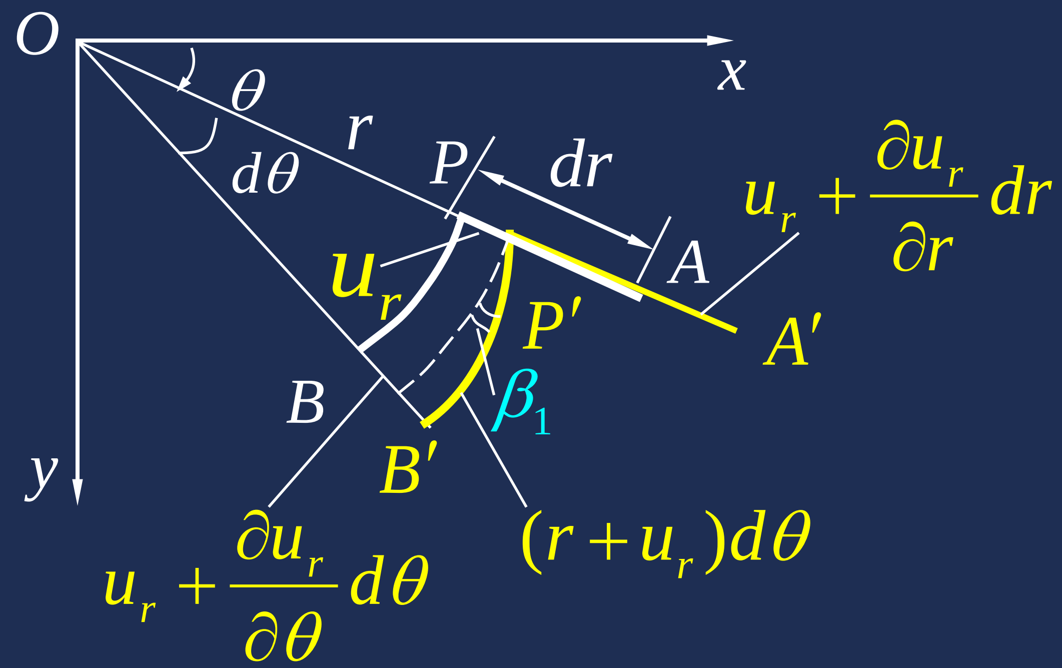
$$\boxed{\varepsilon_{\theta 1} = \frac{u_r}{r}} \qquad \text{(c)}$$

环向线段PB的转角：

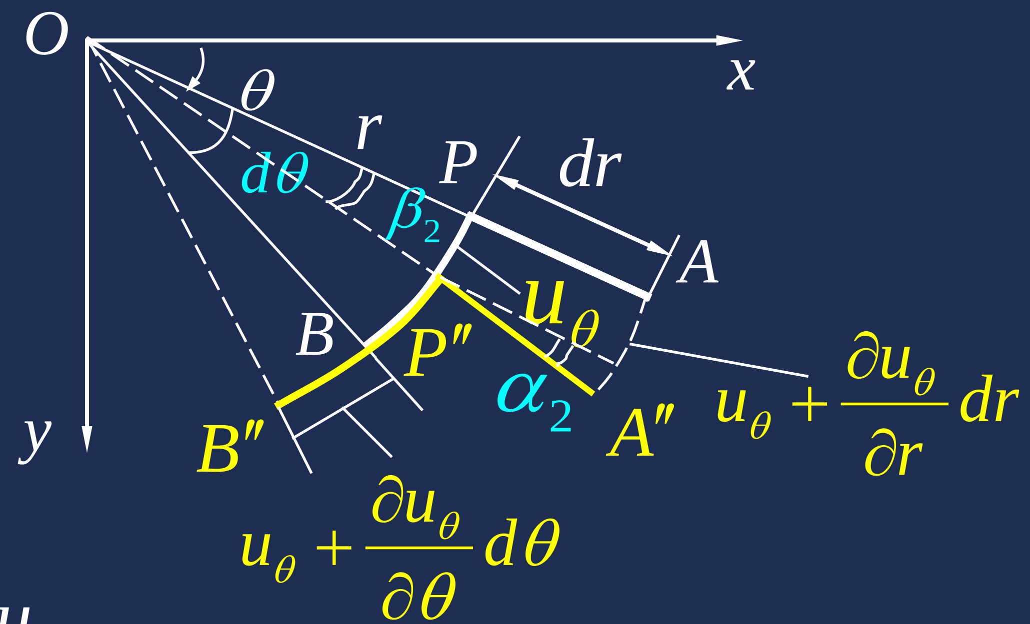
$$\beta_1 = \frac{1}{r}\frac{\partial u_r}{\partial \theta} \qquad \text{(d)}$$

剪应变为：

$$\boxed{\gamma_{r\theta 1} = \alpha_1 + \beta_1 = \frac{1}{r}\frac{\partial u_r}{\partial \theta}} \qquad \text{(e)}$$



(2) 只有环向变形，无径向变形。



径向线段PA的相对伸长：

$$\varepsilon_{r2}=\frac{P''A''-PA}{PA}=\frac{dr-dr}{dr}=0 \qquad \text{(f)}$$

径向线段PA的转角：

$$\alpha_2=\frac{u_\theta+\frac{\partial u_\theta}{\partial r}dr-u_\theta}{dr}=\frac{\partial u_\theta}{\partial r} \qquad \text{(g)}$$

环向线段PB的相对伸长：

$$\varepsilon_{\theta 2}=\frac{P''B''-PB}{PB}=\frac{BB''-PP''}{PB}=\frac{u_\theta+\frac{\partial u_\theta}{\partial \theta}d\theta-u_\theta}{rd\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad \text{(h)}$$

环向线段PB的转角： $\beta_2=-\frac{u_\theta}{r}$ （i）

剪应变为：

$$\gamma_{r\theta 2}=\alpha_2+\beta_2=\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r} \qquad \text{(j)}$$

(3) 总应变

$$\begin{cases} \varepsilon_r = \varepsilon_{r1} + \varepsilon_{r2} = \dfrac{\partial u_r}{\partial r} + 0 = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \varepsilon_{\theta 1} + \varepsilon_{\theta 2} = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \gamma_{r\theta 1} + \gamma_{r\theta 2} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}$$

整理得：

$$\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases} \qquad （4—2）$$

—— 极坐标下的几何方程

2. 物理方程

平面应力情形：

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - \mu\sigma_\theta) \\ \varepsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \mu\sigma_r) \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} = \dfrac{2(1+\mu)}{E}\tau_{r\theta} \end{cases} \quad (4-3)$$

平面应变情形：

$$\begin{cases} \varepsilon_r = \dfrac{1-\mu^2}{E}\left(\sigma_r - \dfrac{\mu}{1-\mu}\sigma_\theta\right) \\ \varepsilon_\theta = \dfrac{1-\mu^2}{E}\left(\sigma_\theta - \dfrac{\mu}{1-\mu}\sigma_r\right) \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} = \dfrac{2(1+\mu)}{E}\tau_{r\theta} \end{cases} \quad (4-4)$$

弹性力学平面问题极坐标求解的基本方程：

平衡微分方程：

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\sigma_r - \sigma_\theta}{r} + k_r = 0 \\ \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{2\tau_{r\theta}}{r} + k_\theta = 0 \end{cases} \qquad （4—1）$$

几何方程：

$$\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases} \qquad （4—2）$$

物理方程：

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - \mu\sigma_\theta) \qquad \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} = \dfrac{2(1+\mu)}{E}\tau_{r\theta} \\ \varepsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \mu\sigma_r) \end{cases} \qquad （4—3）$$

(平面应力情形)

边界条件：

位移边界条件：$(u_r)_s = \bar{u}_r$，$(u_\theta)_s = \bar{u}_\theta$

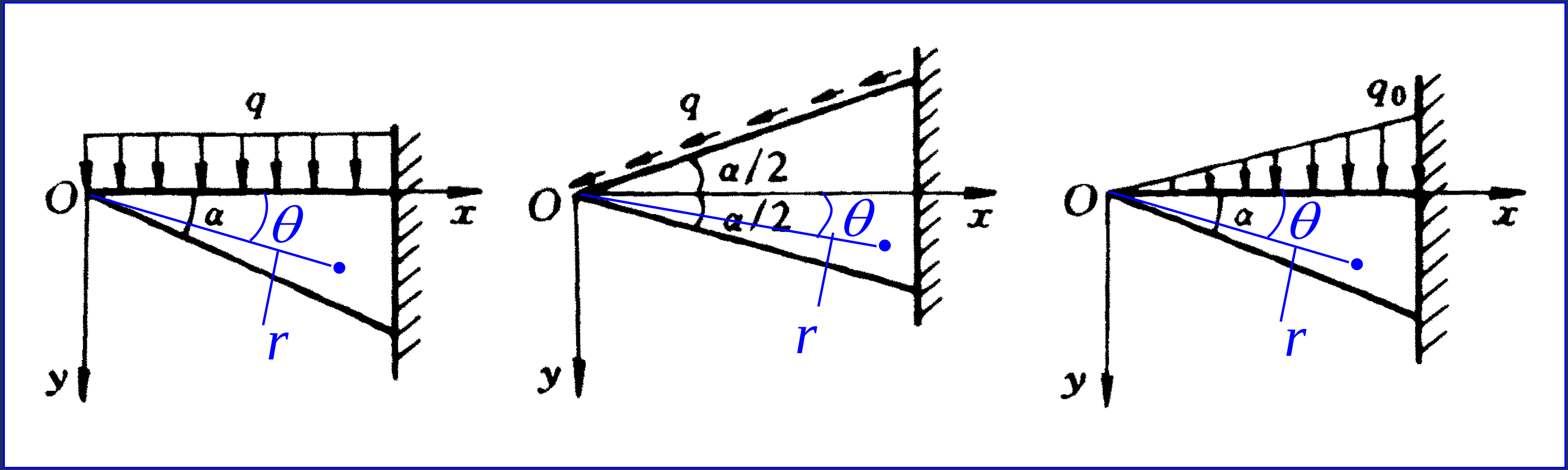
应力边界条件：

$$l(\sigma_r)_s + m(\tau_{r\theta})_s = \bar{k}_r$$

$$l(\tau_{r\theta})_s + m(\sigma_\theta)_s = \bar{k}_\theta$$

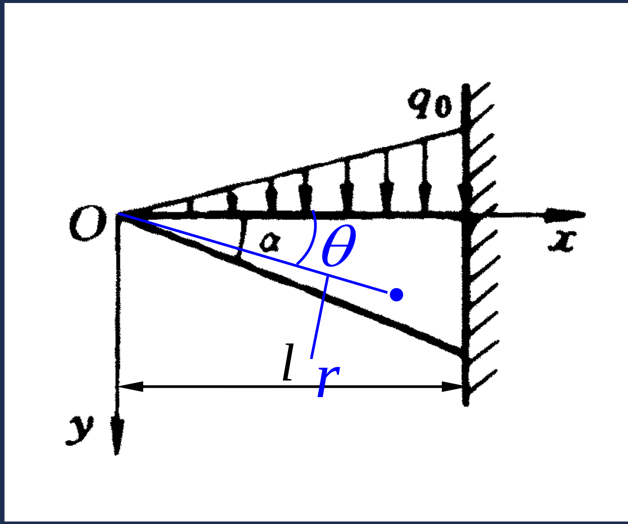
$\bar{u}_r, \bar{u}_\theta$ 为边界上已知位移，$\bar{k}_r, \bar{k}_\theta$ 为边界上已知的面力分量。

（位移单值条件）



$$\begin{cases} \sigma_\theta \big|_{\theta=0^\circ} = -\dfrac{r}{l} q_0 \\ \tau_{r\theta} \big|_{\theta=0^\circ} = 0 \end{cases}$$

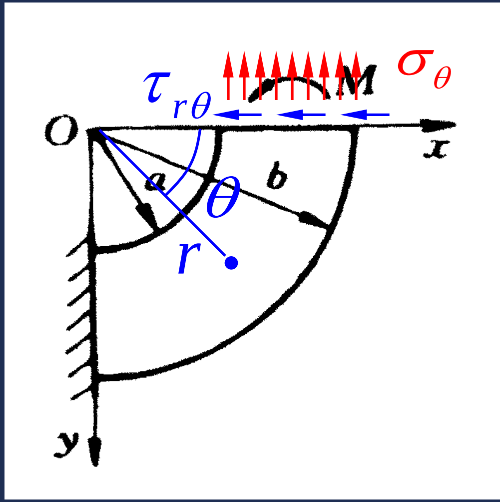
$$\begin{cases} \sigma_\theta \big|_{\theta=\alpha} = 0 \\ \tau_{r\theta} \big|_{\theta=\alpha} = 0 \end{cases}$$



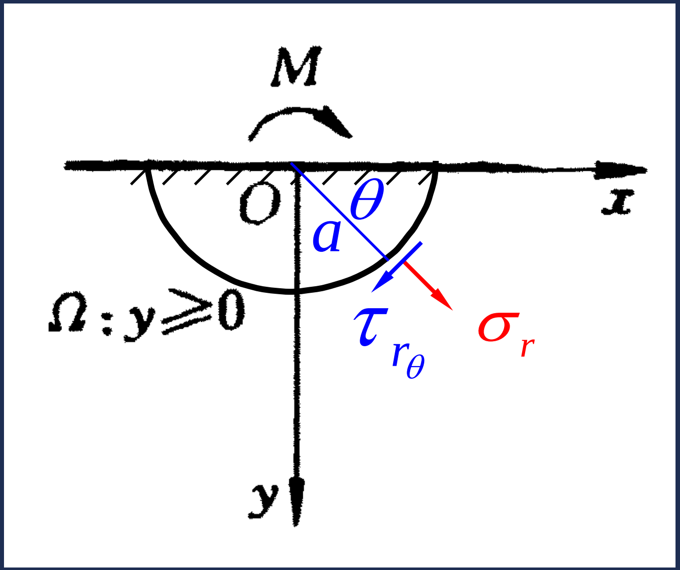
$$\begin{cases} \sigma_r \big|_{r=a} = 0 \\ \tau_{r\theta} \big|_{r=a} = 0 \end{cases}$$

$$\begin{cases} \sigma_r \big|_{r=b} = 0 \\ \tau_{r\theta} \big|_{r=b} = 0 \end{cases}$$

$$\begin{cases} \int_a^b \sigma_\theta \big|_{\theta=0^\circ} dr = 0 \\ \int_a^b \tau_{r\theta} \big|_{\theta=0^\circ} dr = 0 \\ \int_a^b \sigma_\theta \big|_{\theta=0^\circ} r dr = M \end{cases}$$

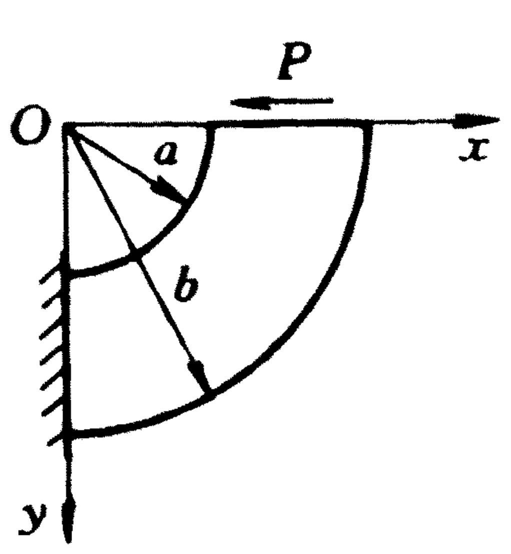


$$\begin{cases}\sigma_\theta\big|_{\theta=0^\circ}=0\\ \tau_{r\theta}\big|_{\theta=0^\circ}=0\end{cases}\qquad\begin{cases}\sigma_\theta\big|_{\theta=180^\circ}=0\\ \tau_{r\theta}\big|_{\theta=180^\circ}=0\end{cases}$$



取半径为 a 的半圆分析，由其平衡得：

$$\begin{cases}\sum F_x=0\\ \int_0^\pi\left(\sigma_r\big|_{r=a}\cos\theta-\tau_{r\theta}\big|_{r=a}\sin\theta\right)a\,d\theta=0\\ \sum F_y=0\\ \int_0^\pi\left(\sigma_r\big|_{r=a}\sin\theta+\tau_{r\theta}\big|_{r=a}\cos\theta\right)a\,d\theta=0\\ \sum M_O=0\\ \int_0^\pi\tau_{r\theta}\big|_{r=a}a\cdot a\,d\theta+M=0\end{cases}$$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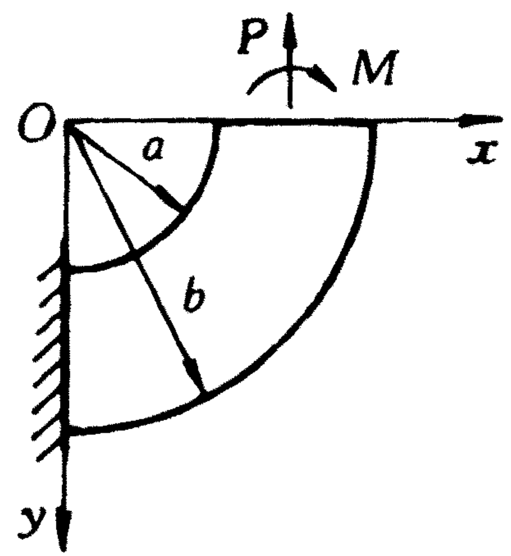
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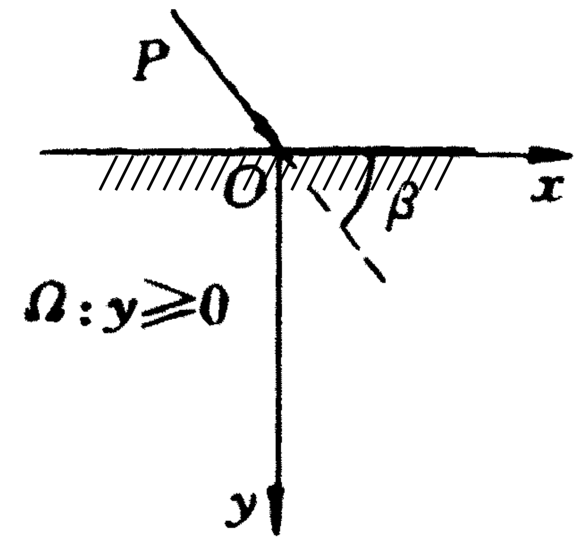
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§ 4-3 极坐标中的应力函数与相容方程

1. 直角坐标下变形调方程（相容方程）

$$\left\{\begin{aligned}
&\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} && \text{(2-22)}\\
&\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)(\sigma_x+\sigma_y)=-(1+\mu)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right) && \text{(2-23)}\\
& && \text{（平面应力情形）}\\
&\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0 && \text{(2-25)}\\
&\nabla^4 \varphi=\frac{\partial^4 \varphi}{\partial x^4}+2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}+\frac{\partial^4 \varphi}{\partial y^4}=0 && \text{(2-27)}
\end{aligned}\right.$$

应力的应力函数表示：

$$\sigma_x=\frac{\partial^2 \varphi}{\partial y^2}-Xx \qquad \sigma_y=\frac{\partial^2 \varphi}{\partial x^2}-Yy \qquad \tau_{xy}=-\frac{\partial^2 \varphi}{\partial x \partial y} \qquad \text{(2-26)}$$

$$\varphi=\varphi(x,y)$$

2. 极坐标下的应力分量与相容方程

方法1：（步骤）

（1）利用极坐标下的几何方程，求得应变表示的相容方程：

$$\frac{1}{r}\frac{\partial^2\varepsilon_r}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varepsilon_\theta}{\partial\theta}\right)-\frac{\partial\varepsilon_r}{\partial r}=\frac{1}{r}\frac{\partial^2(r\cdot\gamma_{r\theta})}{\partial r\partial\theta}$$

（2）利用极坐标下的物理方程，得应力表示的相容方程：

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right)(\sigma_r+\sigma_\theta)=0$$

（常体力情形）

（3）利用平衡方程求出用应力函数表示的应力分量：

$$\sigma_r=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\qquad\sigma_\theta=\frac{\partial^2\varphi}{\partial r^2}\qquad\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

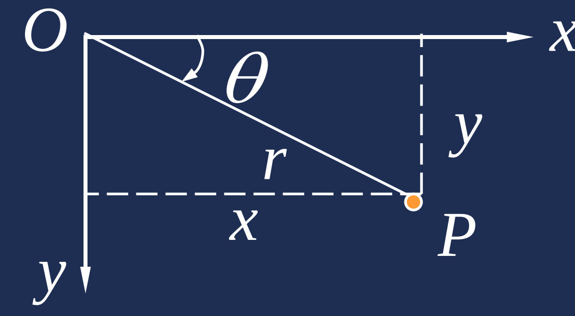
（4）将上述应力分量代入应力表示的相容方程，得应力函数表示的相容方程：

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial\theta^2}\right)^2\varphi=0$$

（常体力情形）

方法2：（用极坐标与直角坐标之间的变换关系求得到）

（1）极坐标与直角坐标间的关系：



$$\begin{cases} r^2 = x^2 + y^2 & \theta = \arctan \dfrac{y}{x} \\ x = r\cos\theta & y = r\sin\theta \end{cases}$$

$$\varphi = \varphi(r,\theta)$$

$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta & \dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta \\ \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r} & \dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r} \end{cases}$$

（2）应力分量与相容方程的坐标变换：

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

应力分量的坐标变换

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \varphi}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial \varphi}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \cos\theta \dfrac{\partial \varphi}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial \varphi}{\partial \theta} = \left[\cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}\right]\varphi \\ \dfrac{\partial \varphi}{\partial y} = \dfrac{\partial \varphi}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial \varphi}{\partial \theta}\dfrac{\partial \theta}{\partial y} = \sin\theta \dfrac{\partial \varphi}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial \varphi}{\partial \theta} = \left[\sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta}\right]\varphi \end{cases}$$

$$\frac{\partial^2\varphi}{\partial x^2}=\left[\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right] \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)$$

$$=\cos^2\theta\frac{\partial^2\varphi}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial\varphi}{\partial r}$$

$$+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \qquad \text{(a)}$$

$$\frac{\partial^2\varphi}{\partial y^2}=\left[\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]\left[\sin\theta\frac{\partial\varphi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}\right] \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)$$

$$=\sin^2\theta\frac{\partial^2\varphi}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial\varphi}{\partial r}$$

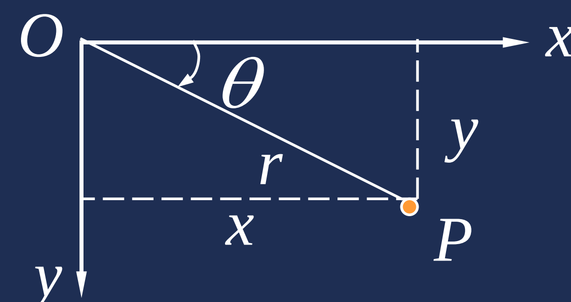
$$-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \qquad \text{(b)}$$

$$\frac{\partial^2\varphi}{\partial x\partial y}=\underbrace{\left[\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]}_{\frac{\partial}{\partial x}}\underbrace{\left[\sin\theta\frac{\partial\varphi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\varphi}{\partial\theta}\right]}_{\frac{\partial\varphi}{\partial y}}$$

$$=\sin\theta\cos\theta\frac{\partial^2\varphi}{\partial r^2}+\frac{\cos^2\theta-\sin^2\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{\sin\theta\cos\theta}{r}\frac{\partial\varphi}{\partial r}$$

$$-\frac{\cos^2\theta-\sin^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\qquad\text{(c)}$$

由直角坐标下应力函数与应力的关系（2－26）：



$$\begin{cases}\sigma_r=\sigma_x\Big|_{\theta=0}=\dfrac{\partial^2\varphi}{\partial y^2}\Big|_{\theta=0}\\[2ex]\sigma_\theta=\sigma_y\Big|_{\theta=0}=\dfrac{\partial^2\varphi}{\partial x^2}\Big|_{\theta=0}\\[2ex]\tau_{r\theta}=\tau_{xy}\Big|_{\theta=0}=-\dfrac{\partial^2\varphi}{\partial x\partial y}\Big|_{\theta=0}=\dfrac{1}{r^2}\dfrac{\partial\varphi}{\partial\theta}-\dfrac{1}{r}\dfrac{\partial^2\varphi}{\partial\theta^2}=-\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta}\right)\end{cases}$$

当$\theta=0$时，$\begin{cases}x\rightarrow r\\ y\rightarrow\theta\end{cases}$

$$\sigma_r = \sigma_x\big|_{\theta=0} = \frac{\partial^2\varphi}{\partial y^2}\bigg|_{\theta=0} = \sin^2\theta\frac{\partial^2\varphi}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial\varphi}{\partial r}$$

$$-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\bigg|_{\theta=0} \Rightarrow \sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\sigma_\theta = \sigma_y\big|_{\theta=0} = \frac{\partial^2\varphi}{\partial x^2}\bigg|_{\theta=0} = \cos^2\theta\frac{\partial^2\varphi}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial\varphi}{\partial r}$$

$$+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\bigg|_{\theta=0} \Rightarrow \sigma_\theta = \frac{\partial^2\varphi}{\partial r^2}$$

$$\tau_{r\theta} = \tau_{xy}\big|_{\theta=0} = -\frac{\partial^2\varphi}{\partial x\partial y}\bigg|_{\theta=0} = -\sin\theta\cos\theta\frac{\partial^2\varphi}{\partial r^2} - \frac{\cos^2\theta-\sin^2\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}$$

$$+\frac{\sin\theta\cos\theta}{r}\frac{\partial\varphi}{\partial r} + \frac{\cos^2\theta-\sin^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\bigg|_{\theta=0} \Rightarrow \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

极坐标下应力分量计算公式：

$$
\left\{
\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \\
\sigma_\theta &= \frac{\partial^2\varphi}{\partial r^2} \\
\tau_{r\theta} &= \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)
\end{aligned}
\right.
\qquad (4-5)
$$

可以证明：式（4—5）满足平衡方程（4—1）。

说明：式（4—5）仅给出体力为零时的应力分量表达式。

相容方程的坐标变换

相容方程的坐标变换

$$\frac{\partial^2\varphi}{\partial x^2}=\cos^2\theta\frac{\partial^2\varphi}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial\varphi}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\qquad\text{(a)}$$

$$\frac{\partial^2\varphi}{\partial y^2}=\sin^2\theta\frac{\partial^2\varphi}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial\varphi}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\qquad\text{(b)}$$

将式(a)与(b)相加，得

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\Rightarrow\quad\nabla^2\varphi=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi$$

得到极坐标下的 Laplace 微分算子：

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

极坐标下的相容方程为：

$$\nabla^2\nabla^2\varphi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi=0$$

$$\Rightarrow\quad \boxed{\nabla^4\varphi=\nabla^2\nabla^2\varphi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\varphi=0}\qquad（4—6）$$

说明：　方程（4—6）为常体力情形的相容方程。

结论： 弹性力学极坐标求解归结为

（1） 由问题的条件求出满足式（4—6）的应力函数 $\varphi(r,\theta)$

$$\nabla^4\varphi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\varphi=0 \qquad (4-6)$$

（2） 由式（4—5）求出相应的应力分量：$\sigma_r,\sigma_\theta,\tau_{r\theta}$

$$\sigma_r=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\qquad \sigma_\theta=\frac{\partial^2\varphi}{\partial r^2}\qquad \tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \qquad (4-5)$$

（3） 将上述应力分量 $\sigma_r,\sigma_\theta,\tau_{r\theta}$ 满足问题的边界条件：

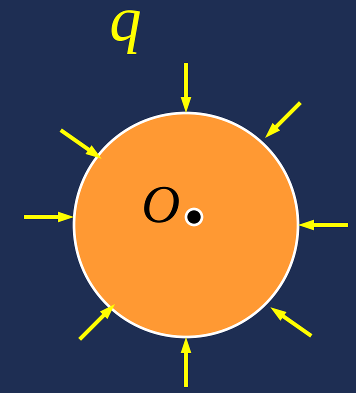
位移边界条件：$(u_r)_s=\bar{u}_r,\ (u_\theta)_s=\bar{u}_\theta$

应力边界条件：
$$\begin{cases} l(\sigma_r)_s+m(\tau_{r\theta})_s=\bar{k}_r \\ l(\tau_{r\theta})_s+m(\sigma_\theta)_s=\bar{k}_\theta \end{cases}$$

（位移单值条件）

$\bar{u}_r,\bar{u}_\theta$ 为边界上已知位移，$\bar{k}_r,\bar{k}_\theta$ 为边界上已知的面力分量。

3. 轴对称问题应力分量与相容方程



轴对称问题：$\varphi = \varphi(r) \Longrightarrow \dfrac{\partial \varphi}{\partial \theta} = 0$

由式（4—5）和（4—6）得应力分量和相容方程为：

应力分量：

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{d\varphi}{dr} \\ \sigma_\theta = \dfrac{d^2\varphi}{dr^2} \\ \tau_{r\theta} = 0 \end{cases} \qquad (4-10)$$

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \varphi}{\partial \theta^2} \\ \sigma_\theta = \dfrac{\partial^2 \varphi}{\partial r^2} \\ \tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta}\right) \end{cases} \qquad (4-5)$$

相容方程：

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 \varphi = 0$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \varphi = 0 \qquad (4-6)$$

§ 4-4　应力分量的坐标变换式

(1) 用极坐标下的应力分量表示直角坐标下的应力分量

$$\left\{\begin{aligned}\sigma_x &= \frac{\sigma_r+\sigma_\theta}{2}+\frac{\sigma_r-\sigma_\theta}{2}\cos 2\theta-\tau_{r\theta}\sin 2\theta\\ \sigma_y &= \frac{\sigma_r+\sigma_\theta}{2}-\frac{\sigma_r-\sigma_\theta}{2}\cos 2\theta+\tau_{r\theta}\sin 2\theta\\ \tau_{xy} &= \frac{\sigma_r-\sigma_\theta}{2}\sin 2\theta+\tau_{r\theta}\cos 2\theta\end{aligned}\right. \tag{4—8}$$

(2) 用直角坐标下的应力分量表示极坐标下的应力分量

$$\left\{\begin{aligned}\sigma_r &= \frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos 2\theta+\tau_{xy}\sin 2\theta\\ \sigma_\theta &= \frac{\sigma_x+\sigma_y}{2}-\frac{\sigma_x-\sigma_y}{2}\cos 2\theta-\tau_{xy}\sin 2\theta\\ \tau_{r\theta} &= \frac{\sigma_x-\sigma_y}{2}\sin 2\theta+\tau_{xy}\cos 2\theta\end{aligned}\right. \tag{4—9}$$