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Research Article

Prandtl's Boundary Layer Equation for Two-Dimensional Flow: Exact Solutions via the Simplest Equation Method

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The simplest equation method is employed to construct some new exact closed-form solutions of the general Prandtl's boundary layer equation for two-dimensional flow with vanishing or uniform mainstream velocity. We obtain solutions for the case when the simplest equation is the Bernoulli equation or the Riccati equation. Prandtl's boundary layer equation arises in the study of various physical models of fluid dynamics. Thus finding the exact solutions of this equation is of great importance and interest.

1. Introduction

Many scientific and engineering problems and phenomena are modeled by nonlinear differential equations. Therefore, the study of nonlinear differential equations has been an active area of research from the past few decades. Considerable attention has been devoted to the construction of exact solutions of nonlinear equations because of their important role in the study of nonlinear physical models. For nonlinear differential equations, we do not have the freedom to compute exact (closed-form) solutions and for analytical work we have to rely on some approximate analytical or numerical techniques which may be helpful for us to understand the complex physical phenomena involved. The exact solutions of the nonlinear differential equations are of great interest and physically more important. These exact solutions, if reported, facilitate the verification of complex numerical codes and are also helpful in a stability analysis for solving special nonlinear problems. In recent years, much attention has been devoted to the development of several powerful and useful methods for finding exact analytical solutions of nonlinear differential equations. These methods include the powerful Lie group method [1], the sine-cosine method [2], the tanh method [3, 4], the extended tanh-function method [5], the Backlund transformation method [6], the transformed rational function method [7], the (G'/G)-expansion method [8], the exponential function rational expansion method [9], and the Adomian's decomposition method [10].

Prandtl [11] introduced boundary layer theory in 1904 to understand the flow behavior of a viscous fluid near a solid boundary. Prandtl gave the concept of a boundary layer in large Reynolds number flows and derived the boundary layer equations by simplifying the Navier-Stokes equations to yield approximate solutions. Prandtl's boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets, and liquid jets, the flow past a stretching plate/surface, flow induced due to a shrinking sheet, and so on. These boundary layer equations are usually solved subject to certain boundary conditions depending upon the specific physical model considered. Blasius [12] solved the Prandtl's boundary layer equations for a flat moving plate problem and gave a power series solution of the problem. Falkner and Skan [13] generalized the Blasius boundary layer problem by considering the boundary layer flow over a wedge inclined at a certain angle. Sakiadis [14] initiated the study of the boundary layer flow over a continuously moving rigid surface with a uniform speed. Crane [15] was the first one who studied the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations with parameter $\gamma = 0$. P. S. Gupta and A. S. Gupta [16] extended the Crane's work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow. The numerical solution for a free two-dimensional jet was obtained by Schlichting [17] and later an analytic study was made by Bickley [18]. Riley [19] derived the solution for a radial liquid jet. Recently, the similarity solution of axisymmetric non-Newtonian wall jet with swirl effects was investigated by Kolář [20]. Naz et al. [21] and Mason [22] have investigated the general boundary layer equations for two-dimensional and radial flows by using the classical Lie group approach and very recently Naz et al. [23] have provided the similarity solutions of the Prandtl's boundary layer equations by implementing the nonclassical/conditional symmetry method.

The simplest equation method is a powerful mathematical tool for finding exact solutions of nonlinear ordinary differential equations. It has been developed by Kudryashov [24, 25] and used successfully by many researchers for finding exact solutions of nonlinear ordinary differential equations [26–28]. The purpose of the present work is to find the exact closed-form solutions of Prandtl's boundary layer equation for two-dimensional flow with constant or uniform main stream velocity by the use of simplest equation method.

Prandtl's boundary layer equation for the stream function $\psi(x, y)$ for an incompressible, steady two-dimensional flow with uniform or vanishing mainstream velocity is [29]

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \nu \frac{\partial^3 \psi}{\partial y^3} = 0. \quad (1)$$

Here (x, y) denote the Cartesian coordinates parallel and perpendicular to the boundary $y = 0$ and ν is the kinematic viscosity. The velocity components $u(x, y)$ and $v(x, y)$, in the x and y directions, are related to stream function $\psi(x, y)$ as

$$u(x, y) = \frac{\partial \psi}{\partial y}, \qquad v(x, y) = -\frac{\partial \psi}{\partial x}. \quad (2)$$

By the use of Lie group theoretic method of infinitesimal transformations [1], the general form of similarity solution for (1) is

$$\psi(x, y) = x^{1-\gamma} F(\eta), \quad \eta = \frac{y}{x^{\gamma}}, \quad (3)$$

where γ is the constant determined from the further conditions and η is the similarity variable. By the substitution of (3) into (1), the third-order nonlinear ordinary differential equation in $F(\eta)$ results, namely, in

$$\nu \frac{d^3 F}{d\eta^3} + (1-\gamma) F \frac{d^2 F}{d\eta^2} + (2\gamma - 1)\left(\frac{dF}{d\eta}\right)^2 = 0. \quad (4)$$

Equation (4) gives the general form of Prandtl's boundary layer equation for two-dimensional flow of a viscous incompressible fluid. The boundary layer equation (4) is usually solved subject to certain boundary conditions depending upon the particular physical model considered. Here, we present the exact closed-form solutions of (4) using the simplest equation method. We organize the paper as follows. In Section 2, we describe briefly the simplest equation method. In Section 3, we apply this method to solve nonlinear Prandtl's boundary layer equation for two-dimensional flow. Finally, some closing remarks are presented in Section 4.

2. A Description of the Simplest Equation Method

Here we present a brief description of the simplest equation method for solving nonlinear ordinary differential equations.

Step 1. We first consider a general form of a nonlinear ordinary differential equation:

$$E\left[F, \frac{dF}{d\eta}, \frac{d^2F}{d\eta^2}, \frac{d^3F}{d\eta^3}, \ldots\right] = 0, \quad (5)$$

where F is the dependent variable and η is the independent variable.

Step 2. The basic idea of the simplest equation method consists in expanding the solutions of the previous ordinary differential equation in a finite series:

$$F(\eta) = \sum_{i=0}^{M} A_i (G(\eta))^i, \quad (6)$$

where $G(\eta)$ is a solution of some ordinary differential equations. These ordinary differential equations are called the simplest equations. The main property of the simplest equation is that we know the general solution of the simplest equation or we at least know the exact analytical solutions of the simplest equation. The parameters $A_0, A_1, \ldots, A_M$ are to be determined from the further conditions.

In this paper we use the Bernoulli and Riccati equations as the simplest equations. These equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

For the Bernoulli equation

$$\frac{dG}{d\eta} = dG(\eta) + eG^2(\eta), \quad (7)$$

where d and e are constants independent of η. The solution of (7) is

$$G(\eta) = d\left[\frac{\cosh[d(\eta + C)] + \sinh[d(\eta + C)]}{1 - e\cosh[d(\eta + C)] - e\sinh[d(\eta + C)]}\right], \quad (8)$$

where C is a constant of integration.

For the Riccati equation

$$\frac{dG}{d\eta} = dG(\eta) + eG^2(\eta) + f, \quad (9)$$

where d, e, and f are constants, we will use the solutions

$$G(\eta) = -\frac{d}{2e} - \frac{\theta}{2e} \tanh\left[\frac{1}{2}\theta(\eta + C)\right],$$

$$G(\eta) = -\frac{d}{2e} - \frac{\theta}{2e} \tanh\left(\frac{1}{2}\theta\eta\right) + \frac{\operatorname{sech}(\theta\eta/2)}{C\cosh(\theta\eta/2) - (2e/\theta)\sinh(\theta\eta/2)}, \tag{10}$$

where

$$\theta^2 = d^2 - 4ef > 0, \tag{11}$$

and C is a constant of integration.

Step 3. One of the main steps in using the simplest equation method is to determine the positive number M in (6). The positive number M can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (5).

Step 4. By the substitution of (6) into (5) and with (7) or (9), the left hand side of (5) is converted into a polynomial in $G(\eta)$. Equating each coefficient of the polynomial to zero yields a set of algebraic equations for A_i, d, e, f.

Step 5. By solving the algebraic equations obtained in Step 4 and substituting the results into (6), we obtain the exact solutions of ODE (5).

3. Application of the Simplest Equation Method

In this section, we employ the simplest equation method and obtain exact closed-form solutions of Prandtl's boundary layer equation (4).

3.1. Solutions of Boundary Layer Equation Using the Equation of Bernoulli as the Simplest Equation. The balancing procedure yields $M = 1$. Thus we search for a solution of (4) of the form

$$F(\eta) = A_0 + A_1 G(\eta), \tag{12}$$

where $G(\eta)$ satisfies the Bernoulli equation and A_0 and A_1 are the parameters to be determined.

By the substitution of (12) into (4) and making use of the Bernoulli equation (7) and then equating all coefficients of the functions G^i to zero, we obtain an algebraic system of equations in terms of A_0 and A_1. Solving this system of algebraic equations, we obtain the values of the constants A_0 and A_1. Therefore the solution of Prandtl's boundary layer equation (4) with $\gamma = 2/3$ is given by

$$F(\eta) = -3\nu d - 6\nu e d \left[\frac{\cosh[d(\eta+C)] + \sinh[d(\eta+C)]}{1 - e\cosh[d(\eta + C)] - e\sinh[d(\eta+C)]}\right], \tag{13}$$

and hence the corresponding stream function becomes

$$\psi(x, y) = -3\nu d x^{1/3} - 6\nu e d x^{1/3} \times \left[\left(\cosh\left[d\left(x^{-2/3} y + C\right)\right] + \sinh\left[d\left(x^{-2/3} y + C\right)\right]\right) \times \left(1 - e\cosh\left[d\left(x^{-2/3} y + C\right)\right] - e\sinh\left[d\left(x^{-2/3} y + C\right)\right]\right)^{-1}\right]. \tag{14}$$

Special Cases. By taking $d = -1$ and $e = 1$ in the previous solution, we obtain a special solution given by

$$\psi(x, y) = 3\nu x^{1/3} \coth\left[\frac{1}{2}\left(x^{-2/3} y + C\right)\right]. \tag{15}$$

Likewise, if we take $d = -1$ and $e = -1$, we deduce

$$\psi(x, y) = 3\nu x^{1/3} \tanh\left[\frac{1}{2}\left(x^{-2/3} y + C\right)\right]. \tag{16}$$

3.2. Solutions of Boundary Layer Equation Using the Equation of Riccati as the Simplest Equation. The balancing procedure yields $M = 1$. Thus the solution of (4) is written in the form

$$F(\eta) = A_0 + A_1 G(\eta). \tag{17}$$

By the insertion of (17) into (4) and making use of the Riccati equation (9) and proceeding as above, we obtain algebraic system of equations in terms of A_0 and A_1. Solving this system, we obtain the solutions of Prandtl's boundary layer equation (4) for $\gamma = 2/3$ as

$$F(\eta) = -3\nu d - 6\nu e\left[-\frac{d}{2e} - \frac{\theta}{2e}\tanh\left(\frac{1}{2}\theta(\eta + C)\right)\right],$$

$$F(\eta) = -3\nu d - 6\nu e\left[-\frac{d}{2e} - \frac{\theta}{2e}\tanh\left(\frac{\eta\theta}{2}\right) + \frac{\operatorname{sech}(\theta\eta/2)}{C\cosh(\theta\eta/2) - (2e/\theta)\sinh(\theta\eta/2)}\right], \tag{18}$$

and the solutions for corresponding stream functions are

$$\psi(x, y) = -3\nu d x^{1/3} - 6\nu e x^{1/3}\left[-\frac{d}{2e} - \frac{\theta}{2e}\tanh\left(\frac{1}{2}\theta\left(x^{-2/3} y + C\right)\right)\right], \tag{19}$$

$$\psi(x,y) = -3\nu d x^{1/3} - 6\nu e x^{1/3}\left[-\frac{d}{2e} - \frac{\theta}{2e}\tanh\left(\frac{\theta x^{-2/3}y}{2}\right) + \left(\operatorname{sech}\left(\frac{\theta x^{-2/3}y}{2}\right)\right) \times \left(C\cosh\left(\frac{\theta x^{-2/3}y}{2}\right) - \frac{2e}{\theta} \times \sinh\left(\frac{\theta x^{-2/3}y}{2}\right)\right)^{-1}\right], \tag{20}$$

where $\theta^2 = d^2 - 4ef$ and C is a constant of integration.

By taking $d = 3$, $e = 1$, and $f = 1$ in (19), we deduce a special solution of stream function ψ, given by

$$\psi(x,y) = -9\nu x^{1/3} + 3\nu x^{1/3}\left[3 + \sqrt{5}\tanh\left(\frac{\sqrt{5}}{2}\left(x^{-2/3}y + C\right)\right)\right]. \tag{21}$$

4. Concluding Remarks

In this study, we have utilized the method of simplest equation for obtaining exact closed-form solutions of the well-known Prandtl's boundary layer equation for two-dimensional flow with uniform mainstream velocity. As the simplest equations, we have used the Bernoulli and Riccati equations. Prandtl's boundary layer equations arise in various physical models of fluid dynamics and thus the exact solutions obtained may be very useful and significant for the explanation of some practical physical models dealing with Prandtl's boundary layer theory. We have also verified that the solutions obtained here are indeed the solutions of Prandtl's boundary layer equation.

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