第五讲：边界层理论

Chapter 17 Introduction to Boundary Layers

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第一节：引言

17.1 INTRODUCTION

应用背景 Background

Turbulent boundary-layer has the major contribution to the large skin-friction drag of the fuselage and wing.

At subsonic cruising speed, approximately half of the total drag is due to the skin-friction drag!

应用背景 Background

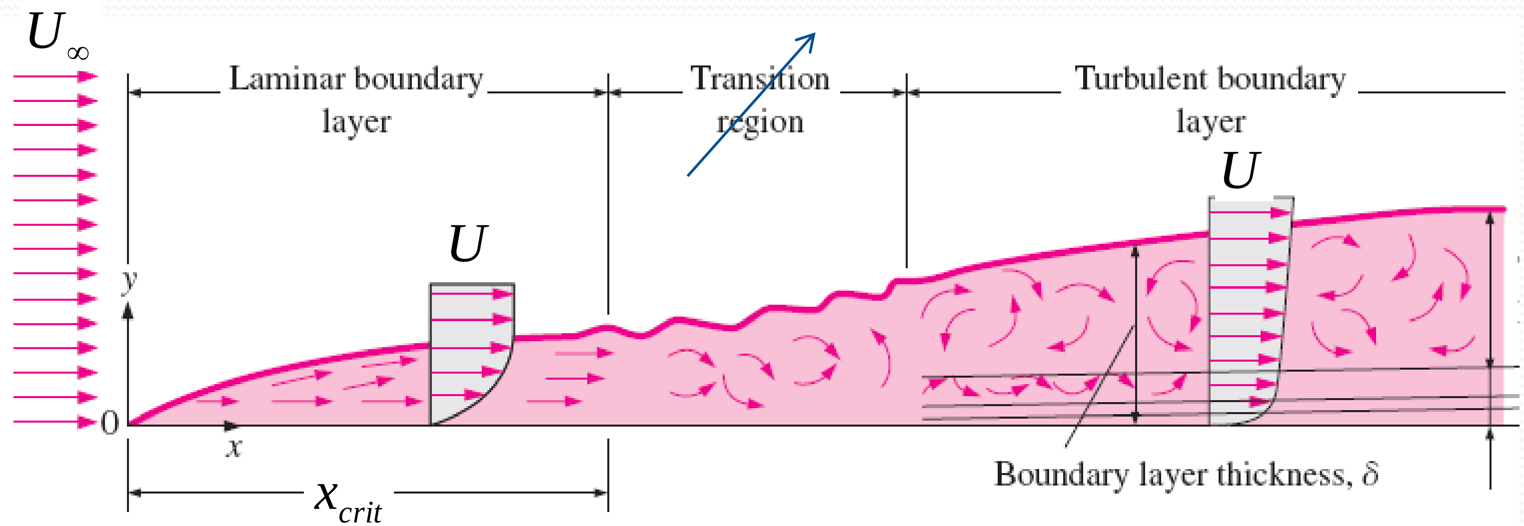
West-east gas pipeline project in China transporting clean natural gas from Xinjiang to the energy-hungry Yangtze river Delta covers 11 provinces and has a total range of around 4,000 km.

A huge amount of energy is needed to pump the natural gas in the long pipelines!

概述 Overview

- 粘性影响存在于流场中的每个点
- 从实用的观点，除浸入流体中物体附近的薄区域内，或者两个不同速度的流层间，其他部位粘性不起作用
- 浸入流体中物体附近的薄区域，该区域粘性效果明显
- 两个不同速度的流层间的边界区域
- 物体附近流动区域内发生的物理过程是表面磨擦、气动加热、流动分离等物理机理
- 计算表面磨擦、气动加热仅需在边界层内进行

边界层流动现象与边界层概念 The concept of the boundary layer



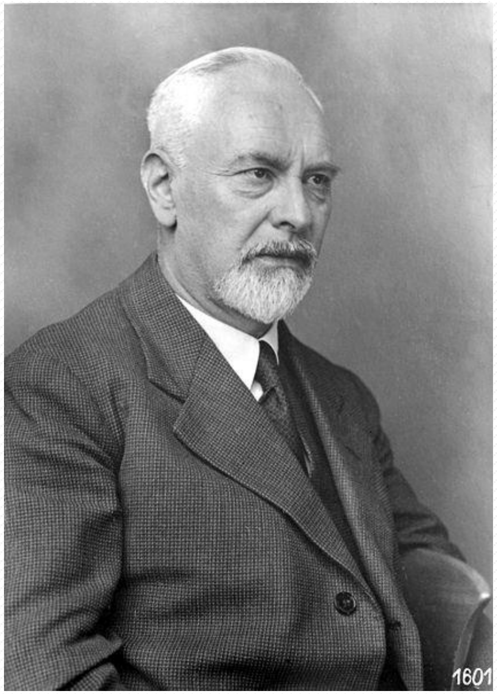
A very satisfactory explanation of the physical process in the boundary layer between a fluid and a solid body could be obtained by the hypothesis of an adhesion（粘附） of the fluid to the walls, that is, by the hypothesis of a zero relative velocity （无滑移）between fluid and wall. If the viscosity was very small and the fluid path along the wall not too long, the fluid velocity ought to resume its normal value at a very short distance from the wall. In the thin transition layer however, the sharp changes of velocity, even with small coefficient of friction, produce marked results.

Ludwig Prandtl, 1904

Prandtl （普朗特）(1875-1953)

Boundary layer theory

现代流体力学之父
空气动力学之父



He founded two institutes in Geottingen, one is now DLR, the other is now Max-Planck Institute for Dynamics and Self-organization.

概述 Overview

- 边界层理论，1904年，德国海德堡，第三届数学大会路德维希. 普朗特提出，是流体力学的巨大进步；
- 粘流分析的革命，解决了计算阻力和流动分离问题；
- N-S方程简化为边界层方程；
- 边界层概念：
  - 与表面接触的流动区域，流动因固壁和流体间的摩擦受阻。
  - 边界层很薄，但对阻力和热传导的影响很大，具有决定性的作用。

超音速流动的边界层纹影图

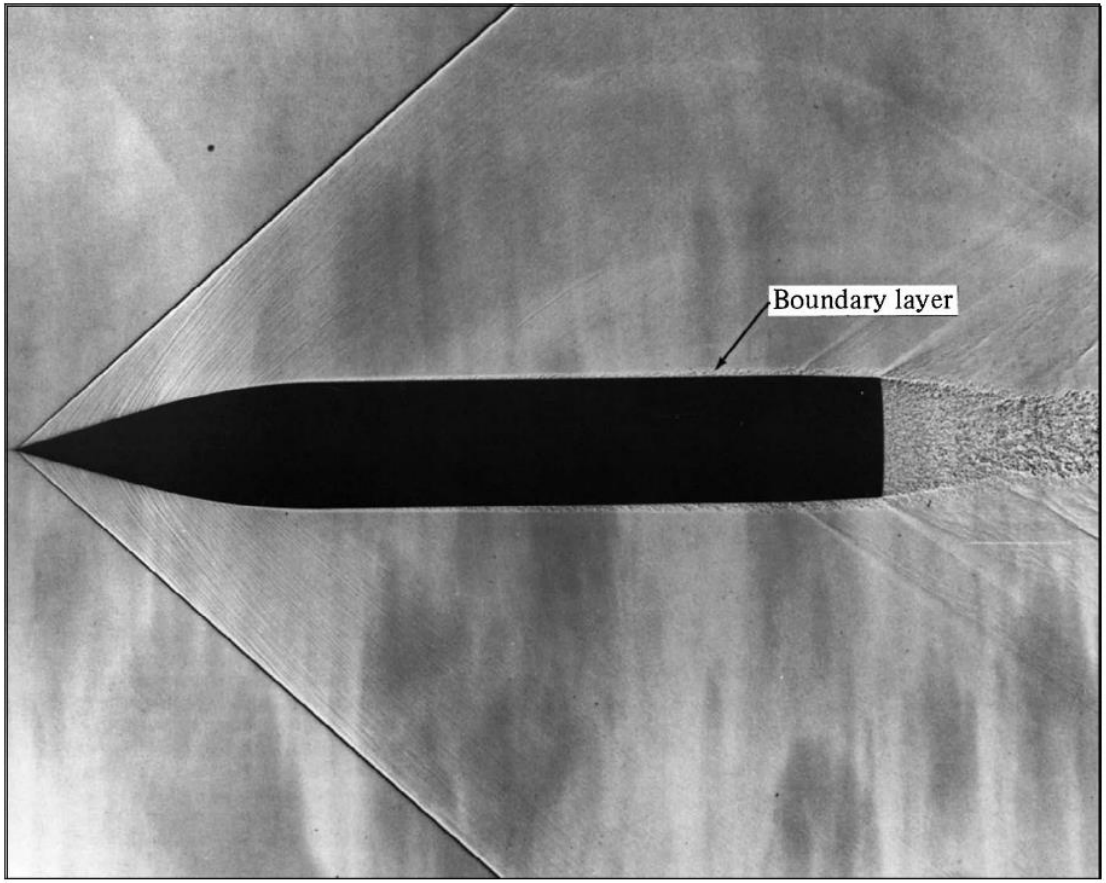


Figure 17.1 The boundary layer on an aerodynamic body. (*Courtesy of the U.S. Army Ballistics Laboratory, Aberdeen, Maryland.*)

本章内容介绍

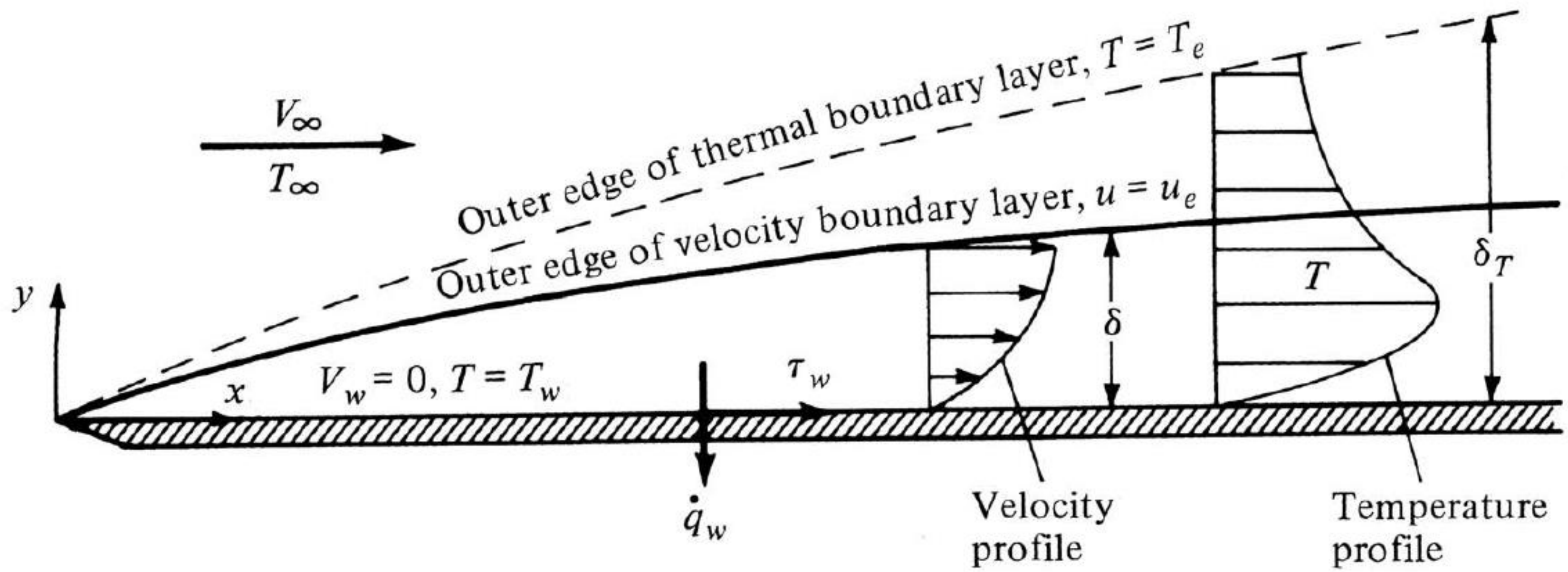
Introduction to boundary layers

- **Boundary Layer Properties**
  - *a.* Velocity and temperature profiles
  - *b.* Shear stress and heat transfer
  - *c.* Boundary layer thickness
  - *d.* Displacement thickness
  - *e.* Momentum thickness
  - *f.* Laminar and turbulent flow
- **The boundary layer equations**

第二节：边界层特性

17.2 BOUNDARY-LAYER PROPERTIES

平板边界层流动



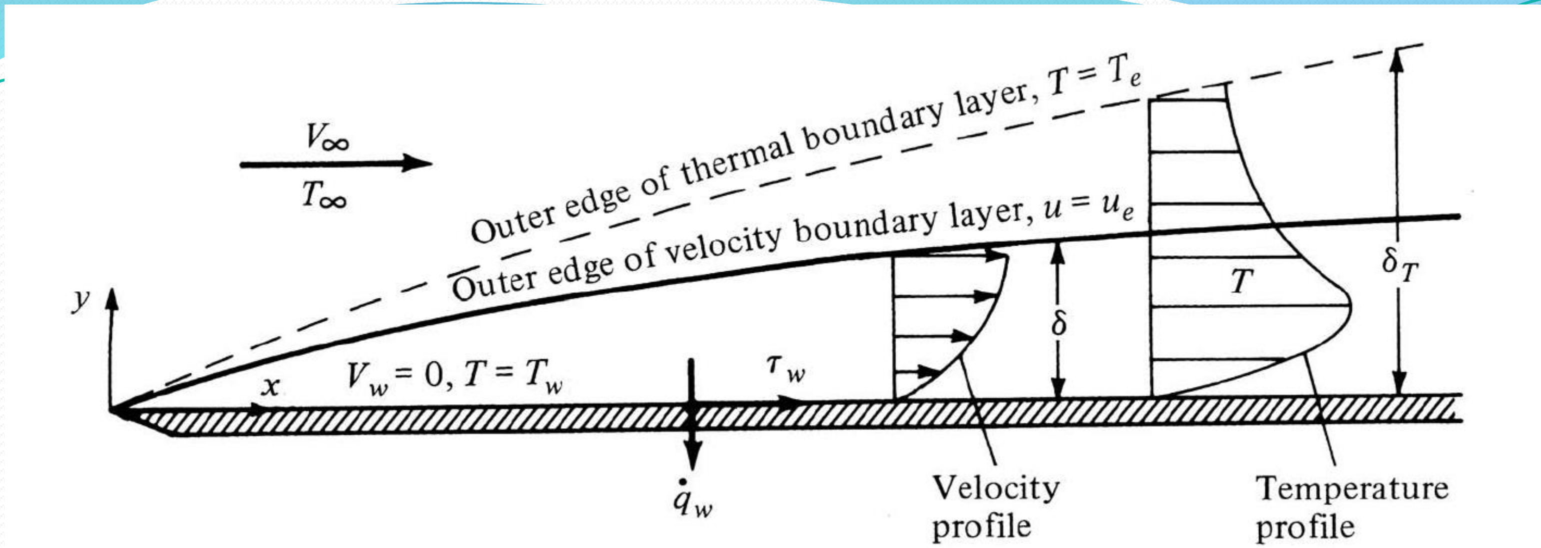
V_∞
T_∞
Outer edge of thermal boundary layer, $T = T_e$
Outer edge of velocity boundary layer, $u = u_e$
y
x
$V_w = 0, T = T_w$
τ_w
$\dot{q}_w$
δ
T
δ_T
Velocity profile
Temperature profile

速度边界层

- 流向速度随 y 增加，直到 $u=u_e$
- $y=\delta, u=0.99u_e$，u_e是自由来流解或无粘解
- δ称为速度边界层厚度
- 速度型，$u=u(y), y=(0, \delta)$，随 x 变化，

温度边界层

- 流向温度随 y 变化，直到 $T=T_e$
- $y=\delta_T, T=0.99T_e, T_e$是自由来流解或无粘解
- δ_T 称为热边界层厚度
- 温度型，$T=T(y), y=(0, \delta_T)$，随 x 变化



Velocity boundary layer thickness d 速度边界层厚度

Thermal boundary layer thickness d_T 温度边界层厚度

- 一般 $\delta_T \neq \delta$, 取决于 Pr, $\delta = \delta(x)$, $\delta_T = \delta_T(x)$

Pr=1, $\delta_T = \delta$; Pr>1, $\delta > \delta_T$; Pr<1, $\delta_T > \delta$;

$$\mathrm{Pr} = \frac{\mu_\infty c_p}{k} \propto \frac{\text{frictional dissipation}}{\text{thermal conduction}}$$

Pr of air is around 0.71

Pr of water is around 5

The consequence of the velocity gradient at the wall is the generation of shear stress at the wall

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_w$$

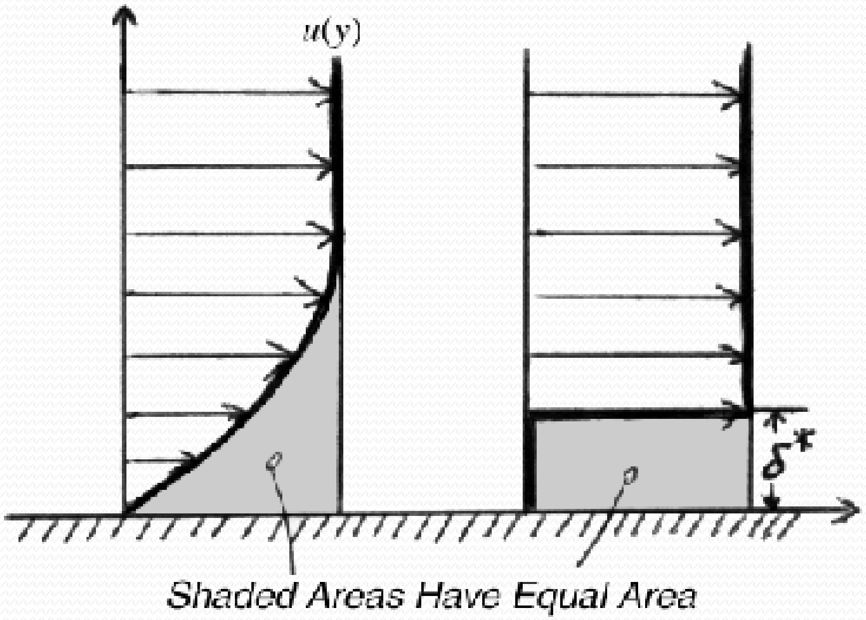
The temperature gradient at the wall generates heat transfer at the wall

$$\dot{q}_w = -k \left(\frac{\partial T}{\partial y} \right)_w$$

One of the central purposes of boundary-layer theory is to compute

$$\tau_w = \tau_w(x) \qquad \dot{q}_w = \dot{q}_w(x)$$

边界层位移厚度



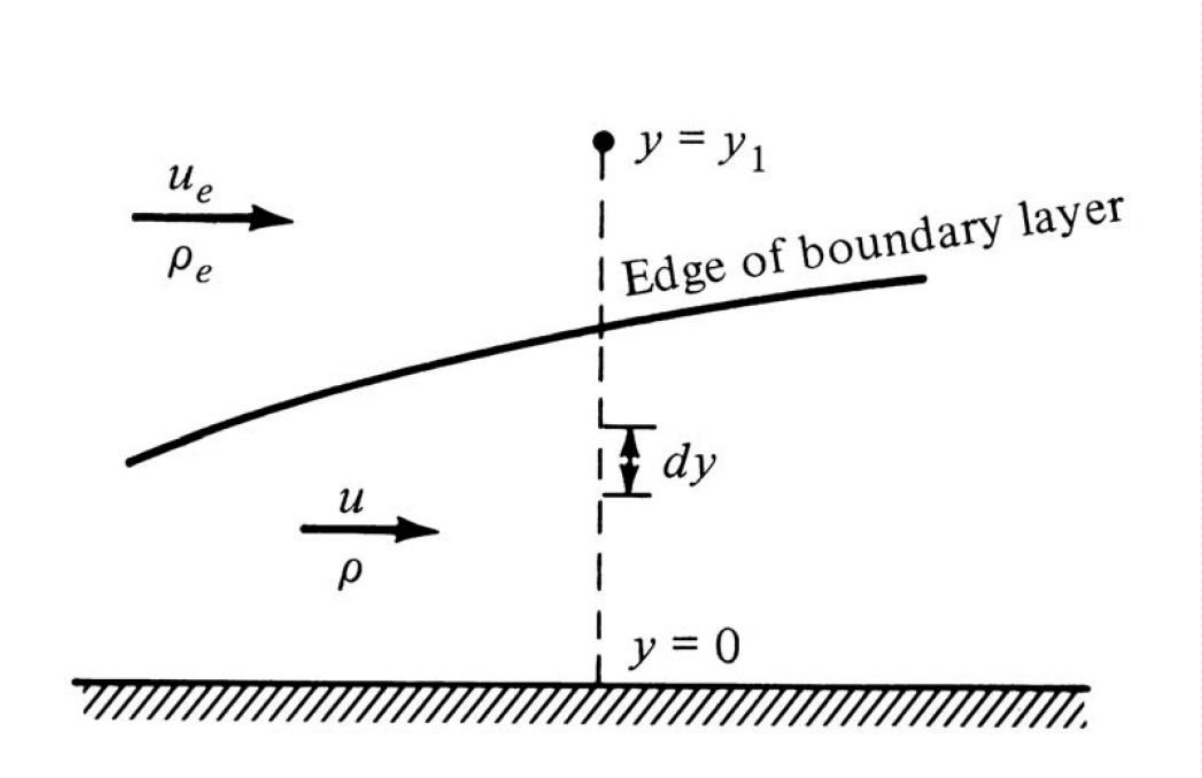
边界层位移厚度定义：

在固体壁面的边界层中，由于流速受到固体壁面的阻滞而降低，使得在这个区域内所通过的流量较之无粘流体流动时所能通过的流量减少，相当于固体壁面向流动内移动了 δ^* 距离后理想流体所通过的流量。这个距离 δ^* 称为边界层位移厚度（displacement thickness）。

$$\delta^* \equiv \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \quad \delta \le y_1 \to \infty$$

边界层位移厚度的物理意义

第一：δ^* index proportional to the "missing mass flow" due to the presence of the boundary layer. 质量流量损失



边界层位移厚度的物理意义

$$A = \text{actual mass flow between 0 and } y_1 = \int_0^{y_1} \rho u \, dy$$

$$B = \begin{array}{l}\text{hypothetical mass flow}\\ \text{between 0 and } y_1 \textit{ if } \text{boundary}\\ \text{layer were not present}\end{array} = \int_0^{y_1} \rho_e u_e \, dy$$

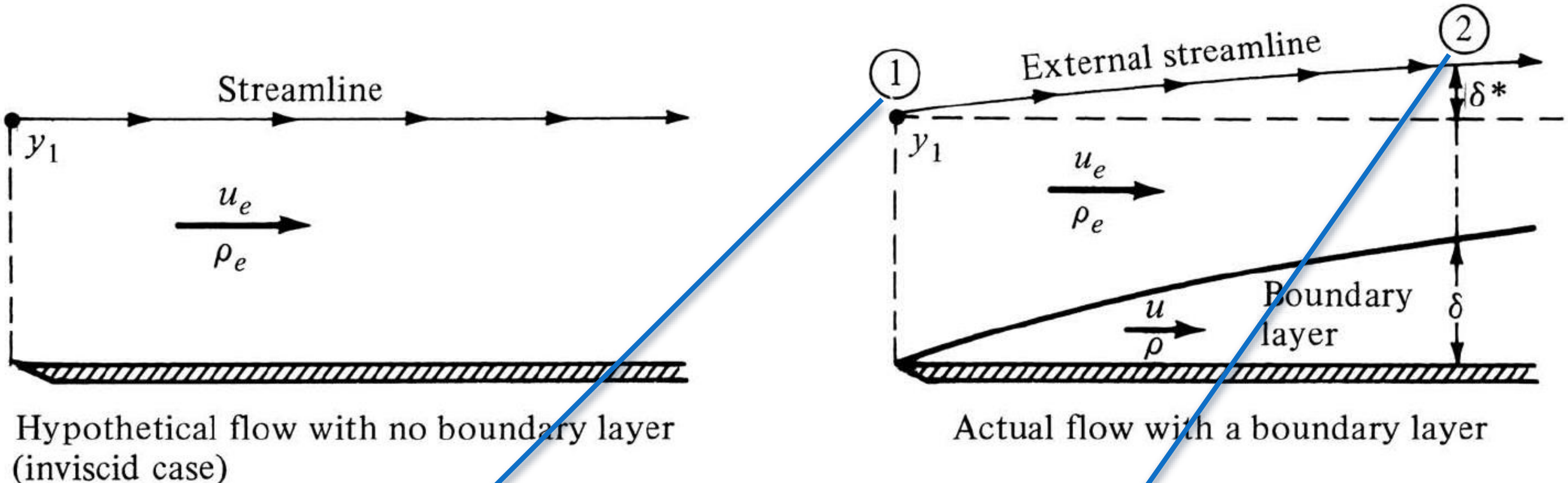
$$B - A = \begin{array}{l}\text{decrement in mass flow due to}\\ \text{presence of boundary layer, that is,}\\ \text{missing mass flow}\end{array} = \int_0^{y_1} (\rho_e u_e - \rho u) \, dy$$

表达损失的质量流量 $= \rho_e u_e \delta^*$

$$\rho_e u_e \delta^* = \int_0^{y_1} (\rho_e u_e - \rho u) \, dy$$

$$\delta^* = \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

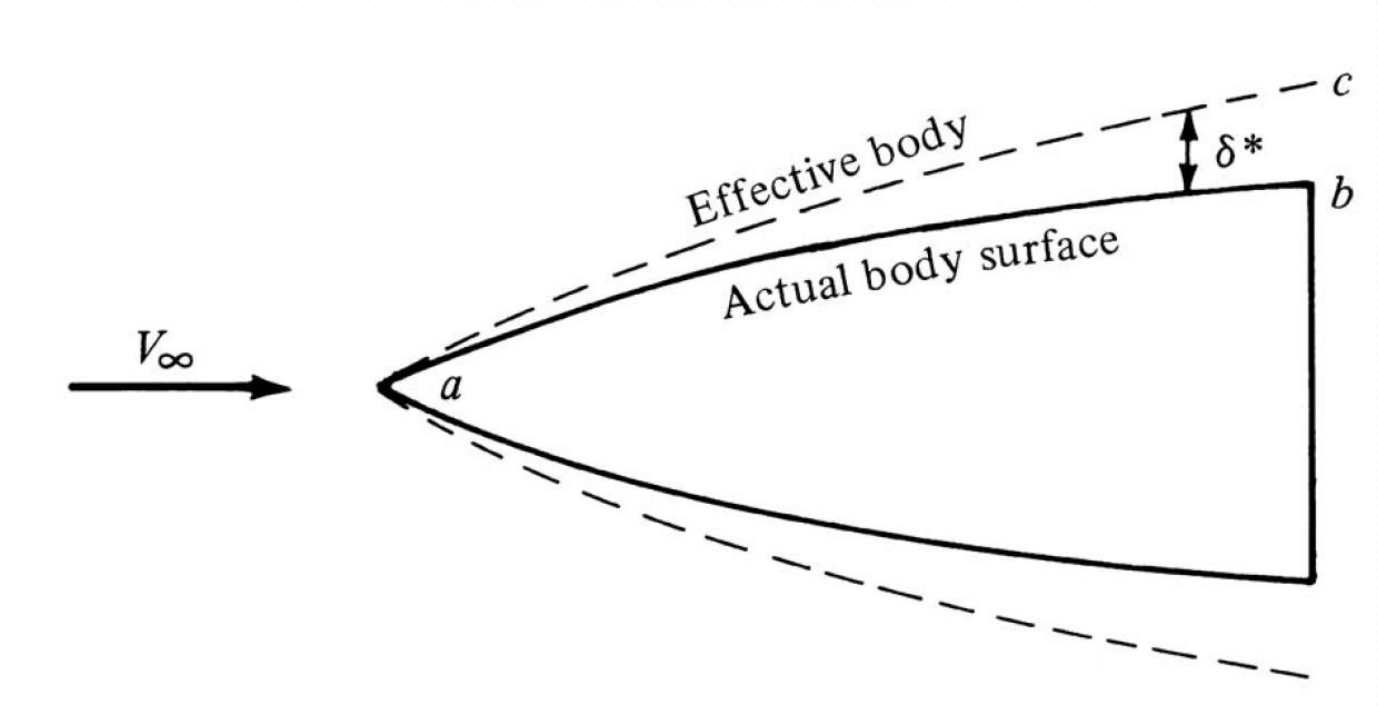
由于边界层的存在，外部无粘流被外移的距离



$$\dot{m} = \int_0^{y_1} \rho_e u_e \, dy$$

$$\dot{m} = \int_0^{y_1} \rho u \, dy + \rho_e u_e \, \delta^*$$

第二：Second interpretation of δ^* gives rise to the concept of an effective body. 等效物体



第二：Second interpretation of δ^* gives rise to the concept of an effective body. 等效物体

- 站位1质量流

$$\dot{m} = \int_0^{y_1} \rho_e u_e \, dy$$

- 站位2质量流

$$\dot{m} = \int_0^{y_1} \rho u \, dy + \rho_e u_e \, \delta^*$$

- 外部和物体表面均为流线

（质量流量相等）

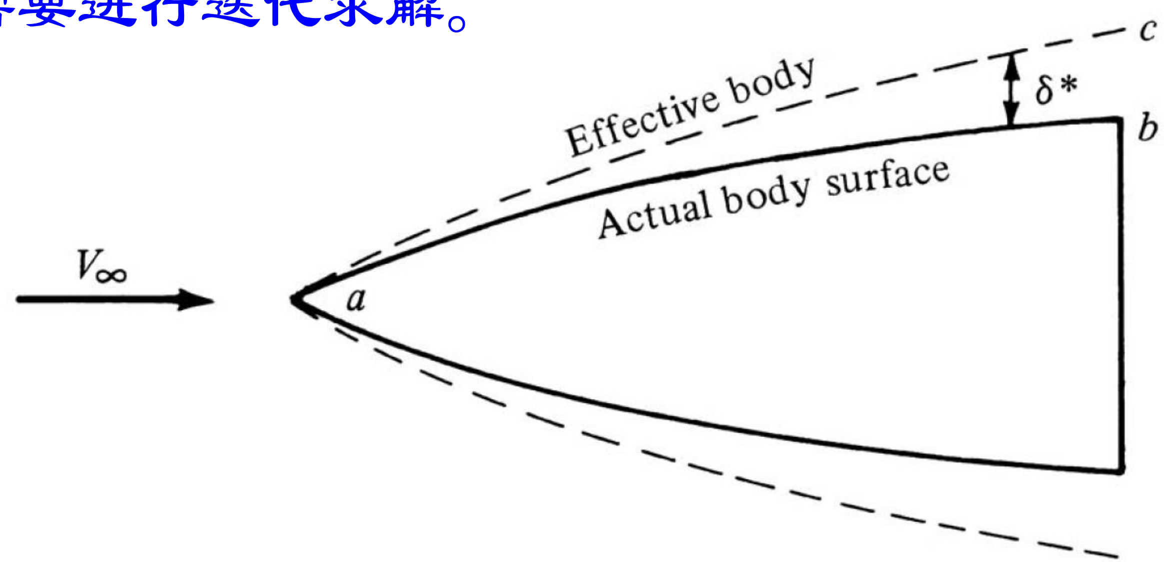
$$\int_0^{y_1} \rho_e u_e \, dy = \int_0^{y_1} \rho u \, dy + \rho_e u_e \, \delta^*$$

$$\delta^* = \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$$

等效物体概念及应用

等效体： 物体表面向外移动边界层位移厚度 δ^*后形成的计算无粘流动的等效物体。

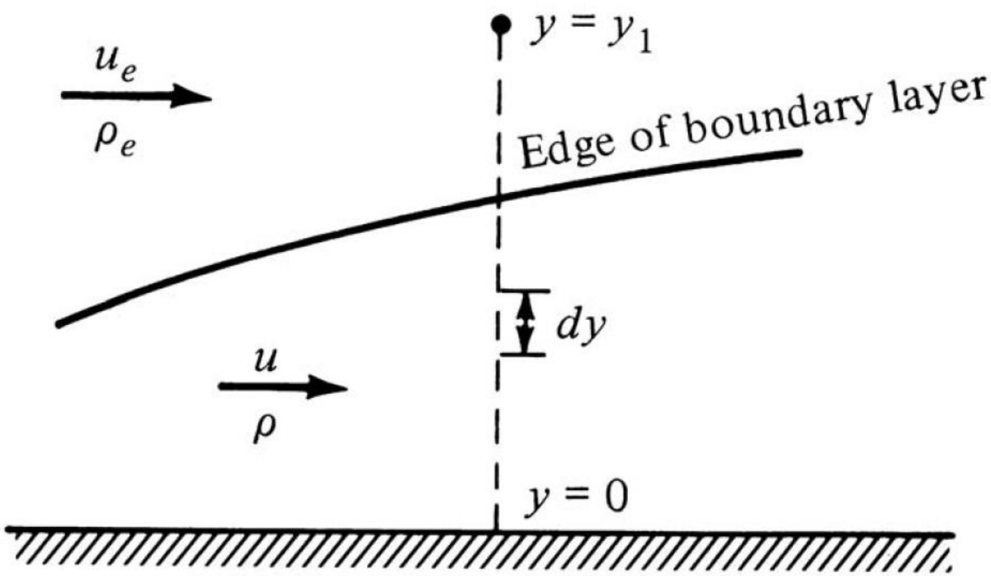
- 求解 δ^*， 需要 u, ρ, 也需要 u_e, ρ_e, T_e 用来求解边界层方程；
- 用无粘流方法求 u_e, ρ_e, T_e 需要 δ^*（等效物体）；
- 因此需要进行迭代求解。



边界层动量厚度 momentum thickness

动量厚度定义:

$$\theta \equiv \int_0^{y_1} \frac{\rho u}{\rho_e u_e}\left(1-\frac{u}{u_e}\right) dy \qquad \delta \le y_1 \to \infty$$



动量厚度的物理意义

- 质量流量： $dm = \rho u \, dy$
- 动量 A： $dm \, u = \rho u^2 \, dy$
- 与 u_e 和质量 **dm** 联系的动量 B： $dm \, u_e = (\rho u \, dy) u_e$
- 动量亏损： **B-A** = $\rho u (u_e - u) \, dy$
- 总的动量亏损 = $\int_0^{y_1} \rho u (u_e - u) \, dy$
- 定义总的动量亏损：

$$\boxed{\text{Missing momentum flow} = \rho_e u_e^2 \theta}$$

$$\rho_e u_e^2 \theta = \int_0^{y_1} \rho u (u_e - u)\, dy$$

$$\theta = \int_0^{y_1} \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e}\right) dy$$

- 动量厚度是正比于边界层存在产生动量亏损的指标
- 是一个在无粘条件下携带亏损动量的假想流管的高度
- 动量厚度与阻力系数

$$\theta(x_1) \propto \frac{1}{x_1} \int_0^{x_1} c_f\, dx = C_f$$

第三节：边界层方程

17.3 THE BOUNDARY-LAYER EQUATIONS

不可压缩二维平板

- x与y轴分别沿物面流动方向和与物面垂直，坐标原点与物面前缘重合。
- 以平板长度L为特征长度；以来流速度U为特征速度。
- 边界层内x方向的尺度为$x \sim L$，x方向的速度分量的量级为$u \sim U$；y方向的尺度为$y \sim \delta$。
- 回顾：Navier-Stokes方程组连续性方程

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

各项量级应相等

$$\frac{\partial v}{\partial y} \sim \left|\frac{\partial u}{\partial x}\right| \sim \frac{U}{L} \tag{2}$$

则

$$v \sim \frac{U}{L}\delta, \quad \frac{\partial v}{\partial x} \sim \frac{U\delta}{L^2}, \quad \frac{\partial^2 v}{\partial x^2} \sim \frac{U\delta}{L^3}, \quad \frac{\partial^2 v}{\partial^2 y} \sim \frac{U}{L\delta} \tag{3}$$

不可压缩二维平板

- 回顾：Navier-Stokes方程组动量方程

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

各项量级估计

$$\frac{\partial u}{\partial x}\sim\frac{U}{L},\quad \frac{\partial u}{\partial y}\sim\frac{U}{\delta},\quad \frac{\partial^2 u}{\partial x^2}\sim\frac{U}{L^2},\quad \frac{\partial^2 u}{\partial y^2}\sim\frac{U}{\delta^2}$$

对于粘性项$\nu\nabla^2\mathbf{u}$，有

$$\left|\frac{\partial^2 u}{\partial x^2}\right|\ll\left|\frac{\partial^2 u}{\partial y^2}\right|,\quad \left|\frac{\partial^2 v}{\partial x^2}\right|\ll\left|\frac{\partial^2 v}{\partial y^2}\right|$$

不可压缩二维平板

假设非定常项$\frac{\partial u}{\partial t}$的量级不会超过惯性项$u\frac{\partial u}{\partial x}$的量级。 局部速度发生变化的时间量级为$L/U$，有

$$\frac{\partial u}{\partial t} \sim u\frac{\partial u}{\partial x} \sim \frac{U^2}{L}, \quad \frac{\partial v}{\partial t} \sim u\frac{\partial v}{\partial x} \sim \frac{U^2}{L^2}\delta$$

压力项与惯性项同量级，有

$$\frac{\partial p}{\partial x} \sim \frac{\rho U^2}{L}$$

边界层内粘性项与惯性项同量级，有

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2}$$

由以上得到

$$\frac{\delta}{L} \sim \sqrt{\frac{\nu}{UL}} = Re^{-1/2}$$

不可压缩二维平板

引入以下无量纲量（这些无量纲量及其导数都是同一量级）

$$x' = x/L, \quad y' = y/\delta = Re^{1/2}y/L, \quad t' = t/(L/U) = Ut/L$$
$$u' = u/U, \quad v' = v/(U\delta/L) = Re^{1/2}v/U, \quad p' = (p - p_0)/(\rho U^2).$$

代入Navier-Stokes方程后得到无量纲化的方程为

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{Re}\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}$$
$$\frac{1}{Re}\left(\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right) = -\frac{\partial p'}{\partial y'} + \frac{1}{Re^2}\frac{\partial^2 v'}{\partial x'^2} + \frac{1}{Re}\frac{\partial^2 v^2}{\partial y'^2}$$

不可压缩二维平板

当Re数是大数时，忽略掉$O(1/Re)$以上的高阶小量，得到近似的边界层方程

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2}$$

$$\frac{\partial p'}{\partial y'} = 0$$

穿过边界层压强沿y方向不变，则

$$p \approx p_e(x), \quad \frac{\partial p}{\partial x} \approx \frac{\partial p_e}{\partial x}$$

外流为无粘流动，由欧拉方程

$$\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x}$$

不可压缩二维平板

(有量纲的)二维边界层方程

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{18}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{19}$$

边界条件

- $x > 0$, $y = 0$ 处，$u = v = 0$，即固壁无滑移条件。
- $x > 0$, $y \to \infty$处，$u(x, y, t) \to U_e(x, t)$，即外流无粘条件。

流动控制方程无量纲化 Nondimensionalize

给定参考量：

ρ_∞, V_∞, p_∞, T_∞ and μ_∞, k_∞

are reference values (say, e.g., freestream values)

c is a reference length (say, length of a flat plate).

$$\rho' = \frac{\rho}{\rho_\infty} \quad u' = \frac{u}{V_\infty} \quad v' = \frac{v}{V_\infty} \quad p' = \frac{p}{p_\infty}$$

$$\mu' = \frac{\mu}{\mu_\infty} \quad x' = \frac{x}{c} \quad y' = \frac{y}{c}$$

$$e' = \frac{e}{c_v T_\infty} \quad k' = \frac{k}{k_\infty} \quad V'^2 = \frac{V^2}{V_\infty^2} = \frac{u^2+v^2}{V_\infty^2} = (u')^2 + (v')^2$$

二维定常无量纲流动控制方程：

$$\frac{\partial \rho' u'}{\partial x'} + \frac{\partial \rho' v'}{\partial y'} = 0$$

连续性方程

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial y'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right]$$

动量方程

$$\rho' u' \frac{\partial v'}{\partial x'} + \rho' v' \frac{\partial v'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial y'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial x'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right]$$

$$\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty^2 \left[\rho' u' \frac{\partial}{\partial x'} (u'^2 + v'^2) + \rho' v' \frac{\partial}{\partial y'} (u'^2 + v'^2) \right]$$

能量方程

$$= \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty} \left[\frac{\partial}{\partial x'} \left(k' \frac{\partial T'}{\partial x'} \right) + \frac{\partial}{\partial y'} \left(k' \frac{\partial T'}{\partial y'} \right) \right] - (\gamma - 1) \left(\frac{\partial (u' p')}{\partial x'} + \frac{\partial (v' p')}{\partial y'} \right)$$

$$+ \gamma(\gamma - 1) \frac{M_\infty^2}{\mathrm{Re}_\infty} \left\{ \frac{\partial}{\partial x'} \left[\mu' v' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] + \frac{\partial}{\partial y'} \left[\mu' u' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] \right\}$$

方程定解条件 —流动边界条件

粘性流动相似参数

通过无量纲流动控制方程得到，为 γ, M_∞, Re, Pr

γ, M_∞, Re 从动量方程和能量方程得出

$$\mathrm{Pr} = \frac{\mu c_p}{k}$$

需从能量方程导出

可压流时需要，为一物性参数，不同气体值不同，标准大气时，为 0.71，与 μ 和 k 一样，是温度 T 的函数。

$$\mathrm{Pr} = \frac{\mu_\infty c_p}{k} \propto \frac{\text{frictional dissipation}}{\text{thermal conduction}}$$

边界层方程 Equations in boundary-layer

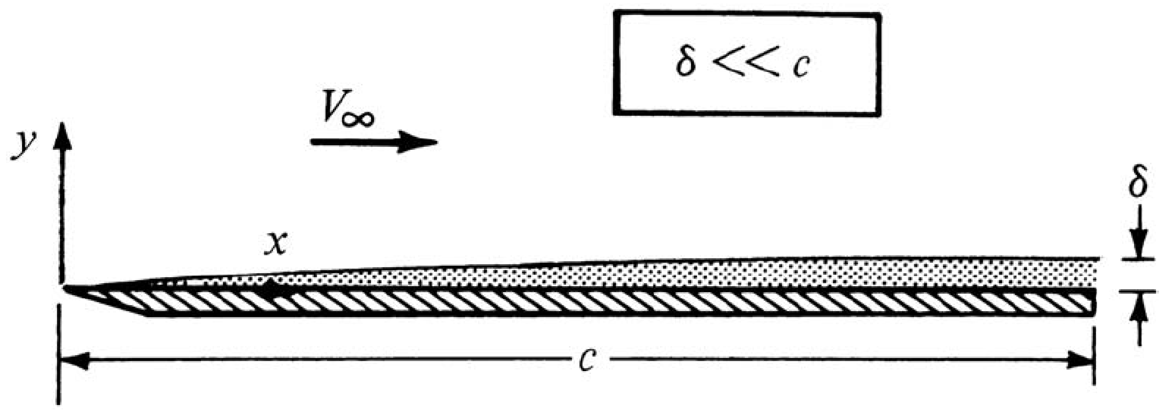


Figure 17.7 The basic assumption of boundary-layer theory: A boundary layer is very thin in comparison with the scale of the body.

In boundary layer $\delta << c$

where δ is the boundary-layer thickness

边界层方程 Equations in boundary-layer

Thus, in boundary layer：

$$x' = \frac{x}{c} \sim O(1) \qquad y' = \frac{y}{c} \sim O(\delta)$$

高雷诺数流动假设

$$u' = \frac{u}{V_\infty} \sim O(1) \qquad \frac{\partial u'}{\partial x'} \sim O(1) \qquad \mathrm{Re}_\infty = \frac{\rho_\infty V_\infty c}{\mu_\infty} \sim \frac{1}{O(\delta^2)}$$

$$v' = \frac{v}{V_\infty} \sim O(\delta) \qquad \frac{\partial v'}{\partial y'} \sim O(1) \qquad M_\infty^2 = \frac{V_\infty^2}{a_\infty^2} \sim O(1)$$

$$\frac{\partial^2 u'}{\partial x'^2} \sim O(1) \qquad \frac{\partial^2 u'}{\partial y'^2} \sim O(\frac{1}{\delta^2}) \qquad \rho' = \frac{\rho}{\rho_\infty} \sim O(1)$$

边界层方程 Equations in boundary-layer

连续性方程：

$$\frac{\partial(\rho' u')}{\partial x'} + \frac{\partial(\rho' v')}{\partial y'} = 0$$

$$\frac{[O(1)][O(1)]}{O(1)} + \frac{[O(1)][O(\delta)]}{O(\delta)} = 0$$

有量纲连续性方程：

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

边界层方程 Equations in boundary-layer

x 方向动量方程：

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial y'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right]$$

$$\rho' u' \frac{\partial u'}{\partial x'} = O(1) \qquad \rho' v' \frac{\partial u'}{\partial y'} = O(1) \qquad \frac{\partial p'}{\partial x'} = O(1)$$

$$\frac{\partial}{\partial y'} \left(\mu' \frac{\partial v'}{\partial x'} \right) = O(1) \qquad \frac{\partial}{\partial y'} \left(\mu' \frac{\partial u'}{\partial y'} \right) = O\left(\frac{1}{\delta^2} \right) \qquad \frac{1}{\mathrm{Re}_\infty} = O(\delta^2)$$

$$O(1) + O(1) = -\frac{1}{\gamma M_\infty^2} O(1) + O(\delta^2) \left[O(1) + O\left(\frac{1}{\delta^2} \right) \right]$$

$$\Rightarrow \quad \rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial y'} \left(\mu' \frac{\partial u'}{\partial y'} \right)$$

边界层方程 Equations in boundary-layer

x 方向有量纲动量方程：

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

y 方向动量方程：

$$\rho' u' \frac{\partial v'}{\partial x'} + \rho' v' \frac{\partial v'}{\partial y'} = -\frac{1}{\gamma M_\infty^2}\frac{\partial p'}{\partial y'} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial x'}\left[\mu'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]$$

$$O(\delta) + O(\delta) = -\frac{1}{\gamma M_\infty^2}\frac{\partial p'}{\partial y'} + O(\delta^2)\left[O(\delta) + O\left(\frac{1}{\delta}\right)\right]$$

$$\Rightarrow \quad \frac{\partial p'}{\partial y'} = O(\delta) = 0 \quad \Rightarrow \quad p = p(x) = p_e(x)$$

y 方向有量纲动量方程： $\dfrac{\partial p}{\partial y} = 0$

边界层方程 Equations in boundary-layer

控制方程：

Continuity: $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

x momentum: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

y momentum: $$\frac{\partial p}{\partial y} = 0$$

Energy: $$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + u \frac{dp_e}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

理想气体状态方程

$$p = \rho R T$$

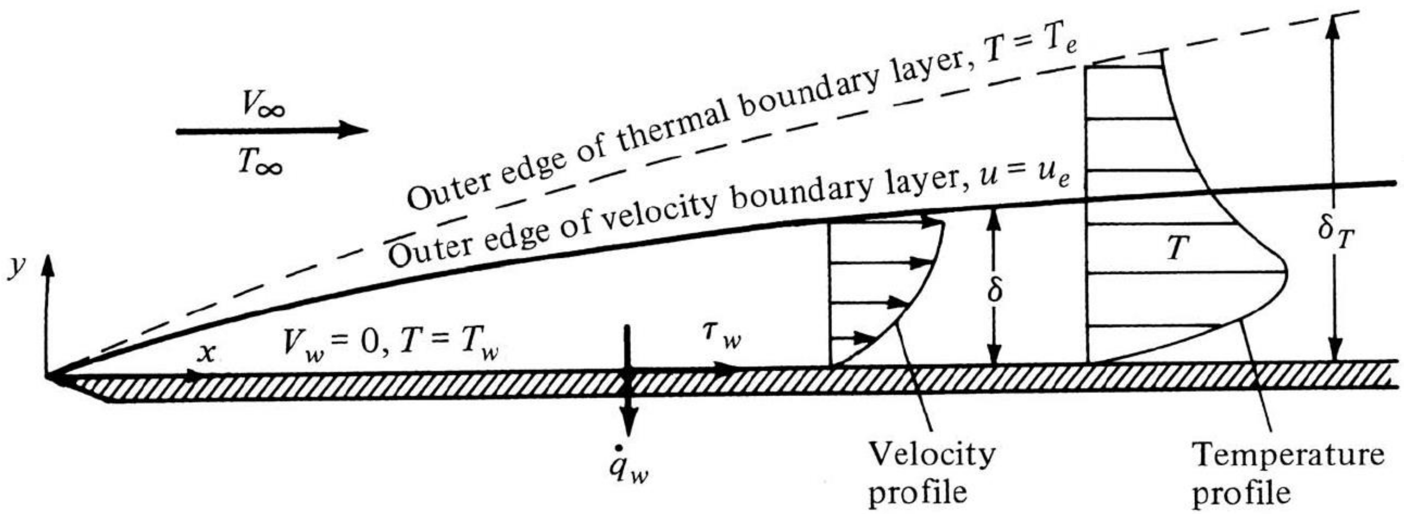
$$h = c_p T$$

定解条件： five unknowns, u, v, ρ, T, and h

At the wall: $y = 0 \quad u = 0 \quad v = 0 \quad T = T_w$

At the boundary-layer edge: $y \to \infty \quad u \to u_e \quad T \to T_e$

边界层流动控制方程是普朗特1904年首先提出的



Velocity boundary layer thickness δ 速度边界层厚度

Thermal boundary layer thickness δ_T 温度边界层厚度

- 一般 $\delta_T \neq \delta$, 取决于 Pr, $\delta = \delta(x)$, $\delta_T = \delta_T(x)$

Pr=1, $\delta_T = \delta$; Pr>1, $\delta > \delta_T$; Pr<1, $\delta_T > \delta$;

$$\mathrm{Pr} = \frac{\mu_\infty c_p}{k} \propto \frac{\text{frictional dissipation}}{\text{thermal conduction}}$$

Pr of air is around 0.71

Pr of water is around 5

边界层方程 Equations in boundary-layer

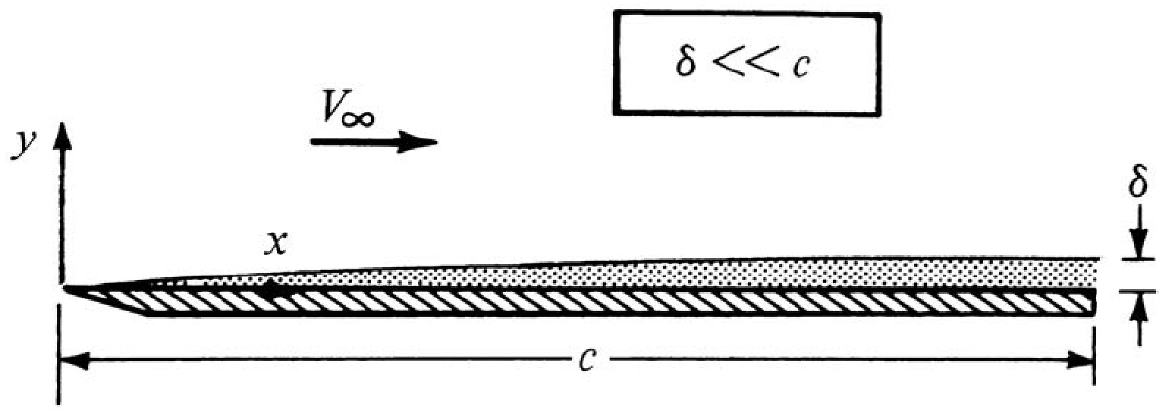


Figure 17.7 The basic assumption of boundary-layer theory: A boundary layer is very thin in comparison with the scale of the body.

In boundary layer $\delta << c$

where δ is the boundary-layer thickness

边界层方程 Equations in boundary-layer

Thus, in boundary layer：

高雷诺数流动假设

$$x' = \frac{x}{c} \sim O(1) \qquad y' = \frac{y}{c} \sim O(\delta)$$

$$u' = \frac{u}{V_\infty} \sim O(1) \qquad \frac{\partial u'}{\partial x'} \sim O(1) \qquad \mathrm{Re}_\infty = \frac{\rho_\infty V_\infty c}{\mu_\infty} \sim \frac{1}{O(\delta^2)}$$

$$v' = \frac{v}{V_\infty} \sim O(\delta) \qquad \frac{\partial v'}{\partial y'} \sim O(1) \qquad M_\infty^2 = \frac{V_\infty^2}{a_\infty^2} \sim O(1)$$

$$\frac{\partial^2 u'}{\partial x'^2} \sim O(1) \qquad \frac{\partial^2 u'}{\partial y'^2} \sim O\left(\frac{1}{\delta^2}\right) \qquad \rho' = \frac{\rho}{\rho_\infty} \sim O(1)$$

边界层方程 Equations in boundary-layer

***x* 方向动量方程：**

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial y'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right]$$

$$\rho' u' \frac{\partial u'}{\partial x'} = O(1) \qquad \rho' v' \frac{\partial u'}{\partial y'} = O(1) \qquad \frac{\partial p'}{\partial x'} = O(1)$$

$$\frac{\partial}{\partial y'} \left(\mu' \frac{\partial v'}{\partial x'} \right) = O(1) \qquad \frac{\partial}{\partial y'} \left(\mu' \frac{\partial u'}{\partial y'} \right) = O\left(\frac{1}{\delta^2} \right)$$

$$O(1) + O(1) = -\frac{1}{\gamma M_\infty^2} O(1) + O(\delta^2) \left[O(1) + O\left(\frac{1}{\delta^2} \right) \right]$$

$$\Rightarrow \quad \frac{1}{\mathrm{Re}_\infty} = O(\delta^2)$$

边界层方程 Equations in boundary-layer

控制方程：

Continuity: $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

x momentum: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

y momentum: $$\frac{\partial p}{\partial y} = 0$$

Energy: $$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + u \frac{dp_e}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

理想气体状态方程

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At the boundary-layer edge: $y \to \infty \quad u \to u_e \quad T \to T_e$

边界层流动控制方程是普朗特1904年首先提出的

第四节：如何求解边界层方程

17.4 HOW DO WE SOLVE THE BOUNDARY-LAYER EQUATIONS?

- The equations are a coupled system of nonlinear partial differential equations for which no general analytical solution has been obtained to date.
- The boundary-layer equations are simpler than the Navier-Stokes equations, especially the boundary-layer y-momentum equation.
- At any axial location along the surface the pressure is constant in the direction normal to the surface.
- For almost one hundred years, engineers and scientists have nudged the boundary layer equations in many different ways, and have come up with reasonable solutions for a number of practical applications.

Solutions of the boundary layer equations can be classified into two groups:

(1) Classical solutions（经典解）, some of which date back to 1908.

(2) Numerical solutions （数值解） obtained by modern computational fluid dynamic techniques.

- In addition to this subdivision based on the solution technique, boundary layer solutions also subdivide on a physical basis into laminar boundary layers（层流边界层） and turbulent boundary layers（湍流边界层）.
- To date, no exact solution has been obtained for turbulent boundary layers, because we still do not have a complete understanding of turbulence.

Finally, we note what is meant by a "boundary-layer solution."

The solution of Equations yields the velocity and temperature profiles throughout the boundary layer.

However, the practical information we want is the solution for the surface shear stress and heat transfer, respectively:

$$\tau_w = \mu_w \left(\frac{\partial u}{\partial y} \right)_w$$

$$\dot{q}_w = k_w \left(\frac{\partial T}{\partial y} \right)_w$$

We first have to solve the boundary-layer equations for the velocity and temperature profiles throughout the boundary layer.

边界层方程的解法

- 经典解和数值解（CFD）
- 层流解和湍流解-流态对解的影响
- 边界层方程的解为速度型和温度型
- 最终转换为剪应力、热传导率（实际关心的量）

第五节：小结

17.5 SUMMARY

- The basic quantities of interest from boundary-layer theory are the velocity and thermal boundary-layer thicknesses, δ and δ_T , respectively, the shear stress at the wall, τ_w, and heat transfer to the surface, $\dot{q}_w$.

the displacement thickness 边界层位移厚度

$$\delta^* \equiv \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \qquad \delta \le y_1 \to \infty$$

the momentum thickness 边界层动量厚度

$$\theta \equiv \int_0^{y_1} \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e}\right) dy \qquad \delta \le y_1 \to \infty$$

- The body shape plus δ^* defines a new effective body seen by the inviscid flow.

- By an order-of-magnitude analysis（量阶分析）, the complete Navier-Stokes equations for two-dimensional flow reduce to the following boundary-layer equations:

Continuity: $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

x momentum: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

y momentum: $$\frac{\partial p}{\partial y} = 0$$

Energy: $$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + u \frac{dp_e}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

- These equations are subject to the boundary conditions:

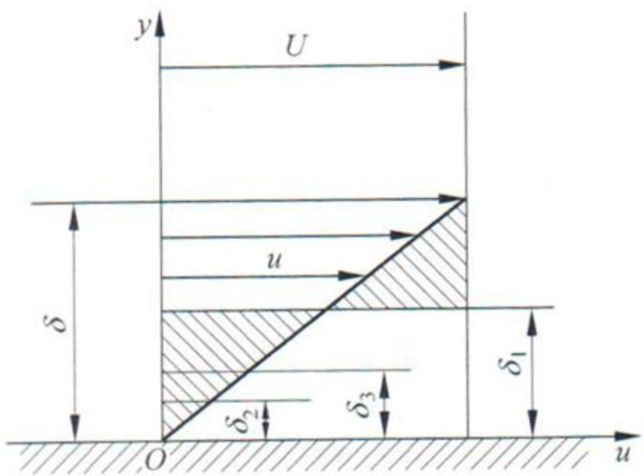
At the wall: $y = 0 \quad u = 0 \quad v = 0 \quad h = h_w$

At the boundary-layer edge: $y \to \infty \quad u \to u_e \quad h \to h_e$

- In the above boundary-layer equations are the assumptions that $\delta \ll c$, *Re* is large, and M_∞ is not inordinately large.

第四次课后作业---题1：

$$\frac{u}{U} = \frac{y}{\delta}$$



对于不可压缩流动，已知边界层内的速度分布：

$$u = \frac{y}{\delta}U$$

求边界层位移厚度和动量厚度：

$$\boxed{\delta^* \equiv \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \quad \delta \le y_1 \to \infty}$$

$$\delta^* = \int_0^{\infty} (1 - \frac{u}{U})dy$$

$$\boxed{\theta \equiv \int_0^{y_1} \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy \qquad \delta \le y_1 \to \infty}$$

$$\theta = \int_0^{\infty} \frac{u}{U}(1 - \frac{u}{U})dy$$

第四次课后作业---题2：

对以下二维定常无量纲能量方程进行量阶分析

（order-of-magnitude analysis）

$$\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty^2 \left[\rho' u' \frac{\partial}{\partial x'}(u'^2 + v'^2) + \rho' v' \frac{\partial}{\partial y'}(u'^2 + v'^2) \right]$$

$$= \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty} \left[\frac{\partial}{\partial x'} \left(k' \frac{\partial T'}{\partial x'} \right) + \frac{\partial}{\partial y'} \left(k' \frac{\partial T'}{\partial y'} \right) \right] - (\gamma - 1) \left(\frac{\partial (u'p')}{\partial x'} + \frac{\partial (v'p')}{\partial y'} \right)$$

$$+ \gamma(\gamma-1) \frac{M_\infty^2}{\mathrm{Re}_\infty} \left\{ \frac{\partial}{\partial x'} \left[\mu' v' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] + \frac{\partial}{\partial y'} \left[\mu' u' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] \right\}$$

推导以下边界层有量纲能量方程：

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y} \right)^2$$

第四次作业：

- 要求：
  - 在A4纸上完成
  - 截止日期：5月26日至5月29日之间
  - 提交方式：电子版（拍照或扫描）
  - 邮件发送，主题“第四次作业_学号_姓名”
  - 例如：第四次作业_2017300300_张三
  - 助教邮箱：912387046@mail.nwpu.edu.cn