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西北工业大学考试试题（卷）

2019 － 2020 学年第 2 学期

开课学院 航空学院 课程 粘性流体力学导论 学时 32

考试日期 2020 年 6 月 23 日 考试时间___小时 考试形式（ 报告 ）（ A ）卷

题号										总分
得分										

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题 1、思考简答

（1）粘性的微观物理机制是什么？

（2）雷诺数（Reynolds number）的含义？

（3）哪种边界层对壁面的剪切力大？

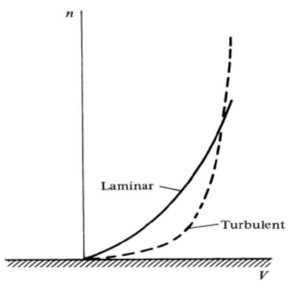


Figure 15.6 Schematic of velocity profiles for laminar and turbulent flows.

$$\left[\left(\frac{\partial V}{\partial n}\right)_{n=0}\right]_{\text{turbulent}} > \left[\left(\frac{\partial V}{\partial n}\right)_{n=0}\right]_{\text{laminar}}$$

答 1：（1）粘性的微观物理机制表述如下。

从微观角度，流体各层之间的摩擦力本质是由于流体分子不停顿地进行不规则的热运动而导致的。这种不规则的分子热运动会使得不同流层间的流体分子质量进行交换，而若流动流体中各层速度彼此不相同的话，邻层中的两个流体分子的动量必然不相同，则这种质量交换会带来动量交换，宏观上则表现为各流层间的作用摩擦力，即流体产生粘性。

（2）雷诺数（Reynolds number）的含义表述如下。

定义雷诺数Re为

$$\mathrm{Re}=\frac{\rho VL}{\mu} \quad \text{或} \quad \mathrm{Re}=\frac{VL}{\nu}$$

其中，ρ表示流体密度，V表示流体运动速度，L表示特征长度，μ表示动力粘性系数，ν表示运动粘性系数，满足关系式$\nu=\frac{\mu}{\rho}$。

雷诺数的大小表示流体的惯性力和粘性力之比。雷诺数越大，惯性力在流体中起到的作用越大，粘性力对流体流动的影响越小。

（3）湍流边界层对壁面的剪切力大。

在贴近壁面的区域内，湍流边界层的速度梯度比层流边界层更大，速度增加更加迅速，对壁面的剪切作用力更大。

即满足以下关系式

$$\left[\left(\frac{\partial V}{\partial n}\right)_{n=0}\right]_{\text{turbulent}}>\left[\left(\frac{\partial V}{\partial n}\right)_{n=0}\right]_{\text{laminar}}$$

题 2、思考简答

在雷诺实验装置中，雷诺为何需要站在高台上？分析说明层流、湍流和流动转捩三种不同的流动现象的特性。

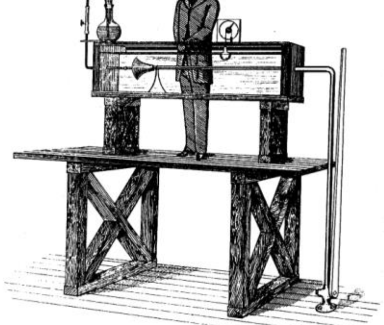
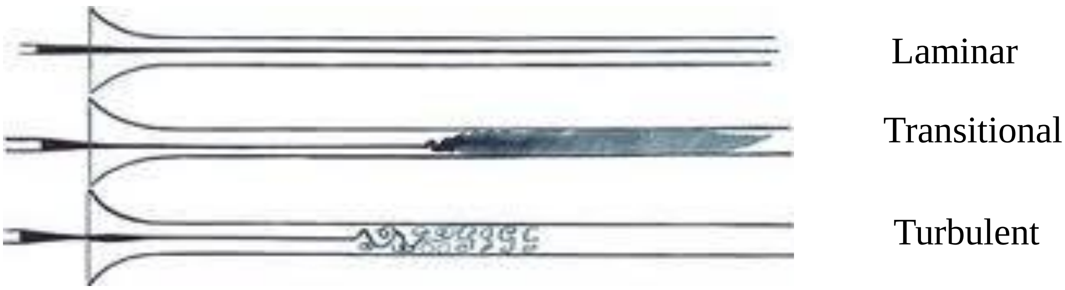


Fig. 9.1. Sketch of Reynolds's dye experiment, taken from his 1883 paper



答 2：

① 在雷诺实验装置中，雷诺需要将有色染料通过细管注入圆管中，以显示圆管中运动流体流动情况。在控制圆管内流体流速变大或变小的过程中，由于当时雷诺没有现代的恒压水泵装置，因此他需要站在高处，通过离地一定高度产来生圆管进口与出口之间的气压差，从而能够驱动管内流体运动，并通过圆管出口处的阀门调节流速。

② 总结层流、湍流和流动转捩三种不同的流动现象特性如下。

层流流动：圆管中流体分层流动，互不混淆，流线和迹线互相平行，呈现出规则平滑的状态；

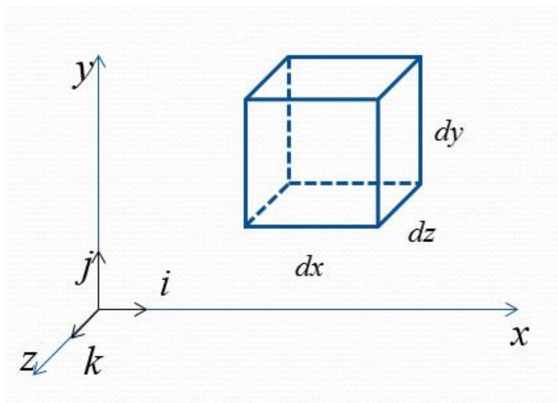
湍流流动：圆管中流体运动趋于紊乱，染色迹线剧烈振荡到破裂，呈现出不规则、较为无序的状态；

流动转捩：是流体流动处于完全层流和完全湍流之间的中间过渡状态。随着雷诺数的增大，圆管内流体流动首先出现周期性的扰动，进而规则性流动被逐渐破坏，逐渐由规则平滑的层流状态转变为不规则的湍流状态。

题 3、理论推导

对于可压缩各向同性牛顿流体，按照牛顿第二定律，采用无限小流体单元模型，推导 y 方向的流动动量方程：

$$r\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[m\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left(l(\nabla\cdot\mathbf{V})+2m\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left[m\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+rf_y$$



解 3：对于可压缩各向同性牛顿流体，推导 y 方向的流动动量方程过程如下：

取一个固定于空间的无限小流体微团模型如图 1 所示。

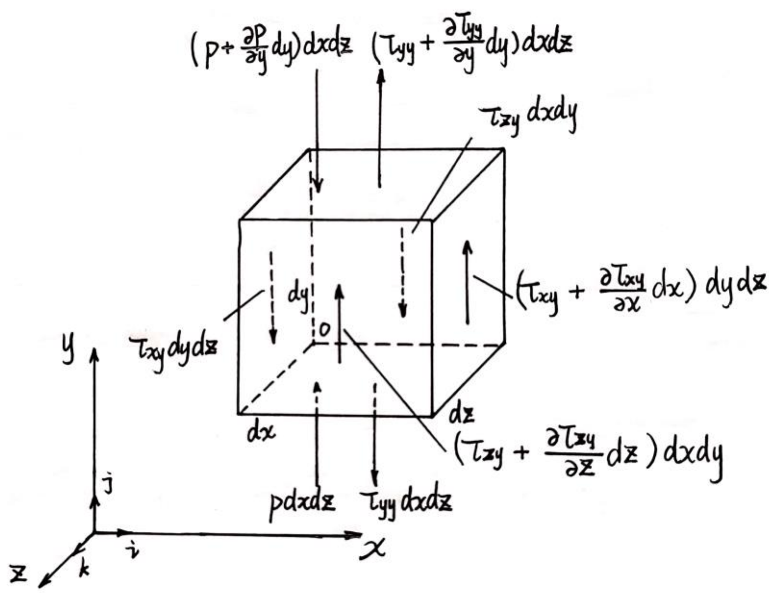


图 1　无限小流体微团模型示意图

考虑该流体微团在 y 方向上的受力，由两部分构成，可表示为

$$F_y = F_{y,surface} + F_{y,body} \tag{1}$$

上式中，第一部分为流体微团在 y 方向所受表面力，有

$$\begin{aligned} F_{y,surface} &= \left[pdxdz - \left(p + \frac{\partial p}{\partial y}dy \right) dxdz \right] \\ &+ \left[\left(\tau_{yy} + \frac{\partial \tau_{yy}}{\partial y}dy \right) dxdz - \tau_{yy}dxdz \right] \\ &+ \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx \right) dydz - \tau_{xy}dydz \right] \\ &+ \left[\left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}dz \right) dxdy - \tau_{zy}dxdy \right] \\ &= \left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} \right) dxdydz \end{aligned} \tag{2}$$

第二部分为流体微团在 y 方向所受体积力，有

$$F_{y,body} = \rho f_y dxdydz \tag{3}$$

其中，ρ 为各向同性牛顿流体密度，f_y 是作用在流体微团 y 方向上的单位质量体积力。

故流体微团在 y 方向上所受合外力为

$$F_y = F_{y,surface} + F_{y,body} = \left(-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} \right) dxdydz + \rho f_y dxdydz \tag{4}$$

另一方面，流体微团的质量 $m = \rho dxdydz$，且 y 方向上的加速度可表示为 $a_y = \frac{Dv}{Dt}$。

根据牛顿第二定律

$$\vec{F} = m\vec{a} \tag{5}$$

考虑流体微团在 y 方向上有

$$F_y = ma_y \tag{6}$$

故可得到对可压缩各向同性牛顿流体微团的 y 方向动量方程

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \tag{7}$$

引入应力与应变率之间的本构关系，对于三维情况下的牛顿流体，满足

$$\begin{cases} \tau_{xy} = \tau_{yx} = \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) \\ \tau_{zy} = \tau_{yz} = \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) \\ \tau_{yy} = \lambda\left(\nabla \cdot \vec{V}\right) + 2\mu \dfrac{\partial v}{\partial y} \end{cases} \tag{8}$$

将上述本构关系式代入 y 方向动量方程，可得

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\lambda\left(\nabla \cdot \vec{V}\right) + 2\mu \frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \rho f_y \tag{9}$$

推导完毕。

题 4、分析推导

给定参考量:

ρ_∞, V_∞, p_∞, T_∞ and μ_∞, k_∞ are reference values (say, e.g., freestream values)

c is a reference length (say, the chord of an airfoil).

$$\rho' = \frac{\rho}{\rho_\infty} \quad u' = \frac{u}{V_\infty} \quad v' = \frac{v}{V_\infty} \quad p' = \frac{p}{p_\infty}$$

$$\mu' = \frac{\mu}{\mu_\infty} \quad x' = \frac{x}{c} \quad y' = \frac{y}{c} \quad t' = \frac{t}{t_\infty} \quad t_\infty = \frac{c}{V_\infty}$$

$$e' = \frac{e}{c_v T_\infty} \quad k' = \frac{k}{k_\infty} \quad V'^2 = \frac{V^2}{V_\infty^2} = \frac{u^2+v^2}{V_\infty^2} = (u')^2 + (v')^2$$

对以下流动控制方程进行无量纲化，分析流动相似参数

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{V} = 0$$

$$r\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(l(\nabla\cdot\mathbf{V}) + 2m\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[m\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + rf_x$$

$$r\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[m\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left(l(\nabla\cdot\mathbf{V}) + 2m\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[m\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + rf_y$$

$$r\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[m\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left(l(\nabla\cdot\mathbf{V}) + 2m\frac{\partial w}{\partial z}\right) + rf_z$$

$$\rho\frac{D(e+V^2/2)}{Dt} = \rho\dot{q} - \nabla\cdot(p\mathbf{V}) + \rho(\mathbf{f}\cdot\mathbf{V}) + \dot{Q}'_{\text{viscous}} + \dot{W}'_{\text{viscous}}$$

解 4:

由题意得，引入参考量（自由来流参数）$\rho_\infty, V_\infty, p_\infty, T_\infty, \mu_\infty, k_\infty$，设 c 为参考长度（例如翼型的弦长），则有无量纲量

$$\rho' = \frac{\rho}{\rho_\infty},\ u' = \frac{u}{V_\infty},\ v' = \frac{v}{V_\infty},\ p' = \frac{p}{p_\infty}$$

$$\mu' = \frac{\mu}{\mu_\infty},\ x' = \frac{x}{c},\ y' = \frac{y}{c},\ t' = \frac{t}{t_\infty},\ t_\infty = \frac{c}{V_\infty}$$

$$e' = \frac{e}{c_v T_\infty},\ k' = \frac{k}{k_\infty},\ V'^2 = \frac{V^2}{V_\infty{}^2} = \frac{u^2 + v^2}{V_\infty{}^2} = \left(u'\right)^2 + \left(v'\right)^2$$

① 由三维形式的连续方程

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0 \tag{1}$$

首先对二维形式的连续方程进行无量纲化

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho u\right)}{\partial x} + \frac{\partial\left(\rho v\right)}{\partial y} = 0 \tag{2}$$

代入无量纲量可得

$$\frac{\partial\left(\rho_\infty \rho'\right)}{\partial\left(\left(c / V_\infty\right)t'\right)} + \frac{\partial\left(\rho_\infty \rho' \cdot V_\infty u'\right)}{\partial\left(cx'\right)} + \frac{\partial\left(\rho_\infty \rho' \cdot V_\infty v'\right)}{\partial\left(cy'\right)} = 0 \tag{3}$$

方程两边同除$\frac{\rho_\infty V_\infty}{c}$，可得

$$\frac{\partial \rho'}{\partial t'} + \frac{\partial\left(\rho' u'\right)}{\partial x'} + \frac{\partial\left(\rho' v'\right)}{\partial y'} = 0 \tag{4}$$

则三维形式的无量纲形式连续方程为

$$\frac{\partial \rho'}{\partial t'} + \nabla' \cdot \left(\rho' \vec{V}'\right) = 0 \tag{5}$$

② 由三维形式的 N-S 方程组动量方程

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\lambda\left(\nabla \cdot \vec{V}\right) + 2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho f_x \tag{6}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\lambda\left(\nabla \cdot \vec{V}\right) + 2\mu \frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \rho f_y \tag{7}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\lambda\left(\nabla \cdot \vec{V}\right) + 2\mu \frac{\partial w}{\partial z}\right] + \rho f_z \tag{8}$$

首先考虑二维粘性可压缩流，对 x 方向的动量方程进行无量纲化，形式为

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\rho f_x \tag{9}$$

其中，方程忽略了粘性正应力τ_{xx}，保留了剪切应力τ_{yx}。

代入无量纲量可得

$$\begin{aligned}&(\rho_\infty\rho')\frac{\partial(V_\infty u')}{\partial((c/V_\infty)t')}+(\rho_\infty\rho')(V_\infty u')\frac{\partial(V_\infty u')}{\partial(cx')}+(\rho_\infty\rho')(V_\infty v')\frac{\partial(V_\infty u')}{\partial(cy')}\\&=-\frac{\partial(p_\infty p')}{\partial(cx')}+\frac{\partial}{\partial(cy')}\left[(\mu_\infty\mu')\left(\frac{\partial(V_\infty v')}{\partial(cx')}+\frac{\partial(V_\infty u')}{\partial(cy')}\right)\right]+(\rho_\infty\rho')f_x\end{aligned} \tag{10}$$

整理可得

$$\begin{aligned}&\rho'\frac{\partial u'}{\partial t'}+\rho'u'\frac{\partial u'}{\partial x'}+\rho'v'\frac{\partial u'}{\partial y'}\\&=-\left(\frac{p_\infty}{\rho_\infty V_\infty{}^2}\right)\frac{\partial p'}{\partial x'}+\left(\frac{\mu_\infty}{\rho_\infty V_\infty c}\right)\frac{\partial}{\partial y'}\left[\mu'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right]+\left(\frac{c}{V_\infty{}^2}\right)\rho' f_x\end{aligned} \tag{11}$$

注意到

$$Eu=\frac{p_\infty}{\rho_\infty V_\infty{}^2}=\frac{\gamma p_\infty}{\gamma\rho_\infty V_\infty{}^2}=\frac{a_\infty{}^2}{\gamma V_\infty{}^2}=\frac{1}{\gamma M_\infty{}^2} \tag{12}$$

$$\frac{\mu_\infty}{\rho_\infty V_\infty c}=\frac{1}{\mathrm{Re}_\infty} \tag{13}$$

其中，M_∞与Re_∞分别代表自由来流的马赫数和雷诺数，Eu代表自由来流的欧拉数，表示流体的压力与惯性力之比，是一个描述动量传递的特征数。

将无量纲方程式左边最后一项中的体积力替换为重力，可得

$$\frac{g_x c}{V_\infty{}^2}=\left(\frac{\sqrt{g_x c}}{V_\infty}\right)^2=\frac{1}{Fr_{x\infty}{}^2} \tag{14}$$

其中，$Fr_{x\infty}$为来流 x 方向的弗劳德数，表示流体在该方向惯性力和重力之比。

故无量纲动量方程可化为

$$\rho'\frac{\partial u'}{\partial t'}+\rho'u'\frac{\partial u'}{\partial x'}+\rho'v'\frac{\partial u'}{\partial y'}=-\frac{1}{\gamma M_{\infty}{}^{2}}\frac{\partial p'}{\partial x'}+\frac{1}{\mathrm{Re}_{\infty}}\frac{\partial}{\partial y'}\left[\mu'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right]+\frac{1}{Fr_{x\infty}{}^{2}}\rho' \tag{15}$$

若忽略体积力（重力），可进一步得到

$$\rho'\frac{\partial u'}{\partial t'}+\rho'u'\frac{\partial u'}{\partial x'}+\rho'v'\frac{\partial u'}{\partial y'}=-\frac{1}{\gamma M_{\infty}{}^{2}}\frac{\partial p'}{\partial x'}+\frac{1}{\mathrm{Re}_{\infty}}\frac{\partial}{\partial y'}\left[\mu'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right] \tag{16}$$

同理可得 y 方向的无量纲动量方程

$$\rho'\frac{\partial v'}{\partial t'}+\rho'u'\frac{\partial v'}{\partial x'}+\rho'v'\frac{\partial v'}{\partial y'}=-\frac{1}{\gamma M_{\infty}{}^{2}}\frac{\partial p'}{\partial y'}+\frac{1}{\mathrm{Re}_{\infty}}\frac{\partial}{\partial x'}\left[\mu'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right] \tag{17}$$

故分析流动相似参数为γ与M_{∞}（或Eu_{∞}）,Re_{∞}。

② 由三维形式的能量方程

$$\rho\frac{D\left(e+V^{2}/2\right)}{Dt}=\rho\dot{q}-\nabla\cdot\left(p\vec{V}\right)+\rho\left(\vec{f}\cdot\vec{V}\right)+\dot{Q}'_{\text{viscous}}+\dot{W}'_{\text{viscous}} \tag{18}$$

完整展开形式如下

$$\begin{aligned}\rho\frac{D\left(e+V^{2}/2\right)}{Dt}&=\rho\dot{q}-\nabla\cdot\left(p\vec{V}\right)\\&\quad+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\&\quad+\frac{\partial\left(u\tau_{xx}\right)}{\partial x}+\frac{\partial\left(u\tau_{yx}\right)}{\partial y}+\frac{\partial\left(u\tau_{zx}\right)}{\partial z}\\&\quad+\frac{\partial\left(v\tau_{xy}\right)}{\partial x}+\frac{\partial\left(v\tau_{yy}\right)}{\partial y}+\frac{\partial\left(v\tau_{zy}\right)}{\partial z}\\&\quad+\frac{\partial\left(w\tau_{xz}\right)}{\partial x}+\frac{\partial\left(w\tau_{yz}\right)}{\partial y}+\frac{\partial\left(w\tau_{zz}\right)}{\partial z}\end{aligned} \tag{19}$$

若忽略质量力做功、体加热和粘性正应力做功，且考虑二维能量方程，有

$$\rho u\frac{\partial\left(e+V^2/2\right)}{\partial x}+\rho v\frac{\partial\left(e+V^2/2\right)}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}+\frac{\partial\left(v\tau_{xy}\right)}{\partial x}+\frac{\partial\left(u\tau_{yx}\right)}{\partial y} \tag{20}$$

将本构关系式$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$代入上式，可得

$$\rho u\frac{\partial\left(e+V^2/2\right)}{\partial x}+\rho v\frac{\partial\left(e+V^2/2\right)}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}+\frac{\partial}{\partial x}\left[\mu v\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu u\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right] \tag{21}$$

代入无量纲量表达式可得

$$\begin{aligned}&(\rho_\infty\rho')(V_\infty u')\frac{\partial\left[c_vT_\infty e'+\left((u')^2+(v')^2\right)\frac{V_\infty{}^2}{2}\right]}{\partial(cx')}+(\rho_\infty\rho')(V_\infty v')\frac{\partial\left[c_vT_\infty e'+\left((u')^2+(v')^2\right)\frac{V_\infty{}^2}{2}\right]}{\partial(cy')}\\&=\frac{\partial}{\partial(cx')}\left((k_\infty k')\frac{\partial(T_\infty T')}{\partial(cx')}\right)+\frac{\partial}{\partial(cy')}\left((k_\infty k')\frac{\partial(T_\infty T')}{\partial(cy')}\right)\\&-\frac{\partial\left[(V_\infty u')(p_\infty p')\right]}{\partial(cx')}-\frac{\partial\left[(V_\infty v')(p_\infty p')\right]}{\partial(cy')}\\&+\frac{\partial}{\partial(cx')}\left\{(\mu_\infty\mu')(V_\infty v')\left[\frac{\partial(V_\infty v')}{\partial(cx')}+\frac{\partial(V_\infty u')}{\partial(cy')}\right]\right\}+\frac{\partial}{\partial(cy')}\left\{(\mu_\infty\mu')(V_\infty u')\left[\frac{\partial(V_\infty v')}{\partial(cx')}+\frac{\partial(V_\infty u')}{\partial(cy')}\right]\right\}\end{aligned} \tag{22}$$

方程两边同除$\frac{\rho_\infty V_\infty c_v T_\infty}{c}$，进一步整理可得

$$\begin{aligned}&\rho'u'\frac{\partial e'}{\partial x'}+\rho'v'\frac{\partial e'}{\partial y'}+\frac{V_\infty{}^2}{2c_vT_\infty}\left[\rho'u'\frac{\partial}{\partial x'}\left(u'^2+v'^2\right)+\rho'v'\frac{\partial}{\partial y'}\left(u'^2+v'^2\right)\right]\\&=\frac{k}{c\rho_\infty V_\infty c_v}\left[\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right)+\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)\right]-\frac{p_\infty}{\rho_\infty c_vT_\infty}\left[\frac{\partial(u'p')}{\partial x'}+\frac{\partial(v'p')}{\partial y'}\right]\\&+\frac{\mu_\infty V_\infty}{c\rho_\infty c_vT_\infty}\left\{\frac{\partial}{\partial x'}\left[\mu'v'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right]+\frac{\partial}{\partial y'}\left[\mu'u'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right]\right\}\end{aligned} \tag{23}$$

注意到

$$\frac{V_\infty{}^2}{2c_vT_\infty}=\frac{(\gamma-1)V_\infty{}^2}{2RT_\infty}=\frac{\gamma(\gamma-1)V_\infty{}^2}{2\gamma RT_\infty}=\frac{\gamma(\gamma-1)V_\infty{}^2}{2a_\infty{}^2}=\frac{\gamma(\gamma-1)}{2}M_\infty{}^2 \tag{24}$$

$$\frac{k_\infty}{c\rho_\infty V_\infty c_v}=\frac{k_\infty\gamma\mu_\infty}{c\rho_\infty V_\infty c_p\mu_\infty}=\gamma\cdot\frac{k_\infty}{\mu_\infty c_p}\cdot\frac{\mu_\infty}{\rho_\infty V_\infty c}=\frac{\gamma}{\Pr_\infty\mathrm{Re}_\infty} \tag{25}$$

其中，$M_\infty,\mathrm{Re}_\infty,\Pr_\infty$分别表示自由来流的马赫数、雷诺数和普朗特数，满足

$\Pr_\infty=\dfrac{\mu_\infty c_p}{k_\infty}$，表示流体摩擦阻力耗散与热传导量的比值。

又有

$$\frac{p_\infty}{\rho_\infty c_vT_\infty}=\frac{(\gamma-1)p_\infty}{\rho_\infty RT_\infty}=\frac{(\gamma-1)p_\infty}{p_\infty}=\gamma-1 \tag{26}$$

$$\frac{\mu_\infty V_\infty}{c\rho_\infty c_vT_\infty}=\frac{\mu_\infty}{\rho_\infty V_\infty c}\left(\frac{V_\infty{}^2}{c_vT_\infty}\right)=\frac{1}{\mathrm{Re}_\infty}(\gamma-1)\frac{V_\infty{}^2}{RT_\infty}=\gamma(\gamma-1)\frac{M_\infty{}^2}{\mathrm{Re}_\infty} \tag{27}$$

则无量纲能量方程可进一步化简为

$$\begin{aligned}&\rho'u'\frac{\partial e'}{\partial x'}+\rho'v'\frac{\partial e'}{\partial y'}+\frac{\gamma(\gamma-1)}{2}M_\infty{}^2\left[\rho'u'\frac{\partial}{\partial x'}\left(u'^2+v'^2\right)+\rho'v'\frac{\partial}{\partial y'}\left(u'^2+v'^2\right)\right]\\&=\frac{\gamma}{\Pr_\infty\mathrm{Re}_\infty}\left[\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right)+\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)\right]-(\gamma-1)\left[\frac{\partial(u'p')}{\partial x'}+\frac{\partial(v'p')}{\partial y'}\right]\\&+\gamma(\gamma-1)\frac{M_\infty{}^2}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial x'}\left[\mu'v'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right]+\frac{\partial}{\partial y'}\left[\mu'u'\left(\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)\right]\right\}\end{aligned} \tag{28}$$

同理可得y方向上的能量方程，并可进一步推广至三维情形，有

$$\begin{aligned}&\rho'\vec{V}'\cdot\nabla'e'+\frac{\gamma(\gamma-1)}{2}M_\infty{}^2\rho'\vec{V}'\cdot\nabla'\left(u'^2+v'^2\right)\\&=\frac{\gamma}{\Pr_\infty\mathrm{Re}_\infty}\nabla'\cdot(k'\cdot\nabla T')-(\gamma-1)\nabla'\cdot\left(p'\vec{V}'\right)+\gamma(\gamma-1)\frac{M_\infty{}^2}{\mathrm{Re}_\infty}\nabla'\cdot\left(\vec{V}'\cdot\tau_{ij}{}'\right)\end{aligned} \tag{29}$$

故分析能量方程流动相似参数为$\gamma,M_\infty,\mathrm{Re}_\infty,\Pr_\infty$。

题 5、分析计算

15.1 Consider the incompressible viscous flow of air between two infinitely long parallel plates separated by a distance h. The bottom plate is stationary, and the top plate is moving at the constant velocity u_e in the direction of the plate. Assume that no pressure gradient exists in the flow direction.

a. Obtain an expression for the variation of velocity between the plates.

b. If T = constant = 320 K, u_e = 30 m/s, and h = 0.01 m, calculate the shear stress on the top and bottom plates.

解 5:

解 15.1

本题描述流动为二维平面定常不可压库埃特（Couette）流动，属于一种定常不可压剪切流动。自主绘制示意图如图 2 所示。

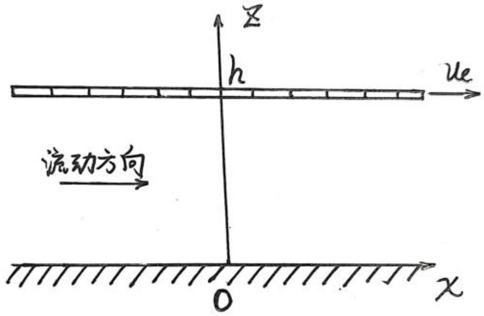


图 2　二维平面定常不可压库埃特流动示意图

a. 首先推导平板之间流场速度型。

由沿流动方向（x 方向）的定常平行剪切流动控制方程

$$-\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=0 \tag{1}$$

由假设可知在流动方向上不存在压强梯度，故有

$$\frac{\partial p}{\partial x}\equiv 0 \tag{2}$$

又由平面流动在 x 和 y 方向上无界，易得流速 u 仅为 z 的函数，故可假设

$u = u(z)$，则得

$$\frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial z^2} = \frac{d^2 u}{dz^2} \tag{3}$$

将式（2）与（3）代入控制方程（1）可得

$$\frac{d^2 u}{dz^2} = 0 \tag{4}$$

求解上述常微分方程可得通解

$$u = u(z) = Az + B \tag{5}$$

其中，A 与 B 为待求常数。

由流动边界条件

$$\begin{cases} u = 0, & z = 0 \\ u = u_e, & z = h \end{cases} \tag{6}$$

代入通解可解出待定系数

$$A = \frac{u_e}{h}, \quad B = 0 \tag{7}$$

则平板之间的流动速度型满足

$$u = u(z) = \frac{u_e}{h} z, \quad 0 \le z \le h. \tag{8}$$

b. 假设流动满足$T = \text{constant} = 320K$，$u_e = 30m/s$，$h = 0.01m$，下面求解上下壁面的剪切应力。

由牛顿粘性定律可得

$$\tau = \mu \frac{du}{dz} \tag{9}$$

将上一问求出的流动速度型代入可得

$$\tau = \mu \frac{u_e}{h} \tag{10}$$

由 Sutherland 公式可得

$$\frac{\mu}{\mu_1} = \left(\frac{T}{288.15}\right)^{\frac{3}{2}} \frac{288.15 + C}{T + C} \tag{11}$$

其中 $\mu_1 = 1.7894 \times 10^{-5}\ kg/(m \cdot s)$ 为参考温度 $T_1 = 288.15\ K$ 时的空气粘性系数，C 为 Sutherland 温度常数，满足 $C = 110.4\ K$，故当 $T = 320\ K$ 可得

$$\begin{aligned}\mu &= \mu_1 \left(\frac{T}{288.15}\right)^{\frac{3}{2}} \frac{288.15 + C}{T + C} \\ &= \left(1.7894 \times 10^{-5}\right)\left(\frac{320}{288.15}\right)^{\frac{3}{2}}\left(\frac{288.15 + 110.4}{320 + 110.4}\right) \\ &= 1.9392 \times 10^{-5}\ kg/(m \cdot s)\end{aligned} \tag{12}$$

则流体上下壁面边界的剪切应力为

$$\tau_{\text{lower}} = \tau_{\text{upper}} = \mu \frac{u_e}{h} = \left(1.9392 \times 10^{-5}\right)\left(\frac{30}{0.01}\right) = 0.05818\ kg/(m \cdot s^2) \tag{13}$$

题 6、分析计算

15.2 Assume that the two parallel plates in Problem 15.1 are both stationary but that a constant pressure gradient exists in the flow direction (i.e., dp/dx = constant).

a. Obtain an expression for the variation of velocity between the plates.

b. Obtain an expression for the shear stress on the plates in terms of dp/dx.

解 6:

解 15.2

本题描述流动为二维平面定常不可压泊萧叶（Poiseuille）流动，也属于一种定常不可压剪切流动。自主绘制示意图如图 3 所示。

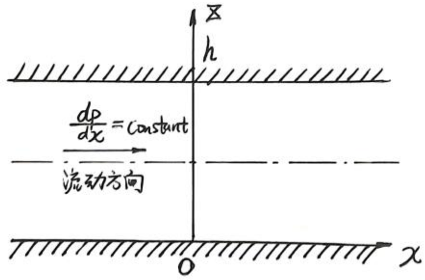


图 3　二维平面定常不可压泊萧叶流动示意图

a. 首先推导平板之间流场速度型。

平面不可压牛顿流体流动的基本控制方程如下。

连续方程：
$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{14}$$

动量方程：
$$\begin{cases}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\dfrac{\mu}{\rho}\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial z^2}\right)\\ \dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+w\dfrac{\partial w}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\dfrac{\mu}{\rho}\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial z^2}\right)\end{cases} \tag{15}$$

其中，ρ 为流体密度，μ 为流体动力粘性系数，两者均为常值。

当时间充分长达到系统平衡状态时，即满足流场定常条件时，有

$$\frac{\partial u}{\partial t}=\frac{\partial w}{\partial t}=0 \tag{16}$$

流动控制方程边界条件为

$$\begin{cases}u=w=0, \quad z=0\\ u=w=0, \quad z=h\end{cases} \tag{17}$$

根据定常条件与边界条件，流动速度型可化为$u=u(z)$，$w=0$

显然，连续方程满足，动量方程可简化为

$$-\frac{1}{\rho}\frac{dp}{dx}+\frac{\mu}{\rho}\frac{d^2u}{dz^2}=0 \tag{18}$$

则可得到

$$\frac{d^2u}{dz^2}=\frac{1}{\mu}\frac{dp}{dx}=\text{constant} \tag{19}$$

求解上述常微分方程，得到通解

$$u=\frac{1}{2\mu}\frac{dp}{dx}z^2+C_1z+C_2 \tag{20}$$

其中，C_1与C_2为待求常数。

代入边界条件，可解得

$$C_1=-\frac{h}{2\mu}\frac{dp}{dx},\quad C_2=0 \tag{21}$$

则该平面流动速度型解为

$$u=-\frac{1}{2\mu}\frac{dp}{dx}z\left(h-z\right)=\frac{1}{2\mu}\frac{dp}{dx}z\left(z-\frac{h}{2}\right)^2-\frac{h^2}{8\mu}\frac{dp}{dx}. \tag{22}$$

即平面定常不可压泊萧叶（Poiseuille）流动平板间的速度剖面为抛物线，最大速度出现在$z=\dfrac{h}{2}$处，即平板之间中央轴线处，有

$$u_{\max}=u\left(z\right)\Big|_{z=\frac{h}{2}}=-\frac{h^2}{8\mu}\frac{dp}{dx}. \tag{23}$$

b. 推导平板壁面处的剪切应力。

应用牛顿剪切应力公式可得

$$\tau_{xz}=\mu\frac{du}{dz}=-\frac{1}{2}\frac{dp}{dx}\left(h-2z\right) \tag{24}$$

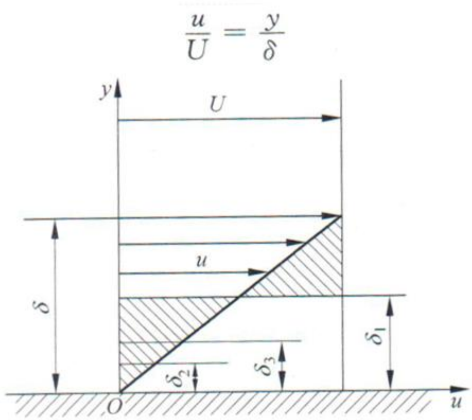
故在上下平板壁面处（$z=0$或$z=h$），剪切应力为

$$\tau_{\text{lower}}=-\frac{h}{2}\frac{dp}{dx},\quad \tau_{\text{upper}}=\frac{h}{2}\frac{dp}{dx} \tag{25}$$

题 7、分析推导

如图所示，对于不可压缩流动，已知边界层内的速度分布：

$$u = \frac{y}{\delta}U$$



求边界层位移厚度和动量厚度：

$$\delta^* = \int_0^{\infty} (1 - \frac{u}{U}) dy$$

$$\theta = \int_0^{\infty} \frac{u}{U}(1 - \frac{u}{U}) dy$$

解 7：对于不可压缩流动，全流场 $\rho \equiv \mathrm{const}$ 。

由定义式可得

① 边界层位移厚度 δ^* 为

$$\begin{aligned} \delta^* &= \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e U}\right) dy, \quad \delta \le y_1 \to \infty \\ &= \int_0^{\delta} \left(1 - \frac{y}{\delta}\right) dy = \left(y - \frac{1}{2\delta} y^2\right)\Bigg|_0^{\delta} = \frac{1}{2}\delta \end{aligned} \tag{1}$$

② 边界层动量厚度θ为

$$\begin{aligned}\theta &= \int_0^{y_1} \frac{\rho u}{\rho_e U}\left(1-\frac{u}{U}\right)dy, \qquad \delta \le y_1 \to \infty \\ &= \int_0^{\delta}\left(\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)dy = \left(\frac{1}{2\delta}y^2-\frac{1}{3\delta^2}y^3\right)\Bigg|_0^{\delta} = \frac{1}{6}\delta\end{aligned} \tag{2}$$

故满足$\delta^* = \dfrac{1}{2}\delta = 3\theta.$

题 8、分析推导

对以下二维定常无量纲能量方程进行量阶分析（order-of-magnitude analysis）

$$\begin{aligned}&\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty^2 \left[\rho' u' \frac{\partial}{\partial x'}(u'^2+v'^2) + \rho' v' \frac{\partial}{\partial y'}(u'^2+v'^2)\right] \\ &\quad = \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty}\left[\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)\right] - (\gamma-1)\left(\frac{\partial(u'p')}{\partial x'} + \frac{\partial(v'p')}{\partial y'}\right) \\ &\qquad + \gamma(\gamma-1)\frac{M_\infty^2}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial x'}\left[\mu' v'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right] + \frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]\right\}\end{aligned}$$

推导以下边界层有量纲能量方程：

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + u\frac{\partial p}{\partial x} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

解 8： 首先对各无量纲量进行量阶分析，引入参考量$V_\infty, \rho_\infty, p_\infty, \mu_\infty, T_\infty, k_\infty$。

在长度为c的平板上，流动边界层厚度为δ，则由边界层特性可得

$$\delta << c$$

考虑二维定常流动连续性方程

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial v} = 0 \tag{1}$$

其无量纲方程形式为

$$\frac{\partial(\rho' u')}{\partial x'}+\frac{\partial(\rho' v')}{\partial v'}=0 \tag{2}$$

由$u'=\dfrac{u}{V_\infty}\sim O(1)$，$\rho'=\dfrac{\rho}{\rho_\infty}\sim O(1)$，$x'=\dfrac{x}{c}\sim O(1)$，$y'=\dfrac{y}{c}\sim O(\delta)$，

代入无量纲定常连续方程可得

$$\frac{[O(1)][O(1)]}{[O(1)]}+\frac{[O(1)][v']}{[O(\delta)]}=0 \tag{3}$$

由上式易知$v'=\dfrac{v}{V_\infty}\sim O(\delta)$。

将对能量方程进行量阶分析所需的无量纲变量量阶列举如下

$$u'=\frac{u}{V_\infty}\sim O(1),\ v'=\frac{v}{V_\infty}\sim O(1),\ \rho'=\frac{\rho}{\rho_\infty}\sim O(1),\ p'=\frac{p}{p_\infty}\sim O(1),$$

$$x'=\frac{x}{c}\sim O(1),\ y'=\frac{y}{c}\sim O(\delta),\ \mu'=\frac{\mu}{\mu_\infty}\sim O(1),\ T'=\frac{T}{T_\infty}\sim O(1),\ h'=\frac{h}{c_pT_\infty}\sim O(1),$$

$$e'=\frac{e}{c_vT_\infty}\sim O(1),\ k'=\frac{k}{k_\infty}\sim O(1),\ v'^2=\frac{v^2}{V_\infty{}^2}=\frac{u^2+v^2}{V_\infty{}^2}=(u')^2+(v')^2\sim O(1)$$

另外，在高雷诺数Re假设下，边界层内满足$\mathrm{Re}_\infty=\dfrac{\rho_\infty V_\infty c}{\mu_\infty}\sim\dfrac{1}{O(\delta^2)}$，且有

无量纲参数$M_\infty{}^2=\dfrac{V_\infty{}^2}{a_\infty{}^2}\sim O(1)$，$\mathrm{Pr}_\infty=\dfrac{\mu_\infty C_p}{k_\infty}\sim O(1)$。

下面对二维定常无量纲能量方程中的各项进行量阶分析。

① 方程左边

a) $\rho' u'\dfrac{\partial e'}{\partial x'}\sim O(1)$，$\rho' v'\dfrac{\partial e'}{\partial y'}\sim O(\delta)\cdot\dfrac{1}{O(\delta)}=O(1)$;

b) $$\rho' u' \frac{\partial\left(u'^2\right)}{\partial x'} \sim O(1),\ \ \rho' u' \frac{\partial\left(v'^2\right)}{\partial x'} \sim O\left(\delta^2\right);$$
$$\rho' v' \frac{\partial\left(u'^2\right)}{\partial y'} \sim O(1),\ \ \rho' v' \frac{\partial\left(v'^2\right)}{\partial y'} \sim O\left(\delta^2\right);$$

② 方程右边

a) $$\frac{\partial}{\partial x'}\left(k' \frac{\partial T'}{\partial x'}\right) \sim O(1),\ \ \frac{\partial}{\partial y'}\left(k' \frac{\partial T'}{\partial y'}\right) \sim O\left(\frac{1}{\delta^2}\right);$$

b) $$\frac{\partial\left(u'p'\right)}{\partial x'} = u' \frac{\partial p'}{\partial x'} + p' \frac{\partial u'}{\partial x'} \sim O(1) + O(1) = O(1),$$
$$\frac{\partial\left(v'p'\right)}{\partial y'} = v' \frac{\partial p'}{\partial y'} + p' \frac{\partial v'}{\partial y'} \sim O(1) + O(1) = O(1);$$

c) $$\frac{\partial}{\partial x'}\left[\mu' v'\left(\frac{\partial v'}{\partial x'}\right)\right] \sim O\left(\delta^2\right),\ \ \frac{\partial}{\partial x'}\left[\mu' v'\left(\frac{\partial u'}{\partial y'}\right)\right] \sim O(1),$$
$$\frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial v'}{\partial x'}\right)\right] \sim O(1),\ \ \frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial u'}{\partial y'}\right)\right] \sim O\left(\frac{1}{\delta^2}\right);$$

故对原无量纲能量方程进行量阶形式变量代换可得

$$\begin{aligned}
&\left[O(1)+O(1)\right] + \frac{\gamma(\gamma-1)}{2} M_\infty{}^2 \left\{\left[O(1)+O\left(\delta^2\right)\right] + \left[O(1)+O\left(\delta^2\right)\right]\right\} \\
&= \frac{\gamma}{\mathrm{Pr}_\infty} O\left(\delta^2\right)\left[O(1)+O\left(\frac{1}{\delta^2}\right)\right] - (\gamma-1)\left[O(1)+O(1)\right] \\
&\quad + \gamma(\gamma-1) M_\infty{}^2 O\left(\delta^2\right)\left\{\left[O\left(\delta^2\right)+O(1)\right] + \left[O(1)+O\left(\frac{1}{\delta^2}\right)\right]\right\}
\end{aligned} \tag{4}$$

忽略$O\left(\delta^2\right)$及以上高阶小量$\left(\delta << c\right)$，则无量纲能量方程可化为

$$\begin{aligned}
&\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty{}^2 \left[\rho' u' \frac{\partial\left(u'^2\right)}{\partial x'} + \rho' v' \frac{\partial\left(u'^2\right)}{\partial y'}\right] \\
&= \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty}\left[\frac{\partial}{\partial y'}\left(k' \frac{\partial T'}{\partial y'}\right)\right] - (\gamma-1)\left[\frac{\partial\left(u'p'\right)}{\partial x'} + \frac{\partial\left(v'p'\right)}{\partial y'}\right] + \gamma(\gamma-1)\frac{M_\infty{}^2}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial u'}{\partial y'}\right)\right]\right\}
\end{aligned} \tag{5}$$

将上式中的无量纲量还原为有量纲量可得

$$\rho u\frac{\partial\left(e+u^2/2\right)}{\partial x}+\rho v\frac{\partial\left(e+u^2/2\right)}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\left[\frac{\partial(up)}{\partial x}+\frac{\partial(vp)}{\partial y}\right]+\frac{\partial}{\partial y}\left[u\left(\mu\frac{\partial u}{\partial y}\right)\right] \tag{6}$$

由 x 方向二维边界层动量微分方程

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{7}$$

将上式方程两边同乘 u，并进行微分代换可得

$$\rho u^2\frac{\partial u}{\partial x}+\rho uv\frac{\partial u}{\partial y}=\rho u\frac{\partial\left(u^2/2\right)}{\partial x}+\rho v\frac{\partial\left(u^2/2\right)}{\partial y}=-u\frac{\partial p}{\partial x}+u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{8}$$

则将式（6）与式（8）相减可得

$$\begin{aligned}\rho u\frac{\partial e}{\partial x}+\rho v\frac{\partial e}{\partial y}&=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\left[\frac{\partial(up)}{\partial x}+\frac{\partial(vp)}{\partial y}\right]+\frac{\partial}{\partial y}\left[u\left(\mu\frac{\partial u}{\partial y}\right)\right]\\&\quad-\left(-u\frac{\partial p}{\partial x}\right)-u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\\&=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2-\nabla\left(\vec{V}\cdot p\right)\end{aligned} \tag{9}$$

将上式中的 e 代换为 $e=h-\dfrac{p}{\rho}$，则可化为

$$\rho u\frac{\partial\left(h-p/\rho\right)}{\partial x}+\rho v\frac{\partial\left(h-p/\rho\right)}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2-\nabla\left(\vec{V}\cdot p\right) \tag{10}$$

移项可得

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\left[\rho\vec{V}\cdot\nabla\left(\frac{p}{\rho}\right)-\nabla\left(\vec{V}\cdot p\right)\right] \tag{11}$$

对上式左边最后一项化简有

$$\rho\vec{V}\cdot\nabla\left(\frac{p}{\rho}\right)-\nabla\left(\vec{V}\cdot p\right)=\rho\vec{V}\cdot\left[\frac{1}{\rho}\cdot\nabla p+p\cdot\left(-\frac{1}{\rho^2}\right)\cdot\nabla\rho\right]-\vec{V}\cdot\nabla p-p\cdot\nabla\vec{V}$$

$$=\vec{V}\cdot\nabla p-\frac{p}{\rho}\cdot\vec{V}\cdot\nabla\rho-\vec{V}\cdot\nabla p-p\cdot\nabla\vec{V} \tag{12}$$

$$=-\frac{p}{\rho}\cdot\vec{V}\cdot\nabla\rho-p\cdot\nabla\vec{V}$$

又由微分形式的边界层连续方程

$$\nabla\cdot\left(\rho\vec{V}\right)=\rho\cdot\nabla\vec{V}+\vec{V}\cdot\nabla\rho=0 \tag{13}$$

由上式解出$\vec{V}\cdot\nabla\rho$，并代入式（12）中，可得

$$\rho\vec{V}\cdot\nabla\left(\frac{p}{\rho}\right)-\nabla\left(\vec{V}\cdot p\right)=-\frac{p}{\rho}\cdot(-\rho)\cdot\nabla\vec{V}-p\cdot\nabla\vec{V}=0 \tag{14}$$

故式（11）可进一步简化为

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{15}$$

上式即为边界层有量纲能量微分方程，推导完毕。

题 9、分析计算

Note: The standard sea level value of viscosity coefficient for air is $\mu = 1.7894 \times 10^{-5}$ kg/(m · s) = 3.7373×10^{-7} slug/(ft · s).

19.1 The wing on a Piper Cherokee general aviation aircraft is rectangular, with a span of 9.75 m and a chord of 1.6 m. The aircraft is flying at cruising speed (141 mi/h) at sea level. Assume that the skin friction drag on the wing can be approximated by the drag on a flat plate of the same dimensions. Calculate the skin friction drag:

a. If the flow were completely laminar (which is not the case in real life)

b. If the flow were completely turbulent (which is more realistic)

Compare the two results.

19.2 For the case in Problem 19.1, calculate the boundary-layer thickness at the trailing edge for

a. Completely laminar flow

b. Completely turbulent flow

19.3 For the case in Problem 19.1, calculate the skin friction drag accounting for transition. Assume the transition Reynolds number = 5×10^5.

19.4 Consider Mach 4 flow at standard sea level conditions over a flat plate of chord 5 in. Assuming all laminar flow and adiabatic wall conditions, calculate the skin friction drag on the plate per unit span.

19.5 Repeat Problem 19.4 for the case of all turbulent flow.

解 9：

解 19.1

① 首先判断来流是否可认为是不可压缩的。

由理想气体状态方程 $p=\rho RT$，可得到自由来流密度为

$$\rho_\infty = \frac{p_\infty}{RT_\infty} = \frac{1.01\times10^5}{(287)(288)} = 1.2219\ kg/m^3 \tag{1}$$

由声速公式可得自由来流声速为

$$a_\infty = \sqrt{\gamma RT_\infty} = \sqrt{(1.4)(287)(288)} = 340.1741\ m/s \tag{2}$$

又巡航来流速度为 $V_\infty = 141\ mi/h = 63.0326\ m/s$，

则在海平面状态下，巡航马赫数为

$$M_{\infty}=\frac{V_{\infty}}{a_{\infty}}=\frac{63.0326}{340.1741}=0.1853<0.3 \tag{3}$$

故自由来流马赫数足够低，可假设来流为不可压缩流。

② 分两种情况计算摩擦阻力。

a. 假设气流完全是层流状态（实际生活中不可能）

由

$$\mathrm{Re}_c=\frac{\rho_{\infty}V_{\infty}c}{\mu_{\infty}}=\frac{(1.2219)(63.0326)(1.6)}{1.7894\times10^{-5}}=6.8867\times10^{6} \tag{4}$$

故层流状态下

$$C_f=\frac{1.328}{\sqrt{\mathrm{Re}_c}}=\frac{1.328}{\sqrt{6.8867\times10^{6}}}=5.0605\times10^{-4} \tag{5}$$

若将机翼所受摩擦阻力近似认为是相同尺寸平板所受摩擦阻力，则有

$$D_f=\frac{1}{2}\rho_{\infty}V_{\infty}{}^{2}SC_f=\frac{1}{2}(1.2219)(63.0326)^{2}(9.75)(1.6)\left(5.0605\times10^{-4}\right)=19.1625\ N \tag{6}$$

实际机翼两面均受摩擦阻力，则完全层流来流条件下，总表面摩擦阻力为

$$D_{\mathrm{laminar}}=2D_f=2(19.1625)=38.3250\ N \tag{7}$$

b. 假设气流完全是湍流状态（更贴近生活实际）

则湍流状态下

$$C_f=\frac{0.074}{(\mathrm{Re}_c)^{\frac{1}{5}}}=\frac{0.074}{\left(6.8867\times10^{6}\right)^{\frac{1}{5}}}=3.1742\times10^{-3} \tag{8}$$

$$D_f=\frac{1}{2}\rho_{\infty}V_{\infty}{}^{2}SC_f=\frac{1}{2}(1.2219)(63.0326)^{2}(9.75)(1.6)\left(3.1742\times10^{-3}\right)=120.1959\ N \tag{9}$$

$$D_{\mathrm{turbulent}}=2D_f=2(120.1959)=240.3918\ N \tag{10}$$

③ 比较来流完全为层流和完全为湍流两种情况下机翼所受摩擦阻力，有

$$\frac{D_{\text{turbulent}}}{D_{\text{laminar}}}=\frac{240.3918}{38.3250}=6.2725 \tag{11}$$

由此可见，平面湍流边界层的壁面摩擦阻力远大于层流。因此，尽量保证飞机飞行时来流与机翼绕流边界层为层流，可有效降低机翼表面摩擦阻力。

解 19.2

① 考虑完全层流与完全湍流两种极端情况下机翼尾部的边界层厚度。

a. 假设来流完全是层流状态

由机翼弦长 $c=1.6\ m$，且 Problem **19.1** 已求得 $\mathrm{Re}_c=6.8867\times10^6$，

故机翼后缘处边界层厚度为

$$\delta_{x=c,\text{laminar}}=\frac{5.0c}{\sqrt{\mathrm{Re}_c}}=\frac{(5.0)(1.6)}{\sqrt{6.8867\times10^6}}=3.048\times10^{-3}\ m=3.048\ mm \tag{12}$$

b. 假设来流完全是湍流状态

由机翼弦长 $c=1.6\ m$，且 Problem **19.1** 已求得 $\mathrm{Re}_c=6.8867\times10^6$，

故机翼后缘处边界层厚度为

$$\delta_{x=c,\text{turbulent}}=\frac{0.37c}{\mathrm{Re}_c^{\frac{1}{5}}}=\frac{(0.37)(1.6)}{\left(6.8867\times10^6\right)^{\frac{1}{5}}}=0.02539\ m=25.393\ mm \tag{13}$$

② 比较来流完全为层流和完全为湍流两种情况下机翼后缘处的边界层厚度，有

$$\frac{\delta_{x=c,\text{turbulent}}}{\delta_{x=c,\text{laminar}}}=\frac{0.02539}{3.048\times10^{-3}}=8.3301 \tag{14}$$

由此可见，平面湍流边界层在相同发展段（机翼后缘）的边界层厚度远大于层流状态。

解 19.3

对于 Problem 19.1 所给情况，考虑一种简单的边界层转捩情形，转捩雷诺数为$\mathrm{Re}_t=5\times10^5$。当$\mathrm{Re}_c\le\mathrm{Re}_t$时，平板边界层近似为层流状态；当$\mathrm{Re}_c\ge\mathrm{Re}_t$时，平板边界层由层流状态近似转捩为层流状态。

首先计算发生转捩点（位置）。

令

$$\mathrm{Re}_x=\frac{\rho_\infty V_\infty x}{\mu_\infty}=\frac{(1.2219)(63.0326)x}{1.7894\times10^{-5}}=5\times10^5 \tag{15}$$

解得：$x=x_t=0.1162\ m$，此位置即为层流边界层和湍流边界层分界处。

a. 考虑机翼近似平板从前缘零位置至$x_t=0.1162\ m$这一段为层流边界层，且有$\mathrm{Re}_t=5\times10^5$。

故层流边界层段总摩擦阻力系数为

$$\begin{aligned}C_{f,\mathrm{laminar}}&=\frac{1}{x_t}\int_0^{x_t}c_f\,dx=\frac{1}{x_t}\int_0^{x_t}\frac{0.664}{\sqrt{\mathrm{Re}_x}}dx\\&=\frac{1}{x_t}(0.664)\sqrt{\frac{\mu_\infty}{\rho_\infty V_\infty}}\int_0^{x_t}x^{-\frac{1}{2}}dx=\frac{1.328}{x_t}\sqrt{\frac{\mu_\infty x_t}{\rho_\infty V_\infty}}\\&=\frac{1.328}{\sqrt{\mathrm{Re}_t}}=\frac{1.328}{\sqrt{5\times10^5}}=1.8781\times10^{-3}\end{aligned} \tag{16}$$

则层流边界层段壁面所受总摩擦阻力（考虑双面）为

$$\begin{aligned}D_{f,\mathrm{laminar}}&=2\times\frac{1}{2}\rho_\infty V_\infty{}^2 b x_t C_{f,\mathrm{laminar}}\\&=(2)\left(\frac{1}{2}\right)(1.2219)(63.0326)^2(9.75)(0.1162)\left(1.8781\times10^{-3}\right)\\&=10.3299N\end{aligned} \tag{17}$$

b. 考虑机翼近似平板从$x_t=0.1162\ m$至后缘$x_c=1.6\ m$这一段湍流边界

层，且有 $\mathrm{Re}_t = 5\times10^5$， $\mathrm{Re}_c = 6.8867\times10^6$。

由当地摩擦阻力系数修正公式 $c_f = \dfrac{\tau_w}{1/2\rho_e u_e^2} = \dfrac{0.02296}{(\mathrm{Re}_x)^{0.139}} = \dfrac{0.02296\mu_\infty^{0.139}}{(\rho_\infty V_\infty x)^{0.139}}$ (18)

则湍流边界层部分平板的平均摩阻系数为

$$\begin{aligned}C_{f,\text{turbulent}} &= \frac{1}{x_c - x_t}\int_{x_t}^{x_c} c_f dx = \frac{1}{x_c - x_t}\int_{x_t}^{x_c} \frac{0.02296\mu_\infty^{0.139}}{(\rho_\infty V_\infty x)^{0.139}} \\ &= 1.8513\times10^{-3}\int_{x_t}^{x_c} x^{-0.139} = \frac{1.8513\times10^{-3}}{0.861}\left(x^{0.861}\right)\Big|_{0.1162}^{1.6} = 2.8857\times10^{-3}\end{aligned} \tag{19}$$

则湍流边界层壁面所受总摩擦阻力（考虑双面）为

$$\begin{aligned}D_{f,\text{turbulent}} &= 2\times\frac{1}{2}\rho_\infty V_\infty{}^2 b\left(x_c - x_t\right)C_{f,\text{turbulent}} \\ &= (2)\left(\frac{1}{2}\right)(1.2219)(63.0326)^2(9.75)(1.6-0.1162)\left(2.8857\times10^{-3}\right) \\ &= 202.6736N\end{aligned} \tag{20}$$

综上 a.b.可得，考虑简单边界层转捩情况下（$\mathrm{Re}_t = 5\times10^5$）时，机翼壁面所受的总摩擦阻力为

$$D_{f,\text{tot}} = D_{f,\text{laminar}} + D_{f,\text{turbulent}} = 10.3299 + 202.6736 = 213.0035\ N \tag{21}$$

考虑转捩情况下，平板所受阻力值介于完全层流状态（$38.3250\ N$）与完全湍流状态（$240.3918\ N$）之间，更加贴近生活、工程实际。

解 **19.4**

由于来流马赫数 $M_\infty = 4 >> 0.3$，故该流动为可压缩流动。

① 首先计算来流速度。

由理想气体状态方程 $p = \rho RT$，可得标准海平面状态下来流密度为

$$\rho_\infty = \frac{p_\infty}{RT_\infty} = \frac{1.01\times10^5}{(287)(288)} = 1.2219\ kg/m^3 \tag{22}$$

由声速公式可得自由来流声速为

$$a_\infty = \sqrt{\gamma RT_\infty} = \sqrt{(1.4)(287)(288)} = 340.1741\ m/s \tag{23}$$

则来流速度为

$$V_\infty = M_\infty a_\infty = (4)(340.1741) = 1360.6964\ m/s \tag{24}$$

② 下面采用 Meador-Smart 参考温度方法求解平板单位展长的摩擦阻力。

在标准海平面状态下，普朗特数为$\Pr = 0.71 = \Pr^*$.

故恢复系数$r = \dfrac{T_{aw} - T_e}{T_0 - T_e} = \sqrt{\Pr} = \sqrt{0.71} = 0.8426$.

查阅 Appendix A，当$M_e = M_\infty = 4$时，$\dfrac{T_0}{T_e} = 4.20$

则有

$$\frac{T_{aw}}{T_e} = 1 + r\left(\frac{T_0}{T_e} - 1\right) = 1 + 0.8426(4.20 - 1) = 3.6963 = \frac{T_w}{T_e} \tag{25}$$

$$\begin{aligned}\frac{T^*}{T_e} &= 0.45 + 0.55\left(\frac{T_w}{T_e}\right) + 0.16r\left(\frac{\gamma-1}{2}\right)M_e^{\,2} \\ &= 0.45 + (0.55)(3.6963) + (0.16)(0.8426)\left(\frac{1.4-1}{2}\right)(4)^2 = 2.9144\end{aligned} \tag{26}$$

故

$$T^* = 2.9144T_e = (2.9144)(288) = 839.3435\ K \tag{27}$$

根据参考温度T^*计算参考密度ρ^*和参考粘性系数μ^*如下

$$\rho^* = \frac{p^*}{RT^*} = \frac{1.01\times10^5}{(287)(839.3435)} = 0.4193\ kg/m^3 \tag{28}$$

由 Sutherland 公式可得

$$\frac{\mu^*}{\mu_0}=\left(\frac{T^*}{T_0}\right)^{\frac{3}{2}}\frac{T_0+110}{T^*+110}=\left(\frac{839.3435}{288}\right)^{\frac{3}{2}}\frac{288+110}{839.3435+110}=2.0858 \tag{29}$$

又由

$$\mu_0=\mu_\infty=1.7894\times10^{-5}\ kg/(m\cdot s) \tag{30}$$

则有

$$\mu^*=2.0858\mu_0=(2.0858)\left(1.7894\times10^{-5}\right)=3.7324\times10^{-5}\ kg/(m\cdot s) \tag{31}$$

即有

$$\mathrm{Re}_c{}^*=\frac{\rho^* V_\infty c}{\mu^*}=\frac{(0.4193)(1360.6964)(5\times0.0254)}{3.7324\times10^{-5}}=1.9413\times10^6 \tag{32}$$

故考虑来流均为层流状态下，有

$$C_f{}^*=\frac{1.328}{\sqrt{\mathrm{Re}_c{}^*}}=\frac{1.328}{\sqrt{1.9413\times10^6}}=9.5312\times10^{-4} \tag{33}$$

因此，考虑单位展长平板壁面的两侧总摩擦阻力为

$$\begin{aligned}D_{f,tot}&=2\times\frac{1}{2}\rho^* V_\infty{}^2 cC_f{}^*\\&=(0.4193)(1360.6964)^2(5\times0.0254)\left(9.5312\times10^{-4}\right)\\&=93.9719\ N\end{aligned} \tag{34}$$

解 19.5

假设流动均为湍流状态。

下面对 Meador-Smart 参考温度方法进行修正，并求解平板单位展长的摩擦阻力。

在标准海平面状态下，普朗特数为 $\mathrm{Pr}=0.71=\mathrm{Pr}^*$.

故恢复系数$r=\dfrac{T_{aw}-T_e}{T_0-T_e}=\Pr^{\frac{1}{3}}=\left(0.71\right)^{\frac{1}{3}}=0.8921$.

查阅 Appendix A，当$M_e=M_\infty=4$时，$\dfrac{T_0}{T_e}=4.20$

则有

$$\frac{T_{aw}}{T_e}=1+r\left(\frac{T_0}{T_e}-1\right)=1+0.8921\left(4.20-1\right)=3.8548=\frac{T_w}{T_e} \tag{35}$$

$$\begin{aligned}\frac{T^*}{T_e}&=0.5\left(1+\frac{T_w}{T_e}\right)+0.16r\left(\frac{\gamma-1}{2}\right){M_e}^2\\&=0.5\left(1+3.8548\right)+\left(0.16\right)\left(0.8921\right)\left(\frac{1.4-1}{2}\right)\left(4\right)^2=2.8841\end{aligned} \tag{36}$$

故

$$T^*=2.8841T_e=\left(2.8841\right)\left(288\right)=830.6308\ K$$

根据参考温度T^*计算参考密度ρ^*和参考粘性系数μ^*如下

$$\rho^*=\frac{p^*}{RT^*}=\frac{1.01\times10^5}{\left(287\right)\left(830.6308\right)}=0.4237\ kg/m^3 \tag{37}$$

由 Sutherland 公式可得

$$\frac{\mu^*}{\mu_0}=\left(\frac{T^*}{T_0}\right)^{\frac{3}{2}}\frac{T_0+110}{T^*+110}=\left(\frac{830.6308}{288}\right)^{\frac{3}{2}}\frac{288+110}{830.6308+110}=2.0725 \tag{38}$$

又由

$$\mu_0=\mu_\infty=1.7894\times10^{-5}\ kg/\left(m\cdot s\right) \tag{39}$$

则有

$$\mu^*=2.0725\mu_0=\left(2.0725\right)\left(1.7894\times10^{-5}\right)=3.7085\times10^{-5}\ kg/\left(m\cdot s\right) \tag{40}$$

即有

$$\mathrm{Re}_c{}^* = \frac{\rho^* V_\infty c}{\mu^*} = \frac{(0.4237)(1360.6964)(5\times 0.0254)}{3.7085\times 10^{-5}} = 1.9744\times 10^{6} \tag{41}$$

故考虑来流均为层流状态下，有

$$C_f{}^* = \frac{0.02667}{\left(\mathrm{Re}_c{}^*\right)^{0.139}} = \frac{0.02667}{\left(1.9744\times 10^{6}\right)^{0.139}} = 3.5560\times 10^{-3} \tag{42}$$

因此，考虑单位展长平板壁面的两侧总摩擦阻力为

$$\begin{aligned} D_{f,tot} &= 2\times \frac{1}{2}\rho^* V_\infty{}^2 c C_f{}^* \\ &= (0.4237)(1360.6964)^2 (5\times 0.0254)\left(3.5560\times 10^{-3}\right) \\ &= 354.2798\ N \end{aligned} \tag{43}$$