1 引言

2 控制方程的无量纲化

2.1 不可压缩流动Navier-Stokes方程组

不可压缩流动的连续性方程和动量方程（不含体积力项）分别为

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{2}$$

对于以上方程组，考虑如下的无量纲化

$$\begin{aligned} &\mathbf{x}/L_0 \to \mathbf{x}^*, \quad t/\left(\frac{L_0}{U_0}\right) \to t^*, \quad \mathbf{u}/U_0 \to \mathbf{u}^*, \\ &p/\left(\frac{m_0 U_0^2}{L_0^3}\right) \to p^*, \quad \rho/\left(\frac{m_0}{L_0^3}\right) \to \rho^* \end{aligned} \tag{3}$$

这里*代表无量纲的量。 L_0，U_0，m_0分别表示特征长度、特征速度和特征质量。

以二维正交笛卡尔坐标系为例，对于连续性方程可以改写为

$$\frac{\partial(U_0 u^*)}{\partial(L_0 x^*)} + \frac{\partial(U_0 v^*)}{\partial(L_0 y^*)} = 0 \tag{4}$$

化简得到

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5}$$

上式表明，无量纲形式的连续性方程可以写为

$$\nabla \cdot \mathbf{u}^* = 0 \tag{6}$$

对于x方向的动量方程可以改写为

$$\begin{aligned} &\frac{\partial(U_0 u^*)}{\partial(L_0/U_0 t^*)} + (U_0 u^*)\frac{\partial(U_0 u^*)}{\partial(L_0 x^*)} + (U_0 v^*)\frac{\partial(U_0 u^*)}{\partial(L_0 y^*)} \\ &= -\frac{1}{m_0/L_0^3 \rho^*}\frac{\partial(m_0 U_0^2/L_0^3 p^*)}{\partial(L_0 x^*)} + \nu\left[\frac{\partial(\frac{\partial U_0 u^*}{\partial L_0 x^*})}{\partial(L_0 x^*)} + \frac{\partial(\frac{\partial U_0 u^*}{\partial L_0 y^*})}{\partial(L_0 y^*)}\right] \end{aligned} \tag{7}$$

化简得到

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*} + \frac{\nu}{U_0 L_0}\left[\frac{\partial(\frac{\partial u^*}{\partial x^*})}{\partial x^*} + \frac{\partial(\frac{\partial u^*}{\partial y^*})}{\partial y^*}\right] \tag{8}$$

上式表明，无量纲形式的动量方程可以写为

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\frac{1}{\rho^*}\nabla p^* + \frac{1}{Re}\nabla^2 \mathbf{u}^* \tag{9}$$

其中Re为雷诺数

$$Re = \frac{U_0 L_0}{\nu} \tag{10}$$

2.2 不可压缩Boussinesq方程组

基于Boussinesq近似，连续性方程、动量方程和能量方程可以分别写为

$$\nabla \cdot \mathbf{u} = 0 \tag{11}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} + g\beta_T (T - T_0)\hat{\mathbf{y}} \tag{12}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T \tag{13}$$

其中T_0为参考温度。$\hat{\mathbf{y}}$表示y方向（假设平行于重力方向）的单位向量。 对于以上方程组，下面以二维正交笛卡尔坐标系为例，采取两组不同的特征尺度进行无量纲化。

1. 选择长度尺度L_0、时间尺度L_0^2/κ、速度尺度κ/L_0、 压强尺度$\rho_0\kappa^2/L_0^2$、温度尺度Δ_T进行无量纲化，则

$$\begin{gathered} \mathbf{x}/L_0 \to \mathbf{x}^*, \quad t/\left(\frac{L_0^2}{\kappa}\right) \to t^*, \quad \mathbf{u}/\left(\frac{\kappa}{L_0}\right) \to \mathbf{u}^*, \\ p/\left(\frac{\rho_0\kappa^2}{L_0^2}\right) \to p^*, \quad (T - T_0)/\Delta_T \to T^* \end{gathered} \tag{14}$$

   对于连续性方程可以改写为

$$\frac{\partial(\kappa/L_0 u^*)}{\partial(L_0 x^*)} + \frac{\partial(\kappa/L_0 v^*)}{\partial(L_0 y^*)} = 0 \tag{15}$$

   化简得到

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{16}$$

上式表明，无量纲形式的连续性方程可以写为

$$\nabla \cdot \mathbf{u}^* = 0 \tag{17}$$

对于x方向的动量方程可以改写为

$$\begin{aligned}
&\frac{\partial(\kappa/L_0 u^*)}{\partial(L_0^2/\kappa t^*)} + (\kappa/L_0 u^*)\frac{\partial(\kappa/L_0 u^*)}{\partial(L_0 x^*)} + (\kappa/L_0 v^*)\frac{\partial(\kappa/L_0 u^*)}{\partial(L_0 y^*)} \\
&= -\frac{1}{\rho_0}\frac{\partial(\rho_0 \kappa^2/L_0^2 p^*)}{\partial(L_0 x^*)} + \nu\left\{\frac{\partial\left[\frac{(\partial\kappa/L_0 u^*)}{\partial(L_0 x^*)}\right]}{\partial(L_0 x^*)} + \frac{\partial\left[\frac{\partial(\kappa/L_0 u^*)}{\partial(L_0 y^*)}\right]}{\partial(L_0 y^*)}\right\} + g\beta_T\Delta_T T^*\hat{\mathbf{y}}
\end{aligned} \tag{18}$$

化简得到

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{\kappa}\left[\frac{\partial(\frac{\partial u^*}{\partial x^*})}{\partial x^*} + \frac{\partial(\frac{\partial u^*}{\partial y^*})}{\partial y^*}\right] + \frac{L_0^3 g\beta_T\Delta_T T^*}{\kappa^2}\hat{\mathbf{y}} \tag{19}$$

上式表明，无量纲形式的动量方程可以写为

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla p^* + Pr\nabla^2\mathbf{u}^* + RaPrT^*\hat{\mathbf{y}} \tag{20}$$

其中Pr为Prandtl数，Ra为Rayleigh数

$$Pr = \frac{\nu}{\kappa}, \quad Ra = \frac{g\beta_T\Delta_T L_0^3}{\nu\kappa} \tag{21}$$

对于温度方程可以改写为

$$\frac{\partial(T^*\Delta_T)}{\partial(L_0^2/\kappa t^*)} + \frac{\kappa}{L_0}u^*\frac{\partial(\Delta_T T^*)}{\partial(L_0 x^*)} + \frac{\kappa}{L_0}v^*\frac{\partial(\Delta_T T^*)}{\partial(L_0 y^*)} = \kappa\left[\frac{\partial\frac{\partial(\Delta_T T^*)}{\partial(L_0 x^*)}}{\partial(L_0 x^*)} + \frac{\partial\frac{\partial(\Delta_T T^*)}{\partial(L_0 y^*)}}{\partial(L_0 y^*)}\right] \tag{22}$$

上式表明，无量纲形式的温度方程可以写为

$$\frac{\partial T^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla T^* = \nabla^2 T^* \tag{23}$$

2. 选择长度尺度L_0、时间尺度$\sqrt{L_0/(\beta_T g\Delta_T)}$、速度尺度$\sqrt{\beta_T g L_0 \Delta_T}$、压强尺度$(\rho_0 g\beta_T\Delta_T L_0)$、温度尺度$\Delta_T$进行无量纲化，则

$$\begin{aligned}
&\mathbf{x}/L_0 \to \mathbf{x}^*, \quad t/\sqrt{L_0/(\beta_T g\Delta_T)} \to t^*, \quad \mathbf{u}/\sqrt{\beta_T g L_0\Delta_T} \to \mathbf{u}^*, \\
&p/\left(\rho_0 g\beta_T\Delta_T L_0\right) \to p^*, \quad \left(T - T_0\right)/\Delta_T \to T^*
\end{aligned} \tag{24}$$

对于连续性方程可以改写为

$$\frac{\partial(\sqrt{\beta_T g L_0\Delta_T}u^*)}{\partial(L_0 x^*)} + \frac{\partial(\sqrt{\beta_T g L_0\Delta_T}v^*)}{\partial(L_0 y^*)} = 0 \tag{25}$$

化简得到

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{26}$$

上式表明，无量纲形式的连续性方程可以写为

$$\nabla\cdot\mathbf{u}^*=0 \tag{27}$$

对于x方向的动量方程可以改写为

$$\begin{aligned}
&\frac{\partial(\sqrt{\beta_T g L_0\Delta_T}u^*)}{\partial\left[\sqrt{L_0/(\beta_T g\Delta_T)}t^*\right]}+\sqrt{\beta_T g L_0\Delta_T}u^*\frac{\partial(\sqrt{\beta_T g L_0\Delta_T}u^*)}{\partial(L_0x^*)}\\
&+\sqrt{\beta_T g L_0\Delta_T}v^*\frac{\partial(\sqrt{\beta_T g L_0\Delta_T}u^*)}{\partial(L_0y^*)}=-\frac{1}{\rho_0}\frac{\partial(\rho_0 g\beta_T\Delta_T L_0p^*)}{\partial(L_0x^*)}\\
&+\nu\left\{\frac{\partial\left[\frac{\partial(\sqrt{\beta_T g L_0\Delta_T}u^*)}{\partial(L_0x^*)}\right]}{\partial(L_0x^*)}+\frac{\partial\left[\frac{\partial(\sqrt{\beta_T g L_0\Delta_T}u^*)}{\partial(L_0y^*)}\right]}{\partial(L_0y^*)}\right\}+g\beta_T\Delta_T T^*\hat{\mathbf{y}}
\end{aligned} \tag{28}$$

化简得到

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{\nu}{\sqrt{g\beta_T\Delta_T L_0^3}}\left[\frac{\partial(\frac{\partial u^*}{\partial x^*})}{\partial x^*}+\frac{\partial(\frac{\partial u^*}{\partial y^*})}{\partial y^*}\right]+T^*\hat{\mathbf{y}} \tag{29}$$

上式表明，无量纲形式的动量方程可以写为

$$\frac{\partial\mathbf{u}^*}{\partial t^*}+\mathbf{u}^*\cdot\nabla\mathbf{u}^*=-\nabla p^*+\sqrt{\frac{Pr}{Ra}}\nabla^2\mathbf{u}^*+T^*\hat{\mathbf{y}} \tag{30}$$

其中Pr为Prandlt数，Ra为Rayleigh数

$$Pr=\frac{\nu}{\kappa},\quad Ra=\frac{g\beta_T\Delta_T L_0^3}{\nu\kappa} \tag{31}$$

对于温度方程可以改写为

$$\begin{aligned}
&\frac{\partial(T^*\Delta_T)}{\partial[\sqrt{L_0/(\beta_T g\Delta_T)}t^*]}+\sqrt{\beta_T g L_0\Delta_T}u^*\frac{\partial(\Delta_T T^*)}{\partial(L_0x^*)}+\sqrt{\beta_T g L_0\Delta_T}v^*\frac{\partial(\Delta_T T^*)}{\partial(L_0y^*)}\\
&=\kappa\left[\frac{\partial\frac{\partial(\Delta_T T^*)}{\partial(L_0x^*)}}{\partial(L_0x^*)}+\frac{\partial\frac{\partial(\Delta_T T^*)}{\partial(L_0y^*)}}{\partial(L_0y^*)}\right]
\end{aligned} \tag{32}$$

化简得到

$$\frac{\partial T^*}{\partial t^*}+\mathbf{u}^*\cdot\nabla T^*=\frac{\kappa}{\sqrt{\beta_T g L_0^3\Delta_T}}\nabla^2T^* \tag{33}$$

上式表明，无量纲形式的温度方程可以写为

$$\frac{\partial T^*}{\partial t^*}+\mathbf{u}^*\cdot\nabla T^*=\sqrt{\frac{1}{PrRa}}\nabla^2T^* \tag{34}$$