《粘性流体力学》第二次作业　　2020年5月8日

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Question：

考虑流动的不可压缩控制方程如下：（引入Boussinesq假设）

$$\begin{cases} \text{连续性方程：} & \nabla\cdot\vec{u}=0 \\ \text{动量方程：} & \dfrac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\dfrac{1}{\rho_0}\nabla p+\nu\nabla^2\vec{u}+g\beta_T(T-T_0)\hat{x} \\ \text{能量方程：} & \dfrac{\partial T}{\partial t}+\vec{u}\cdot\nabla T=\kappa\nabla^2 T \\ \text{（温度形式）} & \end{cases}$$

现利用两种方法对方程组进行无量纲化（Nondimensionalize）：

▶ 第一种无量纲化方法

$$\vec{x}/L_0\to\vec{x}^*,\quad t/\left(\frac{L_0^2}{\kappa}\right)\to t^*,\quad \vec{u}/\left(\frac{\kappa}{L_0}\right)\to\vec{u}^*,$$

$$p/\left(\frac{\rho_0\kappa^2}{L_0^2}\right)\to p^*,\quad (T-T_0)/\Delta_T\to T^*$$

▶ 第二种无量纲化方法

$$\vec{x}/L_0\to\vec{x}^*,\quad t/\sqrt{L_0/(\beta_T g\Delta_T)}\to t^*,\quad \vec{u}/\sqrt{\beta_T g L_0\Delta_T}\to\vec{u}^*,$$

$$p/(\rho_0 g\beta_T\Delta_T L_0)\to p^*,\quad (T-T_0)/\Delta_T\to T^*$$

△ Hint：定义两个无量纲常数

① Prandtl number（普朗特数）：

$$Pr=\frac{\nu}{\kappa}$$

② Rayleigh number（瑞利数）：

$$Ra=\frac{g\beta_T\Delta_T L_0^3}{\nu\kappa}$$

解：A. 利用第一种无量纲化方法。

为简化考虑，分析二维的不可压缩控制方程：

1) 连续性方程

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

即有

$$\frac{\partial(\frac{\kappa}{L_0}u^*)}{\partial(L_0x^*)}+\frac{\partial(\frac{\kappa}{L_0}v^*)}{\partial(L_0y^*)}=0 \tag{2}$$

故化简可得：

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \tag{3}$$

进一步有无量纲连续性方程：

$$\nabla\cdot\vec{u}^*=0 \tag{4}$$

2）动量方程（考虑x方向）为

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]+g\beta_T(T-T_0)\hat{z} \tag{5}$$

进行变量转换有：

$$\frac{\partial\left(\frac{K}{L_0}u^*\right)}{\partial\left(\frac{L_0^2}{K}t^*\right)}+\left(\frac{K}{L_0}u^*\right)\cdot\frac{\partial\left(\frac{K}{L_0}u^*\right)}{\partial(L_0x^*)}+\left(\frac{K}{L_0}v^*\right)\cdot\frac{\partial\left(\frac{K}{L_0}u^*\right)}{\partial(L_0y^*)}=-\frac{1}{\rho_0}\frac{\partial\left(\frac{\rho_0K^2}{L_0^2}p^*\right)}{\partial(L_0x^*)}$$

$$+\nu\left\{\frac{\partial\left[\frac{\partial\left(\frac{K}{L_0}u^*\right)}{\partial(L_0x^*)}\right]}{\partial(L_0x^*)}+\frac{\partial\left[\frac{\partial\left(\frac{K}{L_0}u^*\right)}{\partial(L_0y^*)}\right]}{\partial(L_0y^*)}\right\}+g\beta_T(\Delta_T T^*)\hat{z} \tag{6}$$

化简上式有：

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{\nu}{K}\left[\frac{\partial\left(\frac{\partial u^*}{\partial x^*}\right)}{\partial x^*}+\frac{\partial\left(\frac{\partial u^*}{\partial y^*}\right)}{\partial y^*}\right]$$

$$+\frac{g\beta_T\Delta_T L_0^3}{\nu K}\cdot\frac{\nu}{K}\cdot T^*\hat{z} \tag{7}$$

引入无量纲数 P_r，R_a，并进一步推广有无量纲动量方程：

$$\frac{\partial\vec{u}^*}{\partial t^*}+\vec{u}^*\cdot\nabla\vec{u}^*=-\nabla p^*+P_r\cdot\nabla^2\vec{u}^*+R_a\cdot P_r\cdot T^*\hat{z} \tag{8}$$

3）能量方程（温度形式）考虑x方向为：

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{9}$$

进行变量转换有（$T\to T^*$），有：

$$\frac{\partial(\Delta_T T^*+T_0)}{\partial t}+u\frac{\partial(\Delta_T T^*+T_0)}{\partial x}+v\frac{\partial(\Delta_T T^*+T_0)}{\partial y}=\kappa\left[\frac{\partial^2(\Delta_T T^*+T_0)}{\partial x^2}+\frac{\partial^2(\Delta_T T^*+T_0)}{\partial y^2}\right] \tag{10}$$

化简并结合

$$\frac{\partial T_0}{\partial t}+u\frac{\partial T_0}{\partial x}+v\frac{\partial T_0}{\partial y}=\kappa\left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right) \tag{11}$$

约去左右两式常数 Δ_T 有：

$$\frac{\partial T^*}{\partial t}+u\frac{\partial T^*}{\partial x}+v\frac{\partial T^*}{\partial y}=\kappa\left(\frac{\partial^2 T^*}{\partial x^2}+\frac{\partial^2 T^*}{\partial y^2}\right) \tag{12}$$

再将 t，x，y，u 进行无量纲化有：

$$\frac{\partial T^*}{\partial\left(\frac{L_0^2}{K}t^*\right)}+\left(\frac{K}{L_0}u^*\right)\frac{\partial T^*}{\partial(L_0x^*)}+\left(\frac{K}{L_0}v^*\right)\frac{\partial T^*}{\partial(L_0y^*)}=\kappa\left[\frac{\frac{\partial T^*}{\partial(L_0x^*)}}{\partial(L_0x^*)}+\frac{\frac{\partial T^*}{\partial(L_0y^*)}}{\partial(L_0y^*)}\right] \tag{13}$$

化简可得无量纲能量方程（温度形式）

$$\frac{\partial T^*}{\partial t^*}+u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\kappa\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right) \tag{14}$$

进一步可推得：

$$\frac{\partial T^*}{\partial t^*}+\vec{u}^*\cdot\nabla T^*=\kappa\nabla^2(T^{*2}) \tag{15}$$

∴推导完毕。

B. 利用第二种无量纲化方法

同样地，为简化考虑，分析二维的不可压缩流动方程。

1) 连续性方程

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{16}$$

进行变量转换有：

$$\frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, u^*)}{\partial(L_0 x^*)} + \frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, v^*)}{\partial(L_0 y^*)} = 0 \tag{17}$$

化简可得：

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{18}$$

进一步推广可得无量纲连续方程：

$$\nabla \cdot \vec{u}^* = 0 \tag{19}$$

2) 动量方程，考虑x方向上，有：

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g\beta_T(T-T_0)\hat{z} \tag{20}$$

进行变量转换有：

$$\frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, u^*)}{\partial\left(\sqrt{\frac{L_0}{\beta_T g \Delta_T}}\, t^*\right)} + (\sqrt{\beta_T g L_0 \Delta_T}\, u^*)\frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, u^*)}{\partial(L_0 x^*)} + (\sqrt{\beta_T g L_0 \Delta_T}\, v^*)\frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, u^*)}{\partial(L_0 y^*)}$$

$$= -\frac{1}{\rho_0}\cdot\frac{\partial(\rho_0 g \beta_T \Delta_T L_0 p^*)}{\partial(L_0 x^*)} + \nu\left\{\frac{\partial\left[\frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, u^*)}{\partial(L_0 x^*)}\right]}{\partial(L_0 x^*)} + \frac{\partial\left[\frac{\partial(\sqrt{\beta_T g L_0 \Delta_T}\, u^*)}{\partial(L_0 y^*)}\right]}{\partial(L_0 y^*)}\right\} + g\beta_T(\Delta_T T^*)\hat{z} \tag{21}$$

化简上式有：

$$g\beta_T\Delta_T\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = (g\beta_T\Delta_T)\cdot -\frac{\partial p^*}{\partial x^*} + \frac{\nu\sqrt{\beta_T g L_0 \Delta_T}}{L_0^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) + g\beta_T\Delta_T T^*\hat{z} \tag{22}$$

两边同除 $g\beta_T\Delta_T$ 可得：

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \sqrt{\frac{\nu k}{g\beta_T\Delta_T L_0^3}\cdot\frac{\nu}{k}}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) + T^*\hat{z} \tag{23}$$

引入无量纲数 Ra、Pr，有

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \sqrt{\frac{Pr}{Ra}}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) + T^*\hat{z} \tag{24}$$

进一步推广，有无量纲形式的动量方程.

$$\frac{\partial u^*}{\partial t^*} + \vec{u}^*\cdot\nabla\vec{u}^* = -\nabla p^* + \sqrt{\frac{Pr}{Ra}}\,\nabla^2\vec{u}^* + T^*\hat{z} \tag{25}$$

3）能量方程(温度形式)，考虑x方向有：

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \tag{26}$$

进行变量转换($T\to T^*$)有：

$$\frac{\partial(\Delta_T T^*+T_0)}{\partial t}+u\frac{\partial(\Delta_T T^*+T_0)}{\partial x}+v\frac{\partial(\Delta_T T^*+T_0)}{\partial y}=\kappa\left[\frac{\partial^2(\Delta_T T^*+T_0)}{\partial x^2}+\frac{\partial^2(\Delta_T T^*+T_0)}{\partial y^2}\right] \tag{27}$$

化简上式，并结合

$$\frac{\partial T_0}{\partial t}+u\frac{\partial T_0}{\partial x}+v\frac{\partial T_0}{\partial y}=\kappa\left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right) \tag{28}$$

约去左右两式中常数Δ_T可得：

$$\frac{\partial T^*}{\partial t}+u\frac{\partial T^*}{\partial x}+v\frac{\partial T^*}{\partial y}=\kappa\left(\frac{\partial^2 T^*}{\partial x^2}+\frac{\partial^2 T^*}{\partial y^2}\right) \tag{29}$$

再分别将t，x，y，u进行无量纲化有：

$$\frac{\partial T^*}{\partial\left(\sqrt{\frac{L_0}{\beta_T g\Delta_T}}\,t^*\right)}+\left(\sqrt{\beta_T g L_0\Delta_T}\,u^*\right)\frac{\partial T^*}{\partial(L_0x^*)}+\left(\sqrt{\beta_T g L_0\Delta_T}\,v^*\right)\frac{\partial T^*}{\partial(L_0y^*)}=\kappa\left[\frac{\frac{\partial T^*}{\partial(L_0x^*)}}{\partial(L_0x^*)}+\frac{\frac{\partial T^*}{\partial(L_0y^*)}}{\partial(L_0y^*)}\right] \tag{30}$$

化简上式可得：

$$\frac{\partial T^*}{\partial t^*}+u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{\kappa}{L_0^2}\cdot\sqrt{\frac{L_0}{\beta_T g\Delta_T}}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right) \tag{31}$$

即有

$$\frac{\partial T^*}{\partial t^*}+u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\sqrt{\frac{\nu\kappa}{\beta_T g\Delta_T L_0^3}\cdot\frac{\kappa}{\nu}}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right) \tag{32}$$

引入无量纲数Ra，Pr可化简有

$$\frac{\partial T^*}{\partial t^*}+u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{1}{\sqrt{Ra\cdot Pr}}\left(\frac{\partial^2 T^*}{\partial x^{*2}}+\frac{\partial^2 T^*}{\partial y^{*2}}\right) \tag{33}$$

进一步可推得无量纲化的能量方程(温度形式)：

$$\frac{\partial T^*}{\partial t^*}+\vec{u}^*\cdot\nabla T^*=\frac{1}{\sqrt{Ra\cdot Pr}}\nabla^2 T^* \tag{34}$$

∴推导完毕。