第三讲：动力学相似性

Chapter 15 Introduction to the Fundamental Principles of Viscous Flow

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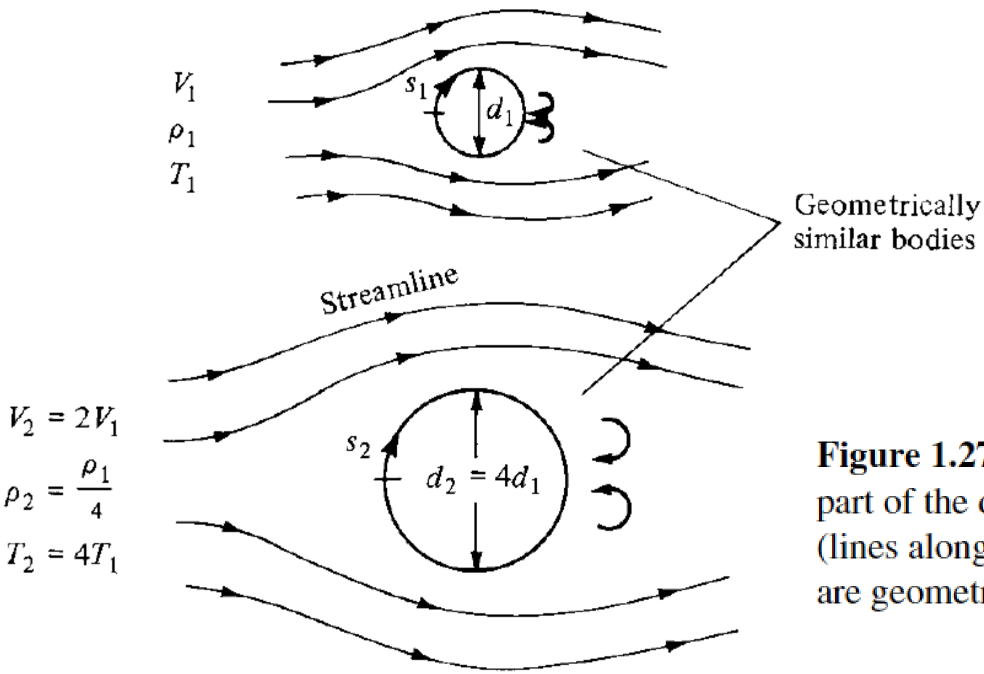
第六节：流动相似准则

1.8 FLOW SIMILARITY

Flow similarity 流动相似

Consider two different flow fields over two different bodies. By definition, different flows are *dynamically similar* if:

1. The streamline patterns are geometrically similar.
2. The distributions of V/V_∞, p/p_∞, T/T_∞, etc., throughout the flow field are the same when plotted against common nondimensional coordinates.
3. The force coefficients are the same.



流动相似实例，两个流动动力学相似，首先流线相似。

Figure 1.27 Example of dynamic flow similarity. Note that as part of the definition of dynamic similarity, the streamlines (lines along which the flow velocity is tangent at each point) are geometrically similar between the two flows.

Flow similarity 流动相似

What are the criteria to ensure that two flows are dynamically similar?

1. The bodies and any other solid boundaries are geometrically similar for both flows.
2. The similarity parameters are the same for both flows.

Implication of the law of similarity

All flows with the same similarity parameters and the same flow geometry are similar --- can be obtained from one another by simply changing the units of measurements of coordinates and velocities, etc.

流动动力学相似是流体力学试验的基本原理。

量纲分析

基本物理量和基本单位

- 力学中的**长度**(米/m)、**时间**(秒/s)、**质量**(千克/kg)
- 热力学中的**温度**(卡尔文/K)、**物质的量**(摩尔/mol)
- 电学中的**电流**(安培/A)
- 光学中的**发光强度**(坎德拉/cd)

- 可以由基本物理量引导出的物理量称为**导出量**。
- 在力学中，基本量纲是长度L、时间T、质量M，而导出量的量纲表示成$\mathsf{L}^{\alpha}\mathsf{T}^{\beta}\mathsf{M}^{\gamma}$，其中$\alpha$, β, γ称为**量纲指数**。
- 若一个物理量的所有量纲指数都为零，称为**无量纲量**。

量纲转化

Table 1: conversion between two systems

variable	**system 1 (SI unit)**	**system 2**	conversion
length	x [m]	x^*	$x = x^* \cdot L_0$
time	t [s]	t^*	$t = t^* \cdot t_0$
mass	m [kg]	m^*	$m = m^* \cdot m_0$
thermodynamic temperature	T [K]	T^*	$T = T^* \cdot T_0$
amount of substance	n [mol]	n^*	$n = n^* \cdot n_0$
electric current	I [A]	I^*	$I = I^* \cdot I_0$

density	ρ [kg $\cdot$ m^{-3}]	ρ^*	$\rho = \rho^* \cdot m_0/L_0^3$
pressure	P [kg $\cdot$ m$^{-1}\cdot$ s^{-2}]	P^*	$P = P^* \cdot \frac{m_0}{L_0 t_0^2}$
macro velocity	u [m $\cdot$ s^{-1}]	u^*	$u = u^* \cdot L_0/t_0$
acceleration	a [m $\cdot$ s^{-2}]	a^*	$a = a^* \cdot L_0/t_0^2$
external force	$G = ma$	G^*	$G = G^* \cdot L_0 m_0/t_0^2$
viscosity	μ [kg $\cdot$ m$^{-1}\cdot$ s^{-1}]	μ^*	$\mu = \mu^* \frac{m_0}{L_0 t_0}$
kinematic viscosity	$\nu = \mu/\rho$ [m$^2\cdot$ s^{-1}]	ν^*	$\nu = \nu^* \cdot L_0^2/t_0$
thermal diffusivity	κ [m^2 $\cdot$ s^{-1}]	κ^*	$\kappa = \kappa^* \cdot L_0^2/t_0$
mass diffusivity	D [m^2 $\cdot$ s^{-1}]	D^*	$D = D^* \cdot L_0^2/t_0$
concentration	C [mol $\cdot$ m^{-3}]	C^*	$C = C^* \cdot n_0/L_0^3$
surface tension	σ	σ^*	$\sigma = \sigma^* \cdot m_0/t_0^2$
heat flux	$\mathbf{J} = -\kappa\nabla T + T\mathbf{u}$	$\mathbf{J}^* = -\kappa^*\nabla T^* + T^*\mathbf{u}^*$	$\mathbf{J} = \mathbf{J}^* \cdot \frac{L_0 T_0}{t_0}$
specific heat	C_v	C_v^*	$C_v = C_v^* \cdot \frac{L_0^2}{t_0^2 T_0}$
thermal expansion coefficient	β_T [K^{-1}]	β_T^*	$\beta = \beta_T^* \cdot \frac{1}{T_H - T_c}$
concentration expansion coefficient	β_c [mol^{-1} $\cdot$ m^3]	β_c^*	$\beta_c = \beta_c^* \cdot \frac{L_0^3}{n_0}$
shear strain rate tensor	$\mathbf{S}$	$\mathbf{S}^*$	$\mathbf{S} = \mathbf{S}^* \cdot \frac{1}{t_0}$
Boltzmann constant	k_B [m$^2\cdot$ kg $\cdot$ s$^{-2}\cdot$ K^{-1}]	k_B^*	$k_B = k_B^* \cdot \frac{L_0^2 m_0}{t_0^2 T_0}$

第七节：动力学相似参数

15.6 SIMILARITY PARAMETERS

First example

The incompressible Naiver-Stokes equation is written as

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

with the scalings

$$\mathbf{x}/L_0 \to \mathbf{x}^*, \quad t/\left(\frac{L_0}{U_0}\right) \to t^*, \quad \mathbf{u}/U_0 \to \mathbf{u}^*,$$

$$p/\left(\frac{m_0 U_0^2}{L_0^3}\right) \to p^*, \quad \rho/\left(\frac{m_0}{L_0^3}\right) \to \rho^*$$

First example

The **continuity equation** is rewritten as

$$\frac{\partial(U_0 u^*)}{\partial(L_0 x^*)} + \frac{\partial(U_0 v^*)}{\partial(L_0 y^*)} = 0$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\nabla \cdot \mathbf{u}^* = 0$$

First example

The x-direction **momentum equation** is rewritten as

$$\frac{\partial(U_0 u^*)}{\partial(L_0/U_0 t^*)} + (U_0 u^*)\frac{\partial(U_0 u^*)}{\partial(L_0 x^*)} + (U_0 v^*)\frac{\partial(U_0 u^*)}{\partial(L_0 y^*)} = -\frac{1}{m_0/L_0^3 \rho^*}\frac{\partial(m_0 U_0^2/L_0^3 p^*)}{\partial(L_0 x^*)} + \nu\left[\frac{\partial(\frac{\partial U_0 u^*}{\partial L_0 x^*})}{\partial(L_0 x^*)} + \frac{\partial(\frac{\partial U_0 u^*}{\partial L_0 y^*})}{\partial(L_0 y^*)}\right]$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*} + \frac{\nu}{U_0 L_0}\left[\frac{\partial(\frac{\partial u^*}{\partial x^*})}{\partial x^*} + \frac{\partial(\frac{\partial u^*}{\partial y^*})}{\partial y^*}\right]$$

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\frac{1}{\rho^*}\nabla p^* + \frac{1}{Re}\nabla^2 \mathbf{u}^*$$

where the Reynolds number is defined as

$$Re = \frac{U_0 L_0}{\nu}$$

三维可压缩N-S方程—粘性流动动量方程

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\rho w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\lambda\nabla\cdot\mathbf{V}+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{\partial v}{\partial t}+\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\rho w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left(\lambda\nabla\cdot\mathbf{V}+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]$$

$$\rho\frac{\partial w}{\partial t}+\rho u\frac{\partial w}{\partial x}+\rho v\frac{\partial w}{\partial y}+\rho w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left(\lambda\nabla\cdot\mathbf{V}+2\mu\frac{\partial w}{\partial z}\right)$$

分析对象：定常二维粘性可压缩流动

x 向动量方程：

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$

式中各量均为量纲量，未考虑质量力 f

忽略了粘性正应力 τ_{xx}

仅保留了剪应力 τ_{yx}

引入无量纲变量

参考量:

ρ_∞, V_∞, p_∞, and μ_∞

are reference values (say, e.g., freestream values)

c is a reference length (say, the chord of an airfoil).

$$\rho' = \frac{\rho}{\rho_\infty} \quad u' = \frac{u}{V_\infty} \quad v' = \frac{v}{V_\infty} \quad p' = \frac{p}{p_\infty}$$

$$\mu' = \frac{\mu}{\mu_\infty} \quad x' = \frac{x}{c} \quad y' = \frac{y}{c}$$

Dimensionless variables

Nondimensionalize

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\left(\frac{p_\infty}{\rho_\infty V_\infty^2}\right) \frac{\partial p'}{\partial x'} + \left(\frac{\mu_\infty}{\rho_\infty V_\infty c}\right) \frac{\partial}{\partial y'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]$$

Noting that

$$\frac{p_\infty}{\rho_\infty V_\infty^2} = \frac{\gamma p_\infty}{\gamma \rho_\infty V_\infty^2} = \frac{a_\infty^2}{\gamma V_\infty^2} = \frac{1}{\gamma M_\infty^2}$$

and

$$\frac{\mu_\infty}{\rho_\infty V_\infty c} = \frac{1}{\mathrm{Re}_\infty}$$

$$a_\infty^2 = \frac{\gamma p_\infty}{\rho_\infty}$$

γ is the ratio between the two specific heat

a_∞ is the sound speed of free stream

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty} \frac{\partial}{\partial y'} \left[\mu' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right]$$

$$\mathrm{Re}_\infty = \frac{\rho_\infty V_\infty c}{\mu_\infty} \qquad M_\infty^2 = \frac{V_\infty^2}{a_\infty^2}$$

M_∞ and Re_∞ are the freestream Mach and Reynolds numbers

无量纲空间 (x', y') 求解无量纲微分方程

相同的几何外形

相同的来流条件 ⟹ 相同的解

相同的控制方程

当 γ，M_∞，Re_∞ 相同，两个物体几何相似（即边界条件相同），则两个流动相似

$$f_1(x', y') \equiv f_2(x', y')$$

γ，M_∞，Re_∞为流动的 三个相似参数.

That u' as a function of x' and y' is the same for the two flows

绕直径大小不同圆柱的流动

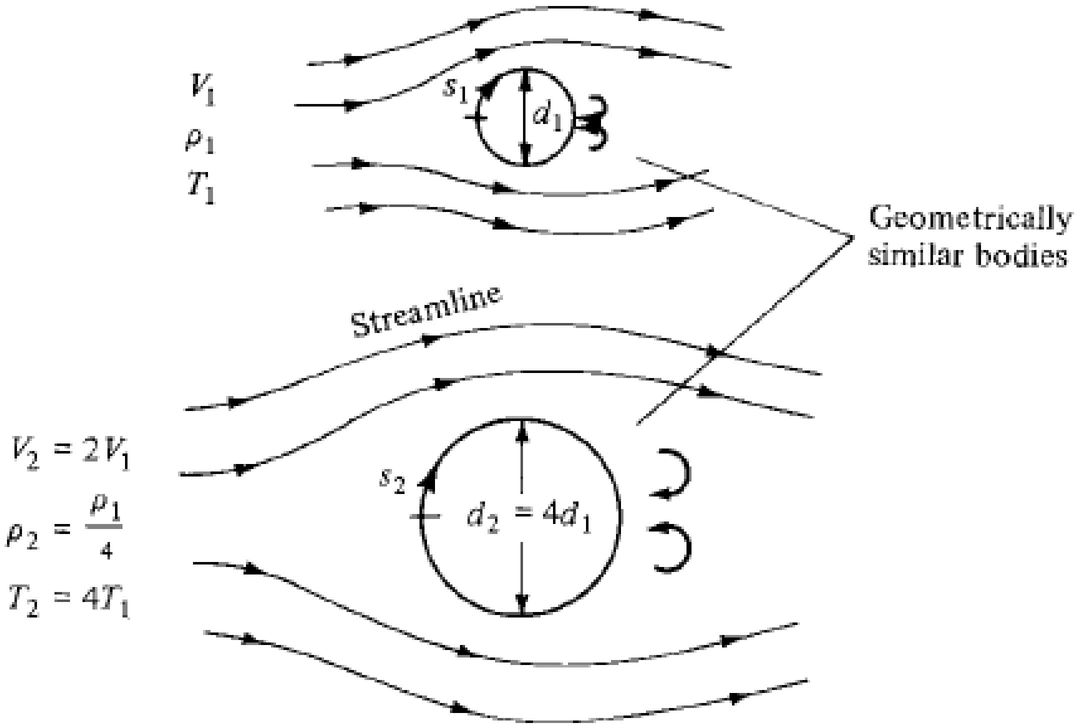


Figure 1.27 Example of dynamic flow similarity. Note that as part of the definition of dynamic similarity, the streamlines (lines along which the flow velocity is tangent at each point) are geometrically similar between the two flows.

绕直径大小不同圆柱的流动

Since $\mu \propto \sqrt{T}$ and $a \propto \sqrt{T}$, then

$$\frac{\mu_2}{\mu_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{4T_1}{T_1}} = 2$$

and

$$\frac{a_2}{a_1} = \sqrt{\frac{T_2}{T_1}} = 2$$

By definition,

$$M_1 = \frac{V_1}{a_1}$$

and

$$M_2 = \frac{V_2}{a_2} = \frac{2V_1}{2a_1} = \frac{V_1}{a_1} = M_1$$

Hence, the Mach numbers are the same. Basing the Reynolds number on the diameter d of the cylinder, we have by definition,

$$\mathrm{Re}_1 = \frac{\rho_1 V_1 d_1}{\mu_1}$$

and

$$\mathrm{Re}_2 = \frac{\rho_2 V_2 d_2}{\mu_2} = \frac{(\rho_1/4)(2V_1)(4d_1)}{2\mu_1} = \frac{\rho_1 V_1 d_1}{\mu_1} = \mathrm{Re}_1$$

绕直径大小不同圆柱的流动

Hence, the Reynolds numbers are the same. Since the two bodies are geometrically similar and M_∞ and Re are the same, we have satisfied all the criteria; the two flows are dynamically similar. In turn, as a consequence of being similar flows, we know from the definition that:

1. The streamline patterns around the two cylinders are geometrically similar.
2. The nondimensional pressure, temperature, density, velocity, etc., distributions are the same around two cylinders.

绕直径大小不同圆柱的流动

1. The streamline patterns around the two cylinders are geometrically similar.
2. The nondimensional pressure, temperature, density, velocity, etc., distributions are the same around two cylinders.

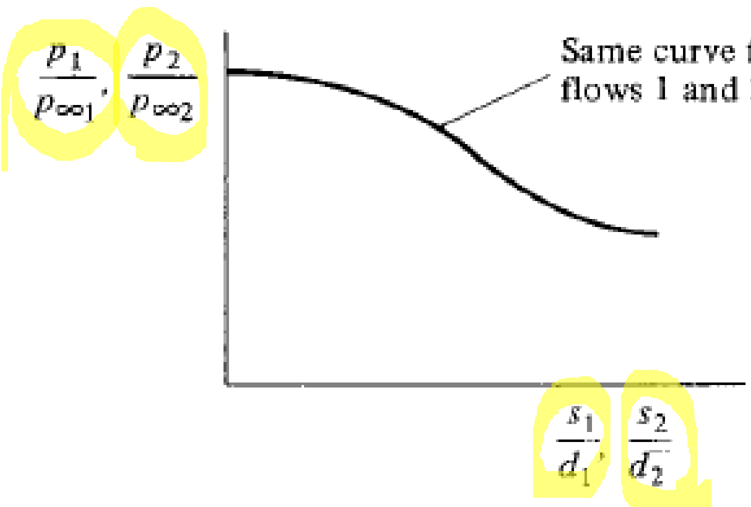


Figure 1.28 One aspect of the definition of dynamically similar flows. The nondimensional flow variable distributions are the same.

粘性流动能量方程：二维定常能量方程无量纲化分析

$$\rho\frac{D(e+V^2/2)}{Dt}=\rho\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$$
$$+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-\nabla\cdot p\mathbf{V}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}$$
$$+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}$$
$$+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}$$

$$\rho u\frac{\partial(e+V^2/2)}{\partial x}+\rho v\frac{\partial(e+V^2/2)}{\partial y}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)-\frac{\partial(up)}{\partial x}$$
$$-\frac{\partial(vp)}{\partial y}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}$$

该方程忽略质量力做功、体加热和粘性正应力做功。

代入应力应变率本构关系

$$
\begin{aligned}
\rho u \frac{\partial(e+V^2/2)}{\partial x} + \rho v \frac{\partial(e+V^2/2)}{\partial y} = {} & \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\
& - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} \\
& + \frac{\partial}{\partial x}\left[\mu v\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \\
& + \frac{\partial}{\partial y}\left[\mu u\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]
\end{aligned}
$$

参考量:

ρ_∞, V_∞, p_∞, T_∞ and μ_∞, k_∞

are reference values (say, e.g., freestream values)

c is a reference length (say, the chord of an airfoil).

Using the same nondimensional variables as before, and introducing

$$e' = \frac{e}{c_v T_\infty} \qquad k' = \frac{k}{k_\infty}$$

$$V'^2 = \frac{V^2}{V_\infty^2} = \frac{u^2 + v^2}{V_\infty^2} = (u')^2 + (v')^2$$

$$
\frac{\rho_\infty V_\infty c_v T_\infty}{c}\left(\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'}\right)
$$

$$
= -\frac{\rho_\infty V_\infty^3}{2c}\left[\rho' u' \frac{\partial}{\partial x'}(u'^2 + v'^2) + \rho' v' \frac{\partial}{\partial y'}(u'^2 + v'^2)\right]
$$

$$
+ \frac{k_\infty T_\infty}{c^2}\left[\frac{\partial}{\partial x'}\left(k' \frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k' \frac{\partial T'}{\partial y'}\right)\right] - \frac{V_\infty p_\infty}{c}\left(\frac{\partial (u' p')}{\partial x'} + \frac{\partial (v' p')}{\partial y'}\right)
$$

$$
+ \frac{\mu_\infty V_\infty^2}{c^2}\left\{\frac{\partial}{\partial x'}\left[\mu' v' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right] + \frac{\partial}{\partial y'}\left[\mu' u' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]\right\}
$$

两边除以 $\dfrac{\rho_\infty V_\infty c_v T_\infty}{c}$

$$
\begin{aligned}
\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} = {} & \frac{V_\infty^2}{2 c_v T_\infty} \left[\rho' u' \frac{\partial}{\partial x'} (u'^2 + v'^2) \right. \\
& \left. + \rho' v' \frac{\partial}{\partial y'} (u'^2 + v'^2) \right] \\
& + \frac{k_\infty}{c \rho_\infty V_\infty c_v} \left[\frac{\partial}{\partial x'} \left(k' \frac{\partial T'}{\partial x'} \right) + \frac{\partial}{\partial y'} \left(k' \frac{\partial T'}{\partial y'} \right) \right] \\
& - \frac{p_\infty}{\rho_\infty c_v T_\infty} \left(\frac{\partial (u' p')}{\partial x'} + \frac{\partial (v' p')}{\partial y'} \right) \\
& + \frac{\mu_\infty V_\infty}{c \rho_\infty c_v T_\infty} \left\{ \frac{\partial}{\partial x'} \left[\mu' v' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] \right. \\
& \left. + \frac{\partial}{\partial y'} \left[\mu' u' \left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] \right\}
\end{aligned}
$$

Examining the coefficients of each term on the right-hand side of Equation

$$\frac{V_\infty^2}{c_v T_\infty} = \frac{(\gamma-1)V_\infty^2}{RT_\infty} = \frac{\gamma(\gamma-1)V_\infty^2}{\gamma RT_\infty} = \frac{\gamma(\gamma-1)V_\infty^2}{a_\infty^2} = \gamma(\gamma-1)M_\infty^2$$

$$\frac{k_\infty}{c\rho_\infty V_\infty c_v} = \frac{k_\infty \gamma \mu_\infty}{c\rho_\infty V_\infty c_p \mu_\infty} = \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty}$$

Prandtl number, $\mathrm{Pr}_\infty \equiv \mu_\infty c_p / k_\infty$, 普朗特数，相似参数

普朗特数的含义：摩擦力引起的能量耗散与热传导传输能量的比值

$$\mathrm{Pr} = \frac{\mu_\infty c_p}{k} \propto \frac{\text{frictional dissipation}}{\text{thermal conduction}}$$

整理方程右边另外两项系数：

$$\frac{p_\infty}{\rho_\infty c_v T_\infty}=\frac{(\gamma-1)p_\infty}{\rho_\infty R T_\infty}=\frac{(\gamma-1)p_\infty}{p_\infty}=\gamma-1$$

$$\frac{\mu_\infty V_\infty}{c\rho_\infty c_v T_\infty}=\frac{\mu_\infty}{\rho_\infty V_\infty c}\left(\frac{V_\infty^2}{c_v T_\infty}\right)=\frac{1}{\mathrm{Re}_\infty}(\gamma-1)\frac{V_\infty^2}{R T_\infty}=\gamma(\gamma-1)\frac{M_\infty^2}{\mathrm{Re}_\infty}$$

$$a_\infty^2=\frac{\gamma p_\infty}{\rho_\infty}=\gamma R T_\infty$$

$$P=\rho RT$$

则方程变为：

$$
\begin{aligned}
&\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} \\
&= \frac{\gamma(\gamma-1)}{2} M_\infty^2 \left[\rho' u' \frac{\partial}{\partial x'}(u'^2 + v'^2) + \rho' v' \frac{\partial}{\partial y'}(u'^2 + v'^2)\right] \\
&\quad + \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty}\left[\frac{\partial}{\partial x'}\left(k' \frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k' \frac{\partial T'}{\partial y'}\right)\right] \\
&\quad - (\gamma-1)\left(\frac{\partial(u'p')}{\partial x'} + \frac{\partial(v'p')}{\partial y'}\right) \\
&\quad + \gamma(\gamma-1)\frac{M_\infty^2}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial x'}\left[\mu' v'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]\right. \\
&\qquad \left. + \frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]\right\}
\end{aligned}
$$

谨记：忽略了质量力做功、体加热和正应力做功。

Implication of the law of similarity

无量纲流动控制方程：

$$\frac{\partial \rho' u'}{\partial x'} + \frac{\partial \rho' v'}{\partial y'} = 0$$

连续性方程

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2}\frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial y'}\left[\mu'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]$$

动量方程

$$\rho' u' \frac{\partial v'}{\partial x'} + \rho' v' \frac{\partial v'}{\partial y'} = -\frac{1}{\gamma M_\infty^2}\frac{\partial p'}{\partial y'} + \frac{1}{\mathrm{Re}_\infty}\frac{\partial}{\partial x'}\left[\mu'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]$$

$$\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty^2 \left[\rho' u' \frac{\partial}{\partial x'}(u'^2 + v'^2) + \rho' v' \frac{\partial}{\partial y'}(u'^2 + v'^2)\right]$$

能量方程

$$= \frac{\gamma}{\mathrm{Pr}_\infty \mathrm{Re}_\infty}\left[\frac{\partial}{\partial x'}\left(k'\frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right)\right] - (\gamma-1)\left(\frac{\partial(u'p')}{\partial x'} + \frac{\partial(v'p')}{\partial y'}\right)$$

$$+ \gamma(\gamma-1)\frac{M_\infty^2}{\mathrm{Re}_\infty}\left\{\frac{\partial}{\partial x'}\left[\mu' v'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right] + \frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]\right\}$$

方程定解条件 —流动边界条件

粘性流动相似参数

通过无量纲流动控制方程得到，为 γ， M_∞， Re， Pr

γ， M_∞， Re 从动量方程和能量方程得出

$$\mathrm{Pr} = \frac{\mu c_p}{k}$$

需从能量方程导出

为一物性参数，不同气体值不同，标准大气时，为 0.71，与 μ 和 k 一样，是温度 T 的函数。

$$\mathrm{Pr} = \frac{\mu_\infty c_p}{k} \propto \frac{\text{frictional dissipation}}{\text{thermal conduction}}$$

第二次作业：

where the Prandtl number is defined as

$$Pr = \frac{\nu}{\kappa}$$

the Rayleigh number is defined as

$$Ra = \frac{g\beta_T \Delta_T L_0^3}{\nu\kappa}$$

- 不可压缩控制方程（Boussinesq假设）

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{u} + g\beta_T(T - T_0)\hat{\mathbf{z}}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa\nabla^2 T$$

- 第一种无量纲化方法

$$\mathbf{x}/L_0 \to \mathbf{x}^*, \quad t/\left(\frac{L_0^2}{\kappa}\right) \to t^*, \quad \mathbf{u}/\left(\frac{\kappa}{L_0}\right) \to \mathbf{u}^*,$$
$$p/\left(\frac{\rho_0\kappa^2}{L_0^2}\right) \to p^*, \quad (T - T_0)/\Delta_T \to T^*$$

- 第二种无量纲化方法

$$\mathbf{x}/L_0 \to \mathbf{x}^*, \quad t/\sqrt{L_0/(\beta_T g\Delta_T)} \to t^*, \quad \mathbf{u}/\sqrt{\beta_T g L_0 \Delta_T} \to \mathbf{u}^*,$$
$$p/(\rho_0 g\beta_T \Delta_T L_0) \to p^*, \quad (T - T_0)/\Delta_T \to T^*$$

第二次作业：

- 要求：
  - 在A4纸上完成
  - 截止日期：5月19日至5月22日之间
  - 提交方式：电子版（拍照或扫描）
  - 邮件发送，主题“第二次作业_学号_姓名”
  - 例如：第二次作业_2017300300_张三
  - 助教邮箱：912387046@mail.nwpu.edu.cn

第八节：粘性流动的解

15.7 SOLUTIONS OF VISCOUS FLOWS

---A PRELIMINARY DISCUSSION

15.7 SOLUTIONS OF VISCOUS FLOWS:

--- A PRELIMINARY DISCUSSION

- The governing continuity, momentum, and energy equations for a general unsteady, compressible, viscous, three-dimensional flow.
- They are nonlinear, coupled, partial differential equations.
- Moreover, they have additional terms—namely, the viscous terms—in comparison to the analogous equations for an inviscid flow.
- Question: how, then, can we make use of the viscous flow equations in order to obtain some practical results?

千禧年大奖难题(Millennium Prize Problems), 又称世界七大数学难题

Main article: Millennium Prize Problems

The institute is best known for establishing the Millennium Prize Problems on May 24, 2000. These seven problems are considered by CMI to be "important classic questions that have resisted solution over the years." For each problem, the first person to solve it will be awarded $1,000,000 by the CMI. In announcing the prize, CMI drew a parallel to Hilbert's problems, which were proposed in 1900, and had a substantial impact on 20th century mathematics. Of the initial 23 Hilbert problems, most of which have been solved, only the Riemann hypothesis (formulated in 1859) is included in the seven Millennium Prize Problems.[1]

For each problem, the Institute had a professional mathematician write up an official statement of the problem, which will be the main standard by which a given solution will be measured against. The seven problems are:

- P versus NP
- The Hodge conjecture
- The Poincaré conjecture—solved, by Grigori Perelman[2]
- The Riemann hypothesis
- Yang–Mills existence and mass gap
- Navier–Stokes existence and smoothness
- The Birch and Swinnerton-Dyer conjecture

美国克雷数学研究所CMI承诺：

任何一个猜想的解答，只要发表在数学期刊上，并经过两年的验证期，解决者就会被颁发一百万美元奖金。

Some of the mathematicians who were involved in the selection and presentation of the seven problems were Atiyah, Bombieri, Connes, Deligne, Fefferman, Milnor, Mumford, Wiles, and Witten.

The solution of viscous flows

We have the following options:

1. Exact solution 精确解
2. Approximate solution 近似解
3. Direct numerical simulation 数值解

Exact solution 精确解

1. There are a few viscous flow problems which, by their physical and geometrical nature, allow many terms in the Navier-Stokes solutions to be precisely zero, with the resulting equations being simple enough to solve, either analytically or by simple numerical methods. Sometimes this class of solutions is called "exact solutions" of the Navier-Stokes equations, because no simplifying *approximations* are made to reduce the equations—just *precise* conditions are applied to reduce the equations. Chapter 16 is devoted to this class of solutions; an example is Couette flow (to be defined later).

Approximate solution 近似解

2. We can simplify the equations by treating certain classes of physical problems for which some terms in the viscous flow equations are small and can be neglected. This is an approximation, not a precise condition. The boundary-layer equations developed and discussed in Chapter 17 are a case in point. However, as we will see, the boundary-layer equations may be simpler than the full viscous flow equations, but they are still nonlinear.

Direct numerical simulation 数值解

3. We can tackle the solution of the full viscous flow equations by modern numerical techniques. For example, some of the computational fluid dynamic algorithms discussed in Chapter 13 in conjunction with "exact" solutions for the inviscid flow equations carry over to exact solutions for the viscous flow equations. These matters will be discussed in Chapter 20.

There are some inherent very important differences between the analysis of viscous flows and the study of inviscid flows:

- First, we have already demonstrated that viscous flows are rotational flows.

  A velocity potential cannot be defined for a viscous flow， a stream function can be defined.

- Second, the boundary condition at a solid surface for a viscous flow is the no-slip condition, the fluid velocity right at the surface is zero:

  $$u = 0 \quad v = 0 \quad w = 0$$

  This is in contrast to the analogous boundary condition for an inviscid flow, the flow-tangency condition at a surface, where only the component of the velocity normal to the surface is zero.

Temperature boundary condition：

- Also, for an inviscid flow, there is no boundary condition on the temperature.
- However, for a viscous flow, the mechanism of thermal conduction ensures that the temperature of the fluid immediately adjacent to the surface is the same as the temperature of the material surface.

$$T = T_w$$

- However, consider the following, more general case, a viscous flow over a surface where heat is being transferred from the gas to the surface.

$$\dot{q}_w = -\left(k\frac{\partial T}{\partial y}\right)_w = 0$$

$$\left(\frac{\partial T}{\partial y}\right)_w = 0$$

In summary, for the wall boundary condition associated with the solution of the energy equation:

1. *Constant temperature wall, where T_w is a specified constant*
   For this given wall temperature, the temperature gradient at the wall $(\partial T/\partial y)_w$ is obtained as part of the flow-field solution and allows the direct calculation of the aerodynamic heating to the wall.

2. *The general, unsteady case, where the heat transfer to the wall $\dot{q}_w$ causes the wall temperature T_w to change, which in turn causes $\dot{q}_w$ to change*
   Here, both T_w and $(\partial T/\partial y)_w$ change as a function of time, and the problem must be solved by treating jointly the viscous flow as well as the thermal response of the wall material (which usually implies a separate thermal conduction heat transfer numerical analysis).

3. *The adiabatic wall case (zero heat transfer), where* $(\partial T/\partial y)_w = 0$
   Here, the boundary condition is applied to the temperature *gradient* at the wall, not to the wall temperature itself. Indeed, the wall temperature for this case is defined as the adiabatic wall temperature T_{aw} and is obtained as part of the flow-field solution.

- Finally, we emphasize again that, from the point of view of applied aerodynamics, the practical results obtained from a viscous flow analysis are the skin friction and heat transfer at the surface.
- However, to obtain these quantities, we usually need a complete solution of the viscous flow field.

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_w$$

$$\dot{q}_w = -k \left(\frac{\partial T}{\partial y} \right)_w$$

第九节：小结

15.8 SUMMARY

1、流体粘性的影响

Shear stress and flow separation are two major ramifications of viscous flow. Shear stress is the cause of skin friction drag D_f, and flow separation is the source of pressure drag D_p, sometimes called form drag. Transition from laminar to turbulent flow causes D_f to increase and D_p to decrease.

2、粘性应力和热传导产生的原因

Shear stress in a flow is due to velocity gradients: for example, $\tau_{yx} = \mu\, \partial u / \partial y$ for a flow with gradients in the y direction. Similarly, heat conduction is due to temperature gradients; for example, $\dot{q}_y = -k\, \partial T / \partial y$, etc. Both μ and k are physical properties of the gas and are functions of temperature.

3、粘性流动控制方程

The general equations of viscous flow are

x momentum:
$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

y momentum:
$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

z momentum:
$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Energy:
$$\begin{aligned}\rho \frac{D(e + V^2/2)}{Dt} &= \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \\ &+ \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \nabla \cdot p\mathbf{V} + \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} \\ &+ \frac{\partial (u\tau_{zx})}{\partial z} + \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z} \\ &+ \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y} + \frac{\partial (w\tau_{zz})}{\partial z}\end{aligned}$$

4、The constitutive law 本构关系

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{xx} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial w}{\partial z}$$

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix}$$

粘性应力张量

second viscosity coefficient
hypothesis made by Stokes

$$\lambda = -\frac{2}{3}\mu$$

5、流动相似参数

The similarity parameters for a flow can be obtained by nondimensionalizing the governing equations; the coefficients in front of the nondimensionalized derivatives give the similarity parameters or combinations thereof. For a viscous, compressible flow, the main similarity parameters are γ, M_∞, Re_∞, and Pr_∞.

6、流动控制方程的解

Exact analytical solutions of the complete Navier-Stokes equations exist for only a few very specialized cases. Instead, the equations are frequently simplified by making appropriate approximations about the flow. In modern times, exact solutions of the complete Navier-Stokes equations for many practical problems can be obtained numerically, using various techniques of computational fluid dynamics.

The physical meaning of Pr

$$\mathrm{Pr} = \frac{\mu c_p}{k}$$

When Pr > 1, viscous boundary layer > the thermal boundary layer
When Pr < 1, viscous boundary layer < the thermal boundary layer

Some examples:

Pr of water at room temperature ~ 6
Pr of air at room temperature ~ 0.7
Pr of mercury at room temperature ~ 0.015