第二讲：粘性流体力学方程介绍

Chapter 15 Introduction to the Equations of Viscous Flow

Lecturer: Ao Xu (徐翱)

Email: axu@nwpu.edu.cn

Office: 航空楼A501

School of Aeronautics

第四节： NAVIER-STOKES方程

15.4 THE NAVIER-STOKES EQUATIONS

Questions: the difference between motions of a solid and a fluid?

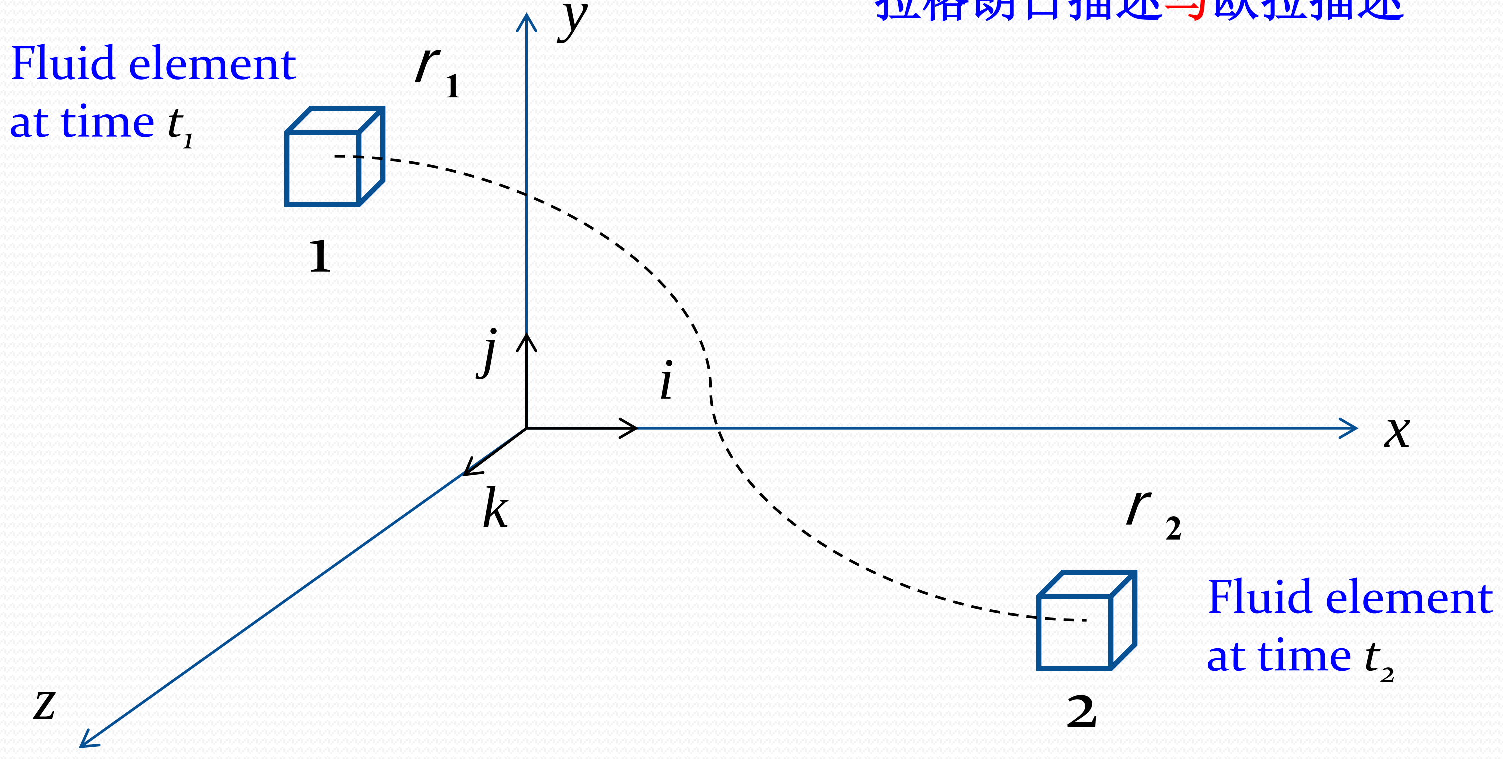
A fluid is any body whose parts yield to (agree) any force impressed on it, and by yielding, are easily moved among themselves.

by Isaac Newton, 1687

流体是指任何物体其部分屈服于（顺从）施加在其上的任何力，并且由于屈服而很容易在它们之间移动。

- Time rate of change of density (temperature, pressure ...) following a moving fluid element

拉格朗日描述与欧拉描述



Since $\rho = \rho(x, y, z, t)$, we can expand this function in a Taylor series about point 1 as follows:

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1)$$
$$+ \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{higher-order terms}$$

Dividing by $t_2 - t_1$, and ignoring the higher-order terms, we have

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial y}\right)_1 \left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial z}\right)_1 \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial t}\right)_1 \tag{2.101}$$

Consider the physical meaning of the left side of Equation (2.101). The term $(\rho_2 - \rho_1)/(t_2 - t_1)$ is the *average* time rate of change in density of the fluid element as it moves from point 1 to point 2. In the limit, as t_2 approaches t_1, this term becomes

$$\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{D\rho}{Dt}$$

Returning to Equation (2.101), note that

$$\lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} \equiv u$$

$$\lim_{t_2 \to t_1} \frac{y_2 - y_1}{t_2 - t_1} \equiv v$$

$$\lim_{t_2 \to t_1} \frac{z_2 - z_1}{t_2 - t_1} \equiv w$$

Thus, taking the limit of Equation (2.101) as $t_2 \to t_1$, we obtain

随体导数

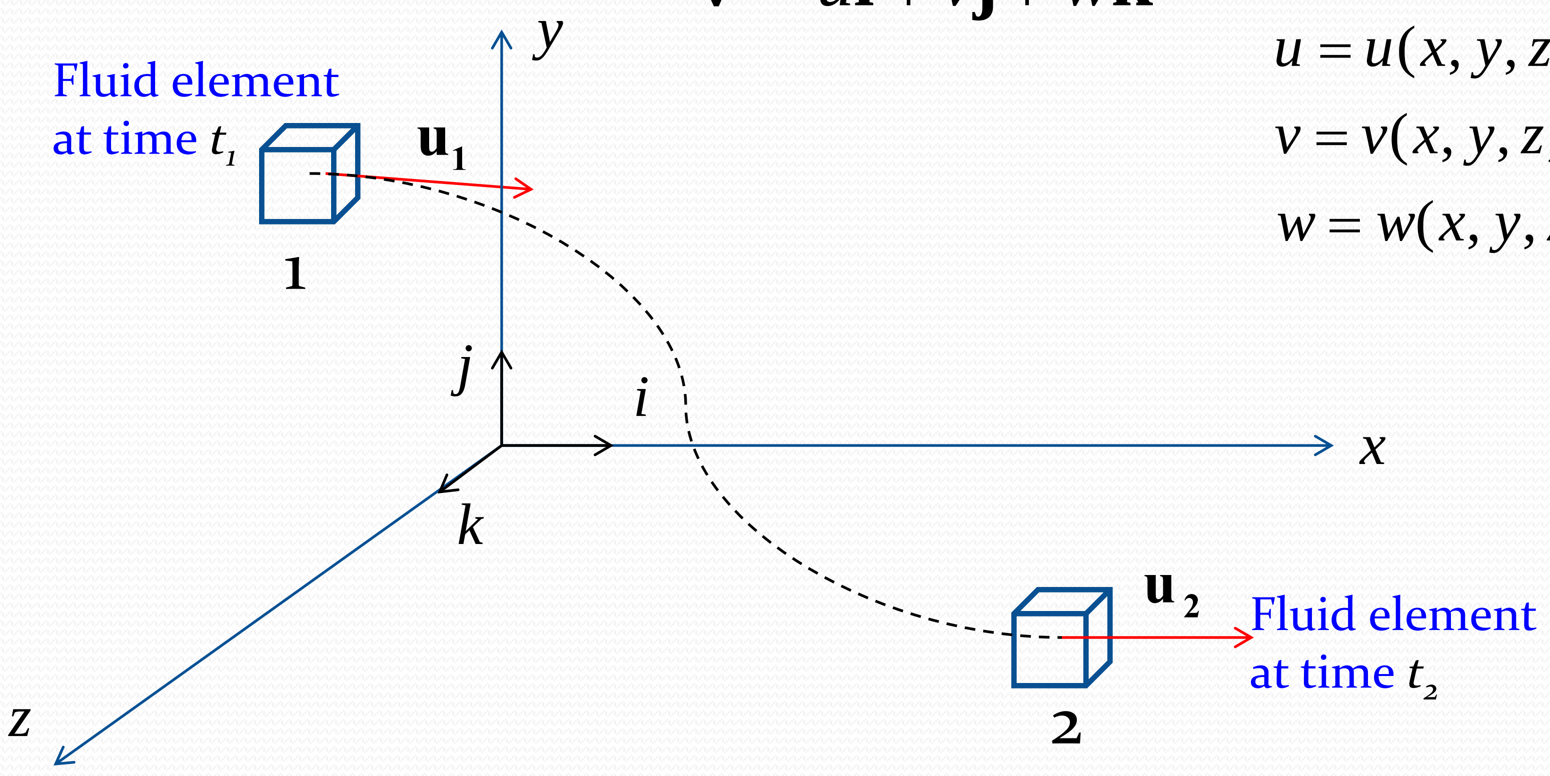
$$\frac{D\rho}{Dt} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t} \tag{2.102}$$

拉格朗日描述 与 欧拉描述

- Time rate of change of velocity following a moving fluid element

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

$$\rho = \rho(x, y, z, t)$$
$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$



Change of Velocity V

$$DV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Time rate
of change of Velocity V

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$$

Local derivative:
time rate of change of velocity at a fixed point

Convective derivative:
time rate of change of velocity due to the movement of the fluid element from one location to another in the flow field where the flow properties are spatially different.

随体导数

Substantial derivative

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}$$

Substantial Derivative 随体导数

- Time rate of change following a moving fluid element

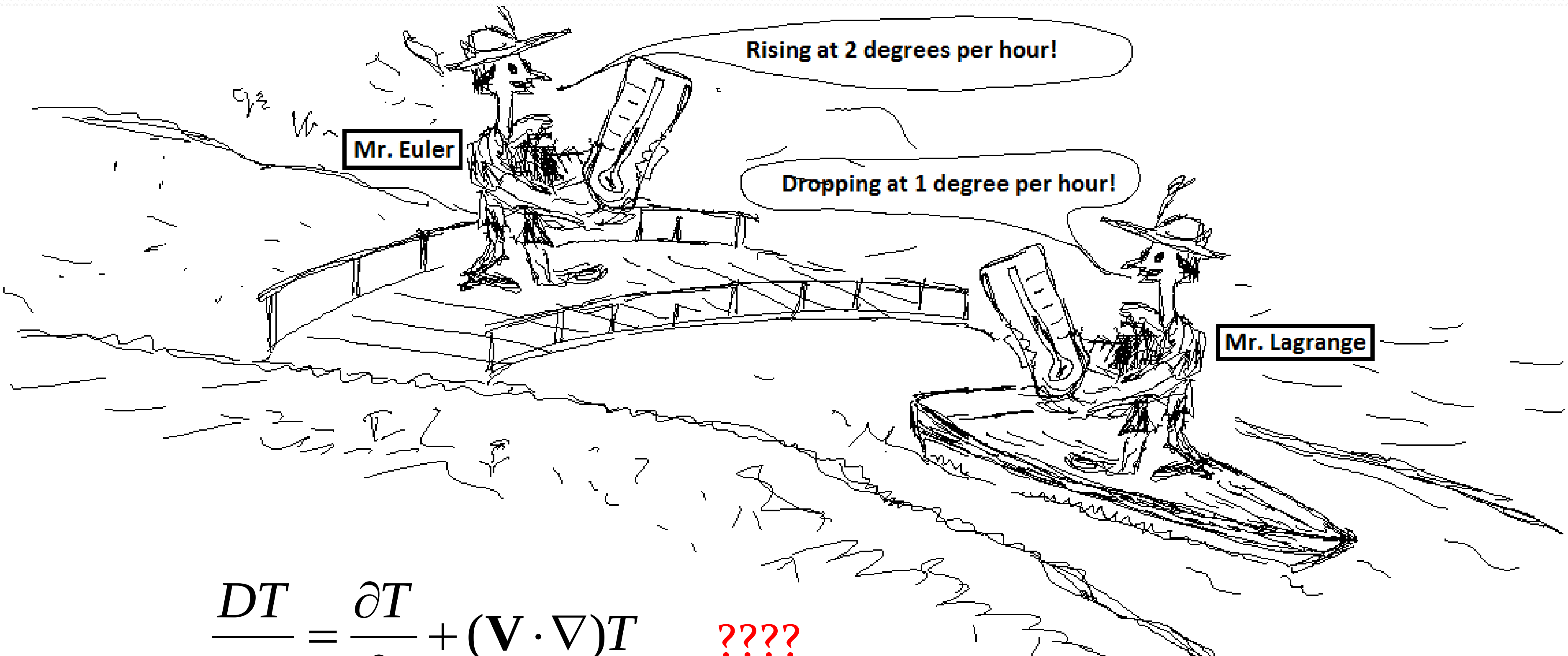
$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z} \quad \text{or} \quad \boxed{\frac{D}{Dt}=\frac{\partial}{\partial t}+(\mathbf{V}\cdot\nabla)}$$

$\frac{D}{Dt}$ Substantial derivative: time rate of change of () following a moving fluid element

$\frac{\partial}{\partial t}$ Local derivative: time rate of change of () at a fixed point

$(\mathbf{V}\cdot\nabla)$ Convective derivative: time rate of change of () due to the movement of the fluid element from one location to another in the flow field where the flow properties are spatially different.

Eulerian frame vs Lagrangian frame



$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T \qquad ????$$


"MATERIAL DERIVATIVE"

Outline :

- Continuity equation
  - mass conservation
- Momentum equations
  - Newton’s second law
- Energy equation
  - energy conservation

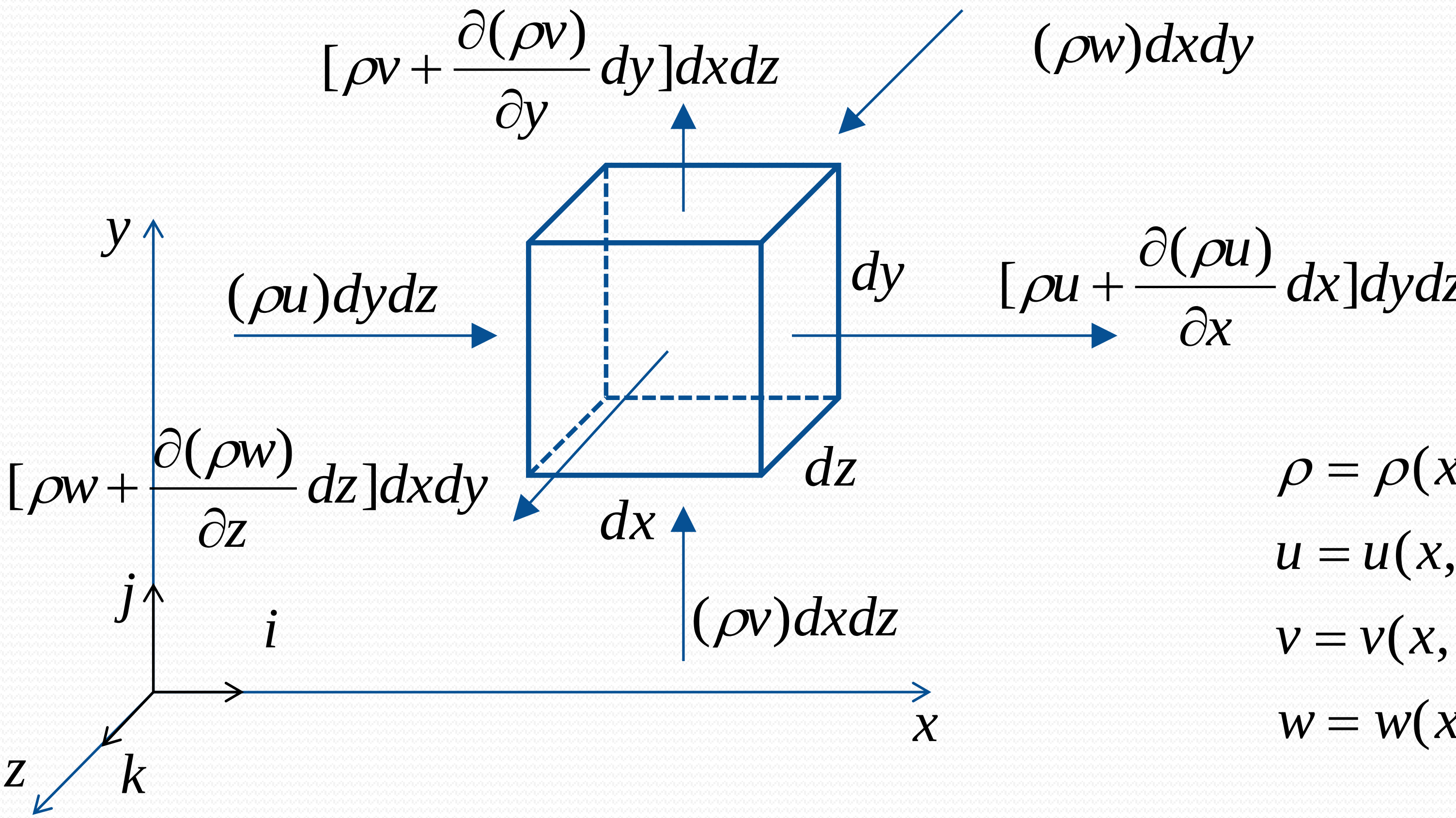
Continuity equation

Physical principle: Mass is conserved

Net mass *flow out* of infinitesimal fluid element	=	Time rate of *decrease* of mass inside infinitesimal fluid element

Continuity equation

- An Infinitesimally Small Element Fixed in Space



$$\rho = \rho(x, y, z, t)$$
$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$

Continuity equation

- Model of an Infinitesimally Small Element Fixed in Space

Net mass flow = $[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}]dxdydz$

Time rate of mass decrease = $-\frac{\partial \rho}{\partial t}(dxdydz)$

Net mass *flow out* of infinitesimal fluid element	=	Time rate of *decrease* of mass inside infinitesimal fluid element

Continuity equation

- Model of an Infinitesimally Small Element Fixed in Space

$$[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}]dxdydz=-\frac{\partial \rho}{\partial t}(dxdydz)$$

$$\frac{\partial \rho}{\partial t}+[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}]=0$$

or

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{V})=0$$

$$\Rightarrow \quad \frac{D\rho}{Dt}+\rho\nabla\cdot\mathbf{V}=0$$

张量运算（选学）

- 张量理论是向量理论的直接延伸
- 向量两种表示方法：符号表示方法、指标表示方法

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 = \sum_{i=1}^{3} A_i\mathbf{e}_i$$

求和约定：在以指标表示的向量或张量表达式中，某一指标在一项中出现两次，则表示该指标取遍$i = 1, 2, 3$的所有值，然后再对不同指标值的结果求和。

$$\mathbf{A} = A_i e_i$$

重复出现的指标称为哑标，单独出现的指标称为自由标。改变哑标的字母并不改变表达式的内容。

张量运算（选学）

在n维空间中，任一向量可以用n个线性无关的基向量的线性组合来表示。

例：三维空间的笛卡尔坐标系x, y, z 中，选择一组正交标准化基$\mathbf{i}, \mathbf{j}, \mathbf{k}$分别为沿$x$, y, z轴的单位矢量，则

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

求和

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \quad \rightarrow \quad w_i = u_i + v_i$$

点乘

$$b = \mathbf{u} \cdot \mathbf{v} \quad \rightarrow \quad b = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Kronecker delta δ_{ij}:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } r \neq j. \end{cases}$$

满足

$$A_i\delta_{ij} = A_j, \qquad \delta_{ij}\delta_{jk} = \delta_{ik}$$
$$\delta_{ii} = 3, \qquad \delta_{ij}\delta_{ij} = 3$$

例1:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

例2:

$$\mathbf{A} \cdot \mathbf{B} = (A_i\mathbf{e}_i) \cdot (B_j\mathbf{e}_j) = A_iB_j(\mathbf{e}_i \cdot \mathbf{e}_j) = A_iB_j\delta_{ij} = A_iB_i$$

张量运算（选学）

张量**T**的分量写成矩阵形式

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

对称张量 $Q_{ij} = Q_{ji}$，　反对称张量 $R_{ij} = -R_{ji}$

张量分解：任意张量T_{ij}可以分解为对称张量和反对称张量之和。

$$T_{ij} = \frac{1}{2}T_{ij} + \frac{1}{2}T_{ji} + \frac{1}{2}T_{ij} - \frac{1}{2}T_{ji}$$

其中，$\frac{1}{2}T_{ij} + \frac{1}{2}T_{ji}$为对称张量；$\frac{1}{2}T_{ij} - \frac{1}{2}T_{ji}$为反对称张量。

张量运算（选学）

标量的梯度为向量

$$\nabla\phi \quad \rightarrow \quad \partial_i\phi$$

向量的梯度为张量

$$\nabla\mathbf{v} \quad \rightarrow \quad \partial_i v_j$$

向量的散度为标量

$$\nabla \cdot \mathbf{v} \quad \rightarrow \quad \partial_i v_i = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$$

张量的散度为向量

$$\nabla \cdot \mathbf{T} \quad \rightarrow \quad \partial_i T_{ij}$$

例1：标量梯度的散度

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi \quad \rightarrow \quad \partial_i \partial_i \phi$$

例2：向量梯度的散度

$$\nabla \cdot (\nabla \mathbf{v}) = \nabla^2 \mathbf{v} \quad \rightarrow \partial_i \partial_i v_j$$

张量运算（选学）

一个张量和一个向量的点积

$$\mathbf{v}\cdot\mathbf{T}=\mathbf{u} \quad \rightarrow \quad v_iT_{ij}=T_{ij}v_i=u_j$$

$$\mathbf{T}\cdot\mathbf{v}=\mathbf{w} \quad \rightarrow \quad T_{ij}v_j=v_jT_{ij}=w_i$$

两个张量的双点积

$$\mathbf{T}:\mathbf{S}=a \quad \rightarrow \quad T_{ij}S_{ji}=\sum_{j=1}^{3}\sum_{i=1}^{3}T_{ij}S_{ji}=a$$

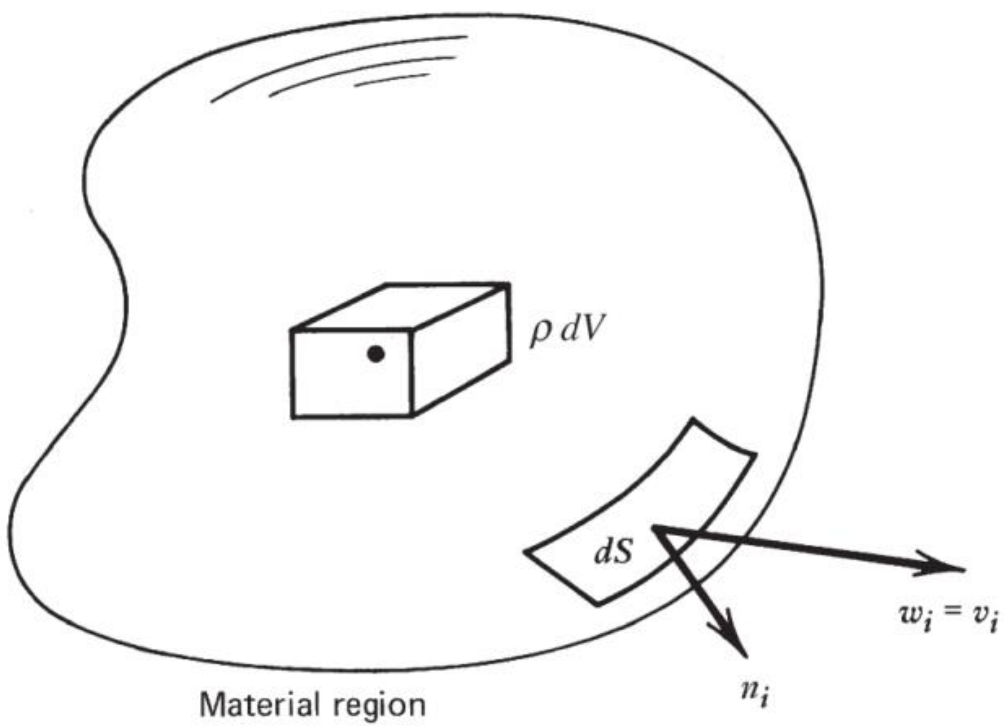
两个向量的并矢积(dyadic product)

$$\mathbf{uv}=\mathbf{T} \quad \rightarrow \quad u_iv_j=T_{ij}$$

严格推导（选学）

- 前提：连续性假设
- 物理原理：质量守恒

$$\frac{dM_{MR}}{dt} = \frac{d}{dt}\int_{MR} \rho dV = 0$$



Leibnitz定理

$$\frac{d}{dt}\int_{R(t)} T_{ij\ldots}(x_i,t)dV = \int_R \frac{\partial T_{ij\ldots}}{\partial t}dV + \int_S n_k w_k T_{ij\ldots}dS$$

$T_{ij\ldots}$可以为任一标量、向量、张量，则

$$\frac{d}{dt}\int_{MR} \rho dV = \int_{MR} \frac{\partial \rho}{\partial t}dV + \int_{MR} n_i v_i \rho dS = 0$$

Gauss定理

$$\int_R \partial_i(T_{jk}\cdots)dV = \int_S n_i T_{jk}\cdots dS$$

$T_{ij\ldots}$可以为任一标量、向量、张量， 则

$$\int_{MR} n_i v_i \rho dS = \int_{MR} \partial_i(\rho v_i)dV$$

严格推导（选学）

$$\int_{MR} \frac{\partial \rho}{\partial t} dV + \int_{MR} \partial_i(\rho v_i) dV = 0$$

连续性方程

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

连续性方程（不可压缩流动）

$$\nabla \cdot \mathbf{v} = 0$$

Momentum equation

Model of an Infinitesimally Small Fluid Element Moving with the Flow

- Newton's second law ($F = ma$) says that the net force on the fluid element equals its mass times the acceleration of the element.

Question: how to apply the Newton's second law on fluid flow?

Newton's second law

$$F = ma$$

$$\boxed{?}$$

$$\frac{DV}{Dt} = \frac{\P V}{\P t} + u\frac{\P V}{\P x} + v\frac{\P V}{\P y} + w\frac{\P V}{\P z}$$

Momentum equation

Infinitesimally Small Fluid Element Moving with the Flow

- Newton's second law ($F = ma$) says that the net force on the fluid element equals its mass times the acceleration of the element.

$$F_x = ma_x$$
$$F_y = ma_y$$
$$F_y = ma_y$$

Momentum equation

Question: what is the source of the force?

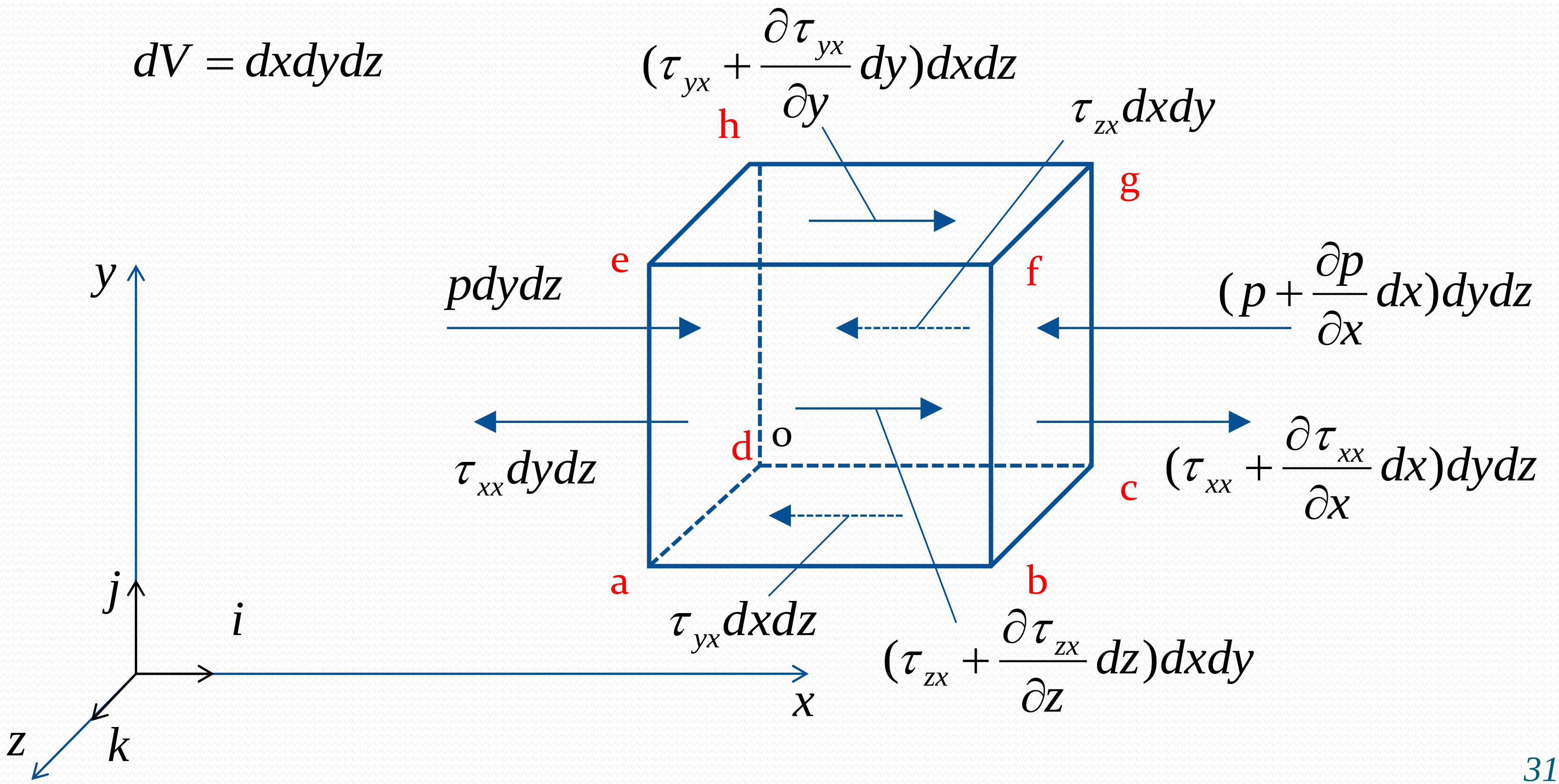
1. Body force 体积力: which act directly on the volumetric mass of the fluid element. Examples are gravitational, electric, and magnetic forces.

2. Surface force 表面力: which act directly on the surface of the fluid element. They are due to two sources: (a) the pressure distribution acting on the surface, imposed by the outside fluid surrounding the fluid element, and (b) the shear and normal stress distributions acting on the surface, also imposed by the outside fluid "tugging (pulling)" or "pushing" on the surface by means of friction.

Momentum equation

Surface forces on the moving fluid element in 3 direction

$$dV = dxdydz$$



Momentum equation

Surface forces on the moving fluid element in *x* direction

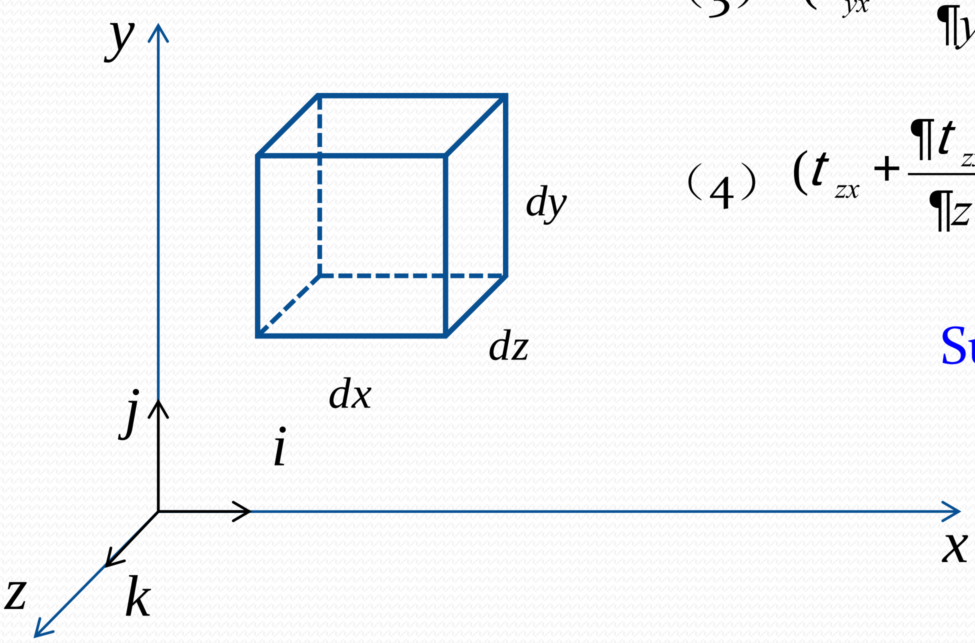
$$dV = dxdydz$$

(1) $$pdydz - \left(p + \frac{\partial p}{\partial x}dx\right)dydz = -\frac{\partial p}{\partial x}dxdydz$$

(2) $$\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}dx\right)dydz - \tau_{xx}dydz = \frac{\partial \tau_{xx}}{\partial x}dxdydz$$

(3) $$\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)dxdz - \tau_{yx}dxdz = \frac{\partial \tau_{yx}}{\partial y}dydxdz$$

(4) $$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz\right)dxdy - \tau_{zx}dxdy = \frac{\partial \tau_{zx}}{\partial z}dzdxdy$$

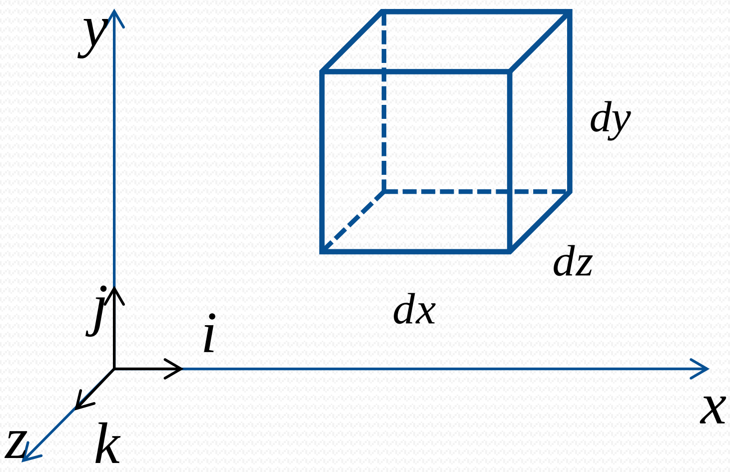


Surface force on x direction：

$$\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)dzdxdy$$

Momentum equation

Body forces on the moving fluid element in *x* direction

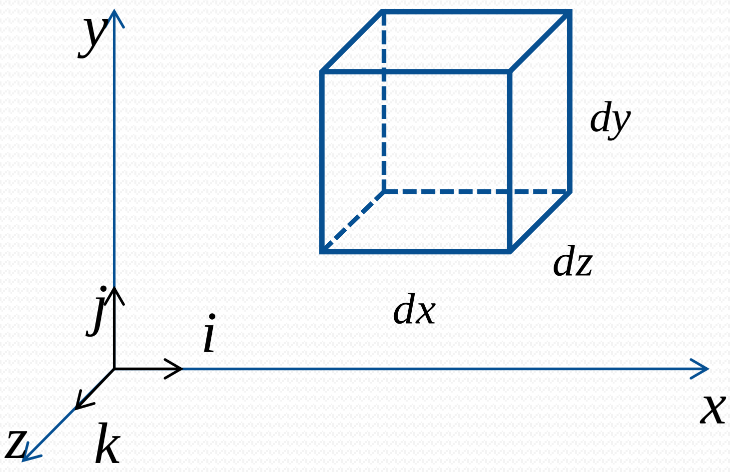


Body force in x direction: $\rho f_x dxdydz$

where ρ is the fluid density, f_x is the body force per unit mass acting on the fluid element.

Momentum equation

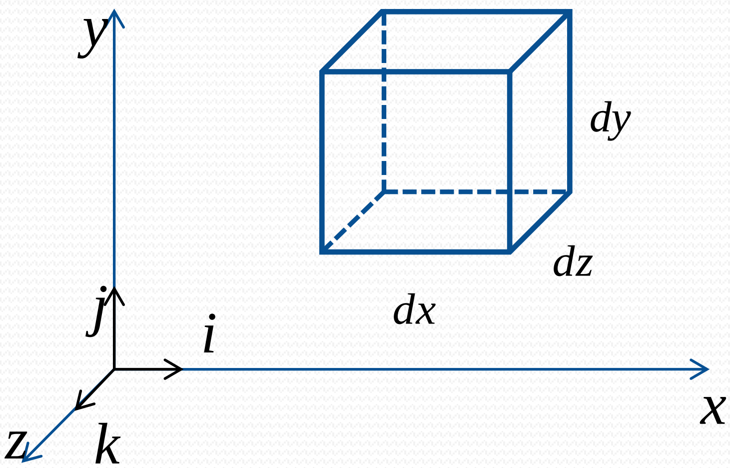
An Infinitesimally Small Element Moving with the fluid



Left-hand side: $F_x = \left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] dxdydz + \rho f_x dxdydz$

Momentum equation

An Infinitesimally Small Element Moving with the fluid



$$m = \rho dxdydz$$

$$a_x = \frac{Du}{Dt}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

Quiz: Draw sketch for the force analysis and Write down the equation for the y and z directions.

Momentum equation

In x direction: $$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

In y direction: $$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

In z direction: $$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

Momentum equations (also called Navier-Stokes equations)

What is the relation between the stress and the strain rate?

What is the relation between the stress（应力） and the strain rate （应变率）?

The constitutive law（本构关系）

Each face of the fluid element experiences both tangential and normal stresses.

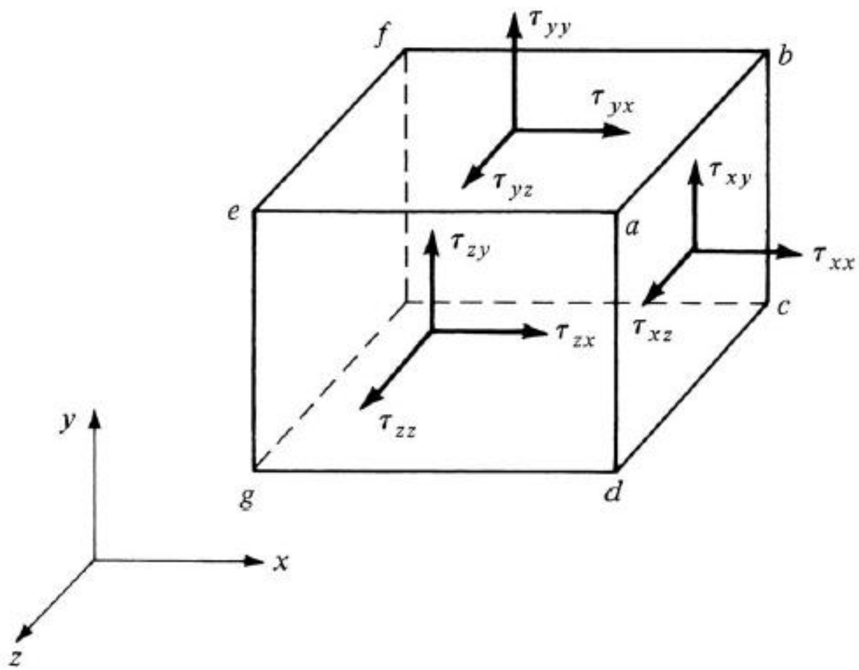


Figure 15.10 Shear and normal stresses caused by viscous action on a fluid element.

The constitutive law 本构关系

Newtonian fluid in two dimensional case:

$$\tau = \mu \frac{\partial u}{\partial y}$$

where μ is the molecular viscosity coefficient.

In Newtonian fluid, the shear stress is proportional to the time rate of strain, i.e., velocity gradient.

Shear stress in two-dimension

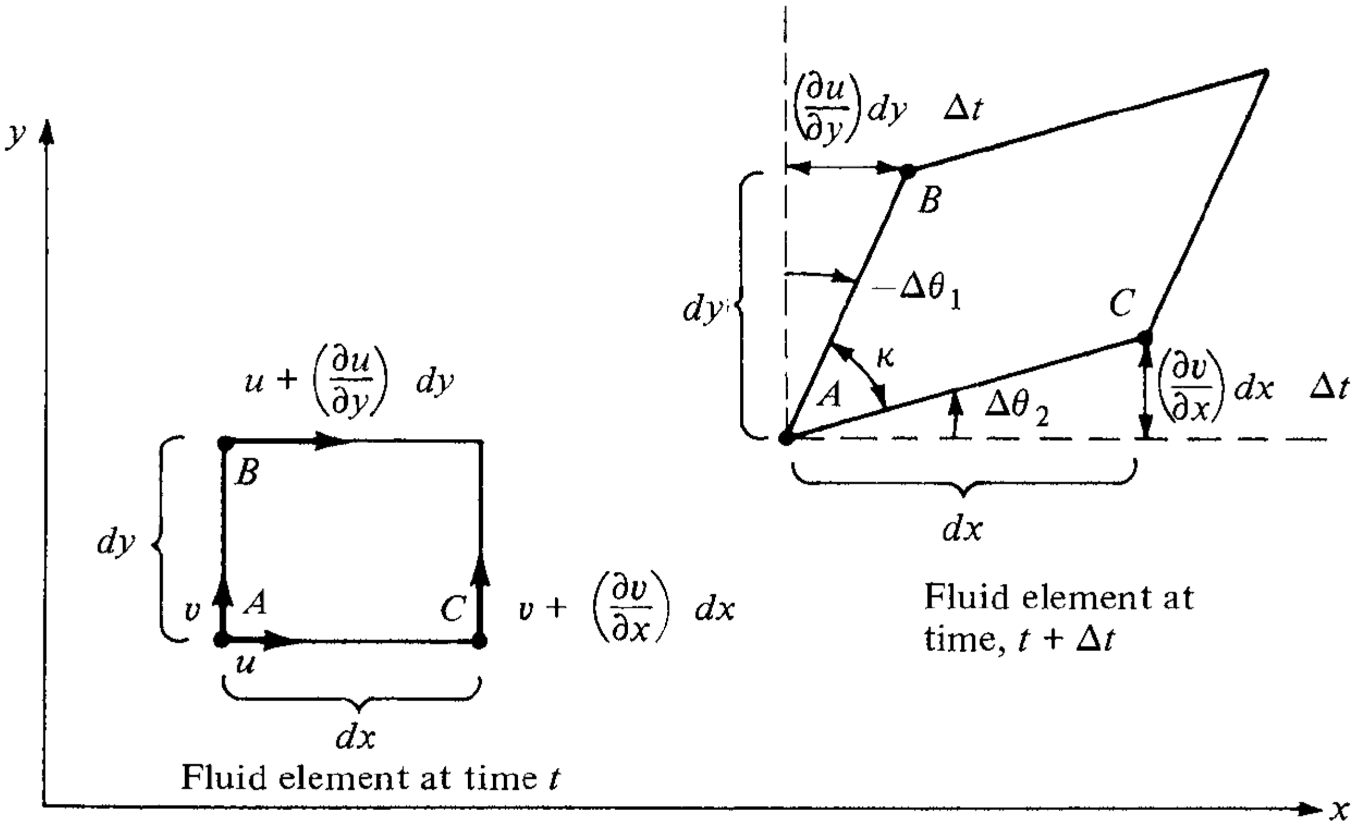


Figure 2.33 Rotation and distortion of a fluid element.

By definition, the *strain* of the fluid element as seen in the xy plane is the change in κ, where positive strain corresponds to a *decreasing* κ.

$$\text{Strain} = -\Delta\kappa = \Delta\theta_2 - \Delta\theta_1$$

In viscous flows , the time rate of strain is ε_{xy},

$$\varepsilon_{xy} \equiv -\frac{d\kappa}{dt} = \frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}$$

we have

$$\varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

In the yz and zx planes, by a similar derivation the strain is, respectively,

$$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

and

$$\varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

The constitutive law 本构关系

Stokes in 1845 obtained 各向同性牛顿流体

$$t_{xx} = /(\nabla \cdot \mathbf{V}) + 2m\frac{\partial u}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\tau_{yy} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial v}{\partial y} \qquad \tau_{xz} = \tau_{zx} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

$$\tau_{zz} = \lambda(\nabla \cdot \mathbf{V}) + 2\mu\frac{\partial w}{\partial z} \qquad \tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

where λ is the second viscosity coefficient.

The hypothesis made by Stokes is that $\lambda = -\frac{2}{3}\mu$

Momentum equation

If the flow is compressible and Newtonian

$$r\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(l(\nabla\cdot\mathbf{V}) + 2m\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[m\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + r f_x$$

$$r\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[m\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left(l(\nabla\cdot\mathbf{V}) + 2m\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[m\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + r f_y$$

$$r\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[m\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[m\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left(l(\nabla\cdot\mathbf{V}) + 2m\frac{\partial w}{\partial z}\right) + r f_z$$

Momentum equation

If the flow is incompressible and Newtonian

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x$$

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 u}{\partial z^2}+\rho f_x$$

In x direction $\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\mu\nabla^2 u+\rho f_x$

In 3 direction $\rho\frac{D\boldsymbol{V}}{Dt}=-\nabla p+\mu\nabla^2\boldsymbol{V}+\rho\boldsymbol{F}$ 动量方程的矢量形式

$$\rho\left(\frac{\partial\boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}\right)=-\nabla p+\mu\nabla^2\boldsymbol{V}+\rho\boldsymbol{F}$$

第五节：粘性流动的能量方程

15.5 THE VISCOUS FLOW ENERGY EQUATION

The viscous flow energy equation

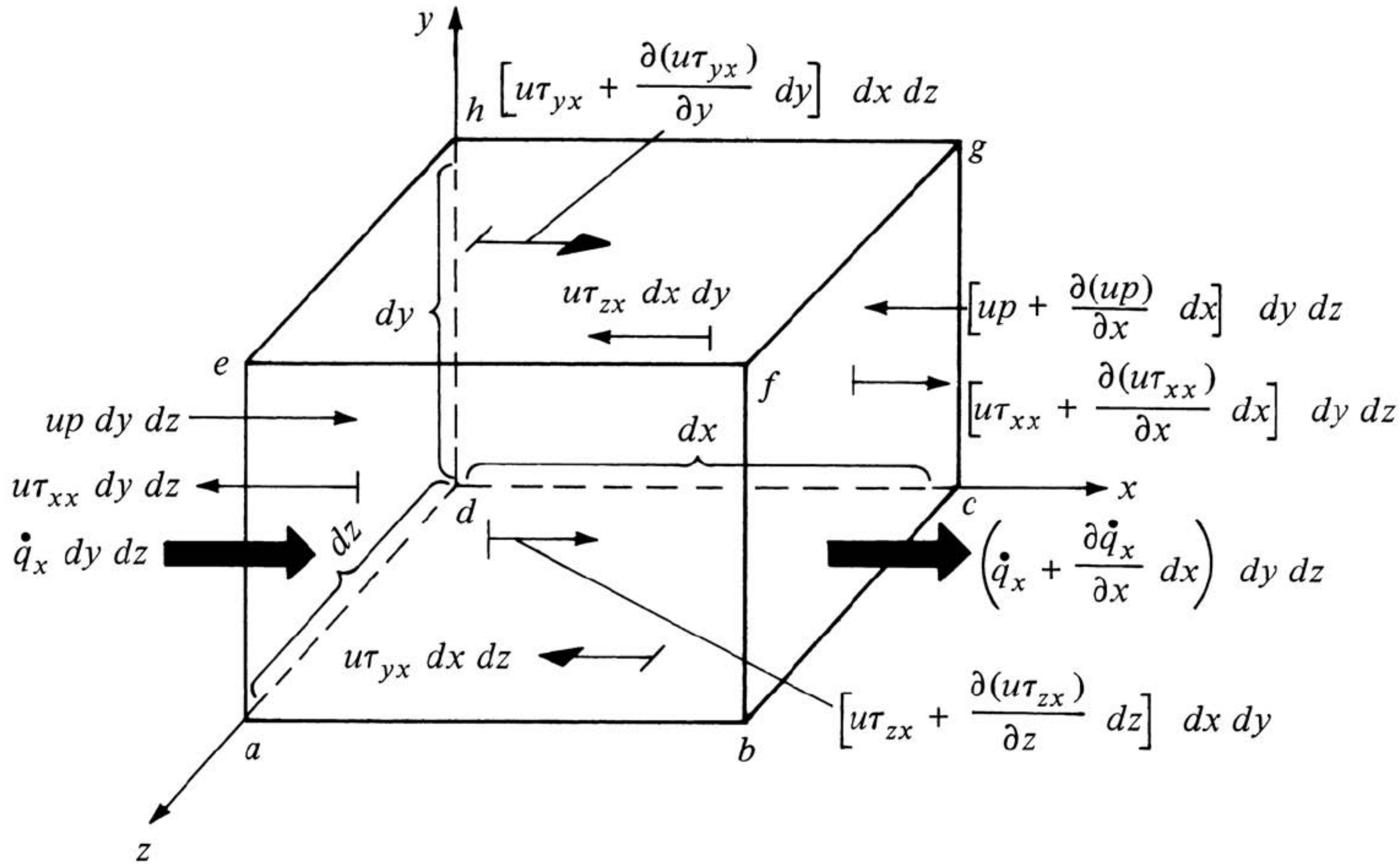
Rate of change of energy inside fluid element	=	net flux of heat into element	+	rate of work done on element due to pressure and stress forces on surface
A	=	B	+	C

We firstly consider C

The rate of work done on the moving fluid element by the forces is simply

$$P = F \times V$$

Energy fluxes associated in the *x* direction are shown



the net rate of work done by pressure in the x direction is

$$\left[up - \left(up + \frac{\partial (up)}{\partial x} dx\right)\right] dy\, dz = -\frac{\partial (up)}{\partial x} dx\, dy\, dz$$

Similarly, the net rate of work done by the shear stresses in the x direction on faces *abcd* and *efgh* is

$$\left[\left(u\tau_{yx} + \frac{\partial (u\tau_{yx})}{\partial y} dy\right) - u\tau_{yx}\right] dx\, dz = \frac{\partial (u\tau_{yx})}{\partial y} dx\, dy\, dz$$

Considering all the forces, the net rate of work done in the x direction is

$$\left[-\frac{\partial (up)}{\partial x} + \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z}\right] dx\, dy\, dz$$

C: Rate of work done on element due to pressure and stress forces on surface

$$C = \left[-\left(\frac{\partial(up)}{\partial x} + \frac{\partial(vp)}{\partial y} + \frac{\partial(wp)}{\partial z}\right) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] dx\, dy\, dz$$

We then consider B

Rate of change of energy inside fluid element	=	net flux of heat into element	+	rate of work done on element due to pressure and stress forces on surface
A	=	B	+	C

B: Net flux of heat into element:

(1) volumetric heating such as absorption or emission of radiation

(2) heat transfer across the surface due to temperature gradients

i.e., thermal conduction

$$\text{Volumetric heating of element} = \rho\dot{q}\,dx\,dy\,dz$$

$$\text{Heating of fluid element by thermal conduction} = -\left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right) dx\,dy\,dz$$

B： Net flux of heat into element

$$B = \left[\rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dx\,dy\,dz$$

Finally we consider A

Rate of change of energy inside fluid element	=	net flux of heat into element	+	rate of work done on element due to pressure and stress forces on surface
A	$=$	B	$+$	C

A: Rate of change of energy inside fluid element

$$A = \rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\, dy\, dz$$

The viscous flow energy equation

$$\rho \frac{D(e + V^2/2)}{Dt} = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \nabla \cdot p\mathbf{V} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}$$

$$\rho \frac{D(e + V^2/2)}{Dt} = \rho \dot{q} - \nabla \cdot (p\mathbf{V}) + \rho(\mathbf{f} \cdot \mathbf{V}) + \dot{Q}'_{\text{viscous}} + \dot{W}'_{\text{viscous}}$$

第一次作业：

- 推导连续性方程，并给出不可压缩流动的连续性方程
- 推导动量方程，并给出不可压缩流动的动量方程
- 要求：
  - 在A4纸上完成
  - 截止日期：5月12日至5月15日之间
  - 提交方式：电子版（拍照或扫描）
  - 邮件发送，主题“第一次作业_学号_姓名”
  - 例如：第一次作业_2017300300_张三
  - 助教邮箱：912387046@mail.nwpu.edu.cn