《粘性流体力学》 第四次作业、第五次作业

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No. 4 Homework

Problem 1:

对于不可压缩流动，已知边界层内的速度分布：

$$u = \frac{y}{\delta} U$$

求边界层位移厚度 δ^* 和动量厚度 θ。

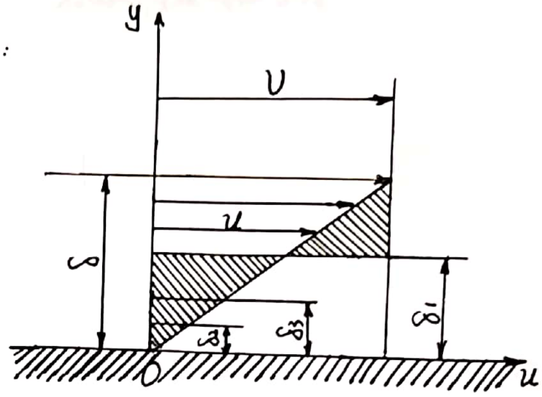

图1 边界层示意图

Hint:

$$\delta^* \equiv \int_0^{y_1} \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy, \quad \delta \le y_1 \to \infty$$

$$\theta \equiv \int_0^{y_1} \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy, \quad \delta \le y_1 \to \infty$$

解1: 对于不可压缩流动，全流场 $\rho \equiv const$（常数）

由定义式可得.

边界层位移厚度满足

$$\delta^* = \int_0^{y_1} \left(1 - \frac{u}{U}\right) dy, \quad \delta \le y_1 \to \infty$$

$$= \int_0^{\delta} \left(1 - \frac{y}{\delta}\right) dy = \left(y - \frac{1}{2\delta} y^2\right)\Big|_0^{\delta} = \frac{1}{2}\delta;$$

边界层动量厚度满足

$$\theta = \int_0^{y_1} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy, \quad \delta \le y_1 \to \infty$$

$$= \int_0^{\delta} \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy = \left(\frac{1}{2\delta} y^2 - \frac{1}{3\delta^2} y^3\right)\Big|_0^{\delta} = \frac{1}{6}\delta.$$

$\therefore$ 故满足 $\delta^* = \frac{1}{2}\delta = 3\theta$.

Problem 2:

对以下二维定常无量纲能量方程进行量阶分析 (order-of-magnitude analysis)

$$\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty^2 \left[\rho' u' \frac{\partial}{\partial x'}(u'^2 + v'^2) + \rho' v' \frac{\partial}{\partial y'}(u'^2 + v'^2)\right]$$

$$= \frac{\gamma}{Pr_\infty Re_\infty}\left[\frac{\partial}{\partial x'}\left(k' \frac{\partial T'}{\partial x'}\right) + \frac{\partial}{\partial y'}\left(k' \frac{\partial T'}{\partial y'}\right)\right] - (\gamma-1)\left[\frac{\partial (u'p')}{\partial x'} + \frac{\partial (v'p')}{\partial y'}\right]$$

$$+ \gamma(\gamma-1)\frac{M_\infty^2}{Re_\infty}\left\{\frac{\partial}{\partial x'}\left[\mu' v'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right] + \frac{\partial}{\partial y'}\left[\mu' u'\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)\right]\right\}$$

推导以下边界层有量纲能量方程：

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

解2. ① 首先对各无量纲变量进行量阶分析。

在长度为c的平板上，流动边界层厚度为δ，则由边界层特性可得

$$\delta \ll c$$

考虑定常，二维流动的连续性方程有

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \qquad (1)$$

其无量纲化方程形式为

$$\frac{\partial(\rho' u')}{\partial x'}+\frac{\partial(\rho' v')}{\partial y'}=0 \qquad (2)$$

由 $u'=\frac{u}{V_\infty}\sim O(1)$，$\rho'=\frac{\rho}{\rho_\infty}\sim O(1)$，$x'=\frac{x}{c}\sim O(1)$，$y'=\frac{y}{c}\sim O(\delta)$；

代入式(2)可得

$$\frac{[O(1)][O(1)]}{[O(1)]}+\frac{[O(1)][v']}{[O(\delta)]}=0 \qquad (3)$$

由上式(3)易知，$v'=\frac{v}{V_\infty}\sim O(\delta)$；

将量阶分析所需的无量纲量阶估算情况列举如下，引入参考量，有

$u'=\frac{u}{V_\infty}\sim O(1)$，$v'=\frac{v}{V_\infty}\sim O(\delta)$，$\rho'=\frac{\rho}{\rho_\infty}\sim O(1)$，$p'=\frac{p}{p_\infty}\sim O(1)$，

$x'=\frac{x}{c}\sim O(1)$，$y'=\frac{y}{c}\sim O(\delta)$，$\mu'=\frac{\mu}{\mu_\infty}\sim O(1)$，$T'=\frac{T}{T_\infty}\sim O(1)$，

$e'=\frac{e}{C_vT_\infty}\sim O(1)$，$k'=\frac{k}{k_\infty}\sim O(1)$，$V'^2=\frac{V^2}{V_\infty^2}=\frac{u^2+v^2}{V_\infty^2}=(u')^2+(v')^2\sim O(1^2+\delta^2)\sim[O(1)]^2\sim O(1)$；

另外，在高雷诺数假设下，边界层内满足 $Re_\infty=\frac{\rho_\infty V_\infty c}{\mu_\infty}\sim\frac{1}{O(\delta^2)}$

又 $M_\infty^2=\frac{V_\infty^2}{a_\infty^2}\sim O(1)$；$Pr_\infty=\frac{\mu_\infty C_p}{k_\infty}\sim O(1)$；

由关系式 $e=h-\frac{p}{\rho}$，结合理想气体状态关系式 $p=\rho RT$ 可得

$$e'=\frac{e}{C_vT_\infty}=\frac{h-\frac{p}{\rho}}{C_vT_\infty}=\frac{h}{C_vT_\infty}-\frac{RT}{C_vT_\infty}=\frac{\gamma h}{C_pT_\infty}-(\gamma-1)T' \qquad (4)$$

故 $e'=\gamma h'-(\gamma-1)T'$，由 $e'\sim O(1)$，$T'\sim O(1)$，则 $h'=\frac{h}{C_pT_\infty}\sim O(1)$；

② 下面对二维定常无量纲能量方程中的各项进行量阶分析。

1) 方程左边

a) $\rho' u' \frac{\partial e'}{\partial x'} \sim O(1)$, $\rho' v' \frac{\partial e'}{\partial y'} \sim O(\delta) \cdot \frac{1}{O(\delta)} = O(1)$;

b) $\rho' u' \frac{\partial (u'^2)}{\partial x'} \sim O(1)$, $\rho' u' \frac{\partial (v'^2)}{\partial x'} \sim O(\delta^2)$;

$\rho' v' \frac{\partial (u'^2)}{\partial y'} \sim O(1)$, $\rho' v' \frac{\partial (v'^2)}{\partial y'} \sim O(\delta^2)$;

2) 方程右边

a) $\frac{\partial}{\partial x'}(k' \frac{\partial T'}{\partial x'}) \sim O(1)$, $\frac{\partial}{\partial y'}(k' \frac{\partial T'}{\partial y'}) \sim O(\frac{1}{\delta^2})$;

b) $\frac{\partial (u'p')}{\partial x'} = u' \frac{\partial p'}{\partial x'} + p' \frac{\partial u'}{\partial x'} \sim O(1) + O(1) = O(1)$,

$\frac{\partial (v'p')}{\partial y'} = v' \frac{\partial p'}{\partial y'} + p' \frac{\partial v'}{\partial y'} \sim O(1) + O(1) = O(1)$;

c) $\frac{\partial}{\partial x'}[\mu' v' (\frac{\partial v'}{\partial x'})] \sim O(\delta^2)$, $\frac{\partial}{\partial x'}[\mu' v' (\frac{\partial u'}{\partial y'})] \sim O(1)$,

$\frac{\partial}{\partial y'}[\mu' u' (\frac{\partial v'}{\partial x'})] \sim O(1)$, $\frac{\partial}{\partial y'}[\mu' u' (\frac{\partial u'}{\partial y'})] \sim O(\frac{1}{\delta^2})$;

则对原方程进行量阶形式变量代换可得：

$$[O(1)+O(1)] + \frac{\gamma(\gamma-1)}{2} M_\infty^2 \{[O(1)+O(\delta^2)] + [O(1)+O(\delta^2)]\}$$
$$= \frac{\gamma}{Pr_\infty} \cdot O(\delta^2) [O(1)+O(\frac{1}{\delta^2})] - (\gamma-1)[O(1)+O(1)] + \gamma(\gamma-1) M_\infty^2 \cdot O(\delta^2) \{[O(\delta^2)+O(1)] + [O(1)+O(\frac{1}{\delta^2})]\} \quad (5)$$

忽略 $O(\delta^2)$ 及以上高阶小量（$\delta \ll 1$），则无量纲能量方程可简化为：

$$\rho' u' \frac{\partial e'}{\partial x'} + \rho' v' \frac{\partial e'}{\partial y'} + \frac{\gamma(\gamma-1)}{2} M_\infty^2 [\rho' u' \frac{\partial (u'^2)}{\partial x'} + \rho' v' \frac{\partial (u'^2)}{\partial y'}]$$
$$= \frac{\gamma}{Pr_\infty Re_\infty} [\frac{\partial}{\partial y'}(k' \frac{\partial T'}{\partial y'})] - (\gamma-1) [\frac{\partial (u'p')}{\partial x'} + \frac{\partial (v'p')}{\partial y'}] + \gamma(\gamma-1) \frac{M_\infty^2}{Re_\infty} \{\frac{\partial}{\partial y'}[\mu' u' (\frac{\partial u'}{\partial y'})]\} \quad (6)$$

③ 将上式中无量纲量还原为有量纲量可得：

$$\rho u \frac{\partial (e+\frac{u^2}{2})}{\partial x} + \rho v \frac{\partial (e+\frac{u^2}{2})}{\partial y} = \frac{\partial}{\partial y}(k \frac{\partial T}{\partial y}) - [\frac{\partial (up)}{\partial x} + \frac{\partial (vp)}{\partial y}] + \frac{\partial}{\partial y}[u(\mu \frac{\partial u}{\partial y})] \quad (7)$$

由已推导的 x-方向边界层动量微分方程：

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(\mu \frac{\partial u}{\partial y}) \quad (8)$$

将上式边界层动量微分方程两边同乘u，并进行微分代换可得.

$$\rho u^2\frac{\partial u}{\partial x}+\rho uv\frac{\partial u}{\partial y}=\rho u\frac{\partial(\frac{1}{2}u^2)}{\partial x}+\rho v\frac{\partial(\frac{1}{2}u^2)}{\partial y}=-u\frac{\partial p}{\partial x}+u\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) \tag{9}$$

则将式(7)与式(9)相减可得.

$$\begin{aligned}\rho u\frac{\partial e}{\partial x}+\rho v\frac{\partial e}{\partial y}&=\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})-\left[\frac{\partial(up)}{\partial x}+\frac{\partial(vp)}{\partial y}\right]+u\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})+\mu(\frac{\partial u}{\partial y})^2\\&\quad-(-u\frac{\partial p}{\partial x})-u\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y})\\&=\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+u\frac{\partial p}{\partial x}+\mu(\frac{\partial u}{\partial y})^2-\nabla(\vec{V}\cdot p)\end{aligned} \tag{10}$$

将上式(10)中e代换有 $e=h-\frac{p}{\rho}$，则可化为

$$\rho u\frac{\partial(h-\frac{p}{\rho})}{\partial x}+\rho v\frac{\partial(h-\frac{p}{\rho})}{\partial y}=\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+u\frac{\partial p}{\partial x}+\mu(\frac{\partial u}{\partial y})^2-\nabla(\vec{V}\cdot p) \tag{11}$$

即有

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+u\frac{\partial p}{\partial x}+\mu(\frac{\partial u}{\partial y})^2+\left[\rho\vec{V}\cdot\nabla(\frac{p}{\rho})-\nabla(\vec{V}\cdot p)\right] \tag{12}$$

对左式最后一项化简有.

$$\begin{aligned}\rho\vec{V}\cdot\nabla(\frac{p}{\rho})-\nabla(\vec{V}\cdot p)&=\rho\vec{V}\cdot\left[\frac{1}{\rho}\nabla p+p\cdot(-\frac{1}{\rho^2})\cdot\nabla\rho\right]-\vec{V}\cdot\nabla p-p\cdot\nabla\vec{V}\\&=\vec{V}\cdot\nabla p-\frac{p}{\rho}\vec{V}\cdot\nabla\rho-\vec{V}\cdot\nabla p-p\cdot\nabla\vec{V}\\&=-\frac{p}{\rho}\vec{V}\cdot\nabla\rho-p\cdot\nabla\vec{V}\end{aligned} \tag{13}$$

又由已推导的边界层连续微分方程有.

$$\nabla\cdot(\rho\vec{V})=\rho\cdot\nabla\vec{V}+\vec{V}\cdot\nabla\rho=0 \tag{14}$$

由上式(14)解出$\vec{V}\cdot\nabla\rho$代入(13)式中，可得.

$$\rho\vec{V}\cdot\nabla(\frac{p}{\rho})-\nabla(\vec{V}\cdot p)=-\frac{p}{\rho}(-\rho\cdot\nabla\vec{V})-p\cdot\nabla\vec{V}=0 \tag{15}$$

将式(15)代入(12)式可得.

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+u\frac{\partial p}{\partial x}+\mu(\frac{\partial u}{\partial y})^2 \tag{16}$$

∴ 上式(16)即为边界层有量纲能量微分方程，推导完毕。