第四讲：流动特例库埃特流动

Chapter 16 A Special Case: Couette Flow

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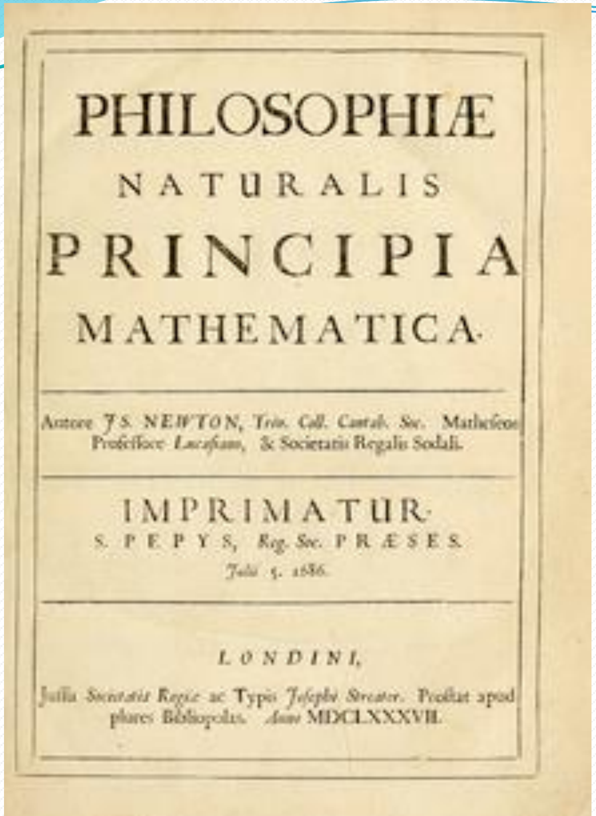

PHILOSOPHIÆ

NATURALIS

PRINCIPIA

MATHEMATICA.

Autore JS. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.

S. PEPYS, Reg. Soc. PRÆSES.

Julii 5. 1686.

LONDINI,

Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

《Philosophiæ Naturalis Principia Mathematica》 (Latin for 《Mathematical Principles of Natural Philosophy》), often referred to as simply the 《Principia》, is a work in three books by Isaac Newton.

《自然哲学的数学原理》

The resistance arising from the want of lubricity in the parts of a fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.

Isaac Newton, 1687,
from Section IX of Book II
of his Principia

流体各部分由于缺乏润滑性而产生的阻力与流体各部分相互分离时的速度成正比。

牛顿流体

第一节：引言

16.1 INTRODUCTION

The solution of viscous flows

We have the following options:

1. Exact solution 精确解
2. Approximate solution 近似解
3. Direct numerical simulation 数值解

本章内容：粘性流动控制方程的精确解

一种特殊流动现象----库埃特流动

基本概念：表面摩擦 skin fraction

热传导 heat transfer

恢复因子 recovery factor

雷诺比拟 Reynolds analogy

定常的平行剪切流动

回顾：均匀流体不可压缩流动的主控方程组:

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

对于定常的平行剪切流动

- 所有流体质点都沿着空间某一确定的方向运动
- 流场中各物理量的分布均不随时间而改变

不失一般性，把流体速度的方向取为x轴，则速度分量可表示为

$$u \neq 0, \quad v = w = 0$$

对于连续方程 $\nabla \cdot \mathbf{u} = 0$其分量形式为

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

简化为

$$\frac{\partial u}{\partial x} = 0$$

对于动量方程 $\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$ 其分量形式为

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

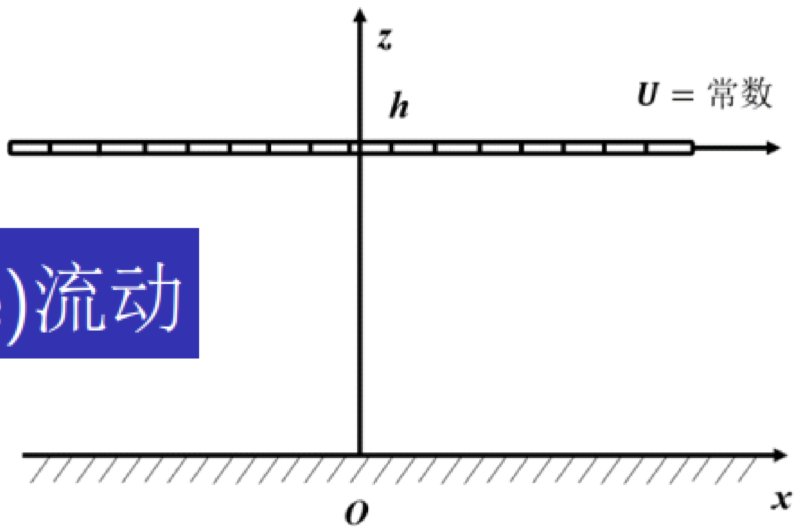
简化为

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = 0$$

定常的平行剪切流动：例1

问题描述

- 在两块无限大的平行板之间充满了流体
- 其中一块平板相对于另一块平板以不变的速度在其自身平面内运动
- 流体在摩擦力作用下发生运动
- 当时间充分长后，流体运动达到定常状态



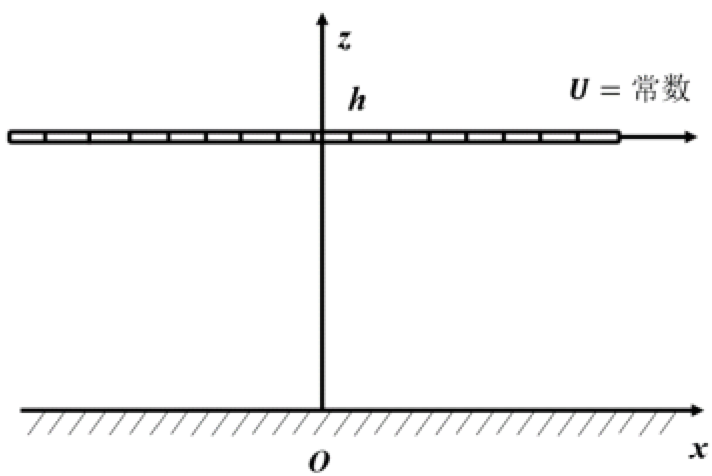
库埃特(Couette)流动

定常平行剪切流动控制方程

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=0$$

1. 无压强梯度推动或阻滞流体运动，则$\partial p/\partial x=0$.
2. 平板在x和y方向无界，则$u=u(z)$.

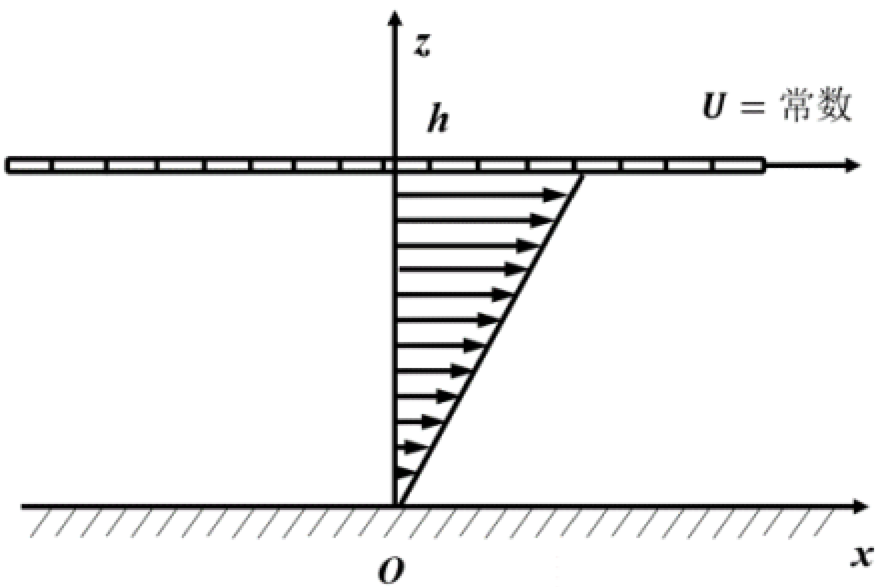
$$\frac{d^2 u}{dz^2}=0,\quad u(z)=Az+B$$



无滑移边界条件

1. $z = 0,\ u = 0$
2. $z = h,\ u = U$

$$u(z) = \frac{U}{h} z, \quad 0 \le z \le h$$



第二节：库埃特流动描述

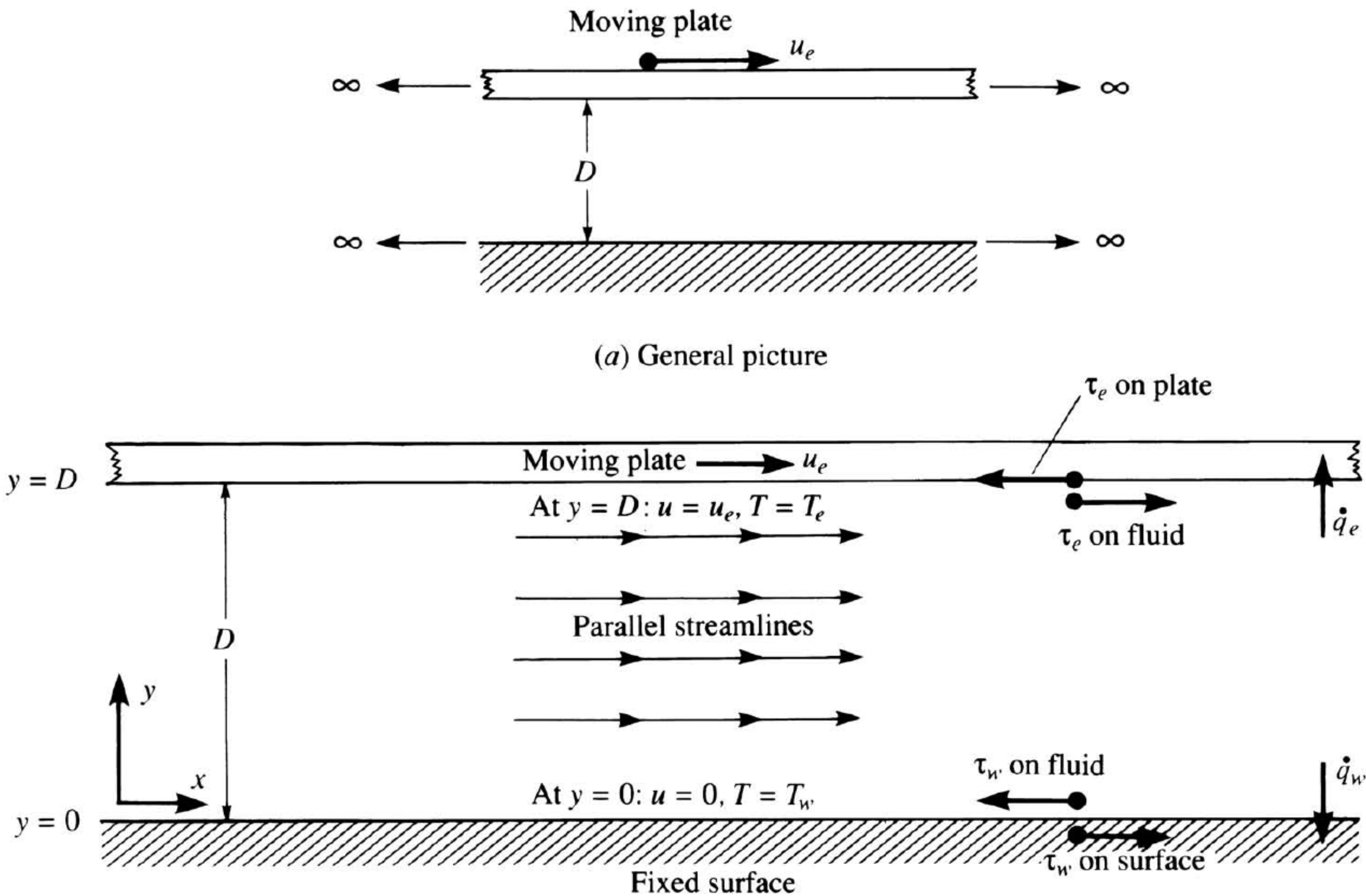
16.2 COUETTE FLOW: GENERAL DISCUSSION

A bit of history

Maurice Marie Alfred Couette (1858-1943) , French physicist and pioneer in fluid dynamics

他主要研究Navier在19世纪20年代的工作，发展了流体“内摩擦”（粘度）的第一个数学处理方法。库埃特流动这种流动类型是为了纪念 Maurice Marie Alfred Couette 而命名的，他是19世纪末法国安格斯大学的一位物理学教授。

库埃特流动: 平行流动 parallel flow



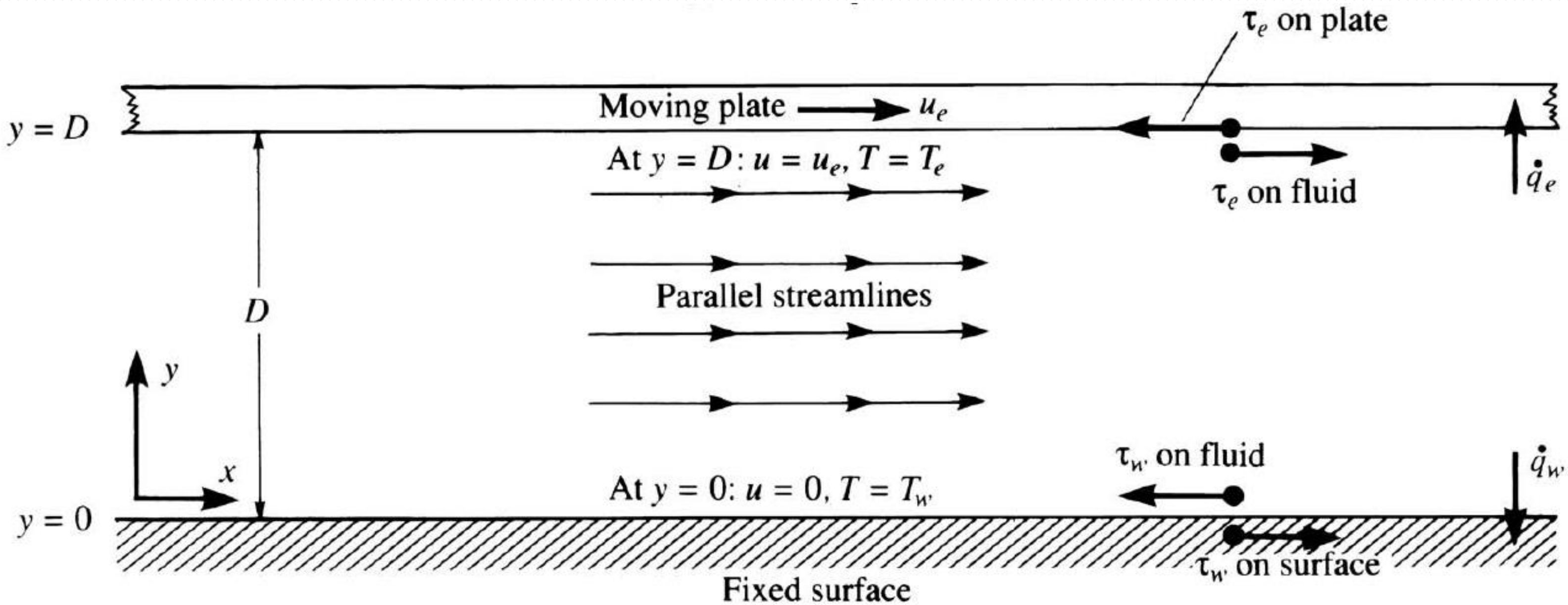
边界条件：

在 y=D 的运动物面处

- u=u_e , T=T_e
- 上界流体和与动平板间的摩擦剪力 τ_e
- 热传导 q_e

在 y=0 的静止物面处

- u=0 , T=T_w
- 下界流体和与动平板间的摩擦剪力 τ_w
- 热传导 q_w



温度场：

1、平板上下温度一般不同，产生温度梯度

2、动能由摩擦消耗变成内能，内能的变化由温度升高显示出来　（粘性耗散）

y 方向的热流量： $$\dot{q}_y = -k\frac{\partial T}{\partial y}$$

热流从温度高的壁面流向温度低的

冷壁：热量传输从流体到壁面

热壁：热量传输从壁面到流体

定常无限长平行流动，任何特性沿 x 方向不变

任何量如果变化，就会变到无穷大或者无穷小

$$v = w = 0$$

二维流动

$$\frac{\partial u}{\partial x} = \frac{\partial T}{\partial x} = \frac{\partial p}{\partial x} = 0$$

无限长平行

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$$

定常流动

动量方程（Momentum Equation）化简

$$\cancel{\rho\frac{\partial u}{\partial t}} + \cancel{\rho u\frac{\partial u}{\partial x}} + \cancel{\rho v\frac{\partial u}{\partial y}} + \cancel{\rho w\frac{\partial u}{\partial z}} = \cancel{-\frac{\partial p}{\partial x}} + \cancel{\frac{\partial}{\partial x}}\left(\lambda\nabla\cdot\mathbf{V} + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial x}}\right)\right]$$

$$\cancel{\rho\frac{\partial v}{\partial t}} + \cancel{\rho u\frac{\partial v}{\partial x}} + \cancel{\rho v\frac{\partial v}{\partial y}} + \cancel{\rho w\frac{\partial v}{\partial z}} = -\frac{\partial p}{\partial y} + \cancel{\frac{\partial}{\partial x}}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left(\cancel{\lambda\nabla\cdot\mathbf{V}} + \cancel{2\mu\frac{\partial v}{\partial y}}\right) + \frac{\partial}{\partial z}\left[\mu\left(\cancel{\frac{\partial w}{\partial y}} + \cancel{\frac{\partial v}{\partial z}}\right)\right]$$

$$\cancel{\rho\frac{\partial w}{\partial t}} + \cancel{\rho u\frac{\partial w}{\partial x}} + \cancel{\rho v\frac{\partial w}{\partial y}} + \cancel{\rho w\frac{\partial w}{\partial z}} = \cancel{-\frac{\partial p}{\partial z}} + \frac{\partial}{\partial x}\left[\mu\left(\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial x}}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\cancel{\frac{\partial w}{\partial y}} + \cancel{\frac{\partial v}{\partial z}}\right)\right] + \cancel{\frac{\partial}{\partial z}}\left(\cancel{\lambda\nabla\cdot\mathbf{V}} + \cancel{2\mu\frac{\partial w}{\partial z}}\right)$$

能量方程（Energy Equation）化简

$$
\begin{aligned}
\rho \frac{D(e+\cancel{V^2}/2)}{Dt} = {} & \cancel{\rho \dot{q}} + \cancel{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\
& + \cancel{\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)} - \nabla\cdot\cancel{p\mathbf{V}} + \cancel{\frac{\partial(u\tau_{xx})}{\partial x}} + \frac{\partial(u\tau_{yx})}{\partial y} \\
& + \cancel{\frac{\partial(u\tau_{zx})}{\partial z}} + \cancel{\frac{\partial(v\tau_{xy})}{\partial x}} + \cancel{\frac{\partial(v\tau_{yy})}{\partial y}} + \cancel{\frac{\partial(v\tau_{zy})}{\partial z}} \\
& + \cancel{\frac{\partial(w\tau_{xz})}{\partial x}} + \cancel{\frac{\partial(w\tau_{yz})}{\partial y}} + \cancel{\frac{\partial(w\tau_{zz})}{\partial z}}
\end{aligned}
$$

描述COUETTE FLOW的流动控制方程

In these equations, for Couette flow

$$v = w = 0 \qquad \frac{\partial u}{\partial x} = \frac{\partial T}{\partial x} = \frac{\partial p}{\partial x} = 0$$

x-momentum equation: $$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

y-momentum equation: $$\frac{\partial p}{\partial y} = 0$$

Energy equation: $$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y}\right) = 0$$

需要说明的问题：

- Couette Flow 的流动控制方程是严格由NS方程得来；
- x 方向动量方程 $\frac{\partial p}{\partial y}=0$ 代表垂直方向没有梯度，对于平行流动 $\frac{\partial p}{\partial x}=0$ ，说明整个流场内部没有压力梯度；
- 无粘流需要压力梯度来推动流动，对比这种粘性流动，是一种可以对流体施加外力的流动；
- 库埃特流动中运动平板对流体产生的剪力维持流体运动。

第三节：不可压库埃特流动

16.3 INCOMPRESSIBLE (CONSTANT PROPERTY) COUETTE FLOW

不可压库埃特流动动量方程的解析解

In the study of viscous flows, a flow field in which ρ, μ, and k are treated as constants is sometimes labeled as "constant property" flow.

x-momentum equation	$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$
For constant μ, this becomes	$\frac{\partial^2 u}{\partial y^2} = 0$
Integrating with respect to y twice	$u = ay + b$

At $y = 0$, $u = 0$; hence, $b = 0$.
At $y = D$, $u = u_e$; hence, $a = u_e/D$.

$$\boxed{u = u_e \left(\frac{y}{D}\right)}$$

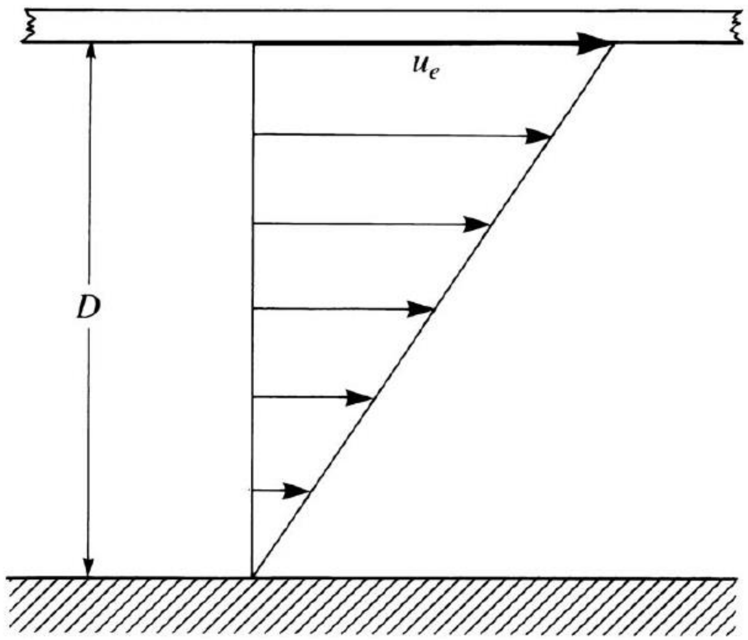


Figure 16.3 Velocity profile for incompressible Couette flow.

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{u_e}{D}$$

$$\boxed{\tau = \mu \left(\frac{u_e}{D}\right)}$$

不可压库埃特流动的特性

由 $\tau_e = \mu(u_e/D)$ 可知：

- u_e 增加，剪切力增加
- 板间距增加，剪切力减小

以上论述限于牛顿流体：

- 符合牛顿内摩擦定律的流体称牛顿流体
- 大部分航空气动问题属于牛顿流体
- 非牛顿流体，血液，有机化合物…

不可压库埃特流动能量方程的解析解

- 傅立叶热传导定律 $\dot{q}_y = -k\dfrac{\partial T}{\partial y}$
- T 变化不大时， ρ, μ, κ 变化不大, 都看成是常数
- 即使很小的温度变化，也会引起明显的热通量
- 为简化研究，认为温度 T 沿 y 方向变化，但忽略 ρ, μ, κ 随温度的很小变化, 认为其为常数 “constant property” 流动

不可压库埃特流动能量方程

$$\frac{k}{\mu}\left(\frac{\partial^2 T}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)=0$$

温度边界条件：

$$\begin{aligned}&\text{At } y=0: &\quad T=T_w\\&\text{At } y=D: &\quad T=T_e\end{aligned}$$

$$\frac{k}{\mu c_p}\frac{\partial^2 h}{\partial y^2}+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)=0$$

$$h=c_pT$$

理想气体的焓值

enthalpy

不可压库埃特流动能量方程

$$\frac{1}{\Pr}\frac{\partial^2 h}{\partial y^2}+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)=0$$

$$\frac{\partial^2 h}{\partial y^2}+\frac{\Pr}{2}\frac{\partial}{\partial y}\left(\frac{\partial u^2}{\partial y}\right)=0$$

$$\Pr=\frac{\mu c_p}{k}$$

Prandtl number

Integrating twice in the y direction

$$h+\left(\frac{\Pr}{2}\right)u^2=ay+b$$

定解条件：

$$\text{At } y=0: \qquad h=h_w \quad \text{and} \quad u=0$$

$$\text{At } y=D: \qquad h=h_e \quad \text{and} \quad u=u_e$$

不可压库埃特流动能量方程的解析解

定解条件：

$$\text{At } y = 0: \qquad h = h_w \quad \text{and} \quad u = 0$$
$$\text{At } y = D: \qquad h = h_e \quad \text{and} \quad u = u_e$$

$$b = h_w$$

$$a = \frac{h_e - h_w + (\Pr/2)u_e^2}{D}$$

$$h = h_w + \left[h_e - h_w + \left(\frac{\Pr}{2}\right)u_e^2\right]\frac{y}{D} - \left(\frac{\Pr}{2}\right)u^2$$

由动量方程的解：

$$u = u_e\left(\frac{y}{D}\right)$$

不可压库埃特流动能量方程的解析解

$$h = h_w + \left[h_e - h_w + \left(\frac{\mathrm{Pr}}{2}\right) u_e^2\right] \frac{y}{D} - \left(\frac{\mathrm{Pr}}{2}\right) u_e^2 \left(\frac{y}{D}\right)^2$$

- Note that h varies *parabolically* 抛物线 with y/D across the flow
- The temperature profile across the flow is also parabolic
- The precise shape of the parabolic curve depends on h_w, h_e

热传导 The heat transfer

$$\dot{q} = -k\frac{\partial T}{\partial y} = -\frac{k}{c_p}\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial y} = \left[h_e - h_w + \left(\frac{\mathrm{Pr}}{2}\right)u_e^2\right]\frac{1}{D} - \mathrm{Pr}\,u_e^2\frac{y}{D^2}$$

$$\Rightarrow \quad \dot{q} = -\mu\left(\frac{h_e - h_w}{\mathrm{Pr}} + \frac{u_e^2}{2}\right)\frac{1}{D} + \mu u_e^2\frac{y}{D^2}$$

- Note that $\dot{q}$ is not constant across the flow， varies linearly with y
- The physical reason for the variation of $\dot{q}$ is viscous dissipation which takes place within the flow, and which is associated with the shear stress in the flow

粘性耗散 Viscous dissipation

$$\mu u_e^2 \frac{y}{D^2} = \tau u_e \left(\frac{y}{D}\right) = \tau u$$

$$\dot{q} = -\mu \left(\frac{h_e - h_w}{\mathrm{Pr}} + \frac{u_e^2}{2}\right)\frac{1}{D} + \tau u$$

- The term τu is viscous dissipation
- Note that, if u_e is negligibly small, then the viscous dissipation is small and can be neglected
- The heat flux becomes constant across the flow

$$\dot{q} \approx -\frac{\mu}{\mathrm{Pr}}\left(\frac{h_e - h_w}{D}\right)$$

上下壁面的热传导

$$\dot{q}_w = \frac{k}{c_p}\left|\frac{\partial h}{\partial y}\right|_w = \frac{\mu}{\mathrm{Pr}}\left|\frac{\partial h}{\partial y}\right|_w$$

Couette flow的解：

$$h = h_w + \left[h_e - h_w + \left(\frac{\mathrm{Pr}}{2}\right)u_e^2\right]\frac{y}{D} - \left(\frac{\mathrm{Pr}}{2}\right)u_e^2\left(\frac{y}{D}\right)^2$$

$$\text{At } y = 0: \qquad \dot{q}_w = \frac{\mu}{\mathrm{Pr}}\left|\frac{h_e - h_w + \frac{1}{2}\mathrm{Pr}\,u_e^2}{D}\right|$$

$$\text{At } y = D: \qquad \dot{q}_w = \frac{\mu}{\mathrm{Pr}}\left|\frac{h_e - h_w - \frac{1}{2}\mathrm{Pr}\,u_e^2}{D}\right|$$

以下分三种情况讨论热流量、焓、温度

（1）忽略粘性耗散 Negligible Viscous Dissipation

$$h = h_w + \left[h_e - h_w + \left(\frac{\mathrm{Pr}}{2}\right)u_e^2\right]\frac{y}{D} - \left(\frac{\mathrm{Pr}}{2}\right)u_e^2\left(\frac{y}{D}\right)^2$$

$$\dot{q} = -\mu\left(\frac{h_e - h_w}{\mathrm{Pr}} + \frac{u_e^2}{2}\right)\frac{1}{D} + \tau u$$

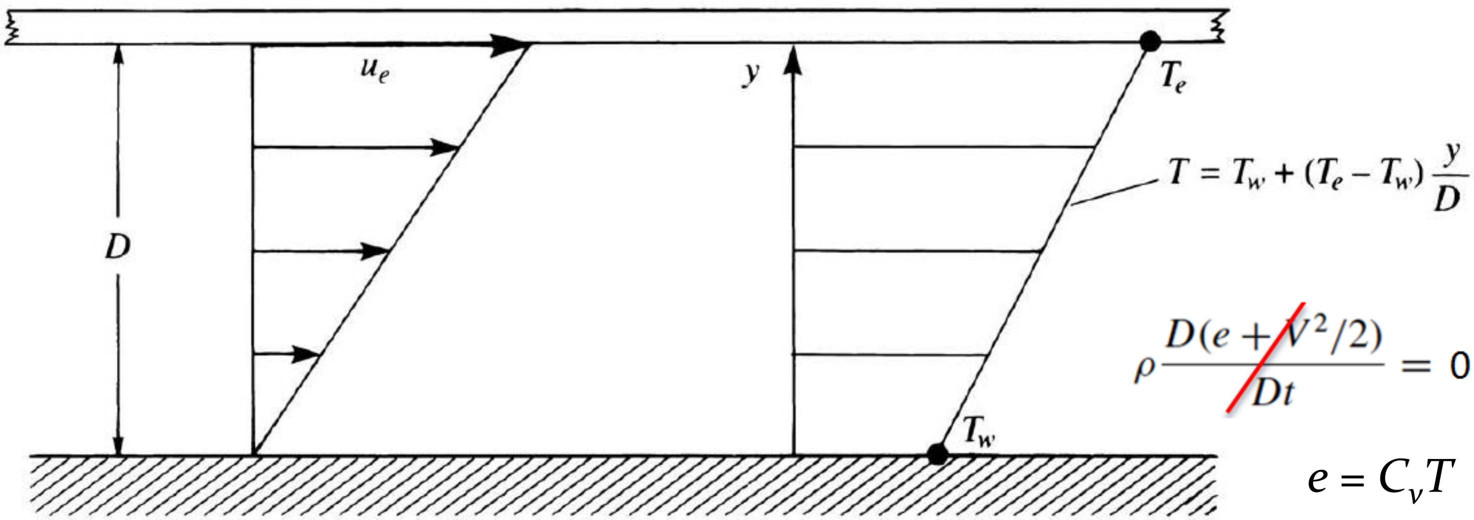
- If u_e is very small, hence τ is very small, then the amount of viscous dissipation is negligibly small.

$$\dot{q} = -\frac{\mu}{\mathrm{Pr}}\left(\frac{h_e - h_w}{D}\right)$$

$$\Rightarrow \quad h = h_w + (h_e - h_w)\frac{y}{D}$$

$$T = T_w + (T_e - T_w)\frac{y}{D} \qquad\qquad h = c_p T$$

忽略粘性耗散温度型——沿 y 轴线性分布



$$\rho \frac{D(e + \cancel{V^2/2})}{Dt} = 0$$

$$e = C_v T$$

Figure 16.4 Couette flow temperature profile for negligible viscous dissipation.

Note that the temperature varies linearly across the flow

上下壁面的热传导

$$\text{At } y = 0: \qquad \dot{q}_w = \frac{\mu}{\Pr}\left|\frac{h_e - h_w + \frac{1}{2}\Pr u_e^2}{D}\right|$$

$$\text{At } y = D: \qquad \dot{q}_w = \frac{\mu}{\Pr}\left|\frac{h_e - h_w - \frac{1}{2}\Pr u_e^2}{D}\right|$$

可以得到：

$$\dot{q}_w = k\left|\frac{T_e - T_w}{D}\right|$$

Heat transfer is the same across the two plates

忽略粘性耗散的几点结论

1. Everything else being equal, the larger the temperature difference across the viscous layer, the greater the heat transfer at the wall. The temperature difference $(T_e - T_w)$ or the enthalpy difference $(h_e - h_w)$ takes on the role of a "driving potential" for heat transfer. For the special case treated here, the heat transfer at the wall is directly proportional to this driving potential.

2. Everything else being equal, the thicker the viscous layer (the larger D is) the smaller the heat transfer is at the wall. For the special case treated here, $\dot{q}_w$ is inversely proportional to D.

3. Heat flows from a region of high temperature to low temperature. For negligible viscous dissipation, if the temperature at the top of the viscous layer is higher than that at the bottom, heat flows from the top to the bottom. In the case sketched in Figure 16.4, heat is transferred from the upper plate into the fluid, and then is transferred from the fluid to the lower plate.

(2) 等壁面温度 Equal Wall Temperatures

$$h = h_w + \left[h_e - h_w + \left(\frac{\Pr}{2}\right)u_e^2\right]\frac{y}{D} - \left(\frac{\Pr}{2}\right)u_e^2\left(\frac{y}{D}\right)^2$$

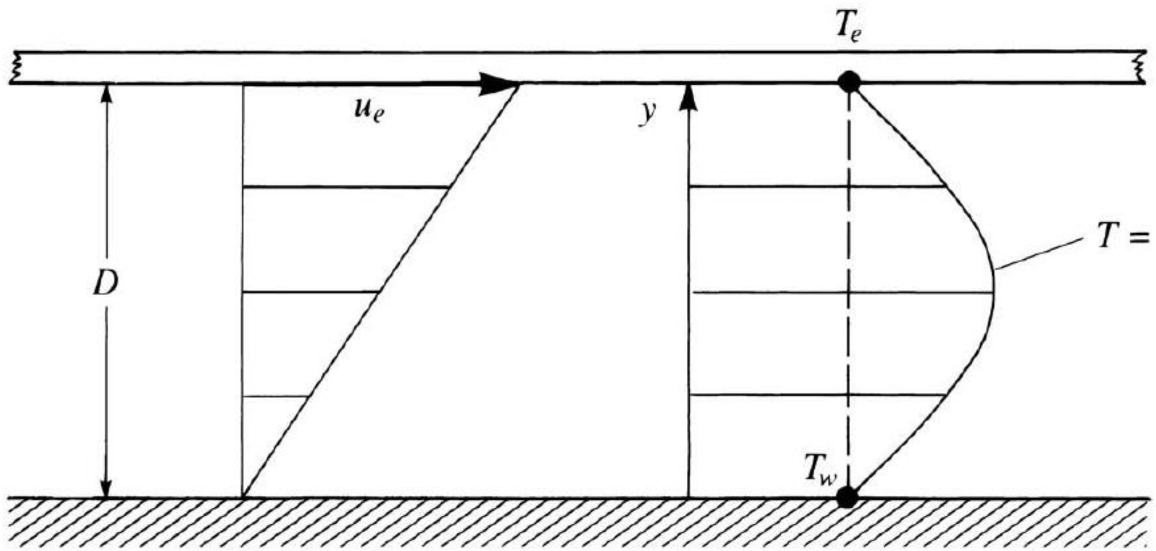
Here we assume that $T_e = T_w$; that is, $h_e = h_w$

$$h = h_w + \frac{1}{2}\Pr u_e^2\left[\frac{y}{D} - \left(\frac{y}{D}\right)^2\right]$$

$$\Rightarrow$$

$$T = T_w + \frac{\Pr u_e^2}{2c_p}\left[\frac{y}{D} - \left(\frac{y}{D}\right)^2\right]$$

等壁面温度的温度型---热量由粘性耗散产生



$$T = T_w + \frac{Pr u_e^2}{2c_p}\left[\frac{y}{D} - \left(\frac{y}{D}\right)^2\right]$$

$$\rho \frac{D(e + \cancel{V^2/2})}{Dt} = 0$$

$$e = C_v T$$

Figure 16.5 Couette flow temperature profile for equal wall temperature with viscous dissipation.

最大温度： $$T_{\max} = T_w + \frac{\Pr u_e^2}{8c_p}$$

上下壁面的热传导：

$$\text{At } y = 0: \qquad \dot{q}_w = \frac{\mu}{2}\frac{u_e^2}{D}$$

$$\text{At } y = D: \qquad \dot{q}_w = \frac{\mu}{2}\frac{u_e^2}{D}$$

- Heat transfer is the same across the two plates
- Question: Since the walls are at equal temperature, where is the heat transfer coming from?
- Answer: Viscous dissipation

$$\dot{q}_w = \tau\left(\frac{u_e}{2}\right)$$

- which further emphasizes that $\dot{q}_w$ is due entirely to the action of shear stress in the flow

等壁面温度的几点结论：

1. Everything else being equal, aerodynamic heating increases as the flow velocity increases. This is why aerodynamic heating becomes an important design factor in high-speed aerodynamics. Indeed, for most hypersonic vehicles, you can begin to appreciate that viscous dissipation generates extreme temperatures within the boundary layer adjacent to the vehicle surface and frequently makes aerodynamic heating the dominant design factor. In the Couette flow case shown here—a far cry from hypersonic flow—we see that $\dot{q}_w$ varies directly as u_e^2.

2. Everything else being equal, aerodynamic heating decreases as the thickness of the viscous layer increases. For the case considered here, $\dot{q}_w$ is inversely proportional to D. This conclusion is the same as that made for the above case of negligible viscous dissipation but with unequal wall temperature.

（3）绝热壁 Adiabatic Wall Conditions (Adiabatic Wall Temperature)

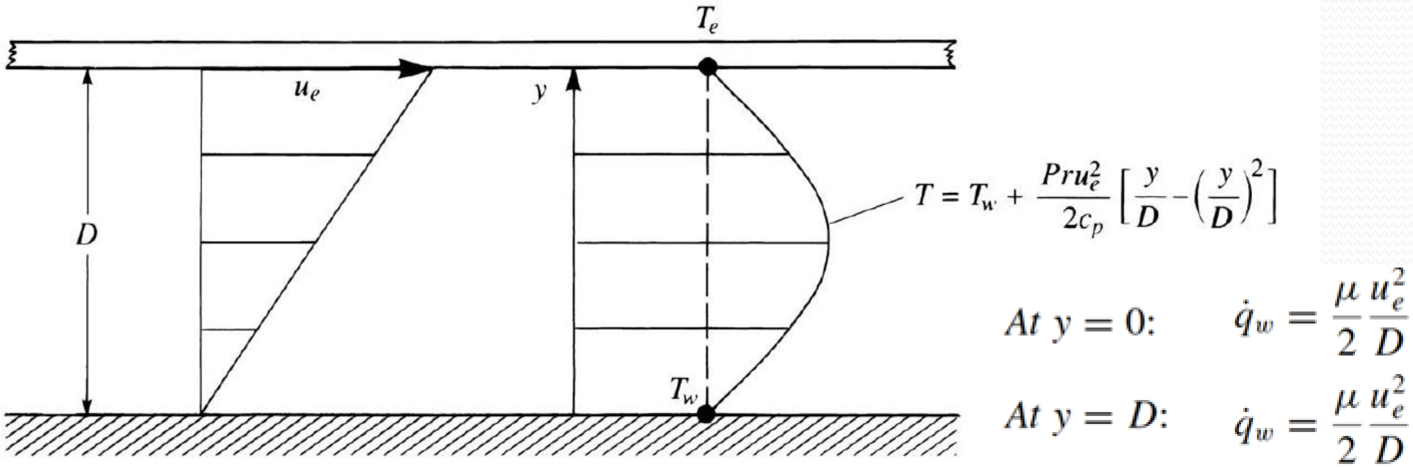


Figure 16.5 Couette flow temperature profile for equal wall temperature with viscous dissipation.

- How can the wall temperature remain fixed at the same time that heat is transferred into the wall?
- There must be some independent mechanism that conducts heat away from the wall at the same rate that the aerodynamic heating is pumping heat into the wall.

假如我们突然停掉冷却装置，会发生什么事？

$$\text{time } t = 0: \qquad \dot{q}_w = \frac{\mu}{2}\frac{u_e^2}{D}$$

$$t > 0: \qquad \dot{q}_w < \frac{\mu}{2}\frac{u_e^2}{D}$$

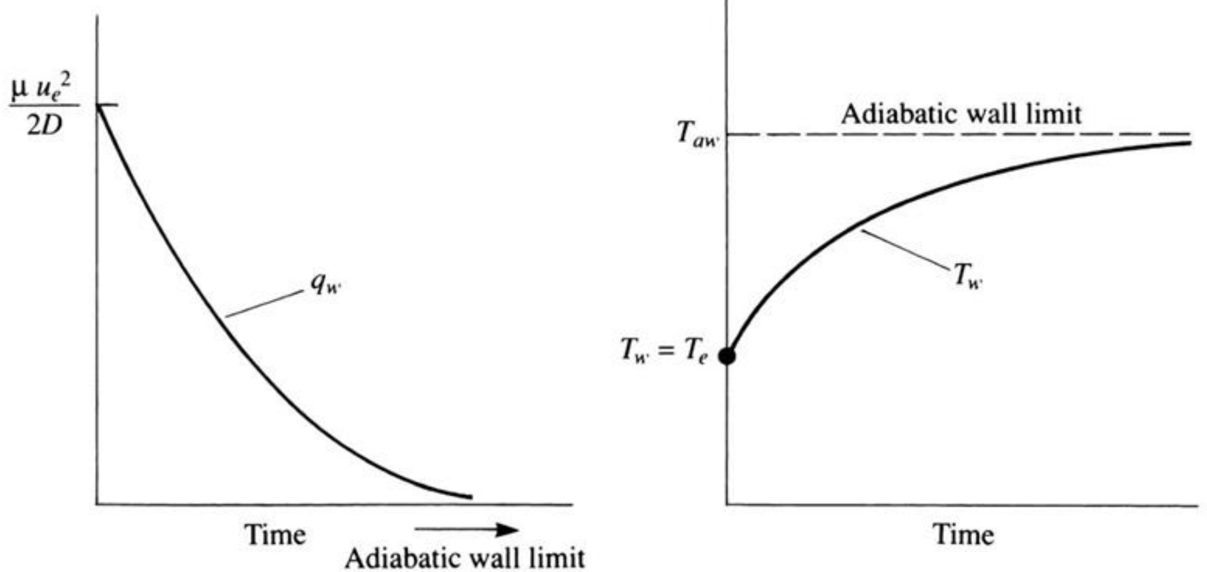


Figure 16.6 Illustration for the definition of an adiabatic wall and the adiabatic wall temperature.

传热率慢慢减小，直到没有温差，温度趋于稳定

绝热壁条件 Adiabatic wall

$$h = h_w + \left[h_e - h_w + \left(\frac{\Pr}{2}\right) u_e^2\right] \frac{y}{D} - \left(\frac{\Pr}{2}\right) u_e^2 \left(\frac{y}{D}\right)^2$$

$$\textit{Adiabatic wall:} \qquad \dot{q}_w = 0 \rightarrow \left(\frac{\partial h}{\partial y}\right)_w = \left(\frac{\partial T}{\partial y}\right)_w = 0$$

with $\partial h/\partial y = 0$, $y = 0$, and $h_w = h_{aw}$,

$$\Rightarrow \quad h_e - h_{aw} + \tfrac{1}{2} \Pr u_e^2 = 0$$

$$\boxed{h_{aw} = h_e + \Pr \frac{u_e^2}{2}}$$

最后的绝热温度是多少？

In turn, the adiabatic wall temperature is given by

$$\boxed{T_{aw} = T_e + \Pr \frac{u_e^2}{2c_p}}$$

温度剖面是什么样子的？

$$h = h_{aw} + \left(h_e - h_{aw} + \mathrm{Pr}\,\frac{u_e^2}{2}\right)\frac{y}{D} - \frac{\mathrm{Pr}}{2}u_e^2\left(\frac{y}{D}\right)^2$$

$$h_e - h_{aw} = -\mathrm{Pr}\,\frac{u_e^2}{2}$$

$$\Rightarrow \quad h = h_{aw} - \mathrm{Pr}\,\frac{u_e^2}{2}\left(\frac{y}{D}\right)^2$$

温度剖面形状 The temperature profile

$$T = T_{aw} - \mathrm{Pr}\,\frac{u_e^2}{2c_p}\left(\frac{y}{D}\right)^2$$

绝热壁的温度剖面

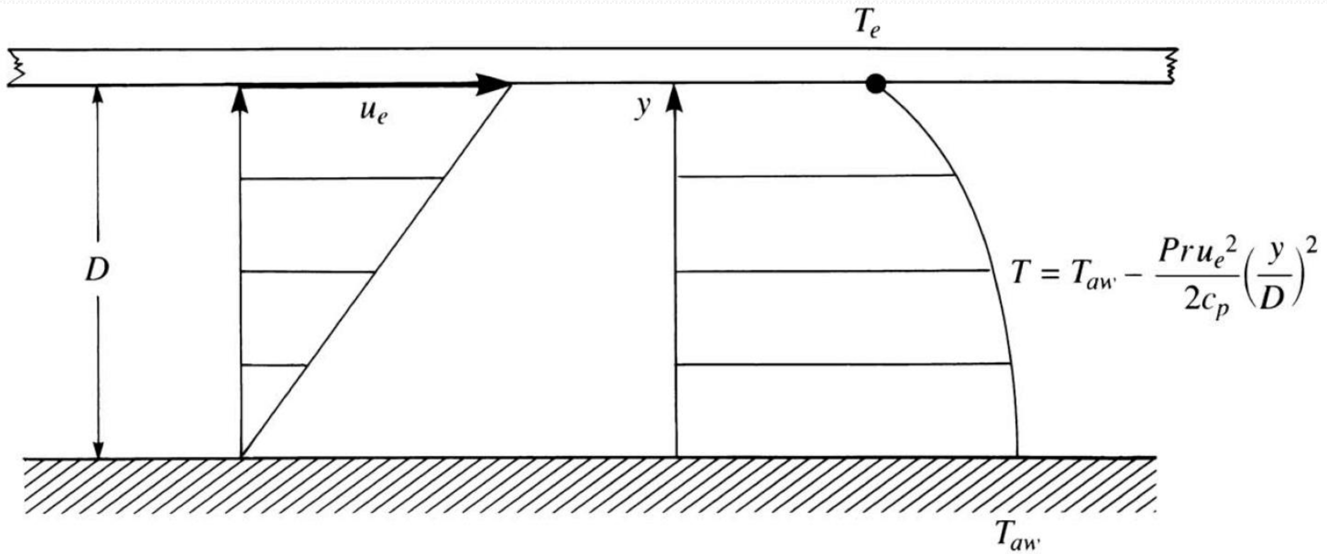


Figure 16.7 Couette flow temperature profile for an adiabatic lower wall.

$$\frac{\partial T}{\partial y} = -\Pr \frac{u_e^2}{c_p D}\left(\frac{y}{D}\right) \qquad \partial T/\partial y = 0 \text{ at } y = 0$$

满足绝热壁条件

Adiabatic wall: $\dot{q}_w = 0 \rightarrow \left(\frac{\partial h}{\partial y}\right)_w = \left(\frac{\partial T}{\partial y}\right)_w = 0$

恢复系数 Recovery Factor

- To define the recovery factor—a useful engineering parameter in the analysis of aerodynamic heating
- The total enthalpy of the flow at the upper plate

$$h_0 = h_e + \frac{u_e^2}{2}$$

- For the special case of Couette flow:

$$h_{aw} = h_e + \Pr \frac{u_e^2}{2}$$

- For any viscous flow

$$h_{aw} = h_e + r\frac{u_e^2}{2} \qquad T_{aw} = T_e + r\frac{u_e^2}{2c_p}$$

r is defined as the recovery factor

$$\boxed{r = \frac{h_{aw} - h_e}{h_0 - h_e} = \frac{T_{aw} - T_e}{T_0 - T_e}}$$

In the special case of Couette flow

$$r = \Pr$$

雷诺类比 Reynolds Analogy

- Reynolds analogy is a relation between the skin friction coefficient and the heat transfer coefficient.
- In our context here, we define the skin friction coefficient as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$

- For Couette flow： $\tau_w = \mu \left(\frac{u_e}{D} \right)$
- Let us define the Reynolds number for Couette flow as

$$\mathrm{Re} = \frac{\rho_e u_e D}{\mu}$$

- Then， skin friction coefficient $c_f = \frac{2}{\mathrm{Re}}$

The skin friction coefficient is a function of just the Reynolds number — a result which applies in general for other incompressible viscous flows

雷诺类比 Reynolds Analogy

- Now let us define a **heat transfer coefficient called the Stanton number**

$$C_H = \frac{\dot{q}_w}{\rho_e u_e (h_{aw} - h_w)}$$

- For Couette flow $\dot{q}_w = \frac{\mu}{\Pr}\left(\frac{h_e - h_w + \frac{1}{2}\Pr u_e^2}{D}\right) = \frac{\mu}{\Pr}\left(\frac{h_{aw} - h_w}{D}\right)$

$$C_H = \frac{(\mu/\Pr)[(h_{aw} - h_w)/D]}{\rho_e u_e (h_{aw} - h_w)} = \frac{\mu/\Pr}{\rho_e u_e D} = \frac{1}{\mathrm{Re}\,\Pr}$$

- Then $\boxed{\frac{C_H}{c_f} = \frac{1}{2}\Pr^{-1}}$ ⟸ 雷诺类比

This is Reynolds analogy， a relation between the heat transfer coefficient and the skin friction coefficient.

A result that applies usually for other incompressible viscous flows.

第四节：可压库埃特流动

16.4 COMPRESSIBLE COUETTE FLOW

可压库埃特流动：

We now assume that u_e is large enough; hence, the changes in temperature within the flow are substantial enough, so that we must treat ρ, μ, and k as variables—this is compressible Couette.

$$v = w = 0 \qquad \frac{\partial u}{\partial x} = \frac{\partial T}{\partial x} = \frac{\partial p}{\partial x} = 0$$

x-momentum equation:

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

y-momentum equation:

$$\frac{\partial p}{\partial y} = 0$$

整个流场中P 为参数

Energy equation:

$$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y}\right) = 0$$

可压库埃特流动控制方程：

The governing equations for compressible Couette flow， with μ and k as variables

动量方程：
$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)=\frac{\partial \tau}{\partial y}=0$$

$$\Rightarrow \quad \tau = \text{const}$$

能量方程：
$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)=0$$

$$\Rightarrow \quad \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\tau\frac{\partial u}{\partial y}=0$$

可压库埃特流动控制方程的解：

The temperature variation of μ is accurately given by Sutherland's law

$$\tau = \mu \frac{\partial u}{\partial y} = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + 110}{T + 110} \left(\frac{\partial u}{\partial y}\right)$$

- Write Equation in terms of the ordinary differential equation that it really is

$$\frac{d}{dy}\left(k \frac{\partial T}{\partial y}\right) + \tau \frac{du}{dy} = 0$$

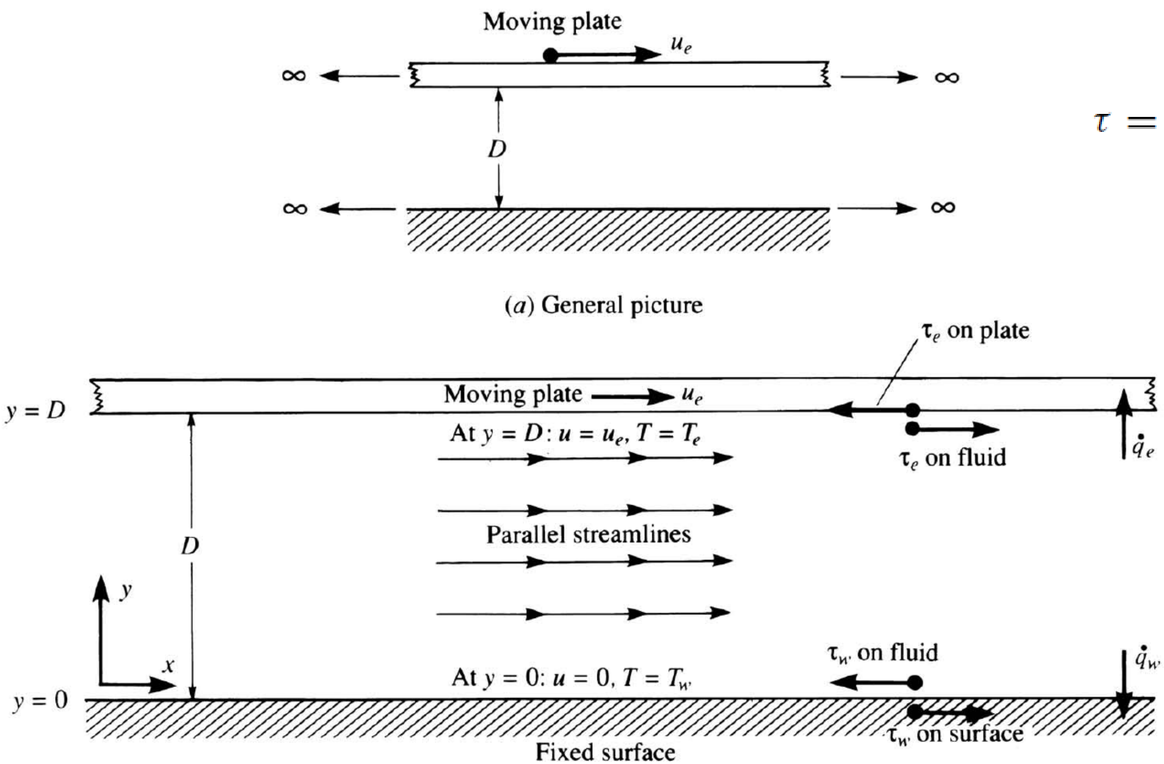
- two-point boundary value problem：

$$y = 0 \quad T = T_w \qquad y = D \quad T = T_e$$

The method involves a double iteration, that is, two minor iterations nested within a major iteration.

两步迭代方法流程，The scheme is as follows:

库埃特流动: 平行流动 parallel flow



$$\tau = \mu \frac{\partial u}{\partial y} = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + 110}{T + 110} \left(\frac{\partial u}{\partial y}\right)$$

$$\frac{du}{dy} = \frac{\tau}{\mu}$$

$$\frac{d}{dy}\left(k \frac{\partial T}{\partial y}\right) + \tau \frac{du}{dy} = 0$$

可压库埃特流动的解—打靶法 Shooting Method :

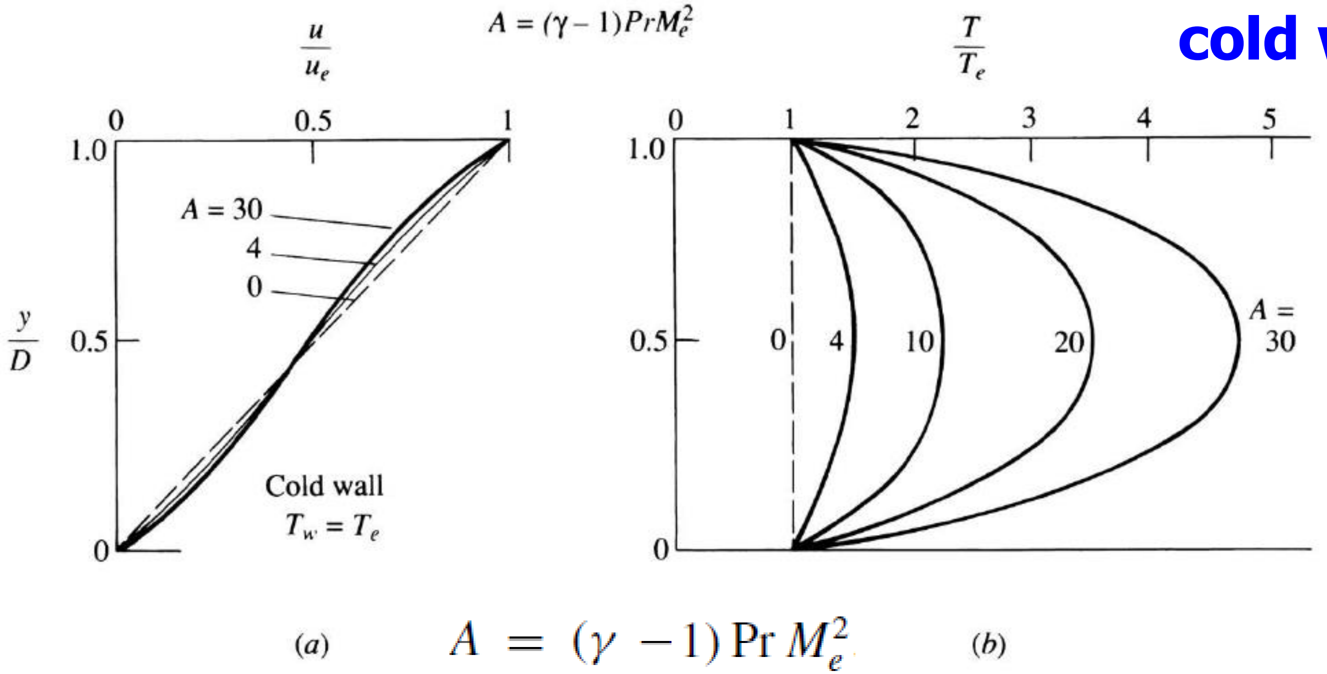
1. Assume a value for τ and $u(y)$ is given by the incompressible result from Equation.
2. Starting at $y = 0$ with the known boundary condition $T = Tw$. Use any numerical Technique, integrate Equation across the flow until $y = D$.
3. Stop the numerical integration when $y = D$ is reached. Check to see if the numerical value of T at $y = D$ equals the specified boundary condition, $T = Te$.
4. From the converged temperature profile obtained by the repetitive iteration in steps 2 and 3, we can obtain $\mu = \mu(y)$.
5. From the definition of shear stress we have $\frac{du}{dy} = \frac{\tau}{\mu}$, Using the assumed value of τ from step 1, and the values of $\mu = \mu(y)$ from step 4, numerically integrate Equation.
6. Stop the numerical integration when $y = D$ is reached. Check to see if the numerical value of u at $y = D$ equals the specified boundary condition, $u = u_e$.
7. Return to step 2, using the new value of τ and the new $u(y)$ obtained from step 6. Repeat steps 2 through 7 until total convergence is obtained.

可压库埃特流动结果

Results for Compressible Couette Flow：

等温壁-冷壁

cold wall case



$$A = (\gamma - 1)\,\mathrm{Pr}\,M_e^2$$

Figure 16.9 Velocity and temperature profiles for compressible Couette flow. Cold wall cases. (*Data Source:* White, Reference 43.)

A = 30 corresponds to Me approximately 10

等温壁-冷壁 cold wall case 流动特性

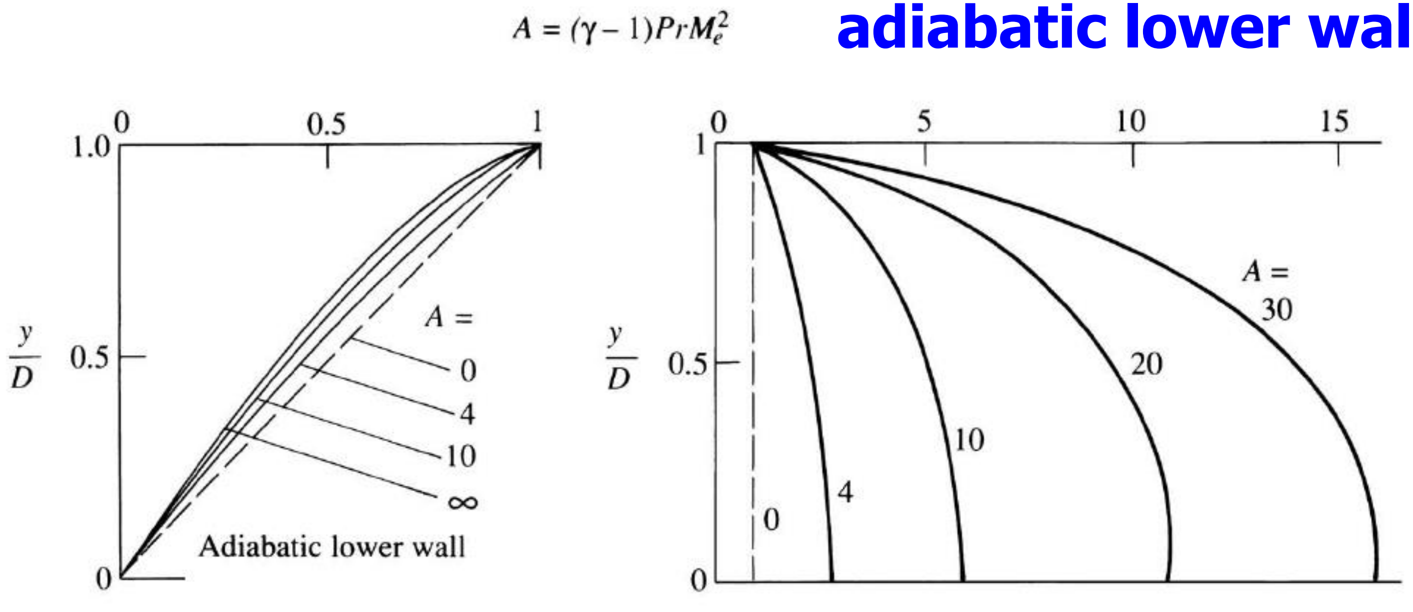
1. The velocity profiles are not greatly affected by compressibility. The velocity profile versus does not change greatly over such a large range of Mach number.
2. There are huge temperature changes in the flow， are due exclusively to viscous dissipation。

可压库埃特流动结果

Results for Compressible Couette Flow：

绝热壁

adiabatic lower wall case



$$A = (\gamma - 1)\,\mathrm{Pr}\, M_e^2$$

Figure 16.10 Velocity and temperature profiles for compressible Couette flow. Adiabatic lower wall. (*Data Source:* White, Reference 43.)

绝热壁 adiabatic lower wall case 流动特性

1. The velocity profiles show a pronounced curvature due to compressibility.

2. There are huge temperature changes in the flow，are due exclusively to viscous dissipation。

解析分析 Some Analytical Considerations :

The recovery factor and Reynolds analogy for compressible case

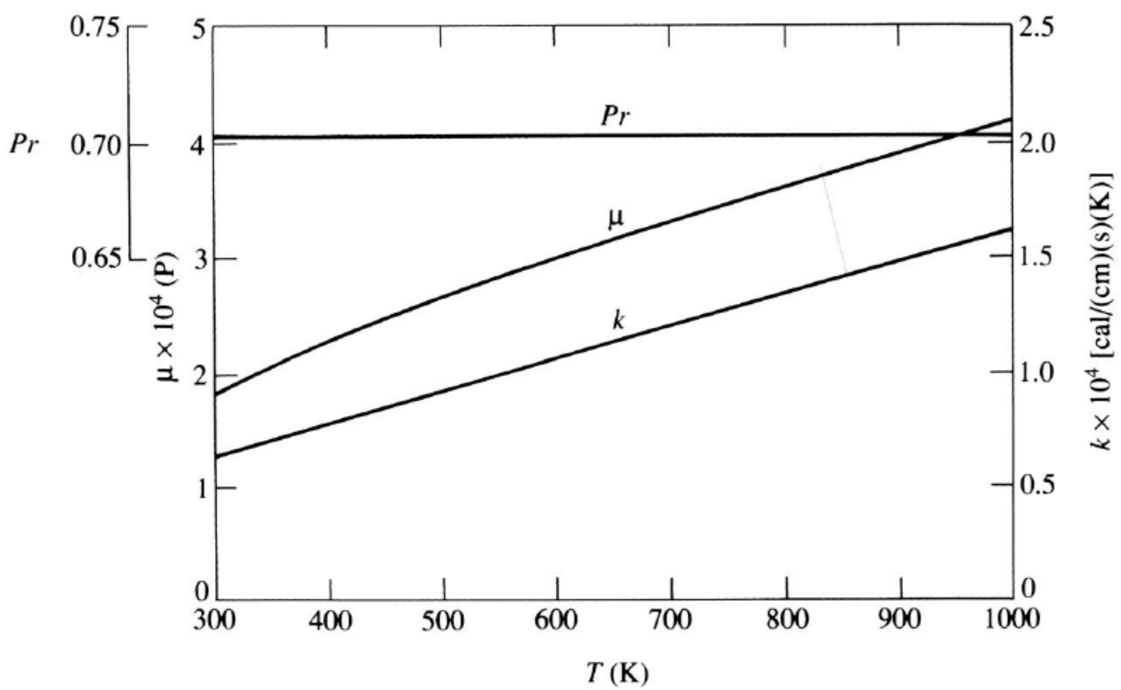


Figure 16.11 Variation of viscosity coefficient, thermal conductivity, and Prandtl number for air as a function of temperature. (*Data Source:* From Schetz, Reference 53.)

$$\mathrm{Pr} = \frac{\mu c_p}{k}$$

Prandtl number

$$\mathbf{Pr \approx 0.71}$$

- Question: How high a Mach number can exist before we would expect to encounter temperatures in the flow above 1000 K?
- Answer: An approximate answer is to calculate that Mach number at which the total temperature is 1000 K. Assuming a static temperature T = 288 K.

$$M = \sqrt{\frac{2}{\gamma - 1}\left(\frac{T_0}{T} - 1\right)} = \sqrt{\frac{2}{0.4}\left(\frac{1000}{288} - 1\right)} = 3.5$$

- A Mach number of 3.5 or less encompasses virtually all operational aircraft today.
- For the case of compressible Couette flow, the assumption of Pr = constant allows the following analysis.

We have this result because that Pr is constant

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)=0$$

Since $T = h/c_p$ and $\mathrm{Pr} = \mu c_p/k$

$$\frac{\partial}{\partial y}\left(\frac{\mu}{\mathrm{Pr}}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)=0$$

$$\frac{\partial}{\partial y}\left[\mu\left(\frac{1}{\mathrm{Pr}}\frac{\partial h}{\partial y}+u\frac{\partial u}{\partial y}\right)\right]=0$$

$$\Rightarrow \quad \frac{1}{\mathrm{Pr}}\frac{\partial h}{\partial y}+u\frac{\partial u}{\partial y}=\frac{a}{\mu}$$

Relation between h and u is obtained

$$\tau = \mu(\partial u/\partial y)$$ τ is constant.

$$\frac{a}{\mu} = \frac{a}{\tau}\frac{\partial u}{\partial y} = b\frac{\partial u}{\partial y}$$ where b is a constant

$$\frac{1}{\Pr}\frac{\partial h}{\partial y} + \frac{\partial(u^2/2)}{\partial y} - b\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad \frac{h}{\Pr} + \frac{u^2}{2} - bu = c$$ $\Pr = \text{constant}$

c is another constant of integration

Apply the boundary condition to obtain the integration constants

At $y = 0$ and $y = D$. At $y = 0$, $h = h_w$ and $u = 0$;

$$c = \frac{n_w}{\mathrm{Pr}}$$

At $y = D$, $h = h_e$ and $u = u_e$;

$$b = \frac{1}{u_e}\left(\frac{h_e - h_w}{\mathrm{Pr}}\right) + \frac{u_e}{2}$$

Inserting b and c into Equation

$$h + \mathrm{Pr}\,\frac{u^2}{2} = h_w + \frac{u}{u_e}(h_e - h_w) + \frac{\mathrm{Pr}}{2}(u u_e)$$

Differentiating Equation with respect to y

$$\frac{\partial h}{\partial y} = -\mathrm{Pr}\,u\frac{\partial u}{\partial y} + \left(\frac{h_e + h_w}{u_e}\right)\frac{\partial u}{\partial y} + \frac{u_e\,\mathrm{Pr}}{2}\frac{\partial u}{\partial y}$$

or

$$\frac{\partial h}{\partial y} = \left(-u\,\mathrm{Pr} + \frac{h_e - h_w}{u_e} + \frac{u_e\,\mathrm{Pr}}{2}\right)\frac{\partial u}{\partial y}$$

Recovery factor for compressible flow

at $y = 0$ for an adiabatic wall

$$(\partial h/\partial y)_w = 0 \quad u = 0 \text{ and } \quad h_w = h_{aw}$$

$$\left(\frac{\partial h}{\partial y}\right)_w = \left(\frac{h_e - h_{aw}}{u_e} + \frac{u_e \Pr}{2}\right)\left(\frac{\partial u}{\partial y}\right)_w = 0$$

Since $(\partial u/\partial y)_w$ is finite

$$\frac{h_e - h_{aw}}{u_e} + \frac{u_e \Pr}{2} = 0$$

or

$$\boxed{h_{aw} = h_e + \Pr \frac{u_e^2}{2}}$$

the recovery factor for compressible Couette flow

$$r = \Pr$$

the recovery factors for the incompressible and compressible cases are the same

Reynolds analogy for compressible flow

Starting from energy equation again

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)=0$$

Recalling that, from the definitions $\dot{q}=k\dfrac{\partial T}{\partial y}$ and $\tau=\mu\dfrac{\partial u}{\partial y}$

then $$\frac{\partial \dot{q}}{\partial y}+\frac{\partial(\tau u)}{\partial y}=0$$

Integrating with respect to y

$$\dot{q}+\tau u=a$$ a is a constant of integration

Evaluating at $y=0$, $u=0$ and $\dot{q}=q_w$.

$$a=\dot{q}_w$$

Hence, $\dot{q} + \tau u = \dot{q}_w$

$$\dot{q}_w = k\frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y}$$

or $$\frac{\dot{q}_w}{\mu} = \frac{k}{\mu}\frac{\partial T}{\partial y} + u\frac{\partial u}{\partial y}$$

Recall that the shear stress is constant

$$\tau = \mu\frac{\partial u}{\partial y} = \tau_w$$

or $$\mu = \frac{\tau_w}{\partial u/\partial y}$$

Also $$\frac{k}{\mu} = \frac{c_p}{\mathrm{Pr}}$$

$$\frac{\dot{q}_w}{\tau_w}\frac{\partial u}{\partial y} = \frac{c_p}{\Pr}\frac{\partial T}{\partial y} + \frac{\partial(u^2/2)}{\partial y}$$

Integrate between the two plates.

$\dot{q}_w$, τ_w, c_p, and Pr are all fixed values:

$$\frac{\dot{q}_w}{\tau_w}\int_0^D \frac{\partial u}{\partial y}dy = \frac{c_p}{\Pr}\int_0^D \frac{\partial T}{\partial y}dy + \int_0^D \frac{\partial(u^2/2)}{\partial y}dy$$

or

$$\frac{\dot{q}_w}{\tau_w}\int_0^{u_e} du = \frac{c_p}{\Pr}\int_{T_w}^{T_e} dT + \int_0^{u_e} d\left(\frac{u^2}{2}\right)$$

which yields

$$\frac{\dot{q}_w}{\tau_w}u_e = \frac{c_p}{\Pr}(T_e - T_w) + \frac{u_e^2}{2}$$

Rearranging that $h = c_p T$

$$\dot{q}_w = \frac{\tau_w}{u_e \Pr}\left(h_e - h_w + \Pr \frac{u_e^2}{2}\right) \qquad h_{aw} = h_e + \Pr \frac{u_e^2}{2}$$

$$\Rightarrow \quad \dot{q}_w = \frac{\tau_w}{u_e \Pr}(h_{aw} - h_w)$$

The skin-friction coefficient and Stanton number ratio is

$$\frac{C_H}{c_f} = \frac{\dot{q}_w/[\rho_e u_e (h_{aw} - h_w)]}{\tau_w/\left(\frac{1}{2}\rho_e u_e^2\right)} = \frac{\dot{q}_w}{\tau_w}\left[\frac{u_e}{2(h_{aw} - h_w)}\right]$$

$$\frac{C_H}{c_f} = \frac{(h_{aw} - h_w)}{u_e \Pr}\left[\frac{u_e}{2(h_{aw} - h_w)}\right]$$

or

$$\boxed{\frac{C_H}{c_f} = \frac{1}{2}\Pr^{-1}}$$

Reynolds analogy:

relation between heat transfer and skin friction coefficients, same for incompressible and compressible flow

第五节：小结

16.5 SUMMARY

库埃特COUETTE FLOW流动小结

- The parallel flow discussed in this chapter illustrates features common to many more complex viscous flows, with the added advantage of lending itself to a relatively straightforward solution.
- The purpose of this discussion has been to introduce many of the basic concepts of viscous flows in a fashion unencumbered by fluid dynamic complexities.
- In particular, we have studied Couette flow and found the following.

1、流体运动的驱动力

The driving force is the shear stress between the moving wall and the fluid. Shear stress is constant across the flow for both incompressible and compressible cases.

2、不可压缩 **Couette flow** 流动的解析解

For incompressible Couette flow, $u = u_e\left(\frac{y}{D}\right)$ $\tau = \mu\left(\frac{u_e}{D}\right)$

3、**Couette flow** 的流动特性

The heat transfer depends on the wall temperatures and the amount of viscous dissipation. For an adiabatic wall, the wall enthalpy is

$$h_{aw} = h_e + r\frac{u_e^2}{2}$$

For incompressible and compressible Couette flow with a constant Prandtl number, holds in both cases;

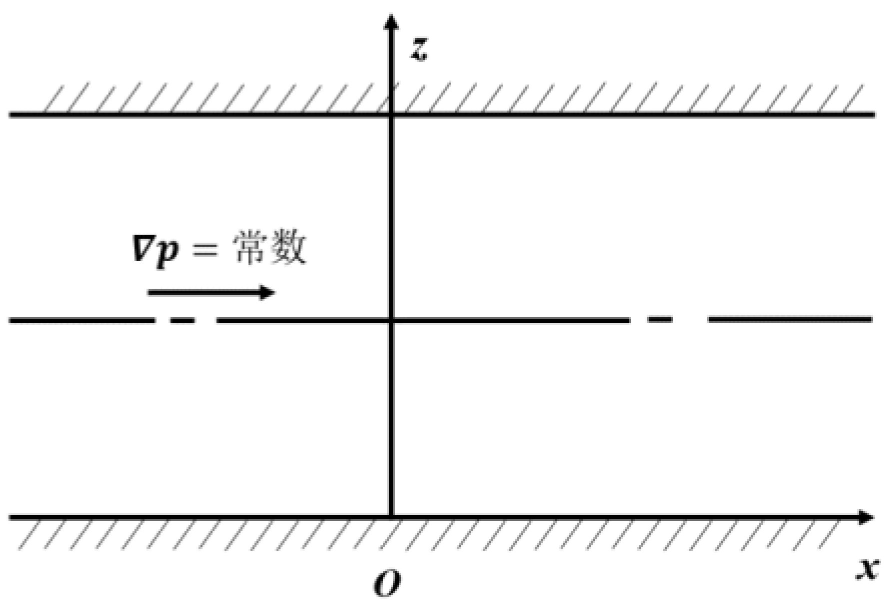
recovery factor $r = \mathrm{Pr}$ Reynolds analogy $\frac{C_H}{c_f} = \frac{1}{2}\mathrm{Pr}^{-1}$

流动由x方向恒定压力驱动，则$\partial p/\partial x = -G$.

第三次作业：

问题描述

- 两块处于相对静止的无限大平行板之间充满流体
- 沿着和平板平行的某一方向在流体内作用一个不变的压强梯度
- 流体在此压力场作用下运动
- 当时间充分长以后，作用在流体上的压力和黏性力达到平衡



第三次作业：

- 要求：
  - 在A4纸上完成
  - 截止日期：5月19日至5月22日之间
  - 提交方式：电子版（拍照或扫描）
  - 邮件发送，主题“第三次作业_学号_姓名”
  - 例如：第三次作业_2017300300_张三
  - 助教邮箱：912387046@mail.nwpu.edu.cn