第七讲：湍流与湍流边界层

Chapter 19 Turbulent Boundary Layers

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The one uncontroversial fact about turbulence is that it is the most complicated kind of fluid motion. 关于湍流的一个不争的事实是它是最复杂的流体运动。

Peter Bradshaw
Imperial College of Science and Technology, London 1978

Turbulence was, and still is, one of the great unsolved mysteries of science, and it intrigued some of the best scientific minds of the day. Arnold Sommerfeld, the noted German theoretical physicist of the 1920s, once told me, for instance, that before he died he would like to understand two phenomena—quantum mechanics and turbulence. Sommerfeld died in 1924. I believe he was somewhat nearer to an understanding of the quantum, the discovery that led to modern physics, but no closer to the meaning of turbulence.

Theodore von Karman, 1967

湍流过去是，现在仍然是，科学中尚未解开的大谜团之一，它激起了当时一些最优秀的科学头脑的兴趣。20世纪20年代著名的德国理论物理学家索默菲尔德曾告诉我，在他去世之前，他想了解两种现象：量子力学和湍流。1924年他去世，我相信他更接近于对量子的理解，这一发现导致了现代物理学，但却没有更接近湍流的意义。

第一节：引言

19.1 INTRODUCTION

湍流特性概述

- 湍流是最复杂的流体运动形式
- 湍流是科学上最大的未解之谜
- 自然趋于向最大无序状态发展：大多边界层流动为湍流
- 湍流特性：

  缺点：摩擦力和气动加热大于层流

  优点：易于保持流动附着于物面，压差阻力小
- 分析依赖于经验数据

流场中流体微团的流动轨迹：迹线

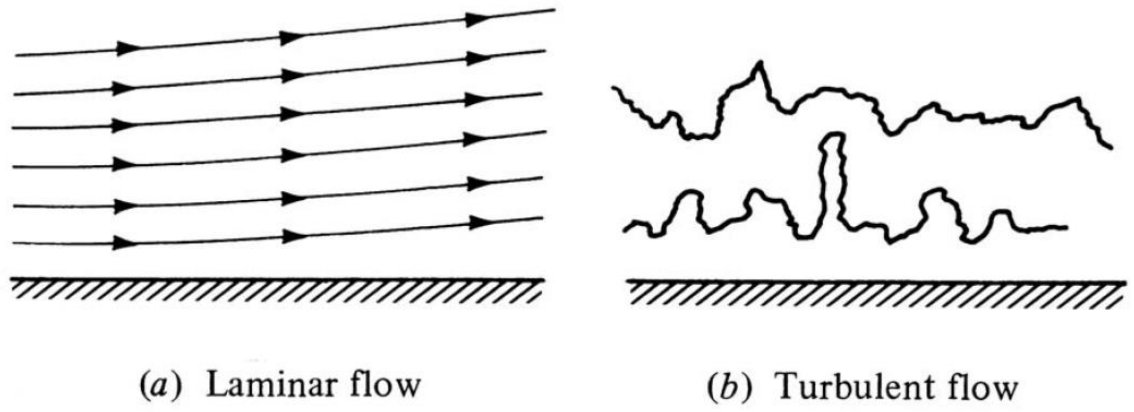


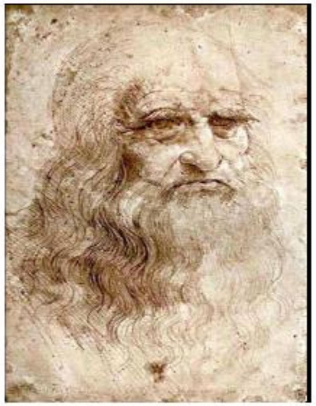
Figure 15.5 Path lines for laminar and turbulent flows.

Is the flow laminar or turbulent?

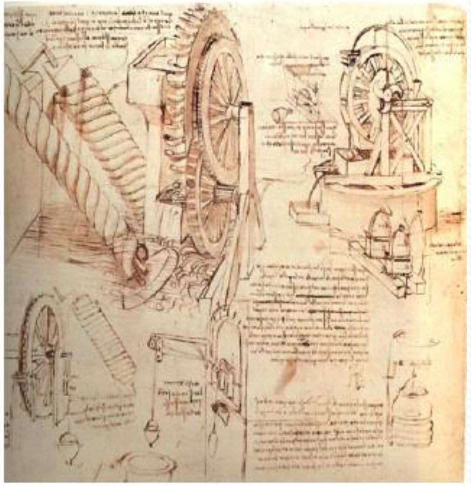
- If the path lines of various fluid elements are smooth and regular, as shown in Figure 15.5a, the flowis called laminar flow.
- In contrast, if the motion of a fluid element is very irregular and tortuous, as shown in Figure 15.5b, the flow is called turbulent flow.

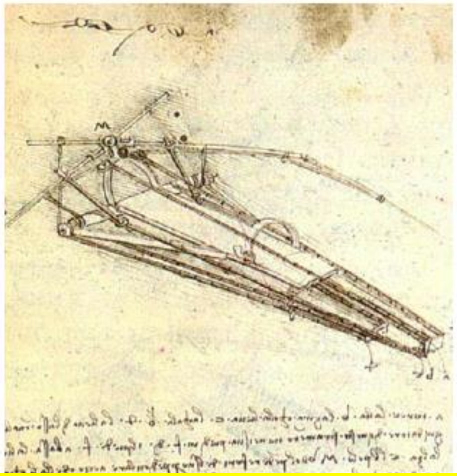
A bit of history

Leonardo Da Vinci 达芬奇 (1452-1519), Italy



LeonardoDaVinci





爱因斯坦认为，达·芬奇的科研成果如果在当时就发表的话，科技可以提前半个世纪。

- 深入、广泛的研究，仍是未解的问题
- 有很多专著、有很多学者花费毕生的精力研究
- 不存在湍流的纯理论。为了获得实际的答案，湍流分析都需要某些类型的经验数据

李政道

海森堡

Also, we note that no *pure theory* of turbulent flow exists. Every analysis of turbulent flow requires some type of *empirical data* in order to obtain a practical answer. As we examine the calculation of turbulent boundary layers in the following sections, the impact of this statement will become blatantly obvious.

第二节：雷诺平均方程

19.2 Reynolds Averaged Equations

瞬时量无法预测，统计量有规律可循

是否可以通过NS方程得到统计量的方程？

不可压缩粘性流动控制方程（层流）

Continuity equation

- Model of an Infinitesimally Small Element Fixed in Space

$$[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}]dxdydz=-\frac{\partial \rho}{\partial t}(dxdydz)$$

$$\frac{\partial \rho}{\partial t}+[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}]=0$$

or

$$\frac{\partial \rho}{\partial t}+\nabla \cdot(\rho \mathbf{V})=0$$

Momentum equation

If the flow is incompressible and Newtonian

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho f_x \quad (1)$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right) + \rho f_x \quad (2)$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial z^2} + \rho f_x \quad (3)$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \rho f_x \quad (4)$$

$$\rho \frac{D\boldsymbol{V}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{V} + \rho \boldsymbol{F} \quad (5)$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{V} + \rho \boldsymbol{F} \quad (6)$$

Reynolds Decomposition and Averaging

A free water jet issuing from a square hole into a pool

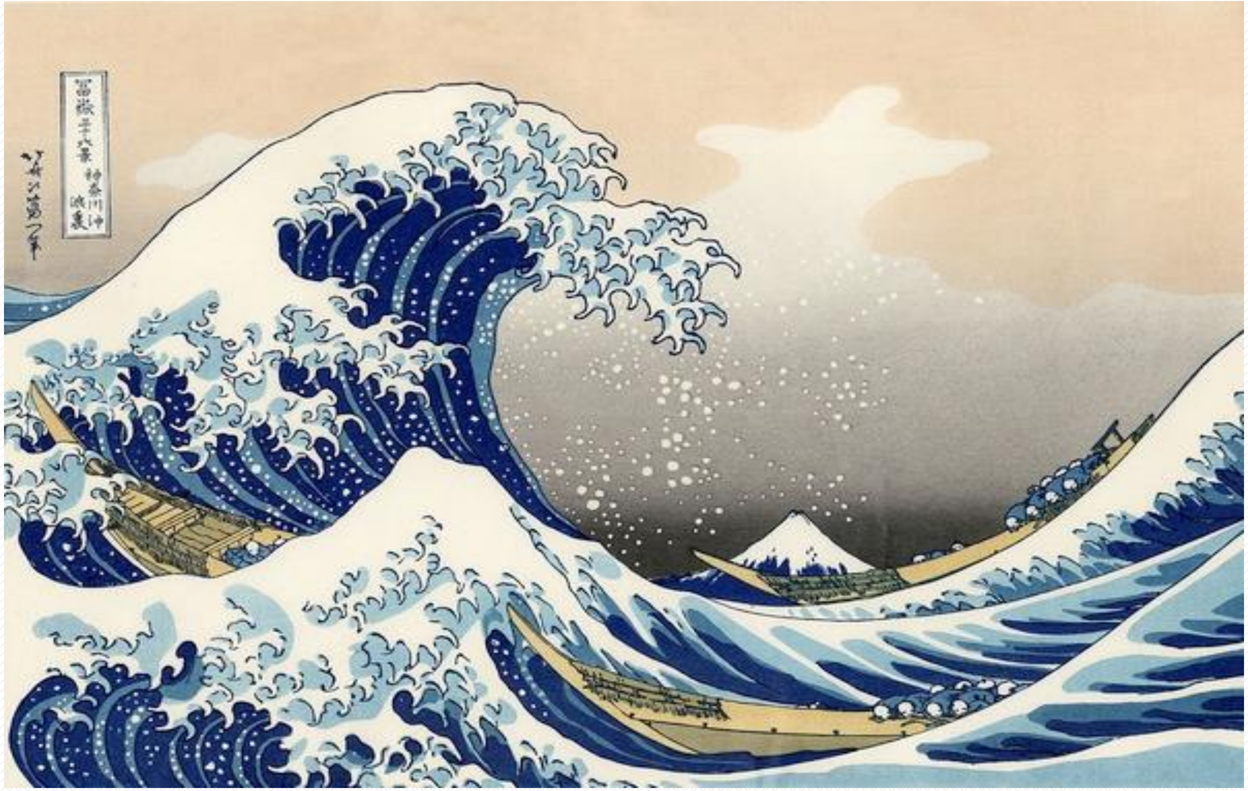
Leonardo da Vinci wrote, “Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion.”

Turbulence = Mean motions + Fluctuations (Eddies)

《星夜》（The Starry Night）作者：文森特·梵高

1889年，布面油画，尺寸 73 cm × 92 cm

现藏于纽约现代艺术博物馆（Museum of Modern Art，MoMA）



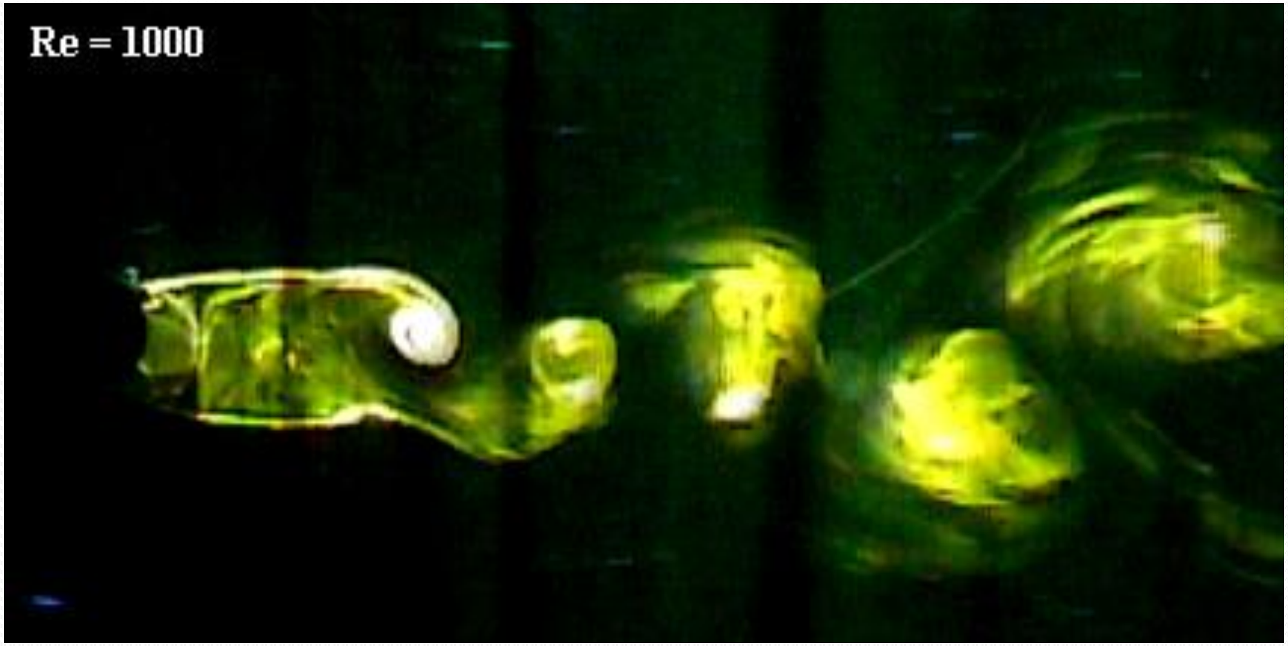
《神奈川冲浪里》是日本浮世绘画家葛饰北斋的著名木版画，于1832年出版，是《富岳三十六景》系列作品之一。它描述巨浪威胁神奈川冲（神奈川外海）的船只，与该系列的其它作品一样，以富士山为背景。图中描绘的惊涛巨浪掀卷着渔船，船工们为了生存而努力抗争的图像，远景是富士山。这幅作品是北斋最有名的作品，也是世界上最有名的日本美术作品之一。

Reynolds Decomposition and Averaging

Mean motion and fluctuations in turbulent flows

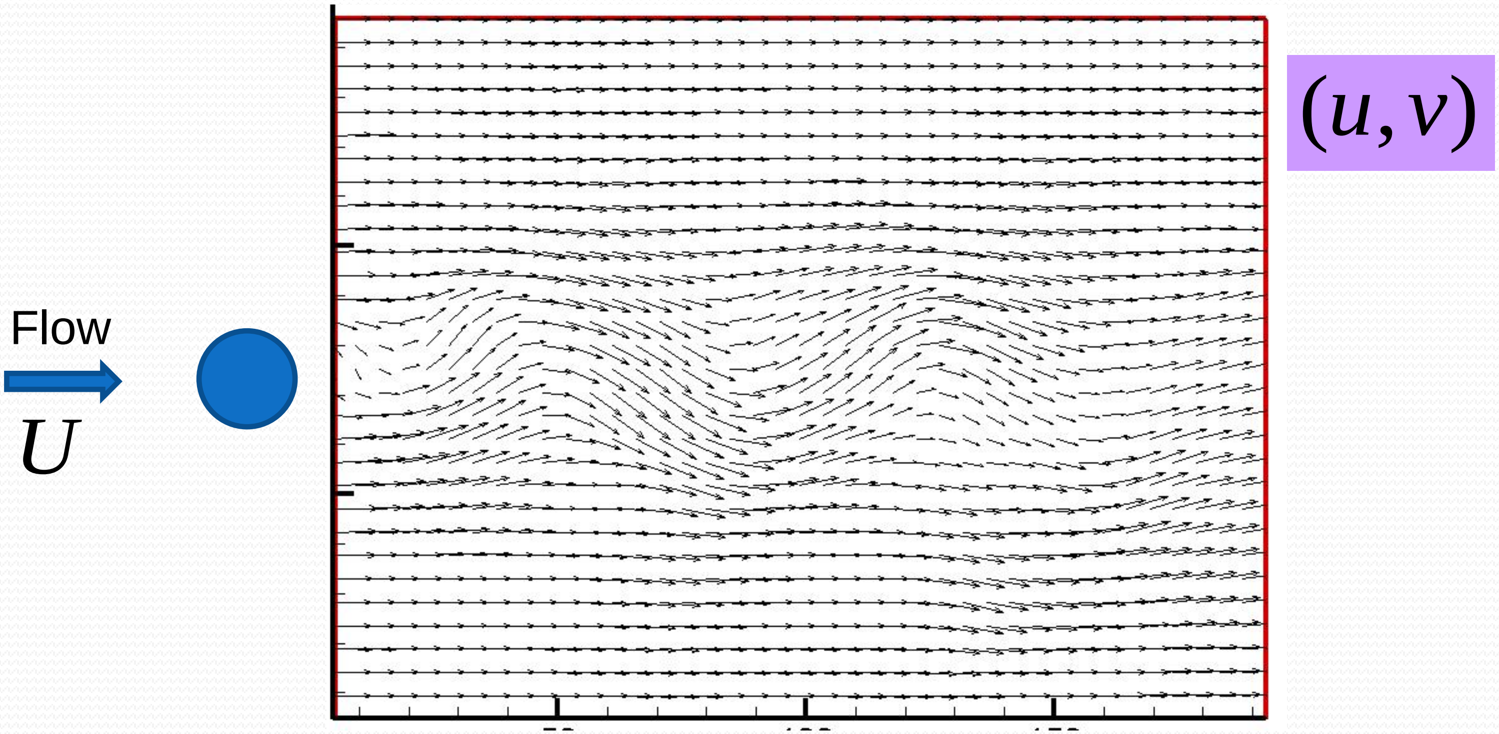
Flow

U



Snapshot of the turbulent wake of a cylinder
(Laser Induced Fluorescence-LIF ?)

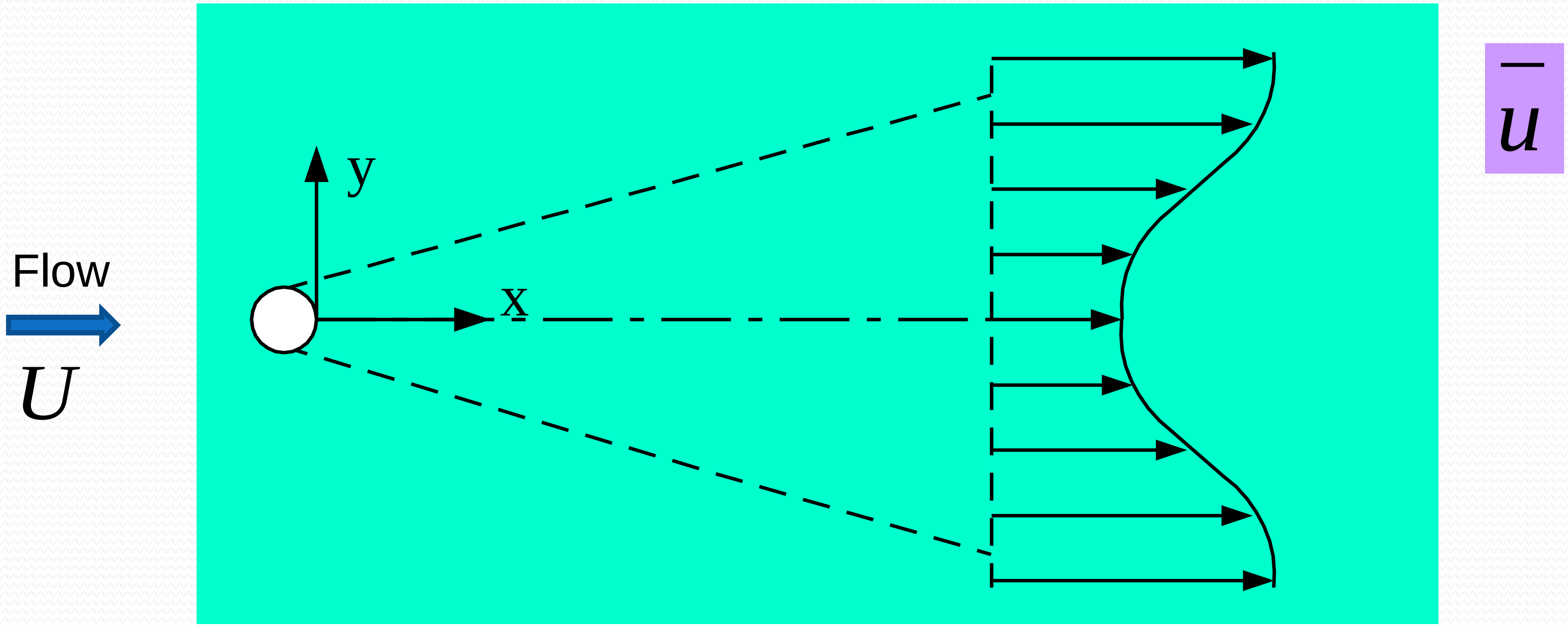
Mean motion and fluctuations in turbulent flows



Instantaneous velocity vector distribution in the turbulent wake of a cylinder
（Particle Image Velocimetry-PIV）

Reynolds Decomposition and Averaging

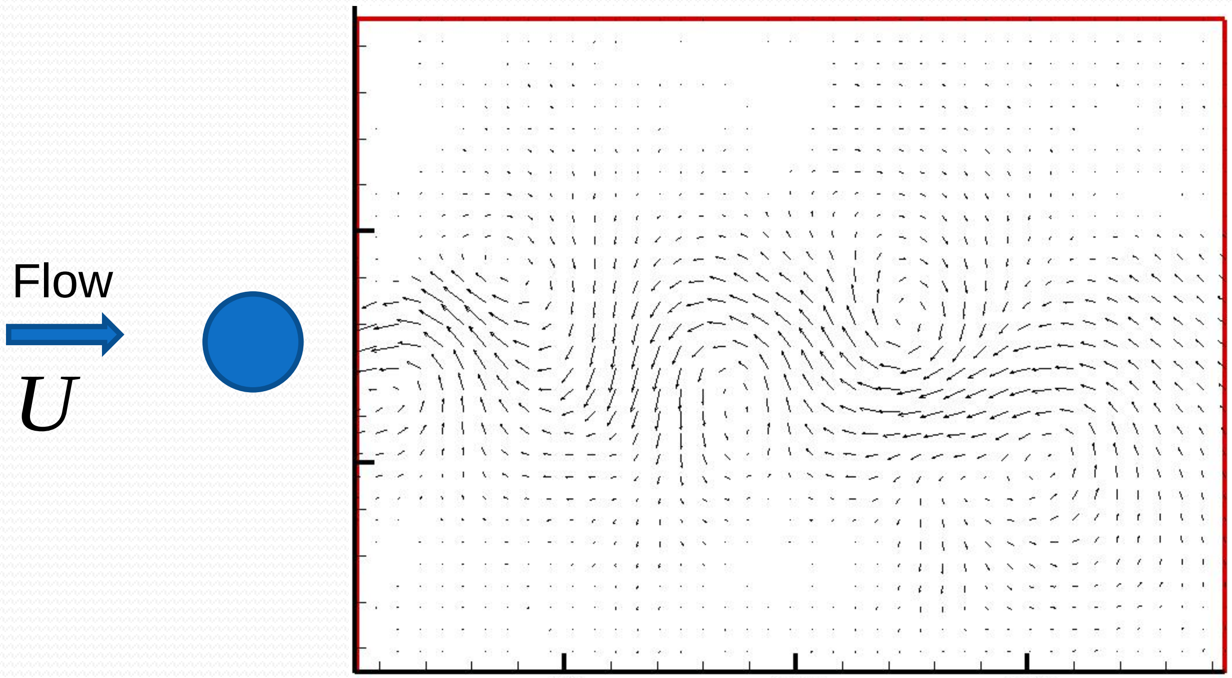
Mean motion and fluctuations in turbulent flows



Streamwise mean (time-averaged) velocity profile in the turbulent wake of a cylinder
(hot-wire technique ?)

Reynolds Decomposition and Averaging

Mean motion and fluctuations in turbulent flows

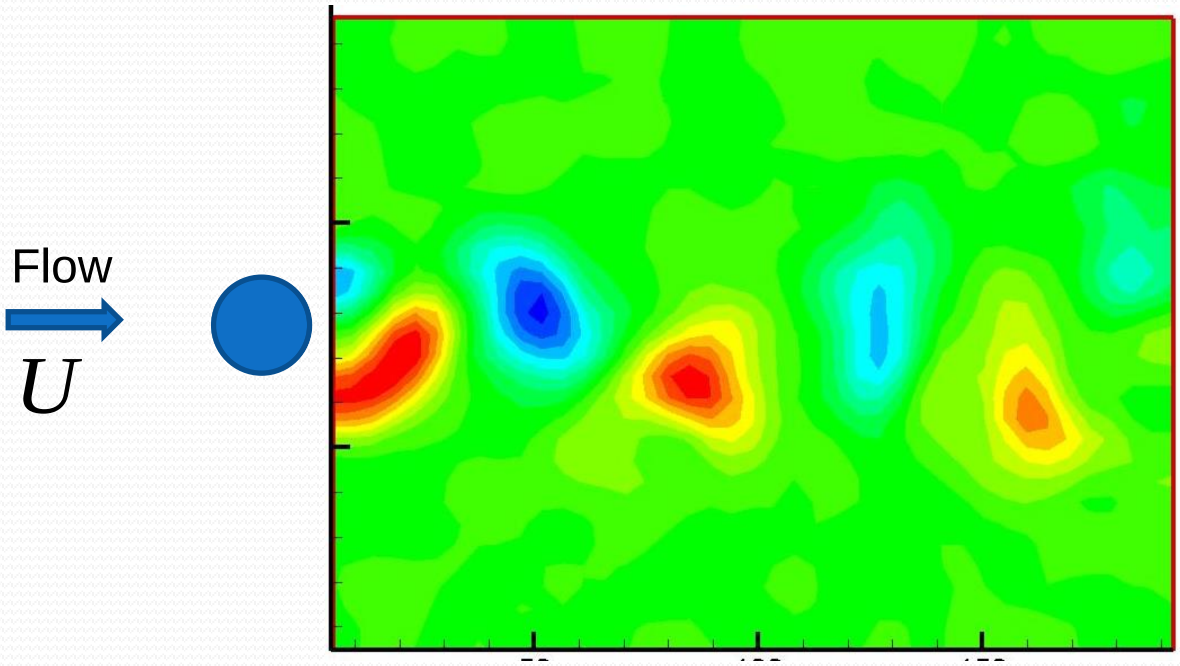


$$(u-U, v-V)$$

Instantaneous velocity vector distribution in the turbulent wake of a cylinder
（Particle Image Velocimetry-PIV）

Reynolds Decomposition and Averaging

Mean motion and fluctuations in turbulent flows



$$\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

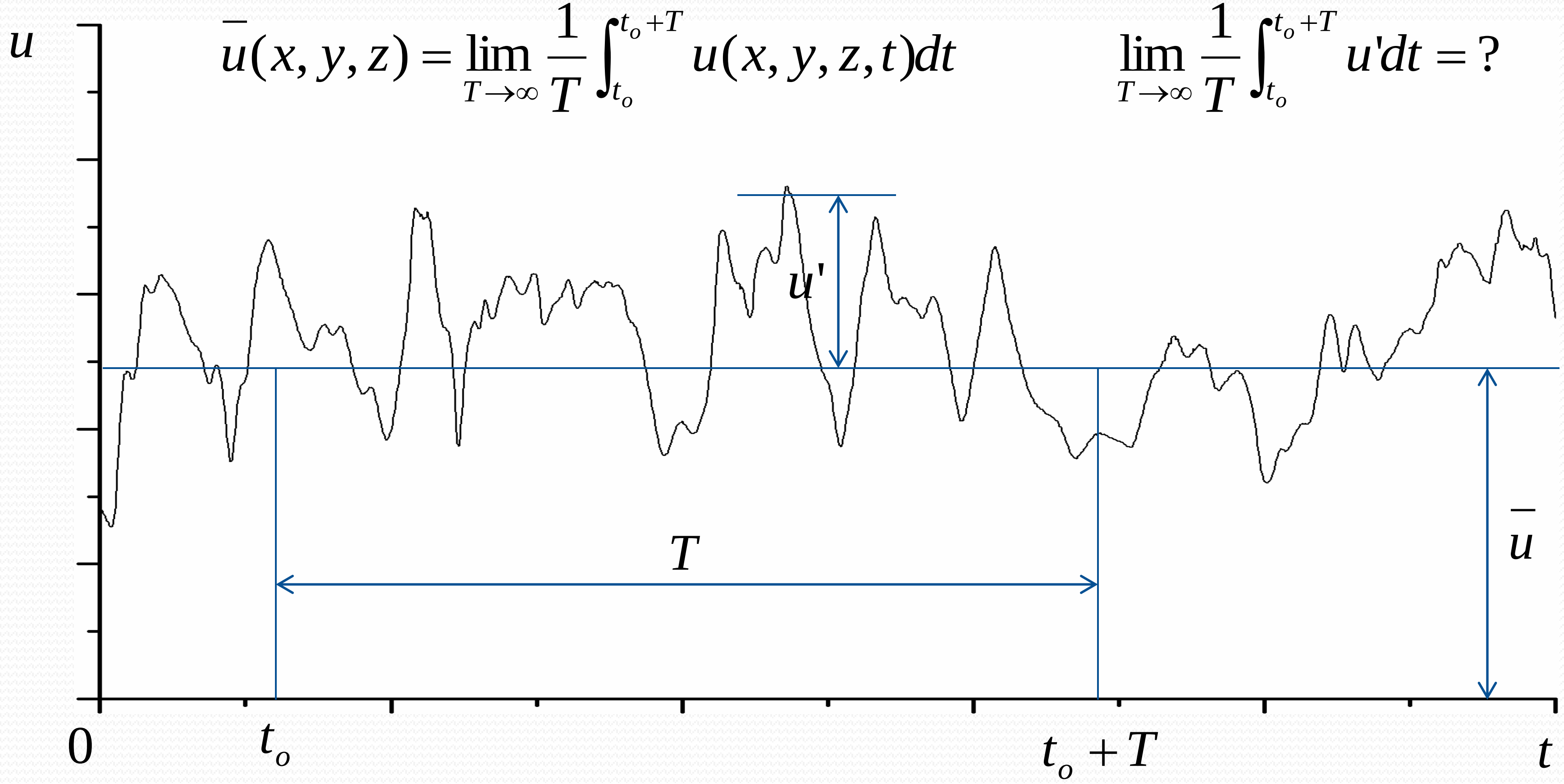
Instantaneous velocity vector distribution in the turbulent wake of a cylinder
（Particle Image Velocimetry-PIV）

Reynolds Decomposition and Averaging

Time Averaging

$$u = \bar{u} + u'$$

$$\bar{u}(x,y,z) = \lim_{T\to\infty} \frac{1}{T}\int_{t_o}^{t_o+T} u(x,y,z,t)\,dt \qquad \lim_{T\to\infty} \frac{1}{T}\int_{t_o}^{t_o+T} u'dt = ?$$



Reynolds Decomposition and Averaging

Time Averaging

$$u = \overline{u} + u'$$

$$v = \overline{v} + v'$$

$$w = \overline{w} + w'$$

$$p = \overline{p} + p'$$

$$\rho = \overline{\rho} + \rho'$$

$$\theta = \overline{\theta} + \theta'$$ Temperature

Reynolds Decomposition and Averaging

Time Averaging

Suppose *f* and *g* are two flow-field quantities (*u*, *v*, *w*, *p*, *density, temperature*, etc.), and *s* is an independent variable (*x*, *y*, *z*, *t*, etc.).

$$\overline{\overline{f}} = \overline{f} \qquad \overline{f'} = 0$$

$$\overline{f + g} = \overline{f} + \overline{g} \qquad \overline{\int f\, ds} = \int \overline{f}\, ds$$

$$\overline{\overline{f} \cdot g} = \overline{f} \cdot \overline{g} \qquad \overline{f \times g} = \overline{f} \times \overline{g} + \overline{f'g'}$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s} \qquad \overline{af} = a\overline{f}$$ *a* is constant

$$\overline{u}(x, y, z) = \lim_{T \to \infty} \frac{1}{T} \int_{t_o}^{t_o + T} u(x, y, z, t) dt$$

Reynolds Decomposition and Averaging

Continuity Equation

For incompressible flow, ρ is constant

$$\nabla \cdot \mathbf{V} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Take average of the left hand side and the right hand side we have:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0$$

Rewrite:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial \left(\bar{u} + u'\right)}{\partial x} + \frac{\partial \left(\bar{v} + v'\right)}{\partial y} + \frac{\partial \left(\bar{w} + w'\right)}{\partial z}$$

$$= \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Reynolds Decomposition and Averaging

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

Fluctuating

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Both the time averaging and fluctuating values of flow quantities satisfy the equation of continuity.

Reynolds Decomposition and Averaging

Momentum Equation (x component)

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Or

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\rho f_x-\frac{\partial p}{\partial x}+\mu\nabla^2 u$$

Decompose the velocity components and the pressure into their time average and fluctuation quantities, and then do the time average over the terms. Eventually, Reynolds equations can be formed as followed:

Reynolds Decomposition and Averaging

We firstly take the average of the left hand side term by term:

$$\overline{\frac{\partial u}{\partial t}} = \overline{\frac{\partial(\bar{u} + u')}{\partial t}} = \overline{\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t}} = \overline{\frac{\partial u'}{\partial t}} = \frac{\partial \overline{u'}}{\partial t} = 0$$

$$\overline{u\frac{\partial u}{\partial x}} = \overline{(\bar{u} + u')\frac{\partial(\bar{u} + u')}{\partial x}} = \overline{(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x})} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}}$$

$$\overline{v\frac{\partial u}{\partial y}} = \overline{(\bar{v} + v')\frac{\partial(\bar{u} + u')}{\partial y}} = \overline{(\bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \bar{u}}{\partial y} + v'\frac{\partial u'}{\partial y})} = \bar{v}\frac{\partial \bar{u}}{\partial y} + \overline{v'\frac{\partial u'}{\partial y}}$$

同样有： $$\overline{w\frac{\partial u}{\partial z}} == \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{w'\frac{\partial u'}{\partial z}}$$

Reynolds Decomposition and Averaging

Momentum Equation (x component)

We then take the average of the right hand side term by term:

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$-\overline{\frac{\partial p}{\partial x}}=-\overline{\frac{\partial(\overline{p}+p')}{\partial x}}=-\overline{\frac{\partial\overline{p}}{\partial x}+\frac{\partial p'}{\partial x}}=-\frac{\partial\overline{p}}{\partial x}$$

$$\overline{\frac{\partial^2 u}{\partial x^2}}=\overline{\frac{\partial^2(\overline{u}+u')}{\partial x^2}}=\overline{\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2 u'}{\partial x^2}}=\frac{\partial^2\overline{u}}{\partial x^2}$$

Reynolds Decomposition and Averaging

Momentum Equation (*x* component)

Now we have the average NS Equation:

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}+\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right]=-\frac{\partial \bar{p}}{\partial x}+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)$$

$$\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}$$

$$=\overline{\frac{\partial u'u'}{\partial x}-u'\frac{\partial u'}{\partial x}}+\overline{\frac{\partial v'u'}{\partial y}-u'\frac{\partial v'}{\partial y}}+\overline{\frac{\partial w'u'}{\partial z}-u'\frac{\partial w'}{\partial z}}$$

$$=\overline{\frac{\partial u'u'}{\partial x}}+\overline{\frac{\partial v'u'}{\partial y}}+\overline{\frac{\partial w'u'}{\partial z}}-\overline{u'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)}$$

$$\boxed{\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0}$$

Reynolds Decomposition and Averaging

Momentum Equation

By using the continuity equation for velocity fluctuations，

Finally we have Reynolds Equation :

$$r\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial x}+m\nabla^{2}\bar{u}-r\left(\frac{\partial \overline{u'^{2}}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

How about other two components?

$$r\left[\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial y}+m\nabla^{2}\bar{v}-r\left(\frac{\partial \overline{u'v'}}{\partial x}+\frac{\partial \overline{v'^{2}}}{\partial y}+\frac{\partial \overline{v'w'}}{\partial z}\right)$$

$$r\left[\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial z}+m\nabla^{2}\bar{w}-r\left(\frac{\partial \overline{u'w'}}{\partial x}+\frac{\partial \overline{v'w'}}{\partial y}+\frac{\partial \overline{w'^{2}}}{\partial z}\right)$$

涡粘性 Eddy viscosity

Questions:

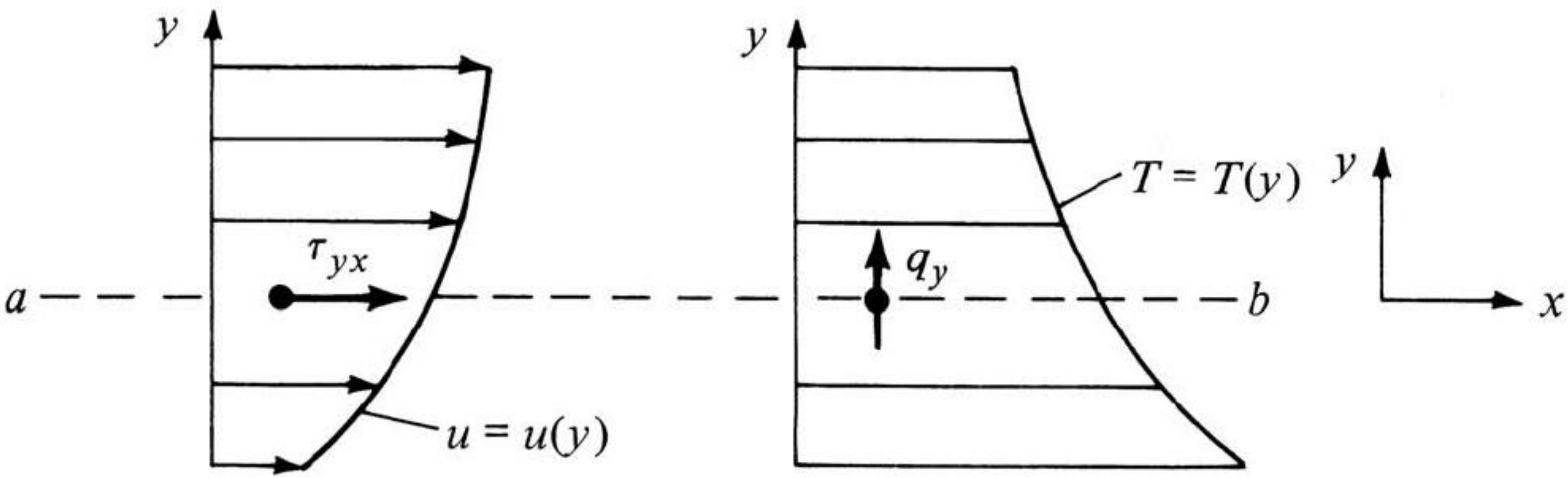
1. Why does the term $\frac{\partial u}{\partial t}$ disappear?
2. What is the physical meaning of the last term on the right-hand side?

$$r\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial x}+m\nabla^2\bar{u}-r\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\begin{aligned}
&m\nabla^2\bar{u}-r\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)\\
&=m\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)-r\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)\\
&=\frac{\partial}{\partial x}\left(m\frac{\partial \bar{u}}{\partial x}-r\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(m\frac{\partial \bar{u}}{\partial y}-r\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(m\frac{\partial \bar{u}}{\partial z}-r\overline{u'w'}\right)
\end{aligned}$$

Momentum transfer

Heat transfer



$$\frac{momentum}{unit\ area \times unit\ time} = \tau_{yx} = \frac{mv}{At} = \mu\frac{\partial u}{\partial y} \qquad \frac{heat}{unit\ area \times unit\ time} = \dot{q}_y = -k\frac{\partial T}{\partial y}$$

μ is the molecular viscosity coefficient, $\boldsymbol{k}$ is the thermal conductivity

How about mass transfer?

湍流粘性和湍流热传导率

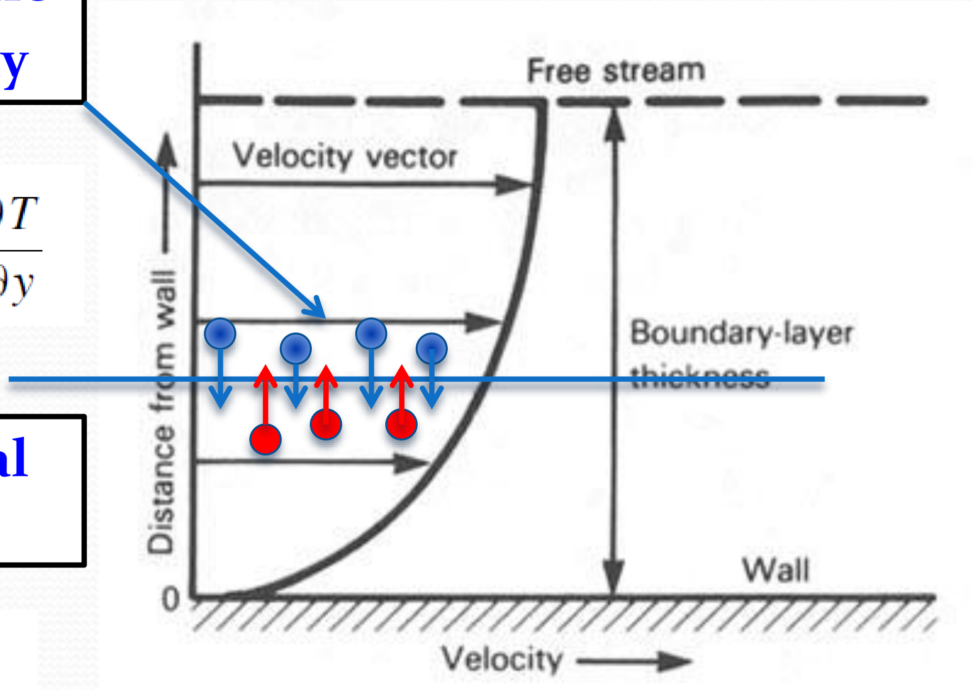
对于湍流，湍流涡或流体微团会产生动量和能量的传输，可以将其简化为涡粘度 ε 和涡热传导率 κ，则应力和应变关系及热传导和温度梯度的关系为，

Laminar: molecule
Turbulence: eddy

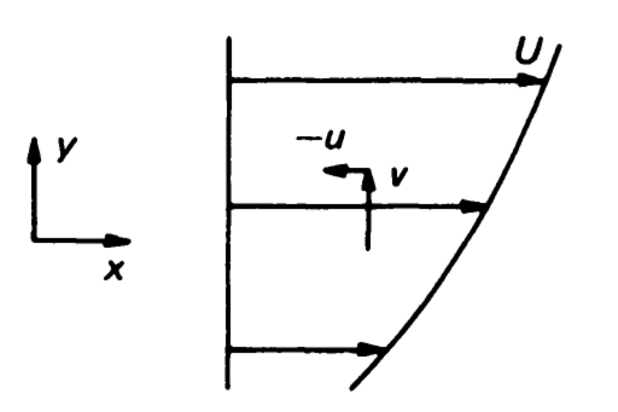
$$\tau_{yx} = (\mu + \varepsilon)\,\frac{\partial u}{\partial y} \qquad \dot{q}_y = -(k + \kappa)\frac{\partial T}{\partial y}$$

Eddy viscosity , Eddy thermal conductivity

$$\varepsilon = \rho l^2 \left|\frac{\partial u}{\partial y}\right| \qquad \kappa = \varepsilon c_p$$



涡粘湍流模型 Eddy viscosity



$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'}\right)$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y}+\mu_T\frac{\partial \bar{u}}{\partial y}\right)$$

$$where \quad -\rho\overline{u'v'}=\mu_T\frac{\partial \bar{u}}{\partial y}$$

$$-\overline{u'v'}=\nu_T\frac{\partial \bar{u}}{\partial y}$$

如何测量？

Recall microscopic picture of molecular viscosity, think about the analogy.

Thermal/Brownian motion vs. turbulent fluctuation

雷诺应力张量 Reynolds Stresses

$$-\rho\begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix} = \begin{pmatrix} \tau'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{xy} & \tau'_{yy} & \tau'_{yz} \\ \tau'_{xz} & \tau'_{yz} & \tau'_{zz} \end{pmatrix}$$

Reynolds Decomposition and Averaging

N-S equation: $$r\left[\frac{\partial u}{\partial t}+\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\right]) = -\frac{\partial p}{\partial x}+m\nabla^2 u$$

Reynolds averaged N-S equation:

$$r\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial x}+m\nabla^2\bar{u} - r\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

- The equations for the mean velocity field depend on the the correlations of the velocity fluctuations.
- This is a manifestation of the closure problem: the equation for a set of statistics contain additional (higher-order) statistics to those in the set considered.
- Such a set of equations---with more unknowns than equations--- is said to be unclosed. It cannot be solved in the absence of separate information to the determine the additional statistics.

中国湍流研究的发展史

I 中国科学家早期湍流研究的回顾

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周培源先生本人的湍流研究工作是 1938 年在昆明西南联大正式展开的。当时跟随他一起作湍流研究的学生有林家翘先生和郭永怀先生。他最早的湍流论文“论雷诺求似应力的方法的推广和湍流的性质”发表于 1940 年的中国物理学报上[9]。他在这篇文章中首次提出除了雷诺平均运动方程以外，还要研究湍流脉动运动方程，并给出了雷诺应力所满足的动力学方程，从而构成了湍流模式理论的奠基性工作。他的这篇文章的另一个贡献是首次提出了四元速度关联用二元速度关联表示的一个假设，与此后不久前苏联科学家密林奥希可夫提出的假设类似。国际上很久不知道周培源先生的工作，直到九十年代经 Lumley 指明，才将首创权归于周培源先生[2]。

Annu. Rev. Fluid Mech. 1995. 27: 1–15

50 YEARS OF TURBULENCE RESEARCH IN CHINA

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INTRODUCTION by J. L. Lumley

周培源

（1902-1993）

求出了雷诺应力等所满足的微分方程，并希望能把边界的影响通过边界条件引入雷诺应力的运算式中，在国际上第一次提出湍流脉动方程。

In this generation there were at least four giants in fluid mechanics from four countries; men who had, each in his own way, tremendous influence inside and outside his country, certainly by contributing to the development of areas of fluid mechanics, but also by providing intellectual and personal leadership; men of such stature that substantial numbers of distinguished workers in fluid mechanics in each country could trace their academic lineage to that great man. I have in mind von Karman in the United States, Kolmogorov in the (then) Soviet Union, G. I. Taylor in the United Kingdom, and Pei-Yuan Chou of China.

(北京大学原校长周培源)

第三节：湍流模型

19.3 TURBULENCE MODELING

不可压缩流动 Reynolds averaged N-S equation RANS:

$$r\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial x}+m\nabla^2\bar{u}-r\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

- It is in the class of what is called an “eddy viscosity” model, where the effects of turbulence in the governing viscous flow equations are included simply by adding an additional term to the transport coefficients.
- In all our previous viscous flow equations, μ is replaced by $(\mu + \mu_T)$ and k by $(k + k_T)$ where μ_T and k_T are the eddy viscosity and eddy thermal conductivity, both due to turbulence.
- In these expressions, μ and k are denoted as the “molecular” viscosity and thermal conductivity, respectively.

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial x}+(\mu+\mu_T)\nabla^2\bar{u}$$

湍流模型

- 求解雷诺平均N-S（RANS）方程（连续方程、动量方程和能量方程）时需要附加湍流模型
- 代数模型：Baldwin-Lomax湍流模型（B-L）
- 涡粘性模型：涡粘性和涡热传导率
- 湍流模型分类：

  代数模型（零方程模型）：B-L，C-S模型

  一方程模型：S-A模型

  两方程模型：K-ε，k−ω模型

Baldwin-Lomax 湍流模型

- Baldwin-Lomax model assumes that the turbulent-boundary layer is split into two layers, an inner and an outer layer, with different expressions for μ_T in each layer:

$$\mu_T = \begin{cases} (\mu_T)_{\text{inner}} & y \le y_{\text{crossover}} \\ (\mu_T)_{\text{outer}} & y \ge y_{\text{crossover}} \end{cases}$$

- Where y is the local normal distance from the wall, and the crossover point from the inner to the outer layer is denoted by $y_{crossover}$.
- By definition, $y_{crossover}$ is that point in the turbulent boundary where $(\mu_T)_{outer}$ becomes less than $(\mu_T)_{inner}$.

For the inner region:

$$(\mu_T)_{\text{inner}} = \rho l^2 |\omega|$$

$$l = ky\left[1 - \exp\left(\frac{-y^+}{A^+}\right)\right]$$

$$y^+ = \frac{\sqrt{\rho_w \tau_w}\, y}{\mu_w}$$

ω is the local vorticity, defined for a two dimensional flow as

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

For the outer region:

$$(\mu_T)_{\text{outer}} = \rho K C_{\text{cp}} F_{\text{wake}} F_{\text{Kleb}}$$

where K and C_{cp} are two additional constants

其中:

$$f(y) = y|\omega|\left[1 - \exp\left(\frac{-y^+}{A^+}\right)\right]$$

$$F_{\text{wake}} = \min\left(y_{\max} F_{\max},\ C_{\text{wk}} y_{\max} U_{\text{dif}}^2 / F_{\max}\right)$$

$$C_{\text{wk}} \text{ is constant}, \quad U_{\text{dif}} = \sqrt{u^2 + v^2}$$

$$F_{\text{Kleb}}(y) = \left[1 + 5.5\left(C_{\text{Kleb}} \frac{y}{y_{\max}}\right)^6\right]^{-1}$$

$A^+ = 26.0$, $C_{cp} = 1.6$, $C_{\text{Kleb}} = 0.3$,
$C_{\text{wk}} = 0.25$, $k = 0.4$, and $K = 0.0168$.

- Note that, like all eddy-viscosity turbulent models, the value of μ_T obtained above is dependent on the flow field properties themselves (for example ω and ρ);
- This is in contrast to the molecular viscosity μ, which is solely a property of the gas itself.
- The molecular values of viscosity coefficient and thermal conductivity are related through the Prandtl number

$$k = \frac{\mu c_p}{\mathrm{Pr}} \qquad k_T = \frac{\mu_T c_p}{\mathrm{Pr}_T}$$

- In lieu of developing a detailed turbulence model for the turbulent thermal conductivity k_T , the usual procedure is to define a “turbulent” Prandtl number， $Pr_T = 1$

Spalart-Allmaras湍流模型（一方程模型）

一方程模式需要求解一个偏微分方程，**Spalart-Allmaras**模型是从经验和量纲分析出发，在伽利略（**Galilean**）不变性原理和分子粘性选择性相关方法的基础上“拼凑”出来的。这种“拼凑”虽然缺乏完备的理论基础，但是却包含了丰富的经验信息。**S-A**模型具有良好的鲁棒性和数值收敛性，它可以很好地模拟绝大部分的附着流动和薄层自由剪切流动。

在**S-A**模型中，湍流粘性系数定义为: $\mu_T = \rho \hat{\upsilon} f_{\upsilon 1}$

其中: $f_{v1} = \dfrac{\chi^3}{\chi^3 + C_{v1}^3}, \chi = \dfrac{\hat{\upsilon}}{\upsilon}, \upsilon = \dfrac{\mu}{\rho}, C_{v1} = 7.1$

参考文献：Spalart P R and Allmaras S R. A One-Equation Turbulence Model for Aerodynamic Flows. AIAA Paper 92-0439, 1992.

是计算湍流粘性系数的工作变量，它满足下面的传输方程：

$$\frac{D\hat{\upsilon}}{Dt}=b_{dif}(\hat{\upsilon})+b_{prod}(S,\hat{\upsilon},d)-b_{dest}(\hat{\upsilon},d)+b_{trip}(d_T)$$

扩散项： $$b_{dif}(\hat{\upsilon})=\frac{1}{\sigma}[\nabla\cdot((\upsilon+\hat{\upsilon})\nabla\hat{\upsilon})+C_{b2}(\nabla\hat{\upsilon})^2]$$

生成项： $$b_{prod}(\hat{\upsilon})=C_{b1}[1-f_{t2}(\chi)]\hat{S}\hat{\upsilon}$$

破坏项： $$b_{dest}(\hat{\upsilon})=\left[C_{w1}f_w(r)-\frac{C_{b1}}{\kappa^2}f_{t2}(\chi)\right](\frac{\hat{\upsilon}}{d})^2$$

移动项： $$b_{trip}=f_{t1}(\chi)\Delta u^2$$

传输方程可以写成：

$$\frac{\mathrm{D}\hat{\upsilon}}{\mathrm{D}t}=C_{b1}(1-f_{t2})\hat{S}\hat{\upsilon}+\frac{1}{\sigma}\nabla\cdot\left[(\upsilon+(1+C_{b2})\hat{\upsilon})\nabla\hat{\upsilon}\right]-\frac{C_{b2}}{\sigma}\hat{\upsilon}\nabla^{2}\hat{\upsilon}-(C_{w1}f_{w}-\frac{C_{b1}}{\kappa^{2}}f_{t2})(\frac{\hat{\upsilon}}{d})^{2}$$

以上各式中的系数取定方法如下：

$$\hat{S}=|\boldsymbol{\omega}|+\frac{\hat{\upsilon}}{\kappa^{2}d^{2}}f_{v2}(\chi)$$

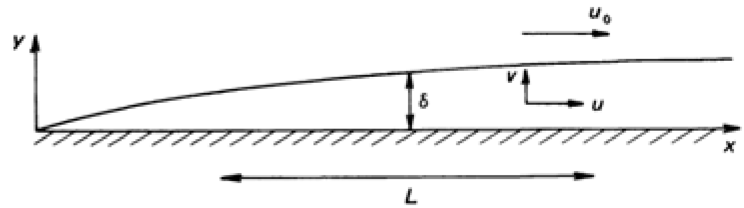
$C_{b1}=0.1355$ ， $\sigma=\frac{2}{3}$ ， $C_{b2}=0.622$ ， $C_{v1}=7.1$ ，

$C_{t3}=1.2$ $C_{t4}=0.5$ ， $\kappa=0.41$ ， $C_{w2}=0.3$ ， $C_{w3}=2.0$

第四节：平板湍流边界层

19.2 RESULTS FOR TURBULENT BOUNDARY LAYERS ON A FLAT PLATE

Averaged Equation for turbulent boundary layer



Boundary layer on flat wall of channel: definition sketch.

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu + \mu_T\right)\frac{\partial \overline{u}}{\partial y}$$

$$\frac{\partial \overline{p}}{\partial y} = 0$$

Reynolds stress $r\overline{u'v'}$

$$-\rho\overline{u'v'} = \mu_T \frac{\partial \overline{u}}{\partial y}$$

湍流平板边界层

- We discuss a few results for the turbulent boundary layer on a flat plate, incompressible and compressible
- To provide a basis of comparison with the laminar results described in the previous section.

Incompressible Turbulent

the boundary-layer thickness：

$$\boxed{\delta = \frac{0.37x}{\mathrm{Re}_x^{1/5}}}$$

比较厚！

$$\delta \propto x^{4/5}$$

Incompressible Laminar

the boundary-layer thickness：

$$\boxed{\delta = \frac{5.0x}{\sqrt{\mathrm{Re}_x}}}$$

比较薄！

$$\delta \propto x^{1/2}$$

壁面摩擦阻力系数比较 $C_f = \dfrac{D_f}{\frac{1}{2}\rho u_e^2 S}$

Incompressible Turbulent

Skin friction drag：

$$\boxed{C_f = \frac{0.074}{\mathrm{Re}_c^{1/5}}}$$

比较大！

Incompressible Laminar

Skin friction drag：

$$\boxed{C_f = \frac{1.328}{\sqrt{\mathrm{Re}_c}}}$$

比较小！

- 经典的对数坐标（log-log）图
- 湍流摩擦阻力系数高于层流
- 湍流C_f - Re曲线斜率小于层流
- Re 相同，M_∞增加，C_f减小；M_∞增加从0到10，湍流C_f减小一个量级

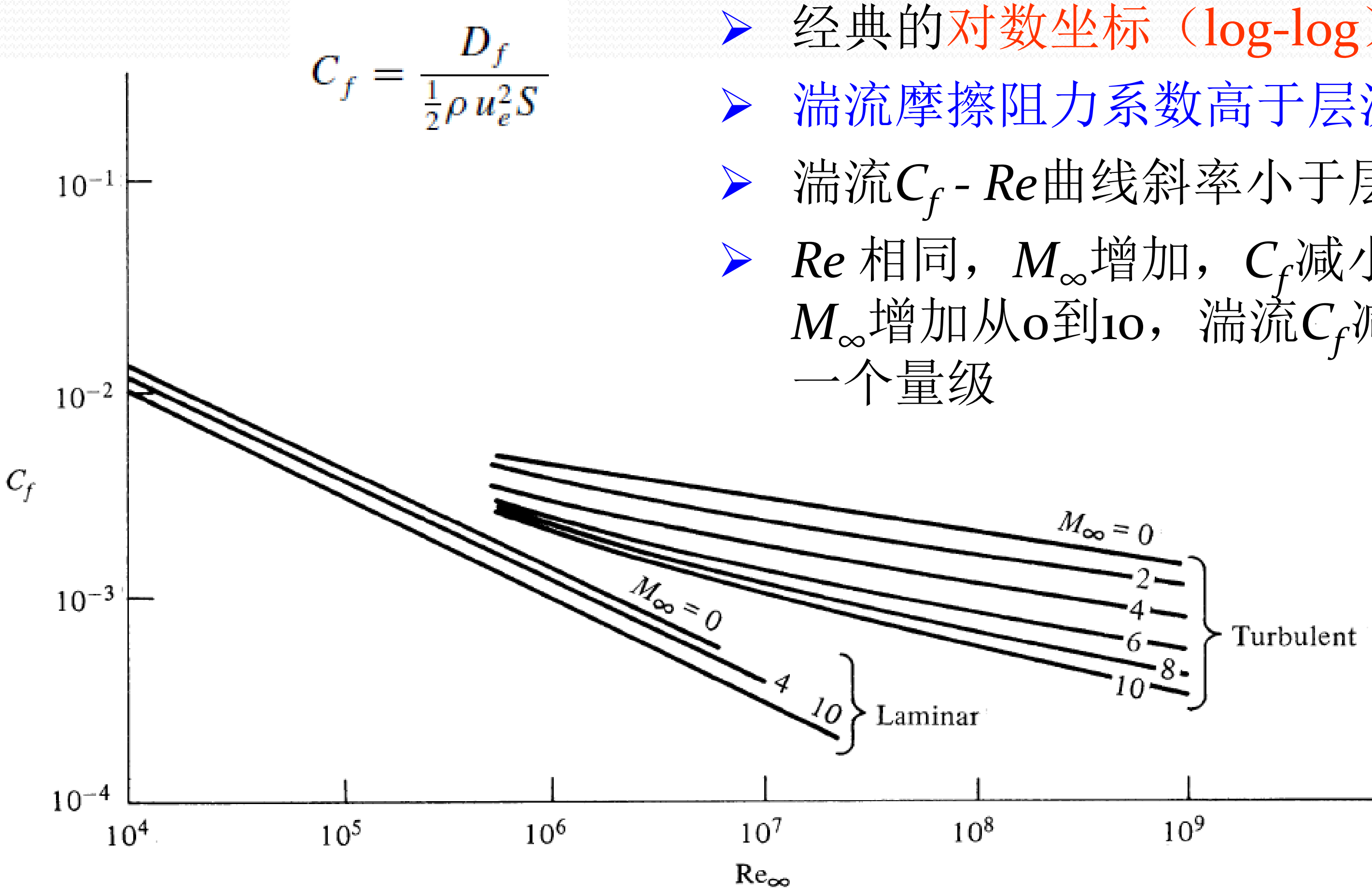


Figure 19.1 Turbulent friction drag coefficient for a flat plate as a function of Reynolds and Mach numbers. Adiabatic wall Pr = 0.75. For contrast, some laminar results are shown. (*Data are from the calculations of van Driest, Reference 47.*)

湍流参考温度方法

层流边界层的参考温度方法可以应用到湍流边界层，

参考温度表达式：

$$\frac{T^*}{T_e} = 1 + 0.032 M_e^2 + 0.58 \left(\frac{T_w}{T_e} - 1 \right)$$

the incompressible turbulent flat plate result for C_f can be modified for compressible turbulent flow as：

$$C_f^* = \frac{0.074}{(\mathrm{Re}_c^*)^{1/5}} \qquad \mathrm{Re}_x^* = \frac{\rho^* u_e x}{\mu^*}$$

$$C_f^* = \frac{D_f}{\frac{1}{2}\rho^* u_e^2 S} \qquad c_f^* = \frac{\tau_w}{\frac{1}{2}\rho^* u_e^2}$$

实例分析：

EXAMPLE 19.2

Consider a flat plate at zero angle of attack in an airflow at standard sea level conditions ($p_\infty = 1.01 \times 10^5$ N/m^2 and $T_\infty = 288$ K). The chord length of the plate (distance from the leading edge to the trailing edge) is 2 m. The planform area of the plate is 40 m^2. At standard sea level conditions, $\mu_\infty = 1.7894 \times 10^{-5}$ kg/(m)(s). Assume the wall temperature is the adiabatic wall temperature T_{aw}. Calculate the friction drag on the plate when the freestream velocity is (*a*) 100 m/s, (*b*) 1000 m/s.

Consider the same flat plate under the same external flow conditions given in Example 18.1. Calculate the friction drag on the plate assuming a *turbulent* boundary layer for a freestream velocity of (*a*) 100 m/s, and (*b*) 1000 m/s.

（a）不可压缩湍流平板边界层

(*a*) For $V_\infty = 100\,\text{m/s}$ $\quad \text{Re}_c = 1.36 \times 10^7$

$$C_f = \frac{0.074}{(\text{Re}_c)^{1/5}} = \frac{0.074}{(1.36 \times 10^7)^{1/5}} = \frac{0.074}{26.71} = 2.77 \times 10^{-3}$$

Also from Example 18.1, we have $\rho_\infty = 1.22\ \text{kg/m}^3$ and $S = 40\ \text{m}^2$. Hence, for one side of the plate,

$$D_f = \tfrac{1}{2}\rho_\infty V_\infty^2 S C_f = \tfrac{1}{2}(1.22)(100)^2(40)(2.77 \times 10^{-3}) = 675.9\ \text{N}$$

The total friction drag taking into account both sides of the plate is

$$D = 2D_f = 2(675.9) = \boxed{1352\ \text{N}}$$

Comparing this result for turbulent flow with the laminar result in Example 1.81*a*, we have

$$\frac{D_{\text{turbulent}}}{D_{\text{laminar}}} = \frac{1352}{175.6} = 7.7$$

（b）可压缩湍流平板边界层

(b) For $V_\infty = 1000$ m/s, $\mathrm{Re}_c = 1.36 \times 10^8$, and $M_\infty = 2.94$.

From Figure 19.1, we have

$$C_f = 1.34 \times 10^{-3}$$

Hence, $D_f = \frac{1}{2}\rho_\infty V_\infty^2 S C_f = \frac{1}{2}(1.22)(1000)^2(40)(1.34 \times 10^{-3}) = 32{,}700 \text{ N}.$

The total friction drag is

$$D = 2(32{,}700) = \boxed{65{,}400 \text{ N}}$$

Again, comparing this result with that from Example 18.1b, we have

$$\frac{D_{\text{turbulent}}}{D_{\text{laminar}}} = \frac{65{,}400}{5026} = 13$$

实例分析：

EXAMPLE 19.2

Repeat Example 19.1*b*, except using the reference temperature method. Assume the plate has an adiabatic wall.

$$\mathrm{Re}_c^* = 3.754 \times 10^7 \quad \text{and} \quad \rho^* = 0.574 \text{ kg/m}^3$$

From Equation (19.3) we have

$$C_f^* = \frac{0.074}{(\mathrm{Re}_c^*)^{1/5}} = \frac{0.074}{(3.754 \times 10^7)^{1/5}} = 2.26 \times 10^{-3}$$

From Equation (19.4),

$$D_f = \tfrac{1}{2}\rho^* u_e^2 S C_f^* = \tfrac{1}{2}(0.574)(1000)^2(40)(2.26 \times 10^{-3}) = 25{,}945 \text{ N}$$

Hence, $\boxed{D = 2(25{,}945) = 51{,}890 \text{ N}}$ $\boxed{65{,}400 \text{ N}}$

Comparing this answer with that obtained in Example 19.1*b*, we find a 20 percent discrepancy between the two methods of calculations. This is not surprising. It simply points out the great uncertainty in making calculations of turbulent skin friction.

参考温度方法修正：

The Meador-Smart Reference Temperature Method for Turbulent

- reference temperature equation for turbulent flow slightly different than that for laminar flow. For a turbulent flow, their equation is： 对参考温度的计算方法进行修正：

$$\frac{T^*}{T_e} = 0.5\left(1 + \frac{T_w}{T_e}\right) + 0.16\,r\left(\frac{\gamma - 1}{2}\right)M_e^2$$

原来的参考温度计算公式：
$$\frac{T^*}{T_e} = 1 + 0.032M_e^2 + 0.58\left(\frac{T_w}{T_e} - 1\right)$$

摩擦系数与阻力系数修正计算公式：

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = \frac{0.02296}{(\mathrm{Re}_x)^{0.139}} \qquad C_f = \frac{D_f}{\frac{1}{2}\rho_\infty V_\infty^2 S} = \frac{0.02667}{(\mathrm{Re}_c)^{0.139}}$$

实例分析：

EXAMPLE 19.3

Repeat Example 9.2 using the Meador-Smart reference temperature method.

For turbulent flow, the recovery factor is approximately

$$r = \mathrm{Pr}^{1/3} = (0.71)^{1/3} = 0.892$$

$$\frac{T_{aw}}{T_e} = 1 + r\left(\frac{T_0}{T_e} - 1\right)$$

For $M_e = 2.94$,

$$\frac{T_0}{T_e} = 2.74 \qquad \frac{T_{aw}}{T_r} = 1 + 0.892(1.74) = 2.55$$

The Meador-Smith equation then becomes

$$\frac{T^*}{T_e} = 0.5\left(1 + \frac{T_w}{T_e}\right) + 0.16r\left(\frac{\gamma - 1}{2}\right)M_e^2 = 2.02$$

$$T^* = 2.02\,T_e = 2.02\,(288) = 581.8\text{ K}$$

$$\rho^* = \frac{p}{RT^*} = \frac{1.01 \times 10^5}{(287)(581.8)} = 0.605\text{ kg/m}^3$$

$$\frac{\mu^*}{\mu_0} = \left(\frac{T^*}{T_0}\right)^{3/2} \frac{T_0 + 110}{T^* + 110} = \left(\frac{581.8}{288}\right)^{3/2} \frac{398}{691.8} = 1.651$$

$$\mathrm{Re}_c^* = \frac{\rho^* u_e c}{\mu^*} = \frac{(0.605)(1000)(2)}{2.95 \times 10^{-5}} = 4.1 \times 10^7$$

Meador-Smith choice of the turbulent skin-friction coefficient

$$C_f^* = \frac{0.02667}{(\mathrm{Re}_c^*)^{0.139}} = \frac{0.02667}{(4.1 \times 10^7)^{0.139}} = 2.32 \times 10^{-3}$$

$$D_f = \tfrac{1}{2}\rho^* V_\infty^2 S C_f^* = \tfrac{1}{2}(0.605)(1000)^2(40)(2.32 \times 10^{-3}) = 28070\text{ N}$$

$$\text{Total drag} = D = 2D_f = 2(28070) = \boxed{56140\text{ N}} \qquad \boxed{65{,}400\text{ N}}$$

Note: This result is more accurate it shows a 14 percent discrepancy compared with the result obtained in Example 19.1*b*.

第五节：小结

19.4 FINAL COMMENTS

1、Reynolds averaged N-S equation RANS（不可压缩）

Continuity Equation

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

Momentum Equations：

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial x}+\mu\nabla^{2}\bar{u}-\rho\left(\frac{\partial \overline{u'^{2}}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\rho\left[\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial y}+\mu\nabla^{2}\bar{v}-\rho\left(\frac{\partial \overline{u'v'}}{\partial x}+\frac{\partial \overline{v'^{2}}}{\partial y}+\frac{\partial \overline{v'w'}}{\partial z}\right)$$

$$\rho\left[\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right]=-\frac{\partial \bar{p}}{\partial z}+\mu\nabla^{2}\bar{w}-\rho\left(\frac{\partial \overline{u'w'}}{\partial x}+\frac{\partial \overline{v'w'}}{\partial y}+\frac{\partial \overline{w'^{2}}}{\partial z}\right)$$

2、涡粘湍流模型

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

$$\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} + \mu_T\frac{\partial \bar{u}}{\partial y}\right)$$

$$where \quad -\rho\overline{u'v'} = \mu_T\frac{\partial \bar{u}}{\partial y}$$

$$-\overline{u'v'} = \nu_T\frac{\partial \bar{u}}{\partial y}$$

$$-\rho\begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix} = \begin{pmatrix} \tau'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{xy} & \tau'_{yy} & \tau'_{yz} \\ \tau'_{xz} & \tau'_{yz} & \tau'_{zz} \end{pmatrix}$$

When the continuity, momentum, and energy equations are used to solve a turbulent flow, some type of turbulence model must be used. In the eddy viscosity concept, the viscosity coefficient and thermal conductivity in these equations must be the *sum* of the molecular and turbulent values.

3、Baldwin-Lomax 湍流模型

Baldwin-Lomax model assumes that the turbulent-boundary layer is split into two layers

$$\mu_T = \begin{cases} (\mu_T)_{\text{inner}} & y \leq y_{\text{crossover}} \\ (\mu_T)_{\text{outer}} & y \geq y_{\text{crossover}} \end{cases}$$

$$(\mu_T)_{\text{inner}} = \rho l^2 |\omega|$$

$$(\mu_T)_{\text{outer}} = \rho K C_{\text{cp}} F_{\text{wake}} F_{\text{Kleb}}$$

$$f(y) = y|\omega| \left[1 - \exp\left(\frac{-y^+}{A^+}\right)\right]$$

4、平板边界层

氢气泡技术 by the hydrogen bubble technique

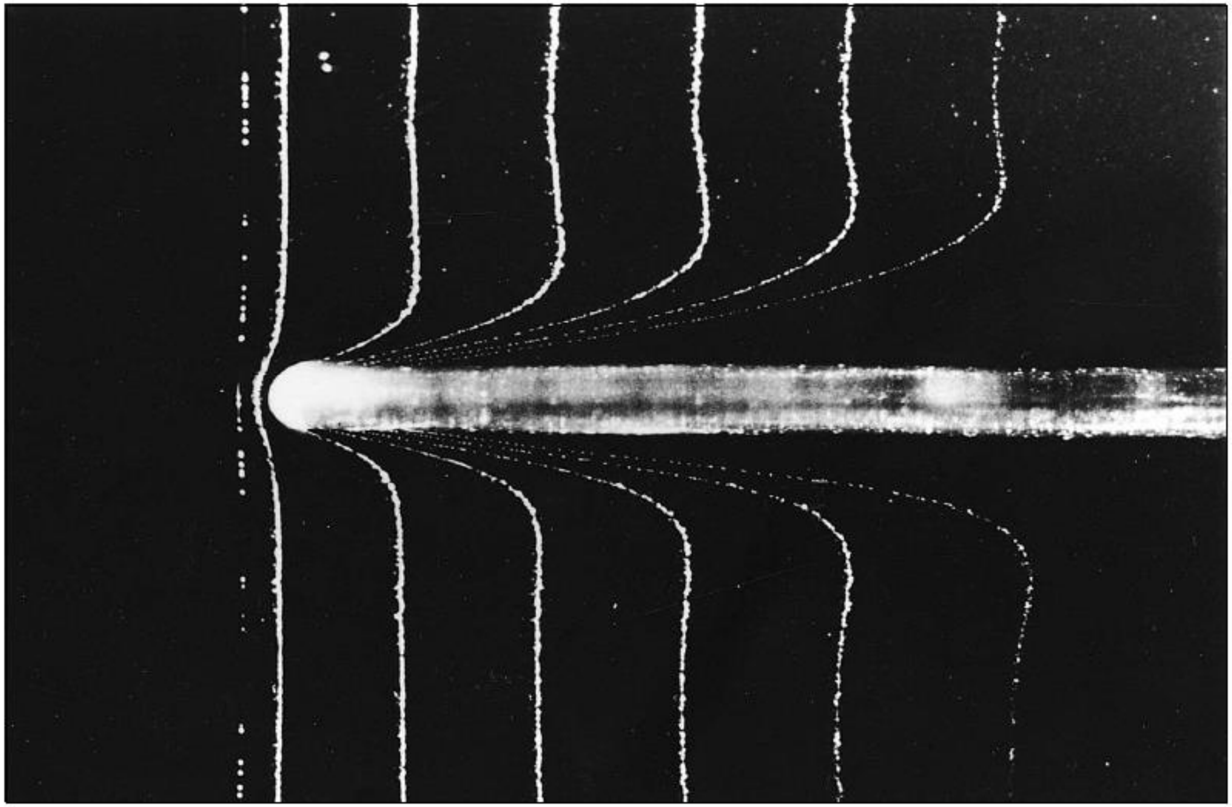


Figure 19.2 Photograph of velocity profiles for the laminar flow over a flat plate. Flow is from left to right. (*Courtesy of Yasuki Nakayama, Tokai University, Japan.*)

湍流边界层方程：

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0$$

Reynolds stress

$$r\,\overline{u'v'}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu+\mu_T\right)\frac{\partial \bar{u}}{\partial y}$$

$$-\rho\overline{u'v'}=\mu_T\frac{\partial \bar{u}}{\partial y}$$

$$\frac{\partial \bar{p}}{\partial y}=0$$

Approximations for the turbulent, incompressible flow over a flat plate are

$$\delta=\frac{0.37x}{\mathrm{Re}_x^{1/5}} \tag{19.1}$$

$$C_f=\frac{0.074}{\mathrm{Re}_c^{1/5}} \tag{19.2}$$

To account for compressibility effects, the data shown in Figure 19.1 can be used, or alternatively the reference temperature method can be employed.