FEM: Theory and Applications (SE 291)
Assignment 01

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For c-code details, Please go-through 'README.txt' in c-program folder 'fem1d'. For flexiblity in ploting, I have also added fem1.m matlab code.

Problem 1

Strong Problem : Find $u(x)$ for BVP.

$$\left.\begin{aligned} -\epsilon u_{xx} + u_x &= 0 \quad \forall\ x \in (a,b) \\ u(a) &= 1 \\ u(b) &= 0 \end{aligned}\right\} \quad \text{— I}$$

Analytical Solution :

put $y = u_x$

$$-\epsilon y_x + y = 0$$

$$\Rightarrow \quad y = c\, e^{x/\epsilon} = u_x$$

$$\Rightarrow \quad u = c_1 + c_2 e^{x/\epsilon} \quad \text{—— } (*)$$

From B.C.s

@ $x = a$, $\quad c_1 + c_2 e^{a/\epsilon} = 1$

@ $x = b$, $\quad c_1 + c_2 e^{b/\epsilon} = 0$

$$\Rightarrow \quad c_1 = + \frac{e^{b/\epsilon}}{e^{b/\epsilon} - e^{a/\epsilon}}, \quad c_2 = \frac{-1}{e^{b/\epsilon} - e^{a/\epsilon}}$$

Hence from $(*)$,

$$\boxed{u = \frac{e^{b/\epsilon} - e^{x/\epsilon}}{e^{b/\epsilon} - e^{a/\epsilon}}}$$

FEM Approach

Step 1. Weak formulation :

let $\quad V := \{ v(x) \in H^1 : v(a) = v(b) = 0 \}$

$\quad U := \{ v(x) \in H^1 : v(a) = 1,\ v(b) = 0 \}$

- Find $u \in U$ s.t.

$$\int_a^b \left(-\epsilon \frac{d^2 u}{dx^2} + \frac{du}{dx} \right) v\, dx = 0 \quad \forall\ v \in V$$

After Integration by part for 1st term

$$B(u,v) = \int_a^b +\epsilon \left(\frac{du}{dx} \cdot \frac{dv}{dx} \right) \cdot dx + \int_a^b \left(\frac{du}{dx} \cdot v \right) dx$$

$$f(v) = 0$$

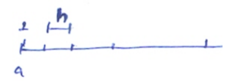
Hence, Weak Problem

Find $u \in U$ s.t.

$$B(u, v) = 0 \quad \forall \; v \in V$$

Here,
$$\left. B(u,v) = \epsilon \int_a^b \left(\frac{du}{dx} \cdot \frac{dv}{dx}\right) dx + \int_a^b \left(\frac{du}{dx} \cdot v\right) dx \right\} -$$

Step 2. Discretization and Basis function



Let $S^h := \text{span}\{\phi_i\} \subset H^1(x)$

$\Rightarrow$ $V^h := \{ v^h(x) \in S^h : v^h(a) = v^h(b) = 0 \}$
$U^h := \{ u^h(x) \in S^h : u^h(a) = 1, \; u^h(b) = 0 \}$

So, from II, we have to find $u^h(x) \in U^h$ (discreate domi

s.t. $B(u^h, v^h) = 0$ ———— (**)

Here,
$$u^h = \sum_{j=1}^{N} u_j \phi_j \quad , \quad v^h = \sum_{i=1}^{N} v_i \phi_i$$

But, $v^h(a) = v^h(b) = 0$

$\Rightarrow$ $v_1 = v_N = 0$

From (**),

$$B\left(\sum_{j=1}^{N} u_j \phi_j \, , \, \sum_{i=2}^{N} v_i \phi_i\right) = 0$$

$\Rightarrow$ Here, v_i — arbitrary scalar ($\because v_i \in v^h$ any v^h)

$\Rightarrow$ $B\left(\sum_{j=1}^{N} u_j \phi_j \, , \, \phi_i\right) = 0 \quad \forall \; i = 2 \dots N-1$

$\Rightarrow$ $\sum u_j \, B(\phi_j, \phi_i) = 0 \quad \forall \; i = 2 \dots N-1$

Also,
$u_1 = 0$ [from U^h definition]
$u_N = 1$

Hence,
$$\left.\begin{aligned} &\sum u_j a_{ij} = 0 \quad \forall \; i = 2 \dots N-1 \\ &u_1 = 1 \\ &u_N = 0 \end{aligned}\right\} \text{———— III}$$

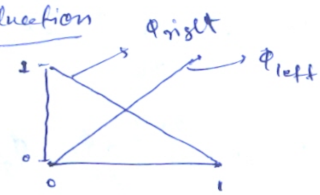
Here,
$$a_{ij} = B(\phi_j, \phi_i)$$

- Here, Problem III involves solving SLE
$$Au = f$$
- Due to linear basis functions as discussed in class, we get following structure:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & * & & & \\ & * & * & * & & \\ & & \ddots & \ddots & \ddots & \\ & & & * & * & * \\ & & & & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{N\times N}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{N\times 1}, \quad f = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{N\times 1}$$

Programming Strategis

- **Refrese element Basis fun and Quad. evaluation**



Refrence element : $\hat{T}(0,1)$, $\hat{x}$

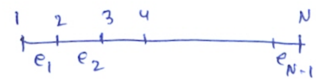
Basis functions :
$$\hat{\phi}_{left}(\hat{x}) = \hat{x}$$
$$\hat{\phi}_{right}(\hat{x}) = 1-\hat{x}$$

$$\frac{d\,\hat{\phi}_{left}(\hat{x})}{d\hat{x}} = 1, \qquad \frac{d\,\hat{\phi}_{right}(\hat{x})}{d\hat{x}} = -1$$

Quad. formula: $\int_0^1 f(x)\,dx = \sum w_i f(x_i)$

for 2 point Quad.
$$\left.\begin{aligned} w_1 &= w_2 = 0.5 \\ x_1 &= 0.5 - 1/2\sqrt{3}, \\ x_2 &= 0.5 + 1/2\sqrt{3} \end{aligned}\right\}$$

2. local A-matrix



- Corresponding to each element, there will be 4 entries.

$$A^i := e_i = [B(\phi_{i-1}, \phi_{i-1}), B(\phi_{i+1}, \phi_i), B(\phi_i, \phi_i), B(\phi_i, \phi_{i-1})]$$

- To evaluate integral, I used reference element (Affine transform)

$$e_k : B(\phi_i, \phi_j) = \underbrace{\int_{e_k} \epsilon \frac{d\phi_i}{dx} \cdot \frac{d\phi_j}{dx}\, dx}_{(I)} + \underbrace{\int_{e_k} \frac{d\phi_i}{dx} \cdot \phi_j\, dx}_{(II)}$$

$$\text{I}: \int_{x_k}^{x_{k+1}} \epsilon \frac{d\phi_i}{dx} \cdot \frac{d\phi_j}{dx}\, dx = (x_{k+1} - x_k)^{-1} \int_0^1 \frac{d\hat{\phi}_i(\hat{x})}{d\hat{x}} \frac{d\hat{\phi}_j(\hat{x})}{d\hat{x}} \cdot dx$$

$$\text{II}: \int_{x_k}^{x^{k+1}} \frac{d\phi_i}{dx} \cdot \phi_j\, dx = \int_0^1 \frac{d\hat{\phi}_i}{d\hat{x}} \hat{\phi}_j\, d\hat{x}$$

- Individual integration for ref. element is being done by 2-point quad formulae.

3. Global-A-matrix : local DoF = 2, Implicit mapping Idea:

For $i = 2 \dots N-1$

$$A_{i,i-1} = B(\phi_{i-1}, \phi_i) = \int_{e_{i-1}} = A^i[2]$$

$$A_{i,i} = B(\phi_i, \phi_i) = \int_{e_{i-1}} + \int_{e_i} = A^{i-1}[3] + A^i[1]$$

$$A_{i,i+1} = B(\phi_{i+1}, \phi_i) = \int_{e_i} = A^{i+1}[4]$$

4. Others

- Stored A into CSR format.
- Used Jacobi method to solve $Au = f$ onto sparse format.
- To vizualize and save the data, I used GNU-Plot + File ~~write~~ writting operation.

RESULTS:

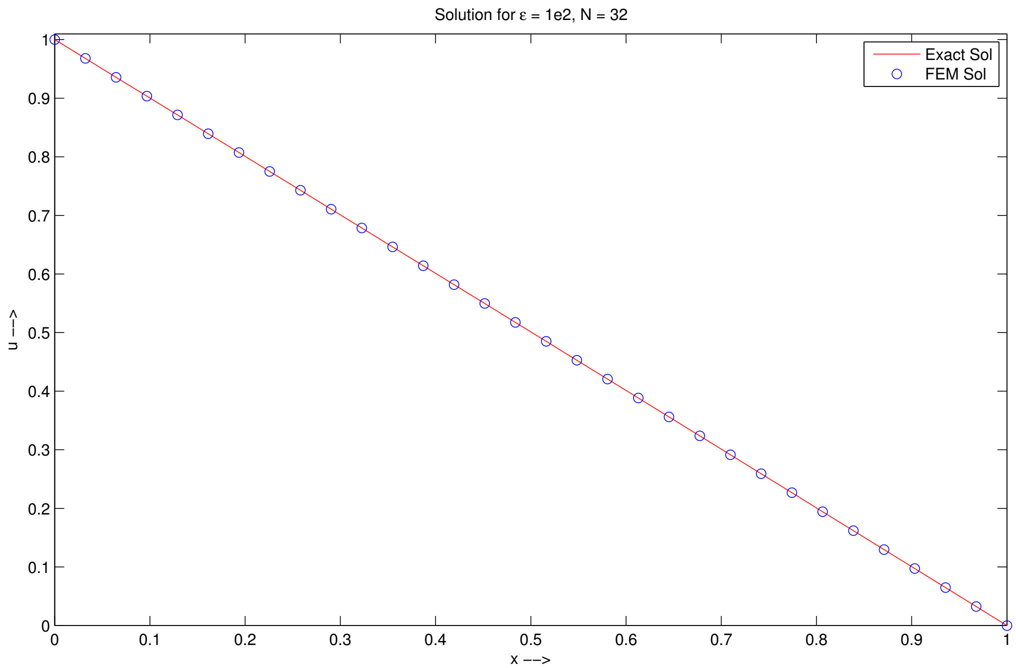


Fig.1. For $\epsilon = 100$, N = 32

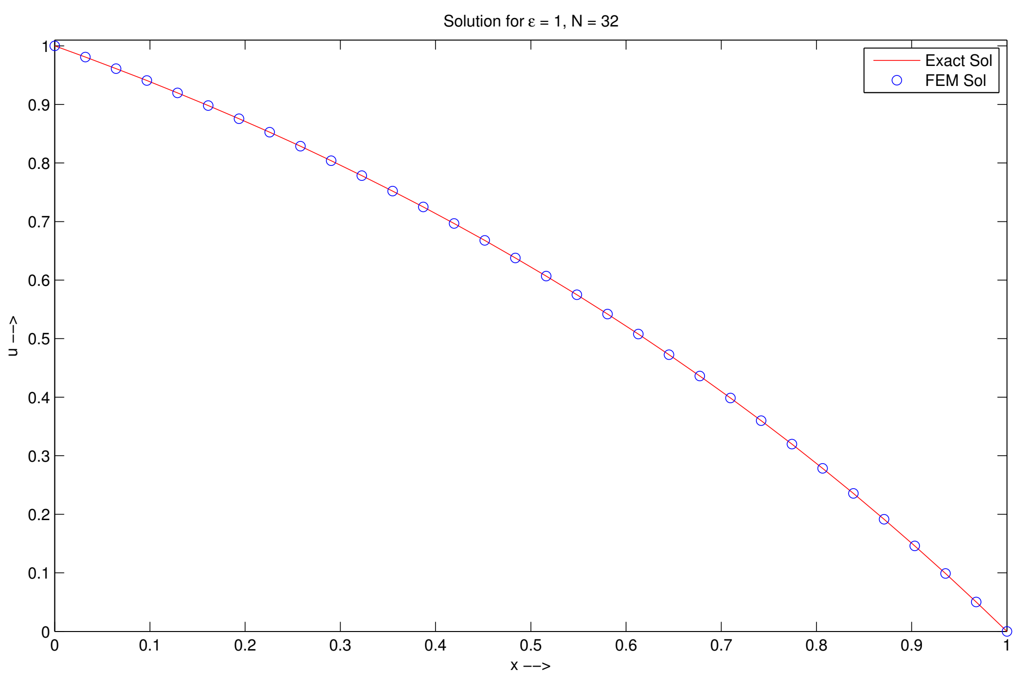


Fig.2. For $\epsilon = 1$, N = 32

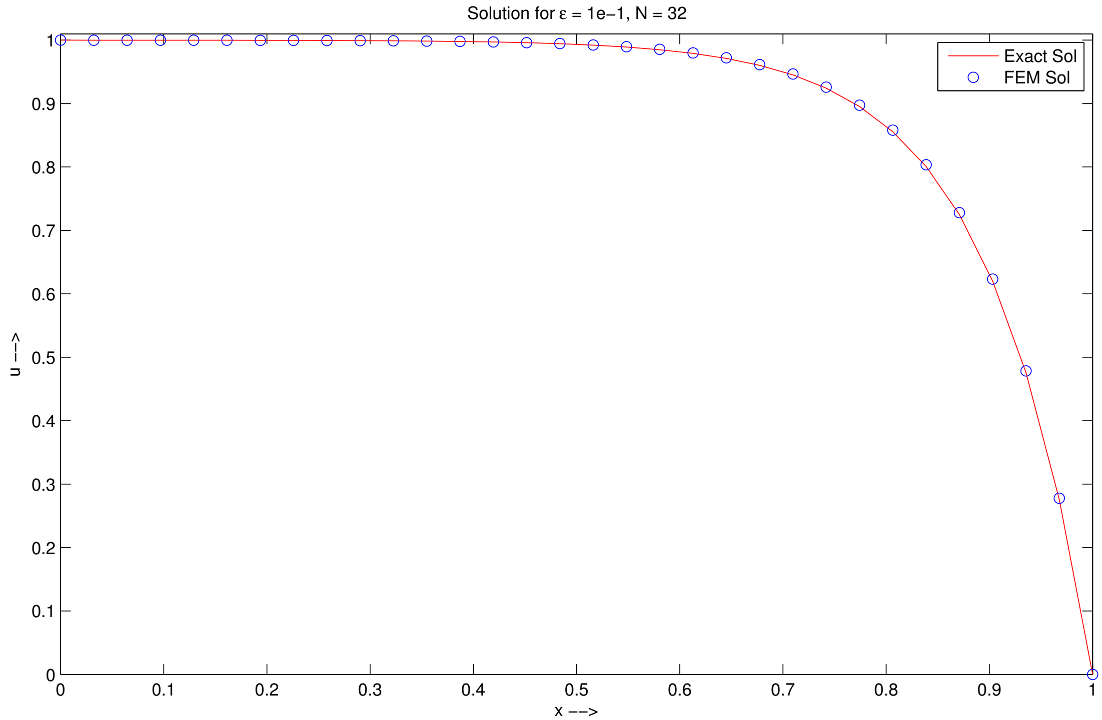


Fig.3. For $\epsilon = 1e-1$, N = 32

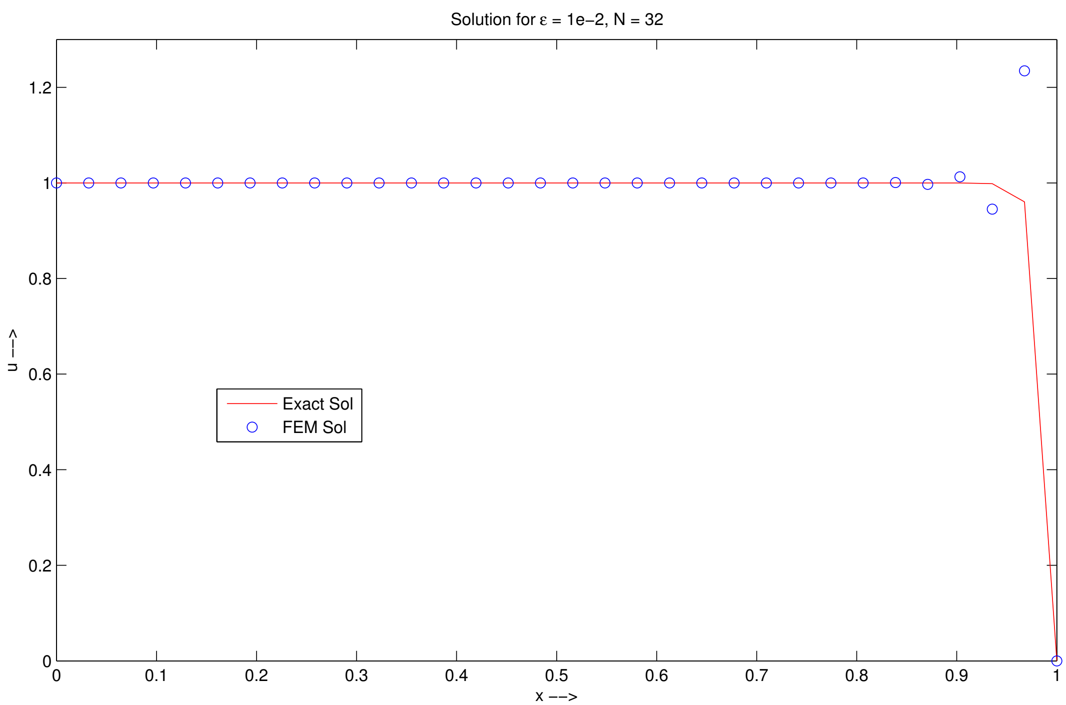


Fig.4. For $\epsilon = 1e-2$, N = 32

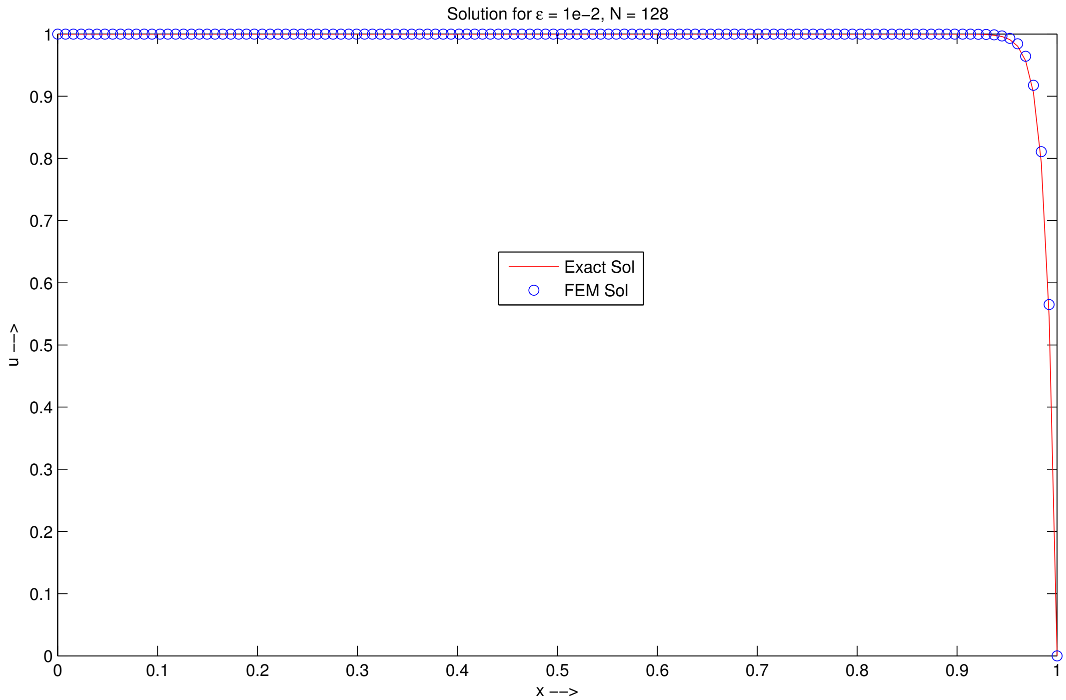


Fig.5. For $\epsilon = 1e-2$, N = 128

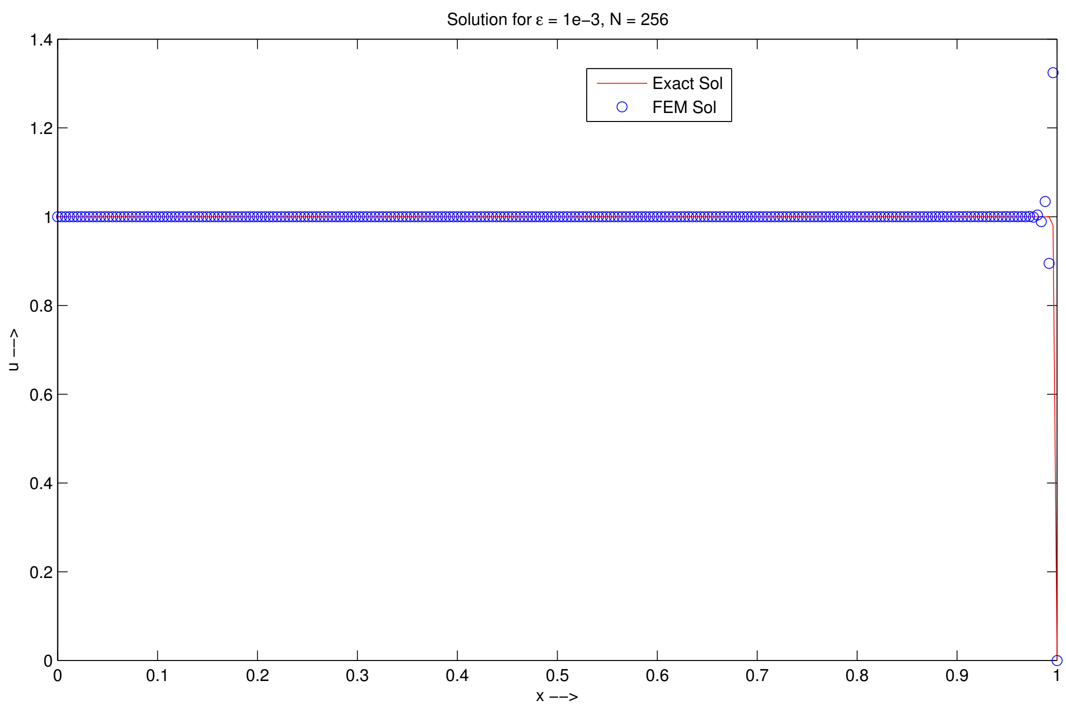


Fig.6. For $\epsilon = 1e-3$, N = 256

Problem 1 continued on next page. . .

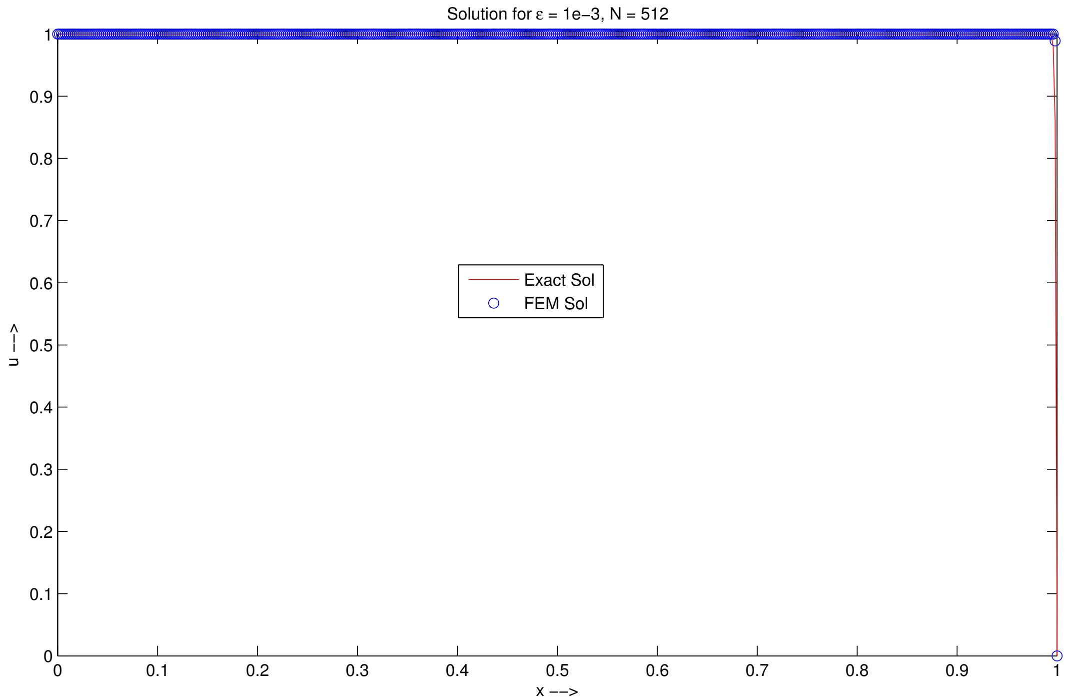


Fig.7. For $\epsilon = 1e - 3$, N = 512

Observations:

- FEM Solution obtained for higher value of ϵ gives very accurate solutions even for few no. of nodes (higher value of h). Fig.1. - Fig.3. Shows plots such accurate results with N = 32.
- Lesser the value of ϵ like for $\epsilon = 10^{-3} to 10^{-8}$, Solution obtained are less accurate. One possible reason behind this could be condition number of system matrix A.
- Choosing less values of h (highr no. of nodes) for lesser value of ϵ gives more accurate results. Reason could be that, In system matrix A, Non-zero enteries constitutes of factor $\frac{\epsilon}{h}$.
- Solution obtained seems to be less accurate for $x = b$ or $x = 1$ in the fig.4. and Fig.6. Steep nature of exact solution near $x = b$ could be the reason.
- For higher values of ϵ slops are becoming more and more streight (almost linear).

END.