5.22 Fri.

目标证明: $\pi(x)=\#\{p\le x\}\sim \frac{x}{\log x}$

$\Updownarrow$ ①

$\psi(x)\sim x$, $\psi(x)=\ln(\mathrm{lcm}(1,\cdots,[x]))$ 增函数

$\Updownarrow$ ②

$\psi_1(x)=\int_1^x \psi(u)\,du \sim \frac{1}{2}x^2$ (只证一边)

② 等价性证明: 取 $\alpha<1<\beta$, $\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} \psi(u)\,du \le \psi(x) \le \frac{1}{(\beta-1)x}\int_x^{\beta x}\psi(u)\,du$

$\psi(x)\le \frac{1}{(\beta-1)x}[\psi_1(\beta x)-\psi_1(x)] \Rightarrow \frac{\psi(x)}{x}\le\frac{1}{\beta-1}\left[\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2-\frac{\psi_1(x)}{x^2}\right]$

$\Rightarrow \limsup_{x\to\infty}\frac{\psi(x)}{x}\le \frac{1}{\beta-1}\left[\frac{1}{2}\beta^2-\frac{1}{2}\right]=\frac{\beta+1}{2}$ 另一边同理 □

• $f_n(u)=\begin{cases}1, & n\le u\\ 0, & \text{otherwise}\end{cases}$ $\quad\Lambda(n)=\begin{cases}\ln p, & n=p^k\\ 0, & \text{otherwise}\end{cases}$

$\psi_1(x)=\int_1^x\psi(u)\,du=\sum_{n=1}^{\infty}\int_1^x\Lambda(n)f_n(u)\,du=\sum_{n\le x}\Lambda(n)\int_n^x du=\sum_{n\le x}\Lambda(n)(x-n)$

$\psi(x)=\sum_{n\le x}\Lambda(n)$

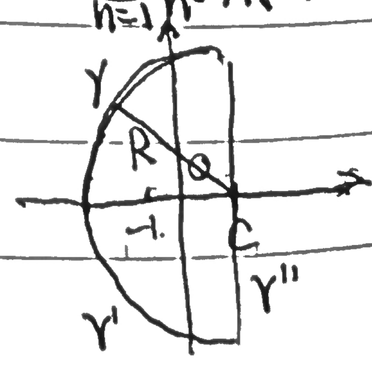
$\psi_1(x)=\sum_{n\le x}\Lambda(n)(x-n)=x\sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right)=x\sum_{n\le x}\Lambda(n)\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}ds$

$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$

$\log\zeta(s)=\sum_{m,p}\frac{p^{-ms}}{m}\Rightarrow\frac{\zeta'(s)}{\zeta(s)}=-\sum(\log p)\cdot p^{-ms}=-\sum_{n=1}^{\infty}\frac{1}{n^s}\Lambda(n)$

• $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s(s+1)}ds=\begin{cases}0, & 0<a\le1\\ 1-\frac{1}{a}, & a\ge1\end{cases}$ $\quad c>0$

计算: 取 $f(s)=\frac{a^s}{s(s+1)}=\frac{e^{s\log a}}{s(s+1)}$



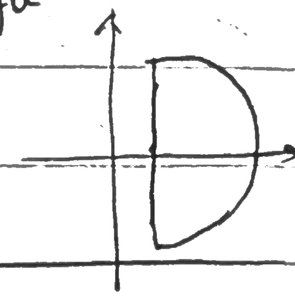
留数.

$\frac{1}{2\pi i}\oint_\gamma f(s)ds = 1+\frac{a^{-1}}{-1} = 1-\frac{1}{a}$

$\gamma=\gamma'+\gamma''$. $s(s+1)\geq \frac{1}{2}R^2$, $\left|\int_{\gamma'} f(s)ds\right| \le \frac{C}{R^2}\cdot\pi R \to 0\ (R\to\infty)$.

$s=\sigma+it$, $\sigma\le C$. 当 $a\geq 1$ 时, $|e^{s\log a}|\le e^{C\log a}$

当 $0<a<1$ 时, $|e^{s\log a}|\to\infty$.



考虑右边圆弧, 无极点 $=0$.

• $\psi_1(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds. = F(s)$

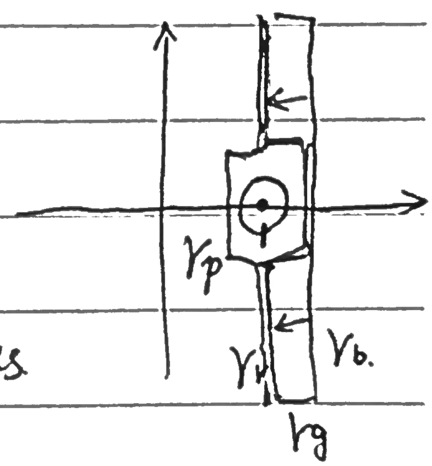
要证: $\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)ds \sim \frac{1}{2}x^2$

$\zeta(s)=\frac{1}{s-1}+\text{holo}..$ $-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1}+\text{holo}.$

$0=\oint_\gamma F(s)ds$, $\gamma=\gamma_r+\gamma_g+\gamma_b$

$F(s)=\underbrace{\frac{x^{s+1}}{s(s+1)}}_{\frac{|x^{\sigma+1}|}{|t|^2}}\underbrace{\left(-\frac{\zeta'(s)}{\zeta(s)}\right)}_{\le A|t|^{\varepsilon}}$. $\to o(t\to\infty)$ $s=\sigma+it$.

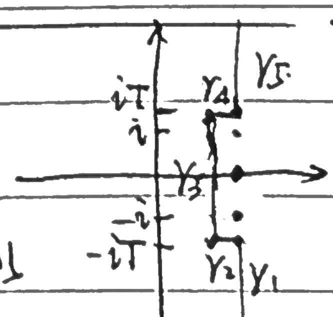
$\int_{\gamma_g}F(s)ds\to 0$.



$\frac{1}{2\pi i}\int_{\gamma_b}F(s) = \frac{1}{2\pi i}\int_{\gamma_r}F(s)ds = \frac{x^2}{2}+\frac{1}{2\pi i}\int_{\gamma_p}F(s)ds$

$\mathrm{Res}_{s=1}F(s)=\frac{x^2}{2}$

$\left|\frac{\zeta'(s)}{\zeta(s)}\right|\le A|t|^{\varepsilon}$



由之前结论: $\sigma\ge 1$. $|\zeta'(s)|\le C_\varepsilon|t|^\varepsilon$. $|\zeta(s)|^{-1}\le C'_\varepsilon|t|^\varepsilon$

$\left|\int_{\gamma_1}F(s)ds\right| \le Cx^2\int_T^\infty \frac{|t|^{2\varepsilon}}{t^2}dt \le \frac{C}{T}x^2 = \varepsilon x^2$, γ_5 类似

$\left|\int_{\gamma_3}F(s)ds\right|$: 令 $\delta=1-\mathrm{Re}(s)$, $|x^{1+s}|=x^{2-\delta}$, 则 $\left|\int_{\gamma_3}F(s)ds\right|\le C_T x^{2-\delta}$

$\left|\int_{\gamma_2}F(s)ds\right| \le C'_T\int_{1-\delta}^1 x^{1+\sigma}d\sigma \le C'_T\frac{x^2}{\log x}$

$\Rightarrow \left|\int_{\gamma_p}F(s)ds\right|\le 2\varepsilon x^2 + C_Tx^{2-\delta}+2C'_T\frac{x^2}{\log x}$, $\frac{\left|\int_{\gamma_p}F(s)ds\right|}{x^2}\le 2\varepsilon+\frac{C_T}{x^\delta}+\frac{2C'_T}{\log x}\to 2\varepsilon\ (x\to\infty)$

$\Rightarrow \frac{1}{2\pi i}\int_{\gamma_b}F(s)\to\frac{x^2}{2}\ (x\to\infty)$. □

No.
Date

$\pi(x) \sim \frac{x}{\log x}$. $\pi(x) \sim Li(x)$.

更好的估计：$\pi(x) = Li(x) + O(\sqrt{x}\log x)$

作业：第七章 Ex. 1, 2, 3, 4, 5, 10, 11