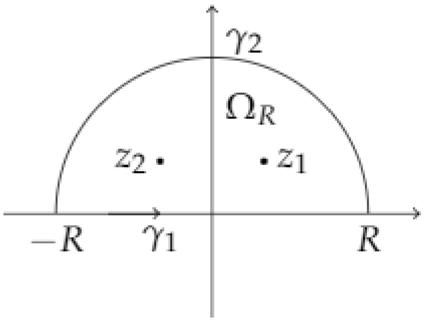
Solutions to homework 3

Exercise 3.2

Proof: Use the contour as in the figure.



The poles of $\frac{1}{1+z^4}$ in Ω_R are $z_1 = e^{\pi i/4}, z_2 = e^{3\pi i/4}$.

$\therefore \int_{\gamma_1+\gamma_2} \frac{dz}{1+z^4} = 2\pi i \left(\mathrm{Res}_{z_1} + \mathrm{Res}_{z_2}\right) \frac{1}{1+z^4}$.

$\mathrm{Res}_{z_1} \frac{1}{1+z^4} = \lim\limits_{z\to z_1} \frac{z-z_1}{1+z^4} = \frac{1}{4} e^{-3\pi i/4}$, $\mathrm{Res}_{z_2} \frac{1}{1+z^4} = \lim\limits_{z\to z_2} \frac{z-z_2}{1+z^4} = -\frac{1}{4} e^{3\pi i/4}$.

$| \int_{\gamma_2} \frac{dz}{1+z^4} | = | \int_0^\pi \frac{iRe^{i\theta} d\theta}{1+R^4 e^{4i\theta}} | \le \frac{R\pi}{R^4-1} \to 0$, as $R \to \infty$.

$\therefore$

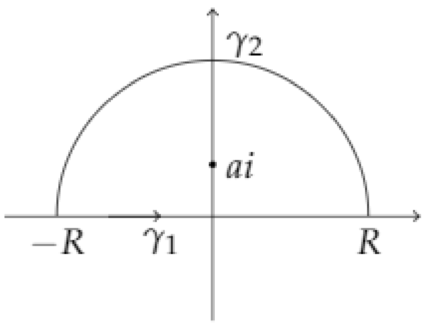
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int_{\gamma_1+\gamma_2} \frac{dz}{1+z^4} = 2\pi i \left(\mathrm{Res}_{z_1} + \mathrm{Res}_{z_2}\right) \frac{1}{1+z^4} = \frac{\sqrt{2}}{2}\pi.$$

Exercise 3.4

Proof:

$$\int_{-\infty}^{\infty} \frac{x sin x}{x^2+a^2} dx = \frac{1}{2i} \left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2} dx - \int_{-\infty}^{\infty} \frac{xe^{-ix}}{x^2+a^2} dx \right) := \frac{1}{2i}\big((1) - (2)\big).$$

For (1), use the contour:



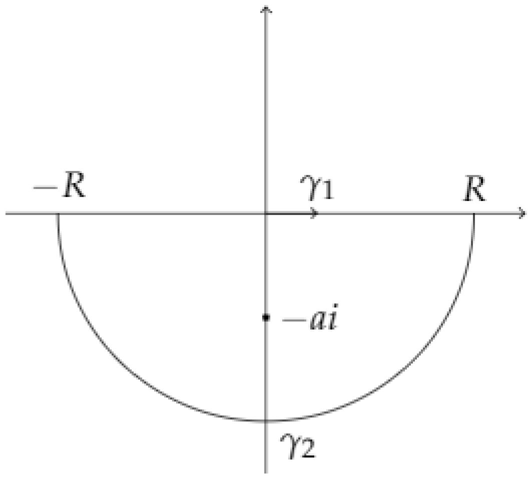
$\int_{\gamma_1+\gamma_2} \frac{ze^{iz}}{z^2+a^2} dz = 2\pi i \,\mathrm{Res}_{ai} \frac{ze^{iz}}{z^2+a^2}$.

$$
\begin{aligned}
|\int_{\gamma_2} \frac{ze^{iz}}{z^2+a^2}dz| &= |\int_0^\pi \frac{iR^2e^{2i\theta}e^{-Rsin\theta+iRcos\theta}d\theta}{R^2e^{2i\theta}+a^2}| \leq \frac{R^2}{R^2-a^2}\int_0^\pi e^{-Rsin\theta}d\theta \\
&= \frac{2R^2}{R^2-a^2}\int_0^{\pi/2} e^{-Rsin\theta}d\theta \leq \frac{2R^2}{R^2-a^2}\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta \\
&= \frac{R\pi}{R^2-a^2}(1-e^{-R}) \to 0, \text{ as } R \to \infty.
\end{aligned}
$$

$\text{Res}_{ai}\frac{ze^{iz}}{z^2+a^2} = \lim_{z\to ai}\frac{(z-ai)ze^{iz}}{z^2+a^2} = \frac{1}{2}e^{-a}$.

$\therefore (1) = \lim_{R\to\infty}\int_{\gamma_1+\gamma_2}\frac{ze^{iz}}{z^2+a^2}dz = 2\pi i\cdot\frac{1}{2}e^{-a} = \pi i e^{-a}$.

Similarly, we use the contour



to compute that $(2) = -\pi i e^{-a}$, hence

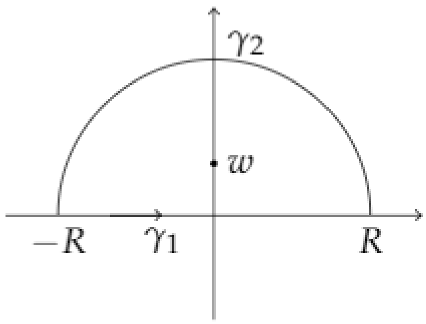
$$
\int_{-\infty}^{\infty}\frac{xsinx}{x^2+a^2}dx = \frac{1}{2i}\cdot(\pi i e^{-a} - (-\pi i e^{-a})) = \pi e^{-a}.
$$

Exercise 3.7

Proof:

$$
\begin{aligned}
\int_0^{2\pi}\frac{d\theta}{(a+cos\theta)^2} &= 2\int_0^\pi \frac{d\theta}{(a+cos\theta)^2} \xlongequal{t=tan\frac{\theta}{2}} 4\int_0^\infty \frac{(1+t^2)dt}{((a-1)t^2+a+1)^2} := 4\int_0^\infty f(t)dt \\
&= 2\int_{-\infty}^{\infty} f(t)dt.
\end{aligned}
$$

Use the contour



,where $w = i\sqrt{\frac{a+1}{a-1}}$, the only pole of the integrand function in the upper half plane,

$$|\int_{\gamma_2} f(z)dz| = |\int_0^{\pi} \frac{(1+R^2e^{2i\theta})Rie^{i\theta}d\theta}{((a-1)R^2e^{2i\theta}+a+1)^2}| \leq \frac{R(R^2+1)}{((a-1)R^2+3-a)^2} \to 0, \text{ as } R \to \infty,$$

$\mathrm{Res}_w f(z) = \frac{d}{dz} f(z)(z-w)^2|_{z=w} = -\frac{i}{2}\frac{a}{(a^2-1)^{3/2}}$,

$\Rightarrow$

$$\int_0^{2\pi} \frac{d\theta}{(a+cos\theta)^2} = 2\int_{-\infty}^{\infty} f(t)dt = 2\lim_{R\to\infty}\int_{\gamma_1+\gamma_2} f(z)dz = 4\pi i\cdot(-\frac{i}{2})\frac{a}{(a^2-1)^{3/2}} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

Exercise 3.8

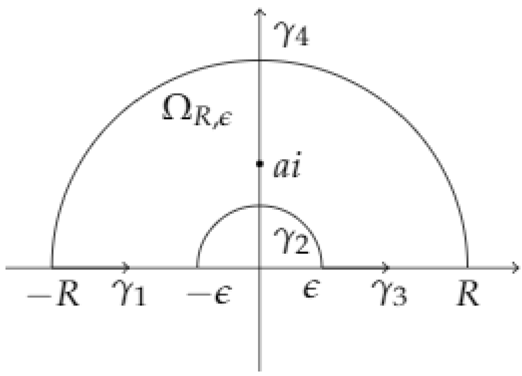
Proof: Use the same contour as in 3.7, and a similar derivation process shows that

$$\int_0^{2\pi} \frac{d\theta}{a+bcos\theta} \xlongequal{t=tan\frac{\theta}{2}} 2\int_{-\infty}^{\infty}\frac{dt}{(a-b)t^2+a+b} = 4\pi i\ \mathrm{Res}_w \frac{1}{(a-b)z^2+a+b} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

,where $w = i\sqrt{\frac{a+b}{a-b}}$ is the only pole of $\frac{1}{(a-b)z^2+a+b}$ in the upper half plane.

Exercise 3.10

Proof: Use the contour



$$\int_{\gamma_1+\gamma_3}\frac{\log z}{z^2+a^2}dz = \int_\epsilon^R \frac{\log z}{z^2+a^2}dz + \int_{-R}^{-\epsilon}\frac{\log(-z)+\pi i}{z^2+a^2}dz = \pi i\int_\epsilon^R \frac{dz}{z^2+a^2} + 2\int_\epsilon^R \frac{\log z}{z^2+a^2}dz \longrightarrow \frac{\pi^2 i}{2a} + 2\int_0^\infty \frac{\log z}{z^2+a^2}dz, \text{ as } \epsilon\to 0, R\to\infty,$$

$$|\int_{\gamma_2}\frac{\log z}{z^2+a^2}dz| = |\int_\pi^0 \frac{i\epsilon e^{i\theta}(\log\epsilon + i\theta)d\theta}{\epsilon^2e^{2i\theta}+a^2}| \leq \frac{\pi(-\epsilon\log\epsilon+\pi\epsilon)}{a^2-\epsilon^2} \longrightarrow 0, \text{ as } \epsilon\to 0,$$

$$|\int_{\gamma_4}\frac{\log z}{z^2+a^2}dz| = |\int_0^\pi \frac{iRe^{i\theta}(\log R + i\theta)d\theta}{R^2e^{2i\theta}+a^2}| \leq \frac{\pi R(\log R+\pi)}{R^2-a^2} \longrightarrow 0, \text{ as } R\to\infty.$$

Note that ai is the only pole of $\frac{\log z}{z^2+a^2}$ in $\Omega_{R,\epsilon}$, we have

$$2\pi i \ \mathrm{Res}_{ai}\frac{\log z}{z^2+a^2} = \lim_{\epsilon\to 0, R\to\infty}\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\frac{\log z}{z^2+a^2}dz = \frac{\pi^2 i}{2a} + 2\int_0^\infty \frac{\log z}{z^2+a^2}dz,$$

as one computes $\mathrm{Res}_{ai}\frac{\log z}{z^2+a^2} = \frac{\pi \log a}{a} + \frac{\pi^2 i}{2a}, \Rightarrow \int_0^\infty \frac{\log z}{z^2+a^2}dz = \frac{\pi}{2a}\log a.$

Exercise 3.14

Proof: Let $g(z) = f(1/z)$, which is holomorphic in $\mathbb{C} - \{0\}$. g is injective since f is injective. By the Casorati-Weierstrass theorem, 0 is not an essential singularity of f **(**Otherwise, the C-W theorem indicates that $g(\mathbb{D} - \{0\})$ is dense in $\mathbb{C}$,where $\mathbb{D}$ is the closed unit disc, but g is open, hence $g(\mathbb{C} - \mathbb{D})$ is open, $\Rightarrow g(\mathbb{D} - \{0\}) \cap g(\mathbb{C} - \mathbb{D}) \neq \emptyset$, contradicting to the injectivity of g**)**.

0 is neither a removable singularity, otherwise f would be a bounded entire function, hence constant. Thus 0 is a pole of f. Suppose it's a pole of degree n, then there exists a polynomial h of degree n, such that $f(1/z) - h(1/z)$ is bounded near 0, which means that $f(z) - h(z)$ is bounded, hence constant. Thus f is a polynomial of degree n. Then injectivity of f implies that deg f=1, and we conclude that $f(z) = az + b$, where $a \neq 0$.

Exercise 3.17

Proof:

(a) $\forall w \in \mathbb{D}$, $|f(z) - w - f(z)| \leq |f(z)| = 1$, if $|z| = 1$. By Rouché's theorem, $f(z) - w$ has the number of zeros with f in $\mathbb{D}$. In particular, $f(z) - w$ has a zero in $\mathbb{D}$ if and only if f has a zero in $\mathbb{D}$. Thus, it suffices to show that f has a zero in $\mathbb{D}$. Assuming the opposite, $g = 1/f$ is holomorphic in $\mathbb{D}$, which, by the maximum modulus principle, has a maximal point, say z_0. Then z_0 is a minimal point of f. Since f is non-constant and is not able to attain its minimum inside $\mathbb{D}$, we have $|z_0| = 1$ and then $|f(z_0)| = 1, \Rightarrow f \equiv 1$, a contradiction!

(b) Use the same arguments.

Exercise 3.18

Proof: See hints.

Exercise 3.19

Proof:

(a) If u attains its maximum at z_0 in Ω. Let $B_r(z_0)$ be a disc in Ω. By mean value formula,

$$u(z_0) = \frac{1}{\pi r^2}\int_{B_r(z_0)} u(x,y)dxdy \leq \frac{1}{\pi r^2}\int_{B_r(z_0)} u(z_0)dxdy = u(z_0),$$

$\Rightarrow u(z) = u(z_0), \forall z \in B_r(z_0).$

Let $\Omega_1 = \{z \in \Omega : u(z) = u(z_0)\}$. The above argument shows that Ω_1 is open. By the continuity of u, Ω_1 is closed. Since Ω is connected and Ω_1 is nonempty, we have $\Omega_1 = \Omega, \Rightarrow u$ is constant, a contradiction!

(b) $\because \bar{\Omega}$ is compact, $\therefore u$ attains a maximum at some $z_0 \in \bar{\Omega}$. If u is constant, then the conclusion is clear. If not, (a) then implies $z_0 \in \bar{\Omega} - \Omega$.

Exercise 3.21

Omitted.