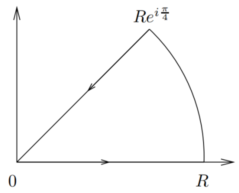
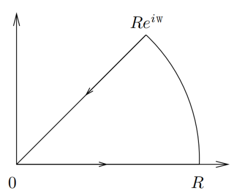
2.1 Prove that
$$\int_0^\infty \sin\left(x^2\right)dx = \int_0^\infty \cos\left(x^2\right)dx = \frac{\sqrt{2\pi}}{4}$$
These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.
[Hint: Integrate the function e^{-z^2} over the path in Figure 14. Recall that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$].
Proof Consider the entire function $f(z) = e^{-z^2}$, take the following path C_R and the circle is denoted by Γ_R.



By Cauchy's theorem
$$0 = \int_{C_R} e^{-z^2}dz = \int_0^R e^{-x^2}dx + \int_{\Gamma_R} e^{-z^2}dz + \int_R^0 e^{-x^2 i}e^{\frac{\pi}{4}i}dx \tag{†}$$
and
$$\begin{aligned}
&\Big|\int_{\Gamma_R} e^{-z^2}dz\Big| = \Big|\int_0^{\frac{\pi}{4}} e^{-R^2(\cos 2\phi+\sin 2\phi)}iRe^{i\phi}d\phi\Big| \\
\le &\int_0^{\frac{\pi}{4}} e^{-R^2\cos 2\phi}Rd\phi \xlongequal{2\phi=\frac{\pi}{2}-\theta} \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin\theta}d\theta \le \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\frac{2\theta}{\pi}}d\theta \\
= &-\frac{R}{2}\cdot\frac{\pi}{2R^2}e^{-\frac{2R^2}{\pi}\theta}\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi}{4R}(1-e^{-R^2}).
\end{aligned}$$
so $\big|\int_{\Gamma_R} e^{-z^2}dz\big| \to 0$ as $R\to+\infty$. Let $R\to+\infty$, from (†) we have
$$\begin{aligned}
\frac{1+i}{\sqrt{2}}\int_0^{+\infty}(\cos x^2 - i\sin x^2)dx &= \int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \\
i.e. \int_0^{+\infty}(\cos x^2 - i\sin x^2)dx &= \int_0^{+\infty} e^{-x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1-i).
\end{aligned}$$
compare real part and imaginary part we will get the conclusion. □

2.3 Evaluate the integrals
$$\int_0^\infty e^{-ax}\cos bx dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin bx dx, \quad a>0$$
by integrating e^{-Az}, $A=\sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$
Proof Consider the entire function $f(z) = e^{-Az}$, take the following path C_R and the circle is denoted by Γ_R.



By Cauchy's theorem
$$0 = \int_{C_R} e^{-Az}dz = \int_0^R e^{-Ax}dx + \int_{\Gamma_R} e^{-Az}dz + e^{i\omega}\int_R^0 e^{-Axe^{i\omega}}dx \tag{†}$$

and

$$\int_0^R e^{-Ax}dx = \frac{1}{A}(1-e^{-AR})$$
$$\int_{\Gamma_R} e^{-Az}dz = \frac{1}{A}(e^{-AR}-e^{-ARe^{i\omega}}) = \frac{1}{A}e^{-AR} - \frac{1}{A}e^{-Ra}(\cos Rb - i\sin Rb)$$
$$\int_0^R e^{-Axe^{i\omega}}dx = \int_0^R e^{-(a+ib)x}dx = \int_0^R e^{-ax}(\cos bx - i\sin bx)dx$$

so we have

$$e^{i\omega}\int_0^R e^{-ax}(\cos bx - i\sin bx)dx = \frac{1}{A} - \frac{1}{A}e^{-Ra}(\cos Rb - i\sin Rb)$$

The right side is equal to $\frac{1}{A}$ when $R\to+\infty$, the left side is

$$\cos\omega\int_0^{+\infty} e^{-ax}\cos bx dx + \sin\omega\int_0^{+\infty} e^{-ax}\sin bx dx$$
$$+i(\sin\omega\int_0^{+\infty} e^{-ax}\cos bx dx - \cos\omega\int_0^{+\infty} e^{-ax}\sin bx dx)$$

so we have

$$\cos\omega\int_0^{+\infty} e^{-ax}\cos bx dx + \sin\omega\int_0^{+\infty} e^{-ax}\sin bx dx = \frac{1}{A}$$
$$\sin\omega\int_0^{+\infty} e^{-ax}\cos bx dx - \cos\omega\int_0^{+\infty} e^{-ax}\sin bx dx = 0$$

Then
$$\int_0^{+\infty} e^{-ax}\cos bx dx = \frac{1}{A}\cos\omega = \frac{a}{a^2+b^2}, \int_0^{+\infty} e^{-ax}\sin bx dx = \frac{1}{A}\sin\omega = \frac{b}{a^2+b^2}.$$

□

2.7 Suppose $f:\mathbb{D}\to\mathbb{C}$ is holomorphic. Show that the diameter $d=\sup_{z,w\in\mathbb{D}}|f(z)-f(w)|$ of the image of f satisfies

$$2\,|f'(0)|\le d$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z)=a_0+a_1z$

Note. In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chapter 4, Book I.

[Hint: $2f'(0)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)-f(-\zeta)}{\zeta^2}d\zeta$ whenever $0<r<1$.]

Proof (1) By corollary 4.2 $f'(0)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta^2}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{-f(-\zeta)}{\zeta^2}\mathrm{d}\zeta$ for any $0<r<1$, then

$$\left|2f'(0)\right| = \left|\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)-f(-\zeta)}{\zeta^2}\mathrm{d}\zeta\right| \le \frac{1}{2\pi}\int_{|\zeta|=r}\frac{|f(\zeta)-f(-\zeta)|}{r^2}\mathrm{d}\zeta \le \frac{d}{2\pi r^2}2\pi r = \frac{d}{r}$$

Let $r\to1$ we have $2\,|f'(0)|\le d$.

(2) For some $f:\mathbb{D}\to\mathbb{C}$ is holomorphic we define:

$$D_r = \sup\{|f(z)-f(w)| : z,w\in D_r(O)\}$$
$$d_r = \sup\{|f(z)-f(w)| : z,w\in D_r(O), |z|=|w|\}$$
$$d_r^* = \sup\{|f(z)-f(-z)|, z\in D_r(O)\}$$

for all $0 < r \le 1$, obviously $D_r \ge d_r \ge d_r^*$. Applying Schwartz's lemma to $F(z) = \frac{1}{d_r^*}(f(rz) - f(-rz)), z \in \mathbb{D}$ yields

$$|f(z) - f(-z)| \le \frac{1}{r} d_r^* |z|, \forall z \in D_r(O) \text{ and } |f'(0)| \le \frac{1}{2r} d_r^* \tag{†}$$

Assume

$$|f'(0)| = \frac{1}{2r} d_r \tag{‡}$$

holds for some $0 < r < 1$ we have $d_r = d_r^*$, after a rotation we may suppose $f'(0) = \frac{1}{2r} d_r$, then $F'(0) = 1, F(z) = z$ by Schwartz's lemma, so $f(rz) - f(-rz) = d_r z, \forall z \in \mathbb{D}$.

We want to show $f'(z)$ is a constant in $\mathbb{D}$. Suppose that $\Im f'(a) \ne 0$ for some $a \in C_r(O)$, consider the function $\varphi(\theta) = |f(ae^{i\theta}) - f(-a)|^2, \theta \in \mathbb{R}$ we have $\varphi'(0) = -2\frac{1}{r}|a|^2 d_r \Im f'(a) \ne 0$, then there exists some θ near 0 such that $\varphi(\theta) > \varphi(0)$. That is

$$d_r \ge |f(ae^{i\theta}) - f(-a)| = \sqrt{\varphi(\theta)} > \sqrt{\varphi(0)} = |f(a) - f(-a)| = d_r$$

contradiction. so $\Im f'(z) = 0$ for all $z \in C_r(O)$, then $\Im f'(z) = 0$ for all $z \in \mathbb{D}$, hence $f'(z) = 0$ is a constant in $\mathbb{D}$.

Next we show $\frac{d_r}{r}$ is a nondecreasing function of $0 < r < 1$, for $0 < r < R < 1$ using Maximum modulus principle repeatly

$$\begin{aligned}
\frac{1}{r} d_r =& \frac{1}{r} \sup\{|f(z) - f(uz)| : z \in D_r(O), |u| = 1\} = \frac{1}{r} \sup\{|f(z) - f(uz)| : z \in C_r(O), |u| = 1\} \\
=& \sup\{\left|\frac{f(z) - f(uz)}{z}\right| : z \in C_r(O), |u| = 1\} \le \sup\{\left|\frac{f(z) - f(uz)}{z}\right| : z \in D_R(O), |u| = 1\} \\
=& \sup\{\left|\frac{f(z) - f(uz)}{z}\right| : z \in C_R(O), |u| = 1\} = \frac{1}{R} \sup\{|f(z) - f(uz)| : z \in C_R(O), |u| = 1\} \\
=& \frac{1}{R} \sup\{|f(z) - f(uz)| : z \in D_R(O), |u| = 1\} = \frac{1}{R} d_R \le \frac{1}{R} d_1
\end{aligned}$$

let $R \to 1$ we get $\frac{1}{r} d_r \le \frac{1}{R} d_R \le d_1$.

Finally, when $|f'(0)| = \frac{1}{2} D_1$ holds, for any $0 < r < 1$ we have

$$\frac{1}{2} D_1 = |f'(0)| \overset{(†)}{\le} \frac{1}{2r} d_r^* \le \frac{1}{2r} d_r \le \frac{1}{2} d_1 \le \frac{1}{2} D_1$$

then (‡) holds and $f'(z) = 0$ is a constant in $\mathbb{D}$. □

2.9 Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $\varphi(z_0) = z_0$ and $\varphi'(z_0) = 1$ then φ is linear.

[Hint: Why can one assume that $z_0 = 0$? Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here we use the standard O notation, where $f(z) = O(g(z))$ as $z \to 0$ means that $|f(z)| \le C|g(z)|$ for some constant C as $|z| \to 0$.]

Proof $f(z) = \varphi(z + z_0) - z_0 : \Omega - z_0 \to \Omega - z_0$ is linear iff φ is linear, so we can assume $z_0 = 0$. Expanding in a power series around 0 and suppose a_n is the first nonzero coefficient with $n > 1$, then $\varphi(z) = z + a_n z^n + O(z^{n+1})$. We show $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$ by induction. $k = 1$ is obvious, if it is true for k it follows that

$$\begin{aligned}
\varphi_{k+1}(z) =& z + k a_n z^n + O(z^{n+1}) + a_n(z + k a_n z^n + O(z^{n+1}))^n + O((z + k a_n z^n + O(z^{n+1}))^{n+1}) \\
=& z + (k+1) a_n z^n + O(z^{n+1})
\end{aligned}$$

Let $r > 0$ such that $D_r \subset \Omega$, by the Cauchy inequalities

$$|\varphi_k^{(n)}(0)| \le \frac{n!}{r^n} \sup_{|z|=r} |\varphi_k(z)|.$$

Suppose Ω is bounded by M, for $\varphi_k^{(n)}(0) = kn!a_n$ we have

$$kn!|a_n| \le \frac{n!M}{r^n}, \text{ then } |a_n| \le \frac{M}{kr^n}.$$

Let $k \to +\infty$ we have $a_n = 0$, thus $\varphi(z) = z$. □

2.11 Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0 (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f\left(Re^{i\varphi}\right) \operatorname{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) d\varphi$$

(b) Show that

$$\operatorname{Re}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}$$

[Hint: For the first part, note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.]

Proof For any $z < R$ fixed, the function $f(\zeta)/(\zeta - w)$ is holomorphic on D_R, so

$$\int_{C_R} \frac{f(\zeta)}{\zeta - w} \mathrm{d}\zeta = 0$$

By Cauchy's integral formular

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

by $|\zeta|^2 = R^2, \zeta = Re^{i\varphi}$ on C_R we have

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{C_R} f(\zeta) \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2 \zeta} \mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \Re(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}) \mathrm{d}\varphi$$

□

2.12 Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x, y) = 0$$

for all $(x, y) \in \mathbb{D}$

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\operatorname{Re}(f) = u$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. [Hint: From the previous chapter we would have $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\operatorname{Re}(F)$ differs from u by a real constant.]

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) u(\varphi) d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}$$

Proof (a) Set $g(z) = 2\frac{\partial u}{\partial z}$, after checking Cauchy-Riemann equations we get $g(z)$ is holomorphic. By theorem 2.1 $g(z)$ has a primitive $F(z)$ on $\mathbb{D}$, let $F(z) = w + iv$ we have $\frac{\partial F}{\partial z} = 2\frac{\partial w}{\partial z}$, then $w - u = c$ for some constant c, then $F(z) - c$ has real part u. By Cauchy-Reimann equations for u, v, we have

$$v(x,y) = \int_{(0,0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y + C_1, (x,y) \in \mathbb{D}$$

where C_1 is a constant.

(b)

$$\begin{aligned}
u(z) =& \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\Re(\frac{e^{i\varphi} + re^{i\varphi}}{e^{i\varphi} - re^{i\varphi}})\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\Re(\frac{e^{i(\varphi-\theta)} + r}{e^{i(\varphi-\theta)} - r})\mathrm{d}\varphi \\
=& \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})\frac{1 - r^2}{1 - 2r\cos(\varphi - \theta) + r^2}\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\varphi})P_r(\theta - \varphi)\mathrm{d}\varphi
\end{aligned}$$

□

2.14 Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denote the power series expansion of f in the open disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Proof Suppose f has Laurent tail $\frac{c}{z-z_0}$ at its pole z_0, then $f(z) - \frac{c}{z-z_0}$ is holomorphic on $\overline{\mathbb{D}}$, then $f(z) - \frac{c}{z-z_0}$ is holomorphic in C_r for some $r > 1$, then $f(z) - \frac{c}{z-z_0}$ has the power series expansion $\sum_{n=0}^{\infty} b_n z^n$, this series is convergence at z_0, so $b_n z_0^n \to 0$ when $n \to \infty$.

For $|z| < 1$ we have

$$\frac{c}{z - z_0} = \frac{-c}{z_0}\frac{1}{1 - \frac{z}{z_0}} = \frac{-c}{z_0}\sum_{n=0}^{\infty}(\frac{z}{z_0})^n$$

so

$$f(z) = \frac{-c}{z_0}\sum_{n=0}^{\infty}(\frac{z}{z_0})^n + \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty}(b_n - \frac{c}{z_0^{n+1}})z^n$$

hence

$$\frac{a_n}{a_{n+1}} = \frac{b_n - \frac{c}{z_0^{n+1}}}{b_{n+1} - \frac{c}{z_0^{n+2}}} = \frac{b_n z_0^{n+2} - cz_0}{b_{n+1}z_0^{n+2} - c} \to \frac{-cz_0}{-c} = z_0, n \to \infty.$$

□

2.15 Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \text{ whenever } |z| = 1,$$

then f is constant.

[Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ whenever $|z| > 1$, and argue as in the Schwarz reflection principle.]

Proof After extension we get an entire function F, we claim it is bounded, if not $F(\infty) = \infty$ then $f(0) = 0$, a contradiction. By Liouville's theorem f is constant. □