Solutions to homework 5

Exercise 3

Proof: By **Theorem 1.7**,

$$\frac{1}{\Gamma(s)}=e^{\gamma s}s\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n},$$

set $s=1/2$, note that $\Gamma(1/2)=\sqrt{\pi}$, we have

$$\begin{aligned}\frac{1}{\sqrt{\pi}}&=\lim_{n\to\infty}\frac{e^{\gamma/2}}{2}\prod_{k=1}^{n}(1+\frac{1}{2k})e^{-1/2k}=\lim_{n\to\infty}\frac{e^{\gamma/2}}{2}e^{-\frac{1}{2}(1+\cdots+1/n)}\prod_{k=1}^{n}(1+\frac{1}{2k})\\&=\frac{1}{2}\lim_{n\to\infty}\frac{1}{\sqrt{n}}\prod_{k=1}^{n}(1+\frac{1}{2k}),\end{aligned}$$

which is equivalent to Walli's product formula.

For the second formula, use **Theorem 1.7** again, we obtain

$$\begin{aligned}\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1/2)}&=\lim_{n\to\infty}e^{\gamma/2}\frac{s+1/2}{2}\prod_{k=1}^{2n}\frac{(1+\frac{s}{k})(1+\frac{2s+1}{2k})}{1+\frac{2s}{k}}e^{-1/2k}\\&=\lim_{n\to\infty}\frac{s+1/2}{2}\frac{1}{\sqrt{2n}}\prod_{k=1}^{2n}\frac{(1+\frac{s}{k})(1+\frac{2s+1}{2k})}{1+\frac{2s}{k}}\\&=\frac{s+1/2}{2}\Big(\lim_{n\to\infty}\prod_{k=n+1}^{2n}(1+\frac{s}{k})(1+\frac{2s+1}{2k})\Big)\cdot\Big(\lim_{n\to\infty}\frac{1}{\sqrt{2n}}\prod_{k=1}^{n}\frac{1+\frac{2s+1}{2k}}{1+\frac{2s}{2k-1}}\Big),\end{aligned}$$

since

$$\begin{aligned}\lim_{n\to\infty}\log\Big(\prod_{k=n+1}^{2n}(1+\frac{s}{k})(1+\frac{2s+1}{2k})\Big)&=\lim_{n\to\infty}\sum_{k=n+1}^{2n}\Big(\log(1+\frac{s}{k})+\log(1+\frac{2s+1}{2k})\Big)\\&=\lim_{n\to\infty}\sum_{k=n+1}^{2n}\Big(\frac{s}{k}+\frac{2s+1}{2k}\Big)=(2s+1/2)\log 2,\end{aligned}$$

we have

$$\lim_{n\to\infty}\prod_{k=n+1}^{2n}(1+\frac{s}{k})(1+\frac{2s+1}{2k})=2^{2s+1/2};$$

and

$$\lim_{n\to\infty}\frac{1}{\sqrt{2n}}\prod_{k=1}^{n}\frac{1+\frac{2s+1}{2k}}{1+\frac{2s}{2k-1}}=\lim_{n\to\infty}\frac{1}{\sqrt{2n}}\frac{2n+1+2s}{2s+1}\frac{(2n-1)!!}{(2n)!!}=\sqrt{\frac{2}{\pi}}\frac{1}{2s+1},$$

hence

$$\frac{\Gamma(2s)}{\Gamma(s)\Gamma(s+1/2)}=\frac{s+1/2}{2}\cdot 2^{2s+1/2}\cdot\sqrt{\frac{2}{\pi}}\frac{1}{2s+1}=\pi^{-1/2}2^{2s-1}.$$

Exercise 4

Proof: $a_n(\alpha) = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}$, by **Theorem 1.7**

$$\frac{1}{\Gamma(\alpha)} = \lim_{n\to\infty} e^{\gamma\alpha}\alpha\prod_{k=1}^{n}(1+\frac{\alpha}{k})e^{-\frac{\alpha}{k}} = \lim_{n\to\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n)}{n!}n^{-\alpha},$$

hence

$$a_n(\alpha) \sim \frac{1}{\Gamma(\alpha)}n^{\alpha-1}.$$

Exercise 6

Proof:

$$\begin{aligned}1+\frac{1}{3}+\cdots+\frac{1}{2n-1} &= \sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{2k} \sim \log 2n+\gamma-\frac{1}{2}(\log n+\gamma)\\ &= \log 2+\frac{1}{2}\log n+\frac{1}{\gamma}.\end{aligned}$$

Exercise 10

Proof: (a) Consider the integral of $f(w)=e^{-w}w^{z-1}$ around the contour. We have $\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(w)\mathrm{d}w = 0$.

- $\int_{\gamma_1}$: $\int_{\gamma_1} f(w)\mathrm{d}w = \int_\epsilon^R e^{-t}t^{z-1}\mathrm{d}t \to \Gamma(z)$, as $R\to\infty, \epsilon\to 0$.
- $\int_{\gamma_2}$:

$$\begin{aligned}|\int_{\gamma_2} f(w)\mathrm{d}w| &= |\int_0^{\pi/2} e^{-Re^{i\theta}}(Re^{i\theta})^{z-1}Rie^{i\theta}\mathrm{d}\theta| \leq \int_0^{\pi/2} e^{-R\cos\theta}R^{\mathrm{Re}(z)}e^{\mathrm{Im}(z)\theta}\mathrm{d}\theta\\ &\leq C\int_0^{\pi/2} e^{-R\cos\theta}R^{\mathrm{Re}(z)}\mathrm{d}\theta = C\int_0^{\pi/2} e^{-R\sin\theta}R^{\mathrm{Re}(z)}\mathrm{d}\theta\\ &\leq C\int_0^{\pi/2} e^{-\frac{2R\theta}{\pi}}R^{\mathrm{Re}(z)}\mathrm{d}\theta = O(R^{-1+\mathrm{Re}(z)}) \to 0, \text{ as } R\to\infty,\end{aligned}$$

  where C depends only on z.
- $\int_{\gamma_3}$: $\int_{\gamma_3} f(w)\mathrm{d}w = -i^z\int_\epsilon^R e^{-it}t^{z-1}\mathrm{d}t \to -i^z\int_0^\infty e^{-it}t^{z-1}\mathrm{d}t$, as $R\to\infty, \epsilon\to 0$.
- $\int_{\gamma_4}$:

$$\begin{aligned}|\int_{\gamma_4} f(w)\mathrm{d}w| &= |\int_0^{\pi/2} e^{-\epsilon e^{i\theta}}(\epsilon e^{i\theta})^{z-1}\epsilon ie^{i\theta}\mathrm{d}\theta| \leq C\int_0^{\pi/2} e^{-\epsilon\cos\theta}\epsilon^{\mathrm{Re}(z)}\mathrm{d}\theta\\ &\leq C\epsilon^{\mathrm{Re}(z)} \to 0, \text{ as } \epsilon\to 0,\end{aligned}$$

  where C depends only on z.

Let $R\to\infty, \epsilon\to 0$ in $\int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(w)\mathrm{d}w = 0$, we have

$$\Gamma(z) - i^z\int_0^\infty e^{-it}t^{z-1}\mathrm{d}t = 0,$$

$\Rightarrow \int_0^\infty e^{-it}t^{z-1}\mathrm{d}t = \Gamma(z)e^{-i\pi z/2}$, i.e.

$$\mathcal{M}(\cos)(z) - i\mathcal{M}(\sin)(z) = \Gamma(z)\cos(\pi\frac{z}{2}) - i\Gamma(z)\sin(\pi\frac{z}{2}),$$

note that $\mathcal{M}(\cos)(z), \mathcal{M}(\sin)(z), \Gamma(z)$ are all real for $0 < z < 1$, thus

$$\mathcal{M}(\cos)(z) = \Gamma(z)\cos(\pi\frac{z}{2}), \mathcal{M}(\sin)(z) = \Gamma(z)\sin(\pi\frac{z}{2}),$$

for $0 < z < 1$ and they are also valid for all $0 < \mathrm{Re}(z) < 1$.

(b) One checks that $\int_0^\infty \sin t\, t^{z-1}\mathrm{d}t$ converges and is analytic in $-1 < \mathrm{Re}(z) < 1$, hence

$$\mathcal{M}(\sin)(z) = \Gamma(z)\sin(\pi\frac{z}{2}), -1 < \mathrm{Re}(z) < 1.$$

Now

$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2} \text{ and } \int_0^\infty \frac{\cos x}{x^{3/2}}\mathrm{d}x = \sqrt{2\pi}$$

follow from the above formula by setting $z = 0$, $z = -\frac{1}{2}$ respectively and that $\lim_{z\to 0} z\Gamma(z) = 1$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

Exercise 12

Proof: (a)

$$|\frac{1}{\Gamma(-k-1/2)}| = |\frac{1}{\Gamma(-1/2)}\frac{(-1)^k(2k+1)!!}{2^k}| = |\frac{(-1)^{k+1}(2k+1)!!}{\sqrt{\pi}2^{k+1}}| > \frac{k!}{2\sqrt{\pi}},$$

since $k!$ is not $O(e^{ck})$, $|\frac{1}{\Gamma(-k-1/2)}|$ is not $O(e^{ck})$, hence $\frac{1}{\Gamma(s)}$ is not $O(e^{c|s|})$.

(b) If such entire function $F(s)$ exists, then $F(s)\Gamma(s)$ is entire with no zeros. Note that $F(s)\Gamma(s)$ has order of growth less than 2, by **Hadamard's factorization theorem**,

$$F(s)\Gamma(s) = e^{Az+B}, \text{ for some } A, B,$$

$\Rightarrow |\frac{1}{\Gamma(z)}| = O(e^{c|z|})$, a contradiction!

Exercise 14

Proof: (a) For $x > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x+1}\log\Gamma(t)\mathrm{d}t = \log\Gamma(x+1) - \log\Gamma(x) = \log x,$$

as a result,

$$\int_x^{x+1}\log\Gamma(t)\mathrm{d}t = x\log x - x + c.$$

(b) Since $\Gamma(x)$ is monotonically increasing for large x,

$$\log\Gamma(n) \leq n\log n - n + c = \int_n^{n+1} \log\Gamma(t)\mathrm{d}t \leq \log\Gamma(n+1),$$

it is readily from the above inequality that $\log\Gamma(n) \sim n\log n + O(n)$.

Problem 2

Proof: For $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}\mathrm{d}x &= \frac{s}{s-1} - s\sum_{n=1}^{\infty}\int_0^1 \frac{x}{(x+n)^{s+1}}\mathrm{d}x \\
&= \frac{s}{s-1} - s\sum_{n=1}^{\infty}\int_0^1 \Big(\frac{1}{(x+n)^s} - \frac{n}{(x+n)^{s+1}}\Big)\mathrm{d}x \\
&= \frac{s}{s-1} - s\sum_{n=1}^{\infty}\Big(\frac{1}{1-s}(\frac{1}{(1+n)^{1-s}} - \frac{1}{n^{1-s}}) + \frac{n}{s}(\frac{1}{(1+n)^s} - \frac{1}{n^s})\Big) \\
&= \frac{s}{s-1} - s\Big(-\frac{1}{1-s} + \frac{1}{s}\sum_{n=1}^{\infty}(\frac{n}{(1+n)^s} - \frac{1}{n^{s-1}})\Big) \\
&= 1 + \sum_{n=1}^{\infty}\frac{1}{(1+n)^s} = \zeta(s),
\end{aligned}$$

that is

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}\mathrm{d}x,\ \text{for } \mathrm{Re}(s) > 1.$$

Since the right side of the above formula is holomorphic in $\mathrm{Re}(s) > 0$, by analytic continuation, the formula is valid in $\mathrm{Re}(s) > 0$.

Problem 3

Proof: Omitted.

For Exercise 15,16,17, see hints.