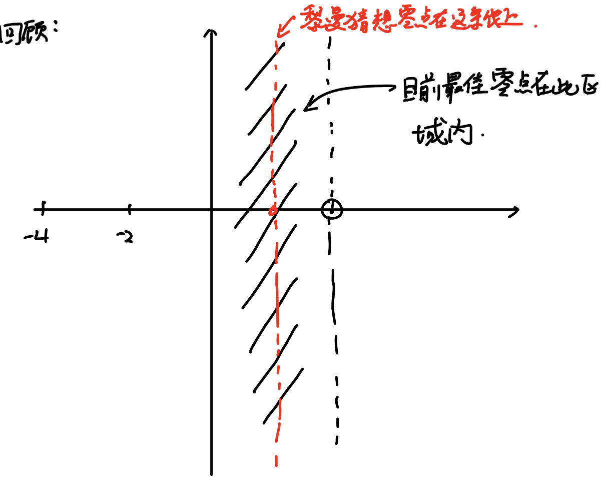
回顾:



上节课的Hadamard 定理.
Poussin

计算黎曼Zeta函数的零点非常困难.

$\frac{1}{2} + \underline{s}\cdot i$

14.134
21.022
25.010
30.424
⋮

$$\zeta = \sum n^{-s} = \prod_p (1-p^{-s})^{-1}$$

$$\ln\zeta(s) = -\sum_p \ln(1-p^{-s}) = \sum_{p,n} \frac{1}{n} p^{-ns}$$
$$= \sum C_n n^{-s}$$

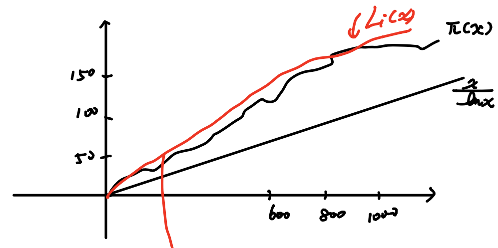
$$\boxed{C_n = \begin{cases} \frac{1}{m} & n = p^m \\ 0 & \text{otherwise} \end{cases}}$$

$$= \int_0^\infty x^{-s}\, d\Pi(x)$$

$$\Pi(x) = \sum_n \frac{1}{n} \pi(x^{\frac{1}{n}})$$

$$\pi(x) = \#\{p \le x\}$$

$$\frac{x}{\ln x}, \quad Li(x) = \int_2^x \frac{dt}{\ln(t)}$$



不一定比π(x)大.

可以证明, x充分大时, [Li(x)-π(x)]正负号反复出现.

这个反例出现的数字非常大.

$$\Pi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\ln\zeta(z)}{z} x^z\, dz$$

$a>1$

$$= -\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty} x^z\, d\left(\frac{\ln\zeta(z)}{z}\right)$$

平凡零点

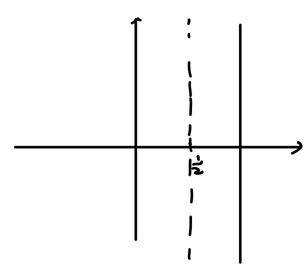
$$\ln\zeta(s) = \ln(\zeta(0)) - \ln(s-1) + \frac{s}{2}\ln\pi - \ln\Gamma\left(\frac{s}{2}+1\right)$$
$$+ \sum_\rho \ln\left(1-\frac{s}{\rho}\right)$$

非平凡零点.

积分 ↓ ↓ ↓

$$\Pi(x) = -\ln 2 + Li(x) + 0$$

$$+ \int_x^\infty \frac{dt}{t(t^2-1)\ln t} - \sum_{\mathrm{Im}\,\rho}\left[Li(x^\rho) + Li(x^{1-\rho})\right]$$



Chebeshev / Merlin Golden function

$$\psi(x) = \sum_{n\le x}\Lambda(n) = \sum_{n=1}^\infty \Lambda(n) f_n(x)$$

$$f(x) = \begin{cases} 1 & n \le x \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(n) = \begin{cases} \ln p & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(x) = \ln\left[\,LCM(1,2,3,\cdots,[x])\,\right]$$

$$= \sum_{p\le x}\left[\frac{\ln x}{\ln p}\right]\cdot \ln p$$

$$\psi(10) = \ln 2\cdot 3 + \ln 3\cdot 2 + \ln 5 + \ln 7$$

$$= \ln\left(LCM(1,2,\cdots,10)\right)$$

$$\boxed{\psi(x) = x - \frac{1}{2}\ln(1-x^{-2}) - \ln(2\pi) - \sum_{\rho}\frac{x^{\rho}}{\rho}}$$

定理: ① $\psi(x) \sim x$

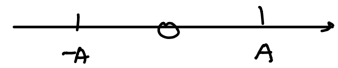
$$\Longleftrightarrow \pi(x) \sim \frac{x}{\ln x}$$

② $\psi(x) \sim x$

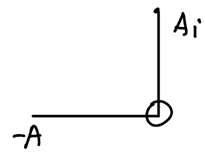
$$\Longleftrightarrow \psi_1(x) \sim \frac{1}{2}x^2$$

$$\psi_1(x) = \int_1^x \psi(u)\,du$$

① $\displaystyle\int_{-A}^{A}\frac{dz}{z}$



②



$$\int_{\gamma}\frac{dz}{z}$$

证明: $\psi(x)\sim x \Rightarrow \pi(x) \sim \dfrac{x}{\ln x}$

$$\psi(x) = \sum_{p\le x}\left[\frac{\ln x}{\ln p}\right]\cdot \ln p \le \sum_{p\le x}\frac{\ln x}{\ln p}\cdot \ln p$$

$$= \sum_{p\le x}\ln x = \pi(x)\cdot \ln(x)$$

$$\frac{\psi(x)}{x} \le \frac{\pi(x)}{\frac{x}{\ln x}}$$

取 $0<\alpha<1$

$$\psi(x) \ge \sum_{p\le x}\ln p \ge \sum_{x^{\alpha}<p\le x}\ln p$$

$$\ge \left(\pi(x) - \pi(x^{\alpha})\right)\cdot \ln x^{\alpha}$$

$$\psi(x) + \alpha\cdot\pi(x^{\alpha})\cdot\ln x \ge \alpha\cdot\pi(x)\ln x$$

$$\frac{\psi(x)}{x} + \alpha\cdot\underbrace{\pi(x^{\alpha})}_{\le x^{\alpha}}\cdot\frac{\ln x}{x} \ge \alpha\cdot\pi(x)\cdot\frac{\ln x}{x}$$

$x\to\infty$

$$1 + 0 \ge \alpha \varliminf_{x\to+\infty}\pi(x)\cdot\frac{\ln x}{x}$$

$$1 \ge \varliminf_{x\to+\infty}\frac{\pi(x)}{\frac{x}{\ln x}}$$

大作业: 与复变有关的问题. 综述性的报告