1. (1) $\left(\frac{dy}{dx}+2y\right)e^{2x}=xe^{x}$

$(ye^{2x})'=xe^{x}$

$ye^{2x}=(x-1)e^{x}+C$

$y=(x-1)e^{-x}+Ce^{-2x}$

(2) $\int \tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\ln(\cos x)+C$

$\therefore\left(\frac{dy}{dx}+y\tan x\right)\frac{1}{\cos x}=\sin 2x\frac{1}{\cos x}$

$\left(y\frac{1}{\cos x}\right)'=2\sin x$

$y\frac{1}{\cos x}=-2\cos x+C$

$y=-2\cos^2 x+C\cos x$

(3) $\frac{dy}{dx}+\frac{2}{x}y=\frac{\sin x}{x}$, $x=0$ 时 $y=0$, $\frac{dy}{dx}$ 未定义.

$\left(\frac{dy}{dx}+\frac{2}{x}y\right)x^2=\frac{\sin x}{x}\cdot x^2$

$(yx^2)'=x\sin x$

$yx^2=-\left(x\cos x-\int\cos x\,dx\right)$

$=-x\cos x+\sin x+C$

$y=-\frac{\cos x}{x}+\frac{\sin x}{x^2}+\frac{C}{x^2}.$

代入 $y(\pi)$,

$\frac{1}{\pi}=\frac{1}{\pi}+\frac{C}{\pi^2}$, $\therefore C=0$

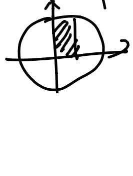
$\therefore y=-\frac{\cos x}{x}+\frac{\sin x}{x^2}$ 且 $x=0$ 时 $y=0$

(4) $\left(\frac{dy}{dx}-\frac{1}{1-x^2}y\right)\sqrt{\left|\frac{x-1}{x+1}\right|}=(1+x)\sqrt{\left|\frac{x-1}{x+1}\right|}$

$=\sqrt{(x-1)(x+1)}$

$\left(y\sqrt{\left|\frac{x-1}{x+1}\right|}\right)'=\sqrt{|(x-1)(x+1)|}=\sqrt{|x^2-1|}=\sqrt{1-x^2}$

$\therefore y\sqrt{\left|\frac{x-1}{x+1}\right|}=1+\frac{1}{2}\arcsin x+\frac{x\sqrt{1-x^2}}{2}$



$y=\sqrt{\frac{1+x}{1-x}}\left(\frac{1}{2}\arcsin x+\frac{x}{2}\sqrt{1-x^2}+1\right)$

2.

(1) $\frac{dy}{dx}=\frac{x^2}{2y}+\frac{y}{2}$

$dy=\frac{x^2dx}{2y}+\frac{y}{2}dx$

$d\left(\frac{y^2}{2}\right)=\frac{x^2}{2}dx+\frac{y^2}{2}dx=\frac{x^2+y^2}{2}dx$

$\therefore \frac{dy^2}{dx}=x^2+y^2$ 是所求.

(2) $dy(x+y^2)=y\cdot dx$

$\frac{dx}{dy}=y+\frac{x}{y}$

(3) $3xy^2dy+y^3dx+x^3dx=0$

$x\,d(y^3)+(x^3+y^3)dx=0$

令 $z=y^3$

$x\,dz+(x^3+z)dx=0$

$\frac{dz}{dx}=-x^2-\frac{z}{x}$

(4) $dy=\frac{1}{\cos y}dx+x\tan y\cdot dx$

$\cos y\,dy=dx+x\sin y\,dx$

$=(1+x\sin y)dx$

$d\sin y=(1+x\sin y)dx$

令 $z=\sin y$

$dz=(1+xz)dx$

$\frac{dz}{dx}=1+xz$

3. $(y'+a(x)y)e^{\int_0^x a(s)ds}\le 0\quad(x\ge 0)$

$\left(ye^{\int_0^x a(s)ds}\right)'\le 0$

$\therefore \varphi(x)e^{\int_0^x a(s)ds}$ 是减函数.

故 $\varphi(x)e^{\int_0^x a(s)ds}\le\varphi(0)$

即 $\varphi(x)\le\varphi(0)e^{-\int_0^x a(s)ds}$

4. 设 $y=C(x)e^{-\int p(x)dx}$

$$C'(x)e^{-\int p(x)dx}+C(x)e^{-\int p(x)dx}(-p(x))+p(x)C(x)e^{-\int p(x)dx}=q(x)$$

$$\therefore C'(x)=q(x)e^{\int p(x)dx}$$

$$C(x)=\int q(x)e^{\int p(x)dx}+C$$

$$\therefore y=\left(\int q(x)e^{\int p(x)dx}+C\right)e^{-\int p(x)dx}$$

原理：不过换元而已.

5. (1) $q(x)\equiv 0 \Rightarrow y=Ce^{-\int_{x_0}^{x}p(s)ds}$

则 $y(x+w)=Ce^{-\int_{x_0}^{x+w}p(s)ds}$

$$=Ce^{-\int_{x_0}^{x}p(s)ds-\int_{x}^{x+w}p(s)ds}$$

$$=Ce^{-\int_{x_0}^{x}p(s)ds}=y(x)$$

(2)

$$y(x)=\left(\int q(x)e^{\int p(x)dx}+C\right)e^{-\int p(x)dx}$$

只需 $y(x_0)=y(x_0+w)$，$\exists x_0$ 就可保证是周期解了

如果周期解不唯一，$\exists u(x),v(x)$ 周期且 $u(x)-v(x)$ 非0，满足齐次方程

$$\frac{dy}{dx}+p(x)y=0 \qquad (*)$$

那么 $u(x)-v(x)$ 由上是以 w 为周期的

如果 $\bar{p}=0$，知确实存在这样的 $u-v$，故周期解不唯一.

如果 $\exists$ 非0的 $\varphi(x)$ 以 w 为周期满足方程 $(*)$，

则 $Ce^{-\int_{x}^{x+w}p(s)ds}=0$，$\forall x$.

$\Leftrightarrow \forall x,\ \int_{x}^{x+w}p(s)ds=0$

$\Leftrightarrow \bar{p}=0$.

6.

$$(ye^{x})'=f(x)e^{x}$$

$$\therefore ye^{x}=\int_{-\infty}^{x}e^{s}f(s)ds+C \qquad (ye^{-\infty}=0)$$

$$y=e^{-x}\int_{-\infty}^{x}e^{s}f(s)ds+Ce^{-x}$$

$$=\left(C+\int_{-\infty}^{x}f(s)\cdot e^{s}ds\right)e^{-x}$$

而 $\left|\int_{-\infty}^{x_0}f(s)e^{s}ds\right|\le|M|\left|\int_{-\infty}^{x_0}e^{s}ds\right|<+\infty$

$\therefore C=0$

$\therefore$ 有界解为 $\left[\int_{-\infty}^{x}f(s)e^{s}ds\right]e^{-x}$

$$y=\int_{-\infty}^{x}f(s)e^{s-x}ds$$

若 $f(x)$ 以 w 为周期，

则 $y(x+w)=\int_{-\infty}^{x+w}f(s)e^{s}ds\ \frac{1}{e^{x+w}}$

令 $s=t+w$

则 $y(x+w)=\int_{-\infty}^{x}f(t+w)e^{t+w}dt\ \frac{1}{e^{x+w}}$

$$=\int_{-\infty}^{x}f(t+w)e^{t}dt\ \frac{1}{e^{x}}$$

$$=\int_{-\infty}^{x}f(t)e^{t}dt\ \frac{1}{e^{x}}=y(x)$$

7. 一致收敛性易得完备

$$\varphi: f\to y=\frac{1}{e^{2a\pi}-1}\int_{x}^{x+2\pi}e^{-a(x-s)}f(s)ds \qquad (a>0)$$

$$\varphi(C_1f_1+C_2f_2)$$

$$=\frac{1}{e^{2a\pi}-1}\int_{x}^{x+2\pi}e^{-a(x-s)}(C_1f_1+C_2f_2)(s)ds$$

$$=C_1\varphi(f_1)+C_2\varphi(f_2)$$

且 $\varphi(f)\le\frac{2\pi}{e^{2a\pi}-1}\|f\|$

$$\le\frac{2\pi}{2a\pi}\|f\|=\frac{1}{a}\|f\|$$