Theorem 1. *Let $G = (V, E, M, \phi, \gamma, s, t)$ be a computation graph, then it holds that:*

$$\begin{aligned} \forall (v_1, m_1) \in (V \backslash \{t\} \times M) \exists ! (v_2, m_2) \in V \times M : \\ (v_1, m_1) \vdash_G (v_2, m_2) \end{aligned} \tag{1}$$

Proof 1. *1. Let $(v_1, m_1) \in (V \backslash \{t\} \times M)$ and $V_{v_1 out} := \{v_2 \in V | (v_1, v_2) \in E\}$*

$$\overset{5}{\Longrightarrow} |V_{v_1 out}| > 0$$

$$\overset{5}{\Longrightarrow} \exists ! (v_1, v_2) \in V_{v_1 out} : \gamma((v_1, v_2)) = 1$$

We choose $m_2 = \phi(v_1)(m_1)$.

$$\Longrightarrow ((v_1, v_2) \in E \wedge \gamma(v_1, v_2) = 1 \wedge m_2 = \phi(v_1)(m_1)$$

$$\overset{6}{\Longleftrightarrow} (v_1, m_1) \vdash_G (v_2, m_2))$$

2. *Let $(v_1, m_1), (v_2, m_2), (v_3, m_3) \in (V \backslash \{t\} \times M)$.*

Suppose

- $(v_2, m_2) \neq (v_3, m_3)$,
- $(v_1, m_1) \vdash_G (v_2, m_2) \wedge (v_1, m_1) \vdash_G (v_3, m_3)$.

$$\overset{6}{\Longrightarrow} (v_1, v_2), (v_1, v_3) \in V_{v_1 out}$$
and $\gamma(v_1, v_2) = \gamma(v_1, v_3) = 1$

Conversely, it holds that:

$$(v_1, v_2), (v_1, v_3) \in V_{v_1 out} \overset{5}{\Longrightarrow}$$

$$\gamma(v_1, v_2) \neq \gamma(v_1, v_3) \vee \gamma(v_1, v_2) = \gamma(v_1, v_3) = 0$$

This contradiction implies:

- $(v_2, m_2) = (v_3, m_3)$
- *or* $(v_1, m_1) \not\vdash_G (v_2, m_2)$
- *or* $(v_1, m_1) \not\vdash_G (v_3, m_3)$

■

Theorem 2. *Let $G = (V, E, M, \phi, \gamma, s, t)$ be a computation graph, then it holds that:*

$$\begin{aligned} \forall (v_1, m_1) \in V \times M : \\ (v_2, m_2) \in \{t\} \times M \implies (v_2, m_2) \not\vdash_G (v_1, m_1) \end{aligned} \tag{2}$$

Proof 2. *Suppose $(v_2, m_2) \in \{t\} \times M, (v_1, m_1) \in V \times M$ and $(v_2, m_2) \vdash_G (v_1, m_1)$.*

$$\overset{6}{\Longrightarrow} (v_2, v_1) \in E$$

Conversely, it holds that:

$$v_2 = t \overset{5}{\Longrightarrow} \forall v_1 \in V : (v_2, v_1) \notin E.$$

This contradiction implies (by theorem 1):
$v_2 \neq t \vee ((v_2, m_2) \not\vdash_G (v_1, m_1))$

■

Corollary 3. *Let $G = (V, E, M, \phi, \gamma, s, t)$ be computation graph, $m_1 \in M$ an initial configuration and $\vdash_G^+$ the transitive hull of $\vdash_G$. Then, it holds that:*

$$\begin{aligned} (s_1, m_1) \vdash_G (s_2, m_2) \vdash_G ... is finite \\ \Longleftrightarrow \exists m_2 \in M : (s, m_1) \vdash_G^+ (t, m_2) \end{aligned} \tag{3}$$

Proof 3. *The corollary is implied by $s \overset{5}{\neq} t$, theorem 1 and theorem 2.*

■

Theorem 4. *There is an algebra $(M, (f_i)_{i \in I})$, so that for all deterministic Turing machines*

$$T := (Z, \Gamma, \Sigma, \delta, z_0, b, t_T) \tag{4}$$

exists an equivalent computation graph

$$G = (V, E, M, \phi, \gamma, s, t_G) \tag{5}$$

so that for each $\alpha z_0 \beta$ exists $m_1 \in M$, so that:

$$\alpha z_0 \beta \vdash_T^+ \alpha' t_T \beta' \iff (s, m_1) \vdash_G^+ (t_G, m_2) \tag{6}$$

Proof 4. *Let $T = (Z, \Gamma, \Sigma, \delta, z_0, b, z_e)$ be an arbitrary deterministic Turing-machine and $w \in \Sigma^*$. Following Schöning[Schöning, 2003], we can assume the existence of an equivalent Turing-machine*

$$T_\epsilon = (Z_\epsilon, \Gamma_\epsilon, \Sigma_\epsilon, \delta_\epsilon, z_0^\epsilon, b_\epsilon, z_e^\epsilon), \tag{7}$$

so that the following holds[Schöning, 2003]:

$$z_0 w \vdash_T^+ \alpha z_e \beta \iff z_0^\epsilon \vdash_{T_\epsilon}^+ \alpha z_e^\epsilon \beta \tag{8}$$

The theorem is proven if we can specify

- *an algebra $(M, (f_i))_{i\ I}$*
- *an initial configuration $m \in M$*
- *a computation graph $G = (V, E, M, \phi, \gamma, s, t)$*

for which it holds that:

$$z_0^\epsilon \vdash_{T_\epsilon}^+ \alpha z_e^\epsilon \beta \iff (s, m) \vdash_G^+ (t, m'), m' \in M \tag{9}$$

Without loss of generality we assume $\Gamma_\epsilon \subset \mathrm{N}n\{0\}$ as well as $b_\epsilon = 1$ and make the following choices:

1. $(N, left, right, write_{a_1}, ..., write_{a_n})$ as algebra, where $a_i \in \Gamma_\epsilon$,
2. $m = c(0, c(b_\epsilon, 0))$ as initial configuration,
3. $\pi_1 : N \to N, n \mapsto c_l^- 1(c_r^- 1(m))$
4. $G = (V, E, \mathrm{N}, \phi, \gamma, s, t)$, as configuration graph, where
   - $V := Z_\epsilon \cup \{s, t\}, Z_\epsilon \cap \{s, t\} = \emptyset$
   - $E := \{(v_1, v_2) \in V^2 | \exists a, b \in \Gamma_\epsilon \exists k \in \{L, N, R\} : \delta_\epsilon(v_1, a) = (v_2, b, k)\} \cup \{(s, z_0^\epsilon), (z_e^\epsilon, t)\}$
   - $\phi(v) = update_v$, where $\phi(s) = identity$
   - $\gamma(v_1, v_2) = transition_{v_1, v_2}$, where $\gamma(s, z_0^\epsilon) = 1$[1]

Depending on a node $v \in V$ $update_v$ extracts the symbol on top of the right stack and applies a $write_i$ to update it in line with the Turing machines transition function:

$$\begin{aligned} update_v(m) = write_i(m) \iff \\ \exists z' \in Z_\epsilon \exists k \in \{L, R, N\} : \\ \delta_\epsilon(v, c_l^- 1(c_r^- 1(m))) = (z', i, k) \end{aligned} \tag{10}$$

The function $transition_{v_1, v_2}$ ensures that the edge (v_1, v_2) of the computation graph can be passed if and only if the corresponding Turing machine transitions from v_1 to v_2 if the symbol on the tape is n.

$$transition_{v_1, v_2}(n) = \begin{cases} 1 & (v_1, v_2) \in E \wedge \\ & \exists b \in \Gamma \exists k \in \{L, R, N\} : \\ & \delta_\epsilon(v_1, n) = (v_2, b, k) \\ & \\ \perp & \text{else} \end{cases} \tag{11}$$

Finally, the theorem follows by induction over $k \geq 1$, where k represents the number of transitions.

■

Theorem 5. *Let $T_G = (V, (transition_i)_{i \in I})$ be the induced transition algebra induced by a directed graph G.*

Then, for t_1, ..., $t_{n-1} \in T^G_{transition}$ it holds that:

$$\begin{aligned} \forall n \geq 2 : \forall (v_1, ..., v_n) \in V^n : \\ connected^n_G(v_1, v_n) \implies v_n = t_{n-1} \circ ... \circ t_1(v_1) \end{aligned} \tag{12}$$

Proof 5. *We prove theorem 5 by induction for all $n \geq 2$.*

- *Base Case ($n = 2$):*

  $connected^2_G(v_1, v_2)$

  $$\implies \exists (v_1, v_2) \in V^2 : (v_1, v_2) \in E$$

  $$\implies \exists i \in I : v_2 = transition_i(v_1)$$

- *Inductive Step ($n + 1 \geq 3$):*

  Suppose $(v_1, ..., v_{n+1}) \in V^{n+1}$ and

  $\forall (v_1, ..., v_n) \in V^n :$
  $connected^n_G(v_1, v_n) \implies v_n = t_{n-1} \circ ... \circ t_1(v_1)$.

  Suppose $connected^{n+1}_G(v_1, v_{n+1}) \implies \forall 1 \leq i \leq n : (v_i, v_{i+1}) \in E$

  We have to make a case distinction:

  1. $suc(v_n) = 1 \overset{(v_n, v_{n+1}) \in E}{\Longrightarrow} transition_\epsilon(v_n) = v_{n+1}$
  2. $suc(v_n) > 1 \overset{(v_n, v_{n+1}) \in E}{\Longrightarrow} transition_{v_{n+1}}(v_n) = v_{n+1}$

  Using the inductive hypothesis and 1) as well as 2), we can conclude:

  $$v_n = t_{n-1} \circ ... \circ t_1(v_1) \wedge \exists i \in I : v_{n+1} = t_i(v_n)$$

  $$\implies v_{n+1} = t_i \circ t_{n-1} \circ ... \circ t_1(v_1)$$

■

Theorem 6. *Let $G = (V, E)$ be a directed graph and*

$$T_G = (V, (transition_i)_{i \in I}) \tag{13}$$

the induced transition algebra. Then for all $v_i, v_j \in V$, $v_i \neq v_j$ it holds that:

$$\begin{aligned} \forall n \geq 0 \forall (v_1, ..., v_{n+2}) \in \{v_i\} \times V^n \times \{v_j\} \\ \forall k \geq 1 : (t_1, ..., t_k) \in T^k_G \implies v_j \neq t_k \circ ... \circ t_1(v_i) \end{aligned} \tag{14}$$

Proof 6. *Suppose $v_i, v_j \in V, v_i \neq v_j$. We assume that there is no path in G connecting v_i and v_j, or formally:*

$$\begin{aligned} (\forall n \geq 2 \forall (v_1, ..., v_n) \in V^n : \\ (v_1 = v_i \wedge v_n = v_j \implies \exists 1 \leq i \leq n-1 : (v_i, v_{i+1}) \notin E) \end{aligned} \tag{15}$$

Furthermore, we assume the existence of a sequence of transitions $t_1, ..., t_k \in T_G$ leading from v_i to v_j, or formally:

$$\exists k \geq 1 \exists (t_1, ..., t_k) \in T^G_{transition} : v_j = t_k \circ ..., \circ t_1(v_i)) \tag{16}$$

The latter results in a tuple

$$(t_k \circ ... t_2 \circ t_1(v_i), ..., t_2 \circ t_1(v_i), t_1(v_i), v_i) \in V^k, \tag{17}$$

[1]A constant function of arity one whose value is 1.

where $t_k \circ ...t_2 \circ t_1(v_i) = v_j$.

For subsequent entries v_i, v_{i+1} *it holds that* $(v_i, v_{i+1}) \in E \vee v_i = v_{i+1}$.

As long as not all subsequent entries are distinct, we iteratively remove one entry from such pairs. In the resulting tuple t_{res}*, subsequent entries are distinct, so that for subsequent entries* v_i, v_{i+1} *it holds that* $(v_i, v_{i+1}) \in E$.

Conversely, we assumed $v_i \neq v_j$ *, implying that:* $t_{res} = (v_j, ..., v_i)$*, which in turn implies a path connecting* v_j *and* v_i*. This contradicts our assumption that no such path exists.*

Consequently, no such sequence of $(t_1, ..., t_k) \in T^G_{transition}$ *can exist.*

■

References

[Schöning, 2003] Uwe Schöning. *Theoretische Informatik - kurzgefasst*. Spektrum Akademischer Verlag, 4. a. (korrig. nachdruck 2003) edition, 2003.