SPECIAL ORTHOGONAL, SPECIAL UNITARY, AND SYMPLECTIC GROUPS AS PRODUCTS OF GRASSMANNIANS

LEK-HENG LIM, XIANG LU, AND KE YE

Abstract. We describe a curious structure of the special orthogonal, special unitary, and symplectic groups that has not been observed, namely, they can be expressed as matrix products of their corresponding Grassmannians realized as involution matrices. We will show that $\mathrm{SO}(n)$ is a product of two real Grassmannians, $\mathrm{SU}(n)$ a product of four complex Grassmannians, and $\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Sp}(2n, \mathbb{C})$ a product of four symplectic Grassmannians over $\mathbb{R}$ or $\mathbb{C}$ respectively.

1. Introduction

By definition, the real, complex, real symplectic, and complex symplectic Grassmannians are respectively

$$
\begin{aligned}
\mathbb{G}(k, \mathbb{R}^n) &= \{k\text{-dimensional real subspaces in } \mathbb{R}^n\},\\
\mathbb{G}(k, \mathbb{C}^n) &= \{k\text{-dimensional complex subspaces in } \mathbb{C}^n\},\\
\mathbb{G}_{\mathrm{Sp}}(2k, \mathbb{R}^{2n}) &= \{2k\text{-dimensional real symplectic subspaces in } \mathbb{R}^{2n}\},\\
\mathbb{G}_{\mathrm{Sp}}(2k, \mathbb{C}^{2n}) &= \{2k\text{-dimensional complex symplectic subspaces in } \mathbb{C}^{2n}\}.
\end{aligned} \tag{1}
$$

These are real smooth manifolds and real affine varieties but the descriptions in (1) are abstract and coordinate-free. To do almost anything with these Grassmannians, one needs a more concrete characterization of these objects. For example, a common approach is to characterize them as homogeneous spaces:

$$
\begin{aligned}
\mathbb{G}(k, \mathbb{R}^n) &\cong \mathrm{O}(n)/\big(\mathrm{O}(k) \times \mathrm{O}(n-k)\big),\\
\mathbb{G}(k, \mathbb{C}^n) &\cong \mathrm{U}(n)/\big(\mathrm{U}(k) \times \mathrm{U}(n-k)\big),\\
\mathbb{G}_{\mathrm{Sp}}(2k, \mathbb{R}^{2n}) &\cong \mathrm{Sp}(2n, \mathbb{R})/\big(\mathrm{Sp}(2k, \mathbb{R}) \times \mathrm{Sp}(2n-2k, \mathbb{R})\big),\\
\mathbb{G}_{\mathrm{Sp}}(2k, \mathbb{C}^{2n}) &\cong \mathrm{Sp}(2n, \mathbb{C})/\big(\mathrm{Sp}(2k, \mathbb{C}) \times \mathrm{Sp}(2n-2k, \mathbb{C})\big).
\end{aligned} \tag{2}
$$

In our recent works [11, 12, 14, 15], largely motivated by computational considerations, we characterized them as submanifolds of matrices:

$$
\begin{aligned}
\mathrm{Gr}(k, \mathbb{R}^n) &:= \{X \in \mathrm{O}(n) : X^2 = I_n,\ \mathrm{tr}(X) = 2k - n\},\\
\mathrm{Gr}(k, \mathbb{C}^n) &:= \{X \in \mathrm{U}(n) : X^2 = I_n,\ \mathrm{tr}(X) = 2k - n\},\\
\mathrm{Gr}_{\mathrm{Sp}}(2k, \mathbb{R}^{2n}) &:= \{X \in \mathrm{Sp}(2n, \mathbb{R}) : X^2 = I_{2n},\ \mathrm{tr}(X) = 4k - 2n\},\\
\mathrm{Gr}_{\mathrm{Sp}}(2k, \mathbb{C}^{2n}) &:= \{X \in \mathrm{Sp}(2n, \mathbb{C}) : X^2 = I_{2n},\ \mathrm{tr}(X) = 4k - 2n\}.
\end{aligned} \tag{3}
$$

These are submanifolds of $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{C})$ respectively and are isomorphic to their abstract counterparts in (1) or homogeneous space counterparts in (2), whether as smooth manifolds or as real affine varieties. We call them the *involution models* for the real, complex, real symplectic, and complex symplectic Grassmannians respectively.

The goal of our article is to state and prove a somewhat curious structure. For any two subsets of $n \times n$ matrices S_1 and S_2, write $S_1 \cdot S_2 = \{X_1X_2 : X_1 \in S_1,\ X_2 \in S_2\}$. Then

$$\begin{aligned}
\operatorname{SO}(n) &= \operatorname{Gr}(\lfloor \tfrac{n}{2} \rfloor, \mathbb{R}^n) \cdot \operatorname{Gr}(\lfloor \tfrac{n}{2} \rfloor, \mathbb{R}^n),\\
\operatorname{SU}(n) &= \operatorname{Gr}(\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^n) \cdot \operatorname{Gr}(\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^n) \cdot \operatorname{Gr}(\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^n) \cdot \operatorname{Gr}(\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^n),\\
\operatorname{Sp}(2n,\mathbb{R}) &= \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{R}^{2n}) \cdot \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{R}^{2n}) \cdot \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{R}^{2n}) \cdot \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{R}^{2n}),\\
\operatorname{Sp}(2n,\mathbb{C}) &= \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^{2n}) \cdot \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^{2n}) \cdot \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^{2n}) \cdot \operatorname{Gr}_{\operatorname{Sp}}(2\lfloor \tfrac{n}{2} \rfloor, \mathbb{C}^{2n}).
\end{aligned} \tag{4}$$

To the best of our knowledge, this neat relation between a classical group and its corresponding Grassmannian has never been observed before, which is somewhat surprising given that these groups and Grassmannians are ubiquitous and have been thoroughly studied. We view these relations as similar to the classical Lie group decompositions of Bruhat, Cartan, Iwasawa, Jordan, Langlands, Levi, et al [2, 9, 3]. Like these classical decompositions, our product $\cdot$ is given by matrix product; but while the factors in these classical decompositions form subgroups, our factors just form subvarieties.

It is well-known that isometries are compositions of involutions, whether over real [10] or complex [7, 8, 17], or in a symplectic setting [1, 4]; see [6] for a general overview. We stress that our results in this article cannot be deduced from these existing results and our proofs are of a distinct nature from those in [1, 4, 6, 7, 8, 10, 17]. On the other hand, over finite-dimensional spaces, the key results in [1, 4, 6, 7, 8, 10, 17] will follow from ours.

Readers familiar with Lie Theory may have observed that the involution models of Grassmannians in (3) are adjoint orbits. But the equalities in (4) do not hold for general adjoint orbits; in fact they do not even hold when $\lfloor n/2 \rfloor$ is replaced with other values. For example we will see in Theorem 2.9 that $\operatorname{SO}(n) \neq \operatorname{Gr}(k_1, \mathbb{R}^n) \cdot \operatorname{Gr}(k_2, \mathbb{R}^n)$ for almost all other values of k_1, k_2.

Our results in (4) are exhaustive in an appropriate sense: For a real or complex vector space $\mathbb{V}$ equipped with an additional structure, the Grassmannian over $\mathbb{V}$ should generally respect the group action that preserves that structure. The four cases in (1) discussed in this article cover the most common type of structures on $\mathbb{V}$ — a nondegenerate bilinear form β. For a positive definite symmetric or Hermitian β, the group action is given by $\operatorname{O}(n)$ or $\operatorname{U}(n)$ and we obtain $\mathbb{G}(k, \mathbb{R}^n)$ or $\mathbb{G}(k, \mathbb{C}^n)$. For a skew-symmetric β, the group action is given by $\operatorname{Sp}(2n, \mathbb{R})$ or $\operatorname{Sp}(2n, \mathbb{C})$ and we obtain $\mathbb{G}_{\operatorname{Sp}}(2k, \mathbb{R}^{2n})$ and $\mathbb{G}_{\operatorname{Sp}}(2k, \mathbb{C}^{2n})$. We leave other, more esoteric, possibilities to future work.

1.1. **Notations and terminologies.** We write $\mathbb{N}$ for the positive integers. For any $S \subseteq \mathbb{C}$, $S^{n\times n}$ is the set of all $n \times n$ matrices with entries in S. We write $\mathsf{S}^2(\mathbb{R}^n)$ for the space of real symmetric matrices. We write I for an identity matrix of appropriate dimension or I_n when the dimension needs to be specified; likewise we write $0_{m,n}$ for an $m \times n$ zero matrix or just 0 when its dimensions are clear from context. We will also define the special matrices

$$I_{m,n} = \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}, \qquad J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

$$K_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

As usual, the orthogonal, special orthogonal, unitary, special unitary, and symplectic groups are

$$\begin{aligned}
\operatorname{O}(n) &= \{X \in \mathbb{R}^{n\times n} : X^{\mathsf{T}} X = I\}, & \operatorname{SO}(n) &= \{X \in \operatorname{O}(n) : \det(X) = 1\},\\
\operatorname{U}(n) &= \{X \in \mathbb{C}^{n\times n} : X^{\mathsf{H}} X = I\}, & \operatorname{SU}(n) &= \{X \in \operatorname{U}(n) :\ \det(X) = 1\},\\
\operatorname{Sp}(2n, \mathbb{F}) &= \{X \in \mathbb{F}^{2n\times 2n} : X^{\mathsf{T}} J_{2n} X = J_{2n}\},
\end{aligned}$$

where the last may be over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Other special sets of interests are $\mathrm{SO}^-(n) \coloneqq \{X \in \mathrm{O}(n) : \det(X) = -1\}$ and $\mathrm{SU}^-(n) \coloneqq \{X \in \mathrm{U}(n) : \det(X) = -1\}$.

Let $d, n, k_1, \dots, k_d \in \mathbb{N}$ and we will always assume that $1 \le k_1, \dots, k_d \le n-1$. We write

$$\begin{aligned} \Phi(k_1,\dots,k_d,\mathbb{R}^n) &= \{X_1\cdots X_d \in \mathrm{O}(n) : X_i \in \mathrm{Gr}(k_i,\mathbb{R}^n),\ i=1,\dots,d\},\\ \Phi(k_1,\dots,k_d,\mathbb{C}^n) &= \{X_1\cdots X_d \in \mathrm{U}(n) : X_i \in \mathrm{Gr}(k_i,\mathbb{C}^n),\ i=1,\dots,d\},\\ \Phi_{\mathrm{Sp}}(2k_1,\dots,2k_d,\mathbb{F}^{2n}) &= \{X_1\cdots X_d \in \mathrm{Sp}(2n,\mathbb{F}) : X_i \in \mathrm{Gr}_{\mathrm{Sp}}(2k_i,\mathbb{F}^{2n})),\ i=1,\dots,d\}. \end{aligned} \tag{5}$$

Henceforth whenever we refer to a "Grassmannian" it will be in sense of the involution models in (3); in particular, a point on a Grassmannian is a matrix. The word "product" would mean matrix product, whether in the sense of two matrices or two subsets of matrices.

2. Product of two real Grassmannians

We will study a few properties of $\Phi(k_1,k_2,\mathbb{R}^n)$ with the goal of determining its dimension, from which we will deduce our required result that $\Phi(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \mathbb{R}^n) = \mathrm{SO}(n)$. We remind the reader that an alternative description of the involution model is as

$$\mathrm{Gr}(k,\mathbb{R}^n) = \{X \in \mathsf{S}^2(\mathbb{R}^n) : X^2 = I,\ \mathrm{tr}(X) = 2k-n\} \tag{6}$$

since an involution is orthogonal if and only if it is symmetric.

We begin with an observation about a product of points from two Grassmannians.

Lemma 2.1. *Let $Y \in \mathrm{O}(n)$ and $k_1,k_2 \in \mathbb{N}$. Then the following are equivalent:*

(i) *$Y \in \Phi(k_1,k_2,\mathbb{R}^n)$.*
(ii) *$QYQ^\mathsf{T} \in \Phi(k_1,k_2,\mathbb{R}^n)$ for all $Q \in \mathrm{O}(n)$.*
(iii) *$XYX = Y^\mathsf{T}$ and $\mathrm{tr}(YX) = 2k_1 - n$ for some $X \in \mathrm{Gr}(k_2,\mathbb{R}^n)$.*

Proof. Let $Y = X_1X \in \Phi(k_1,k_2,\mathbb{R}^n)$. Then $QYQ^\mathsf{T} = (QX_1Q^\mathsf{T})(QXQ^\mathsf{T}) \in \Phi(k_1,k_2)$ for all $Q \in \mathrm{O}(n)$ and we have (i)⇒(ii). Clearly (ii)⇒(i) with $Q = I$.

Let $Y = X_1X$ with $X_1 \in \mathrm{Gr}(k_1,\mathbb{R}^n)$, $X \in \mathrm{Gr}(k_2,\mathbb{R}^n)$. Then $YX = X_1 \in \mathrm{Gr}(k_1,\mathbb{R}^n)$, and so $(YX)^2 = I$ and $\mathrm{tr}(YX) = 2k_1-n$. Hence (i)⇒(iii). Conversely, if $(YX)^2 = I$ and $\mathrm{tr}(YX) = 2k_1-n$ for some $X \in \mathrm{Gr}(k_2,\mathbb{R}^n)$, then as $(YX)(YX)^\mathsf{T} = I = (YX)^2$, we have $X_1 \coloneqq YX \in \mathsf{S}^2(\mathbb{R}^n)$ and $\mathrm{tr}(X_1) = 2k_1 - n$. Thus $X_1 \in \mathrm{Gr}(k_1,\mathbb{R}^n)$ by (6). This shows (iii)⇒(i). ☐

We next deduce a canonical form for matrices satisfying $I_{k,n-k}YI_{k,n-k} = Y^\mathsf{T}$ that we need in the next result.

Lemma 2.2. *Let $k,n \in \mathbb{N}$ with $2k \le n$. Let $Y \in \mathrm{O}(n)$. Then $I_{k,n-k}YI_{k,n-k} = Y^\mathsf{T}$ and $\mathrm{tr}(YI_{k,n-k}) = r$ if and only if there exist $U \in \mathrm{O}(k)$, $V \in \mathrm{O}(n-k)$, and diagonal matrices $\Sigma_0 \in [0,1]^{k\times k}$, $E_1,E_2 \in \{-1,1\}^{k\times k}$, $E_3 \in \{-1,1\}^{(n-2k)\times(n-2k)}$ such that*

$$\Sigma(I_k - \Sigma)(E_1 - E_2) = 0, \qquad \mathrm{tr}\bigl(\sqrt{I_k - \Sigma_0^2}(E_1 - E_2)\bigr) - \mathrm{tr}(E_3) = r \tag{7}$$

and

$$Y = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \sqrt{I_k - \Sigma^2}E_1 & \Sigma & 0 \\ -\Sigma & \sqrt{I_k-\Sigma^2}E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^\mathsf{T}. \tag{8}$$

Proof. Partition Y as

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \quad Y_1 \in \mathbb{R}^{k\times k}, \quad Y_2 \in \mathbb{R}^{k\times(n-k)}, \quad Y_3 \in \mathbb{R}^{(n-k)\times k}, \quad Y_4 \in \mathbb{R}^{(n-k)\times(n-k)}.$$

If $I_{k,n-k}YI_{k,n-k} = Y^\mathsf{T}$, then

$$\begin{bmatrix} Y_1 & -Y_2 \\ -Y_3 & Y_4 \end{bmatrix} = \begin{bmatrix} Y_1^\mathsf{T} & Y_3^\mathsf{T} \\ Y_2^\mathsf{T} & Y_4^\mathsf{T} \end{bmatrix}$$

and so $Y_1 \in \mathsf{S}^2(\mathbb{R}^k)$, $Y_4 \in \mathsf{S}^2(\mathbb{R}^{n-k})$, and $Y_2 = -Y_3^\mathsf{T}$. Since $Y^\mathsf{T} Y = I$, we get

$$Y_1^2 + Y_2Y_2^\mathsf{T} = I_k, \quad -Y_1Y_2 + Y_2Y_4 = 0, \quad Y_2^\mathsf{T} Y_2 + Y_4^2 = I_{n-k}. \tag{9}$$

Let $Y_2 = U[\Sigma, 0]V^\mathsf{T}$ be a singular value decomposition with $(U, V) \in \mathrm{O}(k) \times \mathrm{O}(n-k)$ and

$$[\Sigma, 0] = \begin{bmatrix} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_k & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{k\times(n-k)}, \quad \sigma_1 \geq \cdots \geq \sigma_k.$$

Note that $\sigma_1, \dots, \sigma_k \in [0, 1]$ as Y_2 is a block of an orthogonal matrix. By (9), we get

$$Y_1^2 = U(I_k - [\Sigma, 0][\Sigma, 0]^\mathsf{T})U^\mathsf{T}, \quad -Y_1Y_2 + Y_2Y_4 = 0, \quad Y_4^2 = V(I_{n-k} - [\Sigma, 0]^\mathsf{T}[\Sigma, 0])V^\mathsf{T}.$$

Write $D := \operatorname{diag}\bigl(\sqrt{1-\sigma_1^2}, \dots, \sqrt{1-\sigma_k^2}\bigr) = \sqrt{I_k - \Sigma^2}$. Then

$$Y_1 = UDE_1U^\mathsf{T}, \quad Y_4 = V\begin{bmatrix} DE_2 & 0 \\ 0 & E_3 \end{bmatrix}V^\mathsf{T},$$

where $E_3 \in \{-1, 1\}^{(n-2k)\times(n-2k)}$ and $E_1, E_2 \in \{-1, 1\}^{k\times k}$ satisfy $\Sigma D(E_1 - E_2) = 0$. Therefore

$$Y = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} DE_1 & \Sigma & 0 \\ -\Sigma & DE_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^\mathsf{T}$$

as required. □

We may now characterize a point in $\Phi(k_1, k_2, \mathbb{R}^n)$.

Theorem 2.3 (Canonical form for product of two real Grassmannians)**.** *Let $k_1, k_2, n \in \mathbb{N}$ with $k_2 \leq k_1 < n$ and $k_1 + k_2 \leq n$. Let $Z \in \mathrm{O}(n)$. Then $Z \in \Phi(k_1, k_2, \mathbb{R}^n)$ if and only if there exist $Q \in \mathrm{O}(n)$ and a pair of diagonal matrices $\Sigma \in [0, 1]^{k_2\times k_2}$, $E \in \{-1, 1\}^{k_2\times k_2}$ such that*

$$Z = Q\begin{bmatrix} \sqrt{I_{k_2} - \Sigma^2}E & \Sigma & 0 \\ -\Sigma & \sqrt{I_{k_2} - \Sigma^2}E & 0 \\ 0 & 0 & I_{n-k_1-k_2,k_1-k_2} \end{bmatrix}Q^\mathsf{T}. \tag{10}$$

Proof. By Lemma 2.1, $Z \in \Phi(k_1, k_2, \mathbb{R}^n)$ if and only if $XZX = Z^\mathsf{T}$ and $\operatorname{tr}(ZX) = 2k_1 - n$ for some $X = RI_{k_2,n-k_2}R^\mathsf{T} \in \mathrm{Gr}(k_2, \mathbb{R}^n)$, $R \in \mathrm{O}(n)$. Let $Y := R^\mathsf{T} ZR$. Then by Lemma 2.2, Y decomposes as in (8) into factors satisfying (7). Setting $Q = R \operatorname{diag}(U, V)$, we obtain

$$Z = Q\begin{bmatrix} \sqrt{I_{k_2} - \Sigma^2}E_1 & \Sigma & 0 \\ -\Sigma & \sqrt{I_{k_2} - \Sigma^2}E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}Q^\mathsf{T}$$

where

$$\Sigma(I_{k_2} - \Sigma)(E_1 - E_2) = 0, \quad \operatorname{tr}\left(\sqrt{I_{k_2} - \Sigma_0^2}(E_1 - E_2)\right) - \operatorname{tr}(E_3) = 2k_1 - n.$$

Since $k_1 + k_2 \leq n$, we may further require that $E_1 = E_2$ and $E_3 = I_{n-k_1-k_2,k_1-k_2}$ so that Z has the desired form. □

We will next develop some basic calculus for working with $\Phi(k_1, k_2, \mathbb{R}^n)$. As will be evident from the proof (but with Theorem 3.1 taking the place of Theorem 2.3), the following properties also hold with $\mathbb{C}^n$ in place of $\mathbb{R}^n$.

Corollary 2.4. *Let $k_1, k_2, n \in \mathbb{N}$ with $k_1, k_2 \leq n$. We have*

$$\begin{aligned} \Phi(n - k_1, k_2, \mathbb{R}^n) &= \Phi(k_1, n - k_2, \mathbb{R}^n) = -\Phi(k_1, k_2, \mathbb{R}^n), \\ \Phi(n - k_1, n - k_2, \mathbb{R}^n) &= \Phi(k_1, k_2, \mathbb{R}^n), \\ \Phi(k_2, k_1, \mathbb{R}^n) &= \Phi(k_1, k_2, \mathbb{R}^n). \end{aligned}$$

Proof. The first three equalities are consequence of the fact that in the involution model (3), $\operatorname{Gr}(n-k,\mathbb{R}^n) = -\operatorname{Gr}(k,\mathbb{R}^n)$. It remains to establish the last equality. For $(X_1, X_2) \in \operatorname{Gr}(k_1,\mathbb{R}^n) \times \operatorname{Gr}(k_2,\mathbb{R}^n)$, we have

$$\Phi(k_2,k_1,\mathbb{R}^n) \ni X_2X_1 = (X_1X_2)^{\mathsf{T}} \in \Phi(k_1,k_2,\mathbb{R}^n)^{\mathsf{T}}.$$

Without loss of generality, let $k_2 \le k_1$. If $k_1 + k_2 \le n$, then it follows from Theorem 2.3 that $\Phi(k_1,k_2,\mathbb{R}^n)^{\mathsf{T}} = \Phi(k_1,k_2,\mathbb{R}^n)$, from which we have

$$\Phi(k_2,k_1,\mathbb{R}^n) = \Phi(k_1,k_2,\mathbb{R}^n)^{\mathsf{T}} = \Phi(k_1,k_2,\mathbb{R}^n).$$

If $n < k_1 + k_2$, then $(n-k_1)+(n-k_2) < n$ and $n-k_1 \le n-k_2$. It follows from Theorem 2.3 that $\Phi(n-k_2,n-k_1,\mathbb{R}^n)^{\mathsf{T}} = \Phi(n-k_2,n-k_1,\mathbb{R}^n)$, and therefore

$$\begin{aligned}\Phi(k_1,k_2,\mathbb{R}^n) &= \Phi(n-k_1,n-k_2,\mathbb{R}^n) = \Phi(n-k_2,n-k_1,\mathbb{R}^n)^{\mathsf{T}}\\ &= \Phi(n-k_2,n-k_1,\mathbb{R}^n) = \Phi(k_2,k_1,\mathbb{R}^n),\end{aligned}$$

as required. $\square$

To reduce cluttering subscripts, we write $k' = k_1$ and $k = k_2$ for the next few intermediate results, and only switch back to k_1 and k_2 for Theorem 2.9, the final result of this section. Let $k', k, n \in \mathbb{N}$ with $k \le k' \le n$. For any pair of diagonal matrices $(\Sigma, E) \in [0,1]^{k\times k} \times \{-1,1\}^{k\times k}$, we define the $\mathrm{O}(n)$-orbit

$$\operatorname{Orb}_k(\Sigma,E) \coloneqq \left\{ Q \begin{bmatrix} \sqrt{I_k-\Sigma^2}E & \Sigma & 0\\ -\Sigma & \sqrt{I_k-\Sigma^2}E & 0\\ 0 & 0 & I_{n-k'-k,k'-k}\end{bmatrix} Q^{\mathsf{T}} \in \mathrm{O}(n) : Q \in \mathrm{O}(n)\right\}.$$

As a reminder, the block matrix in the middle is in $\mathrm{O}(n)$ because of (10). We will find another representative for this orbit from which we may deduce its dimension later.

Lemma 2.5. *Let $\Sigma \in \mathbb{R}^{k\times k}$ be the diagonal matrix*

$$\Sigma = \operatorname{diag}(\sigma_0 I_{m_0},\dots,\sigma_s I_{m_s}, 0_{m_{s+1}})$$

where $1 = \sigma_0 > \sigma_1 > \dots > \sigma_s > 0$ and $m_0 + \dots + m_s + m_{s+1} = k$. Let $m \coloneqq m_0 + \dots + m_s$. Then

$$\operatorname{Orb}_k(\Sigma,E) = \{QFQ^{\mathsf{T}} : Q \in \mathrm{O}(n)\},$$

for some

$$F \coloneqq \begin{bmatrix} 0 & I_{m_0} & 0 & 0 & \cdots & 0 & 0 & 0\\ -I_{m_0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & \sqrt{1-\sigma_1^2}I_{p_1,q_1} & \sigma_1 I_{m_1} & \cdots & 0 & 0 & 0\\ 0 & 0 & -\sigma_1 I_{m_1} & \sqrt{1-\sigma_1^2}I_{p_1,q_1} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \sqrt{1-\sigma_s^2}I_{p_s,q_s} & \sigma_s I_{m_s} & 0\\ 0 & 0 & 0 & 0 & \cdots & -\sigma_s I_{m_s} & \sqrt{1-\sigma_s^2}I_{p_s,q_s} & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & I_{q,n-2m-q}\end{bmatrix} \in \mathbb{R}^{n\times n} \tag{11}$$

where q is an integer with $0 \le q \le n-2m$ and $p_j, q_j \in \mathbb{N}$ satisfy $p_j + q_j = m_j$, $j = 1,\dots,s$.

Proof. We denote $\Sigma_1 \coloneqq \operatorname{diag}(\sigma_0,\dots,\sigma_s)$. By simultaneously permuting rows and columns of

$$\begin{bmatrix} \sqrt{I_k-\Sigma^2}E & \Sigma & 0\\ -\Sigma & \sqrt{I_k-\Sigma^2}E & 0\\ 0 & 0 & I_{n-k'-k,k'-k}\end{bmatrix} \in \operatorname{Orb}_k(\Sigma,E),$$

we may obtain

$$\begin{bmatrix} \sqrt{I_m - \Sigma_1^2} D_1 & \Sigma_1 & 0 & 0 & 0 \\ -\Sigma_1 & \sqrt{I_m - \Sigma_1^2} D_1 & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 & 0 \\ 0 & 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & I_{n-k'-k,k'-k} \end{bmatrix} \tag{12}$$

for some diagonal matrices $D_1 \in \{-1,1\}^{m\times m}$ and $D_2 \in \{-1,1\}^{m_{s+1}\times m_{s+1}}$. We denote by q the number of 1's in the diagonal of $\operatorname{diag}(D_2, D_2, I_{n-k'-k,k'-k})$. Obviously, F is different from (12) by further simultaneously permuting columns and rows. Thus, $\operatorname{Orb}_k(\Sigma, E)$ is the orbit of F under the conjugation of $\mathrm{O}(n)$. □

Proposition 2.6. *Let $k, n, q, m_0, m_{j1}, m_{j2}, 1 \le j \le s$ be as in Lemma 2.5. Then*

$$\dim \operatorname{Orb}_k(\Sigma, E) = \binom{n}{2} - \left[m_0^2 + \sum_{j=1}^{s} (m_{j1}^2 + m_{j2}^2) + \binom{q}{2} + \binom{n-q-2m}{2} \right].$$

Proof. Let F be as in (11). It is sufficient to calculate the dimension of the Lie algebra $\mathfrak{g}_F$ of the isotropy group of F. To this end, we consider $X \in \Lambda^2(\mathbb{R}^n)$ such that

$$XF - FX = 0. \tag{13}$$

We partition F as $F = \operatorname{diag}(Q_0, \dots, Q_{2s+2})$ where

$$Q_j = \begin{cases} \begin{bmatrix} 0 & I_{m_0} \\ -I_{m_0} & 0 \end{bmatrix} & \text{if } j = 0, \\ \begin{bmatrix} \sqrt{1-\sigma_j^2} I_{m_{j1}} & \sigma_j I_{m_{j1}} \\ -\sigma_j I_{m_{j1}} & \sqrt{1-\sigma_j^2} I_{m_{j1}} \end{bmatrix} & \text{if } 1 \le j \le 2s \text{ is odd}, \\ \begin{bmatrix} -\sqrt{1-\sigma_j^2} I_{m_{j2}} & \sigma_j I_{m_{j2}} \\ -\sigma_j I_{m_{j2}} & -\sqrt{1-\sigma_j^2} I_{m_{j2}} \end{bmatrix} & \text{if } 1 \le j \le 2s \text{ is even}, \\ I_q & \text{if } j = 2s+1, \\ I_{n-2m-q} & \text{if } j = 2s+2. \end{cases}$$

We also partition $X = (X_{ij})_{i,j=0}^{2s+2}$ accordingly so that (13) becomes

$$X_{ij} Q_j - Q_i X_{ij} = 0, \quad 0 \le i, j \le 2s+2. \tag{14}$$

We observe that if $i \ne j$, then $\sigma(Q_i) \cap \sigma(Q_j) = \emptyset$. This implies that $X_{ij} = 0$, $0 \le i \ne j \le 2s+2$. In particular, X is a block diagonal skew-symmetric matrix. It is left to consider

$$X_{jj} Q_j - Q_j X_{jj} = 0, \quad 0 \le j \le 2s+2. \tag{15}$$

For $j = 2s+1$ (resp. $j = 2s+2$), the solution space of (15) has dimension $\binom{q}{2}$ (resp. $\binom{n-2m-q}{2}$). For $0 \le j \le 2s$, (15) is of the form

$$\begin{bmatrix} bI_p & aI_p \\ -aI_p & bI_p \end{bmatrix} Y - Y \begin{bmatrix} bI_p & aI_p \\ -aI_p & bI_p \end{bmatrix} = 0 \tag{16}$$

for some $a, b \in \mathbb{R}$ satisfying $0 < a \le 1$ and $a^2 + b^2 = 1$. A direct calculation reveals that the dimension of the solution space of (16) is p^2. Therefore, we conclude that

$$\dim \mathfrak{g}_F = m_0^2 + \sum_{j=1}^{s} (m_{j1}^2 + m_{j2}^2) + \binom{q}{2} + \binom{n-2m-q}{2}.$$

□

Corollary 2.7. *We have*

$$\operatorname{Orb}_k(\Sigma, E) = \operatorname{Orb}_k(\Sigma', E_1')$$

if and only if there is a permutation matrix $P \in \mathbb{R}^{k\times k}$ such that $(\Sigma', E') = P(\Sigma, E)P^{\mathsf{T}}$.

Proof. By Lemma 2.5 and the proof of Proposition 2.6, any element in $\operatorname{Orb}_k(\Sigma, E)$ has eigenvalues

$$\underbrace{\pm i, \dots, \pm i}_{m_0 \text{ copies}}, \quad \underbrace{1, \dots, 1}_{q \text{ copies}}, \quad \underbrace{-1, \dots, -1}_{n-2m-q \text{ copies}},$$
$$\underbrace{\sqrt{1-\sigma_j^2} \pm \sigma_j i, \dots, \sqrt{1-\sigma_j^2} \pm \sigma_j i}_{m_{j1} \text{ copies}}, \quad \underbrace{-\sqrt{1-\sigma_j^2} \pm \sigma_j i, \dots, -\sqrt{1-\sigma_j^2} \pm \sigma_j i}_{m_{j2} \text{ copies}},$$

where $1 \le j \le s$. Conversely, any orthogonal matrix with these eigenvalues must be an element of $\operatorname{Orb}_k(\Sigma, E)$. Therefore,

$$\operatorname{Orb}_k(\Sigma, E) \cap \operatorname{Orb}_k(\Sigma', E') \neq \emptyset$$

if and only (Σ', E') can be obtained from (Σ, E) by a simultaneous permutation of columns and rows. □

Corollary 2.8. *For any distinct real numbers $\sigma_1, \dots, \sigma_k \in (0,1)$ and diagonal matrix $E \in \{-1,1\}^{k\times k}$, we have*

$$\operatorname{Orb}_k(\operatorname{diag}(\sigma_1, \dots, \sigma_k), E) \simeq \operatorname{Flag}(2, 4, \dots, 2k, 2k+q; \mathbb{R}^n).$$

In particular,

$$\dim \operatorname{Orb}_k(\operatorname{diag}(\sigma_1, \dots, \sigma_k), E) = \binom{n}{2} - \left[k + \binom{n-2k-q}{2} + \binom{q}{2}.\right].$$

Proof. The formula for $\dim \operatorname{Orb}_k(\operatorname{diag}(\sigma_1, \dots, \sigma_k), E)$ is an immediate consequence of Proposition 2.6. For $1 > \sigma_1 > \dots > \sigma_k > 0$, it is clear from the proof of Proposition 2.6 that the isotropy group of F under the conjugation of $\mathrm{O}(n)$ is

$$\underbrace{\mathrm{O}(2) \times \dots \times \mathrm{O}(2)}_{k \text{ copies}} \times \mathrm{O}(q) \times \mathrm{O}(n-2k-q),$$

from which we may identify Orb_k with $\mathrm{O}(n)/(\underbrace{\mathrm{O}(2) \times \dots \times \mathrm{O}(2)}_{k \text{ copies}} \times \mathrm{O}(q) \times \mathrm{O}(n-2k-q))$. □

Theorem 2.9 (Special orthogonal group as product of two Grassmannians)**.** *Let $k_1, k_2, n \in \mathbb{N}$ with $k_2 \le k_1 < n$ and $k_1 + k_2 \le n$. We have*

$$\dim \Phi(k_1, k_2, \mathbb{R}^n) = \binom{n}{2} - \binom{k_1 - k_2}{2} - \binom{n - k_1 - k_2}{2}.$$

In particular, $\Phi(k_1, k_2, \mathbb{R}^n)$ is a connected component of $\mathrm{O}(n)$ if and only if $k_1 - k_2, n - (k_1 + k_2) \in \{0, 1\}$. Moreover, we have

$$\Phi(k_1, k_2, \mathbb{R}^n) = \begin{cases} \mathrm{SO}(n) & \text{if } k_1 = k_2,\ n = 2k_2 \text{ or } 2k_2 + 1, \\ \mathrm{SO}^-(n) & \text{if } k_1 = k_2 + 1,\ n = 2k_2 + 1 \text{ or } 2k_2 + 2. \end{cases}$$

Proof. By Theorem 2.3, we have

$$\Phi(k_1, k_2, \mathbb{R}^n) = \bigcup_{(\Sigma, E) \in S} \operatorname{Orb}_{k_2}(\Sigma, E)$$

where S consists of all pairs (Σ, E) of diagonal matrices in $[0,1]^{k_2 \times k_2} \times \{-1,1\}^{k_2 \times k_2}$. Since $k_1 + k_2 \le n$, we have

$$(\operatorname{diag}(\sigma_1, \dots, \sigma_{k_2}), E) \in S$$

for any distinct real numbers $\sigma_1, \dots, \sigma_{k_2} \in (0,1)$ and diagonal matrix $E \in \{-1,1\}^{k_2 \times k_2}$. Moreover, according to Corollary 2.8, we have

$$\dim \operatorname{Orb}_{k_2}(\operatorname{diag}(\sigma_1, \dots, \sigma_{k_2}), E) = \binom{n}{2} - \left[k_2 + \binom{n-(k_1+k_2)}{2} + \binom{k_1-k_2}{2}\right].$$

It is clear that the union of such $\operatorname{Orb}_{k_2}(\operatorname{diag}(\sigma_1, \dots, \sigma_{k_2}), E)$'s contains an open subset of $\Phi(k_1, k_2, \mathbb{R}^n)$. Therefore, we may conclude from Corollary 2.7 that

$$\begin{aligned}\dim \Phi(k_1, k_2, \mathbb{R}^n) &= k_2 + \binom{n}{2} - \left[k_2 + \binom{n-(k_1+k_2)}{2} + \binom{k_1-k_2}{2}\right] \\ &= \binom{n}{2} - \binom{k_1-k_2}{2} - \binom{n-k_1-k_2}{2}.\end{aligned}$$

It is obvious that $\Phi(k_1, k_2, \mathbb{R}^n) = \mathrm{SO}(n)$ (or $\mathrm{SO}^-(n)$) if and only if $k_1 - k_2 \le 1$ and $n - k_1 - k_2 \le 1$. If $k_1 = k_2$, then $I \in \Phi(k_1, k_2, \mathbb{R}^n)$ and $\Phi(k_2, k_2, \mathbb{R}^n) = \mathrm{SO}(n)$. If $k_1 = k_2 + 1$, then Theorem 2.3 indicates that $I_{n-1,1} \in \Phi(k_2+1, k_2, \mathbb{R}^n)$ and $\Phi(k_2+1, k_2, \mathbb{R}^n) = \mathrm{SO}^-(n)$. $\square$

3. Product of two complex Grassmannians

In this section, we consider the product of two complex Grassmannians, $\Phi(k_1, k_2, \mathbb{C}^n)$. Similar as Theorem 2.3, we have the following result for the complex version.

Theorem 3.1. *Let $k_1, k_2, n \in \mathbb{N}$ with $k_2 \le k_1 < n$ and $k_1 + k_2 \le n$. For each $Z \in \mathrm{U}(n)$, $Z \in \Phi(k_1, k_2, \mathbb{C}^n)$ if and only if there exist $Q \in \mathrm{U}(n)$ and a diagonal matrix $D_0 = \operatorname{diag}(d_1, \dots, d_{k_2})$ with $d_j = e^{i\alpha_j}$, $\alpha_j \in [0, \pi]$ such that*

$$Z = Q \begin{bmatrix} D_0 & 0 & 0 \\ 0 & \overline{D}_0 & 0 \\ 0 & 0 & I_{n-k_1-k_2, k_1-k_2} \end{bmatrix} Q^{\mathsf{H}}.$$

Proof. Using the same derivation as in Section 2, we have

$$Z = Q_0 \begin{bmatrix} \sqrt{I_{k_2} - \Sigma^2}E & \Sigma' & 0 \\ -\Sigma' & \sqrt{I_{k_2} - \Sigma'^2}E & 0 \\ 0 & 0 & I_{n-k_1-k_2, k_1-k_2} \end{bmatrix} Q_0^{\mathsf{H}}$$

Unitarily diagonalize Z further, then we have the form above. $\square$

Corollary 2.4 also holds by replacing $\mathbb{R}^n$ with $\mathbb{C}^n$.

We aims to count how many real parameters are needed to describe $\Phi(k_1, k_2, \mathbb{C}^n)$ and thereby deduce the dimension of $\Phi(k_1, k_2, \mathbb{C}^n)$.

The idea is basically same as Section 2. Given a unitary diagonal matrix $D_0 \in \mathbb{R}^{k_2 \times k_2}$ specified in Theorem 3.1, we define

$$\operatorname{Orb}_{\mathrm{U},k_2}(D_0) := \left\{ Q \begin{bmatrix} D_0 & 0 & 0 \\ 0 & \overline{D}_0 & 0 \\ 0 & 0 & I_{n-k_1-k_2, k_1-k_2} \end{bmatrix} Q^{\mathsf{H}} \in \mathrm{U}(n) : Q \in \mathrm{U}(n) \right\}.$$

Lemma 3.2. *Suppose diagonal elements of D_0 are*

$$\underbrace{d_0, \dots, d_0}_{m_0 \text{ copies}}, \dots, \underbrace{d_{s+1}, \dots, d_{s+1}}_{m_{s+1} \text{ copies}},$$

where $d_j = e^{i\alpha_j}$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_s < \alpha_{s+1} = \pi$, and $\sum_{j=0}^{s+1} m_j = k_2$. Let $m := \sum_{j=1}^{s} m_j$. Then

$$\operatorname{Orb}_{\mathrm{U},k_2}(D_0) = \{QFQ^{\mathsf{H}} : Q \in \mathrm{U}(n)\},$$

where

$$F := \mathrm{diag}(d_1 I_{m_1}, \cdots, d_s I_{m_s}, \overline{d}_1 I_{m_1}, \cdots, \overline{d}_s I_{m_s}, I_{q,n-2m-q}) \tag{17}$$

for some integer $n - 2m \geq q \geq 0$, $m \leq k_2$.

Proof. Simply reorder the diagonal elements of $\mathrm{diag}(D_0, \overline{D}_0, I_{n-k_1-k_2,k_1-k_2})$, then we get F. □

Proposition 3.3. *Let* $k_2, n, q, m, m_j, 1 \leq j \leq s$ *be as in Lemma 3.2. Then*

$$\dim \mathrm{Orb}_{\mathrm{U},k_2}(D_0) = n^2 - \left[2\sum_{j=1}^{s} m_j^2 + q^2 + (n-q-2m)^2 \right].$$

Proof. Let F be as in (17). It is sufficient to calculate the dimension of the Lie algebra $\mathfrak{g}_F$ of the isotropy group of F:

$$\mathfrak{g}_F = \{X : \ XF = FX, \ X + X^{\mathsf{H}} = 0\}$$

We rename (17) as $F = \mathrm{diag}(Q_1, \dots, Q_{2s+2})$ where

$$Q_j = \begin{cases} d_j I_{m_j} & \text{if } 1 \leq j \leq s, \\ \overline{d}_j I_{m_j} & \text{if } s+1 \leq j \leq 2s, \\ I_q & \text{if } j = 2s+1, \\ -I_{n-2m-q} & \text{if } j = 2s+2. \end{cases}$$

Note that $\sigma(Q_i) \cap \sigma(Q_j) = \emptyset$ when $i \neq j$. We partition $X = (X_{ij})_{i,j=1}^{2s+2}$ accordingly.

Then following the proof of Lemma 2.5, we have $X_{ij} = 0$ when $i \neq j$ and $X_{ii} + X_{ii}^{\mathsf{H}} = 0$. Therefore, we conclude that

$$\dim \mathfrak{g}_F = 2\sum_{j=1}^{s} m_j^2 + q^2 + (n-2m-q)^2.$$

□

Theorem 3.4. *Let* $k_1, k_2, n \in \mathbb{N}$ *with* $k_2 \leq k_1 < n$ *and* $k_1 + k_2 \leq n$. *We have*

$$\dim \Phi(k_1, k_2, \mathbb{C}^n) = n^2 - k_2 - (n-k_1-k_2)^2 - (k_1-k_2)^2$$

Proof. By Theorem 3.1, we have

$$\dim \Phi(k_1, k_2, \mathbb{C}^n) = \bigcup_{D_0 \in S} \mathrm{Orb}_{\mathrm{U},k_2}(D_0)$$

where $S = \{\mathrm{diag}(d_1, \dots, d_{k_2}) : \ \forall j, \ d_j = e^{i\alpha_j}, \ \alpha_j \in [0, \pi]\}$ consists of all possible choice of D_0.

For any distinct $\alpha_1, \dots, \alpha_{k_2} \in (0, \pi)$, according to Proposition 3.3 we have

$$\dim \mathrm{Orb}_{\mathrm{U},k_2}(\mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_{k_2}})) = n^2 - 2k_2 - (n-k_1-k_2)^2 - (k_1-k_2)^2$$

By a same argument as in the proof of Theorem 2.9, we have

$$\begin{aligned} \dim \Phi(k_1, k_2, \mathbb{C}^n) &= k_2 + \left[n^2 - 2k_2 - (n-k_1-k_2)^2 - (k_1-k_2)^2\right] \\ &= n^2 - k_2 - (n-k_1-k_2)^2 - (k_1-k_2)^2. \end{aligned}$$

□

When $k_2 = 0$, we can verify $\dim \Phi(k_1, 0, \mathbb{C}^n) = 2k_1(n-k_1) = \dim \mathrm{Gr}(k_1, \mathbb{C}^n)$.

4. Product of complex Grassmannians

We can further generalize the permutation invariance in Corollary 2.4 as follows:

Corollary 4.1. *Let $k_1, \dots, k_d, n \in \mathbb{N}$ with $k_1, \dots, k_d < n$.*

(i) *For any permutation $\sigma(\cdot)$ of $[d]$, we have*

$$\Phi(k_{\sigma(1)}, k_{\sigma(2)}, \dots, k_{\sigma(d)}, \mathbb{C}^n) = \Phi(k_1, k_2, \dots, k_d, \mathbb{C}^n), \quad \Phi(n - k_1, k_2, \dots, k_d, \mathbb{C}^n) = -\Phi(k_1, k_2, \dots, k_d, \mathbb{C}^n).$$

(ii) *There exist $k_1', \dots, k_d' \in \mathbb{N}$ such that*

$$n > k_1' \geq k_2' \geq \cdots \geq k_d', \quad k_1' + k_2' \leq n, \quad \Phi(k_1', \dots, k_d', \mathbb{C}^n) = \Phi(k_1, \dots, k_d, \mathbb{C}^n).$$

Proof. For $M \in \Phi(k_1, \dots, k_d, \mathbb{C}^n)$, there exist $X_j \in \mathrm{Gr}(k_j, \mathbb{C}^n)$ such that $M = \prod_{j=1}^d X_j$. For any $1 \leq j \leq d-1$, by Corollary 2.4, there exist $X_j' \in \mathrm{Gr}(k_j, \mathbb{C}^n), X_{j+1}' \in \mathrm{Gr}(k_{j+1}, \mathbb{C}^n)$ such that $X_j X_{j+1} = X_{j+1}' X_j'$. This implies $\Phi(k_1, \dots, k_j, k_{j+1}, \dots, k_d, \mathbb{C}^n) = \Phi(k_1, \dots, k_{j+1}, k_j, \dots, k_d, \mathbb{C}^n)$. Then we have (i) by swapping repeatedly.

Suppose we have sorted $(k_j)_{j=1}^d$ to the descending order $k_1 \geq \cdots \geq k_d$. If $k_1 + k_2 > n$, we replace (k_1, k_2) by $(n - k_1, n - k_2)$, and sort to the descending order again. Repeat this procedure until it stops, then we get the desired $(k_j')_{j=1}^d$ in (ii). □

Definition 4.2. For $n, d \in \mathbb{N}$, we define $K_{n,d} \subseteq \mathbb{N}^d$ as

$$K_{n,d} \coloneqq \{(k_1, \dots, k_d) : \ k_i \in \mathbb{N}, \ k_1 + k_2 \leq n, \ k_1 \geq \cdots \geq k_d\}$$

Each tuple $(k_i)_{i=1}^d$ in $K_{n,d}$ gives a unique representation of $\Phi(k_1, \dots, k_d, \mathbb{C}^n)$.

Here we use the result of [5] on $\Phi(k_1, \dots, k_d, \mathbb{C}^n)$ about the length of $\Phi(k_1, \dots, k_d, \mathbb{C}^n)$ as the product of reflections.

First let's briefly restate the result. We call $R \in \mathrm{U}(n)$ a reflection, if $R \in \mathrm{Gr}(n-1, \mathbb{C}^n)$, i.e., unitarily similar to $\mathrm{diag}(-1, I_{n-1})$. Let $\mathrm{G}(n) \coloneqq \mathrm{SU}(n) \cup \mathrm{SU}^-(n)$, which is a normal subgroup of $\mathrm{U}(n)$ generated by reflections.

Definition 4.3 (Length of a matrix and set of matrices, [5])**.** For $A \in \mathrm{G}(n)$, its length as the product of reflections is defined as

$$l(A) \coloneqq \min \left\{ m \in \mathbb{N} : \ A = \prod_{j=1}^m R_j, \ R_j \in \mathrm{Gr}(n-1, \mathbb{C}^n) \right\}$$

For $S \subseteq \mathrm{G}(n)$, the length of this subset is defined as

$$l(S) \coloneqq \max \{ l(A) : \ A \in S \}$$

Theorem 4.4 ([5])**.** *Let the eigenvalues of $A \in G(n)$ be $\{\lambda_j(A)\}_{j=1}^n$ with $\lambda_j(A) = e^{i\theta_j}$, $0 \leq \theta_1 \leq \cdots \leq \theta_n < 2\pi$. Define $k(A) \coloneqq \frac{\sum_{j=1}^n \theta_j}{\pi}$. Then we have*

$$l(A) = \max(k(A), \ k(A^{\mathsf{H}})).$$

Corollary 4.5. *Let $k, k_1, \dots, k_d, n \in \mathbb{N}$ with $k, k_1, \dots, k_d < n$, $k_1 + k_2 \leq n$, and $k_1 \geq \cdots \geq k_d$.*

(i) *If $A \in \mathrm{Gr}(k, \mathbb{C}^n)$, $l(A) = n - k$. This further implies $l(\mathrm{Gr}(k, \mathbb{C}^n)) = n - k$.*

(ii)

$$l(\Phi(k_1, \dots, k_d, \mathbb{C}^n)) \leq \begin{cases} \sum_{j=1}^d k_j & \text{if } d \text{ is even,} \\ n - k_1 + \sum_{j=2}^d k_j & \text{if } d \text{ is odd.} \end{cases}$$

(iii) $l(\mathrm{SU}^-(n)) = 2n - 1$, *and* $l(\mathrm{SU}(n)) = 2n - 2$.

Proof. (i) and (iii) are straightforward by calculation. By $\Phi(k_1, k_2, \mathbb{C}^n) = \Phi(n - k_1, n - k_2, \mathbb{C}^n)$ and (i), we have $l(\Phi(k_1, k_2, \mathbb{C}^n)) \leq k_1 + k_2$. Apply this to pairs of neighboring indices, then we have (ii). □

We are now ready to show that $\mathrm{SU}(n) = \Phi(k_1, k_2, k_3, k_4, \mathbb{C}^n)$. We will prove any $M \in \mathrm{G}(n)$ can be written as $M = X_1X_2X_3X_4$ with $X_i \in \mathrm{Gr}(k_i, \mathbb{C}^n)$ for certain tuples (k_1, k_2, k_3, k_4). Given the notion of degrees and more symmetries in Grassmannians, this is more than a refinement of [7]'s result on the special linear group.

Lemma 4.6. *For $2 \le n \in \mathbb{N}$, let $K_{n,d}$ be defined in Definition 4.2. If $(k_1, k_2, k_3, k_4) \in K_{n,4}$ and $k_2 + k_4 \ge n - 1$, then*

$$\Phi(k_1, k_2, k_3, k_4, \mathbb{C}^n) = \begin{cases} \mathrm{SU}(n) & \text{if } \sum_{i=1}^4 k_i \text{ is even,} \\ \mathrm{SU}^-(n) & \text{if } \sum_{i=1}^4 k_i \text{ is odd.} \end{cases}$$

Proof. For any $C \in \mathrm{SU}(n)$, we can assume $C = \mathrm{diag}(e^{ic_1}, \ldots, e^{ic_n})$ with $\sum_{i=1}^n c_i = 0$ without loss of generality. By Theorem 3.1, we write

$$\begin{aligned} A &= \mathrm{diag}(e^{ia_1}, \ldots, e^{ia_{k_2}}, e^{-ia_1}, \ldots, e^{-ia_{k_2}}, I_{n-k_1-k_2,k_1-k_2}) && \in \Phi(k_1, k_2, \mathbb{C}^n) \\ B &= P_B\,\mathrm{diag}(e^{ib_1}, \ldots, e^{ib_{k_4}}, e^{-ib_1}, \ldots, e^{-ib_{k_4}}, I_{n-k_3-k_4,k_3-k_4}) P_B^{\mathsf{T}} && \in \Phi(k_3, k_4, \mathbb{C}^n) \end{aligned}$$

where P_B is a permutation matrix. Also we will write $a = \begin{bmatrix} a_1 \\ \vdots \\ a_{k_2} \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_{k_4} \end{bmatrix}$ and $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ below.

Next we show there exists a, b such that $C = AB$ for any $c \in \{x \in \mathbb{R}^n : x^{\mathsf{T}}\mathbf{1}_n = 0\}$. It is equivalent to show the following map $\varphi : \mathbb{R}^{k_2+k_4} \to \mathbb{R}^n$ has $\mathrm{rank}(M) = n - 1$ for some P_B:

$$c = \varphi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = M \begin{bmatrix} a \\ b \end{bmatrix} + v$$

where

$$M = \left[\begin{array}{c|c|c|c} I_{k_2} & & & \\ -I_{k_2} & p_1 & \cdots & p_{k_4} \\ 0_{n-2k_2,k_2} & & & \end{array}\right] \in \{0, 1, -1\}^{n\times(k_2+k_4)}, \quad v \in \{0, \pi\}^n$$

and p_l is the $(k_2 + l)$-th column of M. Note that $[p_1, \ldots, p_{k_4}, v]$ has an one-to-one correspondence between P_B. Each column of M has exactly one 1 and one -1, and each row of $[M, v] \in \mathbb{R}^{n\times(k_2+k_4+1)}$ has at most two nonzero elements.

Now we regard M as the indices matrix for a directed graph $\mathcal{G}$ with n nodes and $k_2 + k_4$ edges. From above we only require each node is involved in at most two edges. Let $\tilde{\mathcal{G}}$ be the underlying undirected graph of $\mathcal{G}$. The graph Laplacian of $\tilde{\mathcal{G}}$ is $L = MM^{\mathsf{T}}$. If the number of edges $k_2 + k_4 \ge n - 1$, it is sufficient for $\tilde{\mathcal{G}}$ to be connected, i.e., $\mathrm{rank}(M) = \mathrm{rank}(L) = n - 1$.

The proof of the $\mathrm{SU}^-(n)$ case is similar and we omit it here. ☐

Lemma 4.7. *For $2 \le k \in \mathbb{N}$,*

$$\mathrm{SU}(2k) = \Phi(k, k, k, k - 2, \mathbb{C}^{2k}).$$

Proof. First we will verify that $\mathrm{SU}(4) = \Phi(2, 2, 2, 0, \mathbb{C}^4)$. $U \in \mathrm{U}(2)$ can be parameterized as:

$$U(x, y, \theta) = \begin{bmatrix} x & y \\ -e^{i\theta}\overline{y} & e^{i\theta}\overline{x} \end{bmatrix}, \quad |x|^2 + |y|^2 = 1, \quad \theta \in [0, 2\pi) \tag{18}$$

Without loss of generality, let

$$\begin{aligned} C &= \mathrm{diag}(e^{ic_1}, \ldots, e^{ic_4}) \in \mathrm{SU}(4) \\ A &= \begin{bmatrix} \mathrm{diag}(e^{ia_1}, e^{ia_2}) & 0 \\ 0 & U_A\,\mathrm{diag}(e^{-ia_1}, e^{-ia_2})U_A^{\mathsf{H}} \end{bmatrix} \in \Phi(2, 2, \mathbb{C}^4), && U_A = U(x_A, y_A, \theta_A) \in \mathrm{U}(2) \\ B &= \begin{bmatrix} I_{1,-1} & 0 \\ 0 & U_B I_{1,-1} U_B^{\mathsf{H}} \end{bmatrix} \in \Phi(2, 0, \mathbb{C}^4), && U_B = U(x_B, y_B, \theta_B) \in \mathrm{U}(2) \end{aligned}$$

The matrix equation $C = AB$ can be reduced to

$$\begin{cases}(\cos c_1 + \cos c_2)|x_A|^2 - 2\cos c_3 |x_B|^2 = \cos c_2 - \cos c_3 \\ (\sin c_1 + \sin c_2)|x_A|^2 + 2\sin c_3 |x_B|^2 = \sin c_2 + \sin c_3\end{cases}, \quad |x_A|^2, |x_B|^2 \in [0,1] \tag{19}$$

For any $(c_i)_{i=1}^4$ with $\sum_{i=1}^4 c_i = 0$, we can readily verify that there exists a permutation of $(c_i)_{i=1}^4$ makes (19) admits a solution.

This construction scheme naturally extends to the $k \geq 3$ cases by letting:

$$\begin{aligned} C &= \exp(\operatorname{diag}(c_1, \dots, c_{2k})) \in \mathrm{SU}(2k) \\ A &= \operatorname{diag}(A_1, A_2) \in \Phi(k, k, \mathbb{C}^{2k}) \\ B &= \operatorname{diag}(B_1, B_2) \in \Phi(k, k-2, \mathbb{C}^{2k}) \end{aligned}$$

where

$$\begin{aligned} A_1 &= \exp(\operatorname{diag}(a_{k-1}, -a_1, a_1, -a_2, a_2, \dots, -a_{k-3}, a_{k-3}, -a_{k-2})) \\ A_2 &= \begin{bmatrix} \exp(\operatorname{diag}(a_{k-2}, a_k)) & 0 \\ 0 & U_A \exp(\operatorname{diag}(-a_{k-1}, -a_k)) U_A^{\mathsf{H}} \end{bmatrix} \\ B_1 &= \exp(\operatorname{diag}(-b_1, b_1, -b_2, b_2, \dots, -b_{k-2}, b_{k-2})) \\ B_2 &= \begin{bmatrix} I_{1,-1} & 0 \\ 0 & U_B I_{1,-1} U_B^{\mathsf{H}} \end{bmatrix} \end{aligned}$$

Then solving $C = AB$ reduces to solving (19) with $(\tilde{c}_i)_{i=1}^4 = (\sum_{i=1}^{2k-3} c_i, c_{2k-2}, c_{2k-1}, c_{2k})$. ☐

Theorem 4.8 (Special unitary group as product of four Grassmannians)**.** *Let $1 \leq k \in \mathbb{N}$ and $2 \leq n \in \mathbb{N}$. Let $K_{n,d}$ be defined in Definition 4.2.*

(i) $\mathrm{SU}^-(2k) = \Phi(k, k, k, k-1, \mathbb{C}^{2k})$.
(ii) $\mathrm{SU}^-(2k+1) = \Phi(k+1, k, k, k, \mathbb{C}^{2k+1})$.
(iii) $\mathrm{SU}(2k) = \Phi(k, k, k, k, \mathbb{C}^{2k}) = \Phi(k, k, k-1, k-1, \mathbb{C}^{2k})$. *If* $k \geq 2$, $\mathrm{SU}(2k) = \Phi(k, k, k, k-2, \mathbb{C}^{2k})$.
(iv) $\mathrm{SU}(2k+1) = \Phi(k, k, k, k, \mathbb{C}^{2k+1})$.

Proof. First we prove (i) and (ii). If $(k_i)_{i=1}^4$ satisfies $\mathrm{SU}^-(n) = \Phi(k_1, k_2, k_3, k_4, \mathbb{C}^n)$, it solves the following inequalities

$$\begin{cases} (k_i)_{i=1}^4 \in K_{n,4} & \text{(by Definition 4.2)} \\ \sum_{j=1}^4 k_j \geq l(\mathrm{SU}^-(n)) = 2n-1 & \text{(by Corollary 4.5)} \\ \dim(\Phi(k_1, k_2, \mathbb{C}^n)) + \dim(\Phi(k_3, k_4, \mathbb{C}^n)) \geq \dim(\mathrm{SU}^-(n)) = n^2 - 1 & \text{(by Theorem 3.4)} \end{cases} \tag{20}$$

When $n = 2k$, (20) has one solution $(k, k, k, k-1)$. When $n = 2k+1$, (20) has one solution $(k+1, k, k, k)$. By Lemma 4.6, the above two solutions are necessary and sufficient, so we proved (i) and (ii).

Next we prove (iii) and (iv). If $\mathrm{SU}(n) = \Phi(k_1, k_2, k_3, k_4, \mathbb{C}^n)$, similar inequalities should be satisfied

$$\begin{cases} (k_i)_{i=1}^4 \in K_{n,4} \\ \sum_{j=1}^4 k_j \geq l(\mathrm{SU}(n)) = 2n-2 \\ \dim(\Phi(k_1, k_2, \mathbb{C}^n)) + \dim(\Phi(k_3, k_4, \mathbb{C}^n)) \geq \dim(\mathrm{SU}(n)) = n^2 - 1 \end{cases} \tag{21}$$

When $n = 2k+1$, (21) has two solutions (k, k, k, k) and $(k+1, k, k, k-1)$. The first solution satisfies Lemma 4.6. The second solution $\Phi(k+1, k, k, k-1, \mathbb{C}^n) \subsetneq \mathrm{SU}(2k+1)$, because $A \in \mathrm{SU}(2k+1) \Leftrightarrow -A \in \mathrm{SU}^-(2k+1)$, and $-\Phi(k+1, k, k, k-1, \mathbb{C}^n) = \Phi(k, k, k, k-1, \mathbb{C}^n) \subsetneq \mathrm{SU}^-(2k+1)$ by (ii). So we proved (iv).

When $n = 2k$, (21) has four solutions (k,k,k,k), $(k,k,k-1,k-1)$, $(k,k,k,k-2)$ and $(k+1,k-1,k-1,k-1)$, where the last two only exist for $k \geq 2$. By checking Lemma 4.6 and Lemma 4.7, the first three solutions recover $\mathrm{SU}(2k)$, so we proved (iii). ☐

Remark 4.9. In Theorem 4.8 and its proof, we almost completely answered the question: given $(k_i)_{i=1}^4 \in [n]^4$, do we have $\Phi(k_1,k_2,k_3,k_4,\mathbb{C}^n) = \mathrm{SU}(n)$ (or $\mathrm{SU}^-(n)$), or $\Phi(k_1,k_2,k_3,k_4,\mathbb{C}^n)$ only forms a proper subset of $\mathrm{SU}(n)$ (or $\mathrm{SU}^-(n)$)? The only case we don't know is $\Phi(k+1,k-1,k-1,k-1,\mathbb{C}^n)$ for $n = 2k$. We leave it as an open question and to our future work.

Also, the decomposition into four Grassmannians is the best possible. Below is a modified algebraic result from [8].

Theorem 4.10. *For any $n = 3l$ with $l \in \mathbb{N}$, there exists $U \in \mathrm{SU}(n)$ such that $U \notin \Phi(k_1,k_2,k_3,\mathbb{C}^n)$ for any $(k_i)_{i=1}^3 \in K_{n,3}$.*

Proof. Let $U = e^{i\frac{2\pi}{3}} I \in \mathrm{SU}(n)$. Suppose $U = X_1X_2X_3$ with $X_i \in \mathrm{Gr}(k_i,\mathbb{C}^n)$, then it leads to the contradiction:

$$\begin{aligned} U^4 &= U(X_1U)X_2(UX_3) \\ &= U(X_2X_3)X_2(X_1X_2) \\ &= X_2(UX_3)X_2(X_1X_2) \\ &= X_2X_1X_2X_2X_1X_2 \\ &= I. \end{aligned}$$

☐

5. Product of symplectic Grassmannians

The notations are as in Section 1.1.

Lemma 5.1. $\mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n}) = \{P \operatorname{diag}(I_{k,n-k}, I_{k,n-k})P^{-1} : P \in \mathrm{Sp}(2n,\mathbb{F})\}$.

Proof. Since each $X \in \mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$ is an involution, it is diagonalizable by some $R \in \mathrm{GL}_n(\mathbb{F})$, and eigenvalues of X are 1 and -1, with multiplicity $2k$ and $2n-2k$ respectively. We further notice that $X \in \mathrm{Sp}(2n,\mathbb{F})$, this indicates the existence [16] of some $P \in \mathrm{Sp}(2n,\mathbb{F})$ such that $X = P\operatorname{diag}(I_{k,n-k}, I_{k,n-k})P^{-1}$. ☐

Proposition 5.2. *Let $k, n \in \mathbb{N}$ with $k \leq n$. Then*

$$\begin{aligned} \mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n}) &\simeq \mathrm{Sp}(2n,\mathbb{F})/\big(\mathrm{Sp}(2k,\mathbb{F}) \times \mathrm{Sp}(2(n-k),\mathbb{F})\big) \\ &\simeq \{\mathbb{V} \subseteq \mathbb{F}^{2n} : \ \mathbb{V} \textit{ is a } 2k\textit{-dimensional symplectic subspace}\} \end{aligned}$$

Proof. The second identification is classical [13]. Let $D \coloneqq \operatorname{diag}(I_{k,n-k}, I_{k,n-k})$. We consider a map

$$\varphi : \mathrm{Sp}(2n,\mathbb{F}) \to \mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n}), \quad \varphi(P) = PDP^{-1}.$$

According to Lemma 5.1, φ is surjective. Thus, the adjoint action of $\mathrm{Sp}(2n,\mathbb{F})$ on $\mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$ is transitive. The first identification follows immediately if we prove $\mathrm{Stab}_D(\mathrm{Sp}(2n,\mathbb{F})) \simeq \mathrm{Sp}(2k,\mathbb{F}) \times \mathrm{Sp}(2(n-k),\mathbb{F})$. To that end, we partition $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \in \mathrm{Stab}_D(\mathrm{Sp}(2n,\mathbb{F}))$ by $n \times n$ matrices P_1, P_2, P_3 and P_4. Since $PD = DP$, we have $P_iI_{k,n-k} = I_{k,n-k}P_i$ for each $1 \leq i \leq 4$. We further partition each P_i as $P_i = \begin{bmatrix} X_i & Y_i \\ Z_i & W_i \end{bmatrix}$ where $X_i \in \mathbb{F}^{k\times k}, Y_i \in \mathbb{F}^{k\times(n-k)}, Y_i \in \mathbb{F}^{(n-k)\times k}$ and $W_i \in \mathbb{F}^{(n-k)\times(n-k)}$. Hence we have $Y_i = 0$ and $Z_i = 0_{n-k,k}$. On the other hand, $P^\mathsf{T} J_{2n} P = J_{2n}$ implies that $P_1^\mathsf{T}P_3 - P_3^\mathsf{T}P_1 = P_2^\mathsf{T}P_4 - P_4^\mathsf{T}P_2 = 0$ and $P_1^\mathsf{T}P_4 - P_3^\mathsf{T}P_2 = I$. Therefore, we obtain

$$X_1^\mathsf{T}X_3 = X_3^\mathsf{T}X_1, \quad X_2^\mathsf{T}X_4 = X_4^\mathsf{T}X_2, \quad X_1^\mathsf{T}X_4 - X_3^\mathsf{T}X_2 = I_k, \tag{22}$$

$$W_1^\mathsf{T}W_3 = W_3^\mathsf{T}W_1, \quad W_2^\mathsf{T}W_4 = W_4^\mathsf{T}W_2, \quad W_1^\mathsf{T}W_4 - W_3^\mathsf{T}W_2 = I_{n-k}. \tag{23}$$

We observe that (22) is equivalent to $\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathrm{Sp}(2k,\mathbb{F})$ and (23) is equivalent to $\begin{bmatrix} W_1 & W_2 \\ X_3 & W_4 \end{bmatrix} \in \mathrm{Sp}(2(n-k),\mathbb{F})$. Therefore, we may conclude that $\mathrm{Stab}_D(\mathrm{Sp}(2n,\mathbb{F})) \simeq \mathrm{Sp}(2k,\mathbb{F})\times\mathrm{Sp}(2(n-k),\mathbb{F})$. $\square$

As a consequence of Proposition 5.2, $\mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$ serves as the involution model of the Grassmannian of $2k$-dimensional symplectic subspaces in $\mathbb{F}^{2n}$. Moreover, for each $X \in \mathrm{Sp}(2n,\mathbb{R})$, we write $X = P\operatorname{diag}(I_{k,n-k}, I_{k,n-k})P^{-1}$ and denote by $p_1,\dots,p_{2n}$ column vectors of P. Let $\mathbb{V}_X$ be the $2k$-dimensional subspace spanned by $p_1,\dots,p_k,p_{n+1},\dots,p_{n+k}$.

Corollary 5.3. *The subspace $\mathbb{V}_X$ is well-defined.*

Proof. According to the proof of Proposition 5.2, if

$$X = P\operatorname{diag}(I_{k,n-k}, I_{k,n-k})P^{-1} = P'\operatorname{diag}(I_{k,n-k}, I_{k,n-k})P'^{-1},$$

then P and P' are only differed by an element in $\mathrm{Sp}(2k,\mathbb{R})\times\mathrm{Sp}(2(n-k),\mathbb{R})$. This implies that

$$\operatorname{span}\{p_1,\dots,p_k,p_{n+1},\dots,p_{n+k}\} = \operatorname{span}\{p'_1,\dots,p'_k,p'_{n+1},\dots,p'_{n+k}\}.$$

Hence $\mathbb{V}_X$ is well-defined. $\square$

We consider two maps

$$\psi_1 : \mathrm{Gr}(k,\mathbb{C}^n) \to \mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{R}^{2n}), \quad \psi_1(A+iB) = \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$
$$\psi_2 : \mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{R}^{2n}) \to \mathrm{Gr}(2k,\mathbb{R}^{2n}), \quad \psi_2(X) = 2V_XV_X^{\mathsf{T}} - I_{2n},$$

where $A,B\in\mathbb{R}^{n\times n}$ and $V_X \in \mathrm{V}(2n,2k)$ is a $2n\times 2k$ orthonormal matrix whose column vectors form an orthogonal basis of $\mathbb{V}_X$.

Corollary 5.4. *Both ψ_1 and ψ_2 are well-defined and injective.*

Proof. The well-definedness and injectivity of ψ_1 follows from the inclusion $\mathrm{U}(n)\subseteq\mathrm{Sp}(2n,\mathbb{R})$. Since $\mathbb{V}_X$ is well-defined by Corollary 5.3, the matrix $2V_XV_X^{\mathsf{T}} - I_{2n}$ does not depend on the choice of V_X. This implies that ψ_2 is well-defined. The injectivity follows from the observation that the involution $2V_XV_X^{\mathsf{T}} - I_{2n}$ is uniquely determined by $\mathbb{V}_X$. $\square$

We recall that there is a sequence of inclusions of homogeneous space models [13, Proposition 11]:

$$\frac{\mathrm{U}(n)}{\mathrm{U}(k)\times\mathrm{U}(n-k)} \subseteq \frac{\mathrm{Sp}(2n,\mathbb{R})}{\mathrm{Sp}(2k,\mathbb{R})\times\mathrm{Sp}(2(n-k),\mathbb{R})} \subseteq \frac{\mathrm{O}(2n)}{\mathrm{O}(2k)\times\mathrm{O}(2n-2k)} \tag{24}$$

which is equivalent to

$$\mathbb{G}(k,\mathbb{C}^n) \subseteq \mathbb{G}_{\mathrm{Sp}}(2k,\mathbb{R}^{2n}) \subseteq \mathbb{G}(2k,\mathbb{R}^{2n}) \tag{25}$$

with notations as in (1). In fact, the sequence of inclusions

$$\mathrm{Gr}(k,\mathbb{C}^n) \xhookrightarrow{\psi_1} \mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{R}^{2n}) \xhookrightarrow{\psi_2} \mathrm{Gr}(2k,\mathbb{R}^{2n})$$

exactly corresponds to (24) (resp. (25)), if we identify $\mathrm{Gr}(k,\mathbb{C}^n)$, $\mathrm{Gr}_{\mathrm{Sp}}(2k,\mathbb{R}^{2n})$ and $\mathrm{Gr}(2k,\mathbb{R}^{2n})$ with their homogeneous models (resp. subspace models), respectively.

Lemma 5.5. *If $X\in\mathrm{Sp}(2n,\mathbb{F})$ is generic, then up to a sign, $X = Y_1Y_2$ for some $Y_1,Y_2\in \mathrm{Gr}_{\mathrm{Sp}}(2\lfloor n/2\rfloor,\mathbb{F}^{2n})$.*

Proof. Without loss of generality, we may assume X is diagonal. We write $X = \operatorname{diag}(D,D_1)$ where D and D_1 are $n\times n$ diagonal matrices. Since $X\in\mathrm{Sp}(2n,\mathbb{F})$, we must have $D_1 = D^{-1}$. We write $D = \operatorname{diag}(d_1,\dots,d_n)$. Suppose $m := \lfloor n/2\rfloor$. We write

$$\begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} = \prod_{j=1}^{m} \begin{bmatrix} D_j & 0 \\ 0 & D_j^{-1} \end{bmatrix}$$

where $D_j = \operatorname{diag}(\underbrace{1,\dots,1}_{(2j-2)\text{ copies}}, d_{2j-1}, d_{2j}, \underbrace{1,\dots,1}_{(2m-2j)\text{ copies}})$ for $1 \le j \le m-1$ and

$$D_m = \begin{cases} \operatorname{diag}(\underbrace{1,\dots,1}_{(2m-2)\text{ copies}}, d_{2m-1}, d_{2m}) & \text{if } n = 2m \\ \operatorname{diag}(\underbrace{1,\dots,1}_{(2m-2)\text{ copies}}, d_{2m-1}, d_{2m}, d_{2m+1}) & \text{if } n = 2m+1 \end{cases}$$

We observe that for each $X_j := \begin{bmatrix} D_j & 0 \\ 0 & D_j^{-1} \end{bmatrix}$, there is a subspace $\mathbb{V}_j \subseteq \mathbb{F}^{2n}$ restricts to which X_j is symplectic. Moreover, we may take $\mathbb{V}_j$ so that $\dim \mathbb{V}_j = 4$ or 6 by construction. Therefore, it is sufficient to prove for $n = 2$ and $n = 3$. Since X is clearly similar to X^{-1}, X is a product of two symplectic involutions by [4, Theorem 8] and [1, Theorem 2]. For $n = 2$, $-\operatorname{Gr}_{\mathrm{Sp}}(0,\mathbb{F}^4) = \operatorname{Gr}_{\mathrm{Sp}}(4,\mathbb{F}^4) = \{I_4\}$. Hence $X = Y_1Y_2$ for $Y_1, Y_2 \in \operatorname{Gr}_{\mathrm{Sp}}(2,\mathbb{F}^4)$ unless $X^2 = I_4$. For $n = 3$, we still have $-\operatorname{Gr}_{\mathrm{Sp}}(0,\mathbb{F}^6) = \operatorname{Gr}_{\mathrm{Sp}}(6,\mathbb{F}^6) = \{I_6\}$. Moreover, we also have $\operatorname{Gr}_{\mathrm{Sp}}(2,\mathbb{F}^6) = -\operatorname{Gr}_{\mathrm{Sp}}(4,\mathbb{F}^6)$. Thus we obtain $X = \pm Y_1Y_2$ for some $Y_1, Y_2 \in \operatorname{Gr}_{\mathrm{Sp}}(2,\mathbb{F}^6)$ unless $X^2 = I_6$. By the genericity of X, we may conclude that $X = \pm Y_1Y_2$ for some $Y_1, Y_2 \in \operatorname{Gr}_{\mathrm{Sp}}(2\lfloor n/2\rfloor,\mathbb{F}^{2n})$ when $n = 2, 3$. □

Lemma 5.6. *Let G be a Lie group and let U be an open dense subset of G. If $U = U^{-1}$, then we have $G = UU$.*

Proof. Clearly, U^{-1} is an open dense subset of G. For each $g \in G$, gU^{-1} is also an open dense subset of G. Therefore, there exists some $x \in gU^{-1} \cap U$. This implies that $x = gy$ for some $y \in U^{-1}$ and hence $g = xy^{-1} \in UU$. □

Theorem 5.7 (Symplectic group as product of four Grassmannians)**.** *For any positive integer n, we have*

$$\operatorname{Sp}(2n,\mathbb{F}) = \begin{cases} \Phi_{\mathrm{Sp}}(2k,2k,2k,2k,\mathbb{F}^{2n}) & \text{if } n = 2k \\ \Phi_{\mathrm{Sp}}(2k,2k,2k,2k,\mathbb{F}^{2n}) \bigcup \Phi_{\mathrm{Sp}}(2k+2,2k,2k,2k,\mathbb{F}^{2n}) & \text{if } n = 2k+1 \end{cases}$$

Proof. Let $k := \lfloor n/2 \rfloor$. We claim that every $X \in \operatorname{Sp}(2n,\mathbb{F})$ can be decomposed as $X = \pm Y_1Y_2Y_3Y_4$ for some $Y_1,\dots,Y_4 \in \operatorname{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$. We notice that if $n = 2k$ then $Y \in \operatorname{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$ if and only if $-Y \in \operatorname{Gr}_{\mathrm{Sp}}(2n-2k,\mathbb{F}^{2n}) = \operatorname{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$. If $n = 2k+1$ then $Y \in \operatorname{Gr}_{\mathrm{Sp}}(2k,\mathbb{F}^{2n})$ if and only if $-Y \in \operatorname{Gr}_{\mathrm{Sp}}(2n-2k,\mathbb{F}^{2n}) = \operatorname{Gr}_{\mathrm{Sp}}(2k+2,\mathbb{F}^{2n})$. Thus, it is left to prove the claim. By Lemma 5.5, there is an open dense subset U of $\operatorname{Sp}(2n,\mathbb{F})$ such that each $X \in U$ is a product of two elements in $\operatorname{Gr}_{\mathrm{Sp}}(2\lfloor n/2\rfloor,\mathbb{F}^{2n})$, up to a sign. By Lemma 5.6, we have $UU = \operatorname{Sp}(2n,\mathbb{F})$. The desired decomposition of an element in $\operatorname{Sp}(2n,\mathbb{F})$ follows immediately. □

6. Conclusion

By using the involution model of Grassmannians, we decompose the special orthogonal group and the special unitary group into product of Grassmannians. We fully characterized a product of two Grassmannians, and calculated its dimension formula. In addition, we determine the relation between principal angles and the product of two Grassmannians. In summary, we have shown that

(a) Every $Q \in \operatorname{SO}(n)$ can be written as $Q = X_1X_2$ with $X_1, X_2 \in \operatorname{Gr}(\lfloor n/2\rfloor,\mathbb{R}^n)$.
(b) Every $Q \in \operatorname{SU}(n)$ can be written as $Q = X_1X_2X_3X_4$ with $X_1,X_2,X_3,X_4 \in \operatorname{Gr}(\lfloor n/2\rfloor,\mathbb{C}^n)$.
(c) Every $Q \in \operatorname{Sp}(2n,\mathbb{F})$ can be written as $Q = X_1X_2X_3X_4$ with $X_1,X_2,X_3,X_4 \in \operatorname{Gr}_{\mathrm{Sp}}(2\lfloor n/2\rfloor,\mathbb{F}^{2n})$.

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Computational and Applied Mathematics Initiative, Department of Statistics, University of Chicago, Chicago, IL 60637
Email address: `lekheng@uchicago.edu`

Department of Statistics, University of Chicago, Chicago, IL 60637
Email address: `xiangl1@uchicago.edu`

KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: `keyk@amss.ac.cn`