# An Optimal Iterative Algorithm for Extracting MUCs in a Black-box Constraint Network

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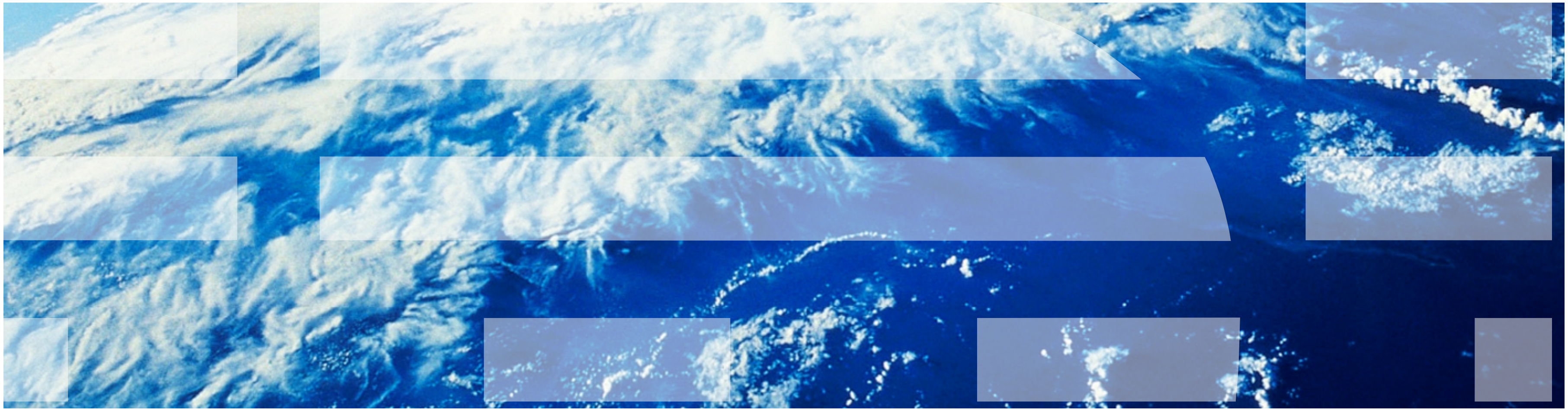


Aug. 22, 2014

# Objective

- Given an unfeasible Constrained Optimization problem M, extract a **Minimal Unsatisfiable Core (MUC)** that is, a minimal subset (in the sense of set inclusion) of unfeasible constraints $X \subseteq M$

- Example: x, y, z in { 1, 2, 3 }
  constraint C1: AllDifferent(x,y,z)
  constraint C2: z == x + 2y
  constraint C3: y < x
  constraint C4: z == 2x

- This problem is unfeasible. A possible MUC is {C1,C2}.

# Objective

- Algorithm developed in the context of providing a Conflict Refiner for CP models in IBM ILOG CPLEX Optimization Studio: when designing optimization models, it is usual to face unfeasible instances (due to errors in the model or the data). Identifying a MUC generally helps to explain the unfeasibility
- Given the complexity of the automatic search, we consider the engine as a **black-box** whose answer is Yes/No for the feasibility of a subset of constraints $X \subseteq M$

# Abstracted problem

- Let U a finite set of cardinality n and P an **upward-closed** (or monotonous) property on its powerset $2^U$, that is, such that:

  $$(X \subseteq Y \subseteq U) \wedge P(X) \Rightarrow P(Y)$$

- Minimal subset: $X \subseteq U$ is a **minimal subset** satisfying P iff:

  $$P(X) \text{ and } \forall Y \subset X, \neg P(Y)$$

- In our original problem:
  - U ≡ Set of constraints in the model
  - P(X) ≡ Subset of constraints X is unfeasible

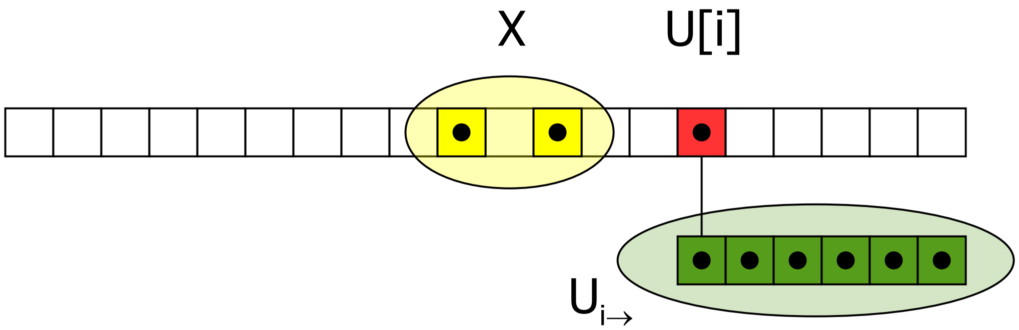
# Abstracted problem

- Problem definition:
  - Let U such that P(U), **find a minimal subset X satisfying P**
  - The complexity of the resolution algorithm is measured as the **number of property checks**. As this is an homogeneous measure we can use a more fine grain complexity comparison than the traditional *big O* comparison. We use a comparison *on the order of* :

$$f(n) \sim g(n) \text{ means } \lim_{n \to \infty} (f(n)/g(n)) = 1$$

# A family of iterative algorithms

- Elements of U are stored in an array of size n
- Array is shuffled so as to rule out any particular structure
- Algorithm will select the “rightmost” minimal set in the array
- Scan elements from left to right
- Grows a current subset X



# A family of iterative algorithms

- Next element U[i] to be added to current subset X is the one with the smallest index i such that P(X U $U_{i\to}$)
- When a new element U[i] is added, property P can be checked on current subset X (if P(X), algorithm can be stopped)
- Algorithms in this family differs according to the way to find the next element U[i]

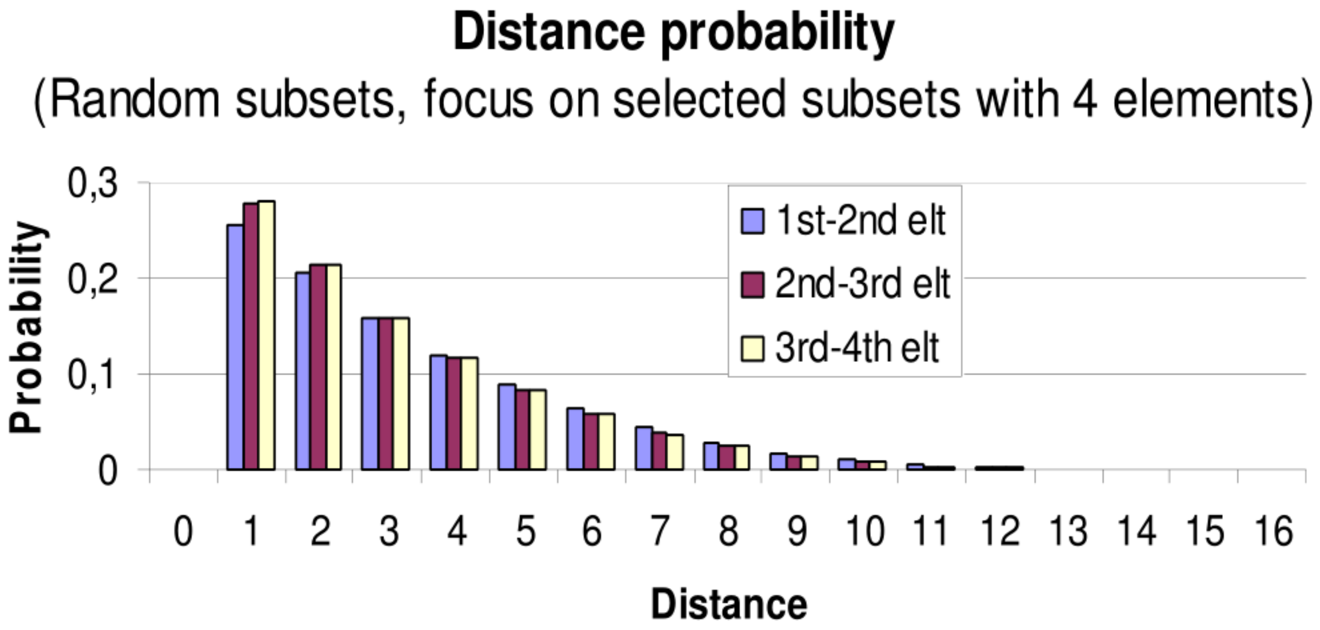
# Position of consecutive elements

- In the particular case of a single minimal subset of size m **uniformly distributed** in U, the probability p(k) that the distance between the ith and i+1th element is k:
  - –Does not depend on the position i
  - –Exponentially decreases with k

$$p(k) = \binom{n-k}{m-1} / \binom{n}{m}$$

# Position of consecutive elements

- Experiments in more general cases

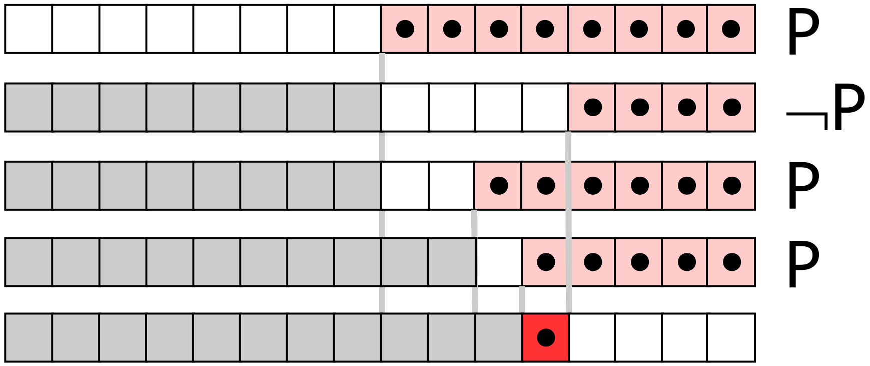


–Next element of the subset is often **close** to the current one

–Estimated distance between consecutive elements can be **learned**

# Dichotomy

- A dichotomy algorithm DC has been proposed in [Hemery&al-06]. It performs a dichotomic search in order to find next element U[i].



# Dichotomy

- This algorithm is efficient for small minimal subsets. It is optimal for minimal subsets of size 1 with a complexity in ~$\log_2(n)$.
- On the other side of the spectrum, it is not efficient for large minimal subsets. For a minimal subset of size n it may require $O(n \log_2(n))$ property checks.
- Our algorithm uses a dichotomic search:
  - To select the first element
  - For the following elements: after the acceleration has reduced the scope of the dichotomy

# Acceleration / Progression

- The probabilistic study of the distance between consecutive elements suggests that it may pay off to look for the next element close to the current one

- Starting from last added element at index i, an acceleration algorithm tries $i+1, i+2, i+4, \ldots, i+2^k$ until first index $j=i+2^k$ such that $\neg P(X \cup U_{j\rightarrow})$

- The algorithm then applies a dichotomy on the reduced index segment $[i+2^{k-1}, i+2^k)$

# Acceleration / Progression

- A similar approach is used in [Marques-Silva&al-13]
- Main differences of our algorithm:
  - The first iteration uses dichotomy, not acceleration
  - The initial step in the acceleration is learned (see below), it is not necessarily 1
  - Up to a certain size, property is checked on current subset X each time a new element is added
- These differences allow showing optimality of the proposed algorithm for small minimal subsets

# Estimation (of distance)

- The probabilistic study of the distance between consecutive elements suggests that the distribution of this distance does not depend much on which element is considered
- Let *s* denote the size of the initial acceleration step (acceleration algorithm tries i+*s*,i+2*s*,i+4*s*,...,i+$2^k$*s*)
- For the first element, we take *s=n* so the search boils down to a pure dichotomic search (line 4 on Algorithm 1)
- For the next elements, the initial acceleration step is computed as the average distance between past elements (line 10 on Algorithm 1)
- Note that in case of a minimal subset of size n, after first element is found (at position 1), the initial acceleration step is always 1

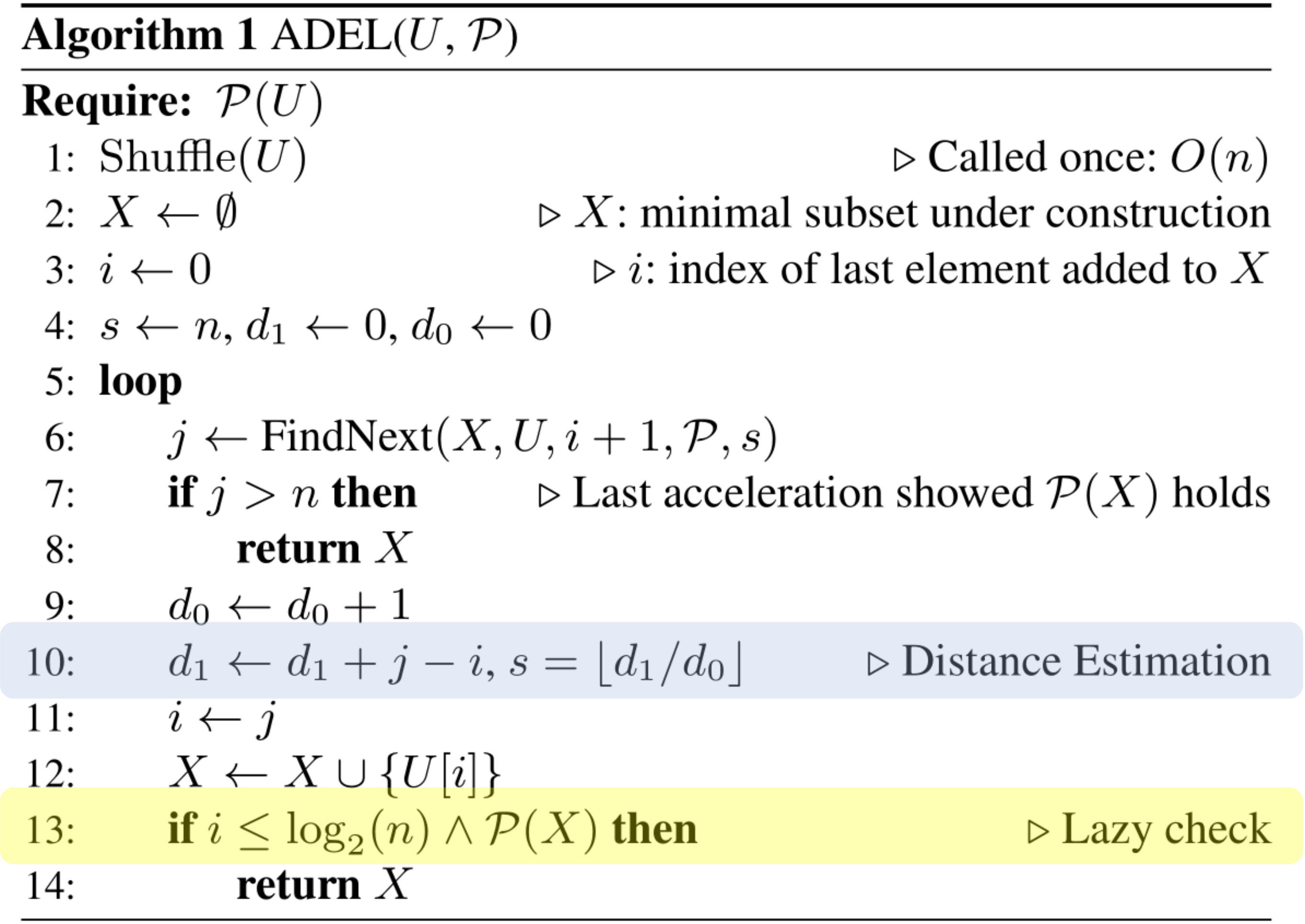
# Lazy checks

- Property P(X) may be checked when a new element is added to X, this allows stopping the search as soon as X satisfies P
- This is efficient for small minimal subsets. Typically, for a minimal subset of size 1, once the initial dichotomy has selected an (the) element, search will be stopped immediately
- This is inefficient for large subsets. Typically, for a minimal subset of size n, n-1 useless checks P(X) will be performed

# Lazy checks

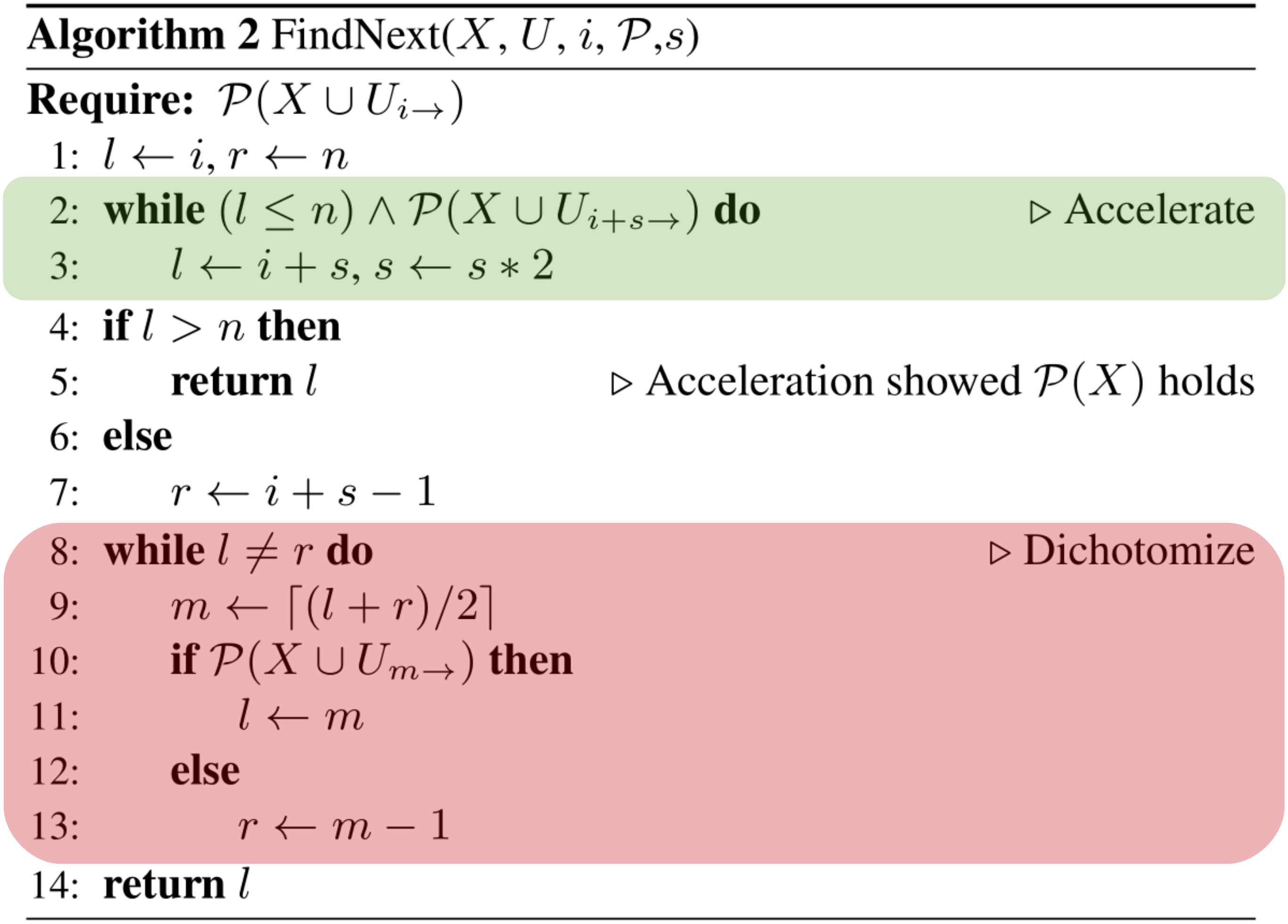
- When the last element of the minimal subset is added to X, if we do not check P(X), it will take the acceleration about O($\log_2(n)$) additional checks to show that X satisfies P(X)
- The idea is to check P(X) only for the first $\log_2(n)$ elements. For larger subsets, the O($\log_2(n)$) price to pay for letting the acceleration show the property will not exceed the O($\log_2(n)$) we already payed to check the first $\log_2(n)$ elements so the algorithm can stop checking P(X) (line 13 of Algorithm 1)

# Detailed ADEL algorithm



```
Algorithm 1 ADEL($U$, $\mathcal{P}$)
Require: $\mathcal{P}(U)$
Shuffle($U$)                                  ▷ Called once: $O(n)$
$X \leftarrow \emptyset$                      ▷ $X$: minimal subset under construction
$i \leftarrow 0$                              ▷ $i$: index of last element added to $X$
$s \leftarrow n$, $d_1 \leftarrow 0$, $d_0 \leftarrow 0$
loop
    $j \leftarrow$ FindNext($X, U, i+1, \mathcal{P}, s$)
    if $j > n$ then                           ▷ Last acceleration showed $\mathcal{P}(X)$ holds
        return $X$
    $d_0 \leftarrow d_0 + 1$
    $d_1 \leftarrow d_1 + j - i$, $s = \lfloor d_1/d_0 \rfloor$     ▷ Distance Estimation
    $i \leftarrow j$
    $X \leftarrow X \cup \{U[i]\}$
    if $i \leq \log_2(n) \wedge \mathcal{P}(X)$ then     ▷ Lazy check
        return $X$
```

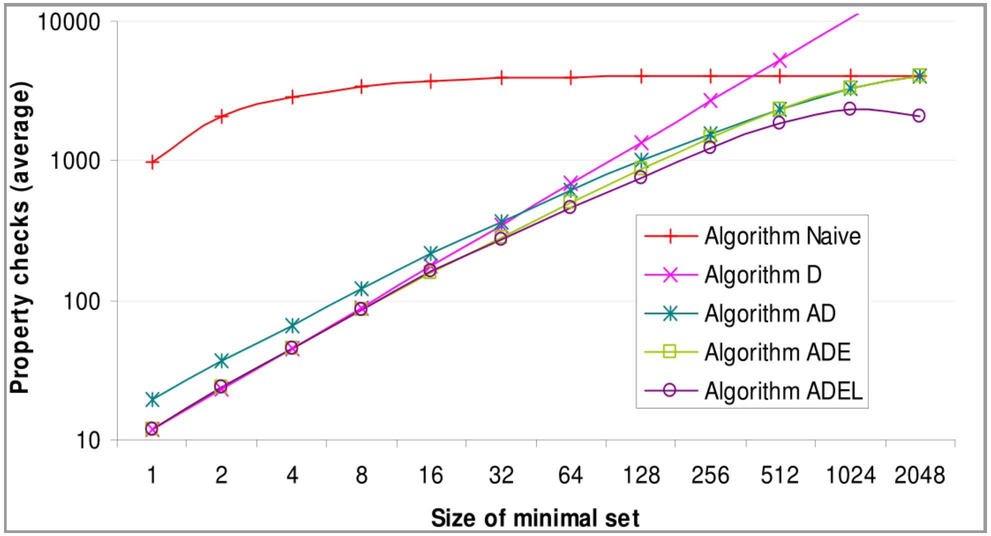
# Detailed ADEL algorithm

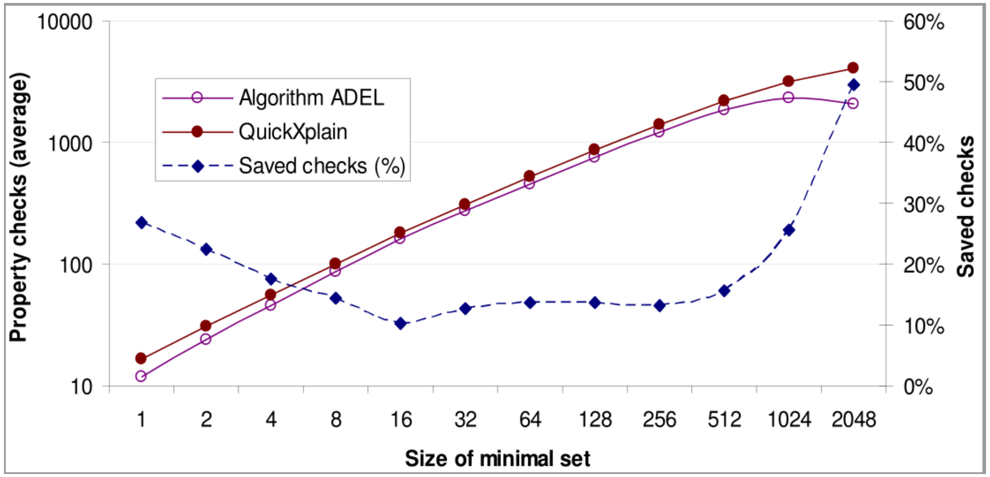


**Algorithm 2** FindNext($X$, $U$, $i$, $\mathcal{P}$,$s$)

```
Require: $\mathcal{P}(X \cup U_{i\rightarrow})$
$l \leftarrow i, r \leftarrow n$
while $(l \leq n) \wedge \mathcal{P}(X \cup U_{i+s\rightarrow})$ do        ▷ Accelerate
    $l \leftarrow i + s, s \leftarrow s * 2$
if $l > n$ then
    return $l$                    ▷ Acceleration showed $\mathcal{P}(X)$ holds
else
    $r \leftarrow i + s - 1$
while $l \neq r$ do                                                   ▷ Dichotomize
    $m \leftarrow \lceil (l + r)/2 \rceil$
    if $\mathcal{P}(X \cup U_{m\rightarrow})$ then
        $l \leftarrow m$
    else
        $r \leftarrow m - 1$
return $l$
```

# Results


10000
1000
100
10
Property checks (average)
1
2
4
8
16
32
64
128
256
512
1024
2048
Size of minimal set
Algorithm Naive
Algorithm D
Algorithm AD
Algorithm ADE
Algorithm ADEL


10000
1000
100
10
Property checks (average)
60%
50%
40%
30%
20%
10%
0%
Saved checks
1
2
4
8
16
32
64
128
256
512
1024
2048
Size of minimal set
Algorithm ADEL
QuickXplain
Saved checks (%)

# Results

- Algorithm ADEL is **optimal** in both extreme cases:
  - For small subsets: ~$\log_2(n)$ for minimal subset of size 1
  - For large subsets: ~n for minimal subset of size n
- Compared to QuickXplain [Junker-04]:
  - For small subsets, it performs 1.5 times less checks
  - For large subsets, it performs twice less checks
- It behaves continuously in between these extreme cases and outperforms all variants as well as QuickXplain
- ADEL algorithm is used as the implementation of the Conflict Refiner functionality for CP models in IBM ILOG CPLEX Optimization Studio since version 12.5

# References

[Junker-04] U. Junker. *QuickXplain: Preferred explanations and relaxations for over-constrained problems*. In Proc. AAAI-04. 2004.

[Hemery&al-06] F. Hemery, C. Lecoutre, L. Sais and F. Boussemart. *Extracting MUCs from constraint networks*. In Proc. ECAI-06. 2006.

[Marques-Silva&al-13] J. Marques-Silva, M. Janota and A. Belov. *Minimal Sets over Monotone Predicates in Boolean Formulae*. In Proc. CAV-13. 2013.