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Prime Implicates Based Inconsistency Characterization

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Abstract. Measuring inconsistency is recognized as an important issue for handling inconsistencies [5, 6]. Based on prime implicates canonical representation, we first characterize the conflicting variables allowing us to refine an existing inconsistency measure. Secondly, we propose a new measure, to circumscribe the internal conflicts in a knowledge base. This measure is proved to satisfy a new but weaker form of dominance.

1 Introduction

Inspired by the example given in [8], suppose that there are n groups of people polling on a set of policies $\{p_1, ..., p_m\}$. The poll result γ_i of each group is a set of propositional formulae. For example, $\{p_1 \wedge \neg p_2, p_1 \vee p_3\}$ expresses that in this group there's one voter who votes p_1 but votes against p_2, and the other voter supports either p_1 or p_3. Now consider the results of two groups: $\gamma_1 = \{p_1 \wedge p_2, \neg p_2\}$, $\gamma_2 = \{p_1, \neg p_1 \vee p_2, \neg p_2, p_2\}$, which are both inconsistent. Then we can use different measures to compare γ_1 and γ_2. By ID_4 [4], γ_1 contains one unit of inconsistency, which seems reasonable because the conflict is merely on p_2 within this group, but ID_4 treats γ_2 equivalently even though there are indeed conflicts within two subgroups. In contrast, ID_{MUS} [8] considers that both poll results have two units of inconsistency because p_1 and p_2 are all involved in at least one subgroup with conflicts. In short, ID_4 ignores some inconsistencies for γ_2, while ID_{MUS} overestimates inconsistency in γ_1, never mentioning ID_Q [3] that is always equal or larger than ID_{MUS} [8]. To improve these language-based measures, we propose a new notion, called *conflicting variables*, from which we derive a new measure ID^c_{MUS} that can distinguish γ_1 and γ_2. We further refine the notion of MUS and propose a new one called *DMUS* (for deduced MUS), which leads to an interesting measure that satisfies a new restrictive but more intuitive dominance property, called *weak dominance*.

2 Preliminaries

In this paper, we consider the propositional language $\mathcal{L}$ built over a finite set of propositional symbols $\mathcal{P}$ using classical logical connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$. A knowledge base (KB) K consists of a finite set of consistent propositional formulae. Sometimes, a propositional formula can be in conjunctive normal form (CNF). K is inconsistent if $K \vdash \bot$, where $\vdash$ is the classical consequence relation. For a set S, $|S|$ denotes its cardinality.
The *Minimal Unsatisfiable Subset (MUS)* of K is defined as follows:

Definition 1 (MUS)**.** *Let K be a KB and $M \subseteq K$. M is a minimal unsatisfiable (inconsistent) subset (MUS) of K iff $M \vdash \bot$, and $\forall M' \subsetneq M$, $M' \nvdash \bot$. The set of all minimal unsatisfiable subsets of K is denoted $MUSes(K)$.*

Definition 2. *I_{MI} value is defined as: $I_{MI}(K) = |MUSes(K)|$.*

Definition 3 (Dominance)**.** *Given a KB K and two formulae α, β. An inconsistency measure I satisfies dominance if the following condition holds: if $\alpha \vdash \beta$ and $\alpha \nvdash \bot$, then $I(K \cup \{\beta\}) \leq I(K \cup \{\alpha\})$.*

The dominance states that if we substitute a consistent formula by its logical consequence, the inconsistency value does not increase.

Definition 4 (Prime Implicate)**.** *A clause π is a* prime implicate *of ϕ if $\phi \vdash \pi$ holds, and for every clause π', if $\phi \vdash \pi'$ and $\pi' \vdash \pi$ hold, then $\pi' \equiv \pi$ holds. $PI(\phi)$ denotes the set of prime implicates of ϕ.*

2.1 Paraconsistent Semantics

Different from classical two-valued semantics, multi-valued semantics (4-valued [1], Quasi Classical [2]), use four elements: *true, false, unknown* and *both*, written by t, f, N, B, respectively. The four truth values together with the ordering $\preceq$ defined below form a lattice $FOUR = (\{t, f, B, N\}, \preceq)$: $f \preceq N \preceq t, f \preceq B \preceq t, N \npreceq B, B \npreceq N$. The operator $\neg$ is defined as $\neg t = f, \neg f = t, \neg B = B$, and $\neg N = N$. A 4-valued interpretation $\mathcal{I}$ is a *4-model* of a KB K, denoted $\mathcal{I} \models_4 K$ if for each formula $\phi \in K$, $\phi^{\mathcal{I}} \in \{t, B\}$.

Paraconsistent semantics lead to different inconsistency measures. Let $\mathcal{I}$ be an interpretation under i-semantics ($i = 4, Q$). Then, *Conflict*$(K, \mathcal{I}) = \{p \in Var(K) \mid p^{\mathcal{I}} = B\}$ is called the *conflicting set* of $\mathcal{I}$ with respect to K. Intuitively, in terms of size-wise minimality, the larger the size of the conflicting set in i-models of K, the more inconsistent K is.

Definition 5 (ID_4 and ID_Q)**.** *The 4- and Q-semantics based inconsistency degrees are defined as:*

$$ID_i(K) = \min_{\mathcal{I} \models_i K} \frac{|Conflict(K, \mathcal{I})|}{|Var(K)|}, \text{ where } i \in \{4, Q\}.$$

3 Measuring Conflicts by Variables

In this section, we introduce the notion of conflicting variables to capture the variables that are really in conflicts. For instance, if $K = \{p \wedge q, \neg p\}$, p is expected to be a conflicting variable, but not q.

Definition 6 (Conflicting Variable)**.** *Let K be a KB and $p \in Var(K)$. p is a* conflicting variable *if there exists, $\mathcal{S} \subseteq K$, $\mathcal{S}' \subseteq K$, and two sets of formulae $\mathcal{D}$ and $\mathcal{D}'$ satisfying the following conditions:*

1. *$|\mathcal{S}| = |\mathcal{D}|$, $\forall \alpha \in \mathcal{D}$, $\exists!\ \phi \in \mathcal{S}$ s.t. $\phi \vdash \alpha$ and $PI(\alpha) \subseteq PI(\phi)$;*
2. *$\mathcal{D} \nvdash \bot$ and $\mathcal{D} \cup \{p\} \vdash \bot$;*
3. *$|\mathcal{S}'| = |\mathcal{D}'|$, $\forall \alpha \in \mathcal{D}'$, $\exists!\ \phi \in \mathcal{S}'$ s.t. $\phi \vdash \alpha$ and $PI(\alpha) \subseteq PI(\phi)$;*

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4. $\mathcal{D}' \nvdash \bot$ *and* $\mathcal{D}' \cup \{\neg p\} \vdash \bot$*;*
5. $\mathcal{D} \cup \mathcal{D}'$ *is a MUS.*

We denote by ConfV(K) *the set of conflicting variables of* K.

Intuitively, a conflicting variable p is a variable such that both its associated literals are logically entailed by sets of consistent logical consequences of K. A particular attention should be paid on the necessary condition $PI(\alpha) \subseteq PI(\phi)$.

Lemma 1. *Given a CNF KB K, ConfV*$(K) =$ *Var*$(MUSes(K))$.

However, the conclusion does not hold for arbitrary KBs. For example, consider $K = \{p \wedge q, \neg p\}$. Clearly, $MUSes(K) = \{K\}$, and *Var*$(MUSes(K)) = \{p, q\}$, but *ConfV*$(K) = \{p\}$. However, it always holds:

Proposition 2. *For any KB K, ConfV*$(K) \subseteq$ *Var*$(MUSes(K))$*. If* K *is a MUS in CNF, then ConfV*$(K) =$ *Var*(K).

Lemma 3. *If* K *is inconsistent, then* K *has at least one conflicting variable, that is* $ConfV(K) \neq \emptyset$.

Let us first recall an interesting inconsistency measure proposed recently in [8], defined as $ID_{MUS}(K) = \frac{|Var(MUSes(K))|}{|Var(K)|}$.

Example 1. *Consider the KB* $K = \{p \wedge (q \vee r), \neg p, \neg q \vee \neg r, s \vee t\}$. K *involves the MUS* $\mathsf{M} = \{p \wedge (q \vee r), \neg p\}$*. So,* $ID_{MUS}(K) = \frac{|\{p,q,r\}|}{|Var(K)|} = 3/5$*. However, we can clearly observe that the variables* q *and* r *are not involved in the conflict of* M.

This example motivates us to exploit the notion of conflicting variables to define a finer inconsistency measure:

Definition 7. *For a given KB K, the* ID^c_{MUS} *measure is defined as:* $ID^c_{MUS}(K) = \frac{|ConfV(K)|}{|Var(K)|}$.

That is, instead of simply taking all variables appearing in the MUSes of K, we only consider conflicting variables that contribute to inconsistency.

Example 2 (Example 1 contd.)**.** *We have ConfV*$(K) = \{p\}$*, so* $ID^c_{MUS}(K) = 1/5$.

The following result shows the relationship among the above inconsistency measures.

Proposition 4. $ID_4(K) \leq ID^c_{MUS}(K) \leq ID_{MUS}(K)$.

4 MUS Based Logical Deduction

In [7], the authors show that I_{MI} measure does not satisfy the dominance by using the following counterexample: $K = \{p, q \wedge r, \neg q\}$, $\alpha = p \wedge r \wedge (\neg p \vee q)$, and $\beta = \neg p \vee q$. Obviously, $K \cup \{\alpha\}$ has one MUS, but $K \cup \{\alpha\}$ has two MUSes.

Since dominance seems too strong to be satisfied by certain inconsistency measure, let us consider a weaker dominance property.

Definition 8 (Weak Dominance)**.** *Let* K *be a KB. An inconsistency measure* I *satisfies the weak dominance (in short* w*-dominance) if for all formulae* α *and* β *such that* $\alpha \nvdash \bot$*,* $\alpha \vdash \beta$ *and* $PI(\beta) \subseteq PI(\alpha)$*, then* $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$.

Contrary to the classical dominance property, w-$dominance$ requires the condition that $PI(\beta) \subseteq PI(\alpha)$.

In the previous example where $K = \{p, p \wedge r, \neg q\}$, $\alpha = p \wedge r \wedge (\neg p \vee q)$ and $\beta = \neg p \vee q$, we have $PI(\beta) = \{\neg p \vee q\} \nsubseteq PI(\alpha) = \{p, q, r\}$. Thus, under the weak dominance, it is not required any more that $K \cup \{\alpha\} \geq K \cup \{\beta\}$.

Proposition 5. I_{MI} *measure does not satisfy w-*$dominance$.

The previous property shows that I_{MI} does not satisfy the weak dominance property. Next we introduce a new notion, *deduced MUS*, to have an inconsistency measure satisfying the w-$dominance$.

Definition 9 (deduced MUS)**.** *Let* K *be a KB and* $M = \langle \mathcal{S}, \mathcal{D} \rangle$ *such that* $\mathcal{S} = \{\phi_1, \ldots, \phi_m\} \subseteq K$ *and* $\mathcal{D} = \{\alpha_1, \ldots, \alpha_m\}$ *a set of formulae (*$\mathcal{S}$ *is called the support of* $\mathcal{D}$*).* M *is a MUS modulo logical deduction (noted DMUS) of* K *if:*

1. $\forall (1 \leq i \leq m)$, $\phi_i \vdash \alpha_i$ *and* $PI(\alpha_i) \subseteq PI(\phi_i)$
2. $\{\alpha_1, \ldots, \alpha_m\}$ *is a MUS;*
3. $\forall \alpha \in \{\alpha_1, \ldots, \alpha_m\}$ *there is no* α' *such that*
   (a) α' *is weaker that* α *(*$\alpha \vdash \alpha'$ *but* $\alpha' \nvdash \alpha$*);*
   (b) $(\mathcal{D} \setminus \{\alpha\}) \cup \{\alpha'\}$ *is a MUS.*

Intuitively, the first condition allows us to capture the conflicts between sub-formulae (α_i) deduced from the original formulae (ϕ_i) of K. The second is to circumscribe the real conflicts inside KB by avoiding introducing useless conflicts. The third condition constrains the set of deduced formulae to be a classical MUS, while the last condition allows us reduce the number of possible deduced MUSes by considering only the weakest deduced formulae.

We can further define a binary relation between DMUSes to avoid redundancies.

Definition 10. *Let* K *be a KB. Two DMUSes* $M = \langle \mathcal{S}, \mathcal{D} \rangle$*, and* $M' = \langle \mathcal{S}', \mathcal{D}' \rangle$ *are equivalent, noted* $M \approx_d M'$*, iff* $\mathcal{S} = \mathcal{S}'$*, and* $\bigcup_{\alpha \in \mathcal{D}} PI(\alpha) = \bigcup_{\alpha \in \mathcal{D}'} PI(\alpha)$.

Proposition 6. $\approx_d$ *is an equivalence relation over DMUSes.*

From this definition, two DMUSes belong to the same equivalence class if they share the same support (the same subset of formulae from K) and they admit the same prime implicates representation.

Based on the equivalence relation $\approx_d$, the set of DMUSes can be partitioned into a set of equivalent classes $DMUSes(K)^{\approx_d}$. Such partition allows us to define the following inconsistency measure.

Definition 11. *Let* K *be a KB.* $I_{DM}(K) = |DMUSes(K)^{\approx_d}|$.

Proposition 7. *The* I_{DM} *measure does not satisfy dominance, but it satisfies w-dominance.*

Proposition 8. *Let* K *be a KB. We have* $I_{MI}(K) \leq I_{DM}(K)$.

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