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Multiobjective Prices of Stability and Anarchy for Multiobjective Games

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Abstract. We generalize the prices of stability and anarchy to *multiobjective* games. In the singleobjective case, the loss of overall efficiency induced by selfish behaviors is a deeply studied subject. While the price of anarchy bounds *above* the loss of efficiency of equilibria, the price of stability bounds it *below*. In a *multiobjective* game, each agent individually decides an action, creating an overall alternative which is valued by each agent using his *multiobjective* valuation function and compared by the Pareto-preference. We issue the study of *multiobjective* prices of stability and anarchy by discussing several possible definitions.

1 Introduction

Many contexts involve numerous agents who all have individual perspectives, behaviors and decisions. They can be economic agents in a market, software agents using the Internet, or traveling agents in a routing network. Game theory models these contexts by games where the *scalar* utility (or cost) each agent earns is an individual function of all the agents' actions (the overall alternative). Each agent decides what is the best for him-self, potentially resulting into an overall alternative which individually satisfies all the agents: an *equilibrium*. One fundamental paradox in game theory is that an equilibrium can be *overall inefficient*. Bounding the loss of overall efficiency of equilibria is of tremendous interest: it indicates the games' efficiency. This *loss of efficiency* is bounded below by the *price of stability* (PoS), and above by the *price of anarchy* (PoA) [8].

Game theory relies on *scalar* utilities, but in practice the agents' preferences are based on different conflicting objectives which hang in the balance for each decision: financial utility, time cost, goods, personal pleasure, resources costs, etc. Instead of *scalar* utilities (or costs) endowed with complete orders, one can model the costs (or utilities) of agents by utility-*vectors* endowed with a (less assumpting) partial order, the Pareto-preference.

In parallel, about the efficiency of economic systems, the folk's theorem says "Making money is creating utility to others". However, there are concrete examples to motivate a refined *multiobjective* analysis of the overall alternatives: Companies indulge themselves to the sole objective of making money, and a country's economic efficiency is solely analyzed from the point of view of gross domestic product. It induces dangerous losses of efficiencies that a singleobjective monetary aggregation does not reflect: non-sustainability of productions, poor quality of life for existing workers, sale of addictive and carcinogenic products. This appeals for a more refined analysis of the loss of efficiency of equilibria, motivating a consistent definition of *multiobjective* prices of stability and anarchy.

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Related Work. *PoA.* (See [7, 6] and the references therein.) In atomic routing games and finite congestion games where the edges (or resources) have affine costs (with respect to the agent-congestion), when the agents have weighted traffics, the PoA is 2.618, and when their traffics are unitary, 2.5 [2, 3]. (To the best of our knowledge, the proofs do not survive multiobjective.) When the edges have polynomial costs of degree d, the PoA is $(\phi_d)^{d+1}$ where ϕ_d is the solution of $(x+1)^d = x^{d+1}$ [1]. *PoS.* (See [10] and the references therein.) In a local connection game [4], each agent is a node in a network and decides what edges to build between him and the other nodes, in order to minimize the number of edges he builds and to minimize the total distance towards the other nodes. Here the payoffs are biobjective by nature, but linearly scalarized by the existing works: the build cost of one edge is a parameter α and the distance cost of one edge is unitary. For $1 < \alpha < 2$, the PoS is conjectured to be 1 and bounded above by $4/3$, otherwise the PoS is 1. *Multiobjective games* were defined early [9]. In the general (non-convex) case, linear scalarizations miss many Pareto-equilibria. Using a Tchebycheff norm enables to reach the whole set of Pareto-equilibria [11].

2 Preliminaries

Let $N = \{1, \dots, n\}$ denote the *set of agents*. Let A^i denote agent i's *action-set*. Given a subset of the players $M \subseteq N$, let $A^M \ni a^M$ denote $\prod_{i \in M} A^i$. Let $A^N \ni a^N$ (or $A \ni a$, for simplicity) denote the *set of overall alternatives*. Given $a \in A$, a^M denotes the restriction of a to M, and a^{-i} the *restriction* of a to $N \setminus \{i\}$. Let $\mathcal{O} = \{1, \dots, p\}$ denote the *set of objectives*, and let $\mathbb{R}_+^p$ denote the *multiobjective valuation space* $]0, +\infty[^p$. Agent i's ***MO valuation*** *function* (utility or cost) is denoted by $u^i : A \to \mathbb{R}_+^p$. Therefore, on the overall alternative a, for agent i, the valuation on the k-th objective is $u_k^i(a)$. Let $u^\Sigma : A \to \mathbb{R}_+^p$ denote the *utilitarian valuation* $\sum_{i \in N} u^i(a) \in \mathbb{R}_+^p$ of the alternatives.

Definition 1. A *multiobjective game* (MOG) $\Gamma = \big(N, \{A^i\}_{i \in N}, \mathcal{O}, \{u^i\}_{i \in N}\big)$ is defined by a set of agents N, an action-set A^i for each agent, a set of objectives $\mathcal{O}$, and an MO valuation function $u^i : A \to \mathbb{R}_+^p$ for each agent.

To model selfish behaviors, we endow MO valuations with the less assumptive preference: we only assume a costs *minimization* setting.

Definition 2. *Pareto-preferences.* Given cost vectors $x, y \in \mathbb{R}_+^p$,
$x \precsim y$ denotes: $\forall k \in \mathcal{O}, x_k \leq y_k$
$x \prec y$ denotes: $x \precsim y$ and $\exists k \in \mathcal{O}, x_k < y_k$
$x \prec_s y$ denotes: $\forall k \in \mathcal{O}, x_k < y_k$

Given a set Y of cost vectors from $\mathbb{R}_+^p$, the subset $\text{EFF}[Y] \subseteq Y$ of *Pareto-efficient* vectors is defined by:

$$\text{EFF}[Y] = \{y^* \in Y \text{ s.t. } \forall y \in Y, \text{ not } (y \prec y^*)\}$$

This partial order over cost-vectors enables to define as efficient all the best compromises between objectives. For $p = 1$ objective, EFF corresponds to $\min$. Minimizing positive linear combinations of the objectives does not enable to reach Pareto-efficient cost-vectors which are not on the boundary of the convex hull in the objective space, while these efficient decisions can make sense for a rational agent preferring balanced outcomes.

Given a function $f : A \to Z$, the *image set* $f(E)$ of a subset $E \subseteq A$ is defined by $f(E) = \{f(a)|a \in E\} \subseteq Z$. We assume individual satisfaction upon Pareto-efficient actions [9, 11], which encompass all rational preferences.

Definition 3. *Pareto-equilibrium (PE).* Given a MOG Γ, an overall alternative $a \in A$ is a Pareto-equilibrium (denoted by $a \in \text{PE}(\Gamma)$) iff all agents individually have an efficient action:

$$\forall i \in N, \quad u^i(a^i, a^{-i}) \in \text{EFF}[u^i(A^i, a^{-i})]$$

For $|\mathcal{O}| = 1$, Pareto-equilibria are pure-strategy Nash equilibria (NE), which will be denoted by $\text{NE}(\Gamma)$.

3 Defining Efficiency in Multiobjective Games

Let us first recall the definitions of the *prices of stability* and *anarchy*:

Definition 4. Given an SO game Γ, the *SO price of stability* is: $\alpha^{\text{LB}}(\Gamma) = \min[u^\Sigma(\text{NE}(\Gamma))] \;/\; \min[u^\Sigma(A)]$ and the *SO price of anarchy* is: $\alpha^{\text{UB}}(\Gamma) = \max[u^\Sigma(\text{NE}(\Gamma))] \;/\; \min[u^\Sigma(A)]$.

While in the SO case, the sets $\min$ and $\max$ are scalars, in the MO case, EFF is a non-trivial set of vectors. Consequently, the prices of stability and anarchy blossom into multiple different concepts.

Definition 5. Given a MOG $\Gamma = \left(N, \{A^i\}_{i\in N}, \mathcal{O}, \{u^i\}_{i\in N}\right)$, $\mathcal{E} = u^\Sigma(\text{PE}(\Gamma))$ are the equilibria-outcomes and $\mathcal{F} = \text{EFF}[u^\Sigma(A)]$ the efficient-outcomes. We define the following ratios to efficiency. γ, δ and r stand for game-theoretic, decision-theoretic and rough:

$$\gamma^{\text{UB}}(\Gamma) = \min\{\rho \in \mathbb{R}_+ : \forall y \in \mathcal{E},\ \exists x^{(y)} \in \mathcal{F},\ y \precsim \rho x^{(y)}\}$$
$$\gamma^{\text{LB}}(\Gamma) = \max\{\rho \in \mathbb{R}_+ : \forall y \in \mathcal{E}, \forall x \in \mathcal{F}, \text{not}\,(y \prec_s \rho x)\}$$

$$\delta^{\text{ALG}}(\Gamma) = \min\{\rho \in \mathbb{R}_+ : \forall x \in \mathcal{F},\ \exists y^{(x)} \in \mathcal{E},\ y^{(x)} \precsim \rho x\}$$
$$\delta^{\text{DEC}}(\Gamma) = \min\{\rho \in \mathbb{R}_+ : \exists x \in \mathcal{F},\ \exists y \in \mathcal{E},\ y \precsim \rho x\}$$
$$\delta^{\text{LB}}(\Gamma) = \max\{\rho \in \mathbb{R}_+ : \forall x \in \mathcal{F},\ \forall y \in \mathcal{E},\ \text{not}\,(y \prec_s \rho x)\}$$

$$r^{\text{UB}}(\Gamma) = \min\{\rho \in \mathbb{R}_+ : \forall y \in \mathcal{E},\ \forall x \in \mathcal{F},\ y \precsim \rho x\}$$
$$r^{\text{LB}}(\Gamma) = \max\{\rho \in \mathbb{R}_+ : \forall y \in \mathcal{E},\ \forall x \in \mathcal{F},\ \rho x \precsim y\}$$

Semantics of Definition 5. The expressions $y \precsim \rho x$ state that the loss of efficiency from x to y is bounded by a multiplicative ρ on all the objectives. While the sets in Definition 5 correspond to valid bounds ρ, there is no need to bound above by both ρ and $\rho' > \rho$. This is why we take $\min$ (or $\max$ for LBs).

(γ^{UB}) In a game-theoretic context, nature consists in a game which selects freely an equilibrium. One is concerned about giving a guarantee over whatever equilibrium the game could output ($\forall y \in \mathcal{E}$). To measure the efficiency of an equilibrium outcome y, let us show flexibility and respect the choices of the agents, by comparing y to the nearest efficient outcome $x^{(y)} \in \mathcal{F}$.

(δ^{ALG}) In a decision-theoretic context, nature consists in a total refinement of the decision maker's preferences over the efficient alternatives (guarantee $\forall x \in \mathcal{F}$), and a coordinator can advise the closest equilibrium outcome $y^{(x)} \in \mathcal{E}$. It is the same ratio as in multiobjective approximation algorithms. The ratio δ^{DEC} is relevant when the coordinator does not care about the decision maker's preferences, but only about the overall efficiency of the selected equilibrium.

($\gamma^{\text{LB}} = \delta^{\text{LB}} = \delta^{\text{DEC}}$) In a game-theoretic context, it is interesting to know what efficiency ratio cannot be realized. Negating $\exists x^{(y)} \in \mathcal{F},\ y \precsim \rho x$ gives the desired definition. The same can be done in a decision-theoretic context, and since universal quantifiers commute, it ends up to be the same concept. Moreover: $\gamma^{\text{LB}} = \delta^{\text{LB}} = \delta^{\text{DEC}}$.

Remark 1. Fixing a MOG Γ, we have: $r^{\text{UB}} \geq \gamma^{\text{UB}}$, $r^{\text{UB}} \geq \delta^{\text{ALG}}$, $\gamma^{\text{UB}} \geq \gamma^{\text{LB}}$, $\delta^{\text{ALG}} \geq \gamma^{\text{LB}}$, and $\gamma^{\text{LB}} \geq r^{\text{LB}}$.

4 Prospects

Many games appeal for a multiobjective study of their efficiency: MO routing games, local connection games, matching markets, etc.

Bounding rough bounds as r^{LB} or r^{UB} is uninformative. One could focus on a theoretical study of γ^{UB} and $\gamma^{\text{LB}} = \delta^{\text{LB}} = \delta^{\text{DEC}}$. As a first step, it is a common multiobjective methodology to relate MO concepts with linearly scalarized concepts. In optimization, it enables to inherit quite easily from the SO literature. These relations are not obvious in our MO PoS/PoA setting. (Mainly because linear scalarizations of the objectives miss efficient decisions.) Tchebycheff scalarizations of the objectives enable a complete characterization of Pareto-equilibria. However, they could suffer of the weights' asymmetry, and the corresponding SO Games lack of known results.

If there are no general results, then natural assumptions can be made. In optimization, the biobjective case is known to be easier to deal with. It could also be the case with regard to the efficiency of biobjective games. The notion of potential was transposed to multiobjective games [5], which could allow more results.

Using ratio-vectors $\rho \in \mathbb{R}_+^p$ (instead of a real ratio $\rho \in \mathbb{R}_+$) enables to be more informative on the different inefficiencies of the different objectives. However, it defines non-trivial *sets* of approximation ratio-vectors, which theoretical study seems difficult (experiments could be more interesting).

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