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'Being a Manifold' as the Topological Primitive of Mereotopology

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Abstract. Mereotopology is an approach to modeling space that allows to formalize human spatial intuition without reference to points. A predominant formal theory in this area is the Regions Connection Calculus (RCC) introduced in 1992. RCC has an original fault: it relies on the notion of (Euclidean) point for the interpretation of its primitive, i.e., the connection relation C. In this paper we show that in the natural structures for mereotopology, RCC is a theory of manifolds in disguise. It follows that RCC can be reformulated without reference to any notion of point both at the syntactic and at the semantic levels.

1 INTRODUCTION

The problem of representing space in qualitative fashion (QSR) has been studied from a variety of perspectives: knowledge representation and reasoning, artificial intelligence, cognitive science, logic, linguistics and philosophy. Representation formalisms in QSR have received increasing attention from the 90s when a series of formal theories started to be coherently motivated, developed, compared and applied, see [10, 15, 23, 8] for reviews.

The Region Connection Calculus (RCC) [18] is one of the most popular approaches to model relationships among regions in space. RCC is a mereotopological theory, that is, it takes extended regions as individuals and constrains their topological relationships, but cannot model richer (metrical or geometrical) information. Technically, RCC is the theory of the binary relation C (connection) where $C(x, y)$ holds for x, y, interpreted as regular sets in the given space (typically $\mathbb{R}^2$ or $\mathbb{R}^3$), when their closures share at least one point.

RCC falls within the tradition of region-based or mereological approaches yet, as said, its semantics explicitly depends on the notion of point. This is unsatisfactory from the cognitive and the philosophical viewpoints and, as N. M. Gotts pointed out in a more general discussion, "[u]sing an interpretation expressed in terms of point-sets might seem inconsistent with the spirit underlying the RCC approach" [12, pg.5]. Although point-based connection is an expressive relation, it is doubtful that mereotopology should depend on it. To the best of our knowledge, the cognitive adequacy of the different forms of connection has not been settled, nonetheless some experiments [19] show the cognitive relevance of alternative forms of connections.

This paper aims to show that RCC (more precisely, RCC8 with universe, complement, binary sum and product) is expressively equivalent to a mereotopological theory in which the notion of point has no role whatsoever, and that in this way it becomes acceptable from the philosophical and cognitive viewpoints. In this sense, this paper is a follow-up of [4], although it departs from it in important aspects as we will clarify. Note that point-free semantics for RCC have already been studied, notably in [21, 20]. The approach proposed by Stell and Worboys is based on Boolean connection algebras, i.e., it applies structures based on lattice theory. From these results, we know that the semantics of RCC can be given independently from the notion of point making the system philosophically more coherent. This result solves the problem to some degree but at the cost of loosing the cognitive appeal of the theory. Indeed, the structures based on (different versions of) Boolean algebras are not inspired by cognitive results or motivations in the area of spatial modeling.

Instead of changing RCC semantics, in this paper we introduce a different mereotopological theory, which does not depend on points in any sense, and compare its expressivity with that of RCC. Our theory exploits two primitives: the binary relation of *parthood* (P) and the unary relation of *manifold region* (R_{Mfd}). In particular, we show that these, in the discussed structures, are sufficient to define the connection relation C of RCC. The proof is divided in two steps. First, we show that C can be defined from P and R_{Mfd} in the mereological substructures described in [15] where regions are finitely decomposable. Then, we generalize the result to the structure of open regular regions in $\mathbb{R}^2$. The overall proof is based on standard logical and topological arguments. All the definitions are explicitly stated.

The fact that in the important models of RCC our theory is equivalent to RCC, leads to the claim that RCC can be restated as a truly point-free theory. This, in our view, strengthens the foundations of that system.

In the next section we motivate the study of different types of regions in mereotopology. Section 3 introduces the language of RCC and relevant definitions in this system. Section 4 defines the mereotopological structures that we will use for our comparison. The next section introduces the language $\mathcal{L}_{Mfd}$ of our theory. Section 6 defines point-connection from $\mathcal{L}_{Mfd}$ in the weak structures. Section 7 defines point-connection in the strongest structure. The concluding section discusses the results we have reached and points to future work.

2 POINTS, REGIONS AND THICKNESS

Points, as informally defined in Euclidean geometry, i.e., elements without parts (and in this very sense atomic), are central under the view that entities should be understood by decomposition *ad infinitum* and by subsequent reconstruction from the atoms. This view is quite hard to pursue and requires sophisticated mathematical techniques, from set-theory to modern topology and geometry, to consistently construct suitable regions of space from points. This is,

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roughly speaking, the view underlying the notion of Euclidean or point-based space representation.

However, when aiming to represent space from the qualitative and cognitive viewpoints, the notion of region becomes more constrained. Things that attract our attention, the so-called everyday objects, present important regularities; they tend to occupy *extended* regions of space that are *everywhere thick*. Decomposition of everywhere thick regions is still fundamental as shown in experimental cognitive theories [2], but the notion of what counts as basic element under this view is quite different than in Euclidean geometry. The intuition is that a region is everywhere thick when there is a small enough object that can move to any place within the region without going through the region boundary. This informal intuition may be associated with different scenarios in the real world and surely depends on ontological and epistemical views as codified into notions like granularity, boundary and movement among others. However, everywhere thick regions, in contrast to regions that can be disconnected by the removal of a single point, are cognitively strong, and any region-based formalism for the representation of space has to be capable to characterize them.

In classical topology [14] and in standard mereotopology [7], the switch from the point-based to the cognitive-based representation of space is often understood as the result of a change of focus from regions that are (point-)connected, technically *self-connected*, to those that are everywhere thick or *strongly self-connected*. This view, adopted for the reconstruction of point-free mereotopology in [4], turns out to be misleading. Indeed, as cognitive research tells us, strong self-connection does not suffice to the cognitive agent. By analyzing the basic elements of these approaches, one notices that the decomposition of a region requires much more than strong self-connection itself.

For instance, Biederman's primitives (*geons*) are blocks, cylinders, wedges, and cones of different sections and lengths [2]. These are spatially rich elements and involve properties that go beyond classical topology and mereotopology, e.g., parallelism (needed in the detection of cylinders) has the strength of affine geometry [16, 3]. When these primitives are topologically analyzed, one notices that they all belong to the class of manifolds: in three-dimensional space, the space experienced by humans, geons are homeomorphic (topologically isomorphic) to spheres, cubes and tetrahedron, i.e., belong to the class of 3-simplexes. In dimension one, a simplex (or 1-simplex) is a segment; a 2-simplex, i.e. a simple in dimension two, is a triangle; a 3-simplex a tetrahedron; and in general a n-simplex is a figure with $n + 1$ vertices such that no three of them are aligned. An homeomorphism [14] between topological spaces X, Y (not necessarily distinct) is a bijective open continuous function $f : X \to Y$, i.e., one-to-one, surjective, continuous with the inverse function f^{-1} also continuous. Homeomorphisms form an equivalence class. Let n be the dimension of the space, simplexes, convex polyhedra, spheres and ovals in that space are in the same homeomorphic class, single toroidal polyedra form a second class, double toroidal polyedra a third etc. Horn toroidal and degenerate toroidal polyhedra (Fig.1) form further classes.

This analysis leads to two outcomes. On the one hand, we have the *homeomorphic option*: take the topological division of regions in homeomorphic classes and develop an approach to space representation by taking as primitive entities those that are homeomorphic to n-simplexes, for n the dimension of the space. This choice, however, seems too demanding since, as said above, homeomorphism is sensitive to holes: balls, single toroidal polyhedra, double toroidal polyhedra etc. belong to different classes. Yet, theories à la Biederman do not take holes into considerations. A second option consists in taking a larger class that distinguishes only between regular and degenerated regions: simplex, toroidal, double toroidals (etc.) polyhedra (top of Fig. 1) belong to a class, and degenerated and horn toroidal polyhedra (bottom of Fig. 1) to another. In this paper, we take this latter classification and use the set of non-degenerated thick regions (simplexes, balls and toroidal phlyhedra), generally called manifolds, to interpret our new topological primitive.

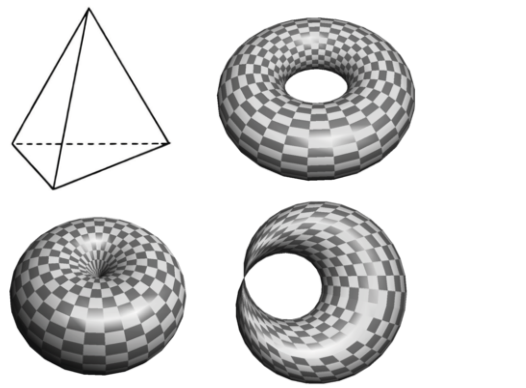

Figure 1. From top left (clockwise): tetrahedron (3-simplex), toroidal, degenerate toroidal and horn toroidal polyhedra. [Produced with Grapher©]

From now on, we will use the term 'region' to mean a generic element of the class of non-empty regular open regions in $\mathbb{R}^n$ and its subclasses (as specified).

3 SIMPLE AND MANIFOLD REGIONS IN RCC

Topologically speaking a self-connected region x is a region that cannot be divided in two subregions whose closures share no point. In mereotopology the notion of self-connection, formally written *PntC*, is definable from the point-connection relation C of RCC [18]:

D1 $PntC_{\mathrm{RCC}}(a) \stackrel{\mathrm{def}}{=} \forall b, c\, [a = b + c \to C(b, c)]$
[self-connected region in RCC*]*

The predicate of strong self-connection is known in the literature under different names: *SR* (simple region) [5, 6, 13]; *ICON* (interior connection) [11, 1]; *SSC* (strong connection) [7]; *SCON* (strongly self connection) [9]; *FSC* (firmly self-connection) [23]; and C_b (strong connection) [22]. *SR* can be defined in RCC as follows [1]:

D2 $SR_{\mathrm{RCC}}(x) \stackrel{\mathrm{def}}{=} \forall y \exists z\, [NTPP(y, x) \to (P(y, z) \wedge NTPP(z, x) \wedge PntC_{\mathrm{RCC}}(z))]$
[simple region in RCC*]*

where $NTPP(y, x)$ is an RCC relation that holds whenever y is included in the interior of x.

From these definitions it follows that by enriching mereology with the unary predicate *SR* we obtain a theory not stronger than RCC. The relationship between the system generated by parthood (P) and simple region (*SR*) and the mereotopology RCC has been studied in [4] where it is shown that they are expressively equivalent in $\mathbb{R}^2$.

Furthermore, in $\mathbb{R}^3$, RCC defines the relation of finger connectivity $FCON(x, m)$ [12], which stands for "x has finger connectivity m" ($m \in \mathbb{N}$). For $m = 1$ and the restriction to finite regions, this relation holds when x is in the topological class of *simplex regions*, i.e., the homeomorphic class of simplexes, convex polyhedra, spheres etc. The definition of *FCON*, based on the notion of dissecting-graphs, is quite complicated and not fully satisfactory (see discussion in [12]). RCC, via *FCON*, can define a notion of topological simplex or manifold without holes as follows:

D3 $R_{Mfd_{\mathrm{RCC}}}(x) \stackrel{\mathrm{def}}{=} FCON(x, 1)$ *[simplex region in* RCC*]*

From these definitions it follows that the relation of parthood (P) enriched with the simple region predicate SR (with the intended interpretation), gives a system within RCC, i.e., a mereotopology. By enriching P with the manifold region predicate $R_{Mfd_{\mathrm{RCC}}}$, we again obtain a mereotopology within the expressivity of RCC. The relationship between the system generated by $P + SR$ and the mereotopology RCC has been studied in [4] where it is shown that they are expressively equivalent in $\mathbb{R}^2$ with some qualification. However, the extension of this result to $\mathbb{R}^n$ (for $n > 2$) cannot be proven with the techniques developed in that paper. In this paper we take a different road and show that, under the same qualifications, the system generated by $P + R_{Mfd}$ can define RCC in all dimensions $n \geq 1$. This solves the initial problem of furnishing RCC in a completely point-free fashion. The approach we take here improves the content of [4] in two ways: it allows to overcome an error in definition D16 (finger strong-connectivity) in that paper[2] and provides a much simpler technique to reconstruct the results from SR.

4 MEREOTOPOLOGICAL STRUCTURES

Our goal is to show that the mereotopology based on the primitives P and R_{Mfd} can define the primitive of RCC. Since RCC is not capable of characterizing its intended models, essentially substructures of $\mathbb{R}^2$ and $\mathbb{R}^3$ and their extension in $\mathbb{R}^n$, we begin by fixing the semantic structures within which we compare the two mereotopologies. Note also that, since our goal is a semantic comparison, we do not need to investigate nor compare the axiomatizations of these systems.

The mereotopological structures we list in this section have been introduced and discussed by Pratt-Hartmann [15, Sect. 2.3]. These, as the author argues, correspond to a *region-based model of space* much better than the whole of regular open sets. The main argument is that among the regular open sets of $\mathbb{R}^n$ there are various pathological sets whose existence does not seem cognitively justified nor needed in qualitative knowledge representation [17]. Notwithstanding this observation, $\mathbb{R}^n$ is one of the intended structures of RCC and thus we add it to the comparison.

In the following definitions we always assume $n > 0$; for a full presentation see [15, Sect. 2.3]; for background topological notions see [14] or any modern topology textbook.

Definition 1 *Let u be a subset of the topological space $\mathbb{R}^n$. We say that u is regular open in $\mathbb{R}^n$ if u is equal to the interior of its closure. We denote the set of regular open subsets of $\mathbb{R}^n$ by $RO(\mathbb{R}^n)$.*

Definition 2 *A set $u \subseteq \mathbb{R}^n$ is said to be* semi-algebraic *if for some m there exists a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in the first order language with arithmetic signature $\langle \leq, +, \cdot, 0, 1 \rangle$ (interpreted over $\mathbb{R}^n$ in the usual way) and m real numbers $b_1, \ldots, b_m$ such that $u = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \mathbb{R}^n \models \phi(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$. We denote the set of regular open semi-algebraic sets in $\mathbb{R}^n$ by $ROS(\mathbb{R}^n)$.*

Definition 3 *A basic polytope in $\mathbb{R}^n$ is the product, in $RO(\mathbb{R}^n)$, of finitely many half-spaces. A polytope in $\mathbb{R}^n$ is the sum, in $RO(\mathbb{R}^n)$, of any finite set of basic polytopes. We denote the set of open polytopes in $\mathbb{R}^n$ by $ROP(\mathbb{R}^n)$.*

Definition 4 *A basic rational polytope in $\mathbb{R}^n$ is the product, in $RO(\mathbb{R}^n)$, of finitely many rational half-spaces. A rational polytope in $\mathbb{R}^n$ is the sum, in $RO(\mathbb{R}^n)$, of any finite set of basic rational polytopes. We denote the set of rational polytopes in $\mathbb{R}^n$ by $ROQ(\mathbb{R}^n)$.*

Lemma 5 *[15, Sect. 2.3]*
For $n > 0$, $ROQ(\mathbb{R}^n) \subsetneq ROP(\mathbb{R}^n) \subsetneq ROS(\mathbb{R}^n) \subsetneq RO(\mathbb{R}^n)$.

Definition 6 *Let X be a topological space. A* mereotopology over X *is a Boolean sub-algebra M of $RO(X)$ such that, if o is an open subset of X and $p \in o$, there exists $r \in M$ such that $p \in r \subseteq o$. We refer to the elements of M as regions. If M is a mereotopology such that any component of a region in M is also a region in M, then we say that M* respects components*.*

Lemma 7 *[15, Lemma 2.12,13,16].*
For $n > 0$, the structures $ROS(\mathbb{R}^n), ROP(\mathbb{R}^n)$ and $ROQ(\mathbb{R}^n)$ are finitely decomposable mereotopologies and respect components.

The structures $ROS(\mathbb{R}^n), ROP(\mathbb{R}^n), ROQ(\mathbb{R}^n)$ and $RO(\mathbb{R}^n)$, the last being the strongest, furnish the semantic frameworks for the comparison of mereotopological systems carried out in this paper.

5 THE LANGUAGE $\mathcal{L}_{Mfd}$

From Section 3 we know that C, the RCC connection relation, can define P and a notion of manifold. The correspondence to the topological notion of general manifold (with or without boundaries) has been studied in [12] but a full characterization is still lacking and seems quite complicated to achieve. Here we show that, starting from the notion of manifold, one can reconstruct the whole RCC theory in all the relevant mereotopological structures.

Let us fix the language $\mathcal{L}_{Mfd}$ as that of first-order logic with two non-logical symbols: the binary P and the unary R_{Mfd}. The language is interpreted over the structures in the previous section, namely, $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n), ROS(\mathbb{R}^n)$ and $RO(\mathbb{R}^n)$. The interpretation function I and the semantics of the language is as usual on the logical part of the language. The primitive P is interpreted as set inclusion: $P(x, y)$ holds when $x^I \subseteq y^I$. For the interpretation of R_{Mfd}, let us call $\mathbb{MR}_n$ the set of connected manifold regions in $\mathbb{R}^n$ [14], i.e. the regions of $\mathbb{R}^n$ that are everywhere locally homeomorphic to a ball or to half-ball in $\mathbb{R}^n$ (these are called *manifolds with boundary*).[3] Manifolds can be finite or infinite. Thus, $\mathbb{MR}_2$ contains the disks (2-spheres), squares, annuli and the half-planes of $\mathbb{R}^2$ among others. $\mathbb{MR}_3$ contains the balls (3-spheres), cubes, non-degenerated toroidal regions and the half-spaces of $\mathbb{R}^3$; and so on. Any region delimited by a simple closed line is in $\mathbb{MR}_2$ (since homeomorphic to a disk) and any region delimited by a simple closed surface is in $\mathbb{MR}_3$ since homeomorphic to a ball. From $A \in \mathbb{MR}_n$, one cannot conclude about the closure of the complement, i.e., whether $\overline{\mathbb{U} \setminus A}$, is in $\mathbb{MR}_n$ or not: an annulus is in $\mathbb{R}^2$ but the closure of the complement is disconnected and so not in $\mathbb{MR}_2$. Instead, a closed disk D and the region $\overline{\mathbb{R}^2 \setminus D}$ are both in $\mathbb{MR}_2$. Let S be one of the structures $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n), ROS(\mathbb{R}^n)$ and $RO(\mathbb{R}^n)$. The primitive R_{Mfd}

[2] The constraints set in definition D16 of [4] do not suffice for the purpose of the construction. The problem was first discovered by Carola Eschenbach (personal communication).

[3] We always talk about regular regions. Thus, in dimension 3 we refer to toroidal polyhedra, not to tori (a torus is the surface of a toroidal polyhedron).

is interpreted in S as the set $\mathbb{MR}_n \cap X$ for X collecting the closures of the regions in S. Thus, $R_{Mfd}(x)$ holds when $\overline{x^I} \in \mathbb{MR}_n \cap X$. Further examples of R_{Mfd} in $\mathbb{R}^2$ are conic sections, (infinite) open stripes as well as any other convex regular region.

In $\mathcal{L}_{Mfd}$ mereological sum (+), difference (−), product (·), complement (*comp*) and the universal region ($\mathbb{U}$) are defined as usual from P. Parthood with this set of definitions (and its standard axiomatization) corresponds to a mereology known as Closed Extensional Mereology [7].

6 CONNECTION IN $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n), ROS(\mathbb{R}^n)$

Mfd-connection is the form of connection arising from the manifold region predicate

D4 $C_{Mfd}(a,b) \stackrel{\text{def}}{=} \exists c,d\,[P(c,a) \wedge P(d,b) \wedge R_{Mfd}(c+d)]$ *[Mfd-connected]*

In RCC, strong-connection is defined by a similar formula: $SC(a,b) \stackrel{\text{def}}{=} \exists c,d\,[P(c,a) \wedge P(d,b) \wedge SR_{\text{RCC}}(c+d)]$
and, although R_{Mfd} and SR_{RCC} are quite different, we have

Proposition 8 *In the structures of Section 4, $C_{Mfd}(a,b) \equiv SC(a,b)$.*

External Mfd-connection holds when two regions are Mfd-connected without overlapping

D5 $EC_{Mfd}(a,b) \stackrel{\text{def}}{=} C_{Mfd}(a,b) \wedge \neg O(a,b)$ *[externally Mfd-connected]*

where overlap means to have some region in common, i.e., $O(x,y)$ if and only if $\exists z(P(z,x) \wedge P(z,y))$.

In $\mathcal{L}_{Mfd}$ a simple region is a region such that for each two manifold parts it contains a manifold overlapping them, Fig. 2.

D6 $SR(a) \stackrel{\text{def}}{=} \forall b,c\,[P(b,a) \wedge P(c,a) \rightarrow \exists d\,[P(d,a) \wedge R_{Mfd}(d) \wedge O(b,d) \wedge O(c,d)]]$ *[simple region]*

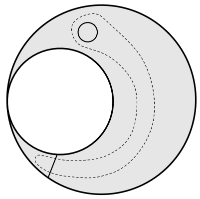

Figure 2. The region in gray is simple (but not manifold), D6.

Point-connection in $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n)$ and $ROS(\mathbb{R}^n)$ is defined by the following formula (Fig. 3)

D7 $C'_{Mfd}(a,b) \stackrel{\text{def}}{=} C_{Mfd}(a,b) \vee \exists a',b'\,[P(a',a) \wedge P(b',b) \wedge R_{Mfd}(a') \wedge R_{Mfd}(comp(a')) \wedge R_{Mfd}(b') \wedge R_{Mfd}(comp(b')) \wedge SR(comp(a'+b')) \wedge \neg R_{Mfd}(comp(a'+b'))]$
[R_{Mfd} point-connection in $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n)$ and $ROS(\mathbb{R}^n)$]

Referring to Fig. 3, the formula says that two regions a,b are point-connected if they are Mfd-connected or there are manifold subregions of a and b (with manifold complements) such that the complement of their sum is not a manifold (inf Fig. 3, the universe U minus $a+b$). The motivation for this definition is that any point of contact is a point in which the boundaries of two regions meet: if there is an isolated point of contact A and we select subregions that include A, then at A the sum of these subregions is homeomorphic to neither a ball nor a half-ball of $\mathbb{R}^n$. Also, from the condition on the complement of their sum, the only reason to not be a manifold is the existence of a point of contact between a' and b' that falsifies the condition on manifolds.

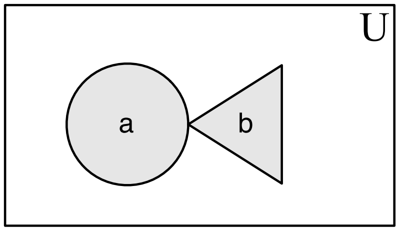

Figure 3. Defining connection C in $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n)$ and $ROS(\mathbb{R}^n)$, D7 (here $a' = a$ and $b' = b$). In 1D it holds for point-connection; in 2D for point- and line-connection; in 3D for point- line- and surface-connection; etc.

Theorem 9 *In $ROQ(\mathbb{R}^n), ROP(\mathbb{R}^n)$ and $ROS(\mathbb{R}^n)$, $C(a,b)$ holds in* RCC *if and only if $C'_{Mfd}(a,b)$ holds in $\mathcal{L}_{Mfd}$.*

Proof.
[Notation: capital letters stand for points in $\mathbb{R}^n$.]
(Left to right) Let $C(a,b)$ and consider regions r,s of $ROQ(\mathbb{R}^n)$ (or $ROP(\mathbb{R}^n)$ or $ROS(\mathbb{R}^n)$, respectively) such that $a^I = r$ and $b^I = s$. From the interpretation of C in RCC, we have: $\overline{r} \cap \overline{s} \neq \emptyset$. We show that, with these assumptions, relationship $C'_{Mfd}(a,b)$ holds as well. The result is obvious when a,b are Mfd-connected (from the definition) or $n = 1$ (in dimension one Mfd-connection and point-connection coincide since manifolds are segments). Assume a,b are not Mfd-connected and $n > 1$. Let us write ∂x for the boundary of a region x. From $C(a,b)$ and $a^I = r$ and $b^I = s$, $\partial r \cap \partial s$ is non-empty. Since regions in $ROQ(\mathbb{R}^n)(ROP(\mathbb{R}^n)$ or $ROS(\mathbb{R}^n)$, respectively) have the finite decomposability property (Lemma 7) there exist a region component of r and a region component of s that have a common boundary, call the latter ∂_*. Again from Lemma 7, these components are also elements of the given structure. Let P be a point in ∂_*; since P is a boundary point, its neighbour is homeomorphic to half-ball in $\mathbb{R}^n$. These neighbours can be taken to be manifold regions. Then, there exist a manifold region r' which is part of r and has P as boundary point. Choose a part s' of s, analogously. It remains to prove that region $\overline{u} = \overline{r'} + \overline{s'}$ is not a manifold, i.e. it cannot be the interpretation of a region satisfying R_{Mfd}. If $\overline{u}$ is a manifold, then P, being a boundary point, must be homeomorphic to a half-ball of $\mathbb{R}^n$. As part of $\partial r'$, p is homeomorphic to a half-ball and, as part of $\partial s'$, it is homeomorphic to a (different) half-ball. No matter how we choose these half-balls, they are connected only at P. Thus any neighbour of P in u is homeomorphic to the sum of two distinct sectors of a sphere centered at P. Thus, $\overline{u}$ is not a manifold. (In particular, for $n = 2$, u is homeomorphic to the sum of two disks with P their only common point; for $n = 3$ to a (part of a) horn toroidal polyhedron with P its degenerate hole.)

(Right to left) Since $C_{Mfd}(a,b)$ already implies $C(a,b)$, assume not $C_{Mfd}(a,b)$. Let $a'^I = r \in \mathbb{R}^n$ and $b'^I = s \in \mathbb{R}^n$. From $R_{Mfd}(a')$ and $R_{Mfd}(b')$, $\overline{r}, \overline{s}$ are manifolds with boundary in $\mathbb{R}^n$. Since $\overline{r} + \overline{s}$ is the sum of two manifolds but is not a manifold itself and since the sum of a finite number of disconnected manifolds is still a manifold, the definition (the homeomorphic condition) can fail only at a point that belongs to both $\overline{r}$ and $\overline{s}$. But in this case $\overline{r}$ and $\overline{s}$ share an (interior or boundary) point from which $C(a', b')$, and then $C(a, b)$. □

7 CONNECTION IN $RO(\mathbb{R}^n)$

In the previous section we have seen that Mfd-connection can define point-connection in the structures proposed in [15] as *the* standard mereotopological structures. The proof takes advantage of a common property of these structures, namely finite decomposability. Nonetheless, the *intended* interpretation of most mereotopological theories, among which RCC, is the full range of open regions and most importantly the values $n = 2$ and 3, that is, $RO(\mathbb{R}^2)$ and $RO(\mathbb{R}^3)$. We actually show that $\mathcal{L}_{Mfd}$ can define RCC point-connection in all the structures $RO(\mathbb{R}^n)$ for $n > 0$.

First, note that C'_{Mfd} immediately extends to any pair of manifold regions in $RO(\mathbb{R}^n)$. That is, it holds even for manifold regions of $\mathbb{R}^n$ which are not in $ROS(\mathbb{R}^n)$. This follows from the fact that to verify $C'_{Mfd}(a, b)$ for a, b manifold regions in $RO(\mathbb{R}^n)$ and not in $ROS(\mathbb{R}^n)$, it suffices to consider manifold subregions which have as boundary point the same point shared by a and b. It follows that we do not need to introduce any restriction on the form of manifold regions in $RO(\mathbb{R}^n)$ to apply C'_{Mfd}. To prove the full result, we need to show that we can define point-connection C for any pair of regions in $RO(\mathbb{R}^n)$, not just the manifold ones.

We now introduce a new clause that takes into account pairs of regions in $RO(\mathbb{R}^n)$ at least one of which is not a manifold and not in $ROS(\mathbb{R}^n)$. To state it positively, we extend the previous definition to cover the case of a region connected to another via a point which is not on the boundary of any of its manifold subregions. See [15, Sect. 2] for a discussion of these special regions of $RO(\mathbb{R}^n)$ and Figures 4 and 5 for examples.

First of all, note that all definitions in Section 6, with the only exception of C'_{Mfd}, capture the intended meaning in any of our mereotopological structures, in particular $RO(\mathbb{R}^n)$. The definition of C'_{Mfd} needs to be enriched to include the condition for point-connection in $\mathbb{R}^n$. Here is the definition of connection generalized to all the structures of Section 4.

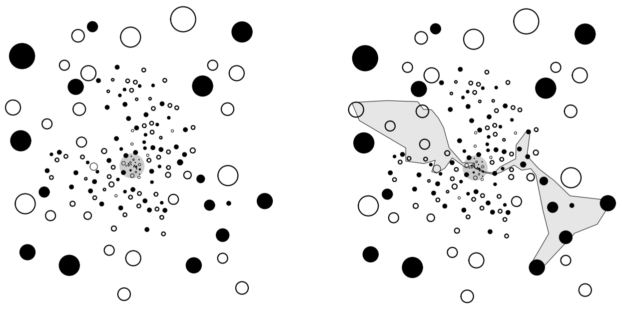

Figure 4. The scattered region formed by the black disks and the scattered region formed by the white disks are point-connected at the center of the shaded area via infinite converging chains of disks, D8. [Image from [4]]

D8 $C''_{Mfd}(a, b) \stackrel{\text{def}}{=} C'_{Mfd}(a, b) \vee \exists c, d\ [(R_{Mfd}(c) \wedge R_{Mfd}(d) \wedge O(c, a) \wedge O(d, b) \wedge C'_{Mfd}(c, d) \wedge R_{Mfd}(comp(c)) \wedge R_{Mfd}(comp(d)) \wedge \forall c', d'\ [(R_{Mfd}(c') \wedge c' \cdot a = c \cdot a \wedge R_{Mfd}(d') \wedge d' \cdot b = d \cdot b) \rightarrow C'_{Mfd}(c', d')]]$
[R_{Mfd} point-connection]

Definition D8 says that $a, b \in \mathbb{R}^n$ are point-connected when Mfd-connected or there are two manifold regions (c, d) with manifold complement, such that the first overlaps a, the second b, and they are point-connected according to C'_{Mfd}. Furthermore, it is required that any pair of manifold regions c', d' that overlap the same parts of a and b, respectively, are also point-connected according to C'_{Mfd}.
Note: the condition that c, d's complements are manifold regions forces the complement of c (and of d) to be Mfd-connected and is particularly important in cases like in Fig. 5: if c and d overlap but a and b do not, this condition guarantees that c' and d' can be chosen so that they do not overlap.

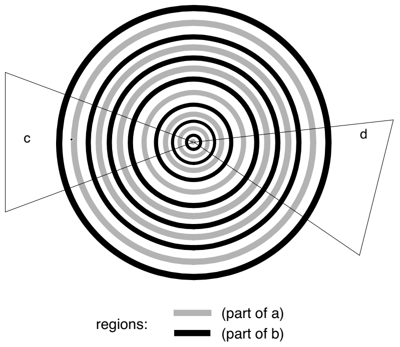

Figure 5. The region formed by the gray annuli and the region formed by the black annuli have the (common) center of the annuli as point of connection.

Theorem 10 *In $RO(\mathbb{R}^n)$,*
$C(a, b)$ holds in RCC *if and only if $C''_{Mfd}(a, b)$ holds in $\mathcal{L}_{Mfd}$.*

Proof. (Left to right.) If a, b are Mfd-connected we are done. Otherwise, let $a^I = r$ and $b^I = s$ (r, s regular open regions of $\mathbb{R}^n$) and assume that (1) r, s are connected according to RCC and that (2) they have only point(s) of connection which do not belong to any of their simple subregions or boundaries of these, i.e. $C'_{Mfd}(a, b)$ fails. Let P be one of these points of connection for r, s. From RCC, we know P is not a point at infinity.

Since P is a limit point of r, we can find a sequence of points of r converging to P. Let R_i be such a sequence. Similarly, let S_i be a sequence of points in s that converges to P. Let $r_i \in \mathbb{MR}_n$ be a neighbour of R_i with $r_i \subseteq r$. Similarly, let $s_i \in \mathbb{MR}_n$ be a neighbour in s of point S_i.
Let r' be a regular open region of $\mathbb{R}^n$ such that $\cup_i r_i \subseteq r'$, $\overline{r'} \in \mathbb{MR}_n$ and $\mathbb{U} \setminus r' \in \mathbb{MR}_n$. Note that r' exists: for t an open sphere with $\overline{r} \cap \overline{t} = \emptyset$ then $\mathbb{U} \setminus t$ satisfies all the conditions for r'. Analogously, choose a regular open region $s' \subset \mathbb{R}^n$ such that $\cup_i s_i \subseteq s'$, $\overline{s'} \in \mathbb{MR}_n$ and $\mathbb{U} \setminus s' \in \mathbb{MR}_n$. Now let $c^I = r'$ and $d^I = s'$. By construction, $C'_{Mfd}(c, d)$. It remains to show that any other pair of manifold regions c', d' such that $c' \cdot a = c \cdot a$ and $d' \cdot b = d \cdot b$, satisfy $C'_{Mfd}(c', d')$ as well. Let c', d' be such a pair of regions and let $c'^I = r^*$ and $d'^I = s^*$. Since $c' \cdot a = c \cdot a$ and $d' \cdot b = d \cdot b$, we have $\cup_i r_i \subseteq r^*$ and $\cup_i s_i \subseteq s^*$, thus P is a limit point for c', d'. Since c', d' are manifold-regions, we conclude $C'_{Mfd}(c', d')$.

(Right to left) By contradiction, suppose that $C''_{Mfd}(a, b)$ holds and $C(a, b)$ fails. Also, from Theorem 9 we can assume $C'_{Mfd}(a, b)$ fails as well. We show that these assumptions lead to a contradiction.
Fix c, d as in D8. Since $C'_{Mfd}(a, b)$ fails, let $a' = a \cdot c$ and $b' = b \cdot d$, then $C''_{Mfd}(a, b)$ if and only if $C''_{Mfd}(a', b')$. Without loss of generality, let $a' = a$ and $b' = b$, i.e., $P(a, c)$, $P(b, d)$. We can also assume that c, d do not overlap: we can find non-ovelapping c', d' for D8 since a, b themselves do not overlap. Let $a^I = r$, $b^I = s$, $c^I = v$

and $d^I = w$. Then, $r \subseteq v$ and $s \subseteq w$. From the semantics of RCC, if $C(a,b)$ fails, then $\overline{r} \cap \overline{s} = \emptyset$. From Theorem 9 and $C'_{Mfd}(c,d)$, it follows that $\overline{v} \cap \overline{w} \neq \emptyset$. Let P be an isolated point in $\overline{v} \cap \overline{w}$ and $P \notin \overline{r}$ (the cases $P \notin \overline{s}$ and $P \notin \overline{r} \cup \overline{s}$ follow easily). Then, there is a neighbour u_P of P with $u_P \cap \overline{r} = \emptyset$. Also, since $v \in \mathbb{M}\mathrm{R}_n$, we can choose u_P such that $\overline{v \setminus u_P} \in \mathbb{M}\mathrm{R}_n$. Now let c' be such that $c'^I = v \setminus u_P$. By construction, $R_{Mfd}(c')$ and $c' \cdot a = c \cdot a$. If Q is another isolated point in $\overline{v} \cap \overline{w}$ we can proceed as before to find c' and/or d' such that Q is not a point of connection for suitable c', d'. Since there are at most denumerable isolated points and we can look for pairwise disconnected neighbours u_P, if c, d are connected by only isolated points, then there exist $c', d' \in \mathbb{M}\mathrm{R}_n$ such that $P(a,c') \wedge P(b,d')$ but not $C'_{Mfd}(c',d')$; thus, $C''_{Mfd}(a,b)$ fails. Contradiction.
Assume now that there is one or more connected sets z (connected in the sense of the topology of $\mathbb{R}^n$) such that $z \subseteq \overline{v} \cap \overline{w}$ and $z \cap \overline{r} = \emptyset$ (again, the cases $z \cap \overline{s} = \emptyset$ and $z \cap (\overline{r} \cup \overline{s}) = \emptyset$ follow easily). Let u_z be a set which is neighbour for each point in z such that $u_z \cap \overline{r} = \emptyset$. (e.g. take $u_z = \cup_i u_{Z_i}$ where Z_i is selected as before for each $Z_i \in z$). Again, we can choose u_z such that $\overline{v \setminus u_z} \in \mathbb{M}\mathrm{R}_n$. As before, we can find c' (d' or both, resp.ly) such that c', d (c, d' or c', d') do not have any point of z as point of connection. Since z is a set of points, it might happen that c' turns out to be disconnected and thus not in $\mathbb{M}\mathrm{R}_n$. Due to condition $R_{Mfd}(comp(c))$ of D8, it follows that c' can be extended to a connected c^* such that $R_{Mfd}(c^*)$ and $P(c', c^*)$ (from which $c^* \cdot a = c \cdot a$) without adding new points of connection with d'. Since the connected sets z are at most denumerable and, by definition, pairwise disconnected, we have shown that there exist c', d' such that $C'_{Mfd}(c', d')$ fails. Contradiction.
We conclude that $C(a,b)$ follows from $C''_{Mfd}(a,b)$.

□

8 CONCLUSIONS AND FUTURE WORK

RCC was presented in the 1990s as as point-free theory of space but it suffers from a conceptual dependence on points at the semantic level. We have shown how to reformulate this mereotopology overcoming this fault. By solving this problem we shed some light on the foundations of RCC, help to understand its connection to general topology and, perhaps more importantly for artificial intelligence, foster its use in cognitive contexts. We recall that point-free semantics for RCC have been already provided [21, 20]. That approach is based on Boolean connection algebras, i.e., it uses structures based on lattice theory but without any direct cognitive relevance. Our result also addresses an observation by Gotts on the interpretation of RCC in terms of point-sets [12, pg.5] since it suggests to understand space in terms of manifold regions and their relationships. Manifolds, which naturally correspond to the regions we experience in everyday life, can be introduced in algebraic fashion without recurring to set-theory. The outcome is that all the formulas that are true under the point-based semantics of RCC are equally true under the manifold-based semantics given by an algebraic approach. One can rely on manifolds as the basic entities for spatial understanding, and still model manifolds as sets of points. This observation allows us to regain the traditional viewpoint of general topology (including the direct connections to Euclidean geometry) while reducing manifold to mathematical objects in $\mathbb{R}^n$. Yet, the step into the point-based representation is now a choice that the knowledge engineer is free to make or refute.

In [4] we proved a similar result for $P{+}SR$ limited to the mereological structures in $\mathbb{R}^2$. The definition of point-connection in that system is quite complex and a generalization to $\mathbb{R}^3$ is hard to achieve, if possible at all. Homotopy and homology classifications suggest that homeomorphic classes provide an interesting set of new primitives for mereotopology. In the future, we plan to investigate further the connection between mereotopology and modern topological classifications. Hopefully, this will also lead to complete the work of Gotts on the definition of manifolds within RCC [12].

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