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The Margin of Victory in Schulze, Cup, and Copeland Elections: Complexity of the Regular and Exact Variants

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Abstract. The margin of victory is a critical measure for the robustness of voting systems in terms of changing election outcomes due to errors in the ballots or fraud in using electronic voting machines. Applications include risk-limiting post-election audits so as to restore the trust in the correctness of election outcomes.

Continuing the work of Xia [24], we show that the margin of victory problem is NP-complete for Schulze and cup elections. We also consider the exact variant of this problem, which we show to be complete for DP in Schulze, cup, and Copeland elections.

Keywords. margin of victory, computational complexity, Schulze elections, cup elections, Copeland elections, computational social choice

1. Introduction

The computational aspects of voting, as a common method of preference aggregation, are a central topic in the field of computational social choice, mainly due to the wide range of applications in multiagent systems (see, e.g., the bookchapter by Brandt et al. [1]). Much of the work so far has focused on winner determination and various types of manipulative attacks (including manipulation [4], bribery [5], and control [6]). Other properties of voting systems have been studied extensively from a social-choice perspective, but much less so in terms of their computational complexity.

We are concerned with one such property: the *robustness of elections*. When voters cast their votes using voting machines in political elections, errors might occur in various ways (be it by accident or with malicious intent [23]), leading to incorrect vote counts. How many errors are affordable before the election outcome (i.e., the winner set) changes? The *margin of victory* is a central concept to measure the robustness of voting systems in terms of changing election outcomes due to errors in the ballots, or due to fraud. It is defined as the smallest number of votes that need to be changed in a given election so as to change its winner set. The higher the margin of victory is, the more robust is the election. In political elections, post-election audits are used to restore the trust in the correctness of election outcomes via “verifiable paper records” [10], and if too many mismatches are found, an extremely costly recount of all votes is in order. Risk-limiting audit methods [20,21,15] do not require to recount *all* votes, while limiting the risk that the result might still be wrong. A critical parameter for this is the margin of victory.

These issues have been intensively studied from the point of view of political sciences, employing mainly statistical methods and focusing on plurality voting [18,19], scoring rules, approval voting, range voting, and single transferable vote (STV) [15,3,8]. Xia [24] was the first to study the margin of victory in terms of its computational complexity while other notions of robustness in voting have been studied by Procaccia et al. [13] and Shiryaev et al. [17] from a computational perspective. In particular, Xia showed that while this problem is efficiently solvable for scoring rules, approval voting, plurality with run-off, and Bucklin voting, it is NP-complete for Copeland, STV, maximin, and ranked pairs, and he also studied the approximability of these NP-complete problems. Continuing this line of research, we study the complexity of the margin of victory for Schulze and cup elections (a.k.a. sequential majority), establishing NP-completeness as well. The Schulze rule is a particularly attractive voting system due to its many desirable axiomatic properties [16]; its computational properties have also drawn much attention recently (see, e.g., [12]).

While our result for cup voting requires a novel reduction, NP-hardness of the margin of victory problem for Schulze is a straightforward consequence of the NP-hardness result for destructive bribery due to Parkes and Xia [12], along with the connection between these two problems due to Xia [24]. We add to this connection by showing that, for multi-winner voting systems, destructive bribery can be easy, yet the margin of victory problem can still be hard.

The main technical contribution of this paper is the study of the *exact* variant of the margin of victory problem for Schulze, cup, and Copeland elections for which we obtain DP-completeness results. Exact variants of NP-hard problems from a variety of areas are known to be DP-complete (often with rather involved proofs; see, e.g., [22] and the survey [14]), including the exact variants of social welfare optimization problems in multiagent resource allocation [9]. In the exact margin of victory problem, we not only ask whether the margin of victory of a given election meets or exceeds some given threshold, but we ask whether or not it falls into a predetermined interval. It is known that the size of this interval does not matter in terms of the problems' complexity (see, e.g., [22]), so we can fine-tune it to whatever accuracy we desire, even to just one integer, and that is how we will define this problem.

2. Preliminaries

Elections and voting systems: An *election* is a pair (C,V), where C is a finite set of candidates and V is a list of votes (or ballots) expressing the voters' preferences over the candidates in C. The form of the ballots depends on the voting system used; we focus on ballots that are (strict) linear orders of the candidates in C. For example, if $C = \{a,b,c\}$ is our candidate set, a ballot could be of the form $b > c > a$ meaning that this voter (strictly) prefers b to c, and (strictly) prefers c to a. Throughout this paper, we will omit the greater-than sign, so the above preference would be written as $b\,c\,a$.

For a given election (C,V) and two candidates $a,b \in C$, let $D_V(a,b)$ denote the number of votes in V that prefer a to b minus the number of votes in V that prefer b to a. If $D_V(a,b) > 0$, we say that a (strictly) beats b in pairwise comparison. Given an election

(C,V), define the *weighted majority graph for* (C,V), denoted by $\text{WMG}(C,V)$, to be the weighted, directed graph G with vertex set C and edges between any two distinct vertices, where the weight of an edge (a,b) is $D_V(a,b)$ and $D_V(a,b) = -D_V(b,a)$ holds by definition.

A *voting system* $\mathscr{E}$ is a (set of) rule(s) for how to determine the winner(s) of an election (C,V) based on the ballots in V. We will denote the *set of winners of* (C,V) *under* $\mathscr{E}$ by $\mathscr{E}(C,V)$. In particular, we will consider the following voting systems.

Let α, $0 \leq \alpha \leq 1$, be a fixed rational number. In *Copeland*$^\alpha$ elections, given an election (C,V), $D_V(a,b)$ is determined for every pair $(a,b) \in C \times C$. Each candidate a receives one point for every pairwise comparison she (strictly) wins (i.e., whenever $D_V(a,b) > 0$), and gets α points for every tie (i.e., whenever $D_V(a,b) = 0$). All candidates with the highest score are the Copeland$^\alpha$ winners of (C,V).

In *Schulze* elections, construct the weighted majority graph $G = \text{WMG}(C,V)$ from a given election (C,V). The *strength of a path from a to b in G* is defined as the smallest weight any edge on this path has. For each pair (a,b) of candidates, $P(a,b)$ denotes the strength of a *strongest* path from a to b (i.e., of a path with the greatest minimum edge weight among all paths from a to b). All candidates $a \in C$ with $P(a,b) \geq P(b,a)$ for all $b \in C \smallsetminus \{a\}$ are the Schulze winners of (C,V). Note that a candidate $a \in C$ is the unique Schulze winner of (C,V) if and only if $P(a,b) > P(b,a)$ for all $b \in C \smallsetminus \{a\}$.

In *cup* (or *sequential majority*) elections, an election is defined by specifying (C,V) and, additionally, a voting tree T (i.e., a complete binary tree with as many leaves as there are candidates in C, where we assume that C contains enough dummy candidates so as to satisfy $\|C\| = 2^k$ for some k, and all dummy candidates are ranked worst in V), and a schedule that assigns the candidates to the leaves of T. Determine the value of $D_V(a,b)$ for each pair of candidates, a and b, that are siblings in the tree. The winner of the pairwise comparison is assigned to the parent node. This procedure is continued until the cup winner is assigned to the root. The schedule is known beforehand and whenever ties occur, they are broken by a beforehand fixed tie-breaking rule.

Complexity theory: We assume that the reader is familiar with the basic notions of the complexity classes P and NP and hardness and completeness with respect to the polynomial-time many-one reduction, denoted by $\leq_{\text{m}}^{\text{p}}$. Papadimitriou and Yannakakis [11] introduced the complexity class $\text{DP} = \{A \smallsetminus B \,|\, A, B \in \text{NP}\}$, the class of differences of any two NP problems, which is, together with coDP, also known as the second level of the boolean hierarchy over NP (see [2]). It is well-known (see, e.g., [14]) that DP contains the exact variants of many NP-complete problems, such as the following DP-complete problem EXACT VERTEX COVER (XVC) that asks for a given undirected graph $G = (A,E)$ and a positive integer k whether $\tau(G) = k$, i.e., whether the size of a smallest vertex cover in G is exactly k.[1] Changing the question to whether $\tau(G) \leq k$ gives the well-known NP-complete VERTEX COVER problem (see, e.g., [7]).

[1] A *vertex cover* of an undirected graph $G = (A,E)$ is a subset $A' \subseteq A$ that contains at least one vertex of each edge. The size of a smallest vertex cover is denoted by $\tau(G)$. (As V is already used to denote voter lists, we will use A to denote vertex sets in graphs, although the common notation is V.)

3. The Margin of Victory and Destructive Bribery

In this section, we will give the formal definitions of the investigated problems and continue the work of Xia [24] by drawing further connections between the margin of victory and the complexity of destructive bribery. The margin of victory is defined as follows.

Definition 1 *For a given voting system $\mathscr{E}$ and a given $\mathscr{E}$ election (C,V), we define the* margin of victory *to be the smallest integer ℓ such that the winner set can be changed by changing ℓ votes in V, while the other votes remain unchanged. We will use the notation* $\text{MoV}(\mathscr{E},(C,V)) = \ell$.

Just as Xia [24], we will focus on the decision version of this problem denoted by $\mathscr{E}$-MARGIN OF VICTORY ($\mathscr{E}$-MOV) for a given voting system $\mathscr{E}$, that asks for an $\mathscr{E}$ election (C,V) and a positive integer k whether $\text{MoV}(\mathscr{E},(C,V)) \leq k$ holds.

The MOV problem is closely related to the standard bribery scenario in elections that was defined by Faliszewski et al. [5]. In particular, in $\mathscr{E}$-DESTRUCTIVE UNWEIGHTED BRIBERY ($\mathscr{E}$-DUB), we are given an $\mathscr{E}$ election (C,V), a designated candidate $p \in C$, and a positive integer k, and we ask whether it is possible to prevent p from being a unique $\mathscr{E}$ winner by bribing at most k voters (i.e., by changing their votes).

The above problem is defined in the so-called *unique-winner model*. By changing the question to whether the designated candidate can be prevented from being *an* $\mathscr{E}$ winner by bribing at most k voters, the problem would be defined in the so-called *nonunique-winner model* (a.k.a. the *co-winner model*). When analyzing the relationship between the bribery problem and the MOV problem, one has to pay close attention on whether the voting system at hand always selects unique winners or whether the winner set may contain more than one candidate. The following result, due to Xia [24], deals with the former type of voting rules and displays the close connection between the margin of victory and the standard bribery scenario in elections.

Proposition 2 (Xia [24]) *Let $\mathscr{E}$ be a voting system that always selects a unique winner of an election in deterministic polynomial time, and satisfies $\mathscr{E}$-MOV $\neq \emptyset$.*[2] *Then $\mathscr{E}$-MOV and $\mathscr{E}$-DUB are $\leq_m^p$-equivalent, i.e., $\mathscr{E}$-MOV $\leq_m^p$ $\mathscr{E}$-DUB and $\mathscr{E}$-DUB $\leq_m^p$ $\mathscr{E}$-MOV.*

For voting rules that may select more than one winner, however, we now show that the above equivalence does not hold in general, unless P could be shown to equal NP. Note that the voting rule we will construct for showing the following theorem is not neutral.[3] Whether there exists a neutral voting rule satisfying the same properties as the one we construct is an interesting open question.

Theorem 3 *There exists a voting system $\mathscr{K}$ such that $\mathscr{K}$-DUB $\in P$ but $\mathscr{K}$-MOV is* NP-*complete.*

Proof Sketch. Due to space constraints we only sketch the proof by providing the voting system $\mathscr{K}$ that always outputs at least two winners if there are at least two candi-

[2]As is common, we view a decision problem such as $\mathscr{E}$-MOV as the language of yes-instances.

[3]A voting rule is called *neutral* if the outcome does not depend on the candidates' naming, i.e., if any two candidates are swapped in each vote, the outcome changes accordingly.

dates. For an election (C,V) with $C=\{p\}\cup C'$, the winnerset in $\mathscr{K}$ is determined as follows:

$$\mathscr{K}(C,V)=\begin{cases} p & \text{if } C=\{p\} \\ \{p\}\cup cup(C',V) & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathscr{K}$-DUB $\in$ P. NP-hardness of $\mathscr{K}$-MoV immediately follows from the NP-hardness of cup-MoV, which we will show in Theorem 5. ❑

4. Margin of Victory in Schulze and Cup Voting

Xia [24] established complexity results for the margin of victory problem for various voting rules, including all scoring protocols, STV, and Copeland elections. We now turn to the complexity of this problem in Schulze and cup elections.

Theorem 4 *For Schulze elections,* MoV *is* NP*-complete.*

Proof. NP-hardness directly follows from the NP-hardness result for Schulze-DUB shown by Parkes and Xia [12] and the fact that $\mathscr{E}$-DUB $\leq_m^p$ $\mathscr{E}$-MoV for each voting system $\mathscr{E}$ with $\mathscr{E}$-MoV $\neq \emptyset$. Membership of Schulze-MoV in NP is easy to see. ❑

Theorem 5 *For cup elections,* MoV *is* NP*-complete.*

Proof. This result follows from the NP-hardness of cup-DUB, which we will show by a reduction from the well-known NP-complete problem Vertex Cover (recall its definition from Section 2) using the so-called *UV technique* introduced by Faliszewski et al. [6]. To do so, let $G=(A,E)$ be an undirected graph with vertex set $A=\{a_1,a_2,\dots,a_n\}$ and edge set $E=\{e_1,e_2,\dots,e_m\}$, and let $k\in\mathbb{N}$. We construct the cup election (C,V) with $C=\{c,d\}\cup E\cup P\cup T$, where $P=\{p_1,p_2,\dots,p_m\}$ and T is a set of dummy candidates that will be used to ensure that the voting tree is balanced (we will come to that later). Let $N_a=\{e\in E \mid e\cap\{a\}\neq\emptyset\}$ be the set of edges incident to vertex $a\in A$.

V contains $2m(n+k-3)+6n+6k-3$ voters whose preferences are listed in Table 1. When a set of candidates, say $Z\subseteq C$, is given in a voter's preference, then we assume that the candidates in Z are ordered with respect to a (tacitly assumed) fixed order, while $\overleftarrow{Z}$ denotes that the candidates are ordered in reverse. In particular, we fix the order of the candidates in P to be $p_1>p_2>\cdots>p_m$.

The dummy candidates in T are always positioned at the bottom of each voter's preference, so they lose every pairwise comparison to the candidates in $C\setminus T$. This implies that their position in the schedule is irrelevant, so we will omit them in Figure 1.

For the sake of readability and clarity, we will omit the dummy candidates in our further arguments, and we will use the voting tree and schedule shown in Figure 1. (To transform this tree into a complete binary tree (i.e., into a legal voting tree), the dummy candidates in T have to be added to the three subtrees with the roots d,c, and X, respectively.) Since the height of the tree is in $\mathscr{O}(\log m)$, we have in total a polynomial number of leaves, which ensures that the reduction is in fact polynomial-time computable.

Table 1. List of votes V for the proof of Theorem 5

#	Preference
one vote for each $a \in A$	$c\ d\ N_a\ P\ (E \smallsetminus N_a)\ T$ $P\ c\ d\ (\overleftarrow{E \smallsetminus N_a})\ \overleftarrow{N_a}\ T$
k votes	$c\ d\ P\ E\ T$ $c\ d\ P\ \overleftarrow{E}\ T$
$2(n+k-2)$ votes	$c\ E\ P\ d\ T$ $c\ \overleftarrow{E}\ P\ d\ T$
$n+k-3$ votes for each $i \in \{1,\dots,m\}$	$c\ (P \smallsetminus \{p_i\})\ p_i\ e_i\ (E \smallsetminus \{e_i\})\ d\ T$ $d\ (\overleftarrow{E \smallsetminus \{e_i\}})\ p_i\ e_i\ (\overleftarrow{P \smallsetminus \{p_i\}})\ c\ T$
one vote	$P\ c\ d\ E\ T$

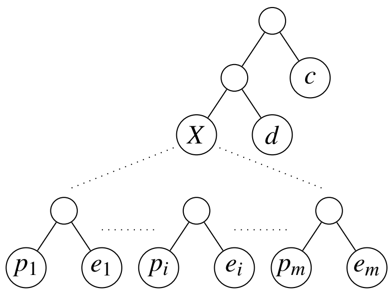

Figure 1. Voting tree of the cup election $(C \smallsetminus T, V)$ without dummies

In this election, we have the following pairwise comparisons between the candidates in $C \smallsetminus T$:

$D_V(c,d) > 4k,\ D_V(c,P) = D_V(d,P) = 2k-1,\ \ D_V(c,E) = D_V(d,E) = 2n+2k+1,$

$$D_V(p_i,p_j) \begin{cases} > 4k \text{ if } i < j \\ \leq 0 \ \ \text{if } i \geq j, \end{cases} \qquad D_V(p_i,e_j) = \begin{cases} -2n-2k+5 & \text{if } i \neq j \\ -1 & \text{if } i = j. \end{cases}$$

Thus, c is the unique cup winner of this election. We claim that G has a vertex cover of size at most k if and only if c can be prevented from being a unique cup winner by changing at most k votes.

From left to right: Assume that $A' \subseteq A$ is a vertex cover of size k. Change the preferences of those k voters corresponding to A' in the first voter group from $c\,d\,N_a\,P\,(E \smallsetminus N_a)\,T$ to $P\,c\,d\,N_a\,(E \smallsetminus N_a)\,T$. Since A' is a vertex cover we have that due to these changes each $e_i \in E$ has one vote where she is positioned behind all candidates in P. So we have that each p_i wins her first pairwise comparison against e_i by one point. In the subelection corresponding to the subtree with root X (recall Figure 1), the relevant pairwise comparisons are among the candidates in P and due to the fixed ordering of these candidates in the votes, p_1 is the winner of this subelection. Both c and d have lost k votes in comparison to p_1 due to the bribe, so p_1 wins both pairwise comparisons and is thus the unique cup winner of this election. So c has been successfully prevented from winning.

From right to left: Assume that c can be prevented from being a unique winner by bribing at most k voters. Due to the scores only candidates from P have a chance to prevent c from being a unique winner, so the following has to hold for the bribed election: A candidate from P, say p_1, has to be the winner of the subelection corresponding to the subtree with root X and p_1 has to win the pairwise comparisons against both d and c. For the latter to hold, all k bribed votes have to have p_1 positioned behind d and c (before the bribe). For the former to hold, no candidate in E may win her first contest, which implies that every $p_i \in P$ has to win the pairwise comparison against the corresponding candidate $e_i \in E$. So the votes that are bribed also have to rank the candidates in E better than those in P before the bribe is conducted. With this we see that the k bribed votes have to be from the first voter group and that the vertices corresponding to these votes have to form a vertex cover of size k to ensure that each $e_i \in E$ loses the first pairwise comparison. ❑

5. Exact Margin of Victory in Schulze, Copeland, and Cup Voting

In this section, we present our complexity results for the exact variants of the margin of victory problem, which for a given voting system $\mathscr{E}$ is denoted by $\mathscr{E}$-EXACT MARGIN OF VICTORY ($\mathscr{E}$-XMOV) and asks for a given $\mathscr{E}$ election (C,V) and a positive integer k whether $\text{MOV}(\mathscr{E},(C,V)) = k$. In particular, we will consider this problem for the two systems studied in the previous section, Schulze and cup, and also for Copeland$^{\alpha}$ voting. (Note that Xia [24] proved that Copeland$^{\alpha}$-MOV is NP-complete.) Due to space constraints we only provide a proof sketch for the following result in Schulze elections.

Theorem 6 *For Schulze elections,* XMOV *is* DP*-complete.*

Proof Sketch. For showing DP-hardness we provide a reduction from the DP-complete problem XVC. Let $G = (A,E)$ be an undirected graph with vertex set $A = \{a_1, a_2, \dots, a_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$, and let k be a positive integer. Without loss of generality, we assume that $6 \le k \le n$ and that $k-1 \bmod 5 = 0$. Let $U = E_1 \cup E_2 \cup E_3$ be the marked union of three copies of E, which are denoted by $E_i = \{e_{i1}, e_{i2}, \dots, e_{im}\}$ for $i \in \{1,2,3\}$, and let $N_a = \{e_{ij} \mid e_j \cap \{a\} \neq \emptyset \text{ and } i \in \{1,2,3\}\}$ denote the set of all edges in U that are incident to vertex $a \in A$. We define the Schulze election (C,V), where $C = \{c,d,e,f,g,h,p\} \cup U$, and V is a list of $40n + 324k - 132$ voters, whose preferences are specified in Table 2. When a set of candidates $Z \subseteq C$ is given in a preference, we assume that the candidates in Z are ordered with respect to a (tacitly assumed) fixed order.

Figure 2 shows a subgraph of the weighted majority graph of this election that only contains edges relevant for the argumentation. Table 3 shows the weights of the relevant strongest paths in (C,V); hence, c is the unique Schulze winner in this election. We will elaborate on some useful properties of the constructed election: Since candidate c is the unique Schulze winner, the winner set can only be changed by achieving $P(c,x) \le P(x,c)$ for at least one candidate $x \in C \setminus \{c\}$. Since $P(c,x) - 2k \ge 12k - 2k > \frac{12(k-1)}{5} + 2k \ge P(x,c) + 2k$ holds for all candidates $x \in C \setminus \{c,p\}$, only p can tie with c when no more than k votes can be changed. So it suffices to focus on the paths leading from c to p, and vice versa. From p to c, the only reasonable path is $((p,d),(d,e),(e,f),(f,g),(g,c))$.

Table 2. List of votes V in the proof of Theorem 6

#	Preference	Type
one vote for each $a \in A$	$h > c > g > f > e > N_a > p > d > (U \setminus N_a)$	1
	$c > g > e > f > d > N_a > p > (U \setminus N_a) > h$	1
	$h > c > g > f > d > e > N_a > p > (U \setminus N_a)$	1
	$c > f > g > e > d > N_a > p > (U \setminus N_a) > h$	1
	$h > g > c > f > e > d > N_a > p > (U \setminus N_a)$	1
n votes	$d > p > e > f > g > c > U > h$	2
	$h > p > d > f > e > g > c > U$	2
	$p > e > d > f > g > c > U > h$	2
	$h > p > d > e > g > f > c > U$	2
	$p > d > e > f > c > g > U > h$	2
$12(k-1)/10$ votes	$h > p > d > e > f > g > c > U$	2
	$p > d > e > f > g > c > U > h$	2
$3k - 3 + 12(k-1)/10$ votes	$h > e > f > g > c > p > d > U$	3
	$d > c > g > f > e > p > U > h$	3
$6k + 12(k-1)/10$ votes	$h > p > g > c > f > e > d > U$	2
	$c > d > e > f > g > p > U > h$	3
	$h > p > c > f > g > e > d > U$	2
	$g > d > e > f > c > p > U > h$	3
	$h > p > g > c > e > f > d > U$	2
	$f > d > e > c > g > p > U > h$	3
$6k$ votes	$c > h > g > e > f > p > d > U$	3
	$d > p > f > e > g > c > h > U$	2
	$h > g > c > e > f > p > d > U$	3
	$d > p > f > e > c > h > g > U$	2
$5n + 41k - 20$ votes	$h > d > c > g > E_1 > E_2 > p > E_3 > f > e$	4a
	$e > f > E_2 > E_1 > p > E_3 > g > c > d > h$	4b
	$h > d > c > f > E_1 > E_3 > p > E_2 > g > e$	5a
	$e > g > E_3 > E_1 > p > E_2 > f > c > d > h$	5b
	$h > d > c > e > E_2 > E_3 > p > E_1 > f > g$	6a
	$g > f > E_3 > E_2 > p > E_1 > e > c > d > h$	6b

Table 3. Weights of the strongest paths in (C,V) in the proof of Theorem 6

x	d	e	f	g	h	p	U
$P(c,x)$	$12k$	$12k$	$12k$	$12k$	$12k$	$6k-6$	$> 12k$
$P(x,c)$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$	$\frac{12(k-1)}{5}$

We can argue that when at most k changes are allowed, we have that $P(p,c) \leq \frac{12(k-1)}{5} + \frac{8(k-1)}{5} = 4(k-1)$ holds in this new election. The following property can be shown for completing the proof:

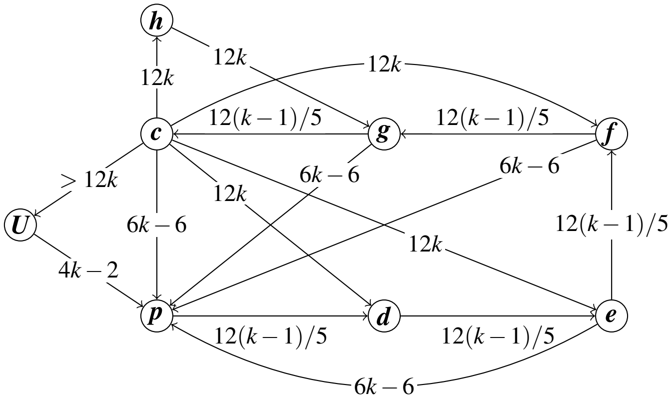

Figure 2. Subgraph of the WMG(C,V) of the Schulze election (C,V) in the proof of Theorem 6

$$\textsc{MoV}(\text{Schulze},(C,V)) \begin{cases} = k-1 & \text{if } \tau(G) < k \\ = k & \text{if } \tau(G) = k \\ > k & \text{otherwise,} \end{cases} \tag{1}$$

where, recall, $\tau(G)$ denotes the size of a smallest vertex cover in G. We show the first case in (1) in detail: Assume that $\tau(G) < k$ and let $A' \subseteq A$ be a vertex cover in G of size $k-1$. For each vertex $a \in A$, there are five voters in V of type 1 (recall Table 2), which only differ in the ordering of the candidates $\{c,d,e,f,g,p\}$. If for each $a \in A'$ one of the five type-1 votes is changed such that $p > d > e > f > g > c > \cdots$ holds in this vote and these changes are carefully conducted while ensuring that all five votes are changed equally often, it can be achieved that $P(c,p) = 4k-4 = P(p,c)$. So we have that $\textsc{MoV}(\text{Schulze},(C,V)) \leq k-1$. By changing at most $k-2$ votes in this election, one could achieve that $P(c,p) \geq 4k-2 > 4k-4 \geq P(p,c)$, so c would remain the unique winner of the changed election. Thus we have that $\textsc{MoV}(\text{Schulze},(C,V)) \geq k-1$, which gives $\textsc{MoV}(\text{Schulze},(C,V)) = k-1$. ❑

For the other two voting systems, we have the same complexity results, and we omit their (similar) proofs due to space constraints.

Theorem 7 *For both cup and Copeland*$^{\alpha}$*,* XMoV *is* DP*-complete.*

6. Conclusions and Open Questions

Continuing the work of Xia [24], we have shown that the margin of victory problem is NP-complete for Schulze and cup elections, and its exact variant is DP-complete for Schulze, cup, and Copeland elections. For future research, it would be interesting to study the approximability of the margin of victory also for Schulze and cup. Furthermore the following variant of the MoV problem might be worthwile to analyze: Instead of counting the number of votes that have to be changed entirely to change an election's outcome, the overall number of swaps (or other changes in the votes depending on the voting system used) leading to a different winner set could be counted. This version would, e.g., model accidental errors in votes more naturally and allow a more fine-grained analysis.

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