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Near Fairness in Matroids

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Abstract. This article deals with the fair allocation of indivisible goods and its generalization to matroids. The notions of fairness under consideration are equitability, proportionality and envy-freeness. It is long known that some instances fail to admit a fair allocation. However, an almost fair solution may exist if an appropriate relaxation of the fairness condition is adopted. This article deals with a matroid problem which comprises the allocation of indivisible goods as a special case. It is to find a base of a matroid and to allocate it to a pool of agents. We first adapt the aforementioned fairness concepts to matroids. Next we propose a relaxed notion of fairness said to be near to fairness. Near fairness respects the fairness up to one element. We show that a nearly fair solution always exists and it can be constructed in polynomial time in the general context of matroids.

1 Introduction

The problem of allocating indivisible goods has been widely studied in computer science [8, 3, 16, 11, 2]. It is defined on a set S of m indivisible goods and n agents. Each agent $i \in \{1, ..., n\}$ has a non-negative utility $u_i(e) \geq 0$ for each good $e \in S$. For the purposes of notation, $u_i(S')$ designates the utility of a subset of goods $S' \subseteq S$ and $[n]$ means $\{1, ..., n\}$. The utilities are additive, i.e. $u_i(S') = \sum_{i \in S'} u_i(e)$ for all $S' \subseteq S$ and $u_i(\emptyset) = 0$ for all $i \in [n]$. We suppose that all the instances are normalized to 1, i.e. $u_i(S) = 1$ for all $i \in [n]$, this means that ideally each agent wants to possess all the goods. The objective is to find an allocation or a partition of S into $\{S_1, S_2, ..., S_n\}$ so that S_i is the share of agent $i \in [n]$. For the sake of fairness, we consider three notions, namely equitability, proportionality and envy-freeness. An allocation $\{S_1, ..., S_n\}$ is equitable (as defined in [6, 3]) if $u_i(S_i) = u_j(S_j)$ for all $i \neq j$, it is proportional if $u_i(S_i) \geq \frac{1}{n}$ for all $i \in [n]$, and it is envy-free if $u_i(S_i) \geq u_i(S_j)$ for all $i \neq j$.

In [12, 13], Gourvès *et al.* introduce a problem that generalizes the allocation of indivisible goods. This problem is defined on a matroid $\mathcal{M} = (X, \mathcal{F})$ where X is a set of elements and $\mathcal{F}$ a family of subsets of X. A feasible solution, called a *base* in what follows, is a member of $\mathcal{F}$ of maximal size. More details on matroids are given in Section 2. Given $n \geq 2$ agents with non-negative and additive utilities $u_i(e) \geq 0$ for all $e \in X$ and for all $i \in [n]$, the objective is to find a base B partitioned into n parts $\{B_1, ..., B_n\}$ so that B_i is the part of agent $i \in [n]$. This problem has two main difficulties, first to find a base B and secondly to partition it into n parts. For the sake of fairness, it would be interesting to use the fairness notions defined for resource allocation problems. However, these definitions are not directly applicable on matroids, an appropriate definition is needed. In this paper, we aim to adapt the definitions of equitability, proportionality and envy-freeness to the general matroid problem.

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Matroid theory is a well established field with nice intersections with combinatorial optimization [20]. Beyond the allocation of indivisible goods, the matroid problem may models several situations, some of which are quoted in [15, 12, 13]. As an example, we mention the following application.

Example 1 *A national museum is going to open new branches in several provincial towns. Some items from the stock of the main museum will be transferred to the branches. The stock is partitioned in categories (statue, painting, pottery, etc.) and for each category there is an upper bound on the number of pieces that the curator of the main museum agrees to lend. How should we allocate the items? From the point of view of the citizens whose town accommodates a branch, the allocation should be fair. Indeed, nobody wants to see his branch less attractive than another. This problem can be modeled by the* partition matroid *as explained in Section 2.*

It is known that an allocation of indivisible goods which is equitable or proportional or envy-free may not always exist (when all goods must be allocated). Indeed consider an instance with two agents and only one good. Allocating this good to one of the two agents will generate envy and no equitability to the other one, moreover proportionality can not be reached since the good is indivisible. Over and above, the existence of equitable, proportional or envy-free allocations can not be decided in polynomial time. Demko and Hill [8] show that deciding the existence of a proportional allocation of indivisible goods is NP-complete. Markakis and Psomas [17] strengthen this result by proving that it is NP-complete to decide if there is an allocation where every person receives a bundle of value at least $\frac{1}{cn}$ for any constant $c \geq 1$. Lipton *et al.* [16] prove the NP-completeness of deciding the existence of envy-free allocations. In the same way, one can show that deciding the existence of an equitable allocation is also NP-complete.

Due to these negative situations, one can think of relaxing these notions of fairness. To this end, we propose a relaxed notion said to be *near* of fairness. Near fairness respects the fairness up to one element. The idea of reaching fairness up to one bundle appears in [5] where Budish introduces *envy bounded by a single good*. He says that an allocation satisfies envy bounded by a single good if for any pair of agents $i \neq j$, either i does not envy j or there exists a good in the share of agent j such that by removing it, i becomes not envious of j. Likewise, Parkes *et al.* [19] study a multi-resource allocation problem with demands. They say a mechanism is envy-free up to one bundle if for every vector of bundles b, it outputs an allocation such that for all $i, j \in [n]$, $u_i(S_i) \geq u_i(S_j - b_i)$. In the same way, we propose some definitions to reach equitability, proportionality and envy-freeness up to one element. These definitions allow on the one hand the assurance of the existence of solutions which will satisfy these fairness notions, and on the other hand, finding them in polynomial time. Our definitions are adapted to the general problem on matroids.

The contribution of this paper is first a redefinition of the notions of equitability, proportionality and envy-freeness in the general context of matroids, we will also introduce the notion of jealousy-freeness which is close to equitability. Then we will propose relaxed definitions of these notions up to one element called near jealousy-freeness, near proportionality and near envy-freeness and we will highlight some relations between these concepts. We propose polynomial time algorithms for solving near jealousy-freeness and near proportionality on matroids. We also present a decentralized polynomial time algorithm for solving near envy-freeness when there are two agents. The proposed solutions are deterministic.

The paper is organized as follows: Section 1.1 makes an overview of related works. Section 2 presents the setting. Equitability and jealousy-freeness are the subject of Section 3, proportionality is considered in Section 4 and envy-freeness in Section 5. Future works are discussed in Section 6. Due to space limitations, some proofs are omitted.

1.1 Related work

The present paper is related to the problem of allocating indivisible goods with additive utilities [8, 16, 1, 2, 17] and its generalization to matroids [12, 13]. We consider the notions of equitability, proportionality and envy-freeness as defined in [6, 3].

There are several papers that deal with envy-freeness in the allocation of indivisible goods [16, 7, 2, 18]. Instead of finding envy-free allocations, the problem of minimizing envy has received much attention [16, 18]. Lipton *et al.* [16] prove that solving and approximating the problem of finding an allocation so that envy is minimized are NP-hard. They also investigate allocations with bounded envy and they present a polynomial time algorithm that constructs allocations in which the envy is bounded by a marginal utility.

Since proportional allocations do not always exist, one can think of relaxing proportionality and reaching a value smaller than $\frac{1}{n}$ [8, 17, 12, 13]. Demko and Hill [8] show the existence of an allocation in which the utility of each agent i for his share is at least $V_n(\alpha) \in [0, \frac{1}{n}]$ where V_n is a nonincreasing function of α defined as the largest utility of an agent for a single good. Markakis and Psomas [17] strengthen this result by presenting a polynomial time algorithm that constructs an allocation where the utility of each agent for his share is at least $V_n(\alpha_i) \geq V_n(\alpha)$ where α_i is the largest utility of agent i for a single good. Gourvès *et al.* [12] extend the centralized algorithm of Markakis and Psomas [17] to the general problem on matroids. They prove that the utility of each agent for his part is at least $W_n(\alpha_i) \geq V_n(\alpha_i)$ where W_n is a new function of α_i, defined as the largest utility of agent i for a single element of the matroid. In [13], they present a decentralized algorithm returning a base of a matroid where each agent i has at least $V_n(\alpha_i)$ when $n \leq 8$ agents.

This paper deals with relaxed notions of fairness which allow the notions to be satisfied up to one element. This idea is not new. Budish [5] introduces *envy bounded by a single good* in the context of allocating indivisible goods without envy. Parkes *et al.* [19] study an allocation problem where each elementary endowment is not a single good but a bundle of heterogeneous resources. They propose solutions which satisfy several notions of fairness simultaneously, including envy-freeness up to one bundle. The work of Demko and Hill [8] for allocating indivisible goods also evokes the fairness up to one element. For two agents, they show how to allocate all but one goods in a deterministic manner and arrange a lottery for the remaining good to ensure an expected utility of $\frac{1}{2}$ for both agents.

2 The setting

A matroid $\mathcal{M} = (X, \mathcal{F})$ consists of a finite set X of m elements and a collection $\mathcal{F}$ of subsets of X such that:

(i) $\emptyset \in \mathcal{F}$;
(ii) if $F_2 \subseteq F_1$ and $F_1 \in \mathcal{F}$ then $F_2 \in \mathcal{F}$;
(iii) for every couple $F_1, F_2 \in \mathcal{F}$ such that $|F_1| < |F_2|$, there exists $x \in F_2 \backslash F_1$ such that $F_1 \cup \{x\} \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called *independent sets*. Deciding whether a subset of elements is independent is done with a routine called *independence oracle*. In what follows we delibarately neglect its time complexity when the time complexity of an algorithm is given.

Inclusion-wise maximal independent sets are called *bases*. All the bases of a matroid $\mathcal{M}$ have the same cardinality $r(\mathcal{M})$, defined as the *rank* of $\mathcal{M}$. Without loss of generality, we suppose that for all $x \in X$, $\{x\} \in \mathcal{F}$, so each $x \in X$ belongs to at least one base (from axiom (iii) of matroids).

Matroids are known to model several structures in combinatorial optimization. For instance, the *graphic matroid* is defined on the set of edges of a graph G, and the independent sets are the forests of G (subsets of edges without cycles). A base of the graphic matroid is a spanning tree if the graph G is connected.

The *partition matroid* is defined on a set X partitioned into k disjoint sets $X_1, \ldots, X_k$ for $k \geq 1$. Given k integers $b_i \geq 0$ $(i = 1, ..., k)$, the independent sets are all the sets $F \subseteq X$ satisfying $|F \cap X_i| \leq b_i$ for all $i = 1, ..., k$. Example 1 can be modeled with a partition matroid. X is the stock of the national museum which is partitioned in k categories. Each b_i is the maximum number of items of X_i that the curator agrees to transfer.

The *free matroid* is defined on a set X, each subset $F \subseteq X$ is independent and the unique base is X. The problem of allocating indivisible goods can be seen as the free matroid such that X is the set of goods. An allocation of the goods on n agents is a partition of X into $X_1, ..., X_n$ so that X_i is the share of agent i. Another modelization with the partition matroid is given in [12, 13].

We often use $+$ instead of $\cup$, and $-$ instead of $\backslash$. We also use the shorthand notations $F + e$ for $F \cup \{e\}$ and $F - e$ for $F \setminus \{e\}$.

The bases of a matroid satisfy the following properties.

Theorem 1 *[4] Let $\mathcal{M}$ be a matroid. Let A and B be bases of $\mathcal{M}$ and let $x \in A - B$. Then there exists $y \in B - A$ such that both $A - x + y$ and $B - y + x$ are bases of $\mathcal{M}$.*

Theorem 2 *[14] Let A and B be bases of a matroid $\mathcal{M}$, and let $\{A_1, \ldots, A_n\}$ be a partition of A. There exists a partition $\{B_1, \ldots, B_n\}$ of B such that $A - A_i + B_i$, $1 \leq i \leq n$ are all bases.*

Consider the additive utility function $u : X \to \mathbb{R}^+$. A classical optimization problem is to find a base B that maximizes $u(B) = \sum_{e \in B} u(e)$. This problem is solved by the well-known polynomial time GREEDY algorithm [9] which takes as input $\mathcal{M}$ and u and it outputs a base of $\mathcal{M}$ of maximum utility u. The maximum utility of a base is denoted by $OPT_u(\mathcal{M})$.

We are given a matroid $\mathcal{M} = (X, \mathcal{F})$, an independent set $F \in \mathcal{F}$ and an additive utility function u. The *completion* of F, denoted by $\mathsf{C}(\mathcal{M}, F)$, consists of all sets $E \subseteq X$ such that $F \cup E$ is a base of $\mathcal{M}$. The *maximal completion* of F with respect to u, denoted by $\mathsf{C}^{\max}(\mathcal{M}, F, u)$, consists of all sets $E \subseteq X$ such that $F \cup E$ is a base of $\mathcal{M}$ and $u(E)$ is maximum. We suppose $\mathsf{C}(\mathcal{M}, B)$ is never empty.

In particular, $\mathsf{C}(\mathcal{M}, B) = \{\emptyset\}$ for any base B of $\mathcal{M}$. An element of $\mathsf{C}^{\max}$ can be found with GREEDY [9].

Given a matroid $\mathcal{M} = (X, \mathcal{F})$ and an independent set $X' \in \mathcal{F}$, the *contraction* of $\mathcal{M}$ by X', denoted by $\mathcal{M}/X'$, is the structure $(X \setminus X', \mathcal{F}')$ where $\mathcal{F}' = \{F \subseteq X \setminus X' : F \cup X' \in \mathcal{F}\}$. It is well-known that $\mathcal{M}/X'$ is also a matroid.

Lemma 1 refers to Lemma 1 in [12].

Lemma 1 *[12] Let $\mathcal{M} = (X, \mathcal{F})$ be a matroid, $u : X \to \mathbb{R}^+$ an additive utility function, B^* a maximum base according to u and $F \in \mathcal{F}$ such that $|F| > 0$. If $F' \subseteq B^*$ denotes the $r(\mathcal{M}) - |F|$ elements with minimum utility u of B^*, then $OPT_u(\mathcal{M}/F) \geq u(F')$.*

2.1 The model

We are given a matroid $\mathcal{M} = (X, \mathcal{F})$ where $X = \{x_1, ..., x_m\}$, a set N of n agents and a utility $u_i(x) \in \mathbb{R}^+$ for every $(i, x) \in N \times X$. Any agent i's utility for a subset F of X is denoted by $u_i(F)$ and defined as $\sum_{x \in F} u_i(x)$. As a convention $u_i(\emptyset) = 0$.

A solution to our problem is called a *base allocation*. A base allocation is a base B of $\mathcal{M}$ partitioned into n subsets $\{B_1, ..., B_n\}$. B_i is called the part (or the share) of agent i. Our work deals with the existence and the computation of nearly fair base allocations where fairness $\in$ {equitability, proportionality, envy-freeness}.

The maximum utility of an agent $i \in N$ for a base of $\mathcal{M}$ is denoted by $OPT_i(\mathcal{M})$. We assume that for every agent i, there exists a base $G^i \in \mathcal{F}$ such that $u_i(G^i) = OPT_i(\mathcal{M}) = 1$ and $u_i(F) \leq 1, \forall F \in \mathcal{F}$. This property can be satisfied after a simple normalization of the instance.

In Example 1, the curator of each branch of the national museum assigns a utility to every item of the stock. One seeks a fair allocation of the items. Using the partition matroid model, one can satisfy the bounds on the maximum number of items transferred per category.

3 Equitability and Jealousy

Definition 1 (Equitability) *A base allocation is equitable if $u_i(B_i) = u_j(B_j)$ for each pair (i, j) of agents.*

There is not always an equitable base allocation even in the special case of allocating indivisible goods. Moreover, deciding if there exists an equitable base allocation is NP-complete. This is done by a reduction of PARTITION (problem [SP12] in [10])

In order to relax equitability, let us introduce the notion of *jealousy*.

Definition 2 (Jealousy-Freeness)

- *Agent i is jealousy-free when $u_i(B_i) \geq u_j(B_j)$ for all $j \in N$;*
- *A base allocation is jealousy-free when everybody is jealousy-free.*

A jealousy-free base allocation is equivalent to an equitable base allocation. Let us introduce a relaxation of jealousy-freeness up to one element.

Definition 3 (Near Jealousy-Freeness)

- *Agent i is nearly jealousy-free when $u_i(B_i) \geq u_j(B_j - x)$ for all $x \in B_j$ and for all $j \in N$;*
- *A base allocation is nearly jealousy-free when every agent is nearly jealousy-free.*

Proposition 1 *Agent i is nearly jealousy-free if and only if $u_i(B_i) \geq u_j(B_j) - \min_{x \in B_j} u_j(x)$ for all $j \in N$.*

Theorem 3 *A nearly jealousy-free base allocation always exists and it can be computed within $O(nm \ln m)$ time.*

Proof Let us prove that the solution returned by Algorithm 1 is nearly jealousy-free. In Algorithm 1, ties are broken by choosing the agent (Step 3) or the element (Step 4) with minimum index.

Algorithm 1: Near Jealousy-Freeness

Data: A matroid $\mathcal{M} = (X, \mathcal{F})$, $n = |N|$ agents and a profile of utilities $(u_i)_{i \in N}$
Result: A nearly jealousy-free base allocation B

1 $B \leftarrow \emptyset$ and $B_i \leftarrow \emptyset$ for every $i \in N$
2 **while** B is not a base of $\mathcal{M}$ **do**
3 find $i \in N$ such that $u_i(B_i) = \min_{j \in N} u_j(B_j)$
4 find $x \in X \setminus B$ such that $B + x \in \mathcal{F}$ and $u_i(x)$ is maximum
5 $B \leftarrow B + x$ and $B_i \leftarrow B_i + x$
6 **return** $B = \{B_1, \ldots, B_n\}$

Obviously, the algorithm finds a base of $\mathcal{M}$ and its time complexity is dominated by the n sortings of the elements of X which are done within $O(m \ln m)$ time (Step 4).

Let $i, j \in N$ be two agents and without loss of generality, assume $u_i(B_i) \geq u_j(B_j)$. Consider the last element $x \in X$ added to B_i. This element corresponds to an iteration where the parts of agents i and j were $B'_i = B_i - x$ and $B'_j \subseteq B_j$, respectively. By construction, agent i has been selected to receive a new element x thus we have $u_i(B'_i) \leq u_j(B'_j)$. Every newly inserted element (Step 4) is the largest within the set of elements which can be added and satisfy the independence property (i.e. $B'_i + x \in \mathcal{F}$). Thus $u_i(x) = \min_{y \in B_i} u_i(y)$. Therefore $u_i(B_i) - \min_{y \in B_i} u_i(y) = u_i(B'_i) \leq u_j(B'_j) \leq u_j(B_j)$, which means that agents i and j are both nearly jealousy-free. □

Example 2 *Consider the partition matroid $\mathcal{M} = (X, \mathcal{F})$ defined by $X = \bigcup_{i=1}^{3} X_i$ with $X_1 = \{e_1\}$, $X_2 = \{e_2\}$, $X_3 = \{e_3, e_4, e_5\}$ and $\mathcal{F} = \{F \subseteq X : |F \cap X_1| \leq 1, |F \cap X_2| \leq 1, |F \cap X_3| \leq 2\}$. There are 3 agents, their utilities are given in the following table.*

i	$u_i(e_1)$	$u_i(e_2)$	$u_i(e_3)$	$u_i(e_4)$	$u_i(e_5)$
1	0.4	0.25	0.2	0.15	0.15
2	0.3	0.2	0.2	0.2	0.3
3	0.25	0.25	0.25	0.25	0.25

There is no jealousy-free base allocation. However Algorithm 1 gives the base allocation $B_1 = \{e_1\}$, $B_2 = \{e_5\}$ and $B_3 = \{e_2, e_3\}$ which is nearly jealousy-free.

4 Proportionality

Definition 4 (Proportionality)

- *A base allocation is proportional for agent i when $u_i(B_i) \geq \frac{1}{n}$;*
- *A base allocation is proportional when it is proportional for every agent.*

A proportional base allocation does not always exist. In addition deciding the existence of a proportional base allocation of a given matroid is NP-complete since it generalizes the existence of a proportional allocation of indivisible goods which is NP-complete [8]. We propose a relaxation of proportionality up to one element.

Definition 5 (Near Proportionality)

- *A base allocation is nearly proportional for agent i if there exists $f \in B$ and $e \in X$ such that $(B - f) + e$ is a base and $u_i((B_i - f) + e) \geq \frac{1}{n}$;*
- *A base allocation is nearly proportional when it is nearly proportional for every agent.*

A base allocation is nearly proportional for agent i if it is either proportional or there is some element $f \in B$ such that by removing it, agent i can add a new element e in his part and the base allocation becomes proportional. The element f is not necessarily in B_i (in this case, $(B_i - f) + e = B_i + e$).

A nearly proportional base allocation always exists and it can be found with Algorithm 2 which uses Algorithm 3. One can use GREEDY (see Section 2) for Step 6 of Algorithm 2. The way the elements of F_2 are allocated in Step 7 of Algorithm 2 has no incidence on the analysis of the resulting base allocation.

Algorithm 2: Near Proportionality

Data: A matroid $\mathcal{M} = (X, \mathcal{F})$, $n = |N|$ agents and a profile of utilities $(u_i)_{i \in N}$
Result: A nearly proportional base allocation B

1 $N_1 \leftarrow \{i \in N : \max_{x \in X} u_i(x) \leq \frac{1}{n}\}$ and $N_2 \leftarrow N - N_1$
2 $F_1 \leftarrow \emptyset$ and $F_2 \leftarrow \emptyset$
3 **if** $N_1 \neq \emptyset$ **then**
4 $\quad F_1 \leftarrow$ Algorithm 3 $(\mathcal{M}, n, N_1, (u_i)_{i \in N_1})$
5 **if** $N_2 \neq \emptyset$ **then**
6 $\quad$ let F_2 be any base of $\mathcal{M}/F_1$
7 $\quad$ allocate randomly the elements of F_2 to the agents of N_2
8 **return** $B = F_1 \cup F_2$

Algorithm 3:

Data: A matroid $\hat{\mathcal{M}}$, an integer $\hat{n}$, a set of agents $\hat{N}$ and a profile of utilities $(u_i)_{i \in \hat{N}}$

1 **for all** $i \in \hat{N}$ **do**
2 $\quad$ let G^i be the result of GREEDY$(\hat{\mathcal{M}}, u_i)$ where $G^i = \{e^i_1, \ldots, e^i_{r(\hat{\mathcal{M}})}\}$ and $u_i(e^i_1) \geq \ldots \geq u_i(e^i_{r(\hat{\mathcal{M}})})$
3 $\quad OPT_i(\hat{\mathcal{M}}) \leftarrow u_i(G^i)$
4 $\quad$ let k_i be the largest index such that $u_i(\{e^i_1, \ldots, e^i_{k_i}\}) \leq \frac{OPT_i(\hat{\mathcal{M}})}{\hat{n}}$
5 pick any $\ell \in \hat{N}$ such that $k_\ell \leq k_i$, for all $i \in \hat{N}$
6 $B_\ell \leftarrow \{e^\ell_1, \ldots, e^\ell_{k_\ell}\}$
7 **if** $|\hat{N}| = 1$ **then**
8 $\quad$ **return** B_ℓ
9 **else**
10 $\quad$ **return** $B_\ell \cup$ Algorithm 3$\left(\hat{\mathcal{M}}/B_\ell, \hat{n}-1, \hat{N} - \ell, (u_j)_{j \in \hat{N}-\ell}\right)$

Theorem 4 *A nearly proportional base allocation always exists and it can be computed within $O(n^2 m \ln m)$ time.*

Proof Let us analyze B, the solution returned by Algorithm 2. As a notation $n = |N|$ and $n_i = |N_i|$ for $i = 1, 2$. Hence $n = n_1 + n_2$.

By construction, there exists an element valued at least $1/n$ for every agent in N_2. If $N_2 \neq \emptyset$ then take any agent $j \in N_2$, B_j denotes its part. Let $e \in X$ such that $u_j(e) \geq 1/n$ and let B' be a base that contains e (every element belongs to a base). If $e \in B$, then $(B - e) + e$ is a base and $u_j((B_j - e) + e) \geq u_j(e) \geq 1/n$. Otherwise $e \in B' - B$ and by Theorem 1, there exists $f \in B$ such that $(B - f) + e$ is base. Moreover $u_j((B_j - f) + e) \geq u_j(e) \geq 1/n$. As a consequence, B is nearly proportional for every agent $j \in N_2$.

Now we focus on the agents of N_1 and we suppose $n_1 \geq 1$. Their parts are built with Algorithm 3 which is recursive. Algorithm 3 is executed $|N_1|$ times and each time, exactly one agent of N_1 receives his part (see Step 6).

In order to show that B is nearly proportional for every member of N_1, we follow the order by which the agents are allocated their respective parts. For the sake of simplicity, and without loss of generality, let us name these agents $1, \ldots, n_1$. That is, agent $i \in N_1$ is allocated B_i during the i-th recursive call of Algorithm 3.

We first show that for all $i, i' \in N_1$ satisfying $i \leq i'$, it holds that

$$OPT_{i'}(\mathcal{M}/(\bigcup_{j<i} B_j)) \geq \frac{n - i + 1}{n} \tag{1}$$

The proof is by induction on i. When $i = 1$ then one has to show that $OPT_{i'}(\mathcal{M}) \geq 1$ for all $i' \in N_1$, which is true by the normalization assumption. Suppose (1) holds for $i = 1..p$. Consider the p-th recursive call of Algorithm 3 in which agent p is eventually allocated B_p. In this case $\hat{\mathcal{M}} = \mathcal{M}/(\bigcup_{j<p} B_j)$, $\hat{n} = n - p + 1$ and $\hat{N} = \{p, \ldots, n_1\}$. Because of Step 5, one has $k_{i'} \geq k_p$ for all $i' > p$; $u_{i'}(G^{i'}) = OPT_{i'}(\hat{\mathcal{M}})$ by Step 3; $u_{i'}(\{e^{i'}_1, \ldots, e^{i'}_{k_{i'}}\}) \leq \frac{OPT_{i'}(\hat{\mathcal{M}})}{\hat{n}}$ by Step 4. It follows that

$$u_{i'}(G^{i'}) - u_{i'}(\{e^{i'}_1, \ldots, e^{i'}_{k_{i'}}\}) \geq \frac{\hat{n} - 1}{\hat{n}} OPT_{i'}(\hat{\mathcal{M}}) \tag{2}$$

Use Lemma 1 with $B^* = G^{i'}$, $F = B_p$, $F' = G^{i'} \setminus \{e^{i'}_1, \ldots, e^{i'}_{k_{i'}}\}$ and $u = u_{i'}$. We get that $OPT_{i'}(\hat{\mathcal{M}}/B_p) \geq u_{i'}(G^{i'} \setminus \{e^{i'}_1, \ldots, e^{i'}_{k_{i'}}\})$. By the additivity of $u_{i'}$, and the fact that $G^{i'} \supseteq \{e^{i'}_1, \ldots, e^{i'}_{k_{i'}}\}$, we deduce that $OPT_{i'}(\hat{\mathcal{M}}/B_p) \geq u_{i'}(G^{i'}) - u_{i'}(\{e^{i'}_1, \ldots, e^{i'}_{k_{i'}}\})$. Use (2) and $\hat{n} = n - p + 1$ to get that $OPT_{i'}(\hat{\mathcal{M}}/B_p) \geq \frac{\hat{n}-1}{\hat{n}} OPT_{i'}(\hat{\mathcal{M}}) = \frac{n-p}{n-p+1} OPT_{i'}(\hat{\mathcal{M}})$. By the induction hypothesis, we know that $OPT_{i'}(\hat{\mathcal{M}}) = OPT_{i'}(\mathcal{M}/\bigcup_{j<p} B_j) \geq \frac{n-p+1}{n}$. It follows that $OPT_{i'}(\mathcal{M}/\bigcup_{j<p+1} B_j) = OPT_{i'}(\hat{\mathcal{M}}/B_p) \geq \frac{n-p}{n-p+1}\frac{n-p+1}{n} = \frac{n-p}{n}$. The property holds for $i = p + 1$ and Inequality (1) is proved.

Now, it remains to show that B is nearly proportional for any agent $i \in N_1$. During the i-th call of Algorithm 3, agent i eventually receives B_i. At this moment $\hat{\mathcal{M}} = \mathcal{M}/(\bigcup_{j<i} B_j)$, $\hat{n} = n - i + 1$ and $\hat{N} = \{i, \ldots, n_1\}$. At Step 2 of Algorithm 3, G^i is an optimal base for $\hat{\mathcal{M}}$ according to u_i and then $u_i(G^i) = OPT_i(\hat{\mathcal{M}})$. By Inequality (1) we get that $u_i(G^i) \geq \frac{n-i+1}{n}$. By construction B_i consists of the k_i largest elements of G^i according to u_i, that is $\{e^i_1, \ldots, e^i_{k_i}\}$. If $k_i = |G_i|$ then $u_i(B_i) = u_i(G^i) \geq \frac{n-i+1}{n} \geq \frac{1}{n}$ (the last inequality is due to $i \leq n$); B is nearly proportional for agent i. Now suppose $k_i < |G_i|$. We have $u_i(\{e^i_1, \ldots, e^i_{k_i}\}) \leq \frac{OPT_i(\hat{\mathcal{M}})}{\hat{n}}$ by Step 4 and there exists $e^i_{k_i+1}$ such that $u_i(\{e^i_1, \ldots, e^i_{k_i+1}\}) > OPT_i(\hat{\mathcal{M}})/\hat{n}$. Since $OPT_i(\hat{\mathcal{M}}) = u_i(G^i) \geq \frac{n-i+1}{n}$ and $\hat{n} = n - i + 1$, we get that $u_i(\{e^i_1, \ldots, e^i_{k_i+1}\}) > 1/n$.

If $e^i_{k_i+1} \in B$, then $(B - e^i_{k_i+1}) + e^i_{k_i+1}$ is a base of $\mathcal{M}$ and $u_i((B_i - e^i_{k_i+1}) + e^i_{k_i+1}) \geq u_i(\{e^i_1, \ldots, e^i_{k_i+1}\}) \geq 1/n$. Otherwise $e^i_{k_i+1} \notin B$ and by Theorem 1, there exists $f \in B$ such that $(B - f) + e^i_{k_i+1}$ is a base of $\mathcal{M}$. Moreover $u_i((B_i - f) + e^i_{k_i+1}) \geq$

$u_i(\{e^i_1, \ldots, e^i_{k_i+1}\}) \geq 1/n$, so B is nearly proportional for agent i.

To conclude, the time complexity of Algorithm 2 is due to the $O(n^2)$ calls of GREEDY whose time complexity is $O(m \ln m)$. □

Example 3 *The instance of Example 2 does not have a proportional base allocation. However a nearly proportional base allocation can be found with Algorithm 2. At Step 1, $N_1 = \{2, 3\}$ and $N_2 = \{1\}$. Then we apply Algorithm 3 on $\mathcal{M}$, $n = 3$, N_1 and $(u_i)_{i \in N_1}$.*

In the first iteration of Algorithm 3, $G^2 = \{e_1, e_5, e_2, e_3\}$ and $G^3 = \{e_1, e_2, e_3, e_4\}$, so $k_2 = k_3 = 1$, $\ell = 2$ and $B_2 = \{e_1\}$.

The second iteration of Algorithm 3 is on $\hat{\mathcal{M}} = \mathcal{M}/\{e_1\} = (X - e_1, \mathcal{F}')$ with $\mathcal{F}' = \{F \subseteq X - e_1 : |F \cap X_2| \leq 1, |F \cap X_3| \leq 2\}$, $\hat{n} = 2$, $\hat{N}_1 = \{3\}$ and u_3. We get that $G^3 = \{e_2, e_3, e_4\}$, $k_3 = 1$, $\ell = 3$ and $B_3 = \{e_2\}$ such that $u_3(B_3) = 0.25 < \frac{OPT_3(\hat{\mathcal{M}})}{\hat{n}} = \frac{u_3(G^3)}{2} = \frac{0.75}{2}$. Finally, $F_1 = B_2 \cup B_3$.

For example, the set $F_2 = \{e_3, e_4\}$ is a base of $\mathcal{M}/F_1$ that can be obtained with GREEDY. *It remains to allocate the elements of F_2 to the unique agent of N_2 which is agent 1, so $B_1 = F_2$. Finally, the resulting base is $B = F_1 \cup F_2 = \{B_1, B_2, B_3\}$ and it is not difficult to see that B is nearly proportional.*

5 Envy-Freeness

Definition 6 (Envy-Freeness)

- *Agent i is envy-free when $u_i(B_i) \geq u_i(B'_i)$ for all $B'_i \in \mathsf{C}(\mathcal{M}, B - B_j)$ and for all $j \in N$;*
- *A base allocation is envy-free when every agent is envy-free.*

If agent i is given the opportunity to complete $B - B_j$ into a base, like agent j does with his part B_j, then any such subset B'_i would not be better than B_i, from agent i's point of view. For the special case of allocating indivisible goods, B_j is the unique way of completing $B - B_j$. Therefore Definition 6 in this context establishes that agent i would not be better off with the part of agent j, which is consistent with the standard notion of envy-freeness. Indeed we could have defined envy-freeness as $u_i(B_i) \geq u_i(B_j)$ for each pair (i, j) of agents however Definition 6 is stronger.

Note that if the base allocation $B = \{B_1, \ldots, B_n\}$ is envy-free then

$$u_i(B_i) = OPT_i(\mathcal{M}/(B - B_i)) \text{ for all } i \in N \qquad (3)$$

One can prove that envy-freeness implies proportionality and the equivalence for two agents also holds for the matroid problem as it is done for the allocation of divisible / indivisible goods.

An envy-free base allocation does not always exist and deciding the existence of an envy-free base allocation is NP-complete even in the special case of allocating indivisible goods [16]. Let us present a relaxation of envy-freeness up to one element.

Definition 7 (Near Envy-Freeness)

- *Agent i is nearly envy-free with respect to agent j when $u_i(B_i) \geq u_i(B'_i)$ for all $B'_i \in \mathsf{C}(\mathcal{M}, (B - B_j) + e)$ where $e \in X \setminus (B - B_j)$ and $(B - B_j) + e \in \mathcal{F}$;*
- *Agent i is nearly envy-free when for all $j \in N$, either he does not envy agent j or he is nearly envy-free with respect to agent j;*
- *A base allocation is nearly envy-free when every agent is either envy-free or nearly envy-free.*

Proposition 2 *Agent i is nearly envy-free with respect to agent j if and only if $u_i(B_i) \geq u_i(D) - \min_{x \in D} u_i(x)$ for all $D \in \mathsf{C}(\mathcal{M}, B – B_j)$.*

Proof Assume that $B = \{B_1, \ldots, B_n\}$ satisfies Definition 7 and let $D \in \mathsf{C}(\mathcal{M}, B - B_j)$. Suppose $e = \arg\min_{x \in D} u_i(x)$. We have $(B - B_j) + e \in \mathcal{F}$ and $D - e \in \mathsf{C}(\mathcal{M}, (B - B_j) + e)$. Hence by Definition 7, $u_i(B_i) \geq u_i(D - e) = u_i(D) - u_i(e) = u_i(D) - \min_{x \in D} u_i(x)$.
Conversely, let $e \in X \setminus (B - B_j)$ with $(B - B_j) + e \in \mathcal{F}$ and let $B'_i \in \mathsf{C}(\mathcal{M}, (B - B_j) + e)$. Set $D = B'_i + e$; we have $u_i(e) \geq \min_{x \in D} u_i(x)$. Hence, $u_i(B_i) \geq u_i(D) - \min_{x \in D} u_i(x) \geq u_i(D) - u_i(e) = u_i(B'_i)$. □

Definition 7 is stronger than *envy bounded by a single good* [5] in the context of allocating indivisible goods. Indeed if agent i is envious of agent j then near envy-freeness means that by removing any good from the part of j, agent i becomes not envious of j. Envy bounded by a single good means there exists some good in the share of j such that by removing it, agent i eliminates his envy for j.

Given a nearly envy-free base allocation, one can construct in polynomial time a nearly envy-free base allocation satisfying Equalities (3).

Lemma 2 *A nearly envy-free base allocation satisfying Equalities (3) is also nearly proportional.*

Note that there exist examples showing that near-envy freeness without Equalities (3) does not imply near proportionality.

Near proportionality does not imply near envy-freeness even for two agents as shown in the following example.

Example 4 *Let the free matroid $\mathcal{M} = (X, \mathcal{F})$ defined by $X = \{e_1, e_2, e_3\}$ and $\mathcal{F} = \{F \subseteq X\}$. The utilities are: $u_1(e_1) = 0.4$, $u_1(e_2) = u_1(e_3) = 0.3$ and $u_2(e_i) = \frac{1}{3}$ for all $i \in \{1, 2, 3\}$. Consider the base allocation $B = \{B_1, B_2\}$ with $B_1 = \{e_3\}$ and $B_2 = \{e_1, e_2\}$. B is nearly proportional since $u_1((B_1 - e_2) + e_2) = 0.6 > \frac{1}{2}$ and $u_2(B_2) = \frac{2}{3} > \frac{1}{2}$ but it is not nearly envy-free because $u_1(B_1) = 0.3 < u_1(B_2) - \min_{e \in B_2} u_1(e) = 0.4$.*

5.1 Near Balance

We introduce the concept of *near balance* with respect to utility u, inspired of the *local optimality* in the work of [1] in the context of allocating indivisible goods.

Definition 8 (Near Balance) *Given a matroid $\mathcal{M} = (X, \mathcal{F})$, a utility function $u : X \to I\!R^+$ and an integer $n \geq 2$, a base allocation $B = \{B_1, \ldots, B_n\}$ with $u(B_1) \geq \cdots \geq u(B_n)$ is* nearly balanced *with respect to utility function u when*

$$i < j \Rightarrow u(B_j) \geq u(B_i) - \min_{y \in B_i} u(y) \qquad \forall i, j \in [1..n] \qquad (4)$$

Corollary 1 *A nearly balanced base allocation always exists and it can be computed in polynomial time.*

Proof Near balance corresponds to near jealousy-freeness in the special case where the agents have the same utility function. The result follows from Theorem 3 and Algorithm 1. □

5.2 Near Envy-Freeness for two agents

Algorithm 4 presents a decentralized algorithm that constructs a nearly envy-free base allocation for two agents. Algorithm 4 draws on the protocol *Divide and Choose* in [13] for finding a base allocation for two agents with guarantees on agents' utilities.

Algorithm 4: Near Envy-Freeness for two agents

Data: A matroid $\mathcal{M} = (X, \mathcal{F})$, agents 1 and 2, u_1, u_2
Result: A nearly envy-free base allocation

1. Agent 1 computes a base G^1 maximum for u_1 (with GREEDY) and he partitions it into G_1^1 and B_2^1 in such a way as to be near balanced.
2. Agent 2 computes $V^i \in \mathsf{C}^{\max}(\mathcal{M}, G_i^1, u_2)$ for $i = 1, 2$ (with GREEDY) then agent 2 chooses $V^k = \arg\max_{i=1,2} u_2(V^i)$ and agent 1 takes G_k^1 for $k \in \{1, 2\}$.

Theorem 5 *When there are two agents, a nearly envy-free base allocation always exists and it can be computed in polynomial time.*

Proof Let $N = \{1, 2\}$. Let G^1 and G^2 be two bases that maximize u_1 and u_2, respectively. Suppose G^1 is partitioned in $G_1^1 \cup G_2^1$ in such a way that G_1^1 and G_2^1 are nearly balanced with respect to u_1.

Let $V^1 \in \mathsf{C}^{max}(\mathcal{M}, G_1^1, u_2)$ and $V^2 \in \mathsf{C}^{max}(\mathcal{M}, G_2^1, u_2)$. We have that $G_1^1 \cup V^1$ and $G_2^1 \cup V^2$ are two bases of $\mathcal{M}$. In addition, V^1 and V^2 are bases of $\mathcal{M}/G_1^1$ and $\mathcal{M}/G_2^1$, respectively. Note that $V^1 \cup V^2$ is not necessarily an independent set. Let us suppose without loss of generality that $u_2(V^1) \geq u_2(V^2)$. By construction, $u_2(V^1) \geq u_2(D)$ for every $D \in \mathsf{C}(\mathcal{M}, G_1^1)$. From now on we analyze the solution $B = \{G_1^1, V^1\}$.

The agents clearly satisfy Equalities 3, so we only need to prove that agent $i \in \{1, 2\}$ is nearly envy-free with respect to agent $3 - i$.

Let us prove that agent 2 is envy-free with respect to agent 1. In other words, $u_2(V^1) \geq u_2(D)$ for every $D \in \mathsf{C}(\mathcal{M}, B \setminus G_1^1) = \mathsf{C}(\mathcal{M}, V^1)$. Let $D \in \mathsf{C}(\mathcal{M}, V^1)$; hence $D \cup V^1$ is a base.

Let G^2 be a maximum base with respect to u_2. Using Theorem 2 with $G^1 = G_1^1 \cup G_2^1$ and G^2, we get that G^2 admits a bi-partition $G_1^2 \cup G_2^2$ such that $G_1^2 \cup G_1^1$ and $G_2^2 \cup G_2^1$ are two bases of $\mathcal{M}$. Thus G_1^2 and G_2^2 belong to $\mathsf{C}(\mathcal{M}, G_1^1)$ and $\mathsf{C}(\mathcal{M}, G_2^1)$, respectively. Since V^1 and V^2 belong to $\mathsf{C}^{max}(\mathcal{M}, G_1^1, u_2)$ and $\mathsf{C}^{max}(\mathcal{M}, G_2^1, u_2)$, respectively, we get that $u_2(V^1) \geq u_2(G_1^2)$ and $u_2(V^2) \geq u_2(G_2^2)$. Summing up these two inequalities gives $u_2(V^1) + u_2(V^2) \geq u_2(G_1^2) + u_2(G_2^2) = u_2(G^2) \geq u_2(D \cup V^1)$. Hence $u_2(V^2) \geq u_2(D)$. Now, since $u_2(V^1) \geq u_2(V^2)$ by hypothesis, the result follows.

Let us prove that agent 1 is nearly envy-free with respect to agent 2. G^1 is partitioned in $G_1^1 \cup G_2^1$ such that G_1^1 and G_2^1 are nearly balanced with respect to u_1. We are going to prove that agent 1 is nearly envy-free using Proposition 2. Let $D \in \mathsf{C}(\mathcal{M}, G_1^1)$. We have $G_1^1 \cup D$ is a base and then we deduce that $u_1(G_1^1) + u_1(G_2^1) \geq u_1(G_1^1 \cup D)$. Hence,

$$u_1(G_2^1) \geq u_1(D) \tag{5}$$

We distinguish two cases: either $u_1(G_1^1) \geq u_1(G_2^1)$ or $u_1(G_1^1) < u_1(G_2^1)$. Obviously, if $u_1(G_1^1) \geq u_1(G_2^1)$, then agent 1 is envy-free because by Inequality (5) we get that $u_1(G_1^1) \geq u_1(G_2^1) \geq u_1(D)$.

Now we study the second case, i.e. $u_1(G_2^1) > u_1(G_1^1)$. Since $\{G_1^1, G_2^1\}$ is a nearly balanced base by construction, we have $u_1(G_1^1) \geq u_1(G_2^1) - \min_{x \in G_2^1} u_1(x)$. Let us prove that $u_1(G_2^1) - \min_{x \in G_2^1} u_1(x) \geq u_1(D) - \min_{x \in D} u_1(x)$ which will conclude the proof.

Let $e = \arg\min_{x \in G_2^1} u_1(x)$ and $f = \arg\min_{x \in D} u_1(x)$ and by contradiction, suppose $u_1(D - f) > u_1(G_2^1 - e)$. Using axiom (iii) of a matroid, there is $y \in G^1 - ((D + G_1^1) - f) = G_2^1 - (D - f)$ such that $B' = (D - f) + G_1^1 + y$ is a base. It must be $u_1(y) \geq u_1(e)$ because $y \in G_2^1$ and e is by definition minimum with respect to u_1 in G_2^1. Hence $u_1(B') = u_1(D - f) + u_1(G_1^1) + u_1(y) > u_1(G_2^1 - e) + u_1(G_1^1) + u_1(y) = u_1(G^1) - u_1(e) + u_1(y) \geq u_1(G^1)$ which is a contradiction with G^1 is a maximum base for u_1. □

6 Discussion and future work

Though a fair solution may not exist, a deterministic solution satisfying fairness up to one element is guaranteed to exist and it can be computed in polynomial time. For equitability and proportionality, the result is achieved with centralized algorithms which work for any number of agents. Is it possible to obtain the same result with a decentralized algorithm? Concerning envy-freeness, the result is obtained for two agents, with a decentralized algorithm. It remains to be seen whether a nearly envy-free base allocation can be found in polynomial time when there are $n \geq 3$ agents, and to investigate decentralized algorithms for near equitability / proportionality. We conjecture that a nearly envy-free base allocation always exists. Another perspective is to combine near fairness with other standard criteria like Pareto Optimality.

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