Counting Linear Extensions Is #P-Complete

(Extended Abstract)

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Abstract. We show that the problem of counting the number of linear extensions of a given partially ordered set is #P-complete. This settles a long-standing open question and contrasts with recent results giving randomized polynomial-time algorithms for *estimating* the number of linear extensions. One consequence is that computing the volume of a rational polyhedron is strongly #P-hard.

We also show that the closely related problems of determining the average height of an element x of a given poset, and of determining the probability that x lies below y in a random linear extension, are #P-complete.

1. Introduction.

The problem of determining the number of linear extensions of a partially ordered set is fundamental in the theory of ordered sets, and is of interest in computer science by virtue of its connections with sorting. For instance, at each stage of any comparison-based sorting algorithm, current information can be expressed as a partial ordering of the data set, any linear extension of which is a possible "solution". If it were easy to compute the number of linear extensions of a poset, one could determine in a sequential sort which is the optimum pair of elements to compare next. (Kahn and Saks [5] have shown that there is always a pair whose comparison splits the set of linear extensions in no more lopsided a fashion than $3/11 : 8/11$, but their proof gives no way of finding such a pair short of computing numbers of linear extensions.)

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Recently randomized polynomial time algorithms have been given which approximate the number of linear extensions to within an abritrary tolerance. In 1989 Dyer, Frieze and Kannan [2], by applying their work on approximating the volumes of convex bodies to the order polytope of a poset, became the first to obtain such an algorithm, a more efficient version of which is now available thanks to the bounds on conductance of the linear extension graph achieved by Karzanov and Khachiyan [6].

On the other hand, the problem of determining the number of linear extensions *exactly* has long been suspected of being #P-complete, thus presumably very difficult (especially in view of Toda's result [11], which implies that one call to a #P oracle suffices to solve any problem in the polynomial hierarchy in deterministic polynomial time).

As far as we know the first to publish the conjecture was Linial [8], who called it "a most intriguing problem in this field." Lovász [9, p. 61] has also mentioned the problem, and apparently many others have considered it. It has, however, resisted analysis, despite the substantial number of related counting problems which have by now been shown to be #P-complete. These problems include computing the volume of a general convex body in Euclidean space [1], which has now been observed by Khachiyan [7] to be #P-complete "in the strong sense" as a consequence of our result.

Our method is direct, showing that with the help an oracle which counts linear extensions, a turing machine can count the number of satisfying assignments to an instance of 3-SAT in polynomial time. This contrasts with other #P-completeness results, as in [8,10], which have utilized the machinery developed in Valiant [13]. Our technique has now been applied by Feigenbaum and Kahn [3] to show that a problem called "POMSET language size" is complete for the class SPAN-P.

2. Preliminaries.

A partially ordered set (or poset) is a set P equipped with an irreflexive transitive relation $<$. An antichain in P is a set of elements (vertices) of P such that no pair is related by $<$.

A *linear extension* of a partially ordered set P on n vertices is a linear ordering $\prec$ of the vertex set such that $x \prec y$ whenever $x < y$ in P. Equivalently, a linear extension of P is a bijection λ from the set of vertices of P to $\{1, \ldots, n\}$ such that $\lambda(x) < \lambda(y)$ whenever $x < y$ in P. We shall be making implicit use of both forms of the definition.

For a poset P, let $N(P)$ denote the number of linear extensions of P. We shall mostly be concerned with the following enumeration problem.

Linear Extension Count
Input. A partially ordered set P.
Output. The number $N(P)$ of linear extensions of P.

Linear Extension Count belongs to the class #P which consists of all counting problems whose solutions are the number of accepting states of some nondeterministic polynomial time Turing machine. In this paper, we shall make use of the basic fact, proved in [12], that the following problem is #P-complete.

3-SAT Count
Input. A propositional formula I in 3-conjunctive normal form.
Output. The number $s(I)$ of satisfying assignments for I.

The main result in Valiant [12] is that computing the permanent of a matrix (equivalently, counting the number of complete matchings in a bipartite graph) is #P-complete. This remains the outstanding example of a case where a decision problem is in P, but the corresponding enumeration problem is #P-complete.

Our main result is another example of this phenomenon, even more extreme since the decision problem is trivial: every poset has a linear extension.

Theorem 1. *Linear Extension Count is #P-complete.*

We shall make use of the following fact concerning the distribution of primes, easily deduced e.g. from Hardy and Wright [4, Chapter 22].

Lemma. *For any $n \geq 150$, the product of the set of primes strictly between n and n^2 is at least $n!\,2^n$.*

3. Proof of Theorem 1.

Suppose we have an oracle $\mathcal{O}(t)$ which, when presented with a partially ordered set P of size at most t, returns in unit time the number of linear extensions of P.

We shall give an algorithm which solves the problem 3-SAT Count in time polynomial in the number m of variables and the number n of clauses, making use of the oracle $\mathcal{O}(t)$, where $t = (7n + m)^3$.

Thus, let I be an instance of 3-SAT Count, consisting of m variables and n clauses which are conjunctions of three literals. For convenience, we set $M = 7n + m$. We may and shall assume that $M \geq 150$, so that the conclusion of the lemma holds for M.

Let P_I be the partially ordered set defined from I as follows. The points of P_I consist of a vertex h_x corresponding to each variable x in the instance, and seven vertices for each clause. If x, y and z are the variables in some particular clause, then each of the seven vertices corresponding to that clause is placed above a different non-empty subset of $\{h_x, h_y, h_z\}$. There are no other comparabilities in P_I. (See Figure 1.)

Let L_I be the number of linear extensions of P_I. Since the size of P_I is just M, this number can certainly be calculated using $\mathcal{O}(M^3)$.

Let S_0 be the set of primes strictly between M and M^2. By the lemma, their product is at least $M!\,2^m$. Since L_I is at most $M!$, there is a set S of primes strictly between M and M^2, none of which divide L_I, whose product is at least 2^m.

Let p be a prime in S. We now define a partially ordered set $Q_I(p)$ as follows. (See Figure 2.)

Figure 1. The relations in P_I corresponding to a clause involving the variables x, y and z.

There are two special vertices a and b which are used to divide linear extensions of the poset into three sections. The section below a will be referred to as the *bottom* section; that between a and b is the *middle* section, and the section above b is the *top*. Some of the other vertices will be bound into one particular section; others will be free to appear in different sections, depending on the linear extension.

Below a in $Q_I(p)$ is an antichain U of size $(m+1)(p-1)$. This antichain is divided into sets of size $p-1$, one set U_x corresponding to each variable x in the instance, and one extra set U_0.

Similarly, between a and b is an antichain V of size $(n+1)(p-1)$. Again this is divided into sets of size $p-1$, with a set V_c corresponding to each clause c, and one other set V_0.

Next, for each variable x in the instance, we have two corresponding *literal vertices* which we shall refer to as x and $\overline{x}$. The literal vertices x and $\overline{x}$ are incomparable with both a and b, and are above all elements in the set U_x corresponding to the variable x.

Finally, we have eight *clause vertices* $c_1, c_2, \ldots, c_8$ for each clause c of the instance. If x, y and z are the three variables involved in the clause c, then there is a clause vertex above each triple of literal vertices consisting of one element from each of $\{x, \overline{x}\}$, $\{y, \overline{y}\}$, $\{z, \overline{z}\}$. The clause vertex c_i which is above that triple of literals which actually constitutes the clause c is also above b; the other clause vertices are above each element of the antichain V_c corresponding to c. Thus all clause vertices are above a.

Figure 2. The poset $Q_I(p)$. Here the ovals represent antichains of size $p-1$. The only clause vertices shown here are those corresponding to the clause $xy\overline{z}$.

The total number of vertices in the poset $Q_I(p)$ is thus

$$2 + (p-1)(n+m+2) + 2m + 8n < p(7n+m) \le M^3.$$

So the number of linear extensions of $Q_I(p)$ can be found using the oracle $\mathcal{O}(M^3)$. We next investigate how this number is related to the number of satisfying assignments for I.

We shall partition the set of linear extensions of $Q_I(p)$ according to which of the literal and clause variables occur in each of the three sections marked off by a and b.

We define a *configuration* to be a partition ϕ of the literal and clause vertices into three sets B^ϕ, M^ϕ and T^ϕ. Let Φ denote the set of all configurations. We say a linear extension of $Q_I(p)$ *respects* a configuration $\phi = (B^\phi, M^\phi, T^\phi)$ if $B^\phi \prec a \prec M^\phi \prec b \prec T^\phi$ in the linear extension. The set of linear extensions respecting a configuration ϕ is denoted L^ϕ.

We say that a configuration is *consistent* if L^ϕ is non-empty, which is the case whenever the information $B^\phi \prec a \prec M^\phi \prec b \prec T^\phi$ is consistent with the partial order $Q_I(p)$. Also, if L^ϕ is non-empty, it is just the set of linear extensions of the partial order P^ϕ defined by adding to $Q_I(p)$ the relations given by $B^\phi < a < M^\phi < b < T^\phi$ and taking the transitive closure.

Thus we have

$$N(Q_I(p)) = \sum_{\phi \in \Phi} N(P^\phi).$$

We shall prove that the only configurations which contribute to this sum, mod p, are those where B^ϕ contains exactly one literal vertex for each variable, M^ϕ contains exactly one clause vertex for each clause, and T^ϕ contains the remaining literal and clause vertices. Furthermore, this is only possible when the set of literal vertices in T^ϕ corresponds to a satisfying assignment for I, and each satisfying assignment gives rise to exactly one such consistent configuration. Finally, when $N(P^\phi)$ is not divisible by p, it is equal to a readily calculable constant.

As a first step towards proving these assertions, let us remark that, for any consistent configuration ϕ, the vertices a and b are comparable with every other vertex in P^ϕ. Let P_B^ϕ be the poset induced on the elements below a in P^ϕ, P_M^ϕ the poset induced on the elements between a and b, and P_T^ϕ the poset induced on the elements above b. Now we have

$$N(P^\phi) = N(P_B^\phi)\, N(P_M^\phi)\, N(P_T^\phi).$$

Thus $N(P^\phi)$ is divisible by p precisely when one of these three terms is.

A consistent configuration ϕ is said to be *feasible* if neither $N(P_B^\phi)$ nor $N(P_M^\phi)$ is divisible by p.

Let ϕ be any feasible configuration. We consider first the bottom section P_B^ϕ of the poset P^ϕ. This consists of the antichain U of size $(p-1)(m+1)$, together with some of the literal vertices. The elements of U_0 are isolated in this poset, as are the elements of

U_x for any x such that neither of the two associated literal vertices x and $\overline{x}$ is in B^ϕ. Let $k = |P_B^\phi|$, and let $r \geq p-1$ be the number of isolated vertices.

A linear extension of P_B^ϕ can be considered as a choice of positions among the heights $1, 2, \ldots, k$ for each of the r isolated vertices, together with a linear extension of the poset induced on the remaining vertices. Hence $N(P_B^\phi)$ is divisible by $k(k-1)\ldots(k-r+1)$. Since ϕ is feasible, this quantity is not divisible by p, and so $r = p-1$, and $k \equiv -1 \pmod{p}$. Since k lies between $(p-1)(m+1)$ and $(p-1)(m+1)+2m$, and $p > m$, this implies that exactly m literal vertices are in B^ϕ—one for each variable.

Therefore the poset P_B^ϕ consists of $p-1$ isolated elements, and m components consisting of one literal vertex above an antichain of size $p-1$. We claim that the number of linear extensions of a poset P of this form is exactly

$$\big(p(m+1)-1\big)!/p^m.$$

To see this, for each variable x, let A_x be the event that a randomly chosen ordering of the vertices of P has the literal vertex associated with x above all "its" $p-1$ vertices. The probability of each A_x is $1/p$, and the m events are independent. Moreover, an ordering of the vertices is a linear extension of P iff each A_x occurs.

The above product $\big(p(m+1)-1\big)!$ has just m terms which are multiples of p, and none which are mutiples of higher powers of p, so $\big(p(m+1)-1\big)!/p^m$ is not divisible by p.

To summarize, if ϕ is feasible, then B^ϕ contains exactly one literal vertex for each variable, and $N(P_B^\phi) = \big(p(m+1)-1\big)!/p^m \not\equiv 0 \pmod{p}$.

We now move up and consider the middle section of P^ϕ. The argument in this case is essentially identical to that for the bottom section.

We assume once more that the configuration ϕ is feasible. We are now concerned with the middle section P_M^ϕ of the poset P^ϕ. This consists of the antichain V of size $(p-1)(n+1)$, together with some of the literal and clause variables, say j of them. Note that $0 \leq j \leq 7n+m < p$. Each of the $p-1$ elements of V_0 is isolated in P_M^ϕ, as are all the elements of V_c for any clause c, none of whose associated clause vertices c_i are in M^ϕ.

Arguing exactly as for the bottom section, we see that, for each clause c, at least one of the vertices c_i associated with c appears in M^ϕ, and that the total number of vertices in P_M^ϕ is congruent to $-1 \pmod{p}$. The only possibility is that exactly n of the literal and clause vertices are in the middle section. Thus M^ϕ contains no literal vertices and exactly one clause vertex for each clause.

Again essentially as for the bottom case, we have that, if P_M^ϕ is of this form, then

$$N(P_M^\phi) = \big(p(n+1)-1\big)!/p^n \not\equiv 0 \pmod{p}.$$

We know that, in each feasible configuration, every variable x has one of its associated literal vertices l_x appearing in B^ϕ, and the other, h_x, in T^ϕ. Thus each feasible configuration induces an assignment $h(\phi)$ of true literals for the instance I, consisting of the literals h_x. We shall show that an assignment that satisfies the instance corresponds to just one feasible configuration, whereas an assignment that does not satisfy the instance corresponds to no feasible configurations. This will imply that the number of feasible configurations is equal to the number of satisfying assignments.

Suppose ϕ is feasible, and let c be a clause involving variables x, y and z. Then seven of the eight associated clause vertices c_i are above at least one of h_x, h_y or h_z, and are therefore themselves in T^ϕ. Therefore it is the eighth clause vertex which appears in M^ϕ, namely that c_i whose designated triple of literal vertices is $\{l_x, l_y, l_z\}$. Therefore the assignment $h(\phi)$ determines ϕ uniquely. If, however, the set $\{l_x, l_y, l_z\}$ corresponds exactly to the set of literals in the clause c, then this chosen clause vertex is above b in $Q_I(p)$, and therfore is necessarily in T^ϕ.

In other words, if any clause is not satisfied by the assignment $h(\phi)$, then ϕ is not feasible, a contradiction. Conversely, if h is any satisfying assignment, then $h = h(\phi)$ for some feasible ϕ, namely the configuration where B^ϕ consists of the literal vertices corresponding to false literals, and M^ϕ consists of the clause vertices which are above only "false" literal vertices.

The next observation is that, if ϕ is a feasible configuration, then the poset P_T^ϕ is isomorphic to the auxiliary poset P_I. Indeed, each variable x is represented by the literal h_x in T^ϕ, and each clause by seven of the eight associated clause vertices. If x, y and z are the variables involved in a clause c, then every non-empty subset of $\{h_x, h_y, h_z\}$ has one clause vertex c_i above just the elements of that subset.

We are now in a position to count the number of linear extensions of $Q_I(p)$, mod p. We know that non-feasible configurations contribute nothing to this sum, and feasible configurations are in 1–1 correspondence with satisfying assignments for I. Moreover, for each feasible configuration ϕ,

$$\begin{aligned} N(P^\phi) &= N(P_B^\phi)\,N(P_M^\phi)\,N(P_T^\phi) \\ &= \big(p(m+1)-1\big)!/p^m \cdot \big(p(n+1)-1\big)!/p^n \cdot L_I, \end{aligned}$$

and none of the three terms making up this product, which we denote by N_0, is divisible by p. (In the case of L_I, this is by definition of the set S of primes we are using.)

In other words, for each feasible configuration ϕ, $N(P^\phi)$ is equal to some N_0 depending on p, n and m but not ϕ. Therefore

$$N(Q_I(p)) \equiv N_0 \cdot s(I) \pmod{p}.$$

Furthermore, N_0 is not divisible by p, and can be calculated quickly.

The oracle $\mathcal{O}(M^3)$ enables us to find the number of linear extensions of $Q_I(p)$ for each prime p in our set S. This then enables us to find $s(I) \pmod{p}$ for every $p \in S$.

Since the product of the primes in S is greater than 2^m, and $s(I)$ is at most 2^m, we can then find the value of $s(I)$.

This completes the proof of Theorem 1. $\square$

Let us make a few remarks about the above proof. Firstly, it is known to be #P-complete to compute the number of satisfying assignments for a Boolean formula in 2-conjunctive normal form, even under very restrictive conditions: see [8,10]. We chose to reduce from 3-SAT Count for reasons of familiarity, as there would be no significant simplification of the proof obtained from using 2-SAT instead.

Note that our construction proves # P-completeness for Linear Extension Count for posets of height at most 5. In fact, the construction can be altered slightly so as to get the height down to 3. We strongly suspect that Linear Extension Count for posets of height 2 is still #P-complete, but it seems that an entirely different construction is required to prove this.

4. Related Problems.

We now discuss the implications of Theorem 1 for two closely related problems. We refer the reader to the survey [14] for background.

For x and y incomparable elements of a poset P, $\Pr(x \prec y)$ denotes the probability that x precedes y in a randomly chosen linear extension of P, where all linear extensions are equally likely. Thus $\Pr(x \prec y)$ can be written as $N(P \cup (x,y))/N(P)$, where $P \cup (x,y)$ is the poset obtained from P by adding the relation $x < y$ and taking the transitive closure.

If x is a vertex in a poset P, and $\prec$ is a linear extension of P, then the *height* of x in $\prec$ is the number of elements below x in $\prec$, plus one. The *average height* $H_P(x)$ of x in P is the average over all linear extensions $\prec$ of P of the height of x in $\prec$.

Strictly speaking, the problems of evaluating these proabilities and average heights do not belong to the class $\#P$, as they are not enumeration problems. However, in view of the following theorem, they may effectively be regarded as a #P-complete problems.

Theorem 2. *The problems of evaluating* $\Pr(x \prec y)$ *and the average height of* x *in a poset* P *are polynomially equivalent to Linear Extension Count.*

Proof. Omitted.

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