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Fuzzy Logic for Preferences expressible by convolutions

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1 INTRODUCTION

In this paper we introduce a new extension of the Predicate Pavelka Logic [4], extended by P. Hajek [1], with the convolution integrals – to be called Fuzzy Integral Logic (*FLI*) suitable to express the preferences expresible by convolutions from the initial functions. We base on some observation from [2] and [3] and in Rossi *et al.* [5].

2 LOGICAL BACKGROUND OF THE ANALYSIS

We begin our construction of *FLI* with some introductory remarks on many-valuated Łukasiewicz Propositional Logic *LukPL*, Rational Pavelka (Propositional) Logic (*RPL*) and Rational Pavelka (Predicate) Logic (*RPL*∀).

Łukasiewicz propositional calculus *LukPL* is based on a language with the following connectives and constants: $\rightarrow$, $\neg$, $\leftrightarrow$, $\wedge$ (weak conjunction), $\otimes$ (strong conjunction), $\vee$ (weak disjunction), $\oplus$ (strong disjunction) and propositional constants $\overline{0}$ and $\overline{1}$. interpreted as follows: $\|\neg(\phi)\| = 1 - x$, $\| \rightarrow (\phi, \psi)\| = \min\{1, 1 - x + y\}$, $\| \wedge (\phi, \psi)\| = \min\{x, y\}$, $\| \vee (\phi, \psi)\| = \max\{x, y\}$, $\| \otimes (\phi, \psi)\| = \max\{0, x + y - 1\}$, and $\| \oplus (\phi, \psi)\| = \min\{1, x + y\}$ for any $x, y \in$ MV-algebra A. *LukPL* could be characterized as an extension of basic fuzzy logic BL by the axiom :$\neg\neg\phi \rightarrow \phi$. The axiomatics of *LukPL* and BL could be found in [1]. *LukPL* is complete with respect to: all MV-algebras (*general completeness*).

Rational Pavelka Logic *RPL* extends *LukPL* and is based on the language of *LukPL* extended by new constants: $\widehat{r_1}, \widehat{r_2}, \widehat{r_3}, \ldots$, representing in the language $\mathcal{L}$(*FLI*) the rational numbers: $r_1, r_2, \ldots, s_1, s_2, \ldots$ etc. from the universe of MV-algebras.

Axioms of RPL are the axioms of *LukPL* plus the following ones:

$$\widehat{r} \rightarrow \widehat{s} = \widehat{r \rightarrow s}, \quad \neg\widehat{r} = \widehat{1 - r}$$

Example: $\widehat{0.6} \rightarrow \widehat{0.5} = \widehat{0.6 \rightarrow 0.5} = 1 - \widehat{0.6 + 0.5} = \widehat{0.9}; \widehat{\neg 0.3} = \widehat{0.7}$ because of Łukasiewicz's implication.

Rational Pavelka Predicate Logic $RPL\forall$ extends the propositional language of RPL with general and existential quantifers and by adding to them the axioms of first order predicate logic. The only inferential rule in $RPL\forall$ in *Modus Ponens*.

Completeness of $RPL\forall$ is usually expressed in the terms of the *Truth degree* of ϕ over T, which is defined as follows:

$$\|\phi\|_T = inf\{\|\phi\|_M | M \models T \; over \;\; MV - algebra \; [0, 1]\} \quad (1)$$

and the so-called *Provability degree* defined as follows:

$$|\phi|_T = sup\{r \; rational \; |T \vdash (\widehat{r} \rightarrow \phi)\} \quad (2)$$

Completeness in such terms is equivalent to the condition: $\|\phi\|_T = |\phi|_T$ for each ϕ of fixed language of theory T.

Examples: If T is complete: its tautologies of T have both truth and provability degree 1; the contratautologies – 0.

3 NEW FUZZY LOGIC OF INTEGRALS FLI

We want to introduce now *FLI* based on $\mathcal{L}(RPL\forall)$ as a system able to express the convolution of type: $\int_0^x f(t)g(x - t)dt$ for $f, g \in L(\mathbf{R})$. We will aim at the expressing typical properties of convolutions such as: commutativity, distributivity and associativity wrt a scalar multiplication in our language $\mathcal{L}$(*FLI*). The approach is based at the definition of the integral $\int f d\mu$ (for each function f: $M \rightarrow [0, 1]$, M- countable or finite and a probability measure μ on M) defined as $\Sigma_{m \in M} f(m)\mu(m)$ and denoted shortly by $\int f dx$.

3.1 Syntax

The alphabet of $\mathcal{L}$(*FLI*) consists of:

- propositional variables: $\phi, \chi, \psi, \ldots x, y, \ldots, \phi_t, \phi_{x-t}, \chi_t, \chi_{x-t}$
- rational constant names: $\widehat{r_1}, \widehat{r_2}, \ldots \widehat{0}, \widehat{1}$,
- quantifiers: $\forall, \exists | \int ()dx, \int\int ()dxdy, \int_0^x ()dt$
- operations: $\rightarrow, \neg, \vee, \wedge, \bullet, \oplus, \otimes, =$.

Set of formulae FOR: The class of well-formed formulae *FOR* of $\mathcal{L}$(*FLI*) form *propositional variables* and *rational constants* as *atomic* formulae. The next - formulae obtained from given $\phi, \chi \in FOR$ by operations $\neg, \vee, \wedge, \rightarrow, \oplus, \otimes, \forall, \exists, \int ()dx$ and the formulae obtained from $\phi_i, \chi_i \in FOR$ by operations $\neg, \vee, \wedge, \rightarrow, \oplus, \otimes, \bullet, \forall, \exists, \int_0^x ()dt$. Finally, formulae obtained from $\phi_i \in$ *FOR* and rational numbers by operations $\neg, \vee, \wedge, \rightarrow, \oplus, \otimes, \bullet$ belong to *FOR* as well. These classes of formulae exhaust the list of *FOR* of $\mathcal{L}$(*FLI*).

Example: $\int_0^x \phi_t \bullet \chi_{x-t} dt \rightarrow \widehat{r} \in FOR$, but $\int \phi dx \rightarrow \int_0^x \chi_t dt$ does not.

Axioms: We extend now the list of the following axioms considered by Hajek in [1]:

$$\int (\neg\phi)dx = \neg \int \phi dx, \int(\phi \rightarrow \chi)dx \rightarrow (\int \phi dx \rightarrow \int \chi dx)$$

$$\int (\phi \otimes \chi)dx = ((\int \phi dx \rightarrow \int (\phi \wedge \chi)dx) \rightarrow \int \chi dx))$$

$$\int\int \phi dxdy = \int\int \phi dydx^5 \text{ (Fubini theorem):}$$

by new axioms defining the algebraic properties of convolutions:

$$\int_0^x \phi_t \bullet \chi_{x-t}dt = \int_0^x \phi_{x-t} \bullet \chi_t dt$$

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[5] If both sides are defined.

associativity wrt the scalar multiplication:

$$\widehat{r}\int_0^x \phi_t \bullet \chi_{x-t}dt = \int_0^x (\widehat{r}\phi_t) \bullet \chi_{x-t}dt, (\text{r}- - constant)$$

distributivity:

$$\int_0^x \phi_t \bullet (\chi_{x-t} \oplus \psi_t)dt = \int_0^x \phi_t \bullet \chi_{x-t}dt \oplus \int_0^x \phi_t \bullet \psi_{x-t}dt$$

As inference rules we assume *Modus Ponens,* generalization rule (like in *RPL*$\forall$) and two new specific rules: $\frac{\phi}{\int \phi dx}, \frac{\phi \to \chi}{\int \phi dx \to \int \chi dx}$ and the same rules for indexed formulae and convolution integrals.

3.2 Semantics

Our intention is to interpret the axioms in another type of models, so-called *weak probabilistic model* for *FLI*. For this purpose we introduce the definition of the integrals I to interpret the $\int$-formulae of $\mathcal{L}(FLI)$ Let Alg be an algebra of the functions from $M \neg \emptyset$ to [0,1] containing each rational constant function $r \in [0,1]$ and closed on $\Rightarrow$(See: [1], p. 240). We define *weak integrals* on *Alg* as a mapping I: $f \in Alg \mapsto If(x) \in [0,1]$ satisfying the conditions:

$$I(1-f)dx = 1 - Ifdx,\ I(f \Rightarrow g)dx \leq (Ifdx \Rightarrow Igdx) \quad (3)$$

$$I(f \otimes_{Sem} g)dx = Ifdx + Igdx - I(f \wedge g)dx \quad (4)$$

$$I(Ifdx)dy = I(Ifdy)dx \quad (5)$$

We extend the above conditions, also considered in [1] by the following ones: (the commutativity of convolution:)

$$I_0^x(f(t)g(x-t)dt = I_0^x g(t)f(x-t)dt \quad (6)$$

$$rI_0^x f(t)g(x-t)dt = I_0^x(rf(t))g(x-t)dt \quad (7)$$

$$\begin{aligned} I_0^x f(t) \star (g(x-t) \oplus_{\text{sem}} h(x-t))dt = \\ \min\{1, I_0^x f(t) \star g(x-t)dt + I_0^x f(t) \star h(x-t)dt\} \end{aligned} \quad (8)$$

Interpretation Let assume that $Int = (\triangle, \|\phi\|)$ with $\triangle \neg \emptyset$ and the truth-value interpretation–function: $\|\|$ of formulae of $\mathcal{L}(RPL\forall)$. We inductively expand now this interpretation for new elements of the grammar $\mathcal{L}$(*FLI*).as below.

syntax ($\phi \in \mathcal{L}(FLI)$)	**fuzzy semantics** ($\|\phi\|_{FLI}$)
ϕ_i	$f(i)$ for $i \in \{x, t, x-t, t-x\}$
$\int \phi dx, (\int_0^x \phi dx$	$Ifdx, (I_0^x fdx)$
$\phi_i \bullet \chi_i$	$\|\phi_i\| \star \|\chi_i\|$ (i like above)
$\|\phi \otimes \chi\|, (\|\phi_i \otimes \chi_i\|)$	$\|\phi\| \otimes_{Sem} \|\chi\|, (\|\phi_i\| \otimes_{Sem} \|\chi_i\|)$
$\int_0^x \phi_t \bullet \chi_{x-t}dt$	$I_0^x g(t) \star f(x-t))dt$
$\widehat{r}\int_0^x dt$	$\|\widehat{r}\| \star \|I_0^x f\|dt = rI_0^x fdt$

The other operators of FLI are interpreted like in $RPL\forall$ for formulae of ϕ and ϕ_i–type, especially: $\|\phi \oplus \chi\| = \|\phi\| \oplus_{sem} \|\chi\| = min\{1, \|\phi\| + \|\chi\|\}$.

Definition of the model: We define a *weak probabilistic model* M as a *n*-tuple of the form: $M = \langle |M|, \{r_0, r_1, ..\}, f_i, If_idx, I_0^x f_i g_j dt, \mathbf{0}, \mathbf{1}, \star\rangle$ where $|M|$ is a countable (or finite) set $\{r_0, r_1, \ldots\}, f_i$ are respectively: a set of rational numbers belonging to $|M|$, and atomic integrable functions. If_i are integrals on the algebra Alg of subsets of $|M|$ and $I_0^x f_i g_j dt$ are convolutions of f_i and g_j expressing *preferences*, and **0** and **1** are distinguished *min* and *max* elements (*the best, the worst preferences*)). $\star$ is a convolution multiplication.

Completeness theorem for *FLI*: *For each theory T over predicate $\mathcal{L}$(FLI) and for each formula $\phi \in \mathcal{L}(FLI)$ it holds: $|\phi|_T = \|\phi\|_T$.*

Undecidability theorem for *FLI:* *The set $TAUT_{FLI}$ of tautologies of the system FLI is undecidable and it is $\prod_2$ - hard.*

4 *FLI* AS LOGIC OF PREFERENCES

Here we justify that FLI could be recognized as a logic for preferences on a base of a short physical example.

Example: Consider that a compressible fluid A flows through two tubes P_1 and P_2. Assume that both tubes are joined at the end and form one tube P with the same diameter as the tubes P_1 and P_2. Assume that the fluid A has a density ρ_1 in P_1 and ρ_2 in P_2. Let f_{ρ_1} and f_{ρ_2} be the probability density functions for ρ_1 and ρ_2 (*resp.*). We are interested in a computation of the probability density function f for the density ρ of the fluid A in a joined tube P. We prefer, however, that f should be smaller than a fixed $\alpha \in [0,1]$. Our task is to build a weak probabilistic model for this preference.

Sketch of the solution: The densities ρ_1, ρ_2 of the fluid A are (together) independent random variables, hence the new density $\rho = \rho_1 + \rho_2$ and the new probability density function f for $\rho_1 + \rho_2$ is defined as follows:

$$f(x) = \int_0^x f_{\rho_1}(\rho)f_{\rho_2}(x-\rho)d\rho \quad (9)$$

Because (12) has to be smaller than α (in accordance with the preference of the physicists) we obtain:

$$\textit{Pref(x)} : f(x) = \int_0^x f_{\rho_1}(\rho)f_{\rho_2}(x-\rho)d\rho < \alpha(\alpha \in [0,1]) \quad (10)$$

as a condition for the considered preference and after reformulation:

$$f(x)_{\textit{Pref}} = \frac{1}{\alpha}\int_0^x f_{\rho_1}(\rho)f_{\rho_2}(x-\rho)d\rho < 1 \quad (11)$$

Let moreover $F(x)_{\textit{Pref}}$ - defined as follow:

$$F(x)_{\textit{Pref}} = \frac{1}{\alpha}I_0^x f_{\rho_1}(\rho)f_{\rho_2}(x-\rho)d\rho < 1 \quad (12)$$

be a semantic integral representing $f(x)_{\textit{Pref}}$ in a newly constructed weak possibilistic model **M**. (12) ensures that the condition $F(x_{\textit{Pref}}) \in [0,1]$. After having the appropriate computation we can obtain from (12) a condition for a domain, say $|M|$. which must be countable, thus we take–for any case–a set $|M| \bigcap Q$ as our domain of **M**. In result, the required model will be as follows: $\langle |M| \bigcap Q, f_{\rho_1}(t), f_{\rho_2}, \frac{1}{\alpha}I_0^x f_{\rho_1}(\rho)f_{\rho_2}(x-\rho)d\rho, \mathbf{0}, \mathbf{1}, \star\rangle$.

5 DISCUSSION, CONCLUSIONS AND FUTURE WORK

The new complete and undecidable extensions of Pavelka-Hajek logic for convolutions has just been given. It has been also explained why *FLI* could be recognized as the logic for preferences.

It seems to be promising to expand the *FLI* towards a system able to express the problems of automatic generation of hypotheses based on empirical data (GUHA). Finally, *FLI* seems to be implementable inso-called π-*calculus*.

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