A note on time-indexed formulations for the resource-constrained project scheduling problem

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Abstract

The classical time-indexed 0–1 linear programming formulations for the resource-constrained project scheduling problem involve binary variables indicating whether an activity starts precisely at or before a given time period. In the literature, references to less classical "on/off" formulations, that involve binary variables indicating whether an activity is in progress during a time period, can also be found. These formulations were not compared to the classical ones in terms of linear programming (LP) relaxations. In this paper we show that the previously proposed on/off formulations are weaker than the classical formulations and we obtain a stronger on/off formulation via non singular transformations of the classical formulations. We also remark that additional time-indexed formulations, presented as appealing in the literature, are in fact either weaker or equivalent to the classical ones.

1 Introduction

Time-indexed, also called discrete-time integer linear programming (ILP) formulations have been widely studied for single machine, parallel machine and resource-constrained scheduling problems [8, 9, 3, 10, 11, 12]. This is due to their relatively strong LP relaxations and to their ability for being extended to various constraints and objectives. In this note, we consider the resource-constrained project scheduling problem (RCPSP), see e.g. [2]. In this problem, $V = \{0, 1, \dots, n, n+1\}$ denotes a set of $n+2$ activities. Each activity $i \in V$ has a duration p_i, with $p_0 = p_{n+1} = 0$. There is a set E of precedence constraints and a set $\mathcal{R}$ of resources. Each resource has a constant availability R_k. Each activity i requires $r_{ik} \geq 0$ units of each resource $k \in \mathcal{R}$ during all its processing time. The RCPSP consists in assigning a start time $S_i \geq 0$ to each activity $i \in V$ such that the total project duration, or makespan, is minimized while precedence and resource constraints are satisfied, which can be conceptually formulated as follows

$$\min S_{n+1} \tag{1}$$
$$S_j - S_i \geq p_i \qquad \forall (i,j) \in E \tag{2}$$
$$\sum_{i \in V | S_i \leq t < S_i + p_i} r_{ik} \leq R_k \qquad \forall t \geq 0 \tag{3}$$
$$S_i \geq ES_i \qquad \forall i \in V \tag{4}$$
$$S_i \leq LS_i \qquad \forall i \in V \tag{5}$$

Under the hypothesis that $(i, n+1) \in E$, $\forall i \in V \setminus \{n+1\}$ and $p_{n+1} = 0$, S_{n+1} represents the makespan. $ES_i \geq 0$ denotes the earliest start time of activity i while $LS_i \geq ES_i + p_i$ denotes its latest start time. If they are not part of the input data, these values can be obtained via longest path computations. Let UB denote an uper bound on the makespan. Assuming that $(0, i) \in E$, $\forall i \in V \setminus \{0\}$ and $p_0 = 0$, ES_i can be set ot the length of the longest path between 0 and i in graph $G(V, E)$ where each “precedence edge” $(i, j) \in E$ is valuated by p_i. Symmetrically, LS_i can be set to UB minus the length of the longest path between i and $(n + 1)$ in the same graph. We will also use notation $LC_i = LS_i + p_i$ for the latest completion time of activity i and $EC_i = ES_i + p_i$ for the earliest completion time of activity i.

We consider the scheduling horizon as a set $H = \{0, \ldots, UB\}$ of consecutive integer values starting with $t = 0$. If all data are integer, the set of start times can be restricted to H and constraints (4,5) can be replaced by

$$S_i \in H \cap [ES_i, LS_i] \quad \forall i \in V \tag{6}$$

We identify time t with the time period defined by interval $[t, t + 1)$. Hence, by convention, we mean that an activity is in progress at time t if its start time verifies $S_i \leq t$ and $S_i + p_i \geq t + 1$. An activity starts at time t iff $S_i = t$ and an activity completes at time t if $S_i + p_i = t$.

Time-indexed formulations are ILP formulations that involve time-indexed (or discrete time) variables of type v_{it} indicating a particular status of activity i at time t (generally started, completed or in progress).

In section 2 we briefly recall the well-known results on time-indexed pulse and step formulations. In Section 3, we present a time-indexed on-off formulation obtained from the others by non singular transformations. We show that the formulation is stronger than the ones previously proposed in the literature and based on the same type of variables. Section 4 makes additional remarks on the strengths of other related time-indexed formulations encountered in the project scheduling litterature. Concluding remarks are drawn in section 5.

2 Known results on step and pulse time-indexed formulations

The more standard time-indexed formulation for the RCPSP ([9, 3]) is based on binary variable x_{it}, $\forall i \in V$, $\forall t \in H$ such that $x_{it} = 1$ iff activity i starts at time t. As for a given activity, all variables x_{it} are equal to 0, except for time $t = S_i$, we refer to this type of variables as pulse variables. The basic discrete time formulation based on pulse start variables (DT) then comes as follows:

$$(DT)\min \quad \sum_{t \in H} t x_{n+1,t} \tag{7}$$

$$\sum_{t \in H} t x_{jt} - \sum_{t \in H} t x_{it} \geq p_i \qquad \forall i \in V, j \in V \setminus \{i\} \tag{8}$$

$$\sum_{i \in V} \sum_{\tau = t - p_i + 1}^{t} r_{ik} x_{i\tau} \leq R_k \qquad \forall t \in H, \forall k \in \mathcal{R} \tag{9}$$

$$\sum_{t \in H} x_{it} = 1 \qquad \forall i \in V \tag{10}$$

$$x_{it} = 0 \qquad \forall i \in V, \forall t \in H^+ \setminus \{ES_i, \ldots, LS_i\} \tag{11}$$

$$x_{it} \in \{0, 1\} \qquad \forall i \in V, \forall t \in \{ES_i, \ldots, LS_i\} \tag{12}$$

Objective (7) minimizes the makespan. Constraints (8) are the precedence constraints. They are a direct translation of constraints (2), observing that, according to the above-definition of the pulse variables, $S_i = \sum_{t \in H} t x_{it}$. Constraints (9) are the resource constraints, translating conceptual resource constraints (3) as, since start time are integer, i is in process at time t iff i starts at a time $\tau \in H \cap [t - p_i + 1, t]$. Constraints (10) state that each activity can only be started once in the scheduling horizon. Constraints (11) set to 0 all variables outside of set $H^+ \cap [ES_i, LS_i]$ where H^+

is defined, for ease of notation, as the extension of H to a sufficient set of negative integers. (12) defines the pulse decision variables.

A formulation having a stronger LP relaxation was proposed by [3]. It aims at replacing constraints (8) by the so-called disaggregated precedence constraints (13).

$$\sum_{\tau=0}^{t-p_i} x_{i\tau} - \sum_{\tau=0}^{t} x_{j\tau} \geq 0 \quad \forall (i,j) \in Er, \forall t \in H \tag{13}$$

These constraints simply model that for each t, $S_j \leq t \Rightarrow S_i \leq t - p_i$.

The disaggregated discrete time formulation based on pulse start variables (DDT) is then obtained by replacing constraints (8) by constraints (13) in formulation (7–12). The formulation is not weaker than (DT) since constraints (8) are implied by constraints (10) and (13) for $0 \leq x_{it} \leq 1$, $\forall i \in V, \forall t \in H$. Moreover, (DDT) has nice properties, for reasons that will be clear below.

We consider another formulation, based now on binary variables ξ_{it} such that $\xi_{it} = 1$ iff activity i starts at time t or before. For a given activity, variable ξ_{it} such that $t < S_i$ are all equal to 0 while the variables such that $t \geq S_i$ are all equal to one. Hence we refer to these type of variables as step variables. With these definitions, the start time can be expressed as:

$$S_i = \sum_{t \in H} t(\xi_{it} - \xi_{i,t-1}) \tag{14}$$

We present only the disaggregated variant of the discrete time formulation based on step variables (SDDT), which can be written:

$$(SDDT) \quad \min \quad \sum_{t \in H} t(\xi_{n+1,t} - \xi_{n+1,t-1}) \tag{15}$$

$$\xi_{i,t-p_i} - \xi_{jt} \geq 0 \qquad \forall (i,j) \in E, \forall t \in H \tag{16}$$

$$\sum_{i \in V} r_{ik}(\xi_{it} - \xi_{i,t-p_i}) \leq R_k \qquad \forall t \in H, \forall k \in \mathcal{R} \tag{17}$$

$$\xi_{i,LS_i} = 1 \qquad \forall i \in V \tag{18}$$

$$\xi_{i,t} - \xi_{i,t-1} \geq 0 \qquad \forall i \in V, \forall t \in H \tag{19}$$

$$\xi_{it} = 0 \qquad \forall i \in V, \forall t \in H^+, t \leq ES_i - 1 \tag{20}$$

$$\xi_{it} \in \{0,1\} \qquad \forall i \in V, \forall t \in \{ES_i, ..., LS_i - 1\} \tag{21}$$

Objective (15) is directly obtained by replacing the start time variable by its expression in function of ξ (14). Disaggregated precedence constraints (16) state that if an activity j is started at time t of before (i.e. $\xi_{jt} = 1$), then activity i has also to be started at time $t - p_i$ or before, which yields Resource constraints (17) follow from the the fact that an activity is in progress at time t iff $\xi_{it} - \xi_{i,t-p_i} = 1$. Indeed, if $t \in [ES_i, ES_i + p_i - 1]$, we have $\xi_{i,t-p_i} = 0$ by definition and i is in progress at time t if and only if $\xi_{it} = 1$. Otherwise, i is in progress at time t iff it has been started at time t but not at time $t - p_i$. Constraints (18) state that each activity has to be started at or before its latest start time LS_i. Constraints (19) define the step function, together with constraints (18). Note that these constraints also set to 1 all variables x_{it} with $t \geq LS_i$. Constraints (20) are just here for ease of notation as noted above, to set to 0 all variables x_{it} with $t < LS_i$. Finally constraints (21) defines the binary step variables.

Although it is presented as new in [6], the step formulation was already presented by Pritsker and Watters [8] and theoretically studied and compared to the pulse formulation by Souza and Wolsey [12] and Sankaran *el al* [11]. This has been also underlined in [7]. If we omit resource constraints in the (SDDT) formulation and if we relax integrality constraints, i.e. considering only constraints (16, 18–20), and $0 \leq \xi_{it} \leq 1$, $\forall i \in V, t \in H$, Souza and Wolsey [12] and Sankaran *el al* [11] observed that the constraint matrix satisfies a sufficient unimodularity condition. It follows from this observation that, without resource-constraints the solution of the LP relaxation of (SDDT) is 0-1. So is the solution of the LP relaxation of (DDT), according to the following remark, made in [12, 11]. The (SDDT) formulation can be obtained directly by applying the following non singular transformation to (DDT) formulation. For all $t \in H$, we have $x_{it} = \xi_{it} - \xi_{it-1}$. Conversly, the inverse transformation defines $\xi_{it} = \sum_{\tau=0}^{t} x_{it}$ and gives the (DDT) formulation from the (SDDT) formulation. Note that, in both cases, this transformation does not change the value of the LP-relaxation. An aggregated discrete-time formulation based on step variables (SDT) could also be defined this way. (DT) and (SDT) formulations are also equivalent, for the same reason, but yield weaker relaxations as fractional solutions can be obtained by solving the LP relaxations without resource constraints [7, 11].

3 The discrete time formulation based on on/off variables

Any non-singular (linear) transformation can be applied on the above-defined formulations to obtain an equivalent one. In this section; we consider "on/off" binary variables μ_{it} where $\mu_{it} = 1$ if activity i is in progress at time t and 0 otherwise. According to our definition of the "in progress" status, an activity with zero duration cannot be in progress, so we will treat activities with zero duration separately. Our model is based on the following observations on the relations between binary variables x_{it}, ξ_{it} and μ_{it}.

As already observed while writing the resource constraints for the (DT), (DDT), (SDT) and (SDDT) models, if $p_i \geq 1$, an activity $i \in V$ is in progress at time $t \in H$ iff $\xi_{it} - \xi_{i,t-p_i} = 1$ and, equivalently, if $\sum_{\tau=t-p_i+1}^{t} x_{i\tau} = 1$. So, for any activity $i \in V$ such that $p_i \geq 1$ and for any time $t \in H$ we define the non singular transformations $\mu_{it} = \xi_{it} - \xi_{i,t-p_i}$ and $\mu_{it} = \sum_{\tau=t-p_i+1}^{t} x_{i\tau}$. To obtain the inverse transformation for ξ_{it} we sum all $\mu_{i\tau}$ for $\tau = t - kp_i$ and $k = 0, \ldots, \lfloor t/p_i \rfloor$, which gives

$$\xi_{it} = \sum_{k=0}^{\lfloor t/p_i \rfloor} \mu_{i,t-kp_i}$$

which means that i is started at t ot before iff it is in progress at time $t - kp_i$ for some $k \in \mathbb{N}$. Futhermore, as $x_{it} = \xi_{it} - \xi_{it-1}$ we obtain the inverse transformation for x_{it},

$$x_{it} = \sum_{k=0}^{\lfloor t/p_i \rfloor} \mu_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} \mu_{i,t-kp_i-1}$$

The start time S_i is then equal to

$$S_i = \sum_{t \in H} t \left(\sum_{k=0}^{\lfloor t/p_i \rfloor} \mu_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} \mu_{i,t-kp_i-1} \right)$$

Considering the particular cases where $p_i = 0$, we change the definitions above with $\xi_{it} = \mu_{it}$. So, by defining $K_{it} = 0$ if $p_i = 0$ and $K_{it} = \lfloor t/p_i \rfloor$ otherwise, we have

$$\xi_{it} = \sum_{k=0}^{K_{it}} \mu_{i,t-kp_i} \text{ and } x_{it} = \sum_{k=0}^{K_{it}} \mu_{i,t-kp_i} - \sum_{k=0}^{K_{i,t-1}} \mu_{i,t-kp_i-1}$$

We have defined a non singular transformation. Substituting variables x_{it} by variables μ_{it} in formulation (DDT), we obtain the formulation (OODDT) below.

$$(OODDT) \min \quad \sum_{t \in H} t(\mu_{n+1,t} - \mu_{n+1,t-1}) \tag{22}$$

$$\sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-(k+1)p_i} - \sum_{k=0}^{K_{jt}} \mu_{j,t-kp_j} \geq 0 \qquad \forall (i,j) \in E, \forall t \in H \tag{23}$$

$$\sum_{i \in V, p_i > 0} r_{ik} \mu_{it} \leq R_k \qquad \forall t \in H, \forall k \in \mathcal{R} \tag{24}$$

$$\sum_{k=0}^{K_{i,LC_i-\phi(i)}} \mu_{i,LC_i-\phi(i)-kp_i} = 1 \qquad \forall i \in V \tag{25}$$

$$\sum_{k=0}^{K_{it}} \mu_{i,t-kp_i} - \sum_{k=0}^{K_{i,t-1}} \mu_{i,t-kp_i-1} \geq 0 \qquad \forall i \in V, \forall t \in H \setminus \{0\} \tag{26}$$

$$\mu_{it} = 0 \qquad \forall i \in V, \forall t \in Z'(i) \tag{27}$$

$$\mu_{it} \in \{0,1\} \qquad \forall i \in V, \forall t \in U'(i) \tag{28}$$

Contraints (23) are the disaggregated precedence constraints, given the expression of start time variables S_i in function of on/off variables $\mu_{i,t}$. Constraints (24) are the resource constraints, which have here a simple expression due to the on/off variables. We define $\phi(i) = 1$ if $p_i \geq 1$ and $\phi(i) = 0$ if $p_i = 0$. With this definition, constraints (25) state that each activity such that $p_i \geq 1$ has to be in progress in exactly one time period among time periods $t = LC_i - 1$, $t = LC_i - 1 - p_i$, $t = LC_i - 1 - 2p_i$.... For activities such that $p_i = 0$, the constraints resorts to constraints (18) as $\mu_{it} = \xi_{it}$. Constraints (26) are obtained by substitution of constraints (19) in (SDDT), or, equivalently, of constraints $x_{it} \geq 0$ on (DDT). They ensure, together with constraints (25) that exactly p_i consecutive variables will be switched-on, i.e. in a non-preemptive fashion (see Theorem 1). Constraints (27) set dummy variables to 0. We define the set of zero variables $Z'(i) = H^+ \setminus \{ES_i, ..., LC_i - 1\}$ if $p_i = 1$ and $Z'(i) = \{t \in H^+ | t \leq ES_i - 1\}$ if $p_i = 0$. Constraints (28) define the binary variables. We define the set of undetermined variables as $U'(i) = \{ES_i, ..., LC_i - 1\}$ if $p_i = 1$ and $U'(i) = \{ES_i, ..., LS_i - 1\}$ if $p_i = 0$.

Klein [6] presented a variant of the on/off formulation based on the formulation of Kaplan [4] for the preemptive RCPSP. This formulation looks like $(OODDT)$ with the following differences. No tasks with duration 0 are allowed. Precedence constraints are replaced by constraints (29). Non-preemption/duration constraints (25–26) are replaced by duration constraints (30) and non-preemption constraints (31).

$$p_i \mu_{jt} - \sum_{q=ES_i}^{t-1} \mu_{it} \leq 0 \quad \forall (i,j) \in E, \forall t \in \{ES_j, \ldots, LC_i - 1\} \tag{29}$$

$$\sum_{t=ES_i}^{LC_i-1} \mu_{it} = p_i \quad \forall i \in V \tag{30}$$

$$p_i(\mu_{it} - \mu_{i,t+1}) - \sum_{q=t-p_i+1}^{t-1} \mu_{iq} \leq 1 \quad \forall i \in V, \forall t \in \{ES_i, \ldots, LC_i - 2\} \tag{31}$$

Precedence constraints (29) state that for an activity j to be in progress at time t, its predecessor i must have been entirely processed during interval $[ES_i, t]$. Duration constraints (30) are straightforward. Non-preemption constraints 31 model the fact that if an activity completes at time $t+1$, in which case the term in factor of p_i is equal to one, the $p_i - 1$ units that precede t must be switched-on. As (OODDT), (SDDT) and (DDT) can be obtained from each other via non singular transformations, they are strictly equivalent and yield the same LP relaxation.

Demeulemeester and Herroelen [5] present another variant. As explained below, the formulation has a slight mistake in the constraints range and we provide here a corrected version, replacing the precedence constraints (29) by exclusive constraints (32) and (33), that are in fact stronger than the Klein's ones. Let (KF) denote this formulation.

$$\mu_{jt} \leq \mu_{i,t-p_i} \quad \forall i \in V, j \in V \setminus \{i\}, \forall t \in H, t \leq EC_i - 1 + p_i \tag{32}$$

$$\mu_{jt} \leq \sum_{q=ES_i+p_i-1}^{t-p_i} \mu_{iq} \quad \forall i \in V, j \in V \setminus \{i\}, \forall t \in H, t \geq EC_i + p_i \tag{33}$$

Constraints (32) state that, to be in progress at time t, an activity j must have its predecessor in process at time $t - p_i$ for any t such that $t - p_i$ falls strictly before the earliest end time of i, $EC_i = ES_i + p_i$. Indeed, if i starts at $t - p_i$ (which allows j to be in process at t) or before, i is necessarily in progress at time $t - p_i$ (i.e. $S_i \leq t - p_i \Leftrightarrow i$ is in progress at time $t - p_i$). If on the other hand, $t - p_i$ exceeds the earliest completion time of i, constraints

(33) state that activity j can only be in progress at time t if its predecessor starts at $t - p_i$ or before which means that it has to be in progress on at least one time period between $ES_i + p_i - 1$ and $t - p_i$.

As already mentioned, there is a slight mistake in the constraint given in [5] as $ES_i + p_i$ was replaced by ES_j in the range of constraints (32) and (33). In the case $ES_i + p_i < ES_j$ this can lead to overconstraining the start time of j. Suppose two activities i and j with $(i, j) \in E$, $p_i = 6$, $ES_i = 0$, $LC_i = 10$, $ES_j = 5$, $LC_i = 16$ and $p_i = 4$. At time $t = 8$, since $t - p_i = 4$ is not strictly before $EC_i = ES_i + p_i = 4$, we are in the range of constraints (33), so j can be in progress at time t either if i is in progress at time $t' = 4$ or $t' = 3$. If we use $ES_j = 5$ instead of $ES_i + p_i = 4$ in the constraints range, $t - p_i$ is strictly before ES_j, which falls in the range of constraints (32) stating that j can be in progress at time t only if i is in progress at time $t' = 4$. This clearly overconstrains the start time of j.

We now focus on the relative strengths of the proposed (OODDT) formulation and the (KF) formulation. We restrict to instances where no activity has a zero duration, otherwise (KF) cannot be used.

Theorem 1. *Formulation (OODDT) is stronger than formulation (KF).*

Proof. Let us consider consider the LP relaxations of (OODDT) and (KF). We first show that constraints (30) are implied by constraints (25) and (26). Let us first rewrite

$$\begin{aligned}
\sum_{t \in H} \mu_{it} &= \sum_{t=ES_i}^{LC_i - 1} \mu_{it} \\
&= \sum_{k=0}^{K_{i,LC_i-1}} \mu_{i,LC_i-1-kp_i} + \sum_{k=0}^{K_{i,LC_i-2}} \mu_{i,LC_i-2-kp_i} + \ldots + \sum_{k=0}^{K_{i,LC_i-p_i}} \mu_{i,LC_i-p_i-kp_i} \\
&= \sum_{k=0}^{K_{i,LC_i-1}} \mu_{i,LC_i-1-kp_i} + \sum_{k=0}^{K_{i,LC_i-1+1}} \mu_{i,LC_i-1+1-kp_i} + \ldots \\
&\quad + \sum_{k=0}^{K_{i,LC_i-1+p_i}} \mu_{i,LC_i-1+p_i-kp_i} \geq p_i,
\end{aligned}$$

which is obtained by remarking that, since $\mu_{i,t} = 0$ for $t \geq LC_i$, for any $x < p_i$, we have $\sum_{k=0}^{K_{i,LC_i-1-x}} \mu_{i,LC_i-1-x-kp_i} = \sum_{k=0}^{K_{i,LC_i-1-x+p_i}} \mu_{i,LC_i-1-x+p_i-kp_i}$. Constraints (25) imply that the first term of the expression is equal to 1.

Constraints (26) recursively imply that each subsequent term is larger than the preceding one, which yields the desired inequality.

Let us now show that constraints (31) are also implied by constraints (25) and (26), given bound constraints $0 \leq \mu_{it} \leq 1$, $\forall i, t$. Suppose that a matrix μ verifies (25, 26) and the bound constraints. For any t, if $\mu_{it} \leq \mu_{i,t+1}$, constraints (31) are always satisfied. The same holds if $p_i = 1$. Suppose that i and t are such that $\mu_{it} > \mu_{i,t+1}$ and $p_i > 1$. Writing constraints (26) for $t+1$, we obtain

$$\sum_{k=0}^{K_{i,t+1}} \mu_{i,t+1-kp_i} \geq \sum_{k=0}^{K_{it}} \mu_{i,t-kp_i}$$

which yields, by extracting $\mu_{i,t+1}$ and $\mu_{i,t}$ from the sums:

$$\sum_{k=0}^{K_{i,t-p_i+1}} \mu_{i,t-p_i+1-kp_i} \geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i}$$

Now by applying constraints (26) for $t - p_i + 1$, $t - p_i + 2$, $\ldots t - 1$ we have

$$\sum_{k=0}^{K_{i,t-1}} \mu_{i,t-1-kp_i} \geq \ldots \geq \sum_{k=0}^{K_{i,t-p_i+2}} \mu_{i,t-p_i+2-kp_i} \geq \sum_{k=0}^{K_{i,t-p_i+1}} \mu_{i,t-p_i+1-kp_i}$$

which implies from the preceding expression that

$$\begin{aligned}
\sum_{k=0}^{K_{i,t-1}} \mu_{i,t-1-kp_i} &\geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i} \\
&\ldots \\
\sum_{k=0}^{K_{i,t-p_i+2}} \mu_{i,t-p_i+2-kp_i} &\geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i} \\
\sum_{k=0}^{K_{i,t-p_i+1}} \mu_{i,t-p_i+1-kp_i} &\geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i}
\end{aligned}$$

Isolating terms $\mu_{i,t-1}, \ldots, \mu_{i,t-p_i+2}, \mu_{i,t-p_i+1}$ on the left hand side we get

$$\mu_{i,t-1} \geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i} - \sum_{k=0}^{K_{i,t-p_i-1}} \mu_{i,t-p_i-1-kp_i}$$

$$\ldots$$

$$\mu_{i,t-p_i+2} \geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i} - \sum_{k=0}^{K_{i,t-2p_i+2}} \mu_{i,t-2p_i+2-kp_i}$$

$$\mu_{i,t-p_i+1} \geq \mu_{i,t} - \mu_{i,t+1} + \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i} - \sum_{k=0}^{K_{i,t-2p_i+1}} \mu_{i,t-2p_i+1-kp_i}$$

Again, constraints (26) applied to $t - p_i$, $t - p_i - 1$, ..., $t - 2p_i + 2$ gives

$$\sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-p_i-kp_i} \geq \sum_{k=0}^{K_{i,t-p_i-1}} \mu_{i,t-p_i-1-kp_i} \geq \cdots \geq$$

$$\sum_{k=0}^{K_{i,t-2p_i+2}} \mu_{i,t-2p_i+2-kp_i} \geq \sum_{k=0}^{K_{i,t-2p_i+1}} \mu_{i,t-2p_i+1-kp_i}$$

which means that all terms after $\mu_{i,t} - \mu_{i,t+1}$ are non negative. Getting rid of these terms and summing up all inequalities we get

$$\sum_{q=t-p_i+1}^{t-1} \mu_{iq} \geq (\mu_{i,t} - \mu_{i,t+1})(p_i - 1)$$

and

$$\sum_{q=t-p_i+1}^{t-1} \mu_{iq} \geq p_i(\mu_{i,t} - \mu_{i,t+1}) - (\mu_{i,t} - \mu_{i,t+1})$$

Since $\mu_{i,t} - \mu_{i,t+1} \leq 1$ from bound constraints, we obtain constraint (31).

$$\sum_{q=t-p_i+1}^{t-1} \mu_{iq} \geq p_i(\mu_{i,t} - \mu_{i,t+1}) - 1$$

We now show that disaggregated precedence constraints (23) are not weaker than Kaplan's precedence constraints (32) and (33).

Consider a precedence constraint (i,j). Writing the disaggregated precedence constraints (23) we get

$$\sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-(k+1)p_i} - \sum_{k=0}^{K_{jt}} \mu_{j,t-kp_j} \geq 0 \implies \mu_{jt} \leq \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-(k+1)p_i}$$

We distinguish two cases. Consider the case where $t \leq EC_i - 1 + p_i$ Since $t - p_i \leq EC_i - 1$ any time $t-(k+1)p_i$ with $k \geq 1$ is strictly before ES_i and so $\sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-(k+1)p_i} = \mu_{i,t-p_i}$ which gives constraints (32). Consider now a time period $t \geq EC_i - 1 + p_i$. All non zero terms of expression $\sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-(k+1)p_i}$ such that $t \geq EC_i - 1$ are also included in expression $\sum_{q=EC_i-1}^{t-p_i} \mu_{iq}$. Furthermore, for the unique $k \geq 1$ such that $t-(k+1)p_i < EC_i - 1$, we have $\mu_{i,EC_i-1} \geq \mu_{i,t-(k+1)p_i}$ by constraints (26). Hence we obtain:

$$\mu_{jt} \leq \sum_{k=0}^{K_{i,t-p_i}} \mu_{i,t-(k+1)p_i} \leq \sum_{q=EC_i}^{t-p_i} \mu_{iq}$$

which yields constraints (33).

Consider now a simple instance with a single activity, no precedence constraints and no resource constraints such that $p_i = 2$, $ES_i = 0$ and $LC_i = H = 3$. Writing the LP relaxation of $(OODDT)$, we obtain:

$$\begin{aligned} \mu_{i0} + \mu_{i2} = 1 &\quad (25) \\ \mu_{i0} \geq 0 &\quad (26), t = 0 \\ \mu_{i1} - \mu_{i0} \geq 0 &\quad (26), t = 1 \\ \mu_{i2} + \mu_{i0} - \mu_{i1} \geq 0 &\quad (26), t = 2 \\ \mu_{i1} - \mu_{i2} - \mu_{i0} \geq 0 &\quad (26), t = 3 \\ 0 \leq \mu_{it} \leq 1 &\quad t = 0,1,2 \end{aligned}$$

Writing the LP relaxation of (KF), we obtain

$$\begin{aligned} \mu_{i0} + \mu_{i1} + \mu_{i2} = 2 &\quad (30) \\ 2\mu_{i0} - 2\mu_{i1} \leq 1 &\quad (31), t = 0 \\ 2\mu_{i1} - 2\mu_{i2} - \mu_{i0} \leq 1 &\quad (31), t = 1 \\ 0 \leq \mu_{it} \leq 1 &\quad t = 0,1,2 \end{aligned}$$

Consider now the fractional solution $\mu_{i0} = \mu_{i1} = \mu_{i2} = \frac{2}{3}$. We observe that this solution satisfies constraints (30) and (31). However, constraints (26) for $t = 2$ and $t = 3$ imply that $\mu_{i1} = \mu_{i2} + \mu_{i0}$, which is violated by the considered solution. $\square$

4 More discrete-time formulations

Klein [6] introduces a variant of the $(SDDT)$ formulation by introducing another step binary variable $\gamma_{it} = 1$ if i completes at time t of after. Observe that we have $\xi_{it} + \gamma_{it} - 1 = \mu_{it}$ for activities with non zero durations and $\xi_{it} + \gamma_{it} - 1 = x_{it}$ for activities with zero duration. Using these non-singular transformations we could obtain aggregated or disaggregated formulations based on γ_{it} variables and equivalent to the ones already presented. Klein [6] introduces a formulation that is weaker but that has an advantage when durations are decision variables. Indeed, in all formulations we presented so far, durations p_i have to be fixed parameters because they are present in the index of variables. Mixing ξ_{it} and γ_{it} allows to get rid of this drawback. We can modify the (SDT) model by adding the following constraints, defining the γ variables and establishing the link with the ξ variables,

$$\sum_{t=ES_i}^{LC_{i-1}} \xi_{it} + \gamma_{it} - 1 = p_i \qquad \forall i \in V \tag{34}$$

$$\gamma_{i,EC_i-1} = 1 \qquad \forall i \in V \tag{35}$$

$$\gamma_{i,t-1} - \gamma_{i,t} \geq 0 \qquad \forall i \in V, \forall t \in H \tag{36}$$

$$\gamma_{it} = 0 \qquad \forall i \in V, \forall t \in H^+, t \geq LC_i \tag{37}$$

$$\gamma_{it} \in \{0,1\} \qquad \forall i \in V, \forall t \in \{EC_i, ..., LC_i - 1\} \tag{38}$$

and replacing resource constraints (17) by

$$\sum_{i \in V} r_{ik}(\xi_{it} + \gamma_{it} - 1) \leq R_k \quad \forall t \in H, \forall k \in \mathcal{R} \tag{39}$$

Bianco and Caramia [1] propose a variant of the step formulation that involves the 0-1 start variable ξ_{it} and another $0-1$ variable ξ'_{it} which equals 1 iff activity i is completed at t or before. Even if it is not mentioned in [1], we have

$$\xi'_{it} = \xi_{t-p_i} \quad \forall i \in V, \forall t \in H \tag{40}$$

Another variable π_{it} is introduced, giving the fraction of activity i that has been performed up to time t. The following constraints are defined:

$$\pi_{it+1} - \pi_{it} = \frac{1}{p_i}(\xi_{it} - \xi'_{it}) \qquad \forall i \in V, \forall t \in H \tag{41}$$

$$\xi'_{it} \leq \pi_{it} \leq \xi_{it} \qquad \forall i \in V, \forall t \in H \tag{42}$$

$$\pi_{it} \geq 0 \qquad \forall i \in V, \forall t \in H \tag{43}$$

Note that if we introduce constraints (40–43) in the SDDT model we obtain a strictly equivalent formulation, as it can be shown that for any feasible value of variable ξ_{it} in the LP relaxation of SDDT, we can obtain values for variables ξ'_{it} immediately through constraints (40) and for variables π_{it} that satisfy (41–43).

The model proposed in [1] finally replaces resource constraints (17) by

$$\sum_{i \in V} r_{ik} p_i (\pi_{i,t+1} - \pi_{i,t}) \leq R_k \quad \forall i \in V, \forall t \in H \tag{44}$$

Remarking that $\pi_{i,t+1} - \pi_{i,t} = \frac{1}{p_i}(\xi_{it} - \xi'_{it}) = \frac{1}{p_i}(\xi_{it} - \xi_{it-d_i})$ we precisely obtain resource constraints (17) .

5 Concluding remarks

We have proposed an on/off time-indexed formulation (OODDT) for the RCPSP, equivalent to the well known pulse (DDT) and step (SDDT) time-indexed formulations in terms of LP-relaxation. The proposed formulation is stronger than the previously proposed on/off formulations. Other time indexed formulations were proposed in the literature without any mention of the relative strengths of their LP relaxations. We remarked that for two of them the LP relaxations are either weaker of equivalent to the three mentioned ones.

We have to acknowledge that the practical performance of a formulation, in terms of integer solving, is not necessarily related to the LP relaxation strength. It is well known that the weak formulation (DT) may outperform the strong formulation (DDT) on some instances. Bianco and Caramia [1] showed through extensive experiments that their formulation generally outperformed (SDDT) in terms of solution time and quality. The way constraints and/or additional redundant variables are introduced and formulated influences the solver performance in terms of memory usage, preprocessing, cutting plane generation and branching. This should not however hide the fact that, in any “new” formulation, constraints that are equivalent, via non singular transformations, to previously proposed ones should be identified and distinguished from actual cutting plane inequalities.

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