A Recursive Algorithm to Generate Balanced Weekend Tournaments

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Abstract

In this paper, we construct a Balanced Weekend Tournament, motivated by the real-life problem of scheduling an n-team double round-robin season schedule for a Canadian university soccer league. In this 6-team league, games are only played on Saturdays and Sundays, with the condition that no team has two road games on any weekend.

The implemented regular-season schedule for $n = 6$ was best-possible, but failed to meet an important "compactness" criterion, as the 10-game tournament required more than five weekends to complete. The motivation for this paper was to determine whether an optimal season schedule, satisfying all of the league's constraints on compact balanced play, could be constructed for sports leagues with $n > 6$ teams.

We present a simple recursive algorithm to answer this question for all even $n > 6$. As a corollary, our construction gives us an explicit solution to a challenging and well-known graph theory question, namely the problem of decomposing the complete directed graph K^*_{2m} into $2m-1$ directed Hamiltonian cycles of length $2m$.

Introduction

Sports scheduling has emerged as a growing field of AI research over the past two decades (Kendall et al. 2010), especially since the introduction of the Traveling Tournament Problem by the head schedulers of Major League Baseball (Easton, Nemhauser, and Trick 2001). Integer programming, constraint programming, metaheuristics, and hybrid methods have been successfully applied to solve complex problems in sports scheduling (Goerigk et al. 2014).

The majority of sports scheduling research deals with the organization and optimization of professional sports leagues, as these leagues generate worldwide audiences and are multi-billion dollar industries. However, a much larger percentage of individuals competing in sports leagues are students playing on varsity teams against other schools, with their games fixed in certain time slots (e.g. weekends).

The author is employed by a Canadian university competing in the Pacific Western Athletic Association (PACWEST), consisting of six soccer teams playing a season-long double round-robin tournament such as this:

Team	1	2	3	4	5	6	7	8	9	10
t_1	$\boldsymbol{t_6}$	t_3	$\boldsymbol{t_5}$	t_2	t_4	t_6	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	$\boldsymbol{t_4}$
t_2	$\boldsymbol{t_5}$	t_6	t_4	$\boldsymbol{t_1}$	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_6}$	$\boldsymbol{t_4}$	t_1	t_3
t_3	t_4	$\boldsymbol{t_1}$	$\boldsymbol{t_6}$	t_5	t_2	$\boldsymbol{t_4}$	t_1	t_6	$\boldsymbol{t_5}$	$\boldsymbol{t_2}$
t_4	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	t_6	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_6}$	t_1
t_5	t_2	$\boldsymbol{t_4}$	t_1	$\boldsymbol{t_3}$	t_6	$\boldsymbol{t_2}$	t_4	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_6}$
t_6	t_1	$\boldsymbol{t_2}$	t_3	$\boldsymbol{t_4}$	$\boldsymbol{t_5}$	$\boldsymbol{t_1}$	t_2	$\boldsymbol{t_3}$	t_4	t_5

Table 1: A 5-weekend 6-team double round robin tournament

Table 1 illustrates an example of a double round-robin tournament, where each team plays in $2(n-1)$ different time slots, with one home game and one road game against each of the other $n-1$ teams. Home games are marked in bold, e.g. t_1 is the home team versus t_6 in slot #1.

In the PACWEST league, the student-athletes on the $n = 6$ teams complete their regular-season games over $n-1 = 5$ weekends, due to academic priorities during the week that prevent weekday games from being scheduled.

The major weakness with this schedule is that four teams, namely t_1, t_2, t_3, t_6, have one weekend where they play two road games (e.g. t_1 in time slots #5 and #6). Playing back-to-back road games is problematic for the teams, given the academic and social commitments of the student-athletes on weekends. There is sometimes the additional cost of having to book a hotel room on the Saturday night, depending on the length of time it would take to make an additional back-and-forth trip in a single weekend.

To mitigate this issue, the PACWEST league approached the author, who had previously helped the Japanese professional baseball league reduce their carbon emissions by developing a regular-season schedule to minimize total travel distance (Hesse 2012).

The challenge for this sports scheduling problem was to have each team play $2(n-1)$ games over $(n-1)$ weekends where every team plays at most one road game in a single weekend, thus ensuring that the student-athletes could make a short trip on either the Saturday or Sunday, and return home immediately following the game. For obvious reasons, such a schedule is applicable to more than just an inter-university soccer league in British Columbia, Canada; given the popularity of soccer (i.e., football) throughout the world, such a schedule would be relevant to any n-team league playing back-to-back games on weekends.

The paper proceeds as follows: we first define the n-team Balanced Weekend Tournament Problem (BWTP). Though no solution exists for $n \leq 6$, we provide a recursive algorithm to generate a solution S_n to the BWTP for all even $n \geq 8$. As a corollary, we show that our construction provides an explicit solution to the Hamiltonian cycle decomposition problem of the complete directed graph K_n^*, which works for all even $n \geq 8$.

Inspired by the p-norm in functional analysis, we then define a similar p-norm to measure the effectiveness of a feasible solution, to show that some schedules are more "balanced" than others. We prove that our recursive construction generates an n-team schedule S_n for which $||S_n||_1 < \frac{7}{4}n^2$ and $||S_n||_p < n^{1+1/p}$ for $p \geq 2$, upper bounds that we conjecture are close to optimal. Finally, at the end of each section of this paper, we conclude with an open problem.

Problem Statement

The schedulers of the PACWEST league requested a double round-robin schedule satisfying the following constraints. Here are the criteria, in decreasing order of importance.

(a) *Each-Venue*: Each pair of teams plays twice, once in each other's home venue.
(b) *Each-Half*: Each pair of teams plays one game in the first half of the season, and one game in the second half.
(c) *No-Repeat*: A team cannot play against the same opponent in two consecutive games.
(d) *No-Two-Road*: Each team plays at most one road game each weekend.
(e) *Compactness*: The tournament takes place over $n - 1$ weekends, with two games each weekend.

We note that condition (e) requires n to be even, since every team must play one game in each time slot.

It is straightforward to create a schedule that satisfies all the conditions except for (c). To do this, we apply the well-known *canonical* schedule (de Werra 1981) to generate a single round-robin tournament for the first half of the season, and use the "English construction" (Kendall et al. 2010) to produce the second half. Here is the construction for $n = 6$.

Team	1	2	3	4	5	6	7	8	9	10
t_1	$\boldsymbol{t_6}$	t_3	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_4}$	t_4	t_6	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$
t_2	$\boldsymbol{t_5}$	t_6	t_4	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_6}$	$\boldsymbol{t_4}$	t_1
t_3	t_4	$\boldsymbol{t_1}$	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_2}$	t_2	$\boldsymbol{t_4}$	t_1	t_6	$\boldsymbol{t_5}$
t_4	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	t_6	t_1	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_6}$
t_5	t_2	$\boldsymbol{t_4}$	t_1	$\boldsymbol{t_3}$	$\boldsymbol{t_6}$	t_6	$\boldsymbol{t_2}$	t_4	$\boldsymbol{t_1}$	t_3
t_6	t_1	$\boldsymbol{t_2}$	t_3	$\boldsymbol{t_4}$	t_5	$\boldsymbol{t_5}$	$\boldsymbol{t_1}$	t_2	$\boldsymbol{t_3}$	t_4

Table 2: A feasible tournament violating the No-Repeat condition

A simple computer program proves that there exist no 6-team tournaments satisfying all five conditions.

Because of this, the PACWEST decided to adopt the almost-optimal 10-game 6-weekend schedule given in Table 3, which met all of the criteria except for (e). In this schedule, two weekends have only a Saturday game and no Sunday game, corresponding to slots 1 and 6.

Team	1	2	3	4	5	6	7	8	9	10
t_1	$\boldsymbol{t_4}$	$\boldsymbol{t_6}$	t_3	$\boldsymbol{t_5}$	t_2	t_4	t_6	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$
t_2	t_3	$\boldsymbol{t_5}$	t_6	t_4	$\boldsymbol{t_1}$	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_6}$	$\boldsymbol{t_4}$	t_1
t_3	$\boldsymbol{t_2}$	t_4	$\boldsymbol{t_1}$	$\boldsymbol{t_6}$	t_5	t_2	$\boldsymbol{t_4}$	t_1	t_6	$\boldsymbol{t_5}$
t_4	t_1	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	t_6	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_6}$
t_5	$\boldsymbol{t_6}$	t_2	$\boldsymbol{t_4}$	t_1	$\boldsymbol{t_3}$	t_6	$\boldsymbol{t_2}$	t_4	$\boldsymbol{t_1}$	t_3
t_6	t_5	t_1	$\boldsymbol{t_2}$	t_3	$\boldsymbol{t_4}$	$\boldsymbol{t_5}$	$\boldsymbol{t_1}$	t_2	$\boldsymbol{t_3}$	t_4

Table 3: The 2016-17 PACWEST Soccer Schedule

For each pair of teams t_i and t_j, let $h(i,j)$ be the slot in which t_i plays a home game against t_j, and $r(i,j)$ be the slot in which t_i plays a road game against t_j. Notice that $|h(i,j) - r(i,j)| = 5$ for all $1 \leq i < j \leq 6$. This is known as a *mirrored* schedule (Ribeiro and Urrutia 2004), and there has been much analysis done on the construction of n-team schedules that require $|h(i,j) - r(i,j)|$ to always equal $n-1$, for all (i,j). Later in the paper, we explain why a mirrored schedule cannot satisfy conditions (d) and (e).

While the PACWEST scheduling problem was solved and implemented, there was little satisfaction as no double round-robin schedule satisfied all five constraints. But inspired by this, we pose the Balanced Weekend Tournament Problem (BWTP), on whether there exists a feasible tournament satisfying these five conditions for $n > 6$.

As mentioned earlier, n must be even. Furthermore, since there are n teams with $\frac{n}{2}$ teams playing at home and $\frac{n}{2}$ teams playing on the road in any given time slot, condition (d) implies that each team must play exactly one home game and one road game every weekend.

A computer search generated numerous solutions to the BWTP for the case $n = 8$, including the following:

Team	1	2	3	4	5	6	7	8	9	10	11	12	13	14
t_1	$\boldsymbol{t_6}$	t_3	$\boldsymbol{t_8}$	t_2	t_4	$\boldsymbol{t_7}$	t_5	$\boldsymbol{t_3}$	t_6	$\boldsymbol{t_4}$	t_8	$\boldsymbol{t_2}$	t_7	$\boldsymbol{t_5}$
t_2	t_8	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_1}$	$\boldsymbol{t_6}$	t_5	t_7	$\boldsymbol{t_8}$	t_4	$\boldsymbol{t_3}$	$\boldsymbol{t_5}$	t_1	t_6	$\boldsymbol{t_7}$
t_3	t_5	$\boldsymbol{t_1}$	$\boldsymbol{t_2}$	t_4	t_7	$\boldsymbol{t_8}$	$\boldsymbol{t_6}$	t_1	$\boldsymbol{t_5}$	t_2	t_6	$\boldsymbol{t_7}$	$\boldsymbol{t_4}$	t_8
t_4	$\boldsymbol{t_7}$	t_2	t_5	$\boldsymbol{t_3}$	$\boldsymbol{t_1}$	t_6	t_8	$\boldsymbol{t_5}$	$\boldsymbol{t_2}$	t_1	t_7	$\boldsymbol{t_8}$	t_3	$\boldsymbol{t_6}$
t_5	$\boldsymbol{t_3}$	t_7	$\boldsymbol{t_4}$	t_6	t_8	$\boldsymbol{t_2}$	$\boldsymbol{t_1}$	t_4	t_3	$\boldsymbol{t_7}$	t_2	$\boldsymbol{t_6}$	$\boldsymbol{t_8}$	t_1
t_6	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_7}$	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	t_4
t_7	t_4	$\boldsymbol{t_5}$	$\boldsymbol{t_6}$	t_8	$\boldsymbol{t_3}$	t_1	$\boldsymbol{t_2}$	t_6	$\boldsymbol{t_8}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_1}$	t_2
t_8	$\boldsymbol{t_2}$	t_6	t_1	$\boldsymbol{t_7}$	$\boldsymbol{t_5}$	t_3	$\boldsymbol{t_4}$	t_2	t_7	$\boldsymbol{t_6}$	$\boldsymbol{t_1}$	t_4	t_5	$\boldsymbol{t_3}$

Table 4: S_8, a solution to the BWTP for $n = 8$

By inspection, we can check that all five criteria of the BWTP are satisfied in the above schedule. To produce this schedule, we enumerate all possible single round-robin tournaments (slots 1 to 7) and determine for each option all possible ways to assign home and away slots so that each team plays one home game and one road game each weekend. We then check whether there exists a complementary single round-robin schedule (slots 8 to 14) for which $h(i,j) \leq 7$ if and only if $r(i,j) \geq 8$ for all (i,j) with $1 \leq i < j \leq 8$.

In the next section, we describe two recursive algorithms that generate a solution to the BWTP for the cases $n = 2k$ and $n = 2k - 2$, given a feasible solution for $n = k$. Combined with the demonstration that there exist feasible solutions for the cases $n = 10$ and $n = 12$, the above two algorithms solve the BWTP for all even values of $n \geq 8$.

Generating S_{2k} and S_{2k-2} from S_k

Let $n = k$ be even. Suppose that S_k is a feasible solution to the BWTP, with teams $t_1, t_2, \ldots, t_k$. We now construct S_{2k}, a feasible solution to the BWTP for the case $n = 2k$. First we create a second copy of the k teams, and label them $u_1, u_2, \ldots, u_k$.

In slots $1 \leq s \leq k$, t_i plays against u_{s+i-1}, with all arithmetic calculated mod k. If s is odd, then t_i is the home team, and if s is even, then t_i is the away team. The schedule for slots $2k+1 \leq s \leq 3k$ is identical, except the home and away teams are switched.

In slots $k+1 \leq s \leq 2k$, we make two copies of the first k columns of S_k. Specifically, if t_i plays at t_j in slot s, then in our schedule S_{2k}, t_i plays at t_j in slot $s+k$, and u_i plays at u_j in slot $s+k$.

In slots $3k+1 \leq s \leq 4k-2$, we make two copies of the last $k-2$ columns of S_k. Specifically, if t_i plays at t_j in slot s, then in our schedule S_{2k}, t_i plays at t_j in slot $s+2k$, and u_i plays at u_j in slot $s+2k$.

To illustrate, Table 5 provides the schedule S_{16}, recursively generated from the schedule S_8 given in Table 4.

Team	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_1	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_8	$\boldsymbol{t_6}$	t_3	$\boldsymbol{t_8}$	t_2	t_4	$\boldsymbol{t_7}$	t_5
t_2	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_8}$	u_1	t_8	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_1}$	$\boldsymbol{t_6}$	t_5	t_7
t_3	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_8	$\boldsymbol{u_1}$	u_2	t_5	$\boldsymbol{t_1}$	$\boldsymbol{t_2}$	t_4	t_7	$\boldsymbol{t_8}$	$\boldsymbol{t_6}$
t_4	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_8}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{t_7}$	t_2	t_5	$\boldsymbol{t_3}$	$\boldsymbol{t_1}$	t_6	t_8
t_5	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_8	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{t_3}$	t_7	$\boldsymbol{t_4}$	t_6	t_8	$\boldsymbol{t_2}$	$\boldsymbol{t_1}$
t_6	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_8}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_4}$	t_3
t_7	$\boldsymbol{u_7}$	u_8	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	t_4	$\boldsymbol{t_5}$	$\boldsymbol{t_6}$	t_8	$\boldsymbol{t_3}$	t_1	$\boldsymbol{t_2}$
t_8	$\boldsymbol{u_8}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{t_2}$	t_6	t_1	$\boldsymbol{t_7}$	$\boldsymbol{t_5}$	t_3	$\boldsymbol{t_4}$
u_1	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	$\boldsymbol{u_6}$	u_3	$\boldsymbol{u_8}$	u_2	u_4	$\boldsymbol{u_7}$	u_5
u_2	t_2	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	u_8	$\boldsymbol{u_4}$	u_3	$\boldsymbol{u_1}$	$\boldsymbol{u_6}$	u_5	u_7
u_3	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	u_5	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	u_4	u_7	$\boldsymbol{u_8}$	$\boldsymbol{u_6}$
u_4	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	$\boldsymbol{u_7}$	u_2	u_5	$\boldsymbol{u_3}$	$\boldsymbol{u_1}$	u_6	u_8
u_5	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_6}$	$\boldsymbol{u_3}$	u_7	$\boldsymbol{u_4}$	u_6	u_8	$\boldsymbol{u_2}$	$\boldsymbol{u_1}$
u_6	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_7}$	u_1	$\boldsymbol{u_8}$	u_7	$\boldsymbol{u_5}$	u_2	$\boldsymbol{u_4}$	u_3
u_7	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_8}$	u_4	$\boldsymbol{u_5}$	$\boldsymbol{u_6}$	u_8	$\boldsymbol{u_3}$	u_1	$\boldsymbol{u_2}$
u_8	t_8	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	$\boldsymbol{u_2}$	u_6	u_1	$\boldsymbol{u_7}$	$\boldsymbol{u_5}$	u_3	$\boldsymbol{u_4}$

Team	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
t_1	$\boldsymbol{t_3}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_8}$	t_6	$\boldsymbol{t_4}$	t_8	$\boldsymbol{t_2}$	t_7	$\boldsymbol{t_5}$
t_2	$\boldsymbol{t_8}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_8	$\boldsymbol{u_1}$	t_4	$\boldsymbol{t_3}$	$\boldsymbol{t_5}$	t_1	t_6	$\boldsymbol{t_7}$
t_3	t_1	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_8}$	u_1	$\boldsymbol{u_2}$	$\boldsymbol{t_5}$	t_2	t_6	$\boldsymbol{t_7}$	$\boldsymbol{t_4}$	t_8
t_4	$\boldsymbol{t_5}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_8	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	$\boldsymbol{t_2}$	t_1	t_7	$\boldsymbol{t_8}$	t_3	$\boldsymbol{t_6}$
t_5	t_4	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_8}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	t_3	$\boldsymbol{t_7}$	t_2	$\boldsymbol{t_6}$	$\boldsymbol{t_8}$	t_1
t_6	$\boldsymbol{t_7}$	u_6	$\boldsymbol{u_7}$	u_8	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	t_4
t_7	t_6	u_7	$\boldsymbol{u_8}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	$\boldsymbol{t_8}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_1}$	t_2
t_8	t_2	u_8	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	t_7	$\boldsymbol{t_6}$	$\boldsymbol{t_1}$	t_4	t_5	$\boldsymbol{t_3}$
u_1	$\boldsymbol{u_3}$	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	u_6	$\boldsymbol{u_4}$	u_8	$\boldsymbol{u_2}$	u_7	$\boldsymbol{u_5}$
u_2	$\boldsymbol{u_8}$	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	u_4	$\boldsymbol{u_3}$	$\boldsymbol{u_5}$	u_1	u_6	$\boldsymbol{u_7}$
u_3	u_1	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{u_5}$	u_2	u_6	$\boldsymbol{u_7}$	$\boldsymbol{u_4}$	u_8
u_4	$\boldsymbol{u_5}$	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{u_2}$	u_1	u_7	$\boldsymbol{u_8}$	u_3	$\boldsymbol{u_6}$
u_5	u_4	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_8	$\boldsymbol{t_7}$	t_6	u_3	$\boldsymbol{u_7}$	u_2	$\boldsymbol{u_6}$	$\boldsymbol{u_8}$	u_1
u_6	$\boldsymbol{u_7}$	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_8}$	t_7	$\boldsymbol{u_1}$	u_8	$\boldsymbol{u_3}$	u_5	$\boldsymbol{u_2}$	u_4
u_7	u_6	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_8	$\boldsymbol{u_8}$	u_5	$\boldsymbol{u_4}$	u_3	$\boldsymbol{u_1}$	u_2
u_8	u_2	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	u_7	$\boldsymbol{u_6}$	$\boldsymbol{u_1}$	u_4	u_5	$\boldsymbol{u_3}$

Table 5: S_{16}, a solution to the BWTP for $n = 16$

We complete the construction by replacing each u_i with t_{k+i}, for each $1 \leq i \leq k$. This gives us our solution S_{2k}.

We now construct S_{2k-2}, a feasible solution to the BWTP for the case $n = 2k-2$. Once again, let $n = k$ be even. Suppose that S_k is a feasible solution to the BWTP. This time, the teams will be labelled $t_1, t_2, \ldots, t_{k-1}, u_1, u_2, \ldots, u_{k-1}$.

In slots $1 \leq s \leq k-2$, t_i plays against u_{s+i}, with all arithmetic calculated mod $k-1$. If s is odd, then t_i is the away team, and if s is even, then t_i is the home team. The schedule for slots $2k-1 \leq s \leq 3k-4$ is identical, except the home and away teams are switched.

In slots $k-1 \leq s \leq 2k-2$, we make two copies of the first k columns of S_k, with the bottom half teams (all the u_i's) inverting home and road positions. Specifically, if t_i plays at t_j in slot s, then in our schedule S_{2k-2}, t_i plays at t_j in slot $s+k-2$, and u_j plays at u_i in slot $s+k-2$.

In slots $3k-3 \leq s \leq 4k-6$, we make two copies of the last $k-2$ columns of S_k, with the bottom half teams inverting home and road positions once again. Specifically, if t_i plays at t_j in slot s, then in our schedule S_{2k-2}, t_i plays at t_j in slot $s+2k-4$, and u_j plays at u_i in slot $s+2k-4$.

Finally, for each $1 \leq i \leq k-1$, whenever t_i is matched up against t_k, replace t_k with u_i. Similarly, whenever u_i is matched up against u_k, then replace u_k with t_i.

To illustrate, Table 6 provides the schedule S_{14}, recursively generated from the schedule S_8 given in Table 4.

Team	1	2	3	4	5	6	7	8	9	10	11	12	13
t_1	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	$\boldsymbol{t_6}$	t_3	$\boldsymbol{u_1}$	t_2	t_4	$\boldsymbol{t_7}$	t_5
t_2	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_1}$	u_2	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_1}$	$\boldsymbol{t_6}$	t_5	t_7
t_3	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_1	$\boldsymbol{u_2}$	t_5	$\boldsymbol{t_1}$	$\boldsymbol{t_2}$	t_4	t_7	$\boldsymbol{u_3}$	$\boldsymbol{t_6}$
t_4	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	$\boldsymbol{t_7}$	t_2	t_5	$\boldsymbol{t_3}$	$\boldsymbol{t_1}$	t_6	u_4
t_5	u_6	$\boldsymbol{u_7}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	$\boldsymbol{t_3}$	t_7	$\boldsymbol{t_4}$	t_6	u_5	$\boldsymbol{t_2}$	$\boldsymbol{t_1}$
t_6	u_7	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	t_1	$\boldsymbol{u_6}$	t_7	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_4}$	t_3
t_7	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	t_4	$\boldsymbol{t_5}$	$\boldsymbol{t_6}$	u_7	$\boldsymbol{t_3}$	t_1	$\boldsymbol{t_2}$
u_1	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	u_6	$\boldsymbol{u_3}$	t_1	$\boldsymbol{u_2}$	$\boldsymbol{u_4}$	u_7	$\boldsymbol{u_5}$
u_2	$\boldsymbol{t_1}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	u_4	$\boldsymbol{u_3}$	u_1	u_6	$\boldsymbol{u_5}$	$\boldsymbol{u_7}$
u_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{u_5}$	u_1	u_2	$\boldsymbol{u_4}$	$\boldsymbol{u_7}$	t_3	u_6
u_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_7	$\boldsymbol{t_6}$	t_5	u_7	$\boldsymbol{u_2}$	$\boldsymbol{u_5}$	u_3	u_1	$\boldsymbol{u_6}$	$\boldsymbol{t_4}$
u_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_7}$	t_6	u_3	$\boldsymbol{u_7}$	u_4	$\boldsymbol{u_6}$	$\boldsymbol{t_5}$	u_2	u_1
u_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_7	$\boldsymbol{u_1}$	t_6	$\boldsymbol{u_7}$	u_5	$\boldsymbol{u_2}$	u_4	$\boldsymbol{u_3}$
u_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{u_4}$	u_5	u_6	$\boldsymbol{t_7}$	u_3	$\boldsymbol{u_1}$	u_2

Team	14	15	16	17	18	19	20	21	22	23	24	25	26
t_1	$\boldsymbol{t_3}$	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	t_6	$\boldsymbol{t_4}$	u_1	$\boldsymbol{t_2}$	t_7	$\boldsymbol{t_5}$
t_2	$\boldsymbol{u_2}$	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_1	t_4	$\boldsymbol{t_3}$	$\boldsymbol{t_5}$	t_1	t_6	$\boldsymbol{t_7}$
t_3	t_1	$\boldsymbol{u_4}$	u_5	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_1}$	u_2	$\boldsymbol{t_5}$	t_2	t_6	$\boldsymbol{t_7}$	$\boldsymbol{t_4}$	u_3
t_4	$\boldsymbol{t_5}$	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{t_2}$	t_1	t_7	$\boldsymbol{u_4}$	t_3	$\boldsymbol{t_6}$
t_5	t_4	$\boldsymbol{u_6}$	u_7	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	t_3	$\boldsymbol{t_7}$	t_2	$\boldsymbol{t_6}$	$\boldsymbol{u_5}$	t_1
t_6	$\boldsymbol{t_7}$	$\boldsymbol{u_7}$	u_1	$\boldsymbol{u_2}$	u_3	$\boldsymbol{u_4}$	u_5	$\boldsymbol{t_1}$	u_6	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_2}$	t_4
t_7	t_6	$\boldsymbol{u_1}$	u_2	$\boldsymbol{u_3}$	u_4	$\boldsymbol{u_5}$	u_6	$\boldsymbol{u_7}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_1}$	t_2
u_1	u_3	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	$\boldsymbol{u_6}$	u_4	$\boldsymbol{t_1}$	u_2	$\boldsymbol{u_7}$	u_5
u_2	t_2	t_1	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	$\boldsymbol{u_4}$	u_3	u_5	$\boldsymbol{u_1}$	$\boldsymbol{u_6}$	u_7
u_3	$\boldsymbol{u_1}$	t_2	$\boldsymbol{t_1}$	t_7	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$	u_5	$\boldsymbol{u_2}$	$\boldsymbol{u_6}$	u_7	u_4	$\boldsymbol{t_3}$
u_4	u_5	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_5}$	u_2	$\boldsymbol{u_1}$	$\boldsymbol{u_7}$	t_4	$\boldsymbol{u_3}$	u_6
u_5	$\boldsymbol{u_4}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_7	$\boldsymbol{t_6}$	$\boldsymbol{u_3}$	u_7	$\boldsymbol{u_2}$	u_6	t_5	$\boldsymbol{u_1}$
u_6	u_7	t_5	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_7}$	u_1	$\boldsymbol{t_6}$	u_3	$\boldsymbol{u_5}$	u_2	$\boldsymbol{u_4}$
u_7	$\boldsymbol{u_6}$	t_6	$\boldsymbol{t_5}$	t_4	$\boldsymbol{t_3}$	t_2	$\boldsymbol{t_1}$	t_7	$\boldsymbol{u_5}$	u_4	$\boldsymbol{u_3}$	u_1	$\boldsymbol{u_2}$

Table 6: S_{14}, a solution to the BWTP for $n = 14$

We complete the construction by replacing each u_i with t_{k-1+i}, for each $1 \leq i \leq k-1$. This gives us our solution S_{2k-2}.

Proof of Correctness

We now justify that the two constructions given in the previous section are indeed feasible, i.e., that S_{2k-2} and S_{2k} are solutions to the Balanced Weekend Tournament Problem.

Consider S_k, a solution to the BWTP for the case $n = k$ satisfying the five conditions given in Section 2. For each $1 \leq i < j \leq k$, let $h(i, j)$ be the slot in which t_i plays a home game against t_j, and $r(i, j)$ be the slot in which t_i plays a road game against t_j.

As S_k is a feasible schedule, we know that the *Each-Venue* condition is satisfied, i.e., $h(i, j)$ and $r(i, j)$ exist for all pairs (i, j). Furthermore, the *Each-Half* condition necessitates that $1 \leq h(i, j) \leq k - 1$ if and only if $k \leq r(i, j) \leq 2k - 2$, and the *No-Repeat* condition implies that $|h(i, j) - r(i, j)| \geq 2$ for all i and j.

We first notice that in both S_{2k-2} and S_{2k}, the *Compactness* condition is trivially satisfied. For the *No-Two-Road* condition, we notice that every team plays exactly one home game and one road game each weekend, in each of our four "blocks". For example, in S_{2k-2}, the four blocks are $s \in [1, k-2]$, $s \in [k-1, 2k-2]$, $s \in [2k-1, 3k-4]$, and $s \in [3k-3, 4k-6]$. The first and third blocks are balanced as we used parity to determine the home and away assignments, while the second and fourth blocks are guaranteed to be balanced as they were recursively constructed from S_k.

We have now dealt with conditions (d) and (e). To justify that conditions (a), (b), (c) are satisfied in S_{2k}, we define $h'(i, j)$ and $r'(i, j)$ for each $1 \leq i < j \leq 2k$.

From our construction for the first and third blocks of S_{2k}, for all $1 \leq i \leq k$ and $k+1 \leq j \leq 2k$, we have $h'(i, j) = (j - i \bmod k) + 1$ and $r'(i, j) = (j - i \bmod k) + 2k + 1$, or $r'(i, j) = (j - i \bmod k) + 1$ and $h'(i, j) = (j - i \bmod k) + 2k + 1$. In both cases, conditions (a), (b), (c) are satisfied, since $|h'(i, j) - r'(i, j)| = 2k$.

From our construction for the second and fourth blocks of S_{2k}, we have for all $1 \leq i < j \leq k$, $h'(i, j) = h(i, j) + k$ if $1 \leq h(i, j) \leq k$ and $h'(i, j) = h(i, j) + 2k$ if $k + 1 \leq h(i, j) \leq 2k - 2$. We also have the same equations relating $r'(i, j)$ to $r(i, j)$. As S_k satisfies condition (b), if $h'(i, j) = h(i, j) + k$, then we must have $r'(i, j) = r(i, j) + 2k$, and vice-versa.

Therefore, for each pair (i, j) with $1 \leq i < j \leq k$, we have $|h'(i, j) - r'(i, j)| = |h(i, j) - r(i, j)| + k$, with the exception of the k pairs (i, j) for which $h(i, j) = k$ or $r(i, j) = k$. In these cases, we have $h'(i, j) = h(i, j) + k$ and $r'(i, j) = r(i, j) + k$, implying that $|h'(i, j) - r'(i, j)| = |h(i, j) - r(i, j)|$.

And if $k+1 \leq i < j \leq 2k$, the same argument shows that $|h'(i, j) - r'(i, j)| = |h(i-k, j-k) - r(i-k, j-k)|$ or $|h'(i, j) - r'(i, j)| = |h(i-k, j-k) - r(i-k, j-k)| + k$, since the indices $t_{k+1}, \ldots, t_{2k}$ are simply $u_1, \ldots, u_k$, and reduces to the case in the previous two paragraphs.

From these equations, we see that conditions (a), (b), (c) are satisfied for all $1 \leq i < j \leq 2k$.

Similarly, we can prove that the three conditions (a), (b), (c) are satisfied in the schedule S_{2k-2}, for all $1 \leq i < j \leq 2k - 2$. Since the case analysis is virtually identical to the S_{2k} case, we omit the details.

To complete the proof that the BWTP has a solution for all $n \geq 8$, it suffices to find a solution for the base cases $n = 8$, $n = 10$, and $n = 12$. The case $n = 8$ was found in Table 4. The next two tables demonstrate computer-generated solutions for $n = 10$ and $n = 12$, thus completing our proof.

Team	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
t_1	$\boldsymbol{t_2}$	t_0	$\boldsymbol{t_3}$	t_6	$\boldsymbol{t_4}$	t_8	$\boldsymbol{t_5}$	t_7	t_9	$\boldsymbol{t_6}$	$\boldsymbol{t_7}$	t_4	$\boldsymbol{t_8}$	t_5	$\boldsymbol{t_9}$	t_2	$\boldsymbol{t_0}$	t_3
t_2	t_1	$\boldsymbol{t_3}$	$\boldsymbol{t_0}$	t_4	t_9	$\boldsymbol{t_5}$	$\boldsymbol{t_7}$	t_6	$\boldsymbol{t_8}$	t_7	t_5	$\boldsymbol{t_6}$	$\boldsymbol{t_9}$	t_8	t_3	$\boldsymbol{t_1}$	$\boldsymbol{t_4}$	t_0
t_3	$\boldsymbol{t_4}$	t_2	t_1	$\boldsymbol{t_5}$	$\boldsymbol{t_6}$	t_0	t_9	$\boldsymbol{t_8}$	$\boldsymbol{t_7}$	t_4	$\boldsymbol{t_9}$	t_7	t_6	$\boldsymbol{t_0}$	$\boldsymbol{t_2}$	t_8	t_5	$\boldsymbol{t_1}$
t_4	t_3	$\boldsymbol{t_5}$	t_9	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_7}$	t_8	$\boldsymbol{t_0}$	t_6	$\boldsymbol{t_3}$	t_0	$\boldsymbol{t_1}$	t_7	$\boldsymbol{t_6}$	$\boldsymbol{t_8}$	t_5	t_2	$\boldsymbol{t_9}$
t_5	$\boldsymbol{t_6}$	t_4	$\boldsymbol{t_7}$	t_3	$\boldsymbol{t_8}$	t_2	t_1	$\boldsymbol{t_9}$	$\boldsymbol{t_0}$	t_8	$\boldsymbol{t_2}$	t_9	t_0	$\boldsymbol{t_1}$	t_7	$\boldsymbol{t_4}$	$\boldsymbol{t_3}$	t_6
t_6	t_5	$\boldsymbol{t_7}$	t_8	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_9}$	t_0	$\boldsymbol{t_2}$	$\boldsymbol{t_4}$	t_1	$\boldsymbol{t_8}$	t_2	$\boldsymbol{t_3}$	t_4	$\boldsymbol{t_0}$	t_9	t_7	$\boldsymbol{t_5}$
t_7	$\boldsymbol{t_8}$	t_6	t_5	$\boldsymbol{t_9}$	$\boldsymbol{t_0}$	t_4	t_2	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_2}$	t_1	$\boldsymbol{t_3}$	$\boldsymbol{t_4}$	t_9	$\boldsymbol{t_5}$	t_0	$\boldsymbol{t_6}$	t_8
t_8	t_7	$\boldsymbol{t_9}$	$\boldsymbol{t_6}$	t_0	t_5	$\boldsymbol{t_1}$	$\boldsymbol{t_4}$	t_3	t_2	$\boldsymbol{t_5}$	t_6	$\boldsymbol{t_0}$	t_1	$\boldsymbol{t_2}$	t_4	$\boldsymbol{t_3}$	t_9	$\boldsymbol{t_7}$
t_9	$\boldsymbol{t_0}$	t_8	$\boldsymbol{t_4}$	t_7	$\boldsymbol{t_2}$	t_6	$\boldsymbol{t_3}$	t_5	$\boldsymbol{t_1}$	t_0	t_3	$\boldsymbol{t_5}$	t_2	$\boldsymbol{t_7}$	t_1	$\boldsymbol{t_6}$	$\boldsymbol{t_8}$	t_4
t_0	t_9	$\boldsymbol{t_1}$	t_2	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_3}$	$\boldsymbol{t_6}$	t_4	t_5	$\boldsymbol{t_9}$	$\boldsymbol{t_4}$	t_8	$\boldsymbol{t_5}$	t_3	t_6	$\boldsymbol{t_7}$	t_1	$\boldsymbol{t_2}$

Table 7: S_{10}, a solution to the BWTP for $n = 10$
(Note: for improved readability, we replaced t_{10} with t_0.)

Team	1	2	3	4	5	6	7	8	9	10	11
t_1	$\boldsymbol{t_2}$	t_{12}	$\boldsymbol{t_3}$	t_6	$\boldsymbol{t_4}$	t_9	$\boldsymbol{t_5}$	t_{11}	$\boldsymbol{t_8}$	t_{10}	$\boldsymbol{t_7}$
t_2	t_1	$\boldsymbol{t_3}$	t_4	$\boldsymbol{t_{12}}$	t_5	$\boldsymbol{t_{11}}$	t_{10}	$\boldsymbol{t_6}$	$\boldsymbol{t_7}$	t_8	$\boldsymbol{t_9}$
t_3	$\boldsymbol{t_4}$	t_2	t_1	$\boldsymbol{t_5}$	t_{12}	$\boldsymbol{t_6}$	t_7	$\boldsymbol{t_8}$	$\boldsymbol{t_{10}}$	t_9	$\boldsymbol{t_{11}}$
t_4	t_3	$\boldsymbol{t_5}$	$\boldsymbol{t_2}$	t_{11}	t_1	$\boldsymbol{t_7}$	t_8	$\boldsymbol{t_{10}}$	$\boldsymbol{t_9}$	t_6	t_{12}
t_5	$\boldsymbol{t_6}$	t_4	$\boldsymbol{t_7}$	t_3	$\boldsymbol{t_2}$	t_{10}	t_1	$\boldsymbol{t_9}$	t_{11}	$\boldsymbol{t_{12}}$	$\boldsymbol{t_8}$
t_6	t_5	$\boldsymbol{t_7}$	t_8	$\boldsymbol{t_1}$	$\boldsymbol{t_9}$	t_3	$\boldsymbol{t_{11}}$	t_2	t_{12}	$\boldsymbol{t_4}$	$\boldsymbol{t_{10}}$
t_7	$\boldsymbol{t_8}$	t_6	t_5	$\boldsymbol{t_9}$	$\boldsymbol{t_{10}}$	t_4	$\boldsymbol{t_3}$	t_{12}	t_2	$\boldsymbol{t_{11}}$	t_1
t_8	t_7	$\boldsymbol{t_9}$	$\boldsymbol{t_6}$	t_{10}	t_{11}	$\boldsymbol{t_{12}}$	$\boldsymbol{t_4}$	t_3	t_1	$\boldsymbol{t_2}$	t_5
t_9	$\boldsymbol{t_{10}}$	t_8	$\boldsymbol{t_{11}}$	t_7	t_6	$\boldsymbol{t_1}$	$\boldsymbol{t_{12}}$	t_5	t_4	$\boldsymbol{t_3}$	t_2
t_{10}	t_9	$\boldsymbol{t_{11}}$	t_{12}	$\boldsymbol{t_8}$	t_7	$\boldsymbol{t_5}$	$\boldsymbol{t_2}$	t_4	t_3	$\boldsymbol{t_1}$	t_6
t_{11}	$\boldsymbol{t_{12}}$	t_{10}	t_9	$\boldsymbol{t_4}$	$\boldsymbol{t_8}$	t_2	t_6	$\boldsymbol{t_1}$	$\boldsymbol{t_5}$	t_7	t_3
t_{12}	t_{11}	$\boldsymbol{t_1}$	$\boldsymbol{t_{10}}$	t_2	$\boldsymbol{t_3}$	t_8	t_9	$\boldsymbol{t_7}$	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_4}$

Team	12	13	14	15	16	17	18	19	20	21	22
t_1	t_4	$\boldsymbol{t_6}$	t_8	$\boldsymbol{t_{12}}$	t_3	$\boldsymbol{t_{11}}$	t_2	$\boldsymbol{t_{10}}$	t_5	$\boldsymbol{t_9}$	t_7
t_2	t_7	t_{11}	$\boldsymbol{t_{10}}$	$\boldsymbol{t_5}$	t_9	t_6	$\boldsymbol{t_1}$	t_3	$\boldsymbol{t_4}$	t_{12}	$\boldsymbol{t_8}$
t_3	t_8	$\boldsymbol{t_9}$	t_6	t_4	$\boldsymbol{t_1}$	$\boldsymbol{t_7}$	t_{11}	$\boldsymbol{t_2}$	t_{10}	t_5	$\boldsymbol{t_{12}}$
t_4	$\boldsymbol{t_1}$	t_{10}	$\boldsymbol{t_{12}}$	$\boldsymbol{t_3}$	t_7	$\boldsymbol{t_8}$	t_5	$\boldsymbol{t_{11}}$	t_2	$\boldsymbol{t_6}$	t_9
t_5	t_9	t_7	$\boldsymbol{t_{11}}$	t_2	$\boldsymbol{t_{10}}$	t_{12}	$\boldsymbol{t_4}$	t_8	$\boldsymbol{t_1}$	$\boldsymbol{t_3}$	t_6
t_6	t_{11}	t_1	$\boldsymbol{t_3}$	t_{10}	$\boldsymbol{t_8}$	$\boldsymbol{t_2}$	t_9	t_7	$\boldsymbol{t_{12}}$	t_4	$\boldsymbol{t_5}$
t_7	$\boldsymbol{t_2}$	$\boldsymbol{t_5}$	t_9	t_8	$\boldsymbol{t_4}$	t_3	$\boldsymbol{t_{12}}$	$\boldsymbol{t_6}$	t_{11}	t_{10}	$\boldsymbol{t_1}$
t_8	$\boldsymbol{t_3}$	t_{12}	$\boldsymbol{t_1}$	$\boldsymbol{t_7}$	t_6	t_4	$\boldsymbol{t_{10}}$	$\boldsymbol{t_5}$	t_9	$\boldsymbol{t_{11}}$	t_2
t_9	$\boldsymbol{t_5}$	t_3	$\boldsymbol{t_7}$	t_{11}	$\boldsymbol{t_2}$	t_{10}	$\boldsymbol{t_6}$	t_{12}	$\boldsymbol{t_8}$	t_1	$\boldsymbol{t_4}$
t_{10}	$\boldsymbol{t_{12}}$	$\boldsymbol{t_4}$	t_2	$\boldsymbol{t_6}$	t_5	$\boldsymbol{t_9}$	t_8	t_1	$\boldsymbol{t_3}$	$\boldsymbol{t_7}$	t_{11}
t_{11}	$\boldsymbol{t_6}$	$\boldsymbol{t_2}$	t_5	$\boldsymbol{t_9}$	t_{12}	t_1	$\boldsymbol{t_3}$	t_4	$\boldsymbol{t_7}$	t_8	$\boldsymbol{t_{10}}$
t_{12}	t_{10}	$\boldsymbol{t_8}$	t_4	t_1	$\boldsymbol{t_{11}}$	$\boldsymbol{t_5}$	t_7	$\boldsymbol{t_9}$	t_6	$\boldsymbol{t_2}$	t_3

Table 8: S_{12}, a solution to the BWTP for $n = 12$

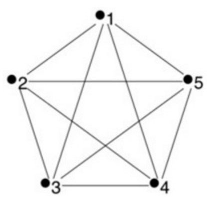
While this recursive construction generates a valid solution S_n for all $n \geq 8$, perhaps there is a more elegant solution that can be determined combinatorially or geometrically. This motivates the first open problem of this paper.

Problem #1: Determine a "canonical" schedule S_n for which a simple function $f(i, j)$ determines the slot in which t_i plays a home game against t_j, where the resulting schedule S_n is a valid solution to the n-team BWTP.

Hamiltonian Decompositions

Let G be a graph with n vertices. We say that G is *Hamiltonian* if it contains a Hamilton cycle, i.e., a cycle passing through each of the n vertices of G. Hamiltonicity is one of the most well-studied concepts in graph theory, due to its applications to the Traveling Salesman Problem. Deciding whether a given graph G is Hamiltonian is one of Karp's 21 celebrated NP-complete problems (Karp 1972).

A generalization of Hamiltonicity is to decide whether a given graph G has a *Hamiltonian decomposition*, i.e., a partition of the edge set of G into disjoint Hamilton cycles.



For example, the complete graph K_5 has a Hamiltonian decomposition, since K_5 can be decomposed into the cycles $1-2-3-4-5-1$ and $1-3-5-2-4-1$. To make our notation easier, we will represent these cycles as (12345) and (13524), respectively.

Clearly, a Hamiltonian decomposition is only possible when every vertex of G has degree r, for some even r. Since the complete graph K_n has degree $n-1$, a natural question is to determine the odd values of n for which K_n has a Hamiltonian decomposition. In 1890, Walecki demonstrated an explicit construction for all odd n (Alspach 2008).

We can generalize this problem by considering Hamiltonian decompositions of *directed* complete graphs K_n^*, where all $n(n-1)$ directed edges, or arcs, are drawn. For all odd n, K_n^* must have a Hamiltonian decomposition by Walecki's Theorem; for example, in the case $n=5$ above, we have the decomposition $\{(12345),(15432),(13524),(14253)\}$, which we get by turning each of the n-cycles in our undirected K_n decomposition into two directed n-cycles.

For even n, the problem of whether K_n^* has a Hamiltonian decomposition was a long-standing open problem in graph theory. A partial solution showed that a Hamiltonian decomposition does not exist for $n=4$ or $n=6$ but does for each $8 \leq n \leq 18$ (Bermond and Faber 1976).

By considering four possible cases for n (i.e., their remainder mod 8), Tillson used difference sequences to prove the existence of a Hamiltonian decomposition of K_n^* for all even values of $n \geq 8$ (Tillson 1980).

As the Tillson construction is both lengthy and complex, a natural question is whether this Hamiltonian decomposition of K_n^* could be generated more simply. We answer this question in the affirmative, as a corollary of our recursive construction to the Balanced Weekend Tournament Problem.

Label the n vertices of K_n^* as $t_1, t_2, \ldots, t_n$, and let S_n be a solution to the n-team BWTP. Consider the $\frac{n}{2}+\frac{n}{2}=n$ matches played on any weekend, i.e., time slots $2s-1$ and $2s$ for each $1 \leq s \leq n-1$. If t_i plays at the home of t_j, this corresponds to the directed edge $t_i \rightarrow t_j$ in K_n^*.

Thus, in Table 4, the games in Slot #1 are represented by the edges $t_2 \rightarrow t_8$, $t_3 \rightarrow t_5$, $t_6 \rightarrow t_1$, and $t_7 \rightarrow t_4$, and the games in Slot #2 are represented by the edges $t_1 \rightarrow t_3$, $t_4 \rightarrow t_2$, $t_5 \rightarrow t_7$, and $t_8 \rightarrow t_6$.

By the definition of an n-team Balanced Weekend Tournament, every team plays one home game and one road game each weekend, i.e., the n edges in slots #1 and #2 must combine to form a directed graph where each vertex has indegree 1 and outdegree 1. In the example above, the resulting digraph is a Hamiltonian cycle.

$$t_1 \rightarrow t_3 \rightarrow t_5 \rightarrow t_7 \rightarrow t_4 \rightarrow t_2 \rightarrow t_8 \rightarrow t_6 \rightarrow t_1$$

Using our earlier notation, we represent this cycle as (13574286). We can readily show that the seven-weekend 8-team BWTP solution in Table 4 has the following Hamiltonian decomposition: (13574286), (12345678), (14625837), (15482763), (16875324), (18473652), and (17264385).

Let S_k be a solution to the k-team BWTP, and let S_{2k-2} be the schedule constructed by our recursive algorithm. We show that if the $k-1$ weekends of S_k generate a Hamiltonian decomposition, then so do the $2k-3$ weekends of S_{2k-2}.

For each $1 \leq i,j \leq k-1$, define an *intra-league* game to be $t_i \rightarrow t_j$ or $u_i \rightarrow u_j$, and an *inter-league* game to be $t_i \rightarrow u_j$ or $t_j \rightarrow u_i$. From our construction (see Table 6), we see there are $k-2$ weekends consisting of exclusively inter-league games, while the other $k-1$ weekends have all intra-league games with the exception of two inter-league games (e.g. $t_2 \rightarrow u_2$ and $u_6 \rightarrow t_6$ in slots #7 and #8).

In Table 6, the $2k-2=14$ inter-league games of slots #1 and #2 trace out a directed cycle with $2k-2=14$ edges.

$$\begin{array}{ccccccccccccccc} & & u_3 & & u_4 & & u_5 & & u_6 & & u_7 & & u_1 & & u_2 \\ & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow \\ t_1 & & t_2 & & t_3 & & t_4 & & t_5 & & t_6 & & t_7 & & t_1 \end{array}$$

Because of how these inter-league games are scheduled (i.e., t_i plays u_{s+i} in slot s, with all arithmetic calculated mod $k-1$), there is no way for these $2k-2$ edges to form anything other than a Hamiltonian cycle; specifically, it is impossible for these $2k-2$ edges to split into two directed $k-1$ cycles, or any other non-Hamiltonian graph with every vertex having indegree 1 and outdegree 1. If the cycle starts at t_1, it must return to t_1 after exactly $2k-2$ steps.

In Table 6, the $2k-2=14$ games of slots #7 and #8 trace out a directed cycle with $2k-2=14$ edges.

$$\begin{array}{ccccccccccccc} t_6 & \rightarrow & t_1 & \rightarrow & t_3 & \rightarrow & t_5 & \rightarrow & t_7 & \rightarrow & t_4 & \rightarrow & t_2 \\ \uparrow & & & & & & & & & & & & \downarrow \\ u_6 & \leftarrow & u_1 & \leftarrow & u_3 & \leftarrow & u_5 & \leftarrow & u_7 & \leftarrow & u_4 & \leftarrow & u_2 \end{array}$$

By our induction hypothesis, the weekend games of S_k decompose into Hamiltonian cycles. For example, the first weekend of S_8 is represented by the 8-cycle (13574286), which, when we remove t_8, turns into the 7-path $[6,1,3,5,7,4,2]$, precisely the sequence described above in both the t and u sequences.

In general, if $[p_1, p_2, \ldots, p_{k-1}]$ is the $(k-1)$-path that arises by removing vertex t_k from a Hamiltonian cycle of S_k, then the corresponding weekend of S_{2k-2} becomes $t_{p_1} \rightarrow t_{p_2} \rightarrow \ldots \rightarrow t_{p_{k-1}} \rightarrow u_{p_{k-1}} \rightarrow \ldots \rightarrow u_{p_2} \rightarrow u_{p_1} \rightarrow t_{p_1}$, i.e., a Hamiltonian cycle with $2k-2$ vertices.

In our recursive construction of S_{2k-2} from S_k, we ensured that there exist two distinct indices p_1 and p_{k-1} for which $t_{p_{k-1}} \to u_{p_{k-1}}$ and $u_{p_1} \to t_{p_1}$, which shows that the weekends of S_{2k-2} consisting of all but two intra-league games forms a Hamiltonian cycle with $2k-2$ vertices.

Thus, we have found an explicit solution to the problem of decomposing the complete directed graph K^*_{2k-2} into $2k-3$ directed Hamiltonian cycles. For example, the columns of Table 6 yield a Hamiltonian decomposition of K^*_{14}.

Unfortunately, a similar argument does not work in our recursive construction of S_{2k}, as this schedule splits into k weekends of only inter-league games and $k-1$ weekends of only intra-league games. Though the former decompose into Hamiltonian cycles of length $2k$, the latter do not; it is easy to see that each of the $k-1$ intra-league weekends (e.g. slots #9 and #10 in Table 5) decompose into two k-cycles, via our induction hypothesis that the $k-1$ weekends of S_k decompose into Hamiltonian cycles of length k.

To fix this, we employ the "cocktail graph" construction (Tillson 1980) where we turn two k-cycles into one $2k$-cycle by identifying edge $u_1 \to v_1$ from cycle C_1 and edge $u_2 \to v_2$ from cycle C_2, and replacing them with the edges $u_1 \to v_2$ and $u_2 \to v_1$. The Tillson construction demonstrates how the edges can be partitioned so that the $2k-1$ subgraphs form a Hamiltonian decomposition of S_{2k}.

Therefore, we have shown how to generate a Hamiltonian decomposition of K^*_n for all even values of $n \geq 8$, although our recursive construction requires edge-switching when $n \equiv 0 \pmod 4$. This inspires the following open problem.

Problem #2: Determine a recursive construction of S_{2k} from S_k for which each of the $2k-1$ weekends directly forms a Hamiltonian cycle, without any switching of edges.

The p-norm

Let S_n be a solution to the BWTP. Recall that $h(i,j)$ is the slot in which t_i plays a home game against t_j, and $r(i,j)$ is the slot in which t_i plays a road game against t_j.

In Table 4, the schedule for $n = 8$ has seven pairs (i,j) with $1 \leq i < j \leq n$ for which $|h(i,j) - r(i,j)| = n-1$, the ideal distance between these pairs of games. As mentioned earlier, the most balanced tournament is mirrored, where $|h(i,j) - r(i,j)| = n-1$ for all pairs (i,j).

Let us define the p-norm of an n-team schedule S_n as

$$||S_n||_p := \left(\sum_{1 \leq i < j \leq n} ||h(i,j) - r(i,j)| - (n-1)|^p \right)^{1/p}.$$

This function, inspired by the p-norm in L^p space, equals 0 if and only if S_n is mirrored. We can check by inspection that in the schedule given in Table 4,

$$||S_8||_p = (7 \cdot 0^p + 12 \cdot 1^p + 7 \cdot 2^p + 2 \cdot 3^p)^{1/p}.$$

This implies that $||S_8||_1 = 32$, $||S_8|_2 = \sqrt{58}$, and $||S_8|_p \to 3$ as $p \to \infty$.

Suppose there exists a schedule S_n that satisfies all the conditions of the BWTP, for which $||S_n||_p = 0$. Let t_i be a team that plays at home in slot #1. Then by the condition that each team plays exactly one home each weekend, as well as the mirroring requirement, t_i plays on the road in slot #2, at home in slot #(n+1), on the road in slot #(n+2), at home in slot #3, on the road in slot #4, at home in slot #(n+3), etc.

Continuing this process, we see that each team that plays at home in slot #1 must play at home in all odd slots and on the road in all even slots. Therefore, if teams t_i and t_j start at home, then they can never play against each other, which is a contradiction. Hence, no schedule can satisfy $||S_n|_p = 0$, i.e., no solution to the BWTP is perfectly balanced. A natural question is to find the S_n for which $||S_n||_p$ is minimized.

Based on our recursive construction, we determine a formula for $||S_{2k}||_p$ and $||S_{2k-2}||_p$ as a function of $||S_k||_p$, and show that our schedule has a small p-norm.

In S_{2k}, define $h'(i,j)$ and $r'(i,j)$ for each $1 \leq i < j \leq 2k$. From our construction, we know that if $1 \leq i \leq k$ and $k+1 \leq j \leq 2k$, then $|h'(i,j) - r'(i,j)| = 2k$. Also, if $1 \leq i < j \leq k$ or $k+1 \leq i < j \leq 2k$, then $|h'(i,j) - r'(i,j)| = |h(i,j) - r(i,j)| + k$, except for k pairs (i,j) for which $|h'(i,j) - r'(i,j)| = |h(i,j) - r(i,j)| \geq 2$. We have:

$$\begin{aligned}
(||S_{2k}||_p)^p &= \sum_{i<j\leq 2k} ||h'(i,j) - r'(i,j)| - (2k-1)|^p \\
&= k^2 \cdot 1^p + 2 \sum_{i<j\leq k} ||h'(i,j) - r'(i,j)| - (2k-1)|^p \\
&< k^2 + 2 \sum_{i<j\leq k} ||h(i,j) - r(i,j)| - (k-1)|^p \\
&\qquad + k \cdot (|2 - (2k-1)|)^p
\end{aligned}$$

Therefore, we have shown that $(||S_{2k}||_p)^p < k^2 + 2\,(||S_k||_p)^p + k \cdot (2k-3)^p$. Similarly, we can prove that $(||S_{2k-2}||_p)^p < (k-1)(k-2) + 2\,(||S_k||_p)^p + (k-1) \cdot (2k-5)^p$.

To derive the latter inequality, notice that in S_{2k-2}, we have $|h'(i,j) - r'(i,j)| = 2k-2$ for all $1 \leq i \leq k-1$ and $k \leq j \leq 2k-2$, provided $j - i \neq k-1$. And for all other pairs (i,j), $|h'(i,j) - r'(i,j)| = |h(i,j) - r(i,j)| + (k-2)$, except for the $(k-1)$ pairs for which $|h'(i,j) - r'(i,j)| = |h(i,j) - r(i,j)| \geq 2$.

From these two inequalities, we use induction to prove that for all $n \geq 8$, $||S_n||_1 < \frac{7}{4}n^2$ and $||S_n||_p < n^{1+1/p}$ for $p \geq 2$. The base cases $n = 8, 10, 12$ follow by inspection, using the schedules given in Tables 4, 7, and 8.

Therefore, our recursive construction produces an infinite set of schedules S_n for which $||S_n|_1$ is bounded by a quadratic function. Given that there are $\binom{n}{2} \sim \frac{n^2}{2}$ pairs of teams, we see each S_n has the property that, on average, each pair of teams plays their two games less than $\frac{7}{2}$ slots apart from the "optimally-balanced" solution that is impossible to attain. But can we do better than a quadratic function? This motivates our final open problem.

Problem #3: Determine whether there exists a function $f(n) = c \cdot n^r$, with $r < 2$, for which some valid schedule S_n satisfies $||S_n||_1 < f(n)$ for all $n \geq 8$.

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