Edmonds polytopes and a hierarchy of combinatorial problems

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Abstract

Let S be a set of linear inequalities that determine a bounded polyhedron P. The closure of S is the smallest set of inequalities that contains S and is closed under two operations: (i) taking linear combinations of inequalities, (ii) replacing an inequality $\Sigma a_j x_j \leq a_0$, where $a_1, a_2, \ldots, a_n$ are integers, by the inequality $\Sigma a_j x_j \leq a$ with $a \geq [a_0]$. Obviously, if integers $x_1, x_2, \ldots, x_n$ satisfy all the inequalities in S, then they satisfy also all inequalities in the closure of S. Conversely, let $\Sigma c_j x_j \leq c_0$ hold for all choices of integers $x_1, x_2, \ldots, x_n$, that satisfy all the inequalities in S. Then we prove that $\Sigma c_j x_j \leq c_0$ belongs to the closure of S. To each integer linear programming problem, we assign a nonnegative integer, called its rank. (The rank is the minimum number of iterations of the operation (ii) that are required in order to eliminate the integrality constraint.) We prove that there is no upper bound on the rank of problems arising from the search for largest independent sets in graphs.
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1. Characterizations and good characterizations

Let us examine the formal structure of the following theorems.

Theorem 1.1 (*Tutte [17]*). *Let G be a (finite undirected) graph. Then the two following conditions are equivalent.*
   (i) *G has a perfect matching* (*that is, a set of pairwise disjoint edges that cover all the vertices of G*),
   (ii) *if an arbitrary set S of vertices is deleted from G, then the number $k_0(G \backslash S)$ of odd components* (*that is, components having an odd number of vertices each*) *of the resulting graph $G \backslash S$ does not exceed $|S|$.*

Theorem 1.2 (*Gallai [11]*). *Let G be a (finite undirected) graph. Then the two following conditions are equivalent*:
   (i) *G is k-colorable*,
   (ii) *the edges of G can be directed in such a way that the resulting directed graph contains no* (*simple directed*) *path having k edges.*

Both of these theorems, asserting the equivalence of (i) and (ii), are characterizations. Yet there is a considerable formal difference between the two. Theorem 1.1 gives necessary and sufficient conditions for the existence of a certain structure (perfect matching in G) in terms of the absence of another structure (a set S with $k_0(G \backslash S) > |S|$). On the other

hand, Theorem 1.2 gives necessary and sufficient conditions for the existence of a certain structure (k-coloring of G) in terms of the existence of another structure (the directions of the edges of G). Another aspect of this difference can be illuminated as follows. It is easy to convince one's supervisor that G has a perfect matching. To do this, one only has to exhibit the matching. (The question of the difficulty of finding the matching is irrelevant for our discussion.) It is equally easy (with the help of Theorem 1.1) to convince the supervisor that G has no perfect matching— one has to exhibit a set S with $k_0(G \backslash S) > |S|$. On the other hand, while it is easy to convice the supervisor that G has a k-coloring, Theorem 1.2 gives no easy way of showing that G has no k-coloring.

Apparently Edmonds [6] has been the first to turn attention to this feature of characterizations; he introduced the term "good characterizations" for the theorems of the first type. Hence Tutte's theorem is a good characterization while Gallai's theorem is not. Needless to say, the words "good characterization" form a nonseparable entity without any reference to the emotional charge of the adjective "good". The statement "Gallai's theorem is not a good characterization" asserts nothing whatsoever about the quality and depth of the theorem.

In our further considerations, the duality theorem of linear programming will play an important role. It expresses the maximum of a linear form $\Sigma c_i x_i$ subject to a set of constraints (primal problem) as a minimum of another form $\Sigma b_i y_i$ subject to other constraints (dual problem). Hence to show that a feasible primal solution $(x_1, x_2, \ldots, x_n)$ is optimal, one only has to exhibit a feasible dual solution $(y_1, y_2, \ldots, y_m)$ with $\Sigma c_i x_i = \Sigma b_i y_i$. In a way, the duality theorem of linear programming is a prototype of a good characterization.

Our last sentence has more into it than meets the eye. Actually, Edmonds [7] has shown how to relate Theorem 1.1 to the duality theorem and made it clear that his approach can be adopted in many different settings. It is the purpose of this paper to study various questions related to Edmonds' technique.

2. Edmonds polytopes

Let G be a graph with vertices $v_1, v_2, \ldots, v_m$ and edges $e_1, e_2, \ldots, e_n$; for each $j = 1, 2, \ldots, m$ we set $S(j) = \{i : v_j \text{ is an endpoint of } e_i\}$. The problem of finding a perfect matching in G can be formulated as the following integer linear programming problem: Maximize

$$(2.1) \qquad \sum_{i=1}^{n} x_i$$

subject to the constraints

$$(2.2) \qquad x_i \geq 0 \quad (i = 1, 2 \ldots, n),$$
$$(2.3) \qquad \sum_{i \in S(j)} x_i \leq 1 \quad (j = 1, 2 \ldots, m),$$
$$(2.4) \qquad x_i = \text{integer} \quad (i = 1, 2 \ldots, n).$$

Obviously, every characteristic vector $(x_1, x_2, \ldots, x_n)$ of a set of pairwise disjoint edges satisfies (2.2)–(2.4). Vice versa, every vector $(x_1, x_2, \ldots, x_n)$ that satisfies (2.2)–(2.4) turns out to be a characteristic vector of a set of pairwise disjoint edges of G. Hence G has a perfect matching if and only if the maximum of (2.1) subject to (2.2)–(2.4) equals $\frac{1}{2}m$.

Because of the integrality constraint (2.4), we cannot express the maximum of (2.1) in terms of the minimum of a dual problem. Besides, if (2.4) is dropped then the maximum of (2.1) can increase. For instance, if G is a triangle then $x_1 = x_2 = x_3 = \frac{1}{2}$ satisfies (2.2), (2.3) and yields $\Sigma x_i = \frac{3}{2}$. However, the maximum of (2.1) subject to (2.2)–(2.4) equals one in this case. Nevertheless, there is a standard way of getting around the inequality constraint. One can think of the polytope P (in the n-dimensional Euclidean space) determined by (2.2), (2.3). The set F of lattice points inside P is finite and its convex hull $E(P)$ is another polytope. A moment's reflection shows that the maximum of (2.1) over F equals the maximum of (2.1) over $E(P)$. Indeed, F is a subset of $E(P)$ while the extremum points of $E(P)$ come from F. More generally, for any polytope P and any linear form $\Sigma c_i x_i$, the problem of maximizing $\Sigma c_i x_i$ over the lattice points inside P reduces into the problem of maximizing $\Sigma c_i x_i$ over $E(P)$. The latter is an ordinary (noninteger) linear

programming problem that offers the advantage of using the duality theorem as long as the list of facets of $E(P)$ is known.

In general, it seems extremely difficult to determine all facets of $E(P)$ from those of P. However, in the above case – when P is defined by (2.2), (2.3) – the list of facets of $E(P)$ is available. Indeed, Edmonds [7] proved that all the inequalities that determine $E(P)$ are (2.2), (2.3), and

$$(2.5) \qquad \sum_{e_i \subset S} x_i \leq k.$$

Here S runs through all sets of $2k + 1$ vertices (k arbitrary) and each edge is interpreted as a two-point set. Now, the maximum of (2.1) subject to (2.2)–(2.4) equals the maximum of (2.1) subject to (2.2), (2.3), (2.5), which is, in turn, equal to the minimum of the corresponding dual program. Therefore Edmonds' theorem (combined with the duality theorem) yields instantly a good characterization of graphs without a perfect matching as follows. A graph G has no perfect matching if and only if there are nonnegative real numbers $a_1, a_2, \ldots, a_m$ and $b(S)$, where S ranges through all odd-cardinality sets of vertices, such that for each edge e with endpoints v_j, v_k the inequality

$$a_j + a_k + \sum_{e \subset S} b(S) \geq 1$$

is satisfied and

$$\sum_{i=1}^{m} a_i + \sum_{S} \tfrac{1}{2}(|S| - 1)b(S) < \tfrac{1}{2}m.$$

Besides, Edmonds [8] proved that the a_i and $b(S)$ can be chosen to be zero or one. Under this added assumption, the above characterization reduces into Tutte's theorem. Of course, Edmonds' theorem is more general; it provides a max-min formula for any *weighted* matching problem. Edmonds also generalized these results to the case of optimum-weighted degree-constrained subgraphs of a given graph (see [7], Section VIII and also [9]). Since these are the only cases when $E(P)$ is a proper subset of P but the description of the facets of $E(P)$ is known, we call $E(P)$ the *Edmonds polytope* of P.

We have seen that the knowledge of the facets of $E(P)$ yields immediately a max-min formula for the corresponding integer linear programming problem. Next, we will study the relations between the facets of $E(P)$ and those of P.

3. The main theorem

It is easy to see how (2.3) and (2.4) imply (2.5). Indeed, let S be any set of $2k + 1$ vertices of G. Summing the inequalities (2.3) for all j with $v_j \in S$ we obtain

$$2 \sum_{e_i \subset S} x_i \leq 2k + 1,$$

or

$$\sum_{e_i \subset S} x_i \leq k + \tfrac{1}{2}.$$

By (2.4), the left-hand side of the last inequality is an integer and so (2.5) follows. This observation leads us to the definition of a closure of a set S of linear inequalities. We shall say that an inequality $\Sigma a_j x_j \leq b$ belongs to the *elementary closure* of S if there are inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m)$$

in S and positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\sum_{i=1}^{m} \lambda_i a_{ij} = a_j = \text{integer} \qquad (j = 1, 2, \ldots, n),$$

$$\left[\sum_{i=1}^{m} \lambda_i b_i \right] \leq b$$

(here $[x]$ denotes the integer part of x). The set of all inequalities belonging to the elementary closure of S will be denoted by $e^1(S)$; for any integer $k > 1$ we define $e^k(S)$ recursively by $e^k(S) = e(S \cup e^{k-1}(S))$. Finally, we set

$$c(S) = \bigcup_{k=1}^{\infty} e^k(S);$$

the set $c(S)$ will be called the *closure* of S. Evidently, all vectors $(x_1, x_2, \ldots, x_n)$ satisfying all the inequalities in S plus the integrality constraint

$$x_i = \text{integer} \qquad (i = 1, 2, \ldots, n)$$

satisfy also all the inequalities in $c(S)$. A converse is given by our next result.

Theorem 3.1. *Let the inequalities*

$$(3.1) \qquad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m)$$

(where a_{ij}, b_i are real numbers) determine a bounded polyhedron in the n-dimensional Euclidean space. Let $c_0, c_1, \ldots, c_n$ be integers such that

$$(3.2) \qquad \sum_{j=1}^{n} c_j x_j \leq c_0$$

holds for any choice of integers $x_1, x_2, \ldots, x_n$ satisfying (3.1). *Then* (3.2) *belongs to the closure of* (3.1).

In the proof, we will use the following auxiliary result.

Lemma 3.2. *Let* (3.1) *and* (3.2) *be as in Theorem* 3.1; *let c be an integer such that $\Sigma c_j x_j < c + 1$ for every choice of reals $x_1, x_2, \ldots, x_n$ satisfying* (3.1). *Then the inequality $\Sigma c_j x_j \leq c$ belongs to the closure of* (3.1).

Proof. Let c^* be the maximum of $\Sigma c_j x_j$ subject to the constraints (3.1). By one of the versions of the duality theorem (see [13, Theorem 8.3.1]), there are nonnegative reals $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\Sigma \lambda_i a_{ij} = c_j (j = 1, 2, \ldots, m)$ and $\Sigma \lambda_i b_i = c^*$. Since $c^* < c + 1$, we have $[c^*] \leq c$ and the conclusion follows.

Proof of Theorem 3.1. Let c^* be the maximum of $\Sigma c_i x_i$ subject to (3.1); set $c = [c^*]$. By Lemma 3.2, the inequality $\Sigma c_i x_i \leq c$ belongs to the closure of (3.1). If $c \leq c_0$, then we are done. Next, we will assume $c > c_0$ and prove that the inequality $\Sigma c_i x_i \leq c - 1$ belongs to the closure of (3.1). Repeating this process $c - c_0$ times, we arrive at the desired conclusion.

Since (3.1) determines a *bounded* polyhedron, there is an integer M with $|x_i| \leq M$ whenever (3.1) is satisfied. By Lemma 3.2, the inequalities

$$-x_i \leq M \quad (i = 1, 2 \ldots, n),$$
$$x_i \leq M \quad (i = 1, 2, \ldots, n)$$

belong to the closure of (3.1). Given a vector $(s_1, s_2, \ldots, s_k)$; where $0 \le k \le n$ and $s_i \in \{-M, -M+1, \ldots, M\}$, we construct a linear form (in the x_i's) $L(s_1, s_2, \ldots, s_k)$ and a number $R(s_1, s_2, \ldots, s_k)$ recursively as follows. Firstly, for the zero-length vector $\emptyset$, we set

$$L(\emptyset) = \sum c_i x_i, \quad R(\emptyset) = c - 1.$$

Secondly, we set

$$L(s_1, s_2, \ldots, s_k) = (M + 1 + s_k)L(s_1, s_2, \ldots, s_{k-1}) - x_k,$$

$$R(s_1, s_2, \ldots, s_k) = (M + 1 + s_k)R(s_1, s_2, \ldots, s_{k-1}) + M.$$

It follows directly from the definition that

$$L(s_1, s_2, \ldots, s_k) = L(s_1, s_2, \ldots, s_{k-1}) + L(s_1, s_2, \ldots, s_k - 1),$$

$$R(s_1, s_2, \ldots, s_k) = R(s_1, s_2, \ldots, s_{k-1}) + R(s_1, s_2, \ldots, s_k - 1)$$

whenever $s_k > -M$. Now, it is easy to establish (by induction on k) that

$$(3.3) \quad \begin{aligned} L(s_1, s_2, \ldots, s_k) &= \sum c_i x_i + \sum_{s_i \ne -M} L(s_1, s_2, \ldots, s_i - 1) - \sum_{s_i = -M} x_i, \\ R(s_1, s_2, \ldots, s_k) &= (c - 1) + \sum_{s_i \ne -M} R(s_1, s_2, \ldots, s_i - 1) + \sum_{s_i = -M} M. \end{aligned}$$

Our next observation is essential for the proof.

Claim. *Let $(s_1, s_2, \ldots, s_k)$ be any vector with $0 \le k \le n$ and $s_i \in \{-M, -M+1, \ldots, M\}$. If*

$$(3.4) \quad \begin{aligned} L(s_1, s_2, \ldots, s_i - 1) &= R(s_1, s_2, \ldots, s_i - 1) & (s_i \ne -M), \\ -x_i &= M & (s_i = -M), \\ \sum c_i x_i &= c, \end{aligned}$$

then $x_i = s_i$ for all $i = 1, 2, \ldots, k$.

Proof of the Claim. We proceed by induction on k. The Claim is trivially true for $k=0$. By the induction assumption, the Claim holds for the vector $(s_1, s_2, \ldots, s_{k-1})$ and so the equations (3.4) with $i \ne k$ imply $x_i = s_i$ for all $i = 1, 2, \ldots, k-1$. If $s_k = -M$ then we are done. If $s_k \ne -M$ then we argue as follows. The equations (3.3) and (3.4) imply that

$$L(s_1, s_2, \ldots, s_{k-1}) = R(s_1, s_2, \ldots, s_{k-1}) + 1.$$

By definition, we have

$$L(s_1, s_2, \ldots, s_k - 1) = (M + s_k)L(s_1, s_2, \ldots, s_{k-1}) - x_k,$$

$$R(s_1, s_2, \ldots, s_k - 1) = (M + s_k)R(s_1, s_2, \ldots, s_{k-1}) + M.$$

Using the last three equations and (3.4) with $i = k$ we deduce $x_k = s_k$ which is the desired result. Thus the Claim is proved.

Now, we are ready for the final coup de grace. Inductively, we shall sweep through the entire set of inequalities

$$(3.5) \quad L(s_1, s_2, \ldots, s_k) \le R(s_1, s_2, \ldots, s_k)$$

in a specified order, and prove that each of these belongs to the closure of (3.1). (In particular, the inequality $L(\emptyset) \le R(\emptyset)$ – which comes last in our ordering – is the one we want.) The linear order is a lexicographic one with each blank – corresponding to $s_{k+1}, s_{k+2}, \ldots, s_n$ – interpreted as $M + 1$. More precisely, we say that (3.5) precedes the inequality $L(t_1, t_2, \ldots, t_r) \le R(t_1, t_2, \ldots, t_r)$ if and only if either $s_j < t_j$ or $r + 1 = j \le k$, where j is the largest subscript with $s_i = t_i$ for all $i < j$.

Case 1. $k = n$. (This case includes the very first inequality in our set, one with $s_1 = s_2 = \cdots = s_n = -M$.) By the induction assumption, all the inequalities

$$(3.6) \qquad L(s_1, s_2, \ldots, s_i - 1) \le R(s_1, s_2, \ldots, s_i - 1) \quad (s_i \ne -M)$$

belong to the closure of (3.1). Moreover, the inequalities

$$(3.7) \qquad - x_i \le M \qquad (s_i = -M),$$

$$(3.8) \qquad \sum c_i x_i \le c$$

belong to the closure of (3.1). Summing up (3.6)–(3.8) and using (3.3) we arrive at the inequality

$$(3.9) \qquad L(s_1, s_2, \ldots, s_n) \le R(s_1, s_2, \ldots, s_n) + 1.$$

This inequality holds for every choice of reals $x_1, x_2, \ldots, x_n$ that obey (3.6)–(3.8). Besides, our Claim implies that equality in (3.9) can occur only if $x_i = s_i$ ($i = 1, 2, \ldots, n$) and $\Sigma c_i x_i = c$. However, these $n + 1$ equations are inconsistent with at least one of the constraints (3.1) – otherwise the assumption of Theorem 3.1 is violated. Therefore

$$L(s_1, s_2, \ldots, s_n) < R(s_1, s_2, \ldots, s_n) + 1$$

holds for any choice of reals $x_1, x_2, \ldots, x_n$ satisfying (3.1), (3.6)–(3.8). By Lemma 3.2, the inequality

$$(3.10) \qquad L(s_1, s_2, \ldots, s_n) \le R(s_1, s_2, \ldots, s_n)$$

belong to the closure of (3.1), (3.6)–(3.8). As (3.6)–(3.8) belong to the closure of (3.1) themselves, we conclude that (3.10) belongs to the closure of (3.1).

Case 2. $k < n$. By the induction assumption, the inequality

$$L(s_1, s_2, \ldots, s_k, M) \le R(s_1, s_2, \ldots, s_k, M)$$

belongs to the closure of (3.1). This inequality can be written as

$$(2M + 1)L(s_1, s_2, \ldots, s_k) - x_k \le (2M + 1)R(s_1, s_2, \ldots, s_k) + M.$$

Besides, the inequality $x_k \le M$ belongs to the closure of (3.1). Adding the last two inequalities and dividing by $2M + 1$ we obtain

$$L(s_1, s_2, \ldots, s_k) \le R(s_1, s_2, \ldots, s_k) + 2M/(2M + 1).$$

Therefore

$$L(s_1, s_2, \ldots, s_k) \le R(s_1, s_2, \ldots, s_k)$$

belongs to the closure of (3.1).

Now, we have proved that all the inequalities (3.5), including $\Sigma c_i x_i \le c - 1$, belong to the closure of (3.1). Repeating this argument $c - c_0$ times (as mentioned above) we prove that (3.2) belongs to the closure of (3.1) and finish thus the proof of Theorem 3.1.

One more remark. It is easy to see that the Edmonds polytope of P can be described by inequalities

$$\sum_{j=1}^{n} a_{ij}^* x_j \le b_i^* \quad (i = 1, 2, \ldots, m^*)$$

where all the a_{ij}^*'s and b_i^*'s are integers. Hence Theorem 3.1 can be restated as follows.

Corollary 3.3. *If* (3.1) *defines a bounded polyhedron P then the closure of* (3.1) *determines* $E(P)$.

4. The Boolean case and branch-and-bound method

Among the integer linear programming problems, those with the constraints

$$x_i = 0 \text{ or } 1 \quad (i = 1, 2, \dots, n)$$

are particularly important. The problems arising from combinatorial considerations have nearly always this form; the x_i's usually represent the characteristic vector of a set satisfying specified conditions. In this section, we turn our attention to these problems. We shall consider polyhedra defined by inequalities

$$
(4.1) \quad
\begin{aligned}
0 \le x_i \le 1 & \quad (i = 1, 2, \dots, n), \\
\sum_{j=1}^{n} a_{ij} x_j \le b_i & \quad (i = 1, 2, \dots, m)
\end{aligned}
$$

and present an alternative proof of Theorem 3.1 within this restricted class. The proof may be found to be more direct and transparent than the one given above; besides, it is related in an amusing way to the branch and bound method. As in the preceding section, we only have to prove the following statement.

Let $c_1, c_2, \dots, c_n$ and c be integers such that:

(i) the inequality $\Sigma c_i x_i \le c$ belongs to the closure of (4.1),
(ii) there are no integers $x_1, x_2, \dots, x_n$ satisfying (4.1) and $\Sigma c_i x_i = c$.

Then the inequality $\Sigma c_i x_i \le c - 1$ belongs to the closure of (4.1).

Actually, we are going to prove that all the inequalities

$$
(4.2) \quad \sum_{i=1}^{n} c_i x_i + \sum_{i \in A} x_i - \sum_{i \in B} x_i \le c - 1 + |A|,
$$

where A, B are disjoint subsets of $\{1, 2, \dots, n\}$, belong to the closure of (4.2). The proof goes by backward induction on $|A| + |B|$; the inequality (4.2) with $A = B = \emptyset$ is the one we want. The induction step is easy. If $|A| + |B| < n$ then there is a subscript $k \notin A \cup B$ and, by the induction assumption, both inequalities

$$
x_k + \sum_A c_i x_i + \sum_A x_i - \sum_B x_i \le c - 1 + |A| + 1,
$$

$$
-x_k + \sum_A c_i x_i + \sum_A x_i - \sum_B x_i \le c - 1 + |A|
$$

belong to the closure of (4.1). Adding them and dividing by two we obtain

$$
\sum c_i x_i + \sum_A x_i - \sum_B x_i \le c - 1 + |A| + \tfrac{1}{2}
$$

and conclude that (4.2) belongs to the closure of (4.1). It remains to verify that all the inequalities (4.2) with $|A| + |B| = n$ belong to the closure of (4.1). Here, we distinguish two cases,

Case 1. $\Sigma_{j \in A} a_{ij} \le b_i$ for all $i = 1, 2, \dots, m$. In this case, we have

$$
\sum_{i \in A} c_i < c
$$

for otherwise (ii) is violated by

$$
x_i = \begin{cases} 1 & i \in A, \\ 0 & i \in B. \end{cases}
$$

Setting $M = \max |c_i|$, we have

$$(M + c_i)x_i \leq M + c_i \quad (i \in A),$$
$$(M - c_i)(-x_i) \leq 0 \quad (i \in B),$$

$$(M - 1)\sum c_i x_i \leq (M - 1)c.$$

Adding these inequalities and dividing by M, we obtain

$$\sum c_i x_i + \sum_A x_i - \sum_B x_i \leq c + |A| + M^{-1}\left(-c + \sum_A c_i\right)$$

and conclude that (4.2) belongs to the closure of (4.1).

   Case 2. $\sum_{j \in A} a_{ij} > b_i$ for some i. Setting $M = \max |a_{ij}|$ we have

$$(M - a_{ij})x_j \leq M - a_{ij} \quad (j \in A),$$
$$(M + a_{ij})(-x_j) \leq 0 \quad (j \in B),$$

$$\sum a_{ij}x_j \leq b_i.$$

Adding these inequalities and dividing by M, we obtain

$$\sum_{j \in A} x_j - \sum_{j \in B} x_j \leq |A| + M^{-1}\left(b_i - \sum_{j \in A} a_{ij}\right)$$

and conclude that

$$\sum_A x_j - \sum_B x_j \leq |A| - 1$$

belongs to the closure of (4.1). Therefore (4.2) also belongs to the closure of (4.1).

   The proof is finished. The reader may have noticed that not all the inequalities (4.2) are required for the induction leading to $\sum c_i x_i \leq c - 1$. Indeed, we can restrict ourselves only to those with

$$A \cup B = \{1, 2, \ldots, |A| + |B|\}.$$

Then the induction is performed along a binary tree with $n + 1$ levels. All the 2^k vertices of the kth level are labelled by distinct zero-one vectors $(z_1, z_2, \ldots, z_k)$ and associated with inequalities

$$\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{k}(2z_i - 1)x_i \leq c - 1 + \sum_{i=1}^{k} z_i.$$

Each vertex labelled $(z_1, z_2, \ldots, z_k)$ with $k < n$ has two successors labelled $(z_1, z_2 \ldots, z_k, 0)$ and $(z_1, z_2, \ldots, z_k, 1)$. The inequality assigned to a parent vertex is obtained by adding the inequalities at its two successors, dividing by two and rounding the right-hand side down to the nearest integer. (Since the right-hand sides of the successors differ in parity, the rounding always cuts down exactly one half.) The inequalities at the terminal vertices are obtained in one of two different ways, according to whether $(z_1, z_2, \ldots, z_n)$ is feasible with respect to (4.1) or not.

   The whole picture rather resembles a binary search (in vain) for a feasible vector $(z_1, z_2, \ldots, z_n)$ that would satisfy $\sum c_i z_i = c$. Actually, it turns out that our method is a translation of the branch-and-bound method [1] into the language of linear inequalities. During the search, we are after the inequality $\sum c_i x_i \leq c - 1$. Therefore we split all possible choices of

integers $x_1, x_2, \ldots, x_n$ into two classes (corresponding to $x_1 = 0$ and $x_1 = 1$) and proceed to prove the inequality in each class separately. The two classes correspond to the two first-level inequalities. Indeed, the inequality

$$-x_1 + \sum c_i x_i \leq c - 1$$

is just another way of saying: "if $x_1 = 0$ then $\Sigma c_i x_i \leq c - 1$; if $x_1 = 1$ then possibly $\Sigma c_i x_i = c$". Similarly, the inequality

$$x_1 + \sum c_i x_i \leq c,$$

reads: "if $x_1 = 0$ then possibly $\Sigma c_i x_i = c$; if $x_1 = 1$ then necessarily $\Sigma c_i x_i \leq c - 1$." The dichotomy between $x_1 = 0$ and $x_1 = 1$ is taken care of by the rounding device. We go on like this, step by step, and require one more x_i at each step to be fixed at a specified value (zero or one) until we hit the level where all the x_i's are fixed. If they are feasible (Case 1) then they cannot satisfy $\Sigma c_i x_i = c$; if they are not feasible (Case 2) then they cannot be reached at all.

Often, it happens that a kth level inequality ($k < n$) belongs to the *elementary* closure of the inequalities (4.1) and $\Sigma c_i x_i \leq c$. In that case we can stop branching out from the corresponding vertex and simplify the proof considerably. In the following section, we illustrate this situation (Example 5.1).

5. Combinatorial applications: independent sets in hypergraphs

Many extremal combinatorial problems can be formulated as problems of finding the largest independent set in a hypergraph. A *hypergraph H* is an ordered pair (V, E), where V is a set and E a collection of subsets of V (see [2]). A set $X \subset V$ is called *independent* (in H) if there is no $A \in E$ with $A \subset X$. If V is finite then the problem of finding the largest set X independent in H is the following zero-one linear programming problem: Maximize $\Sigma_{i \in V} x_i$ subject to

(5.1) $0 \leq x_i \leq 1$ $(i \in V),$
             $\sum_{i \in A} x_i \leq |A| - 1$ $(A \in E),$
(5.2) $x_i = \text{integer}$ $(i \in V).$

Theorem 3.1 guarantees that each inequality $\Sigma_{i \in V} x_i \leq x_0$ valid under constraints (5.1), (5.2) belongs to the closure of (5.1).

Example 5.1. *Hamiltonian circuits in graphs.* The problem of determining whether a given graph $G^* = (V^*, E^*)$ has a Hamiltonian circuit is one of the above kind. The corresponding hypergraph $H = (V, E)$ has $V = E^*$ and the collection E includes two different kinds of sets A:

  (i) those consisting of three distinct edges of G^* that have all one vertex in common;
 (ii) those consisting of circuits in G^* having less than $|V^*|$ edges.

It is quite easy to see that each independent set of size $|V^*|$ in H constitutes a Hamiltonian circuit in G^* and vice versa. Thus G^* has a Hamiltonian circuit if and only if the corresponding zero-one linear programming problem has a feasible solution with

$$\sum_{i \in E^*} x_i = |V^*|.$$

As an example, we consider the Petersen graph with edges enumerated as in Fig. 1. Setting up the linear programming problem, we arrive at constraints

(5.3) $0 \leq x_i \leq 1$ $(i = 1, 2, \ldots, 15),$

ten "star constraints" of the type (i), that is

(5.4) $x_1 + x_2 + x_7 \leq 2,$
             $x_2 + x_3 + x_8 \leq 2,$ etc.,
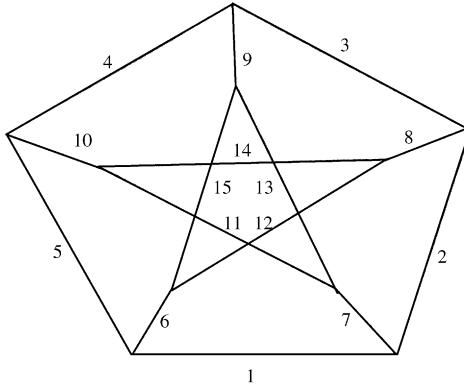
Fig. 1.

and many "circuit constraints" of the type (ii), that is

(5.5) $x_1 + x_2 + x_3 + x_4 + x_5 \leq 4,$
$x_1 + x_2 + x_8 + x_{14} + x_{10} + x_5 \leq 5,$ etc.

It is notoriously well-known that the Petersen graph has no Hamiltonian circuit. Equivalently, one has

(5.6) $$\sum_{i=1}^{15} x_i \leq 9$$

for every choice of integers $x_1, x_2, \ldots, x_{15}$ satisfying (5.3)–(5.5). The integrality constraint is essential here – indeed, setting $x_i = \frac{2}{3}(i = 1, 2, \ldots, 15)$ we satisfy (5.3)–(5.5) and violate (5.6). We are going to show that (5.6) belongs to the closure of (5.3)–(5.5), giving thus a proof of the nonexistence of a Hamiltonian circuit in the Petersen graph.

Let us consider the binary tree in Fig. 2. With each of its vertices (labelled A, B), we associate an inequality

$$\sum_{i=1}^{15} x_i + \sum_{i \in A} x_i - \sum_{i \in B} x_i \leq 9 + |A|.$$

It is not difficult lo prove that the inequalities assigned to the terminal vertices belong to the closure of (5.3)–(5.5). For instance, the inequality corresponding to $A = \{9, 14, 1\}$, $B = \emptyset$ is obtained as the sum of the inequalities

$x_3 + x_4 + x_9 \leq 2,$ $x_8 + x_{12} + x_{14} \leq 2,$
$x_9 + x_{13} + x_{15} \leq 2,$ $x_1 + x_5 + x_6 \leq 2,$
$x_{10} + x_{11} + x_{14} \leq 2,$ $x_1 + x_2 + x_7 \leq 2.$

The inequality corresponding to $A = \{1, 9\}$, $B = \{4, 13, 14\}$ is a sum of the inequalities

$x_{10} + x_{11} + x_{14} \leq 2,$ $x_1 + x_5 + x_6 \leq 2,$
$x_1 + x_2 + x_7 \leq 2,$ $x_9 \leq 1,$ $-x_{14} \leq 0,$

$x_3 + x_8 + x_{12} + x_{15} + x_9 \leq 4.$

Similarly, every other inequality corresponding to a terminal vertex of our tree can be obtained as a sum of a subset of (5.3)–(5.5). As in Section 4, the inequality at each parental vertex (labelled A, B) can be obtained by taking the sum of the two inequalities assigned to its descendants (labelled $A \cup \{k\}$, B and A, $B \cup \{k\}$), dividing by two and rounding the right-hand side down. Thus we conclude that (5.6) belongs to $e^5(S)$ where S is the set of inequalities (5.3)–(5.5).

The application of this technique to the problems of existence of Hamiltonian circuits is discussed in detail in [5]. In particular, [5] contains the following "one-two-three theorem". Given any graph $G = (V, E)$ consider the S

ϕ, {9,14}

{14}, {1,9}

{1,14}, {2,5,9}

{1,2,14}, {5,9}

{1,5,14}, {2,9}

{1,2,5,14}, {9}

{9}, {1,14}

{1,9}, {4,13,14}

{1,13,9}, {4,14}

{1,4,9}, {13,14}

{1,4,9,13}, {14}

{9,14}, {1,4,15}

{9,14,15}, {1,4}

{4,9,14}, {1,15}

{4,9,14,15}, {1}

{1,9,14}, ϕ
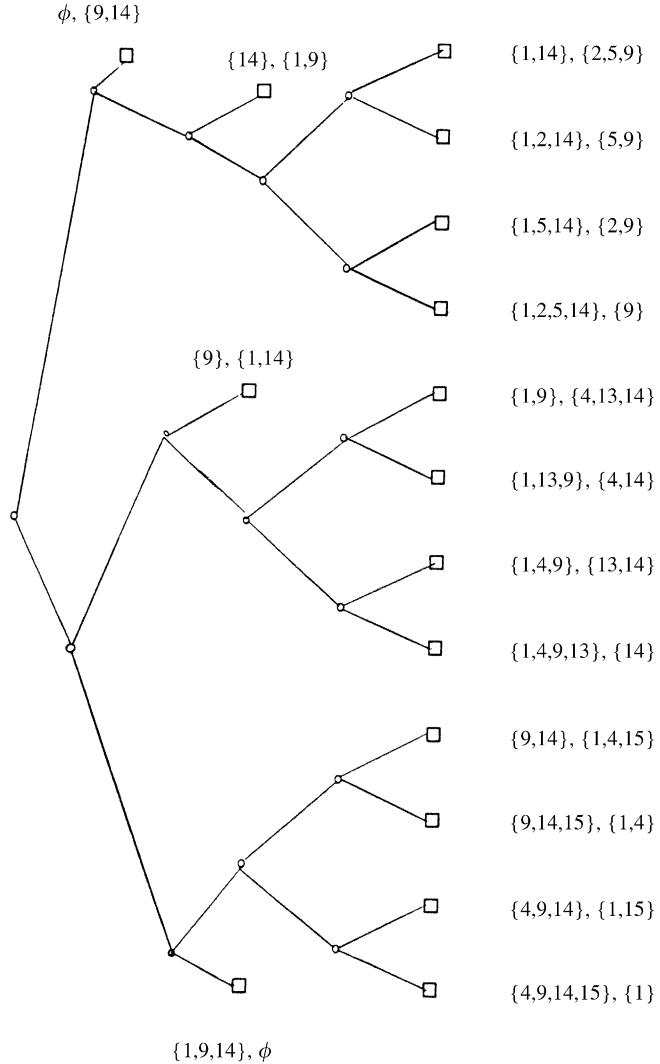
Fig. 2.

of inequalities

$$0 \le x_i \le 1 \qquad\qquad (i \in E),$$
$$\sum_{N \in i} x_i \le 2 \qquad\qquad (v \in V),$$
$$\sum_{I \subset W} x_i \le |W| - 1 \quad (W \subset V, 0 < |W| < |V|).$$

If the maximum of $\Sigma_{i \in E} x_i$ subject to the constraints S and so called "comb inequalities" (which belong to $e^1(S)$) equals $|V|$ then G has the following properties:

  (i) deletion of k vertices from G always results in a graph with at most k components (in other words, G is 1-tough),
 (ii) V can be covered by pairwise disjoint circuits (in other words, G has a 2-factor),
(iii) given any $u, v, w \in V$ there is a circuit in G that passes through all three u, v, w (in other words, G is 3-cyclable).

Example 5.2. *Moser's cube problem.* Let us consider the three-dimensional tick-tack-toe cube with 27 points $(0, 0, 0), (0, 0, 1), \ldots, (2, 2, 2)$. Our objective is to select as many of these 27 points as possible without choosing three collinear ones. Assigning to each point (a, b, c) a variable x_i with $i = 9a + 3b + c + 1$ (see Fig. 3) we arrive at

7	8	9
4	5	6
1	2	3

16	17	18
13	14	15
10	11	12

25	26	27
22	23	24
19	20	21

BOTTOM PLANE   MIDDLE PLANE   TOP PLANE
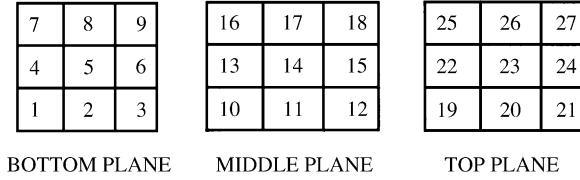
Fig. 3.

the following integer programming formulation of the problem: Maximize $\Sigma_{i=1}^{27} x_i$ subject to

$$0 \le x_i \le 1 \qquad (i = 1, 2, \ldots, 27),$$
$$x_1 + x_2 + x_3 \le 2,$$
$$x_4 + x_5 + x_6 \le 2,$$
$$\vdots$$
(5.7)
$$x_1 + x_5 + x_9 \le 2,$$
$$\vdots$$
$$x_1 + x_{14} + x_{27} \le 2,$$
$$\vdots$$
$$x_i = \text{integer} \qquad (i = 1, 2, \ldots, 27).$$

(Altogether, we have 49 constraints of the form $x_i + x_j + x_k \le 2$, corresponding to 49 collinear triples.) Setting $x_i = \frac{2}{3} (i = 1, 2, \ldots, 27)$ we satisfy all the inequalities (5.7) and obtain $\Sigma x_i = 18$. However, it can be shown that every choice of 17 points out of our 27 always contains a collinear triple. Equivalently, the inequality $\Sigma x_i \le 16$ belongs to the closure of (5.7). This can be shown as follows. We have

$$\tfrac{5}{6}(x_1 + x_2 + x_3) \le \tfrac{5}{3}, \qquad \tfrac{5}{6}(x_1 + x_4 + x_7) \le \tfrac{5}{3},$$
$$\tfrac{5}{6}(x_3 + x_6 + x_9) \le \tfrac{5}{3}, \qquad \tfrac{5}{6}(x_7 + x_8 + x_9) \le \tfrac{5}{3},$$
$$\tfrac{1}{3}(x_1 + x_5 + x_9) \le \tfrac{2}{3}, \qquad \tfrac{1}{3}(x_3 + x_5 + x_7) \le \tfrac{2}{3},$$
$$\tfrac{1}{6}(x_2 + x_5 + x_8) \le \tfrac{1}{3}, \qquad \tfrac{1}{6}(x_4 + x_5 + x_6) \le \tfrac{1}{3}.$$

Adding these inequalities up we conclude that

$$2(x_1 + x_3 + x_7 + x_9) + (x_2 + x_4 + x_6 + x_8) + x_5 \le \left\lceil \tfrac{26}{3} \right\rceil = 8$$

belongs to the closure of (5.7). Multiplying the last inequality by $\frac{4}{5}$ and adding the inequalities

$$\tfrac{2}{5}(x_1 + x_5 + x_9) \le \tfrac{4}{5}, \qquad \tfrac{2}{5}(x_3 + x_5 + x_7) \le \tfrac{4}{5},$$
$$\tfrac{1}{5}(x_2 + x_5 + x_8) \le \tfrac{2}{5}, \qquad \tfrac{1}{5}(x_4 + x_5 + x_6) \le \tfrac{2}{5},$$

we find that

(5.8) $$2(x_1 + x_3 + x_7 + x_9) + (x_2 + x_4 + x_6 + x_8) + 2x_5 \le \left\lceil \tfrac{44}{5} \right\rceil = 8$$

belongs to the closure of (5.7). Now, we set

$$A = x_1 + x_3 + x_7 + x_9 + x_{19} + x_{21} + x_{25} + x_{27},$$

$$B = x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{16} + x_{18} + x_{20} + x_{22} + x_{24} + x_{26},$$

$$C = x_5 + x_{11} + x_{13} + x_{15} + x_{17} + x_{23},$$

$$D = x_{14}.$$

Hence A is the sum of variables assigned to the corners of the cube, B corresponds to edges, C to faces and D to the center of the cube. The inequality (5.8) applies to the points in the bottom horizontal plane. Adding up nine inequalities of this sort (corresponding to nine planes perpendicular to one of the coordinate axes) we obtain

$$6A + 4B + 4C + 6D \leq 72.$$

Adding up all the 12 constraints corresponding to lines that join centers of edges via centers of faces, we obtain

$$2B + 2C \leq 24.$$

Dividing the sum of the last two inequalities by six we arrive at

$$\sum_{i=1}^{27} x_i = A + B + C + D \leq 16$$

which is the desired result.

More generally, one can consider the 3^n-cube and ask for the largest size $f(n)$ of its subset containing no three collinear points. It is easy to show that $f(1) = 2$, $f(2) = 6$, $f(3) = 16$; recently Chandra proved that $f(4) = 43$. It is not difficult to show that $f(n) \geq c. \cdot 3^n / \sqrt{n}$ (see [4]). Moser [16] conjectured that $f(n) = o(3^n)$; this, apparently difficult, problem is still unsettled. Perhaps the technique indicated here could help to solve Moser's conjecture.

6. Combinatorial applications: coloring of hypergraphs

A k-coloring of a hypergraph $H = (V, E)$ is a partition

$$V = C_1 \cup C_2 \cup \ldots \cup C_k$$

such that each C_j is independent in H. In a coloring problem, one asks for the smallest k such that H admits a k-coloring. The coloring problems include the celebrated four-color conjecture as well as the problems of Ramsey's type [3]. At first, it seems that the coloring problems are different from those considered in the previous section. Yet there is an easy way of reducing them to the previous type. Given a hypergraph $H = (V, E)$ and a positive integer k, we consider the hypergraph $H^* = (V^*, E^*)$ where $V^* = V \times \{1, 2, \ldots, k\}$ and E^* includes two kinds of sets A^*:

(i) all the sets $A^* = A \times \{j\}$, where $A \in E, 1 \leq j \leq k$,
(ii) all the couples $A^* = \{(v, i), (v, j)\}$, where $v \in V, i \neq j$.

A moment's reflection shows that H is k-colorable if and only if H^* contains an independent set of size $|V|$.

Example 6.1. *Ramsey's theorem.* It is well-known that, whenever one colors the 15 edges of a complete graph with six vertices by two colors (customarily, red and blue are used), a monochromatic triangle is bound to pop out. Guided by the philosophy explained above, we can formulate this statement as follows. The maximum of

$$T = \sum_{1 \leq i < j \leq 6} (x_{ij} + y_{ij})$$

subject to

$$(6.1) \quad \begin{array}{ll} x_{ij} + x_{jk} + x_{ik} \leq 2, & \\ y_{ij} + y_{jk} + y_{ik} \leq 2, & (1 \leq i < j < k \leq 6) \\ -x_{ij} \leq 0, & \\ -y_{ij} \leq 0, & (1 \leq i < j \leq 6) \\ x_{ij} + y_{ij} \leq 1, & \\ x_{ij} = \text{integer}, y_{ij} = \text{integer} \end{array}$$

does not exceed 14. (Here $x_{ij} = 1$ corresponds to the edge $\{i, j\}$ colored day-glow orange [19] and $y_{ij} = 1$ corresponds to $\{i, j\}$ colored vermilion.)

We proceed to show that $T \leq 14$ belongs to the closure of (6.1). We easily find that $T \leq 15$ does so. Indeed, this is just the sum of all the inequalities $x_{ij} + y_{ij} \leq 1$. Actually, the maximum of T subject to (6.1) equals 15 and can be attained by setting $x_{ij} = y_{ij} = \frac{1}{2}$.

Now, adding up the inequalities

$$x_{12} + x_{13} + x_{23} \leq 2, \quad x_{12} + x_{14} + x_{24} \leq 2,$$
$$x_{13} + x_{14} + x_{34} \leq 2, \quad y_{23} + y_{24} + y_{34} \leq 2,$$
$$x_{ij} + y_{ij} \leq 1 \quad (ij \neq 23, 24, 34),$$
$$T \leq 15,$$

we obtain the inequality

$$2(T + x_{12} + x_{13} + x_{14}) \leq 35.$$

Hence

$$T + x_{12} + x_{13} + x_{14} \leq 17$$

belongs to the closure of (6.1). In the same way, we deduce

$$T + x_{12} + x_{13} + x_{15} \leq 17,$$
$$T + x_{12} + x_{14} + x_{15} \leq 17,$$
$$T + x_{13} + x_{14} + x_{15} \leq 17.$$

Adding up the last four inequalities and $2T \leq 30$ on the top, we obtain

$$3(2T + x_{12} + x_{13} + x_{14} + x_{15}) \leq 98.$$

Therefore

$$2T + x_{12} + x_{13} + x_{14} + x_{15} \leq 32$$

belongs to the closure of (6.1). Similarly, we obtain

$$2T + x_{12} + x_{13} + x_{14} + x_{16} \leq 32, \quad 2T + x_{12} + x_{13} + x_{15} + x_{16} \leq 32,$$
$$2T + x_{12} + x_{14} + x_{15} + x_{16} \leq 32, \quad 2T + x_{13} + x_{14} + x_{15} + x_{16} \leq 32.$$

Adding up the last five inequalities and $2T \leq 30$, we arrive at

$$4(3T + x_{12} + x_{13} + x_{14} + x_{15} + x_{16}) \leq 190,$$

so that

$$3T + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} \leq 47$$

belongs to the closure of (6.1). By the same series of arguments, the inequality

$$3T + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} \leq 47$$

belongs to the closure of (6.1). Adding up these two inequalities and all

$$x_{ij} + y_{ij} \leq 1 \quad (2 \leq i < j \leq 6),$$

we arrive at $7T \leq 104$. Therefore $T \leq 14$ belongs to the closure of (6.1).

The astute reader has noticed that our proof simulates the standard one. We investigated colorings where some of the edges can be left uncolored but no monochromatic triangle occurs; the total number of colored edges is T. We start by observing that $x_{12} = x_{13} = x_{14} = 1$ is incompatible with $T = 15$ (in other words, if in a full coloring all three edges $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ are colored day-glow orange then we run into a contradiction–either one of the triangles 123, 124, 134 is day-glow orange or else 234 is vermilion). Equivalently, $T + x_{12} + x_{13} + x_{14} \leq 17$. Thus only two of the three

edges $\{1, 2\}, \{1, 3\}, \{1, 4\}$ can be colored day-glow orange. Now, symmetry and common sense show that only two of the five edges $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}$ can be colored day-glow orange ($3T + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} \leq 47$). However, the process of getting this inequality from $T + x_{1i} + x_{1j} + x_{1k} \leq 17$ is painfully slow. Similarly, only two out of the five edges $\{1, i\}$ can be colored vermilion ($3T + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} \leq 47$) and so the coloring can never be full ($T \leq 14$). Along the same lines, one can translate the entire proof of Ramsey's theorem into the closure operation language.

7. A hierarchy of combinatorial problems

We have shown that each discrete (integer) linear programming problem can be reduced into a continuous (non-integer) one by a finite number of applications of the elementary closure operation. Hence the process of solving a typical combinatorial (that is, 0-1 linear programming) problem can be seen as consisting of two phases:

(i) generating a number of new linear constraints that belong to the closure of the original ones.

(ii) solving the resulting non-integer linear programming problem. The complexity of phase (i) depends on the divisibility properties of linear combinations of the coefficients in the original constraints. This phase has a number-theoretical character while the other one belongs to the realm of continuous mathematics. A slogan to advertise our Theorem 3.1 might read:

$$\boxed{\text{combinatorics} = \text{number theory} + \text{linear programming}}$$

Now, let us consider a typical integer linear programming problem: Maximize $\Sigma c_j x_j$ subject to the set of S of linear constraints

$$(7.1) \qquad \sum a_{ij} x_j \leq b_i$$

and the integrality constraint

$$(7.1) \qquad x_j = \text{integer} \quad (j = 1, 2, \ldots, n).$$

We shall assume that S defines a *bounded* polyhedron. By Theorem 3.1, there is an integer k such that the maximum of $\Sigma c_j x_j$ subject to S and integrality equals the maximum of $\Sigma c_j x_j$ subject to $e^k(S)$. The smallest such k will be called the *rank* of the problem (7.1). Obviously, the rank of each one-variable problem is at most one. However, the two-variable problems can have an arbitrarily high rank. One of the simplest examples of these is due to Prof. Bondy: maximize y subject to:

$$0 \leq x \leq 1,$$
$$y \geq 0,$$
$$-tx + y \leq 1,$$
$$tx + y \leq t + 1,$$
$$x, y = \text{integer}.$$

(Here t is an arbitrary positive number.)

We can define the rank of any combinatorial problem that can be formulated as a zero-one linear programming problem. However, unless the linear programming formulation is stated explicitly, such a definition can be ambiguous. For instance, the problem of finding the largest independent (stable) set of vertices in a graph $G = (V, E)$ admits at least two different linear programming formulations ($V = \{1, 2, \ldots, n\}$, C is the set of all cliques in G):

(A) maximize $\Sigma_{j=1}^{n} x_j$ subject to:

$$0 \leq x_j \leq 1 \qquad (j \in V),$$
$$x_i + x_j \leq 1 \qquad (\{i, j\} \in E),$$
$$x_j = \text{integer} \quad (j \in V).$$

(B) maximize $\Sigma_{j=1}^n x_j$ subject to:

$$
(7.2) \quad
\begin{aligned}
x_j &\geq 0 && (j \in V), \\
\sum_{j \in A} x_j &\leq 1 && (A \in C), \\
x_j &= \text{integer} && (j \in V).
\end{aligned}
$$

If G is complete then (A) has rank $\geq \log_2(n-1)$ while (B) has rank zero. In the following, we will show that there is no universal upper bound on the rank of (B).

An inequality $\Sigma a_j x_j \leq b$ will be called *positive regular* if $a_j \geq 0$ $(j = 1, 2, \ldots, n)$ and $b \geq \max(a_1, a_2, \ldots, a_n) > 0$. A *strength* of such an inequality is the ratio $(\Sigma a_j) b^{-1}$. A linear inequality will be called *negative regular* if it reads $-x_j \leq 0$.

Lemma 7.1. *Let S be a set of linear inequalities*

$$
\sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m)
$$

where, for each $i = 1, 2, \ldots, n$, the ith inequality reads $-x_i \leq 0$ and, for each $i = n+1, n+2, \ldots, m$, the ith inequality is positive regular of strength $\leq s$. Let the inequality

$$
(7.3) \quad \sum_{j=1}^n a_j x_j \leq b
$$

belong to $e^1(S)$. Then (7.3) can be written as a linear combination of the negative regular inequalities in S and a positive regular inequality of strength $< 2s$ that belongs to $e^1(S)$.

Proof. There are nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$
a_j = \sum_{i=1}^m \lambda_i a_{ij} = -\lambda_j + \sum_{i=n+1}^m \lambda_i a_{ij} = \text{integer} \quad (j = 1, 2, \ldots, n),
$$

$$
b = \left[\sum_{i=1}^m \lambda_i b_j \right] = \left[\sum_{i=n+1}^m \lambda_i b_i \right].
$$

Set

$$
\mu_i =
\begin{cases}
\lambda_i - [\lambda_i], & i = 1, 2, \ldots, n \\
\lambda_i, & i = n+1, n+2, \ldots, m.
\end{cases}
$$

Then all the μ_i's are nonnegative and (7.3) can be written as a sum of inequalities

$$
(7.4) \quad [\lambda_i](-x_i) \leq 0,
$$

$$
(7.5) \quad \sum_{j=1}^n \left(\sum_{i=1}^m \mu_i a_{ij} \right) x_j \leq \left[\sum_{i=1}^n \mu_i b_i \right].
$$

For each $j = 1, 2, \ldots, n$ we have

$$
\sum_{i=1}^m \mu_i a_{ij} = -(\lambda_j - [\lambda_j]) + \sum_{i=n+1}^m \lambda_i a_{ij} = \text{integer}.
$$

Besides, we have $\lambda_j - [\lambda_j] < 1$ and $\Sigma_{i=n+1}^m \lambda_i a_{ij} \geq 0$. Therefore each $c_j = \Sigma_{i=1}^m \mu_i a_{ij}$ is a nonnegative integer. If $c_j = 0$ for all $j = 1, 2, \ldots, n$ then (7.3) is a sum of inequalities (7.4) and we are done. Next, we assume $c_k \geq 1$ for some k. Since $b_i \geq a_{ij}$ for all $i = n+1, n+2, \ldots, m$ and $j = 1, 2, \ldots, n$, we have

$$\sum_{i=n+1}^m \lambda_i b_i \geq \sum_{i=n+1}^m \lambda_i a_{ij} \geq c_j$$

for each j and so $b \geq c_j$ for each j. Hence (7.5) is positive regular and $b \geq 1$. Then $2b = 2[\Sigma \lambda_i b_i] > \Sigma \lambda_i b_i$ and so

$$2s \cdot b > \sum_{i=n+1}^m \lambda_i (sb_i) \geq \sum_{i=n+1}^m \lambda_i \left(\sum_{j=1}^n a_{ij} \right)$$
$$\geq \sum_{i=1}^m \mu_i \left(\sum_{j=1}^n a_{ij} \right) = \sum_{i=1}^n c_j.$$

Hence (7.5) has strength $< 2s$ and the proof is finished.

Let us note that the bound given by Lemma 7.1. is best possible. Indeed, if $n > \epsilon^{-1}$, then the inequality

$$\sum_{i=1}^{2n-1} x_i \leq 1$$

has strength greater than $(2 - \epsilon)n$ but belongs to the elementary closure of inequalities

$$\sum_{i \in A} x_i \leq 1 \quad (A \subset \{1, 2, \ldots, 2n - 1\}, |A| = n)$$

that all have strength n.

A repeated application of Lemma 7.1 yields

Theorem 7.2. *Let S be a set of regular inequalities; let k be a positive integer. Then each inequality that belongs to $e^k(S)$ can be written as a linear combination of negative regular inequalities that belong to S and positive regular inequality that belongs to $e^k(S)$. Besides, if all positive regular inequalities in S have strength $\leq s$ then all positive regular inequalities in $e^k(S)$ have strength $< 2^k \cdot s$.*

Corollary 7.3. *Given any N there is a graph G such that (B) has rank greater than N.*

Proof. Erdös [10] has shown that given any n there is a graph G with more than $2^{n/2}$ vertices that contains neither a complete subgraph with n vertices nor an independent set of n vertices. Let k be the rank of (B) corresponding to G. Then the maximum of Σx_i subject to $e^k(S)$ does not exceed $n - 1$ and so, by Lemma 3.2, the inequality

$$\sum_{i \in V} x_i \leq n - 1$$

belongs to $e^{k+1}(S)$. By Theorem 7.2, the strength of each positive regular inequality in $e^{k+1}(S)$ is smaller than $(n - 1)2^{k+1}$. Hence we have

$$|V| \cdot (n - 1)^{-1} < (n - 1)2^{k+1}$$

and so

$$2^{k+1} > 2^{n/2} \cdot (n - 1)^{-2}.$$

If n is sufficiently large (with respect to N) then the last inequality implies $k > N$ which is the desired conclusion.

The rank of a problem measures the complexity of the first phase of its solution, that is the complexity of its reduction into a continuous problem. It indicates the "degree of discreteness". If discreteness is what makes discrete mathematics

hard, then the rank should relate to computational complexity. There seems to be an indication of such a relation. Indeed, a number of combinatorial problems have rank zero [14,18]; these include integral network flows and bipartite matching problems. Edmonds' theory show that matching problems in general have rank at most one. All these problems of small rank are known to admit polynomial-time algorithms [8;9A;14A]. On the other hand, the class (B) of this section has unbounded rank and belongs to the class of "hard" problems introduced by Cook [5B] and Karp [15A]. One may be tempted to believe that each class of zero-one linear programming problems having a bounded rank possesses a polynomial-time algorithm. If this were true then, in particular, there would be a polynomial-time algorithm searching for the largest independent sets in perfect graphs. (Indeed, these are exactly those graphs for which (B) has rank zero. The proof of the equivalence, based on the theory of anti-blocking polyhedra [10A], has been known to Professor D.R. Fulkerson; it can be found in [16A]. An alternative] proof, based on the results of Lovász [15B,15C] is given in [5A].)

In this context, it may be interesting to note that each class of integer linear programming problems with bounded rank admits a good characterization, More precisely, if (7.1) has optimum x_0 and rank at most k then there is a string of at most $1 + n + \cdots + n^{k+1}$ linear inequalities such that

(i) each of them belongs either to S or to the elementary closure of (at most n of) the preceding ones,
(ii) the last one reads $\Sigma c_i x_i \le x_0$.

Appendix

Relations to Gomory's algorithm

An alternative proof of Theorem 3.2 can be based on Gomory's integer programming algorithm [12]. Here we begin with a set of inequalities

$$(8.1) \qquad x_j \le 0,$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i,$$

$$(8.2) \qquad \sum_{j=1}^{n} c_j x_j \le c_0,$$

where a_{ij}, b_i, c_j are integers, the polyhedron defined by (8.1) is bounded and (8.2) holds for every choice of integers $x_1, x_2, \ldots, x_n$ that satisfy (8.1). Gomory describes a way of generating new constraints, called *cuts*, that are satisfied by every choice of integers $x_1, x_2, \ldots, x_n$ satisfying (8.1) as well as all the previously generated cuts. It turns out that these cuts belong to the closure of (8.1); an account of this is given by Hn [15, Section 13.3]. Gomory proves that, after a finite number of cuts are generated in a systematic fashion, the maximum of $\Sigma c_i x_i$, subject to (8.1) and the added cuts, can be attained by integers $x_1, x_2, \ldots, x_n$. Therefore, by the duality theorem, (8.2) belongs to the closure of (8.1). Now, to prove Theorem 3.1 in its full generality, one has to get rid of the inequalities $x_j \ge 0$ in (8.1) as well as to get around the integrality assumption placed upon a_{ij}, b_i. However, these modifications can be carried out in quite a routine manner.

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References

[1] E. Balas, A note on the branch-and-bound principle, Operations Res. 16 (1968) 443–445.
[2] C. Berge, Graphes et hypergraphs (Dunod, Paris, 1970).

[3] V. Chvátal, Hypergraphs and Ramseyian theorems, Proc. Am. Math. Soc. 27 (1971) 434–440.

[4] V. Chvátal, Remarks on a problem of Moser, Canad. Math. Bull. 15 (1972) 19–21.

[5] V. Chvátal, Edmonds polytopes and weakly Hamiltonian graphs, Math. Programming, to appear.

[5A] V. Chvátal, On certain polytopes associated with graphs, submitted to J. Combin. Theory.

[5B] S.A. Cook, The complexity of theorem-proving procedures, in: Conference record of third ACM symposium on theory of computing (1970) 151–158.

[6] J. Edmonds, Minimum partition of a matroid into independent subsets, J. Res. Nat. Bur. Standards Sect. B 69 (1965) 67–72.

[7] J. Edmonds, Maximum matching and a polyhedron with 0, 1-vertices, J. Res. Nat. Bur. Standards Sect. B 69 (1965) 125–130.

[8] J. Edmonds, Paths, tree and flowers, Canad. J. Math., 17 (1965) 449–467.

[9] J. Edmonds and E.L. Johnson, Matching: A well-solved class of integer linear programs, in: R.K. Guy et al., eds., Combinatorial structures and their applications (Gordon and Breach, New York, 1970).

[9A] J. Edmonds and R.M. Karp, Theoretical improvements in algorithmic efficiency for network flow problems, in: R.K. Guy et al., eds., Combinatorial structures and their applications (Gordon and Breach, New York, 1970).

[10] P. Erdös, Some remarks on the theory of graphs. Bull. Amer. Math. Soc. 53 (1947) 292–294.

[10A] D.R. Fulkerson, Anti-blocking polyhedra. J. Combin. Theory 12 (1972) 50–71.

[11] T. Gallai, On directed paths and circuits, in: P. Erdös and G. Katona, eds., Theory of graphs (Akademiai Kiado, Budapest, 1968).

[12] R.E. Gomory, An algorithm for integer solutions to linear programs, Princeton IBM Math. Report (Nov. 1958); also in: R.L. Graves and P. Wolfe, eds., Recent advances in Mathematical programming (McGraw-Hill, New York, 1963).

[13] M. Hall, Jr., Combinatorial theory (Biaisdell, Waltham, Mass., 1967).

[14] A.J. Hoffman and J.B. Kruskal, Integral boundary points of convex polyhedra, in: H.W. Kuhn and A.W. Tucker, eds., Linear inequalities and related systems, Ann. of Math. Study 38 (Princeton University Press, Princeton, NJ. 1956).

[14A] J. Hopcroft and R.M. Karp, A $n^{5/2}$ algorithm for maximum matchings in bipartite graphs, in: Twelfth annual symposium on switching and automata theory (1971) 122–125.

[15] T.C. Hu, Integer programming and network flows (Addison–Wesley, Reading, Mass., 1969).

[15A] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller et al., eds., Complexity of computer computations (Plenum Press, New York, 1972).

[15B] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (1972) 253–268.

[15C] L. Lovász, A characterization of perfect graphs, J. Combin. Theory 13 (1972) 95–98.

[16] L. Moser, Problem P. 170, Canad. Math. Bull. 13 (1970) 268.

[16A] L.E. Trotter, Jr., Solution characteristics and algorithms for the vertex-packing problem, Thesis, Cornell University (1973).

[17] W.T. Tutte, The factorization of linear graphs, J. London Math. Soc. 22 (1947) 107–111.

[18] A.F. Veinott, Jr., and G.B. Dantzig, Integer extreme points, SIAM Review 10 (1968) 371–372.

[19] T. Wolf, The electric kool-acid acid test (Bantam Books, 1969).