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Analysis of interval-based possibilistic networks

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Abstract. This paper proposes interval-based possibilistic networks. This extension allows to compactly encode and reason with epistemic uncertainty and imprecise beliefs as well as with multiple expert knowledge. We propose a natural semantics based on compatible possibilistic networks. The paper shows finally that computing uncertainty bounds of an event can be done in interval-based networks without extra computational cost.

1 Introduction

Graphical models are compact and powerful tools for modeling and reasoning with uncertain and complex information. However, it is difficult for an agent to provide precise and reliable crisp belief degrees. This has led researchers to develop alternative and flexible formalisms for representing and managing ill-known beliefs. Interval-based possibilistic networks (IPNs) generalize standard possibilistic networks [2][1] to encode ill-know beliefs where these latter are encoded by means of sub-intervals of [0, 1]. More precisely, IPNs allow to compactly encode families of standard joint possibility distributions. Intervals offer more flexibility to represent and to handle incomparable events.

In this paper, we first propose a definition of IPNs which extend standard possibilistic ones and we extend the definition of chain rule for IPNs. We then propose a natural semantics for IPNs based on compatible standard possibilistic networks. We provide precise relations between a set of compatible standard networks and an interval-based distribution induced by the extended chain rule. Lastly, we show that computing the uncertainty bounds of any event, given in terms of guaranteed possibility measure Δ and possibility measure Π can be computed in IPNs without extra computational cost.

2 Interval-based possibilistic networks: Definitions

An IPN $\mathcal{I}$ is a possibilistic network where the graphical component has the same representation while local possibility tables contain intervals allowing to encode some imprecision on the encoded beliefs.

Definition 1. *An IPN $\mathcal{I}$=<G,Θ^I> is a belief network where the uncertainty is represented by intervals. Namely, $\mathcal{I}$ consists of*

1. *a directed acyclic graph G encoding direct independence relationships between variables $I(A_i, U_i, Y)$ and*
2. *a set of local interval-based possibility tables Θ^I where $\forall\theta^I_{a_i|u_i}\in\Theta^I$, $\theta^I_{a_i|u_i}\subseteq[0,1]$.*

In case where all the parameters of the network are singletons, then the network is a standard (pointwise-based) network.

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Example 1. *Figure 1 is an example of an IPN over two Boolean variables A and B.*

A	$\pi(A)$
T	[.8, 1]
F	[.4, 1]

B	A	$\pi(B\|A)$
T	T	[.2, .6]
F	T	[1, 1]
T	F	[.4, 1]
F	F	[.8, 1]

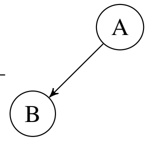

Figure 1. $\mathcal{I}$: Example of an IPN.

Compatible possibilistic networks

Let us define the concept of compatible network.

Definition 2. *Let $\mathcal{I}$=<G,Θ^I> be an IPN. A pointwise-based possibilistic network $\mathcal{G}$=<G,Θ> is compatible with $\mathcal{I}$ iff*

1. *$\mathcal{I}$ and $\mathcal{G}$ have exactly the same graph and*
2. *$\forall\theta_{a_i|u_i}\in\Theta$, $\theta_{a_i|u_i}\in\theta^I_{a_i|u_i}$ with $\theta^I_{a_i|u_i}\in\Theta^I$.*

Coherent interval-based possibilistic networks

In IPNs, we refer to the existence of compatible networks with the concept of coherent IPN defined as follows:

Definition 3. *An IPN $\mathcal{I}$=<G,Θ^I> is coherent iff*

- *there is at least one pointwise-based possibilistic network $\mathcal{G}$ which is compatible with $\mathcal{I}$ and*
- *all the values composing the parameters $\theta^I_{a_i|u_i}$ are feasible. Namely $\forall\theta^I_{a_i|u_i}\in\Theta^I_i$, $\forall\alpha\in\theta^I_{a_i|u_i}$, there exists a compatible network $\mathcal{G}$=<G,Θ> such that $\theta_{a_i|u_i}=\alpha$.*

3 IPNs: 2 alternative semantics

Semantics based on compatible models

This semantics views a coherent IPN $\mathcal{I}$ as a family of compatible pointwise-based networks. Each of these compatible networks encodes a joint pointwise-based distribution we call c-model.

Definition 4. *A possibility distribution π is a c-model for an IPN $\mathcal{I}$=<G, Θ^I> if there exists a standard network $\mathcal{G}$=<G, Θ> compatible with $\mathcal{I}$ such that $\forall a_1..a_n\in\Omega$, $\pi(a_1..a_n)=\min_{i=1}^n(\theta^G_{a_i|u_i})$, where u_i is the parent of a_i in $a_1..a_n$.*

There is another way to define a compatible joint distribution based on the concept of π-model, defined as follows:

Definition 5. *Let $\mathcal{F}_C$ denote the set of c-models of the IPN $\mathcal{I}$. A pointwise-based distribution π is a π-model of the IPN $\mathcal{I}$ iff it retrieves all the parameters of $\mathcal{I}$ and satisfies all the independence relations encoded by $\mathcal{I}$, namely:*

- *Condition 1: $\forall\theta^I_{a_i|u_i}\in\Theta^I$, $\Pi(a_i|u_i)\in\theta^I_{a_i|u_i}$.*
- *Condition 2: $\forall I(A_i, U_i, Y)\in I$, $\Pi(A_i|U_i,Y)=\Pi(A_i|U_i)$.*

Proposition 1. *A distribution π is a π-model of an IPN $\mathcal{I}$=<G, Θ^I> iff π is a c-model of $\mathcal{I}$, namely, $\pi\in\mathcal{F}_C$.*

Semantics based on extending the chain rule

Here we show if an IPN can be seen as an interval-based joint distribution. This latter is defined as follows:

Definition 6. *An interval-based joint distribution π_I is a mapping from Ω to the set of sub-intervals of $[0,1]$ which associates with each interpretation $\omega_i \in \Omega$ an interval $[\alpha_i, \beta_i] \subseteq [0,1]$.*

Definition 7. *A pointwise-based possibility distribution π is compatible with the interval-based one π_I iff $\forall \in \omega_i \Omega$, $\pi(\omega_i) \in \pi_I(\omega_i)$.*

Let $\mathcal{F}_C^I$ denote the set of pointwise-based joint possibility distributions that are compatible with the interval-based distribution π_I. Given an interval-based distribution π_I, the plausibility of an event ϕ is assessed as follows:

Definition 8. *Let $\phi \subseteq \Omega$ be an arbitrary event and let π_I be an interval-based distribution. Then the possibility interval associated with ϕ, computed from π_I, is the interval defined as follows:*

$$\Pi_I(\phi) = [\min_{\pi \in \mathcal{F}_C^I} (\Pi(\phi)), \max_{\pi \in \mathcal{F}_C^I} (\Pi(\phi))]. \quad (1)$$

Similarly, the conditional possibility of $\phi \subseteq \Omega$ given an evidence $\psi \subseteq \Omega$ is defined as an interval as follows:

$$\Pi_I(\phi \mid \psi) = [\min_{\pi \in \mathcal{F}_C^I} (\Pi(\phi \mid \psi)), \max_{\pi \in \mathcal{F}_C^I} (\Pi(\phi \mid \psi))]. \quad (2)$$

A natural question raises here about how to induce from an IPN $\mathcal{I}$ an interval-based joint distribution π_I. Two methods can be considered:

1. Extending the standard min-based chain rule to the interval-based setting directly as follows:

$$\pi_{\mathcal{I}}(a_1, a_2, .., a_n) = [\min_{i=1}^{n} \underline{\pi}(a_i|u_i), \min_{i=1}^{n} \overline{\pi}(a_i|u_i)]. \quad (3)$$

2. Using the compatible networks, a joint interval-based distribution can be directly computed from the compatible networks. Interestingly enough, this is equivalent to the extended chain rule of Equation 3. Namely,

   Proposition 2. *Let $\mathcal{F}_C^I$ denote the set of compatible pointwise-based networks $\mathcal{G}$ with the interval-based one $\mathcal{I}$. Then $\forall \omega \in \Omega$,*

$$\pi_I(\omega) = [\min_{\pi \in \mathcal{F}_C^I} (\Pi(\omega)), \max_{\pi \in \mathcal{F}_C^I} (\Pi(\omega))]. \quad (4)$$

4 Analysis of IPNs semantics

The following proposition relates the set of pointwise-based distributions $\mathcal{F}_C$ induced by the compatible standard networks and the set $\mathcal{F}_C^I$ of pointwise-based distributions compatible with the interval-based possibility distribution π_I.

Proposition 3. *Let $\mathcal{I}$=<G,Θ^I> be an IPN and let π_I be the induced interval-based distribution from $\mathcal{I}$. Then $\mathcal{F}_C \subseteq \mathcal{F}_C^I$ but $\mathcal{F}_C^I \not\subseteq \mathcal{F}_C$.*

Now we answer the following questions:

(1) Given an interval-based possibility distribution π_I, is it possible to build from π_I an IPN $\mathcal{I}$ such that $\forall \omega \in \Omega$, $\pi_I(\omega) = \pi_{\mathcal{I}}(\omega)$?
(2) Given a family of compatible possibility distributions $\mathcal{F}$, can we build an IPN $\mathcal{I}$ such that $\mathcal{F} = \mathcal{F}_C^I$?

The interesting point is that we get the converse of the results obtained in the previous section. Indeed, the statement (2) is false as shown in the following example.

Example 2. *Consider one binary variable A and assume that $\mathcal{F}$ is composed of the following two possibility distributions π_1 and π_2.*

A	$\pi_1(A)$
T	1
F	$.5$

A	$\pi_2(A)$
T	1
F	$.7$

Clearly, it is impossible to build an IPN $\mathcal{I}$ such that $\mathcal{F} = \mathcal{F}_C^I$.

However, statement (1) is true as it is shown in the following:

Proposition 4. *An interval-based possibility distribution π_I can be represented by an IPN $\mathcal{I}$. Namely $\forall \omega_i \Omega$, $\pi_I(\omega_i) = \pi_{\mathcal{I}}(\omega_i)$,*

where $\pi_{\mathcal{I}}(\omega_i)$ is the interval computed from the IPN $\mathcal{I}$ using the interval-based chain rule of Equation 3.

5 Inferring uncertainty bounds from IPNs

We show that for any event $\phi \subseteq \Omega$, the upper bound of the possibility degree $\overline{\Pi}_I(\phi)$ and the guaranteed possibility degree $\Delta_I(\phi)$ can be computed from two special standard networks, hence **the uncertainty interval is computed using only two calls for an inference algorithm in standard networks**.

Proposition 5. *Let $\mathcal{I}$=<G, Θ^I> be an IPN. Let $\mathcal{G}_L$=<G, Θ> be the standard possibilistic network such that i) $\mathcal{I}$ and $\mathcal{G}_L$ have the same graphical structure and ii) $\forall a_i \in D_{A_i}$, $\theta^{G_L}_{a_i|u_i} = \underline{\theta}^I(a_i|u_i)$. Then $\forall \phi \subseteq \Omega$, $\Delta_I(\phi) = \Pi^{G_L}(\phi)$.*

Similarly to computing $\Delta_I(\phi)$, the upper bound $\overline{\Pi}_I(\phi)$ can be computed from the standard network G_U defined as follows:

Proposition 6. *Let $\mathcal{I}$=<G, Θ^I> denote an IPN. Let $\mathcal{G}_U$=<G, Θ> be the standard possibilistic network such that i) $\mathcal{I}$ and $\mathcal{G}_U$ have the same graph and ii) $\forall a_i \in D_{A_i}$, $\theta^{G_U}_{a_i|u_i} = \overline{\theta}^I(a_i \mid u_i)$. Then, $\forall \phi \subseteq \Omega$, $\overline{\Pi}_I(\phi) = \Pi^{G_U}(\phi)$.*

6 Conclusions

This paper provided foundations of interval-based possibilistic networks.It provided two semantics for IPNs where the first one is based on compatible possibilistic networks while the second is based on interval-based joint possibility distributions. The paper related the two semantics and showed that they are not equivalent. The semantics that can be associated with an IPN is the family of distributions induced from the pointwise-based compatible networks. As a first consequence, inference algorithms like the junction tree algorithm [3] cannot directly be adapted for the interval-based setting since such algorithms rely on local distributions unless they are adapted to consider the set of possibility distributions.

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