Compact-MDD: Efficiently Filtering (s)MDD Constraints with Reversible Sparse Bit-Sets

Abstract

Multi-Valued Decision Diagrams (MDDs) are instrumental in modeling combinatorial problems with Constraint Programming. In this paper, we propose a related data structure called sMDD (semi-MDD) where the central layer of the diagrams is non-deterministic. We show that it is easy and efficient to transform any table (set of tuples) into an sMDD. We also introduce a new filtering algorithm, called Compact-MDD, which is based on bitwise operations, and can be applied to both MDDs and sMDDs. Our experimental results show the practical interest of our approach.

1 Introduction

Constraint Programming (CP) is a general and flexible framework for modeling and solving combinatorial constrained problems [Rossi *et al.*, 2006]. Many kind of constraints have been introduced in the literature, but general forms that are based on data structures such as tables, automatas, and MDDs (Multi-valued Decision Diagrams) remain quite popular. For example, over the past decade, many filtering algorithms have been proposed for table and MDD constraints, respectively leading to the state-of-the-art algorithms called Compact-Table [Demeulenaere *et al.*, 2016] and MDD4R [Perez and Régin, 2014]. In this paper, we focus our interest on decision diagrams [Bryant, 1986] for constraint reasoning, which is definitively a hot topic; see, e.g., [Andersen *et al.*, 2007; Hadzic *et al.*, 2008; Hoda *et al.*, 2010; Gange *et al.*, 2011; Bergman *et al.*, 2014; Amilhastre *et al.*, 2014; Bergman *et al.*, 2016; Perez and Régin, 2017; Perez, 2017].

In theory, it is always possible to express a constraint under the form of a table, which simply lists the tuples allowed by the constraint, or an MDD whose paths indicate these tuples. Clearly, tables and MDDs have the same expressive power, but the main advantage of MDDs is their ability to compress the set of tuples, possibly gaining an exponential factor in term of space. When compression is high, it is very relevant to convert tables into MDDs, by using a procedure that identifies similar prefixes and suffixes (in tuples). Unfortunately, it is known that different orderings on the variables (columns of the table) can lead to very different MDDs in term of size, and discovering the optimal order is an NP-hard task.

In this paper, we are interested in using decision diagrams for representing tables (while assuming an arbitrary ordering on the variables). We propose to relax one strong property of MDDs (out-determinism, which is the requirement that two arcs going out from the same node must be labeled differently). In this respect, we propose to refine the compression procedure by targeting a diagram that is no more an MDD. More precisely, the diagram generated by our procedure is an MVD (Multi-valued Variable Diagram) [Amilhastre *et al.*, 2014], and because it admits a particular structure, basically representing two connected MDDs of approximately the same size (height), we shall call this structure an sMDD (semi-MDD).

Our contributions are summarized as follows: (i) a new structure called sMDD, adapted to the filtering of constraints, (ii) a new algorithm for converting any table into an sMDD, (iii) a new filtering algorithm enforcing Generalized Arc Consistency (GAC) on constraints defined by sMDDs, and also MDDs, by relying on bit-set operations, as in [Wang *et al.*, 2016; Demeulenaere *et al.*, 2016], (iv) some experimental results showing that the number of nodes in sMDDs is usually far smaller than in equivalent MDDs, while leading to a faster filtering process compared to previous approaches [Cheng and Yap, 2010; Perez and Régin, 2014].

2 Technical Background

A *constraint network* is composed of a set of variables and a set of constraints. Each *variable* x has an associated (finite) domain $dom(x)$ containing the values that can be assigned to it; this *current* domain is included in the *initial* domain $dom^0(x)$. Each *constraint* c involves an ordered set of variables, called the *scope* of c and denoted by $scp(c)$, and is semantically defined by a *relation* $rel(c)$ containing the tuples allowed for the variables involved in c. The *arity* of a constraint c is $|scp(c)|$. When the domain of a variable x is (becomes) singleton, we say that x is *bound*.

Given a sequence $\langle x_1, \ldots, x_r \rangle$ of r variables, an r-tuple τ on this sequence of variables is a sequence of values $\langle a_1, \ldots, a_r \rangle$, where the individual value a_i is also denoted by $\tau[x_i]$. An r-tuple τ is *valid* on an r-ary constraint c iff $\forall x \in scp(c), \tau[x] \in dom(x)$, and τ is *allowed* by c iff $\tau \in rel(c)$. A *support* on c is a tuple that is both valid on c and allowed by c. A *literal* is a pair (x, a) where x is a variable and a a value. A literal (x, a) is *Generalized Arc-*

Consistent (GAC) on c iff there is a support τ on c such that $\tau[x] = a$. A constraint c is GAC iff any literal (x, a) such that $x \in scp(c)$ and $a \in dom(x)$ is GAC on c.

A directed graph is composed of nodes and arcs. Each arc has an orientation from one node, the *tail* of the arc, to another node, the *head* of the arc. For a given node ν, the set of arcs with ν as tail (resp., head) is called the set of *outgoing* (resp., *incoming*) arcs of ν. A (arc-)labeled directed graph is a directed graph such that a label is associated with each arc. A node is *in-d* (in-deterministic) iff no two incoming arcs have the same label, *in-nd* otherwise. A node is *out-d* (out-deterministic) iff no two outgoing arcs have the same label, *out-nd* otherwise. A directed acyclic graph (DAG) is a (finite) directed graph with no directed cycles. An MVD (Multi-valued Variable Diagrams) [Amilhastre *et al.*, 2014], associated with a constraint of arity r, is a layered DAG, with one special root node at level 0, denoted by ROOT, r layers of arcs, one layer for each variable of the constraint scope $\langle x_1, \dots, x_r\rangle$, and one special sink node at level r, denoted by SINK. The arcs going from level $i-1$ to level i are *on* the variable x_i: any such arc is labeled by a value in $dom^0(x_i)$. A *valid path* in an MVD is a path from the root to the sink such that the label of each involved arc going from level $i-1$ to i is a value in $dom(x_i)$. The set of supports of a constraint c defined by an MVD M corresponds to the valid paths in M. One classical type of MVD is the Multi-valued Decision Diagram (MDD) [Bryant, 1986], which guarantees that each node is out-d (each node at level i has at most $|dom^0(x_i)|$ outgoing arcs, labeled with different values), but possibly in-nd. An example is given in Fig. 1d. We now introduce the data structure studied in this paper.

Definition 1 *A semi-MDD, or sMDD, is an MVD such that each node at a level* $< \lfloor \frac{r}{2} \rfloor$ *is out-d and each node at a level* $> \lfloor \frac{r}{2} \rfloor + 1$ *is in-d.*

This means that in an sMDD, a node at a level $< \lfloor \frac{r}{2} \rfloor$ is possibly in-nd, and a node at a level $> \lfloor \frac{r}{2} \rfloor + 1$ is possibly out-nd. Also, a node at level $\lfloor \frac{r}{2} \rfloor$ or $\lfloor \frac{r}{2} \rfloor + 1$ is possibly both in-nd and out-nd. An example is given in Fig. 2h. An interesting property of sMDDs is that they remain sMDDs after reversing the direction of the arcs (this is not true for MDDs).

A *table constraint* c is such that $rel(c)$ is explicitly defined by listing the tuples that are allowed by c. A MVD (resp., MDD and sMDD) constraint c is such that $rel(c)$ is defined by a MVD (resp., MDD and sMDD).

3 From Tables to Diagrams

The table of an extensional constraint c can be compactly represented by a trie [Gent *et al.*, 2007] in which successive levels are associated with successive variables in the scope of c. A trie can be further reduced by merging nodes[1], so as to obtain an MDD.

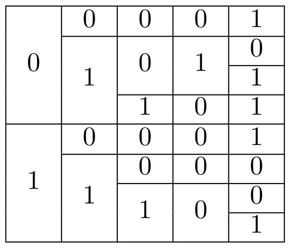
3.1 Generating (Reduced) MDDs

Reduction algorithms for generating diagram decisions, i.e., algorithms for transforming tables (sets of tuples) into BDDs and MDDs, have been proposed in the literature. A first algorithm based on a breadth-first bottom-up exploration, was proposed in [Bryant, 1986] for BDDs, and a second algorithm, using a dictionary and called `mddify`, was proposed in [Cheng and Yap, 2010; 2008] for MDDs. More recently, `pReduce` [Perez and Régin, 2015] has been shown to admit a better worst-case time complexity than `mddify`.

Fig. 1 illustrates the creation of an MDD in the spirit of `pReduce`. Initially, we consider a constraint c defined by the table shown in Fig. 1a. First, the trie corresponding to this table is created[2], Fig. 1b, and a (non-reduced) MDD can be easily derived from this trie, Fig. 1c. Then, the MDD is reduced by successively merging nodes when possible, from bottom to top. Merging is done by finding nodes having similar sets of outgoing arcs. Two sets of outgoing arcs are similar if they have the same cardinality, and for each arc in one set, there is an arc in the other set with the same label (value) and the same head. In our example, you can observe that nodes M, O and P have only one outgoing arc, each one labeled with 1 and reaching SINK. Hence, these nodes can be merged (node MOP in Fig. 1d). The MDD resulting from this iterative merging process is shown in Fig. 1d.

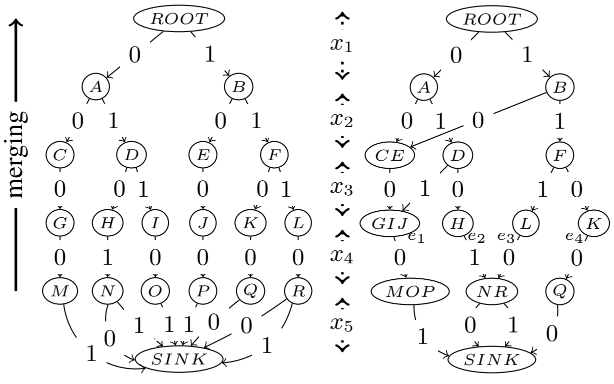
x_1	x_2	x_3	x_4	x_5
0	0	0	0	1
0	1	0	1	0
0	1	0	1	1
0	1	1	0	1
1	0	0	0	1
1	1	0	0	0
1	1	1	0	0
1	1	1	0	1

(a) Table



0	0	0	0	1
	1	0	1	0
				1
		1	0	1
1	0	0	0	1
	1	0	0	0
		1	0	0
				1

(b) Trie



(c) MDD from the Trie (d) Reduced MDD

Figure 1: Reducing a Table into an MDD

3.2 Generating sMDDs

Now, we propose to refine the reduction procedure by targeting a diagram that is an sMDD. The interest is that such structure is expected to contain less nodes (this issue is discussed later), and that efficient algorithms can be defined on

[1]In the spirit of the Hopcroft algorithm for DFA minimization

[2]Here, for simplicity a table structure is kept in Fig. 1b.

sMDDs.

The algorithm we propose is composed of five main steps, and is called `sReduce`. First, the initial table is split in two main parts:

- the p-table (table for the prefixes) corresponding to the first $\lfloor \frac{r}{2} \rfloor$ columns (or variables),
- the s-table (table for the suffixes) corresponding to the last $r - \lfloor \frac{r}{2} \rfloor - 1$ columns (or variables),

At this point, note that all variables, except one, are involved in one of these two partial tables. For example, on our example with $r = 5$, we obtain a p-table with 2 columns (corresponding to x_1 and x_2) and an s-table with 2 columns (corresponding to x_4 and x_5). The missing column (for variable x_3) will be considered in a later stage.

Second, duplicates are removed from the p-table and the s-table, and the p-table and the s-table are lexicographically sorted, respectively using an increasing and decreasing order. Considering again the initial table depicted in Fig. 1a, after these three steps, we obtain the p-table and the s-table shown in Fig. 2a and 2d.

Third, for both the p-table and the s-table, we build some equivalent tables sharing prefixes and suffixes (we call them p-trie and s-trie), and naturally derive equivalent trees from them (we call them p-tree and s-tree). Importantly, the order of the columns is preserved, and we start with a special root node for the p-tree whereas we finish with a special sink node for the s-tree. An illustration is given by figures 2b, 2c, 2e and 2f.

Fourth, for each tuple τ in the initial table, we build an arc between the node in the p-tree corresponding to the end of the prefix of τ and the node in the s-tree corresponding to the start of the suffix of τ: this arc is labeled with the value for the intermediate variable, which was involved neither in the p-table nor in the s-table. We obtain a new diagram, depicted in Fig. 2g, where arcs have been added for x_3.

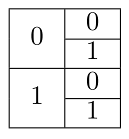
Fifth, "classical" reduction is performed twice. On the one hand, from bottom to top, merging can be conducted by starting from the nodes that were leaves in the p-tree. For merging, the algorithm searches for similarities between sets of outgoing arcs. As an illustration, let us consider nodes C and E in Fig. 2g. These two nodes have both one outgoing arc with the same label 0 and the same head: therefore, they can be merged (node CE in Fig. 2h). On the other hand, from top to bottom, merging can be conducted by starting from the nodes that had no parent in the s-tree. For merging, the algorithm searches now for similarities between sets of incoming arcs. As an illustration, observe how nodes H and J in Fig. 2g can be merged (node HJ in Fig. 2h). The graph obtained after complete reduction is depicted in Fig. 2h.

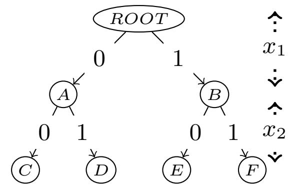
Proposition 1 *The graph obtained after executing* `sReduce` *on any specified table is an sMDD.*

Proof: Before executing merging operations, the diagram (at the end of step 4) is an sMDD, by construction. Merging conducted in the first (bottom-up) pass preserves out-determinism of any node at a level $< \lfloor \frac{r}{2} \rfloor$, while merging conducted in the second (top-down) pass preserves in-determinism of any node at a level $> \lfloor \frac{r}{2} \rfloor + 1$. ■

x_1	x_2
0	0
0	1
1	0
1	1

(a) p-table

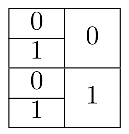

(b) p-trie

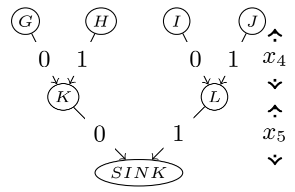


(c) p-tree

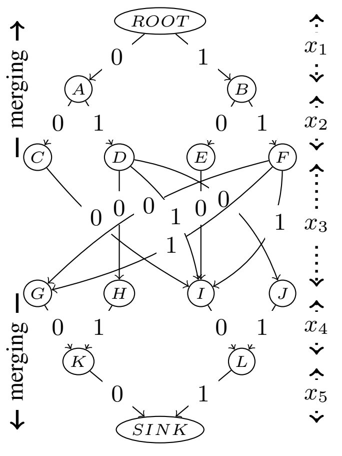
x_4	x_5
0	0
1	0
0	1
1	1

(d) s-table

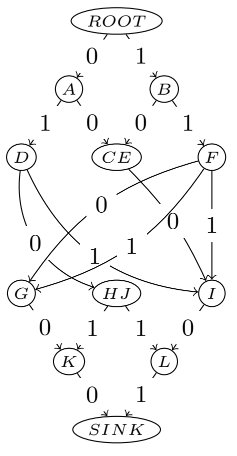

(e) s-trie



(f) s-tree



(g) sMDD



(h) Reduced sMDD

Figure 2: Reducing a Table into an sMDD

The complexity of `sReduce` is XXXXX.

Constraint checking, i.e., checking if a tuple τ is accepted by an sMDD, can be done in $\mathcal{O}(r)$ by performing a bi-directional search. First, the validity of the prefix of the tuple is tested by visiting the sMDD from the root down to one node ν_1 at level $\lfloor \frac{r}{2} \rfloor$. Second, the validity of the suffix is tested by visiting the sMDD from the sink node up to one node ν_2 at level $\lfloor \frac{r}{2} \rfloor + 1$. If both prefix and suffix are valid, it remains to test if there exists an arc between the two nodes ν_1 and ν_2 with the right value in τ.

One interest of sMDDs over MDDs is the potential reduction of the number of nodes. Assuming a domain size d, the number of nodes of the initial trie is in $O(d^r)$ for the MDD while it is in $O(d^{r/2})$ for the sMDD. The gain can thus be exponential in the arity of the constraint although merging renders predictions difficult to establish. On our example, starting from the same table, the MDD has 14 nodes and 19 arcs while the sMDD has 12 nodes and 18 arcs.

4 Compact-MDD

In this section, we describe a new filtering algorithm that can be applied to any type of layered graphs such as MVD, sMDD or MDD. It is called Compact-MDD (or CMDD), and borrows some principles from CT [Demeulenaere *et al.*, 2016] and MDD4R [Perez and Régin, 2014]. Its description is given under the form of an object-oriented programming class in Algorithm 1.

4.1 Data Structures

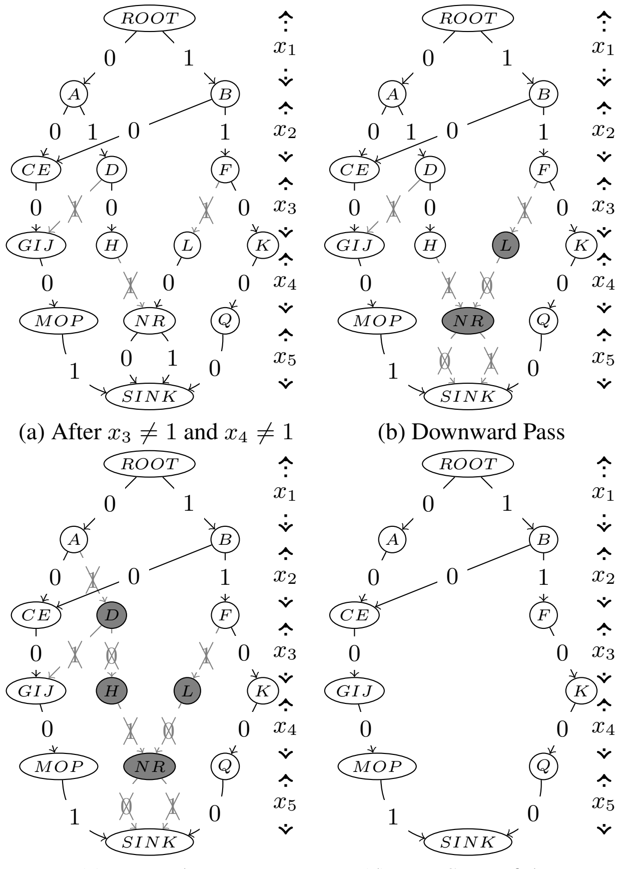
As fields of Class `Constraint-CMDD`, we first find `scp` for representing the scope $\langle x_1, \ldots, x_r \rangle$ of c and `currArcs` for representing the current set of valid arcs of the diagram. More precisely, a reversible sparse bit-set from Class `RSparseBitSet`, as described in [Demeulenaere *et al.*, 2016], is associated with each variable x of *scp*: `currArcs`$[x]$ keeps track of the valid arcs on x. Each arc in the diagram admits an associated bit in `currArcs`: the arc is valid iff the bit is set to 1. Note that this is similar to `currTable` that keeps track of the valid tuples in CT. As an example, for the MDD in Fig. 1d, `currArcs`$[x_2]$ and `currArcs`$[x_3]$ respectively correspond to sequences of 4 and 5 bits (all set to 1, initially). In this data structure, one field is `words`, an array of k-bit words (e.g., $k = 64$), which defines the current value of the bit-set. Each reversible sparse bit-set has another field: a bit-set called `mask` that is useful for performing and recording intermediate computations. Interestingly, operations on `mask` are optimized so as to only consider non-zero words. We now succinctly describe the methods in `RSparseBitSet`. Method isEmpty() simply checks whether the number of non-zero words is different from zero. Method clearMask() sets to zero all words of `mask` whereas Method reverseMask() reverses all words of `mask`. Method addToMask() applies a word by word logical bit-wise *or* operation. Finally, Method intersectIndex() checks if a given bit-set intersects with the current bit-set: it returns the index of the first word where the intersection is non-zero, -1 otherwise. For the sake of simplicity, we shall use `currArcs`$[x][i]$ as a shortcut for `currArcs`$[x]$.`words`[`i`], and `currArcs`$[x]$.intxn as a shortcut for `currArcs`$[x]$.intersectIndex.

We also have three fields $\texttt{S}^{\texttt{val}}$, $\texttt{S}^{\texttt{sup}}$ and `lastSizes` in the spirit of STR2 [Lecoutre, 2011]. The set $\texttt{S}^{\texttt{val}}$ contains variables whose domains have been reduced since the previous call to CMDD on c. To set up $\texttt{S}^{\texttt{val}}$, we need to record the domain size of each variable x right after the execution of CMDD on c: this value is recorded in `lastSizes`$[x]$. The set $\texttt{S}^{\texttt{sup}}$ contains unbound variables whose domains contain each at least one value for which a support must be found.

	e_1	e_2	e_3	e_4
`supports`$[x_4, 0]$	1	0	1	1
`supports`$[x_4, 1]$	0	1	0	0
`arcsT`$[GIJ, x_4]$	1	0	0	0
`arcsT`$[H, x_4]$	0	1	0	0
`arcsT`$[L, x_4]$	0	0	1	0
`arcsT`$[K, x_4]$	0	0	0	1
`arcsH`$[x_4, MOP]$	1	0	0	0
`arcsH`$[x_4, NR]$	0	1	1	0
`arcsH`$[x_4, Q]$	0	0	0	1

Figure 3: Data structures related to arcs on x_4 of Fig. 1d

These two sets allow us to restrict loops on variables to relevant ones. To ease computations, at each level we find three types of precomputed bit-sets: these bit-sets are never modified. First, `supports`$[x, a]$ indicates for each arc on the variable x whether or not the value a is initially supported by this arc (bit set to 1 iff a is supported). Second, `arcsT`$[\nu, x]$ and `arcsH`$[x, \nu']$ indicates for each arc on x whether ν and ν' are respectively the tail and the head of this arc. Fig. 3 displays these structures associated with x_4 in the MDD depicted in Fig. 1d. Finally, we have dynamic bit-sets for handling so-called residues. We shall see their role when describing the algorithm.



(a) After $x_3 \neq 1$ and $x_4 \neq 1$ (b) Downward Pass (c) Upward Pass (d) New State of the MDD

Figure 4: Updating the MDD from Fig. 1d after $x_3 \neq 0 \land x_4 \neq 0$

4.2 Algorithm

The main method in `Constraint-CMDD` is enforceGAC(). After the initialization of the sets $\texttt{S}^{\texttt{val}}$ and $\texttt{S}^{\texttt{sup}}$, calling updateGraph() allows us to update the graph, and more specifically `currArcs` to filter out (indices of) arcs that are no more valid. Once the graph is updated, it is possible to test whether each value has still a support, by calling filterDomains(). If ever a domain wipe-out (failure due to a domain becoming empty) occurs, an exception is thrown during the update of the graph (and so, this is not directly managed in this main method). At the end of enforceGAC(), `lastSizes` is updated in view of the next call.

Updating the Graph. As in MDD4R, the goal of updateGraph() is to remove the arcs that are no more part of a valid path. An arc can be: (i) *trivially* removed when the value of the label of the arc has been removed from the variable domain (since the previous call) (ii) or *untrivially* removed when all paths involving the arc are no more valid. Method updateGraph() follows this observation: it identifies first the arcs that can be trivially removed before identifying those that can be untrivially removed. Fig. 4 illustrates the whole updating process, considering the effect of having two deleted values on the MDD depicted in Fig. 1d. We shall refer to this illustration all along the description of this part of the algorithm.

In Method updateGraph(), after initializing all masks associated with the variables in the scope of the constraint, all arcs that can be trivially removed are handled by calling updateMasks(). For each variable $x \in \mathtt{S}^{\mathtt{val}}$, i.e., each variable x whose domain has changed since the last time the filtering algorithm was called, updateMasks() operates on the associated masks. This method assumes an access to the set of values Δ_x removed from $dom(x)$ since the last call to enforceGAC(). There are two ways of updating the masks (before updating `currArcs` from these masks, later): either incrementally or from scratch after resetting as proposed in [Perez and Régin, 2014]. This is the strategy implemented in updateMasks(), by considering a reset-based computation when the size of the domain is smaller than the number of deleted values. In case of an incremental update (line 16), the union of the arcs to be removed is collected by calling addToMask() for each bit-set (of supports) corresponding to removed values, whereas in case of a reset-based update (line 19), we perform the union of the arcs to be kept. To get masks ready to apply, we just need to reverse them when they have been built from present values. Unlike CT, the update of `currArcs` from the computed masks is not done immediately. This would be redundant knowing that it must be done twice during the next step. Fig. 4a shows in gray the arcs that are added to the masks.

Last but not least, we need now to determine which arcs can be untrivially removed: this is achieved by calling the methods propagateDown() and propagateUp(), which, similarly to MDD4R, perform two passes on the diagram. During the downward (resp., upward) pass, each level is examined from the root (resp., sink) to the sink (resp., root)[3].

In Method propagateDown(), for a specified variable x_i, provided that that some arcs on x_i have been removed (the presence of arcs trivially removed are tested at Line 24 with $x_i \in \mathtt{S}^{\mathtt{val}}$, and the presence of arcs untrivially removed are given by the Boolean variable `localChange`), we have to process (and propagate) them. To start, `currArcs` is first updated (Line 25), and if no more arcs on x_i remain, a backtrack is forced because there is necessarily a domain-wipe-out. If x_i is not the last variable in the scope of the constraint, we have to deal with x_{i+1}. Specifically, every node[4] ν that is the tail of a currently valid arc on x_{i+1} is tested: when there is no more valid arcs on x_i with ν as head, all arcs on x_{i+1} with ν

[3]Actually, we can start propagation from the first and last unbound variables. For experiments, we used this code optimization.

[4]Those are maintained in practice in a reversible sparse-set as in [Perez and Régin, 2014]

Algorithm 1: Class Constraint-CMDD

```
1  Method enforceGAC()
2      $\mathtt{S}^{\mathtt{val}} \leftarrow \{x \in \mathtt{scp} : \mathtt{lastSizes}[x] \neq |dom(x)|\}$
3      $\mathtt{S}^{\mathtt{sup}} \leftarrow \{x \in \mathtt{scp} : |dom(x)| > 1\}$
4      updateGraph()
5      filterDomains()
6      foreach variable $x \in \mathtt{S}^{\mathtt{val}} \cup \mathtt{S}^{\mathtt{sup}}$ do
7          $\mathtt{lastSizes}[x] \leftarrow |dom(x)|$
8  Method updateGraph()
9      foreach variable $x \in \mathtt{scp}$ do
10         $\mathtt{currArcs}[x]$.clearMask()
11     updateMasks()
12     propagateDown($x_1$, false)
13     propagateUp($x_r$, false)
14 Method updateMasks()
15     foreach variable $x \in \mathtt{S}^{\mathtt{val}}$ do
16         if $|\Delta_x| < |dom(x)|$ then // Incremental update
17             foreach value $a \in \Delta_x$ do
18                 $\mathtt{currArcs}[x]$.addToMask($\mathtt{supports}[x, a]$)
19         else                        // Reset-based update
20             foreach value $a \in dom(x)$ do
21                 $\mathtt{currArcs}[x]$.addToMask($\mathtt{supports}[x, a]$)
22             $\mathtt{currArcs}[x]$.reverseMask()
23 Method propagateDown($x_i$, localChange)
24     if $x_i \in \mathtt{S}^{\mathtt{val}}$ or localChange then
25         $\mathtt{currArcs}[x_i]$.removeMask()
26         if $\mathtt{currArcs}[x_i]$.isEmpty() then
27             throw Backtrack
28         if $x_i \neq x_r$ then
29             localChange ← false
30             foreach node $\nu \in \{\nu :$
                 $\mathtt{currArcs}[x_{i+1}]$.intxn($\mathtt{arcsT}[\nu, x_{i+1}]$) $\neq -1\}$ do
31                 $j \leftarrow \mathtt{residuesH}[x_i, \nu]$
32                 if $\mathtt{currArcs}[x_i][j]$ & $\mathtt{arcsH}[x_i, \nu][j] = 0^{64}$ then
33                     $j \leftarrow \mathtt{currArcs}[x_i]$ .intxn($\mathtt{arcsH}[x_i, \nu]$)
34                     if $j \neq -1$ then
35                         $\mathtt{residuesH}[x_i, \nu] \leftarrow j$
36                     else
37                         $\mathtt{currArcs}[x_{i+1}]$ .addToMask($\mathtt{arcsT}[\nu, x_{i+1}]$)
38                         localChange ← true
39             propagateDown($x_{i+1}$, localChange)
40     else if $x_i \neq x_r$ then
41         propagateDown($x_{i+1}$, false)
42 Method propagateUp($x_i$, localChange)
       /* Similar to propagateDown with $x_1$ instead of $x_r$,
          $x_{i-1}$ instead of $x_{i+1}$, inverted use of arcsT and
          arcsH, inverted use of residuesT and residuesH.       */
43 Method filterDomains()
44     foreach variable $x \in \mathtt{S}^{\mathtt{sup}}$ do
45         foreach value $a \in dom(x)$ do
46             $i \leftarrow \mathtt{residues}[x, a]$
47             if $\mathtt{currArcs}[x][i]$ & $\mathtt{supports}[x, a][\mathtt{i}] = 0^{64}$ then
48                 $i \leftarrow \mathtt{currArcs}[x]$.intxn($\mathtt{supports}[x, a]$)
49                 if $i \neq -1$ then
50                     $\mathtt{residues}[x, a] \leftarrow i$
51                 else
52                     $dom(x) \leftarrow dom(x) \setminus \{a\}$
```

as tail are then untrivially removed. In other words, if there is no more valid incoming arc for a node ν at level i, then all outgoing arcs of ν become invalid: this is implemented by the code at Lines 29..38. Note that the search of supporting arcs is improved by keeping track in `residuesH` of the last valid incoming arc, and starting with it. This increases the odds of not testing too many words of `currArcs`. Also, note how the variable `localChange` becomes true as soon as an arc is untrivially removed.

Fig. 4b shows the behavior of downward propagation on our example. For the two first levels, nothing happens (because there were no values removed wrt these two levels). However, at the level of x_3, we can see that all incoming arcs of the node L have been removed. Hence, the outgoing arcs of L are added to the mask associated with the next level, and removed when reaching this level. On the other hand, the node GIJ has still one valid incoming arc. Fig. 4c shows the result of upward propagation (after the downward one has been completed).

Filtering Domains. The process of filtering domains is very similar to that described in CT [Demeulenaere *et al.*, 2016]. This is given by Method filterDomains() in Algorithm 1. For each remaining unbound variable x in $\mathtt{S}^{\mathtt{sup}}$, and each value a in $dom(x)$, the intersection between the valid arcs on x, $\mathtt{currArcs}[x]$, and the arcs labeled with value a, $\mathtt{supports}[x, a]$, determines if a is still supported. An empty intersection means that a can be deleted, at Line 52. This is correct because all "remaining" arcs in $\mathtt{currArcs}[x]$ are necessarily part of a valid path in the graph. The search of supports starts by using `residues`.

Back to our example, remaining arcs as defined by `currArcs` corresponds to the MDD depicted in Fig. 4d. Regarding x_5, $\mathtt{currArcs}[x_5]$ is 1001. Because $\mathtt{supports}[x_5, 0]$ is 0101 and $\mathtt{supports}[x_5, 1]$ is 1010, we can deduce (from bitwise intersections) that both values are still valid for x_5.

We can prove that CMDD enforces GAC (proof omitted, due to lack of space). Concerning the complexity of CMDD XXXXX.

5 Experimental Results

In our system, we have implemented `pReduce`, MDD4R [Perez and Régin, 2014], CT [Demeulenaere *et al.*, 2016], and the two algorithms proposed in this paper, namely, `sReduce` and CMDD. We have conducted an experimentation on the $4,111$ available XCSP3 instances [Boussemart *et al.*, 2016] that only contain table constraints. We have compared the relative efficiency of MDD4R (after executing `pReduce` to convert tables), CMDD^p (i.e., CMDD after executing `pReduce`), CMDD^s (i.e., CMDD after executing `sReduce`) and CT (on the original tables). We have filtered out the instances taking less than 2 seconds or leading to a time out (10 minutes) for all algorithms. Each execution was given 10GB of RAM. Results are reported using performance profiles [Dolan and Moré, 2002].

We first compared `sReduce` with `pReduce`. In term of execution time, XXXX en terme de temps d'execution, dire une pharse ou 2 sur sReduce vs pReduce XXXX. In term of size, Fig. 5 shows two performance profiles that allow us to compare globally the number of nodes and arcs in the generated MDDs and sMDDs for all the tables involved in our benchmark (around $2,500,000$ tables). As we predicted, the number of nodes is significantly reduced in the generated sMDDs (more than a factor 8 for at least 70% of the tables), while the number of arcs tends to be slightly higher.

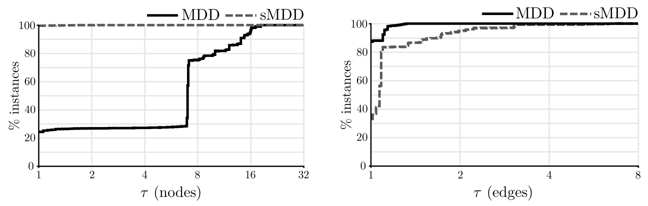


Figure 5: Comparing the size of the generated MDDs and sMDDs

On the left of Fig. 6, execution times of MDD4R, CMDD^p and CMDD^s are compared. Clearly, CMDD outperforms MDD4R, even when it is executed on "simple" MDDs. Using sMDDs just makes it more robust. For example, CMDD^s, CMDD^p and MDD4R are at least 2 times slower than the best (virtual) algorithm on 5%, 20% and 35% of the instances, respectively. On the right of Fig. 6, CT is additionally considered. In general, CT still outperforms decision diagram approaches, but the gap is reduced: 40% of the instances are solved by CMDD^s within a factor 2 compared to the time taken by CT, instead of 5% previously with MDD4R.

It is important to note that these global results do not tell the entire story. Indeed, when the compression is high, using decision diagrams remains the appropriate approach. For example, on the problem instance XXX, the compression ratio is around YYY, and the execution times of MDD4R, CMDD^p and CMDD^s (obtained of course on the same search trees) are respectively ZZZ. This definitively confirms the real interest of approaches based on decision diagrams.

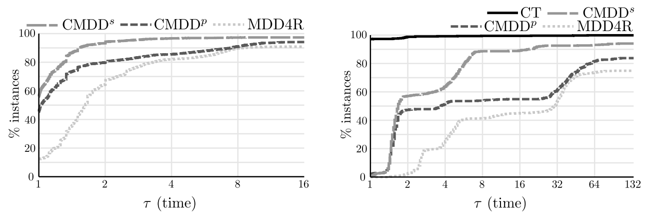


Figure 6: Comparing MDD4R, CMDD^p, CMDD^s and CT

6 Conclusion

We have proposed an original variant of decision diagrams for representing (table) constraints, and have introduced an original efficient filtering algorithm, based on it. The new algorithm, CMDD, outperforms the state-of-the-art algorithm MDD4R, and is close to CT in general. Interestingly, when the compression is high, CMDD becomes the fastest approach. As a future work, we would like to study if sMDDs could be used to represent other types of constraints.

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