– Quality in Quantity –
Relative Direction Constraints using Sector Sets around Oriented Points

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1   Introduction

Qualitative spatial reasoning (QSR) is a way of spatial reasoning about with concepts like the mereotopological relations *inside*, *touching* and *distant* as well as orientational relations like *north* and *inwards* and relative directional relations like *left* and *front*. In general QSR can be understood as any kind of reasoning about non-numerical descriptions of spatial layout and action.

Concerning reasoning about absolute and relative directions the families of $\mathcal{STAR}$ [4] and $\mathcal{OPRA}$ [5] relation sets have gained popularity and have been used as a means to reason about street networks [3, 7] and sailing rules [1]. Both are
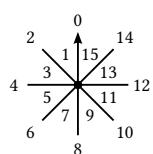
Figure 1: $\mathcal{OPRA}_4$ half-relations.

based on oriented points and sectors regularly dividing the plane around the points, with the distinction that in the former all points share the same orientation, as if they were pointing towards an infinitely distant North Pole, while the latter allows the points to be oriented arbitrarily. This step from absolute to relative directions makes reasoning about a given spatial description significantly harder. While realizability of a given point configuration can be decided in polynomial time when the description is given using only convex disjunctions of a $\mathcal{STAR}$ relation set [4], it becomes NP-hard when using any set of relative directional relations that differentiates *left* and *right* [6].

While the granularity of the $\mathcal{OPRA}$ relations makes them quite useful, from a point of view concerned with application, the set of allowed angles in the $\mathcal{OPRA}$ family seems rather arbitrary and unmotivated. The restriction to an even number of equiangular sectors and the impossibility to mix different granularities, i.e. different numbers of sectors dividing the plane, stem from the prevailing reasoning approach in QSR for which it was designed; this relation algebraic approach deploys stored tables of binary compositions of spatial relations in order to apply the path consistency algorithm to infinite domains – such as half-lines and sectors in the Euclidean plane or meretopological *regions*. The use of composition tables often leads to results that are coarser than path consistency in the actual domain. In this case the algorithm yields the weaker concept of the *algebraic closure* (AC) of a given constraint network. Since the runtime of practical path consistency algorithms is quartic in the number of basic relations, a finer granularity unnecessarily leads to slower reasoning about $\mathcal{OPRA}$ relations using this approach.

We show that these drawbacks can be circumvented by a flexible language that can be layered by a qualitative representation while allowing the actual reasoning to take place on a quantitative level using modern SMT-solvers. This approach also allows for integration of different (even quantitative) constraints. A non-relation algebraic approach to the problem of presenting a realization is given in [2].

2   The $\mathcal{OPUS}$ Relation Set

The $\mathcal{OPUS}$ (Oriented Points Using Sectors) relation set allows to describe relative directions of oriented points in the plane by means of sets of sectors around those oriented points.

Definition 1. *The set of $\mathcal{OPUS}$ relations is the set of binary relations of the following form*

$$Opus(I_1, I_2) \coloneqq \big\{(x,y) \in \mathbb{O}^2 \mid \dot{x} \neq \dot{y} \ \wedge \ \angle\vec{x}(\dot{y} - \dot{x}) \in \bigcup I_1$$
$$\vee \ \dot{x} = \dot{y} \ \wedge \ \angle\vec{x}\,\vec{y} \quad \in \bigcup I_2 \big\},$$

where $\mathbb{O}$ denotes the set of oriented points in the real plane, $\vec{x}$ denotes the orientation of x, $\dot{x}$ denotes the location of x and I_1, I_2 are finite sets of real intervals. The language combining such $\mathcal{OPUS}$ constraints by conjunction and disjunction is simply called the $\mathcal{OPUS}$. For notational convenience we define

$$Opus_{AB}I_1 I_2 \ \overset{def}{\equiv} \ (A,B) \in Opus(I_1, I_2) \ .$$

E.g., we write $Opus_{AB}\big\{[0, \frac{\pi}{4}], [\frac{7}{4}\pi, 2\pi)\big\}\big\{[\pi, \pi]\big\}$ to denote that B, as seen from oriented point A, lies either in the front quadrant including the boundary or exactly at A but oriented backwards. Since the $\mathcal{OPUS}$ distinguishes between the directions *left* and *right* by the relations $Opus(\{(0, \pi)\}, \varnothing)$ and $Opus(\{(\pi, 2\pi)\}, \varnothing)$, deciding realizability of a spatial description using $\mathcal{OPUS}$ relations is as hard as deciding satisfiability in the existential theory of the reals, which is NP-hard [6].

3 $\mathcal{OPUS}$ Triangle Consistency

A spatial description given in the language of the $\mathcal{OPUS}$ can be expressed in terms of the quantifier free first order theory of linear inequalities over the reals, denoted by QF_LRA. Adding angular dependencies, such as that the sum of interior angles

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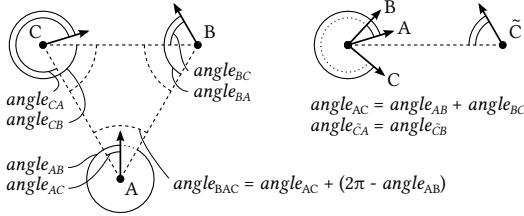
Figure 2: Proper and degenerated oriented point triangles.

equals π in every triangle (*triangle constraint*), we can use an SMT-solver to decide the satisfiability of these angular constraints. Note, however, that these angular dependencies do not suffice to provide a complete decision procedure. Although they supersede algebraic closure, they still allow for false positives when deciding the consistency of $\mathcal{OPUS}$ sentences. However, e.g. a deviated Pappus configuration gets detected instantaneously.

Each $\mathcal{OPUS}$ constraint $Opus_{AB}I_1I_2$ can be directly translated into the following QF_LRA formula:

$$\text{OPair}_{AB}I_1I_2 \stackrel{\text{def}}{\equiv} \begin{pmatrix} \neg same_{AB} \wedge \bigvee_{i \in I_1} \inf i \lesssim angle_{AB} \lesssim \sup i \\ \vee \; same_{AB} \wedge \bigvee_{i \in I_2} \inf i \lesssim angle_{AB} \lesssim \sup i \end{pmatrix}$$

where $\lesssim$ is either strict or not depending on whether the interval is open or closed at the respective end.

In order to express the triangle constraint for a triple ABC of distinct oriented points we first need to define the interior angles through the given angular information and such that they lie in the interval $[0, 2\pi]$. For this we define the following formulas:

$$\begin{aligned}
\text{ang}_{BAC} \quad & \stackrel{\text{def}}{\equiv} \quad \text{if } (angle_{AB} \leq angle_{AC}) \\
& \qquad \text{then } (angle_{BAC} = angle_{AC} - \quad angle_{AB} \quad) \\
& \qquad \text{else } \; (angle_{BAC} = angle_{AC} + (2\pi - angle_{AB}))
\end{aligned}$$

$$\begin{aligned}
\text{tri}_{ABC}^{\text{prop}} \quad & \stackrel{\text{def}}{\equiv} \quad angle_{BAC} \;, \; angle_{CBA} \;, \; angle_{ACB} > 0 \\
& \wedge \quad angle_{BAC} + angle_{CBA} + angle_{ACB} = \pi
\end{aligned}$$

$$\text{tri}_{BAC}^{\text{deg}} \quad \stackrel{\text{def}}{\equiv} \quad angle_{BAC} = \pi \; \wedge \; angle_{CBA} \;, \; angle_{ACB} = 0 \; .$$

With these formulas the triangle $\triangle ABC$ can be described by

$$\begin{aligned}
\text{tri}_{ABC} \quad & \stackrel{\text{def}}{\equiv} \quad (\text{ang}_{BAC} \; \wedge \; \text{ang}_{CBA} \; \wedge \; \text{ang}_{ACB}) \\
& \wedge \; \left(\text{tri}_{ABC}^{\text{prop}} \; \vee \; \text{tri}_{BAC}^{\text{deg}} \; \vee \; \text{tri}_{CBA}^{\text{deg}} \; \vee \; \text{tri}_{ACB}^{\text{deg}} \right).
\end{aligned}$$

Taking account of the orientation of the triangle we arrive at

$$\odot ABC \quad \stackrel{\text{def}}{\equiv} \quad \text{if } \begin{pmatrix} 0 \leq angle_{AC} - \quad angle_{AB} \quad \leq \pi \\ \vee \; 0 \leq angle_{AC} + (2\pi - angle_{AB}) \leq \pi \end{pmatrix}$$
$$\text{then } \left(\text{tri}_{ABC} \right) \quad \text{else } \left(\text{tri}_{CBA} \right)$$

to describe the case where the points A, B, C are distinct. When all points coincide we have to make sure that all relative orientations match:

$$\begin{aligned}
\odot ABC \quad & \stackrel{\text{def}}{\equiv} \quad \text{if} \quad (angle_{AB} + angle_{BC} < 2\pi) \\
& \qquad \text{then} \, (angle_{AC} = angle_{AB} + angle_{BC}) \\
& \qquad \text{else} \; (angle_{AC} = angle_{AB} + angle_{BC} - 2\pi).
\end{aligned}$$

When exactly two points A and B coincide, they also have to lie at the same angle as seen from the third oriented point C:

$$\odot ABC \quad \stackrel{\text{def}}{\equiv} \quad \odot ABC \; \wedge \; (angle_{CA} = angle_{CB}) \, .$$

Now we only need to formulate when to apply these three formulas. This can be done with a straightforward case-by-case analysis over the variables $same_{AB}$, $same_{AC}$ and $same_{BC}$:

$$? ABC \stackrel{\text{def}}{\equiv} \left(\begin{array}{l} \neg same_{AB} \wedge \left(\begin{array}{l} \neg same_{AC} \wedge \begin{pmatrix} \neg same_{BC} \wedge \odot ABC \\ \vee \; same_{BC} \wedge \odot BCA \end{pmatrix} \\ \vee \; same_{AC} \wedge \begin{pmatrix} \neg same_{BC} \wedge \odot ACB \\ \vee \; same_{BC} \wedge \text{False} \end{pmatrix} \end{array} \right) \\ \vee \; same_{AB} \wedge \cdots \end{array} \right)$$

Due to Euclid's First common notion some branches end in False and can simply be left away.

Now the formula $\triangle S$ for the triangle consistency of a given $\mathcal{OPUS}$ description S can be given by translating each constraint $Opus_{AB}I_1I_2$ in the $\mathcal{OPUS}$ sentence S into the corresponding OPair$_{AB}I_1I_2$ formula, restricting the angles of all pairs not occurring in S to lie in $[0, 2\pi]$ and conjuncting the formula $? ABC$ for every unordered triple ABC.

Definition 2. *A sentence S in the language of the $\mathcal{OPUS}$ is called* triangle consistent *if the corresponding formula $\triangle S$ is satisfiable.*

First benchmarks on triangle consistency (TC) and AC show that while the runtime of TC grows significantly faster with the number of points it starts of much lower and naturally is independent of any granularity. In general it allows for efficient and effective reasoning for up to 50 points.

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