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# THE SINGLE MACHINE PROBLEM WITH QUADRATIC PENALTY FUNCTION OF COMPLETION TIMES: A BRANCH-AND-BOUND SOLUTION*

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$N$ jobs are to be sequenced on a single machine, each job carrying a penalty cost which is a quadratic function of its completion time. The objective is to find a sequence which minimizes the total penalty. Criteria are developed for ordering a pair of adjacent jobs in a sequence and these are incorporated into a branch-and-bound procedure.

## 1. Introduction

In this paper consideration is confined to the single machine scheduling problem. Even this limited case is rich in variety, for there are a large number of possible penalty functions, some of which have yielded interesting solution procedures [1]–[9]. In some types, such as the linear penalty case solved by McNaughton [4] and the minimization of maximum lateness solved by Smith [8], jobs can be ordered according to a simple criterion.

Lawler [3] has studied the situation where the penalty function attached to a job is any nondecreasing function of completion time, and the objective is to minimize the maximum penalty associated with any job in the set. Where the objective is to minimize the sum of the penalties associated with the jobs and the penalty functions are nonlinear, the problem appears to be quite difficult. Schild and Fredman [6] have developed an algorithm to deal with this case, but the calculations are complicated and become prohibitive when the number of jobs is large. For the quadratic penalty case, to be considered in the remainder of this paper, the objective is to minimize the function $f = \sum_{i=1}^{n} c_i x_i^2$, where $x_i$ is the completion time of job $i$ and $c_i$ is a cost coefficient. As a preliminary, it is worth illustrating that, in general, two jobs in adjacent positions cannot be ordered without reference to other jobs in the set. For example, consider the problem with $(a_1, c_1) = (1, 15)$, $(a_2, c_2) = (3, 17)$, $(a_3, c_3) = (1, 7)$, where $a_i$ is the processing time of job $i$, $i = 1, 2, 3$. It can easily be verified that the sequence minimizing $\sum c_i x_i^2$ is 123. However, if the first job is changed from (1, 15) to (2, 15), the optimal sequence is changed from 123 to 132. It follows from this that criteria for ordering jobs are unlikely to be simple, though Emmons [2] and Srinivasan [9] have incorporated such criteria into algorithms for the one-machine problem with tardiness penalties. The following section shows how criteria can be developed for quadratic penalty functions.

## 2. An Ordering Criterion for the Quadratic Case

Suppose that jobs $i$ and $j$ occupy adjacent positions in a sequence, and that the job immediately preceding $i$ and $j$ is completed at time $t$. Then the penalties, $P(ij)$ and $P(ji)$, associated with jobs $i$ and $j$ when they are ordered $ij$ and $ji$ respectively are

$$P(ij) = c_i(t + a_i) + c_j(t + a_i + a_j)^2, \quad P(ji) = c_j(t + a_j) + c_i(t + a_j + a_i)^2.$$

Thus $P(ij) < P(ji)$ if

$$c_j a_i(2t + a_i + 2a_j) < c_i a_j(2t + a_j + 2a_i), \quad \text{or}$$

$$c_j a_i(2t + a_i + a_j) + c_j a_i a_j < c_i a_j(2t + a_i + a_j) + c_i a_i a_j. \tag{1}$$

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Sufficient conditions for an ordering $ij$ are, therefore,

$$c_i/a_i > c_j/a_j, \text{ and} \tag{2}$$

$$c_i > c_j. \tag{3}$$

Note that condition (2) is McNaughton's criterion for the linear penalty function case.

Consider the example with $(a_1, c_1) = (1, 8)$, $(a_2, c_2) = (2, 6)$ and $(a_3, c_3) = (3, 5)$. We have $c_1/a_1 > c_2/a_2 > c_3/a_3$ and $c_1 > c_2 > c_3$, hence an optimal solution is given by the sequence 123. In general, however, both criteria will not be met simultaneously for a given pair of jobs.

Suppose, however, when (2) is met but (3) is not met that jobs $i$ and $j$ are ordered according to (2) alone; that is, they are always ordered $ij$ when $c_i/a_i > c_j/a_j$. Then it can be seen from (1) that the maximum reduction in penalty that can occur through interchanging the order $ij$ to $ji$ is

$$(c_j - c_i)a_ia_j. \tag{4}$$

Thus by ordering a set of jobs according to the single criterion (2) and making an adjustment to allow for the potential improvement that could be obtained through interchanges, a lower bound can be found for the optimal solution.

Consider the example of Table 1.

TABLE 1

*A 5-Job, 1-Machine Problem*

| Job ($i$) | Operation time ($a_i$) | Cost coefficient ($c_i$) | $c_i/a_i$ |
|---|---|---|---|
| 1 | 10 | 2 | 1/5 |
| 2 | 4 | 5 | 5/4 |
| 3 | 6 | 7 | 7/6 |
| 4 | 5 | 3 | 3/5 |
| 5 | 7 | 1 | 1/7 |

Ordering the jobs according to decreasing value of $c_i/a_i$ gives the sequence 23415. Jobs 3, 4, 1 and 5 are also in decreasing order of $c_i$, so that criterion (2) is also met and the sequence 3415 is, therefore, optimal for the reduced set of jobs. However, $c_2 < c_3$, so that there is a potential improvement through interchange of jobs 2 and 3 of $(7-5)(4)(6) = 48$. No further improvement is possible, since $c_2 > c_4$. Hence a lower bound to the penalty function $\sum c_i x_i^2$ of any sequence is $f(23415) - 48$, where $f(23415)$ is the penalty associated with the sequence 23415.

Evaluating $f$ as 3729 gives a lower bound to all sequences as 3681. The analysis of the above example is quite simple, as it involves the interchange of a single pair of jobs which are in adjacent positions. Consider a change to the operation times of jobs 4 and 5 to give the problem of Table 2.

TABLE 2

*Modification of the Problem of Table 1*

| Job ($i$) | Operation time ($a_i$) | Cost coefficient | $c_i/a_i$ |
|---|---|---|---|
| 1 | 10 | 2 | 1/5 |
| 2 | 4 | 5 | 5/4 |
| 3 | 6 | 7 | 7/6 |
| 4 | 1 | 3 | 3 |
| 5 | 2 | 1 | 1/2 |

The "primary" ordering according to $c_i/a_i$ gives the sequence 42351, for which the corresponding $c_i$ values are in the order 3, 5, 7, 1, 2. There are, therefore, a number of possible reductions in the total penalty attached to the sequence 42351, and these are obtained by considering the following interchanges:

51 to 15: potential reduction $= (2-1)(10)(2) = 20$,

23 to 32: potential reduction $= (7-5)(4)\ (6) = 48$, followed by

43 to 34: potential reduction $= (7-3)(6)\ (1) = 24$, and finally

42 to 24: potential reduction $= (5-3)(4)\ (1) = 8$.

These interchanges are designed to convert the primary ordering of 42351 to the "secondary" ordering 32415 where jobs are ordered in decreasing value of $c_i$ according to criterion (3). Since $f(42351) = 2202$, a lower bound to the optimal solution of the problem is $2202 - (20 + 48 + 74 + 8) = 2102$.

## 3. Incorporation in a Branch-and-Bound Procedure

Consider, for the problem of Table 2, those sequences which have job 3 and job 5 fixed in the first and second positions respectively. The primary ordering of the remaining jobs is 421 and the secondary ordering is 241. A simple interchange, $42 \rightarrow 24$, is necessary on the primary to obtain the secondary with a potential gain of 8. It follows, therefore, that a lower bound, LB(35), to the penalty function for sequences starting with the pre-sequence 35 is

$$\mathrm{LB}(35) = f(35421) - 8 = 2462 - 8 = 2454.$$

For sequences starting with job 5, the primary and secondary orderings give 54231 and 53241 with interchanges $42 \rightarrow 24$, $43 \rightarrow 34$ and $23 \rightarrow 32$.

$$\mathrm{LB}(5) = f(54321) - (8 + 24 + 48) = 2517 - 80 = 2437.$$

Development of the scheduling tree is carried out as follows: A node at the $r$th level of the tree corresponds to a partial sequence $S = (i_1, i_2, \ldots, i_r)$. The remaining $n - r$ jobs have yet to be scheduled and selection of one of these for the $(r + 1)$th position results in a branch to a descendant node at the $(r + 1)$th level. Branching takes place from the node with the lowest lower bound, irrespective of its level, and the lower

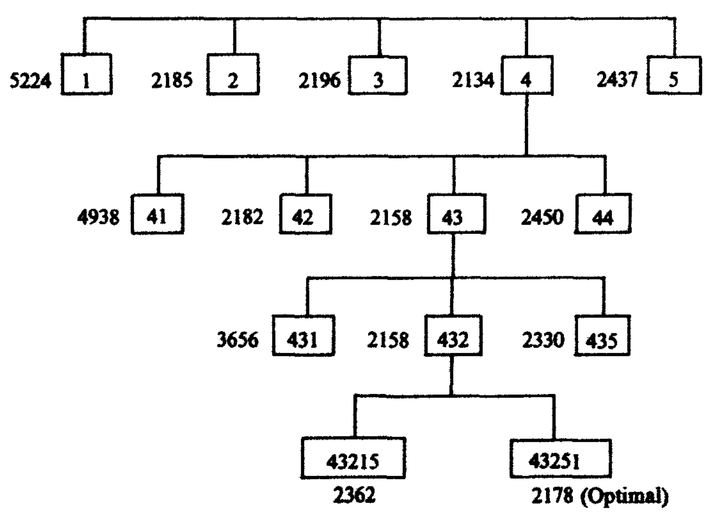


FIGURE 1. Scheduling Tree for Problem of Table 2.

bounds of all the nodes at the next level are calculated. The complete scheduling tree for the problem of Table 2 is shown in Figure 1, an optimal solution 43251 with associated penalty of 2178 being found after the evaluation of 14 nodes.

## 4. Modification of the Cost Function

Suppose that the cost function is changed from $\sum c_i x_i^2$ to the slightly more complex form $\sum (c_i x_i^2 + c_i' x_i)$. It can be shown that, for two adjacent jobs $i$ and $j$, $i$ should precede $j$ if

$$c_j a_i(2t + a_i + a_j) + c_j a_i a_j + c_j' a_i < c_i a_j(2t + a_i + a_j) + c_i a_i a_j + c_i' a_j.$$

The primary ordering is made, as before, according to the decreasing value of the ratio $c_i/a_i$. In the secondary ordering $i$ precedes $j$ if $c_j a_i a_j + c_j' a_i < c_i a_i a_j + c_i' a_j$; that is, if $c_i + c_i'/a_i > c_j + c_j'/a_j$. The secondary ordering is in this case based on a rather different criterion. It is only slightly more complicated to obtain the corresponding potential from interchange. The expression for this becomes $a_i a_j[(c_j + c_j'/a_j) - (c_i + c_i'/a_i)]$.

Consider the example of Table 3 which extends the data of Table 2 to include values of $c_i'$.

TABLE 3

*Extended Problem of Table 2*

| Job($i$) | Operation time ($a_i$) | Cost Coefficients $c_i$ | $c_i'$ | $c_i/a_i$ | $c_i + c_i'/a_i$ |
|---|---|---|---|---|---|
| 1 | 10 | 2 | 18 | 1/5 | $3\frac{4}{5}$ |
| 2 | 4 | 5 | 8 | 5/4 | 7 |
| 3 | 6 | 7 | 1 | 7/6 | $7\frac{1}{6}$ |
| 4 | 1 | 3 | 2 | 3 | 5 |
| 5 | 2 | 1 | 15 | 1/2 | $8\frac{1}{2}$ |

The primary ordering is, as before, 42351, but the secondary ordering is now 53241, with a potential reduction due to interchange from the following pairs:

$$35 \text{ to } 53{:}\ 12(8\tfrac{1}{2} - 7\tfrac{1}{6}) = 16,$$

$$25 \text{ to } 52{:}\ 8(8\tfrac{1}{2} - 7) = 12,$$

$$45 \text{ to } 54{:}\ 2(8\tfrac{1}{2} - 5) = 7,$$

$$42 \text{ to } 24{:}\ 4(7 - 5) = 8,$$

$$43 \text{ to } 34{:}\ 6(7\tfrac{1}{6} - 5) = 13,$$

$$23 \text{ to } 32{:}\ 24(7\tfrac{1}{6} - 7) = 4.$$

Since $f(42351) = 2864$, a lower bound to the penalty cost is $2864 - (16 + 12 + 7 + 8 + 13 + 4) = 2804$. The branch-and-bound procedure is then applied in a similar way to that of the previous section.

## 5. Computational Results and Extensions

The method of calculating the lower bounds which has been described is a simple one, so that solutions to problems of a reasonable size may be found quite quickly. Nevertheless, it is unlikely to be a practicable procedure for large problems without some modification to reduce the number of nodes generated. In this respect, it may be that good solutions can be found by eliminating any backtracking, so that the search is limited to the evaluation of $n + (n - 1) + \cdots + 1 = \frac{1}{2}n(n + 1)$ nodes to produce an approximate, polynomial-bounded, procedure.

A more promising avenue for further work is suggested by the form of expression (1). The condition for an ordering of jobs $i$ and $j$ is dependent on the value of $t$, the time for completion of the jobs immediately prior to $i$ and $j$ in the partial sequence. There exists a critical value of $t$ which, once exceeded, allows an ordering to be made. Now, as the value of $t$ increases with progress through the scheduling tree, it should be possible to fix the orderings of some jobs which initially do not satisfy the criteria (2) and (3), thus reducing the number of nodes generated. This aspect is currently under investigation.[1]

[1] The author is grateful to the referees for their comments and suggestions.

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