Problem-solving by Mixed-Integer Programming

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Setting

- We consider a general Mixed Integer Program in the form

$$\max\{c^T x : Ax \leq b, x \geq 0,\ x_j \in \mathbb{Z}, \forall j \in I\} \tag{1}$$

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- The lecture basically covers the MIP but we will try to discuss when possible how crucial is the LP component (the engine), and how much the whole framework is built on top the capability of effectively solving LPs.

- Roughly speaking, using the LP computation as a tool, MIP solvers integrate the branch-and-bound and the cutting plane algorithms through variations of the general branch-and-cut scheme [Padberg & Rinaldi 1987] developed in the context of the Traveling Salesman Problem.

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   We will run over the first 50+ exciting years of MIP by showing some crucial milestones and we will highlight the building blocks that are making nowadays solvers effective from both a performance and an application viewpoint.

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2. Heuristic nature and capability of MIP solvers.
   Nowadays MIP solvers should **not** be conceived as black-box exact tools. In fact, they provide countless options for their smart use as hybrid algorithmic frameworks, which thing might turn out especially interesting on the applied context. We will review some of those options and possible hybridizations.

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- It looks like a major (crucial) step to get to nowadays MIP solvers has been the ultimate proof that cutting plane generation in conjunction with branching could work in general, i.e., after the success in the TSP context:
  - 1994 Balas, Ceria & Cornuéjols: lift-and-project
  - 1996 Balas, Ceria, Cornuéjols & Natraj: Gomory cuts revisited

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10.0	2005	201	650	1.91
9.0	2003	142	793	2.73
8.0	2002	117	856	3.56
7.1	2001	63	930	4.59
6.5	1999	71	997	7.47
6.0	1998	55	1060	21.30
5.0	1997	45	1069	22.57
4.0	1995	37	1089	26.29
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- Does anybody know which was the key feature of Cplex v. 6.5?

MIP Evolution, Cutting Planes

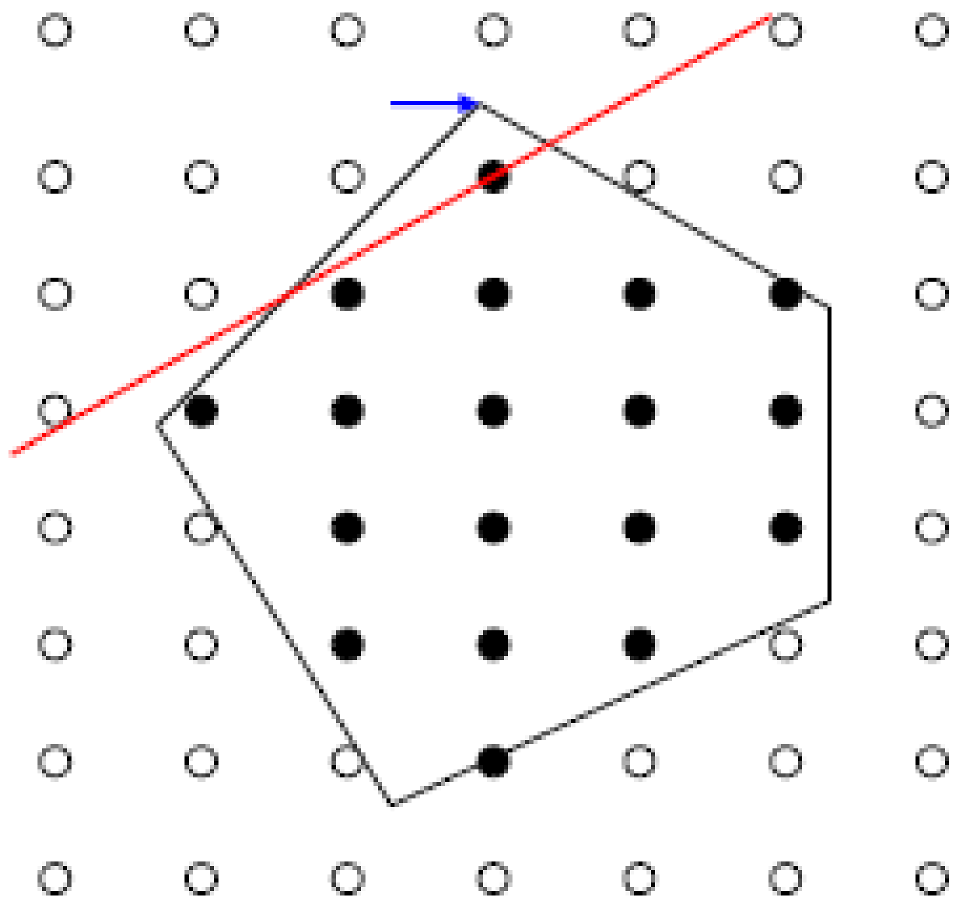

Figure 1: Strengthening the LP relaxation by cutting planes.

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- Moreover, the MIP computation has reached such an effective and stable quality to allow the solution of sub-MIPs in the algorithmic process, the MIPping approach [Fischetti & Lodi 2002]. These sub-MIPs are solved both for cutting plane generation and in the primal heuristic context.

MIP Building Blocks: Preprocessing/Presolving

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     Thus, modern MIP solvers have the capability of cleaning up and strengthen a model so as to create a presolved instance on which the MIP technology is then applied.

MIP Building Blocks: Preprocessing/Presolving (cont.d)

2. Algorithmic preprocessing:
   more sophisticated presolve mechanisms are also able to discover important implications and sub-structures that might be of fundamental importance later on in the computation for both branching purposes and cutting plane generation.

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   As an example, the presolve phase determines the clique table or conflict graph, i.e., groups of binary variables such that no more than one can be non-zero at the same time.
   The so-called conflict graph is then fundamental to separate clique inequalities [Johnson and Padberg 1982] written as

$$\sum_{j \in Q} x_j \leq 1 \tag{2}$$

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   This is currently the component of MIP solvers in which the integration with CP techniques has advanced the most.

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- Given the MIP (1), we are mainly interested in the two sets

$$S := \{Ax \leq b, x \geq 0,\ x_j \in \mathbb{Z}, \forall j \in I\} \tag{3}$$

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- Generality: We are interested in general-purpose cutting planes, those that can be derived without assuming any special structure for the polyhedron P.
- Validity: An inequality $\alpha x \leq \beta$ is said to be valid for S if it is satisfied by all $x \in S$.
- Obtaining a valid inequality for a continuous set: Given P, any valid inequality for it is obtained as $uAx \leq \beta$, where $u \in \mathbb{R}^m_+$ and $\beta \geq ub$. (Farkas Lemma)

MIP Building Blocks: Cutting Planes (cont.d)

- Separation:
  Given a family of valid inequalities $\mathcal{F}$ and a solution $x^* \in P \setminus S$, the **Separation problem for** $\mathcal{F}$ is defined as

  Find an inequality $\alpha x \leq \beta$ of $\mathcal{F}$ valid for S such that $\alpha x^* > \beta$ or show that none exists.

- Iterative strengthening
  1. solve the problem $\{\max c^T x \ : \ x \in P\}$ and get x^*
  2. if $x^* \in S$ then **STOP**
  3. solve the separation problem, add $\alpha x \leq \beta$ to P and go to 1.

- (Almost) all cutting plane classes in the arsenal of nowadays MIP solvers belong to the family of split cuts, i.e., they are separated by exploiting in some way (from easy to complex) a disjunction on the integer variables.

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- A basic rounding argument:
  If $x \in \mathbb{Z}$ and $x \leq f$ $f \notin \mathbb{Z}$, then $x \leq \lfloor f \rfloor$.

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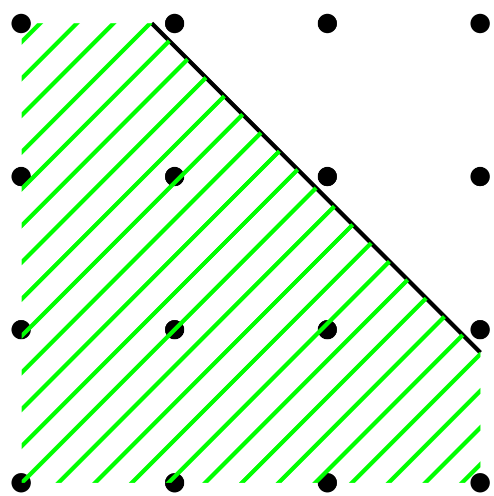
- Example:
  $x \in \mathbb{Z}^2$ such that $x_1 + x_2 \leq 1.9$



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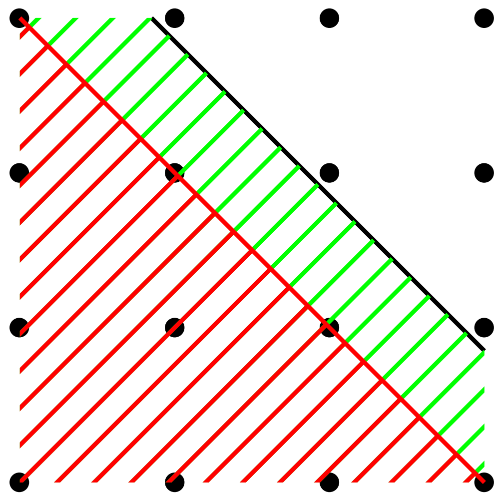
  $x \in \mathbb{Z}^2$ such that $x_1 + x_2 \leq 1.9 \Rightarrow\ x_1 + x_2 \leq \lfloor 1.9 \rfloor = 1$



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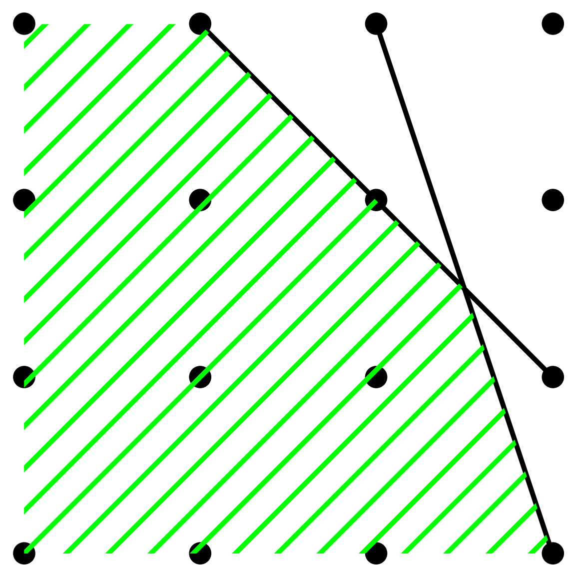

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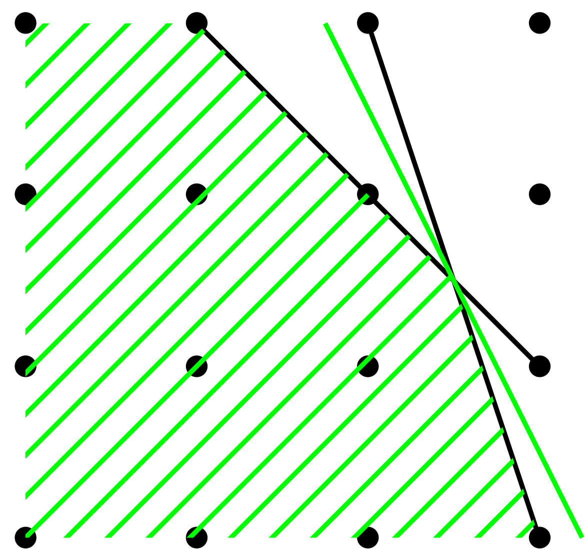

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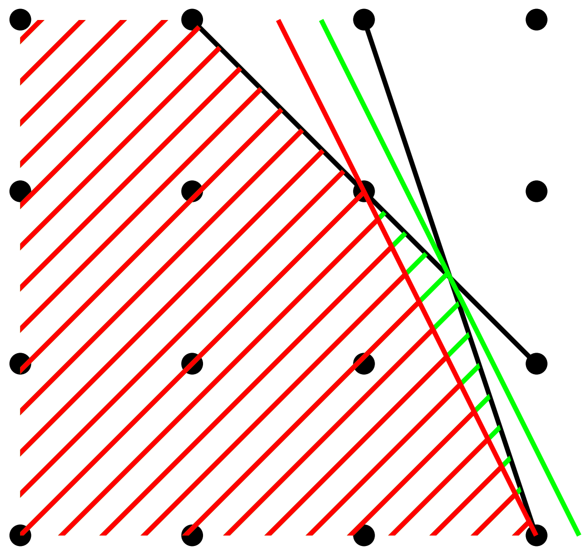

MIP Building Blocks: Branching

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- Usually, for MIP solvers the branching creates two children by using the rounding of the solution of the LP relaxation value of a fractional variable, say x_j, constrained to be integral

$$x_j \leq \lfloor x_j^* \rfloor \quad \textbf{OR} \quad x_j \geq \lfloor x_j^* \rfloor + 1. \tag{5}$$

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- In addition, the LP relaxation is solved at every node to decide if the node itself is worthwhile to be further partitioned: if the LP relaxation value is already not better (bigger) than the incumbent, the node can be safely fathomed.

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- In fact, on average, for a single LP solution Interior Point algorithms perform better than the Simplex algorithm [Rothberg 2010], which is in turn (currently) unbeatable in the iterative context.

MIP Building Blocks: Branching (cont.d)

- Of course, the basic idea of the splitting a node does not require that branching is performed as in (5): i.e., more than two children could be created, and one can branch on more general hyperplanes, or, in general, on any other disjunctive condition.

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- In fact, on average, for a single LP solution Interior Point algorithms perform better than the Simplex algorithm [Rothberg 2010], which is in turn (currently) unbeatable in the iterative context.

- The described branch-and-bound framework requires two independent and important decisions at any step: Node and Variable selection.

MIP Building Blocks: Branching (cont.d)

1. Node selection:
   This is very classical: one extreme is the so called best-bound first strategy in which one always considers the most promising node, i.e., the one with the highest LP value, while the other extreme is depth first where one goes deeper and deeper in the tree and starts backtracking only once a node is fathomed.

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   In order to devise stronger criteria one has to do much more work.

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   The most recent effective and sophisticated method, called reliability branching [Achterberg et al. 2005], integrates strong and pseudocost branchings by defining a reliability threshold, i.e., a level below which the information of the pseudocosts is not considered accurate enough and some strong branching is performed.

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- In other words, heuristics largely impact on the user perception of the quality of a solver, and are fundamental in the real-world context.

- The primal heuristics implemented in the solvers go from very light and easy, as variations of the classical rounding of the LP solution, to much more heavy and complex, like local search and metaheuristics.

- Details on these latter classes of heuristics will be discussed in the second part of the lecture.

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- However, the most intriguing development associated with the availability of multiple cores is the fact of exploiting them for doing different "things", not different nodes. In other words, to run different algorithmic strategies on different cores and/or use them to learn what is the best.

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  - callbacks:
    allow flexibility to accommodate the user code so as to take advantage of specific knowledge

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  3. to embed the solver into custom applications, which would require software to have a callable library.
- Finally, the user may wish to adapt certain aspects of the algorithm, and, as already discussed, this can be achieved by callback functions, or, when the source code is available, through abstract interfaces.

MIP Commercial Software

1. Cplex

Version	12.6
Website	`http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/`
Interfaces	C, C++, Java, `.NET`, Matlab, Python, Microsoft Excel

- Cplex is owned and distributed by IBM.
- A special search algorithm, called *dynamic search* can be used instead of branch-and-cut.
- Cplex is moving to Mixed Integer Non-Linear Programming MINLP, being already able to solve a large portion of quadratic and quadratically-constrained Mixed Integer Programs.

2. Gurobi

Version	5.6
Website	`www.gurobi.com`
Interfaces	C, C++, Java, Python, `.NET`, Matlab

- Gurobi Optimizer contains a relatively new MIP solver that was built from scratch to exploit modern multi-core processing technology.
- Gurobi is also available “on demand” using the Amazon Elastic Compute Cloud.

MIP Commercial Software (cont.d)

3. LINDO

Version	8.0
Website	`www.lindo.com`
Interfaces	C, Visual Basic, Matlab, Ox

- LINDO Systems offers a MIP solver as part of its LINDO API.

4. Mosek

Version	7.0
Website	`www.mosek.com`
Interfaces	C, C++, Java, .NET, Python

- MOSEK ApS is a company specializing in generic mathematical optimization software.
- Mosek suite is especially powerful for MINLP and is available through GAMS.

5. XPRESS-MP

Version	7.7
Website	`http://www.fico.com/en/Products/DMTools/xpress-overview/Pages/Xpress-Optimizer.aspx`
Interfaces	C, C++, Java, .NET, VBA

- A unique feature of XPRESS-MP is that it offers an option to branch into general (split) disjunctions, or to search for special structures on which to branch.

MIP Noncommercial Software

1. BLIS

License	Common Public License
Version	0.93
Website	`https://projects.coin-or.org/CHiPPS`
Language	C++

- Open-source MIP solver available as part of the COIN-OR project.
- Built on top of the COIN-OR High-Performance Parallel Search Framework (CHiPPS), it runs on a distributed memory platforms.
- LPs are solved using the COIN-OR linear programming Solver (Clp).

2. CBC

License	Common Public License
Version	2.8.7
Website	`https://projects.coin-or.org/Cbc`
Language	C++

- Open-source MIP solver distributed under the COIN-OR project and built from many COIN components, including the COIN-OR Clp.

MIP Noncommercial Software (cont.d)

3. GLPK

License	GNU General Public License (GPL)
Version	4.52
Website	`http://www.gnu.org/software/glpk/`
Language	C

- The software distinguishes itself through the large number of community-built interfaces available.

4. lp_solve

License	GNU lesser general public license (LGPL)
Version	5.5.2
Website	`http://lpsolve.sourceforge.net/5.5/`
Language	C

- Open source linear and integer programming solver.

MIP Noncommercial Software (cont.d)

5. MINTO

License	Given as library only
Version	3.1
Website	http://coral.ie.lehigh.edu/minto/
Language	C

- Black-box solver and solver framework for MIP.
- Primary development of the software was done in the 1990's: a whole generation of MIP researchers has been trained with MINTO!

6. SCIP

License	ZIB Academic License
Version	3.0.1
Website	http://scip.zib.de/
Language	C

- Developed and distributed by a team of researchers at Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB).
- SCIP is also a framework for Constraint Integer Programming and branch-cut-and-price, allowing the user significant control of the algorithm.
- Current benchmarks indicate that SCIP is likely the fastest noncommercial MIP solver.

MIP Noncommercial Software (cont.d)

7. SYMPHONY

License	Common Public License
Version	5.2
Website	`http://www.coin-or.org/SYMPHONY/index.htm`
Language	C

- The core solution methodology of SYMPHONY is a customizable branch, cut, and price algorithm that can be executed sequentially or in parallel.
- SYMPHONY has several unique features including the capability to warm start the branch-and-bound process from a previously calculated branch-and-bound tree, even after modifying the problem data.

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S: MIP solvers are open to different "worlds" and nowadays more and more hybrid algorithms.

Trivial Facts (1. and 2.)

Some of the points anticipated above are trivial:

1. The classical user of a MIP solver enforces limits to the computation. The most classical of these limits is the time limit, but other limits are often applied, like number of solutions (sometimes just 1), percentage gap and number of nodes.

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2. Any MIP solver works with tolerances.
   The tolerance is both in the solution of the LPs and on the branching side.
   It is pretty tight for feasibility, thus good solvers certify their solutions as "really feasible".
   It is less strict for optimality. A popular default value for that is 0.01% that, for special applications, is far from acceptable [Koch et al. 2011].
   (Ever noticed the number of nodes left to explore at the end of a run?)

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     * for the same problem (or problems in the same family)
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Performance Variability, Emilie Danna #1

Example: 10 teams, CPLEX 11, Linux

```
Tried aggregator 1 time.
MIP Presolve eliminated 20 rows and 425 columns.
Reduced MIP has 210 rows, 1600 columns, and 9600 nonzeros.
Presolve time =    0.01 sec.
Clique table members: 170.
MIP emphasis: balance optimality and feasibility.
MIP search method: dynamic search.
Parallel mode: none, using 1 thread.
Root relaxation solution time =    0.05 sec.

        Nodes                                  Cuts/
   Node  Left     Objective  IInf  Best Integer     Best Node    ItCnt     Gap

      0     0      917.0000   140                   917.0000     1100
      0     0      924.0000   165                   Cuts: 50     1969
      0     0      924.0000   167                   Cuts: 17     2348
      0     0      924.0000   175                Cliques: 14     2731
*     0+    0                        924.0000       924.0000     2731    0.00%

Clique cuts applied:  16
Zero-half cuts applied:  3
Gomory fractional cuts applied:  1

Solution pool: 1 solution saved.

MIP - Integer optimal solution:  Objective =  9.2400000000e+02
Solution time =    0.41 sec.  Iterations = 2731  Nodes = 0
```

2

Example: 10 teams, CPLEX 11, AIX

ILOG Changing the rules of business

```
Tried aggregator 1 time.
MIP Presolve eliminated 20 rows and 425 columns.
Reduced MIP has 210 rows, 1600 columns, and 9600 nonzeros.
Presolve time =    0.00 sec.
Clique table members: 170.
MIP emphasis: balance optimality and feasibility.
MIP search method: dynamic search.
Parallel mode: none, using 1 thread.
Root relaxation solution time =    0.18 sec.

        Nodes                                         Cuts/
   Node  Left     Objective  IInf  Best Integer    Best Node    ItCnt     Gap

      0     0     917.0000   151                    917.0000     1053
      0     0     924.0000   152                    Cuts: 53     1801
      0     0     924.0000   161                 Cliques: 14     2336
      0     0     924.0000   163                 Cliques: 12     2609
      0     2     924.0000   163                    924.0000     2609
*   100+   96                       952.0000      924.0000    12316    2.94%
   1000   520     926.7273    85      952.0000      924.0000    97832    2.94%
*  1425     0     integral     0      924.0000      924.0000   122948    0.00%

Clique cuts applied:  12
Zero-half cuts applied:  4
Gomory fractional cuts applied:  2

Solution pool: 2 solutions saved.

MIP - Integer optimal solution:  Objective =  9.2400000000e+02
Solution time =   41.39 sec.  Iterations = 122948  Nodes = 1426
```

3

Slightly-less Trivial Facts: 4.

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   Because of the MIP $\mathcal{NP}$-hardness, it is both theoretically and practically hard to recognize problems as good or bad for an idea, then such an idea must be HEURISTICALLY "weakened" to accomplish simultaneously the two given goals.

Benchmarking: Performance Variability

- The performance variability phenomenon has certainly been observed for decades, especially by the Artificial Intelligence and SATisfiability communities.
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- Indeed, the most dangerous effect of performance variability is the misinterpretation of the computational results obtained by testing a scientific idea or even a change in the code that might appear harmless.

- Precisely in the attempt of showing potential mistakes associated with performance variability, yet another very instructive example is discussed in the provocative talk by Fischetti [Fischetti and Monaci 2012].

Performance Variability, Fischetti & Monaci #1

- The computational investigation calls for amending MIP (1) by one single (mysterious) cut obtained by a parametrized lifting procedure that has 9 different variants.
- In other words, 9 copies of any MIP instance are produced, each one differing from the original MIP by 1 valid inequality only.

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  3. picking those requiring more than 10,000 nodes and 100 CPU seconds on the original version of the MIP.

Performance Variability, Fischetti & Monaci #2

	Avg. sec.s	Avg. nodes	Time ratio	Node ratio
Default (no cut)	**533,00**	**64499,09**	**1,00**	**1,00**
Method #1	**397,50**	**37194,89**	**0,75**	**0,58**
Method #2	419,22	44399,47	0,79	0,69
Method #3	468,87	48971,72	0,88	0,76
Method #4	491,77	46348,39	0,92	0,72
Method #5	582,42	58223,10	**1,09**	0,90
Method #6	425,38	43492,35	0,80	0,67
Method #7	457,95	46067,74	0,86	0,71
Method #8	446,89	44481,75	0,84	0,69
Method #9	419,57	41549,07	0,79	0,64

Cases with large speedup

	NO CUT		METHOD #1		
	Time	Nodes	Time	Nodes	Time Speedup
glass4	43,08	118.151	12,95	17.725	3,33
neos-1451294	3.590,27	20.258	102,94	521	34,88
neos-1593097	149,94	10.879	16,12	508	9,30
neos-1595230	1.855,69	152.951	770,6	89.671	2,41
neos-603073	452,4	36.530	130,75	10.017	3,46
neos-911970	3.588,54	5.099.389	3,29	1.767	1.090,74
ran14x18_1	3.287,59	1.480.624	2.066,70	759.265	1,59

Performance Variability, Fischetti & Monaci #4

- When the cutting plane separation procedure is disclosed, we discover that it produces the trivially-valid and redundant linear inequality

$$\sum_{j=1}^{n} x_{\pi^k(j)} \geq -1, \tag{6}$$

where π^k is a random permutation of the nonnegative variables $x_1, \ldots, x_n$.

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- However, there is more. The testbed selection is heavily biased to favor problems in which the reference solver does not perform well.
- Thus, virtually any change in the solution process (due to the column permutation) is a winner.

Overall: does Cplex v.x dominate Cplex v.y ($x > y$, even $x >> y$)?

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Cplex versions	year	better	worse	time
11.0	2007	0	0	1.00
10.0	2005	201	650	1.91
9.0	2003	142	793	2.73
8.0	2002	117	856	3.56
7.1	2001	63	930	4.59
6.5	1999	71	997	7.47
6.0	1998	55	1060	21.30
5.0	1997	45	1069	22.57
4.0	1995	37	1089	26.29
3.0	1994	34	1107	34.63
2.1	1993	13	1137	56.16
1.2	1991	17	1132	67.90

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- There are 17 instances on which Cplex 1.2 (1991) is at least 10% faster than Cplex 11.0 (2007)!

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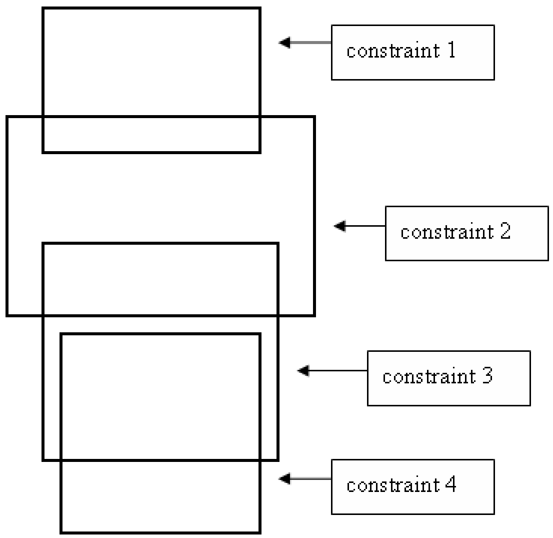

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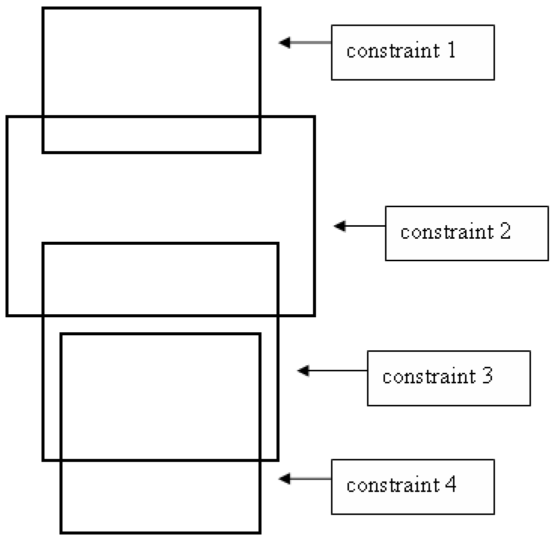

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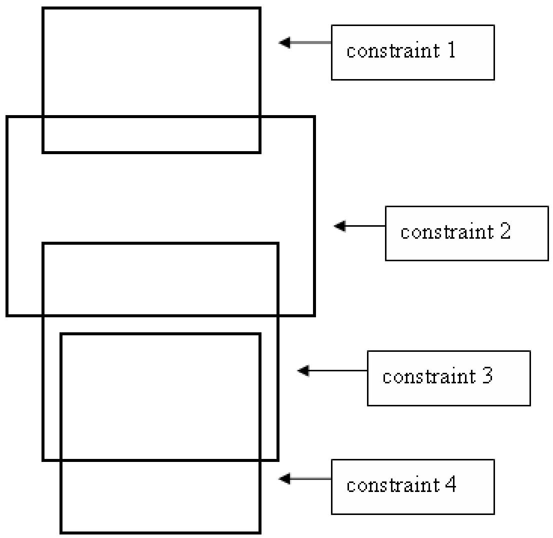

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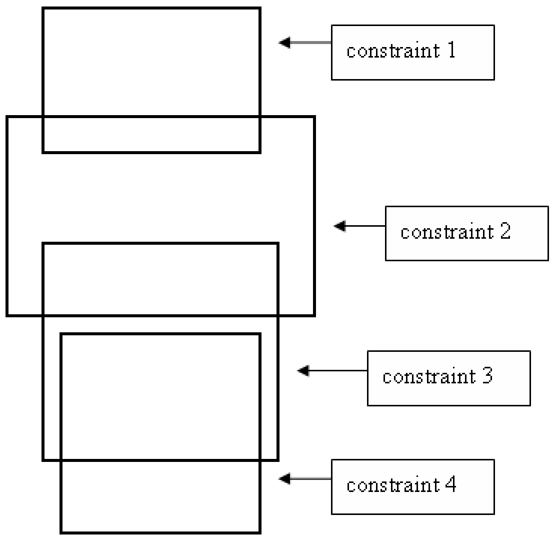

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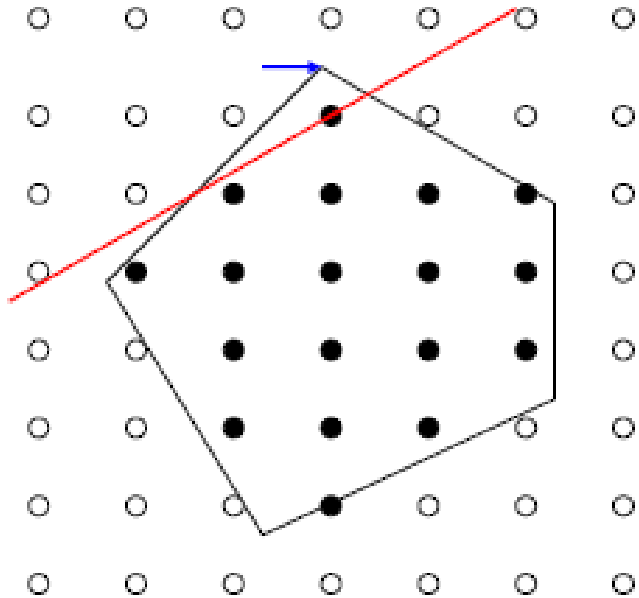
- MIPs do not explicitly contain global constraints, thus, the propagation is applied by LOCALLY comparing constraints/variables, a structurally heuristic process.

- Indeed, random permutations of rows/columns of the MIP generally lead to worse performance of the solvers mostly because of reduced preprocessing effectiveness.

5.B: Cutting Planes



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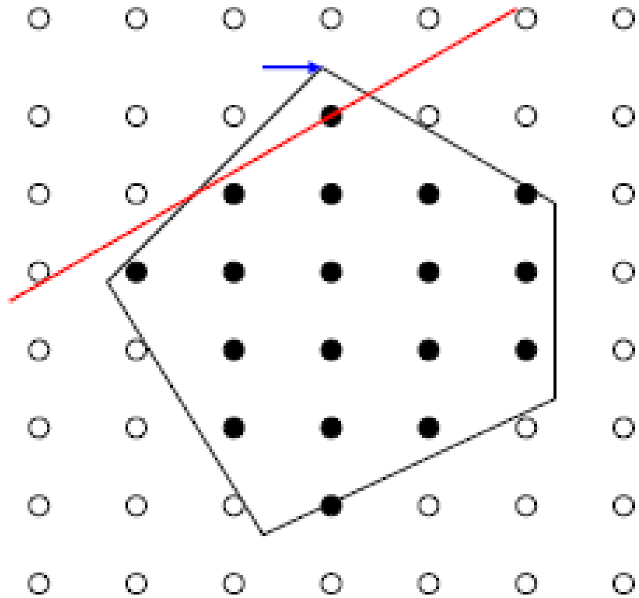

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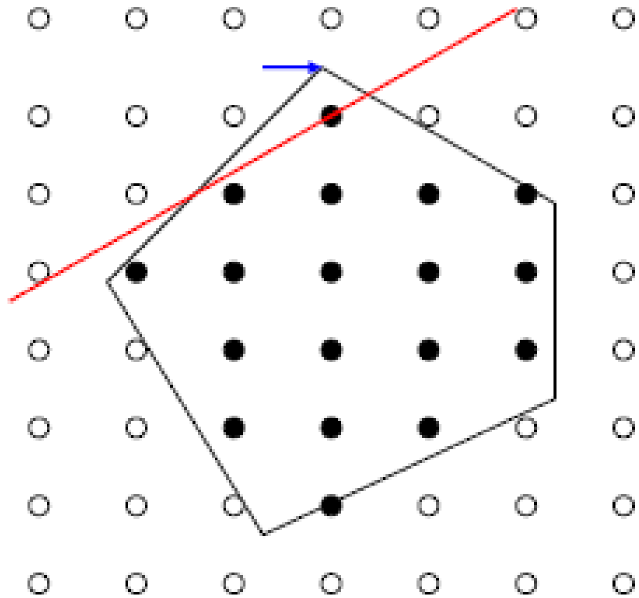

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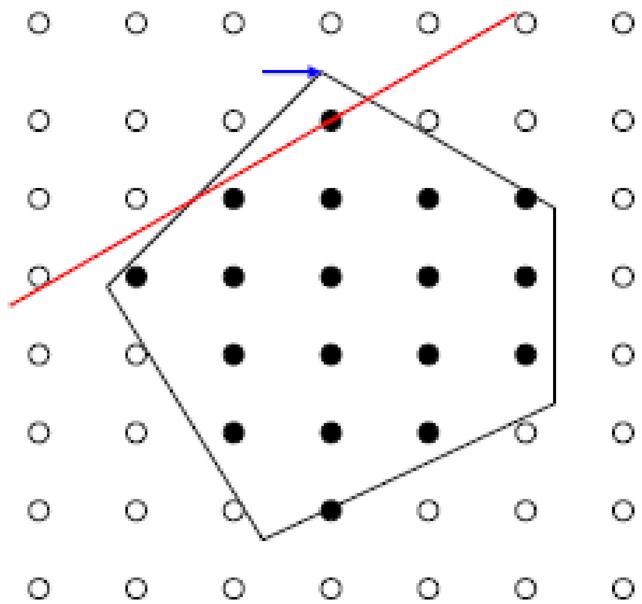

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  - apply the cut separation procedure to such a mixed-integer set (Gomory fractional, Gomory Mixed-Integer, MIR, $\{0, \frac{1}{2}\}$, . . .)
- The most recent heuristic decision that appeared to be highly crucial is the cut selection, i.e., which among all possible separated cutting planes should be added?

5.C: Branching

- The branching phase is by construction (probably) the most delicate part of the branch-and-cut framework currently implemented by commercial and non-commercial MIP solvers.

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  - each LP is not solved to optimality but within a given iteration limit.

- The heuristic side of branching is not limited to the above criteria and has an impact in almost all branching components, as for example in the decision on how to break ties.

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- However, the most surprising impact of primal heuristics in the solver is on MIPping.

  Indeed, the sub-MIPs used to find better solutions and/or to generate cuts are NEVER solved to optimality: the full integration of sophisticated heuristics in the solvers allows to count on the fact that nested calls of the same solvers could produce heuristic solutions fast!

5.D: Primal Heuristics (cont.d)

- The role of (primal) heuristics in MIP solvers is associated with three distinct aspects.

  1. Achieving Integer-Feasibility Quickly.
     Finding a first feasible solution is sometimes the main issue when solving a MIP. This is true theoretically because the feasibility problem for MIP is $\mathcal{NP}$-complete, but also from the user's perspective (as discussed) the solver needs to provide a feasible solution as quick as possible.

5.D: Primal Heuristics (cont.d)

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   Finding a first feasible solution is sometimes the main issue when solving a MIP. This is true theoretically because the feasibility problem for MIP is $\mathcal{NP}$-complete, but also from the user's perspective (as discussed) the solver needs to provide a feasible solution as quick as possible.

2. Reaching (quasi-)Optimality Quickly.
   Of course, once a feasible solution has been found, the challenge becomes getting better and better ones, and local search heuristics come into the play for that.

5.D: Primal Heuristics (cont.d)

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  3. Analyzing Infeasible MIP solutions.
     During the enumeration tree a large amount of (slightly) infeasible solutions is encountered, either as infeasible nodes in the tree itself, or as a result of the application to a primal heuristic. It is then possible to use heuristics to repair these solutions.

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- Of course, the use of MIP for heuristic solution of optimization problems is much older and much more widespread than matheuristics, but this is not the case for metaheuristics.

- Even the idea of designing MIP methods specifically for finding heuristic solutions has innovative traits, when opposed to exact methods that turn into heuristics when enough computational resources are not available.

Matheuristics (cont.d)

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  2. MIP as a paradigm for new metaheuristics.

- In both cases a common key step prescribes to collect possible components of the problem solution and to include them in a MIP formulation, often as columns of a set-partitioning formulation, possibly with additional constraints.

- The resulting restricted MIP formulation is then solved in an exact or heuristic way.

Summary and Conclusions

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  2. reliable: the software tools are stable;
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- All of the above look like solid reasons for developing the skills for using (and, why not, improving on) the MIP technology.