Analysis II HS21

by plaffranchi

2 ODE (ordinary differential equation)

Theorem 2.1.6

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then the ODE $y' = F(x, y)$ has a unique solution f defined on a "largest" open interval I containing x_0 such that $f(x_0) = y_0$.

Definition 2.2.1

Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}_0$. An homogeneous linear ODE of order k on I is of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = 0$ where the coefficients $a_0, \ldots, a_{k-1}$ are complex-valued functions on I, and the unknown is a function $I \to \mathbb{C}$ that is k-times differentiable on I. An equation of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = b$, where $b : I \to \mathbb{C}$ is another function, is called an inhomogeneous linear ODE.

Recognize an ODE

1. no coefficients before the highest derivative
2. all coefficients are continuous
3. no products of y or their derivatives
4. no powers of y or their derivatives
5. no functions depending on y or their derivatives

Proposition 2.3.1

Any solution of $y' + ay = 0$ is of the form $f(x) = z\exp(-A(x))$ where A is a primitive of a. The unique solution with $f(x_0) = y_0$ is $f(x) = y_0 \exp(A(x_0) - A(x))$.

Solving inhomogeneous equations

Case 1: Make a guess. For example $y' = y + x^2$ guess $f(x) = ax^2 + bx + c$, and solve the equation.
Case 2: Use the variation of the constant. Assume $f_p = z(x)\exp(-A(x))$ for $z : I \to \mathbb{C}$. Then $z'(x) = b(x)\exp(A(x)) \implies k(x) = \int b(x)\exp(A(x))dx$.

Definition Linear differential equations with constant coefficients

Let $k \in \mathbb{N}_0$, $a_0, \ldots, a_{k-1} \in \mathbb{C}$ fixed and b a general continuous function, then $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = b$ is such equation.

Solution of hom. diff. eq. with constant coefficients

Look for solutions of the form $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{C}$. Then we have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geqslant 0$ and for all x, which means that

$$\begin{aligned} & f^{(k)}(x) + a_{k-1}f^{(k-1)}(x) + \cdots + a_1f'(x) + a_0f(x) \\ = & e^{\alpha x}\left(\alpha^k + a_{k-1}\alpha^{k-1} + \cdots + a_1\alpha + a_0\right). \end{aligned}$$

This translates into finding the zeros of the characteristic polynomial:

$$\begin{aligned} P(X) = & X^k + a_{k-1}X^k + \cdots + a_1X + a_0 \\ = & (X - \alpha_1) \cdots (X - \alpha_k) = 0 \end{aligned}$$

Imaginary roots

If a root is not real i.e. $\alpha = \beta + i\gamma$, the solution $f(x) = e^{\alpha x}$ does not take real values, but $\overline{\alpha} = \beta - i\gamma$ is also a root, hence we can write $\widetilde{f}_1(x) = e^{\beta x}\cos(\gamma x)$, $\widetilde{f}_2(x) = e^{\beta x}\sin(\gamma x)$ instead of $f_1(x) = e^{\alpha x}$, $f_2(x) = e^{\bar{\alpha} x}$

Multiple roots

Case 1: no multiple roots. Any solution of the equation is of the form $f(x) = z_1e^{a_1x} + \cdots + z_ke^{a_kx}$.
Case 2: multiple roots. Suppose that α is a multiple root of order j with $2 \leqslant j \leqslant k$. Then the k functions $f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = xe^{\alpha x}$, $\cdots$, $f_{\alpha,j-1}(x) = x^{j-1}e^{\alpha x}$ are linearly independent solutions. Taking the union of the functions $f_{\alpha,j}$ for all roots of P, each with its multiplicity, gives a basis of the space of solutions.

3 Differential calculus in $\mathbb{R}^n$

Definition 3.2.1.

Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$. Write $x_k = (x_{k,1}, \ldots, x_{k,n})$. Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The sequence (x_k) converges to $(\to)$ y as $k \to +\infty$ if $\forall \varepsilon > 0$, if $\exists N \geqslant 1$ such that $\forall n \geqslant N$, we have $\|x_k - y\| < \varepsilon$.

Lemma 3.2.2.

$(x_k) \to y$ as $k \to +\infty \iff$ either: (1) $\forall i, 1 \leqslant i \leqslant n$, the sequence of real numbers $(x_{k,i}) \to y_i$. (2) The sequence of real numbers $\|x_k - y\| \to 0$ as $k \to +\infty$.

Definition 3.2.3.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. (1) Let $x_0 \in X$. f is continuous at x_0 if $\forall\varepsilon > 0\ \exists\delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon, \forall x \in X$. (2) f is continuous on X if it is continuous at $x_0\ \forall x_0 \in X$.

Proposition 3.2.4.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$. The function f is continuous at $x_0 \iff \forall (x_k)_k \geqslant 1$ in X s.t. $x_k \to x_0$ as $k \to +\infty$, the sequence $(f(x_k))_k \geqslant 1$ in $\mathbb{R}^m$ converges to $f(x)$.

Definition 3.2.5.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We say that f has the limit y as $x \to x_0$ with $x \neq x_0$ if for every $\varepsilon > 0$, there exists $\delta > 0$, s.t. $\forall x \in X, x \neq x_0$, s.t. $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \varepsilon$. We then write $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y$.

Proposition 3.2.7.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We have $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y \iff \forall (x_k) \in X$ s.t. $x_k \to x$ as $k \to +\infty$, and $x_k \neq x_0$, the sequence $(f(x_k))$ in $\mathbb{R}^m$ converges to y.

Proposition 3.2.9.

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and $p \geqslant 1$ an integer. Let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be continuous functions. Then the composite $g \circ f$ is continuous.

Definition 3.2.11.

(1) A subset $X \subset \mathbb{R}^n$ is bounded if the set of $\|x\|$ for $x \in X$ is bounded in $\mathbb{R}$. (2) A subset $X \subset \mathbb{R}^n$ is closed if for every sequence (x_k) in X that converges in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$. (3) A subset $X \subset \mathbb{R}^n$ is compact if it is bounded and closed.

Proposition 3.2.13.

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed set $Y \subset \mathbb{R}^m$, $f^{-1}(Y) = x \in \mathbb{R}^n : f(x) \in Y \subset \mathbb{R}^n$ is closed.

Theorem 3.2.15.

Let $X \subset \mathbb{R}^n$ be a non-empty compact set and $f : X \to \mathbb{R}$ a continuous function. Then f is bounded and achieves its max and min. I.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \sup_{x\in X} f(x)$, $f(x_-) = \inf_{x\in X} f(x)$.

Definition 3.3.1.

A subset $X \subset \mathbb{R}^n$ is open if, for any $x = (x_1, \ldots, x_n) \in X$, there exists $\delta > 0$ such that the set

$$\{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \text{ for all } i\}$$

is contained in X. In other words: any point of $\mathbb{R}^n$ obtained by changing any coordinate of x by at most δ is still in X.

Proposition 3.3.2.

A set $X \subset \mathbb{R}^n$ is open if and only if the complement $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is closed.

Corollary 3.3.3.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $Y \subset \mathbb{R}^m$ is open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$.

Definition 3.3.5

Let $X \subset \mathbb{R}^n$ be an open set. Let $f : X \to \mathbb{R}^m$ be a function. Let $1 \leqslant i \leqslant n$. We say that f has a partial derivative on X with respect to the i-th variable, or coordinate, if for all $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in X$, the function defined by

$$g(t) = f\left(x_{0,1}, \ldots, x_{0,i-1}, t, x_{0,i+1}, \ldots, x_{0,n}\right)$$

on the set $I = \{t \in \mathbb{R} : (x_{0,1}, \ldots, x_{0,i-1}, t, x_{0,i+1}, \ldots, x_{0,n}) \in X\}$ is differentiable at $t = x_{0,i}$. Its derivative $g'\left(x_{0,i}\right)$ at $x_{0,i}$ is denoted

$$\frac{\partial f}{\partial x_i}\left(x_0\right), \quad \partial_{x_i} f\left(x_0\right), \quad \partial_i f\left(x_0\right)$$

Proposition 3.3.7.

Consider $X \subset \mathbf{R}^n$ open and f, g functions from X to $\mathbf{R}^m$. Let $1 \leqslant i \leqslant n$.
(1) If f and g have partial derivatives with respect to the i-th coordinate on X, then $f + g$ also does, and

$$\partial_{x_i}(f+g) = \partial_{x_i}(f) + \partial_{x_i}(g).$$

(2) If $m = 1$, and if f and g have partial derivatives with respect to the i-th coordinate on X, then fg also does and

$$\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g).$$

Furthermore, if $g(x) \neq 0$ for all $x \in X$, then f/g has a partial derivative with respect to the i-th coordinate on X, with

$$\partial_{x_i}(f/g) = \left(\partial_{x_i}(f)g - f\partial_{x_i}(g)\right)/g^2.$$

Definition 3.3.9.

Let $X \subset \mathbf{R}^n$ open and $f : X \to \mathbf{R}^m$ a function with partial derivatives on X. Write

$$f(x) = \left(f_1(x), \ldots, f_m(x)\right).$$

For any $x \in X$, the matrix

$$J_f(x) = \left(\partial_{x_j} f_i(x)\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$$

with m rows and n columns is called the Jacobi matrix of f at x.