# Analysis II HS21

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## 2 ODE (ordinary differential equation)

### Theorem 2.1.6

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then the ODE $y' = F(x, y)$ has a **unique solution** $f$ defined on a "largest" open interval I containing $x_0$ such that $f(x_0) = y_0$.

### Definition 2.2.1

Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}_0$. An **homogeneous linear ODE of order** $k$ on $I$ is of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = 0$ where the coefficients $a_0, \ldots, a_{k-1}$ are complex-valued functions on $I$, and the unknown is a function $I \to \mathbb{C}$ that is $k$-times differentiable on $I$. An equation of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = b$, where $b : I \to \mathbb{C}$ is another function, is called an **inhomogeneous linear ODE**.

### Recognize an ODE

1. no coefficients before the highest derivative
2. all coefficients are continuous
3. no products of $y$ or their derivatives
4. no powers of $y$ or their derivatives
5. no functions depending on $y$ or their derivatives

### Proposition 2.3.1

Any solution of $y' + ay = 0$ is of the form $f(x) = z\exp(-A(x))$ where $A$ is a primitive of $a$. The unique solution with $f(x_0) = y_0$ is $f(x) = y_0 \exp(A(x_0) - A(x))$.

### Solving inhomogeneous equations

**Case 1:** Make a guess. For example $y' = y + x^2$ guess $f(x) = ax^2 + bx + c$, and solve the equation.
**Case 2:** Use the variation of the constant. Assume $f_p = z(x)\exp(-A(x))$ for $z : I \to \mathbb{C}$. Then $z'(x) = b(x)\exp(A(x)) \implies k(x) = \int b(x)\exp(A(x))dx$.

### Definition Linear differential equations with constant coefficients

Let $k \in \mathbb{N}_0$, $a_0, ..., a_{k-1} \in \mathbb{C}$ fixed and $b$ a general continuous function, then $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1y' + a_0y = b$ is such equation.

### Solution of hom. diff. eq. with constant coefficients

Look for solutions of the form $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{C}$. Then we have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geqslant 0$ and for all $x$, which means that

$$\begin{aligned} &f^{(k)}(x) + a_{k-1}f^{(k-1)}(x) + \cdots + a_1f'(x) + a_0f(x) \\ =&e^{\alpha x}\left(\alpha^k + a_{k-1}\alpha^{k-1} + \cdots + a_1\alpha + a_0\right). \end{aligned}$$

This translates into finding the zeros of the characteristic polynomial:

$$\begin{aligned} P(X) =& X^k + a_{k-1}X^k + \cdots + a_1X + a_0 \\ =& (X - \alpha_1)\cdots(X - \alpha_k) = 0 \end{aligned}$$

### Imaginary roots

If a root is not real i.e. $\alpha = \beta + i\gamma$, the solution $f(x) = e^{\alpha x}$ does not take real values, but $\overline{\alpha} = \beta - i\gamma$ is also a root, hence we can write $\widetilde{f}_1(x) = e^{\beta x}\cos(\gamma x)$, $\widetilde{f}_2(x) = e^{\beta x}\sin(\gamma x)$ instead of $f_1(x) = e^{\alpha x}$, $f_2(x) = e^{\bar{\alpha} x}$

### Multiple roots

**Case 1: no multiple roots.** Any solution of the equation is of the form $f(x) = z_1e^{a_1x} + \cdots + z_ke^{a_kx}$.
**Case 2: multiple roots.** Suppose that $\alpha$ is a multiple root of order $j$ with $2 \leqslant j \leqslant k$. Then the $k$ functions $f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = xe^{\alpha x}$, $\cdots$, $f_{\alpha,j-1}(x) = x^{j-1}e^{\alpha x}$ are linearly independent solutions. Taking the union of the functions $f_{\alpha,j}$ for all roots of $P$, each with its multiplicity, gives a basis of the space of solutions.

## 3 Differential calculus in $\mathbb{R}^n$

### Definition 3.2.1.

Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$. Write $x_k = (x_{k,1}, \ldots, x_{k,n})$. Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The sequence $(x_k)$ **converges to** $(\to)$ $y$ as $k \to +\infty$ if $\forall\varepsilon > 0$, if $\exists N \geqslant 1$ such that $\forall n \geqslant N$, we have $\|x_k - y\| < \varepsilon$.

### Lemma 3.2.2.

$(x_k) \to y$ as $k \to +\infty \iff$ either: (1) $\forall i, 1 \leqslant i \leqslant n$, the sequence of real numbers $(x_{k,i}) \to y_i$. (2) The sequence of real numbers $\|x_k - y\| \to 0$ as $k \to +\infty$.

### Definition 3.2.3.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. (1) Let $x_0 \in X$. $f$ is **continuous at** $x_0$ if $\forall\varepsilon > 0\ \exists\delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon, \forall x \in X$. (2) $f$ is **continuous** on $X$ if it is continuous at $x_0\ \forall x_0 \in X$.

### Proposition 3.2.4.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$. The function $f$ is continuous at $x_0 \iff \forall (x_k)_k \geqslant 1$ in $X$ s.t. $x_k \to x_0$ as $k \to +\infty$, the sequence $(f(x_k))_k \geqslant 1$ in $\mathbb{R}^m$ converges to $f(x)$.

### Definition 3.2.5.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We say that $f$ has the **limit** $y$ as $x \to x_0$ with $x \neq x_0$ if for every $\varepsilon > 0$, there exists $\delta > 0$, s.t. $\forall x \in X, x \neq x_0$, s.t. $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \varepsilon$. We then write $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y$.

### Proposition 3.2.7.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We have $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y \iff \forall (x_k) \in X$ s.t. $x_k \to x$ as $k \to +\infty$, and $x_k \neq x_0$, the sequence $(f(x_k))$ in $\mathbb{R}^m$ converges to $y$.

### Proposition 3.2.9.

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and $p \geqslant 1$ an integer. Let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be continuous functions. Then the composite $g \circ f$ is continuous.

### Definition 3.2.11.

(1) A subset $X \subset \mathbb{R}^n$ is **bounded** if the set of $\|x\|$ for $x \in X$ is bounded in $\mathbb{R}$. (2) A subset $X \subset \mathbb{R}^n$ is **closed** if for every sequence $(x_k)$ in $X$ that converges in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$. (3) A subset $X \subset \mathbb{R}^n$ is **compact** if it is bounded and closed.

### Proposition 3.2.13.

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed set $Y \subset \mathbb{R}^m$, $f^{-1}(Y) = x \in \mathbb{R}^n : f(x) \in Y \subset \mathbb{R}^n$ is closed.

### Theorem 3.2.15.

Let $X \subset \mathbb{R}^n$ be a non-empty compact set and $f : X \to \mathbb{R}$ a continuous function. Then $f$ is bounded and achieves its max and min. I.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \sup_{x\in X} f(x)$, $f(x_-) = \inf_{x\in X} f(x)$.

**Definition 3.3.1.**

A subset $X \subset \mathbb{R}^n$ is **open** if, for any $x = (x_1, \ldots, x_n) \in X$, there exists $\delta > 0$ such that the set

$$\{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \text{ for all } i\}$$

is contained in $X$. In other words: any point of $\mathbb{R}^n$ obtained by changing any coordinate of $x$ by at most $\delta$ is still in $X$.

**Proposition 3.3.2.**

A set $X \subset \mathbb{R}^n$ is open if and only if the complement $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is closed.

**Corollary 3.3.3.**

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $Y \subset \mathbb{R}^m$ is open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$.

**Definition 3.3.5**

Let $X \subset \mathbb{R}^n$ be an open set. Let $f : X \to \mathbb{R}^m$ be a function. Let $1 \leqslant i \leqslant n$. We say that $f$ has a **partial derivative** on $X$ with respect to the $i$-th variable, or coordinate, if for all $x_0 = (x_{0,1}, \ldots, x_{0,n}) \in X$, the function defined by

$$g(t) = f(x_{0,1}, \ldots, x_{0,i-1}, t, x_{0,i+1}, \ldots, x_{0,n})$$

on the set $I = \{t \in \mathbb{R} : (x_{0,1}, \ldots, x_{0,i-1}, t, x_{0,i+1}, \ldots, x_{0,n}) \in X\}$ is differentiable at $t = x_{0,i}$. Its **derivative** $g'(x_{0,i})$ at $x_{0,i}$ is denoted

$$\frac{\partial f}{\partial x_i}(x_0), \quad \partial_{x_i} f(x_0), \quad \partial_i f(x_0)$$

**Proposition 3.3.7.**

Consider $X \subset \mathbb{R}^n$ open and $f, g$ functions from $X$ to $\mathbb{R}^m$. Let $1 \leqslant i \leqslant n$. (1) If $f$ and $g$ have partial derivatives with respect to the $i$-th coordinate on $X$, then $f + g$ also does, and

$$\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g).$$

(2) If $m = 1$, and if $f$ and $g$ have partial derivatives with respect to the $i$-th coordinate on $X$, then $fg$ also does and

$$\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g).$$

Furthermore, if $g(x) \neq 0$ for all $x \in X$, then $f/g$ has a partial derivative with respect to the $i$-th coordinate on $X$, with

$$\partial_{x_i}(f/g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g))/g^2.$$

**Definition 3.3.9. (Jacobi matrix)**

Let $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$ a function with partial derivatives on $X$. Write

$$f(x) = (f_1(x), \ldots, f_m(x)).$$

For any $x \in X$, the matrix

$$J_f(x) = \left(\partial_{x_j} f_i(x)\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$$

with $m$ rows and $n$ columns is called the **Jacobi matrix** of $f$ at $x$.

**Definition 3.3.11 (Gradient, Divergence)**

Let $X \subset \mathbb{R}^n$ be open. (1) Let $f : X \to \mathbb{R}$ be a function. If all partial derivatives of $f$ exist at $x_0 \in X$, then the column vector

$$\begin{pmatrix} \partial_{x_1} f(x_0) \\ \cdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

is called the **gradient** at $x_0$, and is denoted $\nabla f(x_0)$. (2) Let $f = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ be a function with values in $\mathbb{R}^n$ such that all partial derivatives of all coordinates $f_i$ of $f$ exist at $x_0 \in X$. Then the real number

$$\operatorname{Tr}(J_f(x_0)) = \sum_{i=1}^{n} \partial_{x_i} f_i(x_0),$$

the trace of the Jacobi matrix, is called the **divergence** of $f$ at $x_0$, and is denoted $\operatorname{div}(f)(x_0)$.

**Definition 3.4.2.**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$ be a function. Let $u$ be a linear map $\mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in X$. We say that $f$ **is differentiable at** $x_0$ **with differential** $u$ if

$$\lim_{x \to x_0} \frac{1}{\|x - x_0\|}(f(x) - f(x_0) - u(x - x_0)) = 0$$

where the limit is in $\mathbb{R}^m$. We then denote $df(x_0) = u$. If $f$ is differentiable at every $x_0 \in X$, then we say that $f$ is differentiable on $X$.

**Proposition 3.4.4.**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$ be a function that is differentiable on $X$. (1) The function $f$ is continuous on $X$. (2) The function $f$ admits partial derivatives on $X$ with respect to each variable. (3) Assume that $m = 1$. Let $x_0 \in X$, and let $u(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n$ be the differential of $f$ at $x_0$. We then have $\partial_{x_i} f(x_0) = a_i$ for $1 \leqslant i \leqslant n$.

**Proposition 3.4.6.**

Let $X \subset \mathbb{R}^n$ be open, $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^m$ differentiable functions on $X$. (1) The function $f + g$ is differentiable with differential $d(f + g) = df + dg$, and if $m = 1$, then $fg$ is differentiable. (2) If $m = 1$ and if $g(x) \neq 0$ for all $x \in X$, then $f/g$ is differentiable.

**Proposition 3.4.7.**

Let $X \subset \mathbb{R}^n$ be open, $f : X \to \mathbb{R}^m$ a function on $X$. If $f$ has all partial derivatives on $X$, and if the partial derivatives of $f$ are continuous on $X$, then $f$ is differentiable on $X$, with differential determined by its partial derivatives, in the sense that the matrix of the differential $df(x_0)$, with respect to the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, is the Jacobi matrix of $f$ at $x_0$.

**Proposition 3.4.9 (Chain rule)**

Let $X \subset \mathbb{R}^n$ be open, $Y \subset \mathbb{R}^m$ be open, and let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be differentiable functions. Then $g \circ f : X \to \mathbb{R}^p$ is differentiable on $X$, and for any $x \in X$, its differential is given by the composition

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0).$$

In particular, the Jacobi matrix satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$$

where the right-hand side is a matrix product.

**Definition 3.4.11.**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$ a function that is differentiable. Let $x_0 \in X$ and $u = df(x_0)$ be the differential of $f$ at $x_0$. The graph of the affine linear approximation

$$g(x) = f(x_0) + u(x - x_0)$$

from $\mathbb{R}^n$ to $\mathbb{R}^m$, or in other words the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x_0) + u(x - x_0)\}$$

is called the **tangent space** at $x_0$ to the graph of $f$.

**Definition 3.4.13.**

Let $X \subset \mathbb{R}^n$ be an open set and let $f : X \to \mathbb{R}^m$ be a function. Let $v \in \mathbb{R}^n$ be a non-zero vector and $x_0 \in X$. We say that $f$ has **directional derivative** $w \in \mathbb{R}^m$ **in the direction** $v$, if the function $g$ defined on

the set
$$I = \{t \in \mathbb{R} : x_0 + tv \in X\}$$
by
$$g(t) = f(x_0 + tv)$$
has a derivative at $t = 0$, and this is equal to $w$. In other words, this means that the limit
$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(x_0+tv) - f(x_0)}{t}$$
exists and is equal to $w$.

**Proposition 3.4.15.**

Let $X \subset \mathbb{R}^n$ be an open set and let $f : X \to \mathbb{R}^m$ be a differentiable function. Then for any $x \in X$ and non-zero $v \in \mathbb{R}^n$, the function $f$ has a directional derivative at $x_0$ in the direction $v$, equal to $\mathrm{df}(x_0)(v)$.

**Definition 3.5.1.**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$. We say that $f$ is of class $C^1$ if $f$ is differentiable on $X$ and all its partial derivatives are continuous. The set of functions of class $C^1$ from $X$ to $\mathbb{R}^m$ is denoted $C^1(X;\mathbb{R}^m)$. Let $k \geqslant 2$. We say, by induction, that $f$ is of class $C^k$ if it is differentiable and each partial derivative $\partial_{x_i} f : X \to \mathbb{R}^m$ is of class $C^{k-1}$. The set of functions of class $C^k$ from $X$ to $\mathbb{R}^m$ is denoted $C^k(X;\mathbb{R}^m)$.
If $f \in C^k(X;\mathbb{R}^m)$ for all $k \geqslant 1$, then we say that $f$ is of class $C^\infty$. The set of such functions is denoted $C^\infty(X;\mathbb{R}^m)$.

**Proposition 3.5.4 (Mixed derivatives commute)**

$k \geqslant 2$. Let $X \subset \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}^m$ be a function of class $C^k$. Then the partial derivatives of order $k$ are independent of the order in which the partial derivatives are taken: for any variables $x$ and $y$, we have $\partial_{x,y} f = \partial_{y,x} f$ and for any variables $x, y, z$, we have
$$\partial_{x,y,z} f = \partial_{x,z,y} f = \partial_{y,z,x} f = \partial_{z,x,y} f = \cdots$$

**Definition 3.5.9 (Hessian).**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a $C^2$ function. For $x \in X$, the **Hessian matrix** of $f$ at $x$ is the symmetric square matrix
$$\mathrm{Hess}_f(x) = \left(\partial_{x_i,x_j} f\right)_{1\leqslant i,j\leqslant n}.$$
We also sometimes write simply $H_f(x)$.

**Definition 3.7.1 (Taylor polynomials).**

Let $k \geqslant 1$ be an integer. Let $f : X \to \mathbb{R}$ be a function of class $C^k$ on $X$, and fix $x_0 \in X$. The $k$-th Taylor polynomial of $f$ at the point $x_0$ is the polynomial in $n$ variables of degree $\leqslant k$ given by
$$\begin{aligned} T_k f(y; x_0) = & f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)\, y_i + \cdots \\ & + \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1!\cdots m_n!} \frac{\partial^k f}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}(x_0)\, y_1^{m_1}\cdots y_n^{m_n} \end{aligned}$$
where the last sum ranges over the tuples of $n$ non-negative integers such that the sum is $k$.

**Proposition 3.7.3 (Taylor approximation)**

Let $k \geqslant 1$ be an integer. Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ be a function of class $C^k$. For $x_0$ in $X$, if we define $E_k f(x; x_0)$ by
$$f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$$
then we have
$$\lim_{\substack{x\to x_0\\ x\neq x_0}} \frac{E_k f(x;x_0)}{\|x - x_0\|^k} = 0$$

**Proposition 3.8.1.**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a differentiable function. If $x_0 \in X$ is such that $f(y) \leqslant f(x_0)$ for all $y$ close enough to $x_0$ (local maximum at $x_0$ ) or $f(y) \geqslant f(x_0)$ for all $y$ close enough to $x_0$ (local minimum at $x_0$ ). Then we have $df(x_0) = 0$, or in other words $\nabla f(x_0) = 0$, or equivalently $\frac{\partial f}{\partial x_i}(x_0) = 0$ for $1 \leqslant i \leqslant n$.

**Definition 3.8.2 (Critical point)**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a differentiable function. A point $x_0 \in X$ such that $\nabla f(x_0) = 0$ is called a **critical point** of the function $f$.

**Definition 3.8.6 (Non-degenerate critical point)**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ a function of class $C^2$. A critical point $x_0 \in X$ of $f$ is called **non-degenerate** if the Hessian matrix has non-zero determinant.

**Corollary 3.8**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a function of class $C^2$. Let $x_0$ be a non-degenerate critical point of $f$. Let $p$ and $q$ be the number of positive and negative eigenvalues of $\mathrm{Hess}_{\,f}(x_0)$. (1) If $p = n$, equivalently if $q = 0$, the function $f$ has a local minimum at $x_0$. (2) If $q = n$, equivalently if $p = 0$, the function $f$ has a local maximum at $x_0$. (3) Otherwise, equivalently if $pq \neq 0$, the function $f$ does not have a local extremum at $x_0$. One then says that $f$ has a saddle point at $x_0$.

**Proposition 3.9.2**

Let $X \subset \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions of class $C^1$. If $x_0 \in X$ is a local extremum of the function $f$ restricted to the set $Y = \{x \in X : g(x) = 0\}$ then either $\nabla g(x_0) = 0$, or there exists $\lambda_0 \in \mathbb{R}$ such that
$$\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$$
or in other words, there exists $\lambda$ such that $(x_0, \lambda)$ is a critical point of the differentiable function $h : X \times \mathbb{R} \to \mathbb{R}$ defined by $h(x,\lambda) = f(x) - \lambda g(x)$. Such a value $\lambda$ is called a Lagrange multiplier at $x_0$.

**Definition 3.10.1 (Change of variable)**

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ be differentiable. Let $x_0 \in X$. We say that $f$ is a **change of variable** around $x_0$ if there is a radius $r > 0$ such that the restriction of $f$ to the ball
$$B = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$$
of radius $r$ around $x_0$ has the property that the image $Y = f(B)$ is open in $\mathbb{R}^n$, and if there is a differentiable map $g : Y \to B$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_B$.

**Theorem 3.10.2 (Inverse funtion theorem)**

(Inverse function theorem). Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ differentiable. If $x_0 \in X$ is such that $\det(J_f(x_0)) \neq 0$, i.e., such that the Jacobian trix of $f$ at $x_0$ is invertible, then $f$ is a change of variable around $x_0$. Moreover, the Jacobian of $g$ at $x_0$ is determined by
$$J_g(f(x_0)) = J_f(x_0)^{-1}$$
In addition, if $f$ is of class $C^k$, then $g$ is of class $C^k$.

**Theorem 3.10.4 (Implicit Function Theorem).**

Let $X \subset \mathbf{R}^{n+1}$ be open and let $g : X \to \mathbf{R}$ be of class $C^k$ with $k \geqslant 1$. Let $(x_0, y_0) \in \mathbf{R}^n \times \mathbf{R}$ be such that $g(x_0, y_0) = 0$. Assume that

$$\partial_y g(x_0, y_0) \neq 0$$

Then there exists an open set $U \subset \mathbf{R}^n$ containing $x_0$, an open interval $I \subset \mathbf{R}$ containing $y_0$, and a function $f : U \to \mathbf{R}$ of class $C^k$ such that the system of equations

$$\begin{cases} g(x, y) = 0 \\ x \in U, \quad y \in I \end{cases}$$

is equivalent with $y = f(x)$. In particular, $f(x_0) = y_0$. Moreover, the gradient of $f$ at $x_0$ is given by

$$\nabla f(x_0) = -\frac{1}{(\partial_y g)(x_0, y_0)} \nabla_x g(x_0, y_0)$$

where $\nabla_x g = (\partial_{x_1} g, \ldots, \partial_{x_n} g)$