

2 ODE (ordinary differential equation)

Theorem 2.1.6

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be differentiable. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then the ODE $y' = F(x, y)$ has a unique solution f defined on a "largest" open interval I containing x_0 such that $f(x_0) = y_0$.

Definition 2.2.1

Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}_0$. An homogeneous linear ODE of order k on I is of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$ where the coefficients a_0, \dots, a_{k-1} are complex-valued functions on I , and the unknown is a function $I \rightarrow \mathbb{C}$ that is k -times differentiable on I . An equation of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$, where $b : I \rightarrow \mathbb{C}$ is another function, is called an inhomogeneous linear ODE.

Recognize an ODE

1. no coefficients before the highest derivative
2. all coefficients are continuous
3. no products of y or their derivatives
4. no powers of y or their derivatives
5. no functions depending on y or their derivatives

Proposition 2.3.1

Any solution of $y' + ay = 0$ is of the form $f(x) = z \exp(-A(x))$ where A is a primitive of a . The unique solution with $f(x_0) = y_0$ is $f(x) = y_0 \exp(A(x_0) - A(x))$.

Solving inhomogeneous equations

Case 1: Make a guess. For example $y' = y + x^2$ guess $f(x) = ax^2 + bx + c$, and solve the equation.

Case 2: Use the variation of the constant. Assume $f_p = z(x) \exp(-A(x))$ for $z : I \rightarrow \mathbb{C}$. Then $z'(x) = b(x) \exp(A(x)) \implies k(x) = \int b(x) \exp(A(x)) dx$.

Definition Linear differential equations with constant coefficients

Let $k \in \mathbb{N}_0$, $a_0, \dots, a_{k-1} \in \mathbb{C}$ fixed and b a general continuous function, then $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$ is such equation.

Solution of hom. diff. eq. with constant coefficients

Look for solutions of the form $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{C}$. Then we have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geq 0$ and for all x , which means that

$$\begin{aligned} f^{(k)}(x) + a_{k-1}f^{(k-1)}(x) + \dots + a_1f'(x) + a_0f(x) \\ = e^{\alpha x} (\alpha^k + a_{k-1}\alpha^{k-1} + \dots + a_1\alpha + a_0). \end{aligned}$$

This translates into finding the zeros of the characteristic polynomial:

$$\begin{aligned} P(X) &= X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_0 \\ &= (X - \alpha_1) \cdots (X - \alpha_k) = 0 \end{aligned}$$

Imaginary roots

If a root is not real i.e. $\alpha = \beta + i\gamma$, the solution $f(x) = e^{\alpha x}$ does not take real values, but $\bar{\alpha} = \beta - i\gamma$ is also a root, hence we can write $\tilde{f}_1(x) = e^{\beta x} \cos(\gamma x)$, $\tilde{f}_2(x) = e^{\beta x} \sin(\gamma x)$ instead of $f_1(x) = e^{\alpha x}$, $f_2(x) = e^{\bar{\alpha} x}$

Multiple roots

Case 1: no multiple roots. Any solution of the equation is of the form $f(x) = z_1 e^{\alpha_1 x} + \dots + z_k e^{\alpha_k x}$.

Case 2: multiple roots. Suppose that α is a multiple root of order j with $2 \leq j \leq k$. Then the k functions $f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = x e^{\alpha x}$, \dots , $f_{\alpha,j-1}(x) = x^{j-1} e^{\alpha x}$ are linearly independent solutions. Taking the union of the functions $f_{\alpha,j}$ for all roots of P , each with its multiplicity, gives a basis of the space of solutions.

3 Differential calculus in \mathbb{R}^n

Definition 3.2.1.

Let $(x_k)_{k \in \mathbb{N}}$ where $x_k \in \mathbb{R}^n$. Write $x_k = (x_{k,1}, \dots, x_{k,n})$. Let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The sequence (x_k) converges to $(\rightarrow) y$ as $k \rightarrow +\infty$ if $\forall \varepsilon > 0$, if $\exists N \geq 1$ such that $\forall n \geq N$, we have $\|x_k - y\| < \varepsilon$.

Lemma 3.2.2.

$(x_k) \rightarrow y$ as $k \rightarrow +\infty \iff$ either: (1) $\forall i, 1 \leq i \leq n$, the sequence of real numbers $(x_{k,i}) \rightarrow y_i$. (2) The sequence of real numbers $\|x_k - y\| \rightarrow 0$ as $k \rightarrow +\infty$.

Definition 3.2.3.

Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. (1) Let $x_0 \in X$. f is continuous at x_0 if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon$, $\forall x \in X$. (2) f is continuous on X if it is continuous at $x_0 \forall x_0 \in X$.

Proposition 3.2.4.

Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. Let $x_0 \in X$. The function f is continuous at $x_0 \iff \forall (x_k)_k \geq 1$ in X s.t. $x_k \rightarrow x_0$ as $k \rightarrow +\infty$, the sequence $(f(x_k))_k \geq 1$ in \mathbb{R}^m converges to $f(x_0)$.

Definition 3.2.5.

Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We say that f has the limit y as $x \rightarrow x_0$ with $x \neq x_0$ if for every $\varepsilon > 0$, there exists $\delta > 0$, s.t. $\forall x \in X, x \neq x_0$, s.t. $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \varepsilon$. We then write $\lim_{x \rightarrow x_0, x \neq x_0} f(x) = y$.

Proposition 3.2.7.

Let $X \subset \mathbb{R}^n$ and $f : X \rightarrow \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We have $\lim_{x \rightarrow x_0, x \neq x_0} f(x) = y \iff \forall (x_k) \in X$ s.t. $x_k \rightarrow x_0$ as $k \rightarrow +\infty$, and $x_k \neq x_0$, the sequence $(f(x_k))$ in \mathbb{R}^m converges to y .

Proposition 3.2.9.

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and $p \geq 1$ an integer. Let $f : X \rightarrow Y$ and $g : Y \rightarrow \mathbb{R}^p$ be continuous functions. Then the composite $g \circ f$ is continuous.

Definition 3.2.11.

(1) A subset $X \subset \mathbb{R}^n$ is bounded if the set of $\|x\|$ for $x \in X$ is bounded in \mathbb{R} . (2) A subset $X \subset \mathbb{R}^n$ is closed if for every sequence (x_k) in X that converges in \mathbb{R}^n to some vector $y \in \mathbb{R}^n$, we have $y \in X$. (3) A subset $X \subset \mathbb{R}^n$ is compact if it is bounded and closed.