

## 2 ODE (ordinary differential equation)

### Theorem 2.1.6

Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  be differentiable. Let  $x_0 \in \mathbb{R}$  and  $y_0 \in \mathbb{R}^2$ . Then the ODE  $y' = F(x, y)$  has a **unique solution**  $f$  defined on a "largest" open interval  $I$  containing  $x_0$  such that  $f(x_0) = y_0$ .

### Definition 2.2.1

Let  $I \subset \mathbb{R}$  be an open interval and  $k \in \mathbb{N}_0$ . An **homogeneous linear ODE of order  $k$**  on  $I$  is of the form  $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = 0$  where the coefficients  $a_0, \dots, a_{k-1}$  are complex-valued functions on  $I$ , and the unknown is a function  $I \rightarrow \mathbb{C}$  that is  $k$ -times differentiable on  $I$ . An equation of the form  $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$ , where  $b : I \rightarrow \mathbb{C}$  is another function, is called an **inhomogeneous linear ODE**.

### Recognize a linear ODE

1. no coefficients before the highest derivative
2. all coefficients are continuous
3. no products of  $y$  or their derivatives
4. no powers of  $y$  or their derivatives
5. no functions depending on  $y$  or their derivatives

### Proposition 2.3.1

Any solution of  $y' + ay = 0$  is of the form  $f(x) = z \exp(-A(x))$  where  $A$  is a primitive of  $a$ . The unique solution with  $f(x_0) = y_0$  is  $f(x) = y_0 \exp(A(x_0) - A(x))$ .

### Solving inhomogeneous equations

**Case 1:** Make a guess. For example  $y' = y + x^2$  guess  $f(x) = ax^2 + bx + c$ , and solve the equation.

**Case 2:** Use the variation of the constant. Assume  $f_p = z(x) \exp(-A(x))$  for  $z : I \rightarrow \mathbb{C}$ . Then  $z'(x) = b(x) \exp(A(x)) \implies k(x) = \int b(x) \exp(A(x)) dx$ .

### Definition Linear differential equations with constant coefficients

Let  $k \in \mathbb{N}_0$ ,  $a_0, \dots, a_{k-1} \in \mathbb{C}$  fixed and  $b$  a general continuous function, then  $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$  is such equation.

### Solution of hom. diff. eq. with constant coefficients

Look for solutions of the form  $f(x) = e^{\alpha x}$  for  $\alpha \in \mathbb{C}$ . Then we have  $f^{(j)}(x) = \alpha^j e^{\alpha x}$  for all  $j \geq 0$  and for all  $x$ , which means that

$$\begin{aligned} & f^{(k)}(x) + a_{k-1}f^{(k-1)}(x) + \dots + a_1f'(x) + a_0f(x) \\ &= e^{\alpha x} (\alpha^k + a_{k-1}\alpha^{k-1} + \dots + a_1\alpha + a_0). \end{aligned}$$

This translates into finding the zeros  $(\alpha_1, \dots, \alpha_k \in \mathbb{C})$  of the characteristic polynomial:

$$\begin{aligned} P(X) &= X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_0 \\ &= (X - \alpha_1) \dots (X - \alpha_k) = 0 \end{aligned}$$

### Imaginary roots

If a root is not real i.e.  $\alpha = \beta + i\gamma$ , the solution  $f(x) = e^{\alpha x}$  does not take real values, but  $\bar{\alpha} = \beta - i\gamma$  is also a root, hence we can write  $\tilde{f}_1(x) = e^{\beta x} \cos(\gamma x)$ ,  $\tilde{f}_2(x) = e^{\beta x} \sin(\gamma x)$  instead of  $f_1(x) = e^{\alpha x}$ ,  $f_2(x) = e^{\bar{\alpha} x}$ .

### Multiple roots

**Case 1: no multiple roots.** Any solution of the equation is of the form  $f(x) = z_1 e^{a_1 x} + \dots + z_k e^{a_k x}$ .

**Case 2: multiple roots.** Suppose that  $\alpha$  is a multiple root of order  $j$  with  $2 \leq j \leq k$ . Then the  $k$  functions  $f_{\alpha,0}(x) = e^{\alpha x}$ ,  $f_{\alpha,1}(x) = x e^{\alpha x}$ ,  $\dots$ ,  $f_{\alpha,j-1}(x) = x^{j-1} e^{\alpha x}$  are linearly independent solutions. Taking the union of the functions  $f_{\alpha,j}$  for all roots of  $P$ , each with its multiplicity, gives a basis of the space of solutions.

## 3 Differential calculus in $\mathbb{R}^n$

### Definition 3.2.1.

Let  $(x_k)_{k \in \mathbb{N}}$  where  $x_k \in \mathbb{R}^n$ . Write  $x_k = (x_{k,1}, \dots, x_{k,n})$ . Let  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ . The sequence  $(x_k)$  **converges to**  $(\rightarrow) y$  as  $k \rightarrow +\infty$  if  $\forall \varepsilon > 0 \exists N \geq 1$  s.t.  $\forall k \geq N$ , we have  $\|x_k - y\| < \varepsilon$ .

### Lemma 3.2.2.

$(x_k) \rightarrow y$  as  $k \rightarrow +\infty \iff$  either:

- (1)  $\forall i, 1 \leq i \leq n$ , the sequence of real numbers  $(x_{k,i}) \rightarrow y_i$ .

- (2) The sequence of real numbers  $\|x_k - y\| \rightarrow 0$  as  $k \rightarrow +\infty$ .

### Definition 3.2.3.

Let  $X \subset \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}^m$ . (1) Let  $x_0 \in X$ .  $f$  is **continuous at  $x_0$**  if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.  $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon, \forall x \in X$ . (2)  $f$  is **continuous** on  $X$  if it is continuous at  $x_0 \forall x_0 \in X$ .

### Proposition 3.2.4.

Let  $X \subset \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}^m$ . Let  $x_0 \in X$ . The function  $f$  is continuous at  $x_0 \iff \forall (x_k)_k \geq 1$  in  $X$  s.t.  $x_k \rightarrow x_0$  as  $k \rightarrow +\infty$ , the sequence  $(f(x_k))_k \geq 1$  in  $\mathbb{R}^m$  converges to  $f(x_0)$ .

### Definition 3.2.5.

Let  $X \subset \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}^m$ . Let  $x_0 \in X$  and  $y \in \mathbb{R}^m$ . We say that  $f$  has the **limit  $y$**  as  $x \rightarrow x_0$  with  $x \neq x_0$  if for every  $\varepsilon > 0$ , there exists  $\delta > 0$ , s.t.  $\forall x \in X, x \neq x_0$ , s.t.  $\|x - x_0\| < \delta$ , we have  $\|f(x) - y\| < \varepsilon$ . We then write  $\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x) = y$ .

### Proposition 3.2.7.

Let  $X \subset \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}^m$ . Let  $x_0 \in X$  and  $y \in \mathbb{R}^m$ . We have  $\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x) = y \iff \forall (x_k) \in X$  s.t.  $x_k \rightarrow x_0$  as  $k \rightarrow +\infty$ , and  $x_k \neq x_0$ , the sequence  $(f(x_k))$  in  $\mathbb{R}^m$  converges to  $y$ .

### Proposition 3.2.9.

Let  $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$  and  $p \geq 1$  an integer. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow \mathbb{R}^p$  be continuous functions. Then the composite  $g \circ f$  is continuous.

### Definition 3.2.11.

- (1) A subset  $X \subset \mathbb{R}^n$  is **bounded** if the set of  $\|x\|$  for  $x \in X$  is bounded in  $\mathbb{R}$ .
- (2) A subset  $X \subset \mathbb{R}^n$  is **closed** if for every sequence  $(x_k)$  in  $X$  that converges in  $\mathbb{R}^n$  to some vector  $y \in \mathbb{R}^n$ , we have  $y \in X$ .
- (3) A subset  $X \subset \mathbb{R}^n$  is **compact** if it is bounded and closed.

### Proposition 3.2.13.

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a continuous map. For any closed set  $Y \subset \mathbb{R}^m$ ,  $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subset \mathbb{R}^n$  is closed.

**Theorem 3.2.15.**

Let  $X \subset \mathbb{R}^n$  be a non-empty compact set and  $f : X \rightarrow \mathbb{R}$  a continuous function. Then  $f$  is bounded and achieves its max and min. I.e.  $\exists x_+, x_- \in X$  s.t.  $f(x_+) = \sup_{x \in X} f(x)$ ,  $f(x_-) = \inf_{x \in X} f(x)$ .

**Definition 3.3.1.**

A subset  $X \subset \mathbb{R}^n$  is **open** if, for any  $x = (x_1, \dots, x_n) \in X$ , there exists  $\delta > 0$  s.t. the set

$$\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \text{ for all } i\}$$

is contained in  $X$ . In other words: any point of  $\mathbb{R}^n$  obtained by changing any coordinate of  $x$  by at most  $\delta$  is still in  $X$ .

**Proposition 3.3.2.**

A set  $X \subset \mathbb{R}^n$  is open if and only if the complement  $Y = \{x \in \mathbb{R}^n : x \notin X\}$  is closed.

**Corollary 3.3.3.**

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous and  $Y \subset \mathbb{R}^m$  is open, then  $f^{-1}(Y)$  is open in  $\mathbb{R}^n$ .

**Definition 3.3.5**

Let  $X \subset \mathbb{R}^n$  be an open set. Let  $f : X \rightarrow \mathbb{R}^m$  be a function. Let  $1 \leq i \leq n$ . We say that  $f$  has a **partial derivative** on  $X$  with respect to the  $i$ -th variable, or coordinate, if for all  $x_0 = (x_{0,1}, \dots, x_{0,n}) \in X$ , the function defined by

$$g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$$

on the set  $I = \{t \in \mathbb{R} : (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$  is differentiable at  $t = x_{0,i}$ . Its **derivative**  $g'(x_{0,i})$  at  $x_{0,i}$  is denoted

$$\frac{\partial f}{\partial x_i}(x_0), \quad \partial_{x_i} f(x_0), \quad \partial_i f(x_0)$$

**Proposition 3.3.7.**

Consider  $X \subset \mathbb{R}^n$  open and  $f, g$  functions from  $X$  to  $\mathbb{R}^m$ . Let  $1 \leq i \leq n$ .

- (1) If  $f$  and  $g$  have partial derivatives with respect to the  $i$ -th coordinate on  $X$ , then  $f + g$  also does, and

$$\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g).$$

- (2) If  $m = 1$ , and if  $f$  and  $g$  have partial derivatives with respect to the  $i$ -th coordinate on  $X$ , then  $fg$  also does and

$$\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g).$$

Furthermore, if  $g(x) \neq 0$  for all  $x \in X$ , then  $f/g$  has a partial derivative with respect to the  $i$ -th coordinate on  $X$ , with

$$\partial_{x_i}(f/g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g))/g^2.$$

**Definition 3.3.9. (Jacobi matrix)**

Let  $X \subset \mathbb{R}^n$  open and  $f : X \rightarrow \mathbb{R}^m$  a function with partial derivatives on  $X$ . Write

$$f(x) = (f_1(x), \dots, f_m(x)).$$

For any  $x \in X$ , the matrix

$$J_f(x) = (\partial_{x_j} f_i(x))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

with  $m$  rows and  $n$  columns is called the **Jacobi matrix** of  $f$  at  $x$ .

**Definition 3.3.11 (Gradient, Divergence)**

Let  $X \subset \mathbb{R}^n$  be open. (1) Let  $f : X \rightarrow \mathbb{R}$  be a function. If all partial derivatives of  $f$  exist at  $x_0 \in X$ , then the column vector

$$\begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

is called the **gradient** at  $x_0$ , and is denoted  $\nabla f(x_0)$ . (2) Let  $f = (f_1, \dots, f_n) : X \rightarrow \mathbb{R}^n$  be a function with values in  $\mathbb{R}^n$  s.t. all partial derivatives of all coordinates  $f_i$  of  $f$  exist at  $x_0 \in X$ . Then the real number

$$\text{Tr}(J_f(x_0)) = \sum_{i=1}^n \partial_{x_i} f_i(x_0),$$

the trace of the Jacobi matrix, is called the **divergence** of  $f$  at  $x_0$ , and is denoted  $\text{div}(f)(x_0)$ .

**Definition 3.4.2.**

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^m$  be a function. Let  $u$  be a linear map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $x_0 \in X$ . We say that  $f$  is **differentiable at  $x_0$  with differential  $u$**  if

$$\lim_{x \rightarrow x_0} \frac{1}{\|x - x_0\|} (f(x) - f(x_0) - u(x - x_0)) = 0$$

where the limit is in  $\mathbb{R}^m$ . We then denote  $df(x_0) = u$ . If  $f$  is differentiable at every  $x_0 \in X$ , then we say that  $f$  is differentiable on  $X$ .

**Proposition 3.4.4.**

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^m$  be a function that is differentiable on  $X$ . (1) The function  $f$  is continuous on  $X$ . (2) The function  $f$  admits partial derivatives on  $X$  with respect to each variable. (3) Assume that  $m = 1$ . Let  $x_0 \in X$ , and let  $u(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$  be the differential of  $f$  at  $x_0$ . We then have  $\partial_{x_i} f(x_0) = a_i$  for  $1 \leq i \leq n$ .

**Proposition 3.4.6.**

Let  $X \subset \mathbb{R}^n$  be open,  $f : X \rightarrow \mathbb{R}^m$  and  $g : X \rightarrow \mathbb{R}^m$  differentiable functions on  $X$ .

- (1) The function  $f + g$  is differentiable with differential  $d(f + g) = df + dg$ , and if  $m = 1$ , then  $fg$  is differentiable.
- (2) If  $m = 1$  and if  $g(x) \neq 0$  for all  $x \in X$ , then  $f/g$  is differentiable.

**Proposition 3.4.7.**

Let  $X \subset \mathbb{R}^n$  be open,  $f : X \rightarrow \mathbb{R}^m$  a function on  $X$ . If  $f$  has all partial derivatives on  $X$ , and if the partial derivatives of  $f$  are continuous on  $X$ , then  $f$  is differentiable on  $X$ , with differential determined by its partial derivatives, in the sense that the matrix of the differential  $df(x_0)$ , with respect to the canonical basis of  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , is the Jacobi matrix of  $f$  at  $x_0$ .

**Proposition 3.4.9 (Chain rule)**

Let  $X \subset \mathbb{R}^n$  be open,  $Y \subset \mathbb{R}^m$  be open, and let  $f : X \rightarrow Y$  and  $g : Y \rightarrow \mathbb{R}^p$  be differentiable functions. Then  $g \circ f : X \rightarrow \mathbb{R}^p$  is differentiable on  $X$ , and for any  $x \in X$ , its differential is given by the composition

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0).$$

In particular, the Jacobi matrix satisfies

$$J_{g \circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$$

where the right-hand side is a matrix product.

**From Example 3.4.10, 2.36L**

Referring to the proposition above (3.4.9)

- (2) let  $p = 1$ , then it holds that

$$\partial_j(g \circ f)(x) = \sum_{i=1}^m \partial_i g(f(x)) \cdot \partial_j f_i(x)$$

- (3) let  $p = 1$  and  $X \subset \mathbb{R}$ , then it holds that

$$\begin{aligned}(g \circ f)'(t) &= \sum_{i=1}^m (\partial_i g)(f(t)) \cdot f'_i(t) \\ &= \langle \nabla g(f(t)), f'(t) \rangle.\end{aligned}$$

#### Definition 3.4.11.

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^m$  a function that is differentiable. Let  $x_0 \in X$  and  $u = df(x_0)$  be the differential of  $f$  at  $x_0$ . The graph of the affine linear approximation

$$g(x) = f(x_0) + u(x - x_0)$$

from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , or in other words the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x_0) + u(x - x_0)\}$$

is called the **tangent space** at  $x_0$  to the graph of  $f$ .

#### Definition 3.4.13.

Let  $X \subset \mathbb{R}^n$  be an open set and let  $f : X \rightarrow \mathbb{R}^m$  be a function. Let  $v \in \mathbb{R}^n$  be a non-zero vector and  $x_0 \in X$ . We say that  $f$  has **directional derivative**  $w \in \mathbb{R}^m$  **in the direction**  $v$ , if the function  $g$  defined on the set

$$I = \{t \in \mathbb{R} : x_0 + tv \in X\}$$

by

$$g(t) = f(x_0 + tv)$$

has a derivative at  $t = 0$ , and this is equal to  $w$ . In other words, this means that the limit

$$\lim_{\substack{t \rightarrow 0 \\ t \neq 0}} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exists and is equal to  $w$ .

#### Proposition 3.4.15.

Let  $X \subset \mathbb{R}^n$  be an open set and let  $f : X \rightarrow \mathbb{R}^m$  be a differentiable function. Then for any  $x \in X$  and non-zero  $v \in \mathbb{R}^n$ , the function  $f$  has a directional derivative at  $x_0$  in the direction  $v$ , equal to  $df(x_0)(v)$ .

#### Definition 3.5.1.

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^m$ . We say that  $f$  is of class  $C^1$  if  $f$  is differentiable on  $X$  and all its partial derivatives are continuous. The set of functions of class  $C^1$  from  $X$  to  $\mathbb{R}^m$  is denoted  $C^1(X; \mathbb{R}^m)$ . Let  $k \geq 2$ . We say, by induction, that  $f$  is of class  $C^k$  if it is differentiable and each partial derivative  $\partial_{x_i} f : X \rightarrow \mathbb{R}^m$  is of class  $C^{k-1}$ . The set of

functions of class  $C^k$  from  $X$  to  $\mathbb{R}^m$  is denoted  $C^k(X; \mathbb{R}^m)$ .

If  $f \in C^k(X; \mathbb{R}^m)$  for all  $k \geq 1$ , then we say that  $f$  is of class  $C^\infty$ . The set of such functions is denoted  $C^\infty(X; \mathbb{R}^m)$ .

#### Proposition 3.5.4 (Mixed derivatives commute)

$k \geq 2$ . Let  $X \subset \mathbb{R}^n$  be open and let  $f : X \rightarrow \mathbb{R}^m$  be a function of class  $C^k$ . Then the partial derivatives of order  $k$  are independent of the order in which the partial derivatives are taken: for any variables  $x$  and  $y$ , we have  $\partial_{x,y} f = \partial_{y,x} f$  and for any variables  $x, y, z$ , we have

$$\partial_{x,y,z} f = \partial_{x,z,y} f = \partial_{y,z,x} f = \partial_{z,x,y} f = \dots$$

#### Definition 3.5.9 (Hessian).

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}$  a  $C^2$  function. For  $x \in X$ , the **Hessian matrix** of  $f$  at  $x$  is the symmetric square matrix

$$\text{Hess}_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i, j \leq n}.$$

We also sometimes write simply  $H_f(x)$ .

#### Definition 3.7.1 (Taylor polynomials).

Let  $k \geq 1$  be an integer. Let  $f : X \rightarrow \mathbb{R}$  be a function of class  $C^k$  on  $X$ , and fix  $x_0 \in X$ . The  $k$ -th Taylor polynomial of  $f$  at the point  $x_0$  is the polynomial in  $n$  variables of degree  $\leq k$  given by

$$\begin{aligned}T_k f(y; x_0) &= f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) y_i + \dots \\ &\quad + \sum_{m_1 + \dots + m_n = k} \frac{1}{m_1! \dots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}(x_0) y_1^{m_1} \dots y_n^{m_n}\end{aligned}$$

where the last sum ranges over the tuples of  $n$  non-negative integers s.t. the sum is  $k$ .

#### Corollary $T_2 f$ (L)

With  $x, x_0 \in \mathbb{R}^2$ ,  $x_0$  fixed.

$$T_2 f(x; x_0) = f(x_0) + \langle \nabla f(x_0), x \rangle + \frac{1}{2} x^\top \cdot \text{Hesse}_f(x_0) \cdot x$$

#### Proposition 3.7.3 (Taylor approximation)

Let  $k \geq 1$  be an integer. Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}$  be a function of class  $C^k$ . For  $x_0$  in  $X$ , if we define  $E_k f(x; x_0)$  by

$$f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$$

then we have

$$\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

#### Proposition 3.8.1. (Maximum, minimum and derivative)

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}$  a differentiable function. If  $x_0 \in X$  is s.t.

$$f(y) \leq f(x_0) \text{ for all } y \text{ close enough to } x_0 \text{ (local maximum at } x_0 \text{)}$$

or

$$f(y) \geq f(x_0) \text{ for all } y \text{ close enough to } x_0 \text{ (local minimum at } x_0 \text{)}.$$

Then we have  $df(x_0) = 0$ , or in other words  $\nabla f(x_0) = 0$ , or equivalently  $\frac{\partial f}{\partial x_i}(x_0) = 0$  for  $1 \leq i \leq n$ .

#### Definition 3.8.2 (Critical point)

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}$  a differentiable function. A point  $x_0 \in X$  s.t.  $\nabla f(x_0) = 0$  is called a **critical point** of the function  $f$ .

#### Definition 3.8.6 (Non-degenerate critical point)

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^n$  a function of class  $C^2$ . A critical point  $x_0 \in X$  of  $f$  is called **non-degenerate** if the Hessian matrix has non-zero determinant.

#### Definition Positive definite (linalg)

Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix. We say  $A$  is positive definite if  $\langle x, Ax \rangle > 0$ .

#### Proposition Definite matrix and eigenvalues (linalg)

If a matrix is positive definite, all its eigen values are positive.

#### Definition Leading principal minor (linalg)

The  $k$ -th leading principal minor of a matrix  $M$  is the determinant of its upper-left  $k \times k$  sub-matrix.

#### Proposition Leading principal minor and definite matrix (linalg)

A matrix is positive definite  $\iff$  all its leading principal minors (1 to  $n$ ) are positive.

### Corollary 3.8

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}$  a function of class  $C^2$ . Let  $x_0$  be a non-degenerate critical point of  $f$ . Let  $p$  and  $q$  be the number of positive and negative eigenvalues of  $\text{Hess}_f(x_0)$ .

- (1) If  $p = n$ , equivalently if  $q = 0$ , the function  $f$  has a local minimum at  $x_0$ .
- (2) If  $q = n$ , equivalently if  $p = 0$ , the function  $f$  has a local maximum at  $x_0$ .
- (3) Otherwise, equivalently if  $pq \neq 0$ , the function  $f$  does not have a local extremum at  $x_0$ . One then says that  $f$  has a saddle point at  $x_0$ .

### Proposition 3.9.2

Let  $X \subset \mathbb{R}^n$  be open and let  $f : X \rightarrow \mathbb{R}$  and  $g : X \rightarrow \mathbb{R}$  be functions of class  $C^1$ . If  $x_0 \in X$  is a local extremum of the function  $f$  restricted to the set  $Y = \{x \in X : g(x) = 0\}$  then either  $\nabla g(x_0) = 0$ , or there exists  $\lambda_0 \in \mathbb{R}$  s.t.

$$\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$$

or in other words, there exists  $\lambda$  s.t.  $(x_0, \lambda)$  is a critical point of the differentiable function  $h : X \times \mathbb{R} \rightarrow \mathbb{R}$  defined by  $h(x, \lambda) = f(x) - \lambda g(x)$ . Such a value  $\lambda$  is called a Lagrange multiplier at  $x_0$ .

### Definition 3.10.1 (Change of variable)

Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^n$  be differentiable. Let  $x_0 \in X$ . We say that  $f$  is a **change of variable** around  $x_0$  if there is a radius  $r > 0$  s.t. the restriction of  $f$  to the ball

$$B = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$$

of radius  $r$  around  $x_0$  has the property that the image  $Y = f(B)$  is open in  $\mathbb{R}^n$ , and if there is a differentiable map  $g : Y \rightarrow B$  s.t.  $f \circ g = \text{Id}_Y$  and  $g \circ f = \text{Id}_B$ .

### Theorem 3.10.2 (Inverse function theorem)

(Inverse function theorem). Let  $X \subset \mathbb{R}^n$  be open and  $f : X \rightarrow \mathbb{R}^n$  differentiable. If  $x_0 \in X$  is s.t.  $\det(J_f(x_0)) \neq 0$ , i.e., s.t. the Jacobian trix of  $f$  at  $x_0$  is invertible, then  $f$  is a change of variable around  $x_0$ . Moreover, the Jacobian of  $g$  at  $x_0$  is determined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

In addition, if  $f$  is of class  $C^k$ , then  $g$  is of class  $C^k$ .

### Theorem 3.10.4 (Implicit Function Theorem).

Let  $X \subset \mathbb{R}^{n+1}$  be open and let  $g : X \rightarrow \mathbb{R}$  be of class  $C^k$  with  $k \geq 1$ . Let  $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$  be s.t.  $g(x_0, y_0) = 0$ . Assume that

$$\partial_y g(x_0, y_0) \neq 0$$

Then there exists an open set  $U \subset \mathbb{R}^n$  containing  $x_0$ , an open interval  $I \subset \mathbb{R}$  containing  $y_0$ , and a function  $f : U \rightarrow \mathbb{R}$  of class  $C^k$  s.t. the system of equations

$$\begin{cases} g(x, y) = 0 \\ x \in U, \quad y \in I \end{cases}$$

is equivalent with  $y = f(x)$ . In particular,  $f(x_0) = y_0$ . Moreover, the gradient of  $f$  at  $x_0$  is given by

$$\nabla f(x_0) = -\frac{1}{(\partial_y g)(x_0, y_0)} \nabla_x g(x_0, y_0)$$

where  $\nabla_x g = (\partial_{x_1} g, \dots, \partial_{x_n} g)$

### Definition 4.1.1. (parameterized curve, line integral)

- (1) Let  $I = [a, b]$  be a closed and bounded interval in  $\mathbb{R}$ . Let  $f(t) = (f_1(t), \dots, f_n(t))$  be a continuous function from  $I$  to  $\mathbb{R}^n$ , i.e.,  $f_i$  is continuous for  $1 \leq i \leq n$ . Then we define

$$\int_a^b f(t) dt = \left( \int_a^b f_1(t) dt, \dots, \int_a^b f_n(t) dt \right) \in \mathbb{R}^n.$$

- (2) A **parameterized curve** in  $\mathbb{R}^n$  is a continuous map  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  that is piecewise  $C^1$ , i.e., there exists  $k \geq 1$  and a partition

$$a = t_0 < t_1 < \dots < t_{k-1} < t_k = b$$

such that the restriction of  $f$  to  $]t_{j-1}, t_j[$  is  $C^1$  for  $1 \leq j \leq k$ . We say that  $\gamma$  is a parameterized curve, or a path  $\mathbf{x}$ , between  $\gamma(a)$  and  $\gamma(b)$ .

- (3) Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a parameterized curve. Let  $X \subset \mathbb{R}^n$  be a subset containing the image of  $\gamma$ , and let  $f : X \rightarrow \mathbb{R}^n$  be a continuous function. The integral

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \in \mathbb{R}$$

is called the **line integral of  $f$  along  $\gamma$** . It is denoted

$$\int_{\gamma} f(s) \cdot ds, \quad \text{or} \quad \int_{\gamma} f(s) \cdot d\vec{s}.$$

### Definition 4.1.4

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a parameterized curve. An **oriented reparameterization** of  $\gamma$  is a parameterized curve  $\sigma : [c, d] \rightarrow \mathbb{R}^n$  such that  $\sigma = \gamma \circ \varphi$ , where  $\varphi : [c, d] \rightarrow [a, b]$  is a continuous map, differentiable on  $]a, b[$ , that is strictly increasing and satisfies  $\varphi(a) = c$  and  $\varphi(b) = d$ .

### Proposition 4.1.5.

Let  $\gamma$  be a parameterized curve in  $\mathbb{R}^n$  and  $\sigma$  an oriented reparameterization of  $\gamma$ . Let  $X$  be a set containing the image of  $\gamma$ , or equivalently the image of  $\sigma$ , and  $f : X \rightarrow \mathbb{R}^n$  a continuous function. Then we have

$$\int_{\gamma} f(s) \cdot d\vec{s} = \int_{\sigma} f(s) \cdot d\vec{s}.$$

### Definition 4.1.8

Let  $X \subset \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}^n$  a continuous vector field. If, for any  $x_1, x_2$  in  $X$ , the line integral  $\int_{\gamma} f(s) \cdot d\vec{s}$  is independent of the choice of a parameterized curve  $\gamma$  in  $X$  from  $x_1$  to  $x_2$ , then we say that the vector field is conservative.

## 4 Integration in $\mathbb{R}^n$



**Lemma 3.14L)**

A vectorfield  $f : X \rightarrow \mathbb{R}^n$  is conservative  $\iff$  for each parameterized curved  $\gamma$  contained in  $X$  it holds that  $\int_{\gamma} f(s) d\vec{s} = 0$ .

**Theorem 4.1.10**

Let  $X$  be an open set and  $f$  a conservative vector field. Then there exists a  $C^1$  function  $g$  on  $X$  such that  $f = \nabla g$ .

If any two points of  $X$  can be joined by a parameterized curve, then  $g$  is unique up to addition of a constant: if  $\nabla g_1 = f$ , then  $g - g_1$  is constant on  $X$ .

**Definition 3.16L (path-connected space, potential)**

- (1) An open set  $X \subset \mathbb{R}^n$  is **path-connected** if for each pair of points in  $X$  are the endpoints of a parameterized curve.
- (2) A function  $g : X \rightarrow \mathbb{R}$  s.t.  $\nabla g = f$  is called **potential** of  $f$ .

**Proposition 4.1.13**

Let  $X \subset \mathbb{R}^n$  be an open set and  $f : X \rightarrow \mathbb{R}^n$  a vector field of class  $C^1$ . Write  $f(x) = (f_1(x), \dots, f_n(x))$ . If  $f$  is conservative, then we have

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

for any integers with  $1 \leq i \neq j \leq n$ .

**Definition 4.1.15 (star shaped)**

A subset  $X \subset \mathbb{R}^n$  is **star shaped** if there exists  $x_0 \in X$  such that, for all  $x \in X$ , the line segment joining  $x_0$  to  $x$  is contained in  $X$ . We then also say that  $X$  is **star-shaped around**  $x_0$ .

**Theorem 4.1.17**

Let  $X$  be a star-shaped open subset of  $\mathbb{R}^n$ . Let  $f$  be a  $C^1$  vector field s.t.  $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$  on  $X$  for all  $i \neq j$  between 1 and  $n$ . Then the vector field  $f$  is conservative.

**Definition 4.1.20 (curl)**

Definition 4.1.20. Let  $X \subset \mathbb{R}^3$  be an open set and  $f : X \rightarrow \mathbb{R}^3$  a  $C^1$  vector field. Then the curl of  $f$ , denoted  $\text{curl}(f)$ , is the continuous vector field on  $X$  defined by

$$\text{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

where  $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$ .

**Definition Integral on a rectangle**

Let  $R = [a, b] \times [c, d]$  a compact rectangle in  $\mathbb{R}^2$ . Let  $f : R \rightarrow \mathbb{R}$  be bounded, s.t.  $\exists M \geq 0, \forall (x, y) \in R, |f(x, y)| \leq M$ . For each partition  $P_x$  with  $x_0 = a < x_1 < \dots < x_n = b$  of  $[a, b]$  and  $P_y$  with  $y_0 = c < y_1 < \dots < y_n = d$  of  $[c, d]$ , we can subdivide  $R$  into rectangles  $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$  with area  $\mu(R_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$ . Let

$$f_{ij} = \inf_{I_{ij}} f(x, y), \quad F_{ij} = \sup_{I_{ij}} f(x, y)$$

. We can then define an upper and a lower sum, respectively:

$$s(P_x \times P_y) = \sum_{i=1}^n \sum_{j=1}^m f_{ij} \mu(I_{ij})$$

$$S(P_x \times P_y) = \sum_{i=1}^n \sum_{j=1}^m F_{ij} \mu(I_{ij})$$

**Definition Riemann integrable in  $\mathbb{R}^2$** 

Let  $f : R \rightarrow \mathbb{R}$  bounded. Then is  $f$  integrable on  $R$  if

$$\sup_{(P_x, P_y)} s(P_x \times P_y) = \inf_{(P_x, P_y)} S(P_x \times P_y).$$

This value is defined as

$$\int_R f(x, y) d(x, y) \text{ or } \iint_R f(x, y) d(x, y).$$

**Definition Riemann integrable and characteristical functon**

$f$  is on  $A$  integrable if  $f \cdot \mathcal{X}_A$  is integrable on  $R$ , where  $\mathcal{X}_A$  is the characteristical polynomial of  $f$ . We then write  $\int_A f(x, y) d(x, y)$  for  $\int_R f(x, y) \mathcal{X}_A(x, y) d(x, y)$ .

**Riemann integral's properties**

- (1) **Linearity**
- (2) **Positivity** Let  $f, g : A \rightarrow \mathbb{R}$  integrable with  $f \leq g$ . Then it follows

$$\int_A f(x, y) d(x, y) \leq \int_A g(x, y) d(x, y).$$

Furthermore, if  $f \geq 0$ ,  $B \subset A$  and  $f$  on  $B$  integrable:

$$\int_B f(x, y) d(x, y) \leq \int_A f(x, y) d(x, y).$$

- (3) **Triangular inequality** Let  $f : A \rightarrow \mathbb{R}$  integrable (in particular, bounded), then is  $|f|$  integrable and

$$\left| \int_A f(x, y) d(x, y) \right| \leq \int_A |f(x, y)| d(x, y)$$

- (4) **Volume** Let  $R = [a, b] \times [c, d]$ , then  $\int_R d(x, y) = (b - a)(d - c)$ .
- (5) **(O. Stolz 1886)** Let  $f : R \rightarrow \mathbb{R}$  on  $R = [a, b] \times [c, d]$  integrable. We assume that  $y \mapsto f(x, y) \forall x \in [a, b]$  is integrable on  $[c, d]$ . This implies that

$$x \mapsto \int_c^d f(x, y) dy$$

is integrable on  $[a, b]$ , and

$$\int_R f(x, y) d(x, y) = \int_a^b \left( \int_c^d f(x, y) dy \right) dx$$

**Proposition 3.30L**

Let  $R = [a, b] \times [c, d]$  be a compact rectangle and  $f : R \rightarrow \mathbb{R}$  continuous. Then is  $f$  on  $R$  integrable.

**Proposition 3.31L**

Let  $K \subset \mathbb{R}^2$  a compact subset and  $f : K \rightarrow \mathbb{R}$  continuous. Then is  $f$  uniformly continuous.

**Definition 3.32L (null set)**

Let  $X \subset R \subset \mathbb{R}^2$ . Then  $X$  is a null set (in  $\mathbb{R}^2$ ) if  $\forall \epsilon > 0$  there are finite many rectangles  $R_k = [a_k, b_k] \times [c_k, d_k]$  with  $1 \leq k \leq n$ , s.t.

$$X \subset \bigcup_{k=1}^n R_k, \quad \sum_{k=1}^n \mu(R_k) < \epsilon$$

**Lemma 3.33L**

Let  $\varphi : [0, 1] \rightarrow \mathbb{R}^2$  a Lipschitz curve, so that

$$\|\varphi(s) - \varphi(t)\| \leq M \cdot |s - t| \quad \forall s, t \in [0, 1]$$

Then is the image  $\varphi([0, 1]) \subset \mathbb{R}^2$  a null set.

**Proposition 3.35L**

Let  $f : R \rightarrow \mathbb{R}$  bounded. Let

$$X = \{(x, y) \in R : f \text{ is not continuous at } (x, y)\}$$

If  $X$  is a null set, then  $f$  is not integrable on  $R$ .

### Proposition 3.36L

Let  $\varphi_1, \varphi_2 : [a, b] \rightarrow \mathbb{R}$  continuous with  $\varphi_1 \leq \varphi_2 \ \forall x \in [a, b]$ . Let  $A = \{(x, y) : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\}$ . Let  $f : A \rightarrow \mathbb{R}$  continuous. Then is  $f$  on  $A$  integrable and it holds:

$$\int_A f(x, y) d(x, y) = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy.$$

### Lemma 3.37L

Let  $\varphi : [a, b] \rightarrow \mathbb{R}$  continuous. Then is the  $graph(\varphi) = \{(x, \varphi(x), x \in [a, b]\} \subset \mathbb{R}^2$  a null set.

### Definition Border (Rand)

Let  $A \in \mathbb{R}^2$  the **border** (*Rand* in German) is defined as

$$\partial A = \{(x, y) \in \mathbb{R}^2 : \forall \delta > 0 \ C_\delta(x, y) \cap A \neq \emptyset \wedge C_\delta(x, y) \cap (\mathbb{R}^2 \setminus A) \neq \emptyset\}$$

where  $C_\delta(x, y) = ]x - \delta, x + \delta[ \times ]y - \delta, y + \delta[$ .

### Theorem 4.4.2 (Change of variable formula).

Let  $U \subset \mathbb{R}^2$  be compact subsets. Let  $\varphi : U \rightarrow \mathbb{R}^2$  be a continuous map in  $C^1$  and  $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  a continuous function. Furthermore  $B \subset U$ . We assume

- (1)  $\varphi(B) = A$ ;  $A, B$  compact;  $\partial A, \partial B$  null sets.
- (2)  $\varphi : B \setminus N \rightarrow A$  is injective, where  $N \subset B$  is a null set.

Then it follows:

$$\int_A f(x, y) d(x, y) = \int_B f(\varphi(u, v)) |\det(J_\varphi(u, v))| d(u, v).$$

### Example, polar coordinates (L)

Let  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2; (r, \varphi) \mapsto (r \cos \varphi, r \sin \varphi)$ . Then we can convert the integral in cartesian coordinates to polar coordinates in the following manner:

$$\int_{x^2+y^2 \leq R^2} f(x, y) d(x, y) = \int_0^{2\pi} \int_0^R f(r \cos \varphi, r \sin \varphi) r d(r, \varphi)$$

### Definition 3.39L Jordan curve, orientation

A **parameterized Jordan curve** is a parameterized curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  with the following properties

- (1)  $\gamma(a) = \gamma(b)$
- (2)  $\gamma : ]a, b[ \rightarrow \mathbb{R}^2$  is injective
- (3) A Jordan curve in  $\mathbb{R}^2$  is the image of a parameterized Jordan curve.

Let  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . A **base**  $(b_1, b_2)$  of  $\mathbb{R}^2$  is **positive oriented** if for the distinct matrix  $g \in M_{2,2}(\mathbb{R})$  with  $g(e_1) = b_1$  and  $g(e_2) = b_2$ ,  $\det(g) > 0$ . It is negative oriented if  $\det(g) < 0$ .

### Definition 3.40 reguläres Gebiet (L)

A *reguläres Gebiet* is a open bounded subset  $A \subset \mathbb{R}^2$  whose border  $\partial A$  is a finite union of disjoint Jordan curves. Each of this curve is called a border component of  $A$ .

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  a parameterized Jordan curve so that  $\gamma([a, b])$  is a border component of  $A$ . Then  $\gamma$  is positive oriented relative to  $A$  if  $(n(t), \gamma'(t))$  is a positive oriented basis of  $\mathbb{R}^2$ , where  $n(t)$  is the unitary vector orthogonal to  $\gamma'(t)$  and pointing outwards (not to  $A$ ).

### Theorem 4.6.3 (3.41L) Green's formula

Let  $A \subset \mathbb{R}^2$  be a *reguläres Gebiet* and  $F : U \rightarrow \mathbb{R}^2; (x, y) \mapsto \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$  a  $C^1$  vector field, where  $A \cup \partial A \subset U \in \mathbb{R}^2$ . Then it holds

$$\int_A \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) d(x, y) = \int_{\partial A} F(s) ds := \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}$$

## 5 Extra

### Vector field example

A vector field often discussed in the series and seen in some lecture's examples:

$$f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2; f(x, y) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}.$$

This vector field is **not** conservative, even if  $\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}$ . For example, the line integral over the parameterized curve of a "clockwise" closed circle of radius 1 doesn't give 0 as expected, but  $\pi/2$  instead.

We can though define the potentials:

- $g_1 : \mathbb{R}^+ \times \mathbb{R}; (x, y) \mapsto \arctan(y/x)$
- $g_2 : \mathbb{R} \times \mathbb{R}^+; (x, y) \mapsto \arctan(-x/y) = \arctan(y/x) - \pi/2$
- $g_3 : \mathbb{R}^- \times \mathbb{R}; (x, y) \mapsto \arctan(y/x)$
- $g_4 : \mathbb{R} \times \mathbb{R}^-; (x, y) \mapsto \arctan(-x/y) = \arctan(y/x) - \pi/2$