Analysis II HS21

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2 ODE (ordinary differential equation)

Theorem 2.1.6

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then the ODE $y' = F(x, y)$ has a **unique solution** f defined on a "largest" open interval I containing x_0 such that $f(x_0) = y_0$.

Definition 2.2.1

Let $I \subset \mathbb{R}$ be an open interval and $k \in \mathbb{N}_0$. An **homogeneous linear ODE of order** k on I is of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1 y' + a_0 y = 0$ where the coefficients $a_0, \ldots, a_{k-1}$ are complex-valued functions on I, and the unknown is a function $I \to \mathbb{C}$ that is k-times differentiable on I. An equation of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1 y' + a_0 y = b$, where $b: I \to \mathbb{C}$ is another function, is called an **inhomogeneous linear ODE**.

Recognize a linear ODE

1. no coefficients before the highest derivative
2. all coefficients are continuous
3. no products of y or their derivatives
4. no powers of y or their derivatives
5. no functions depending on y or their derivatives

Proposition 2.3.1

Any solution of $y' + ay = 0$ is of the form $f(x) = z\exp(-A(x))$ where A is a primitive of a. The unique solution with $f(x_0) = y_0$ is $f(x) = y_0 \exp(A(x_0) - A(x))$.

Solving inhomogeneous equations

Case 1: Make a guess. For example $y' = y + x^2$ guess $f(x) = ax^2 + bx + c$, and solve the equation.
Case 2: Use the variation of the constant. Assume $f_p = z(x)\exp(-A(x))$ for $z: I \to \mathbb{C}$. Then $z'(x) = b(x)\exp(A(x)) \implies k(x) = \int b(x)\exp(A(x))dx$.

Definition Linear differential equations with constant coefficients

Let $k \in \mathbb{N}_0$, $a_0, \ldots, a_{k-1} \in \mathbb{C}$ fixed and b a general continuous function, then $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_1 y' + a_0 y = b$ is such equation.

Solution of hom. diff. eq. with constant coefficients

Look for solutions of the form $f(x) = e^{\alpha x}$ for $\alpha \in \mathbb{C}$. Then we have $f^{(j)}(x) = \alpha^j e^{\alpha x}$ for all $j \geqslant 0$ and for all x, which means that

$$\begin{aligned} & f^{(k)}(x) + a_{k-1}f^{(k-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) \\ = & e^{\alpha x}\left(\alpha^k + a_{k-1}\alpha^{k-1} + \cdots + a_1\alpha + a_0\right). \end{aligned}$$

This translates into finding the zeros ($\alpha_1, \ldots, \alpha_k \in \mathbb{C}$) of the characteristic polynomial:

$$\begin{aligned} P(X) = & X^k + a_{k-1}X^k + \cdots + a_1 X + a_0 \\ = & (X - \alpha_1)\cdots(X - \alpha_k) = 0 \end{aligned}$$

Imaginary roots

If a root is not real i.e. $\alpha = \beta + i\gamma$, the solution $f(x) = e^{\alpha x}$ does not take real values, but $\bar{\alpha} = \beta - i\gamma$ is also a root, hence we can write $\tilde{f}_1(x) = e^{\beta x}\cos(\gamma x)$, $\tilde{f}_2(x) = e^{\beta x}\sin(\gamma x)$ instead of $f_1(x) = e^{\alpha x}$, $f_2(x) = e^{\bar{\alpha} x}$

Multiple roots

Case 1: no multiple roots. Any solution of the equation is of the form $f(x) = z_1 e^{a_1 x} + \cdots + z_k e^{a_k x}$.
Case 2: multiple roots. Suppose that α is a multiple root of order j with $2 \leqslant j \leqslant k$. Then the k functions $f_{\alpha,0}(x) = e^{\alpha x}$, $f_{\alpha,1}(x) = xe^{\alpha x}$, $\cdots$, $f_{\alpha,j-1}(x) = x^{j-1}e^{\alpha x}$ are linearly independent solutions. Taking the union of the functions $f_{\alpha,j}$ for all roots of P, each with its multiplicity, gives a basis of the space of solutions.

Special form solutions

For $d \geqslant 0$ and Q, Q_1, Q_2 are polynomials of degree d if β is a root of the companion polynomial, or Q, Q_1, Q_2 are polynomials of degree $d + j$ if β is a root of the companion polynomial of multiplicity j.

b(x)	**Solution's form**
$x^d e^{\beta x}$	$Q(x)e^{\beta x}$
$x^d\cos(\beta x)$	$Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)$
$x^d\sin(\beta x)$	$Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)$

Other forms from previous year:

$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a\sin(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$b\cos(\beta x)$	$c\sin(\beta x) + d\cos(\beta x)$
$ae^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\Big(c\sin(\beta x) + d\cos(\beta x)\Big)$
$be^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\Big(c\sin(\beta x) + d\cos(\beta x)\Big)$
$P_n(x)e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x)e^{\alpha x}\sin(\beta x)$	$e^{\alpha x}\Big(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\Big)$
$P_n(x)e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\Big(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x)\Big)$

3 Differential calculus in $\mathbb{R}^n$

Definition 3.2.1.

Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$. Write $x_k = (x_{k,1}, \ldots, x_{k,n})$. Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The sequence (x_k) **converges to** $(\to)$ y as $k \to +\infty$ if $\forall \varepsilon > 0\ \exists N \geqslant 1$ s.t. $\forall k \geqslant N$, we have $\|x_k - y\| < \varepsilon$.

Lemma 3.2.2.

$(x_k) \to y$ as $k \to +\infty \iff$ either:

(1) $\forall i, 1 \leqslant i \leqslant n$, the sequence of real numbers $(x_{k,i}) \to y_i$.

(2) The sequence of real numbers $\|x_k - y\| \to 0$ as $k \to +\infty$.

Definition 3.2.3.

Let $X \subset \mathbb{R}^n$ and $f: X \to \mathbb{R}^m$. (1) Let $x_0 \in X$. f is **continuous at** x_0 if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon, \forall x \in X$. (2) f is **continuous** on X if it is continuous at $x_0\ \forall x_0 \in X$.

Proposition 3.2.4.

Let $X \subset \mathbb{R}^n$ and $f: X \to \mathbb{R}^m$. Let $x_0 \in X$. The function f is continuous at $x_0 \iff \forall (x_k)_k \geqslant 1$ in X s.t. $x_k \to x_0$ as $k \to +\infty$, the sequence $(f(x_k))_k \geqslant 1$ in $\mathbb{R}^m$ converges to $f(x)$.

Definition 3.2.5.

Let $X \subset \mathbb{R}^n$ and $f: X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We say that f has the **limit** y as $x \to x_0$ with $x \neq x_0$ if for every $\varepsilon > 0$, there exists $\delta > 0$, s.t. $\forall x \in X, x \neq x_0$, s.t. $\|x - x_0\| < \delta$, we have $\|f(x) - y\| < \varepsilon$. We then write $\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y$.

Proposition 3.2.7.

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We have $\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = y \iff \forall (x_k) \in X$ s.t. $x_k \to x$ as $k \to +\infty$, and $x_k \neq x_0$, the sequence $(f(x_k))$ in $\mathbb{R}^m$ converges to y.

Proposition 3.2.9.

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and $p \geqslant 1$ an integer. Let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be continuous functions. Then the composite $g \circ f$ is continuous.

Definition 3.2.11.

(1) A subset $X \subset \mathbb{R}^n$ is **bounded** if the set of $\|x\|$ for $x \in X$ is bounded in $\mathbb{R}$.

(2) A subset $X \subset \mathbb{R}^n$ is **closed** if for every sequence (x_k) in X that converges in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$.

(3) A subset $X \subset \mathbb{R}^n$ is **compact** if it is bounded and closed.

Proposition 3.2.13.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. For any closed set $Y \subset \mathbb{R}^m$, $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subset \mathbb{R}^n$ is closed.

Theorem 3.2.15.

Let $X \subset \mathbb{R}^n$ be a non-empty compact set and $f : X \to \mathbb{R}$ a continuous function. Then f is bounded and achieves its max and min. I.e. $\exists x_+, x_- \in X$ s.t. $f(x_+) = \sup_{x \in X} f(x)$, $f(x_-) = \inf_{x \in X} f(x)$.

Definition 3.3.1.

A subset $X \subset \mathbb{R}^n$ is **open** if, for any $x = (x_1, \dots, x_n) \in X$, there exists $\delta > 0$ s.t. the set

$$\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \text{ for all } i\}$$

is contained in X. In other words: any point of $\mathbb{R}^n$ obtained by changing any coordinate of x by at most δ is still in X.

Proposition 3.3.2.

A set $X \subset \mathbb{R}^n$ is open if and only if the complement $Y = \{x \in \mathbb{R}^n : x \notin X\}$ is closed.

Corollary 3.3.3.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $Y \subset \mathbb{R}^m$ is open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$.

Definition 3.3.5

Let $X \subset \mathbb{R}^n$ be an open set. Let $f : X \to \mathbb{R}^m$ be a function. Let $1 \leqslant i \leqslant n$. We say that f has a **partial derivative** on X with respect to the i-th variable, or coordinate, if for all $x_0 = (x_{0,1}, \dots, x_{0,n}) \in X$, the function defined by

$$g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$$

on the set $I = \{t \in \mathbb{R} : (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$ is differentiable at $t = x_{0,i}$. Its **derivative** $g'(x_{0,i})$ at $x_{0,i}$ is denoted

$$\frac{\partial f}{\partial x_i}(x_0), \quad \partial_{x_i} f(x_0), \quad \partial_i f(x_0)$$

Proposition 3.3.7.

Consider $X \subset \mathbb{R}^n$ open and f, g functions from X to $\mathbb{R}^m$. Let $1 \leqslant i \leqslant n$.

(1) If f and g have partial derivatives with respect to the i-th coordinate on X, then $f + g$ also does, and

$$\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g).$$

(2) If $m = 1$, and if f and g have partial derivatives with respect to the i-th coordinate on X, then fg also does and

$$\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g).$$

Furthermore, if $g(x) \neq 0$ for all $x \in X$, then f/g has a partial derivative with respect to the i-th coordinate on X, with

$$\partial_{x_i}(f/g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g))/g^2.$$

Definition 3.3.9. (Jacobi matrix)

Let $X \subset \mathbb{R}^n$ open and $f : X \to \mathbb{R}^m$ a function with partial derivatives on X. Write

$$f(x) = (f_1(x), \dots, f_m(x)).$$

For any $x \in X$, the matrix

$$J_f(x) = \left(\partial_{x_j} f_i(x)\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$$

with m rows and n columns is called the **Jacobi matrix** of f at x.

Definition 3.3.11 (Gradient, Divergence)

Let $X \subset \mathbb{R}^n$ be open. (1) Let $f : X \to \mathbb{R}$ be a function. If all partial derivatives of f exist at $x_0 \in X$, then the column vector

$$\begin{pmatrix} \partial_{x_1} f(x_0) \\ \cdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$$

is called the **gradient** at x_0, and is denoted $\nabla f(x_0)$. (2) Let $f = (f_1, \dots, f_n) : X \to \mathbb{R}^n$ be a function with values in $\mathbb{R}^n$ s.t. all partial derivatives of all coordinates f_i of f exist at $x_0 \in X$. Then the real number

$$\operatorname{Tr}(J_f(x_0)) = \sum_{i=1}^{n} \partial_{x_i} f_i(x_0),$$

the trace of the Jacobi matrix, is called the **divergence** of f at x_0, and is denoted $\operatorname{div}(f)(x_0)$.

Definition 3.4.2.

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$ be a function. Let u be a linear map $\mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in X$. We say that f **is differentiable at** x_0 **with differential** u if

$$\lim_{x \to x_0} \frac{1}{\|x - x_0\|}(f(x) - f(x_0) - u(x - x_0)) = 0$$

where the limit is in $\mathbb{R}^m$. We then denote $df(x_0) = u$. If f is differentiable at every $x_0 \in X$, then we say that f is differentiable on X.

Proposition 3.4.4.

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$ be a function that is differentiable on X. (1) The function f is continuous on X. (2) The function f admits partial derivatives on X with respect to each variable. (3) Assume that $m = 1$. Let $x_0 \in X$, and let $u(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n$ be the differential of f at x_0. We then have $\partial_{x_i} f(x_0) = a_i$ for $1 \leqslant i \leqslant n$.

Proposition 3.4.6.

Let $X \subset \mathbb{R}^n$ be open, $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^m$ differentiable functions on X.

(1) The function $f + g$ is differentiable with differential $d(f + g) = df + dg$, and if $m = 1$, then fg is differentiable.

(2) If $m = 1$ and if $g(x) \neq 0$ for all $x \in X$, then f/g is differentiable.

Proposition 3.4.7.

Let $X \subset \mathbb{R}^n$ be open, $f : X \to \mathbb{R}^m$ a function on X. If f has all partial derivatives on X, and if the partial derivatives of f are continuous on X, then f is differentiable on X, with differential determined by its partial derivatives, in the sense that the matrix of the differential $df(x_0)$, with respect to the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^m$, is the Jacobi matrix of f at x_0.

Proposition 3.4.9 (Chain rule)

Let $X \subset \mathbb{R}^n$ be open, $Y \subset \mathbb{R}^m$ be open, and let $f : X \to Y$ and $g : Y \to \mathbb{R}^p$ be differentiable functions. Then $g \circ f : X \to \mathbb{R}^p$ is differentiable on X, and for any $x \in X$, its differential is given by the composition

$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0).$$

In particular, the Jacobi matrix satisfies

$$J_{g\circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$$

where the right-hand side is a matrix product.

From Example 3.4.10, 2.36L

Referring to the propostion above (3.4.9)

(2) let $p = 1$, then it holds that

$$\partial_j (g \circ f)(x) = \sum_{i=1}^{m} \partial_i g(f(x)) \cdot \partial_j f_i(x)$$

(3) let $p = 1$ and $X \subset \mathbb{R}$, than it holds that

$$\begin{aligned}(g \circ f)'(t) &= \sum_{i=1}^{m} (\partial_i g)(f(t)) \cdot f_i'(t) \\ &= \langle \nabla g(f(t)), f'(t) \rangle .\end{aligned}$$

Definition 3.4.11.

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$ a function that is differentiable. Let $x_0 \in X$ and $u = df(x_0)$ be the differential of f at x_0. The graph of the affine linear approximation

$$g(x) = f(x_0) + u(x - x_0)$$

from $\mathbb{R}^n$ to $\mathbb{R}^m$, or in other words the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x_0) + u(x - x_0)$$

is called the **tangent space** at x_0 to the graph of f.

Definition 3.4.13.

Let $X \subset \mathbb{R}^n$ be an open set and let $f : X \to \mathbb{R}^m$ be a function. Let $v \in \mathbb{R}^n$ be a non-zero vector and $x_0 \in X$. We say that f has **directional derivative** $w \in \mathbb{R}^m$ **in the direction** v, if the function g defined on the set

$$I = \{t \in \mathbb{R} : x_0 + tv \in X\}$$

by

$$g(t) = f(x_0 + tv)$$

has a derivative at $t = 0$, and this is equal to w. In other words, this means that the limit

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(x_0 + tv) - f(x_0)}{t}$$

exists and is equal to w.

Proposition 3.4.15.

Let $X \subset \mathbb{R}^n$ be an open set and let $f : X \to \mathbb{R}^m$ be a differentiable function. Then for any $x \in X$ and non-zero $v \in \mathbb{R}^n$, the function f has a directional derivative at x_0 in the direction v, equal to $df(x_0)(v)$.

Definition 3.5.1.

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^m$. We say that f is of class C^1 if f is differentiable on X and all its partial derivatives are continuous. The set of functions of class C^1 from X to $\mathbb{R}^m$ is denoted $C^1(X; \mathbb{R}^m)$. Let $k \geqslant 2$. We say, by induction, that f is of class C^k if it is differentiable and each partial derivative $\partial_{x_i} f : X \to \mathbb{R}^m$ is of class C^{k-1}. The set of functions of class C^k from X to $\mathbb{R}^m$ is denoted $C^k(X; \mathbb{R}^m)$.
If $f \in C^k(X; \mathbb{R}^m)$ for all $k \geqslant 1$, then we say that f is of class C^∞. The set of such functions is denoted $C^\infty(X; \mathbb{R}^m)$.

Proposition 3.5.4 (Mixed derivatives commute)

$k \geqslant 2$. Let $X \subset \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}^m$ be a function of class C^k. Then the partial derivatives of order k are independent of the order in which the partial derivatives are taken: for any variables x and y, we have $\partial_{x,y} f = \partial_{y,x} f$ and for any variables x, y, z, we have

$$\partial_{x,y,z} f = \partial_{x,z,y} f = \partial_{y,z,x} f = \partial_{z,x,y} f = \cdots$$

Definition 3.5.9 (Hessian).

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a C^2 function. For $x \in X$, the **Hessian matrix** of f at x is the symmetric square matrix

$$\mathrm{Hess}_f(x) = \left(\partial_{x_i,x_j} f\right)_{1\leqslant i,j\leqslant n}.$$

We also sometimes write simply $H_f(x)$.

Definition 3.7.1 (Taylor polynomials).

Let $k \geqslant 1$ be an integer. Let $f : X \to \mathbb{R}$ be a function of class C^k on X, and fix $x_0 \in X$. The k-th Taylor polynomial of f at the point x_0 is the polynomial in n variables of degree $\leqslant k$ given by

$$\begin{aligned}T_k f(y; x_0) =& f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) y_i + \cdots \\ &+ \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(x_0) y_1^{m_1} \cdots y_n^{m_n}\end{aligned}$$

where the last sum ranges over the tuples of n non-negative integers s.t. the sum is k.

Corollary $T_2 f$ (L)

With $x, x_0 \in \mathbb{R}^2$, x_0 fixed.

$$T_2 f(x; x_0) = f(x_0) + \langle \nabla f(x_0), x \rangle + \frac{1}{2} x^\top \cdot \mathrm{Hesse}_f(x_0) \cdot x$$

Proposition 3.7.3 (Taylor approximation)

Let $k \geqslant 1$ be an integer. Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ be a function of class C^k. For x_0 in X, if we define $E_k f(x; x_0)$ by

$$f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$$

then we have

$$\lim_{\substack{x\to x_0\\ x\neq x_0}} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

Proposition 3.8.1. (Maximum, minimum and derivative)

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a differentiable function. If $x_0 \in X$ is s.t.

$f(y) \leqslant f(x_0)$ for all y close enough to x_0 (local maximum at x_0)

or

$f(y) \geqslant f(x_0)$ for all y close enough to x_0 (local minimum at x_0).

Then we have $df(x_0) = 0$, or in other words $\nabla f(x_0) = 0$, or equivalently $\frac{\partial f}{\partial x_i}(x_0) = 0$ for $1 \leqslant i \leqslant n$.

Definition 3.8.2 (Critical point)

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a differentiable function. A point $x_0 \in X$ s.t. $\nabla f(x_0) = 0$ is called a **critical point** of the function f.

Definition 3.8.6 (Non-degenerate critical point)

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ a function of class C^2. A critical point $x_0 \in X$ of f is called **non-degenerate** if the Hessian matrix has non-zero determinant.

Definition Positive definite (linalg)

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. We say A is positive definite if $\langle x, Ax\rangle > 0$.

Proposition Definite matrix and eigenvalues (linalg)

If a matrix is positive definite, all its eigen values are positive.

Definition Leading principal minor (linalg)

The k-th leading principal minor of a matrix M is the determinant of its upper-left $k \times k$ sub-matrix.

Proposition Leading principal minor and definite matrix (linalg)

A matrix is positive definite $\iff$ all its leading principal minors (1 to n) are positive.

Corollary 3.8.7

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a function of class C^2. Let x_0 be a non-degenerate critical point of f. Let p and q be the number of positive and negative eigenvalues of $\operatorname{Hess}_f(x_0)$.

(1) If $p = n$, equivalently if $q = 0$, the function f has a local minimum at x_0.

(2) If $q = n$, equivalently if $p = 0$, the function f has a local maximum at x_0.

(3) Otherwise, equivalently if $pq \neq 0$, the function f does not have a local extremum at x_0. One then says that f has a saddle point at x_0.

Proposition 3.9.2

Let $X \subset \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions of class C^1. If $x_0 \in X$ is a local extremum of the function f restricted to the set $Y = \{x \in X : g(x) = 0\}$ then either $\nabla g(x_0) = 0$, or there exists $\lambda_0 \in \mathbb{R}$ s.t.

$$\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$$

or in other words, there exists λ s.t. (x_0, λ) is a critical point of the differentiable function $h : X \times \mathbb{R} \to \mathbb{R}$ defined by $h(x, \lambda) = f(x) - \lambda g(x)$. Such a value λ is called a Lagrange multiplier at x_0.

Definition 3.10.1 (Change of variable)

Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ be differentiable. Let $x_0 \in X$. We say that f is a **change of variable** around x_0 if there is a radius $r > 0$ s.t. the restriction of f to the ball

$$B = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$$

of radius r around x_0 has the property that the image $Y = f(B)$ is open in $\mathbb{R}^n$, and if there is a differentiable map $g : Y \to B$ s.t. $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_B$.

Theorem 3.10.2 (Inverse funtion theorem)

(Inverse function theorem). Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ differentiable. If $x_0 \in X$ is s.t. $\det(J_f(x_0)) \neq 0$, i.e., s.t. the Jacobian trix of f at x_0 is invertible, then f is a change of variable around x_0. Moreover, the Jacobian of g at x_0 is determined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

In addition, if f is of class C^k, then g is of class C^k.

Theorem 3.10.4 (Implicit Function Theorem).

Let $X \subset \mathbb{R}^{n+1}$ be open and let $g : X \to \mathbb{R}$ be of class C^k with $k \geqslant 1$. Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}$ be s.t. $g(x_0, y_0) = 0$. Assume that

$$\partial_y g(x_0, y_0) \neq 0$$

Then there exists an open set $U \subset \mathbb{R}^n$ containing x_0, an open interval $I \subset \mathbb{R}$ containing y_0, and a function $f : U \to \mathbb{R}$ of class C^k s.t. the system of equations

$$\begin{cases} g(x, y) = 0 \\ x \in U, \quad y \in I \end{cases}$$

is equivalent with $y = f(x)$. In particular, $f(x_0) = y_0$. Moreover, the gradient of f at x_0 is given by

$$\nabla f(x_0) = -\frac{1}{(\partial_y g)(x_0, y_0)} \nabla_x g(x_0, y_0)$$

where $\nabla_x g = (\partial_{x_1} g, \dots, \partial_{x_n} g)$

4 Integration in $\mathbb{R}^n$

Definition 4.1.1. (parameterized curve, line integral)

(1) Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Let $f(t) = (f_1(t), \dots, f_n(t))$ be a continuous function from I to $\mathbb{R}^n$, i.e., f_i is continuous for $1 \leqslant i \leqslant n$. Then we define

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t), \dots, \int_a^b f_n(t)dt\right) \in \mathbb{R}^n.$$

(2) A **parameterized curve** in $\mathbb{R}^n$ is a continuous map $\gamma : [a, b] \to \mathbb{R}^n$ that is piecewise C^1, i.e., there exists $k \geqslant 1$ and a partition

$$a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$$

such that the restriction of f to $]t_{j-1}, t_j[$ is C^1 for $1 \leqslant j \leqslant k$. We say that γ is a parameterized curve, or a path x, between $\gamma(a)$ and $\gamma(b)$.

(3) Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterized curve. Let $X \subset \mathbb{R}^n$ be a subset containing the image of γ, and let $f : X \to \mathbb{R}^n$ be a continuous function. The integral

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \in \mathbb{R}$$

is called the **line integral of** f **along** γ. It is denoted

$$\int_\gamma f(s) \cdot ds, \quad \text{or} \quad \int_\gamma f(s) \cdot d\vec{s}.$$

Definition 4.1.4

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parameterized curve. An **oriented reparameterization** of γ is a parameterized curve $\sigma : [c, d] \to \mathbb{R}^n$ such that $\sigma = \gamma \circ \varphi$, where $\varphi : [c, d] \to [a, b]$ is a continuous map, differentiable on $]a, b[$, that is strictly increasing and satisfies $\varphi(a) = c$ and $\varphi(b) = d$.

Proposition 4.1.5.

Let γ be a parameterized curve in $\mathbb{R}^n$ and σ an oriented reparameterization of γ. Let X be a set containing the image of γ, or equivalently the image of σ, and $f : X \to \mathbb{R}^n$ a continuous function. Then we have

$$\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}.$$

Definition 4.1.8

Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a continuous vector field. If, for any x_1, x_2 in X, the line integral $\int_\gamma f(s) \cdot d\vec{s}$ is independent of the choice of a parameterized curve γ in X from x_1 to x_2, then we say that the vector field is conservative.

Lemma 3.14L)

A vectorfield $f : X \to \mathbb{R}^n$ is conservative $\iff$ for each parameterized curved γ contained in X it holds that $\int_\gamma f(s) d\vec{s} = 0$.

Theorem 4.1.10

Let X be an open set and f a conservative vector field. Then there exists a C^1 function g on X such that $f = \nabla g$.
If any two points of X can be joined by a parameterized curve, then g is unique up to addition of a constant: if $\nabla g_1 = f$, then $g - g_1$ is constant on X.

Definition 3.16L (path-connected space, potential)

(1) An open set $X \subset \mathbb{R}^n$ is **path-connected** if for each pair of points in X are the endpoints of a parameterized curve.

(2) A function $g : X \to \mathbb{R}$ s.t. $\nabla g = f$ is called **potential** of f.

Proposition 4.1.13

Let $X \subset \mathbb{R}^n$ be an open set and $f : X \to \mathbb{R}^n$ a vector field of class C^1. Write $f(x) = (f_1(x), \dots, f_n(x))$. If f is conservative, then we have

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

for any integers with $1 \leqslant i \neq j \leqslant n$.

Definition 4.1.15 (star shaped)

A subset $X \subset \mathbb{R}^n$ is **star shaped** if there exists $x_0 \in X$ such that, for alla $x \in X$, the line segment joining x_0 to x is contained in X. We then also say that X is **star-shaped around** x_0

Theorem 4.1.17

Let X be a star-shaped open subset of $\mathbb{R}^n$. Let f be a C^1 vector field s.t. $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ on X for all $i \neq j$ between 1 and n. Then the vector field f is conservative.

Definition 4.1.20 (curl)

Definition 4.1.20. Let $X \subset \mathbb{R}^3$ be an open set and $f : X \to \mathbb{R}^3$ a C^1 vector field. Then the curl of f, denoted $\operatorname{curl}(f)$, is the continuous vector field on X defined by

$$\operatorname{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

where $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$.

Definition Integral on a rectangle

Let $R = [a,b] \times [c,d]$ a compact rectangle in $\mathbb{R}^2$. Let $f : R \to \mathbb{R}$ be bounded, s.t. $\exists M \geqslant 0, \forall (x,y) \in \mathbb{R}, |f(x,y)| \leqslant M$. For each partition P_x with $x_0 = a < x_1 < \dots < x_n = b$ of $[a,b]$ and P_y with $y_0 = c < y_1 < \dots < y_n = d$ of $[c,d]$, we can subdivide R into rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ with area $\mu(R_{ij}) = (x_i - x_{i-1})(y_j - y_{j-1})$. Let

$$f_{ij} = \inf_{I_{ij}} f(x,y),\ F_{ij} = \sup_{I_{ij}} f(x,y)$$

. We can then define an upper and a lower sum, respectively:

$$s(P_x \times P_y) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} \mu(I_{ij})$$

$$S(P_x \times P_y) = \sum_{i=1}^{n} \sum_{j=1}^{m} F_{ij} \mu(I_{ij})$$

Definition Riemann integrable in $\mathbb{R}^2$

Let $f : R \to \mathbb{R}$ bounded. Then is f integrable on R if

$$\sup_{(P_x, P_y)} s(P_x \times P_y) = \inf_{(P_x, P_y)} S(P_x \times P_y).$$

This value is defined as

$$\int_R f(x,y) d(x,y) \text{ or } \int \int_R f(x,y) d(x,y).$$

Definition Riemann integrable and characteristical funciton

f is on A integrable if $f \cdot \mathcal{X}_A$ is integrable on R , where $\mathcal{X}_A$ is the characteristical polynomial of f. We then write $\int_A f(x,y) d(x,y)$ for $\int_R f(x,y) \mathcal{X}_A(x,y) d(x,y)$.

Riemann integral's properties

(1) **Linearity**

(2) **Positivity** Let $f, g : A \to \mathbb{R}$ integrable with $f \leqslant g$. Then it follows

$$\int_A f(x,y) d(x,y) \leqslant \int_A g(x,y) d(x,y).$$

Furthermore, if $f \geqslant 0$, $B \subset A$ and f on B integrable:

$$\int_B f(x,y) d(x,y) \leqslant \int_A f(x,y) d(x,y).$$

(3) **Triangular inequality** Let $f : A \to \mathbb{R}$ integrable (in particular, bounded), then is $|f|$ integrable and

$$\left| \int_A f(x,y) d(x,y) \right| \leqslant \int_A |f(x,y)|\, d(x,y)$$

(4) **Volume** Let $R = [a,b] \times [c,d]$, then $\int_R d(x,y) = (b-a)(d-c)$.

(5) **(O. Stolz 1886)** Let $f : R \to \mathbb{R}$ on $R = [a,b] \times [c,d]$ integrable. We assume that $y \mapsto f(x,y)\ \forall x \in [a,b]$ is integrable on $[c,d]$. This implies that

$$x \mapsto \int_c^d f(x,y) dy$$

is integrable on $[a,b]$, and

$$\int_R f(x,y) d(x,y) = \int_a^b \left(\int_c^d f(x,y) dy \right) dx$$

Proposition 3.30L

Let $R = [a,b] \times [c,d]$ be a compact rectangle and $f : R \to \mathbb{R}$ continuous. Then is f on R integrable.

Proposition 3.31L

Let $K \subset \mathbb{R}^2$ a compact subset and $f : K \to \mathbb{R}$ continuous. Then is f uniformly continuous.

Definition 3.32L (null set)

Let $X \subset R \subset \mathbb{R}^2$. Then X is a null set (in $\mathbb{R}^2$) if $\forall \epsilon > 0$ there are finite many rectangles $R_k = [a_k, b_k] \times [c_k, d_k]$ with $1 \leqslant k \leqslant n$, s.t.

$$X \subset \bigcup_{k=1}^{n} R_k, \quad \sum_{k=1}^{n} \mu(R_k) < \epsilon$$

Lemma 3.33L

Let $\varphi : [0,1] \to \mathbb{R}^2$ a Lipschitz curve, so that

$$\|\varphi(s) - \varphi(t)\| \leqslant M \cdot |s-t| \quad \forall s,t \in [0,1]$$

Then is the image $\varphi([0,1]) \subset \mathbb{R}^2$ a null set.

Proposition 3.35L

Let $f : R \to \mathbb{R}$ bounded. Let

$$X = \{(x,y) \in \mathbb{R} : f \text{ is in } (x,y) \text{ not continuous}\}$$

If X is a null set, then f is not integrable on R.

Proposition 3.36L

Let $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ continuous with $\varphi_1 \leqslant \varphi_2$ $\forall x \in [a,b]$. Let $A = \{(x,y) : a \leqslant x \leqslant b, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\}$. Let $f : A \to \mathbb{R}$ continuous. Then is f on A integrable and it holds:

$$\int_A f(x,y) d(x,y) = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y) dy.$$

Lemma 3.37L

Let $\varphi : [a,b] \to \mathbb{R}$ continuous. Then is the $graph(\varphi) = \{(x, \varphi(x), x \in [a,b]\} \subset \mathbb{R}^2$ a null set.

Definition Border (Rand)

Let $A \in \mathbb{R}^2$ the **border** (*Rand* in German) is defined as

$$\partial A = \{(x,y) \in \mathbb{R}^2 : \forall \delta > 0 \; C_\delta(x,y) \cap A \neq \emptyset \wedge C_\delta(x,y) \cap \left(\mathbb{R}^2 \setminus A\right) \neq \emptyset\}$$

where $C_\delta(x,y) =]x-\delta, x+\delta[\times]y-\delta, y+\delta[$.

Theorem 4.4.2 (Change of variable formula).

Let $U \subset \mathbb{R}^2$ be compact subsets. Let $\varphi : U \to \mathbb{R}^2$ be a continuous map in C^1 and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ a continuous function. Furthermore $B \subset U$. We assume

(1) $\varphi(B) = A$; A, B compact; $\partial A, \partial B$ null sets.

(2) $\varphi : B \setminus N \to A$ is injective, where $N \subset B$ is a null set.

Then it follows:

$$\int_A f(x,y) d(x,y) = \int_B f(\varphi(u,v)) \left|\det\left(J_\varphi(u,v)\right)\right| d(u,v).$$

Example, polar coordinates (L)

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2; (r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi)$. Then we can convert the integral in cartesian coordinates to polar coordinates in the following manner:

$$\int_{x^2+y^2 \leqslant R^2} f(x,y) d(x,y) = \int_0^{2\pi} \int_0^R f(r\cos\varphi, r\sin\varphi) r d(r,\varphi)$$

Definition 3.39L Jordan curve, orientation

A **parameterized Jordan curve** is a parameterized curve $\gamma : [a,b] \to \mathbb{R}^2$ with the following properties

(1) $\gamma(a) = \gamma(b)$

(2) $\gamma :]a,b] \to \mathbb{R}^2$ is injective

(3) A Jordan curve in $\mathbb{R}^2$ is the image of a parameterized Jordan curve.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A **base** (b_1, b_2) of $\mathbb{R}^2$ is **positive oriented** if for the distinct matrix $g \in M_{2,2}(\mathbb{R})$ with $g(e_1) = b_1$ and $g(e_2) = b_2$, $\det(g) > 0$. It is negative oriented if $det(g) < 0$.

Definition 3.40 *reguläres Gebiet* (L)

A *reguläres Gebiet* is a open bounded subset $A \subset \mathbb{R}^2$ whose border ∂A is a finite union of disjoint Jordan curves. Each of this curve is called a border component of A.
Let $\gamma : [a,b] \to \mathbb{R}^2$ a parameterized Jordan curve so that $\gamma([a,b])$ is a border component of A. Then γ is positive oriented relative to A if $(n(t), \gamma'(t))$ is a positive oriented basis of $\mathbb{R}^2$, where $n(t)$ is the unitary vector orthogonal to $\gamma'(t)$ and pointing outwards (not to A).

Theorem 4.6.3 (3.41L) Green's formula

Let $A \subset \mathbb{R}^2$ be a *reguläres Gebiet* and $F : U \to \mathbb{R}^2; (x,y) \mapsto \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$ a C^1 vector field, where $A \cup \partial A \subset U \in \mathbb{R}^2$. Then it holds

$$\int_A \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) d(x,y) = \int_{\partial A} F(s) ds := \sum_{i=1}^k \int_{\gamma_i} f \cdot d\vec{s}$$

5 Extra

Vector field example

A vector field often discussed in the series and seen in some lecture's examples:

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2; f(x,y) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}.$$

This vector field is **not** conservative, even if $\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}$. For example, the line integral over the parameterized curve of a "clockwise" closed circle of radius 1 doesn't give 0 as expected, but $\pi/2$ instead.
We can though define the potentials:

- $g_1 : \mathbb{R}^+ \times \mathbb{R}; (x,y) \mapsto \arctan(y/x)$
- $g_2 : \mathbb{R} \times \mathbb{R}^+; (x,y) \mapsto \arctan(-x/y) = \arctan(y/x) - \pi/2$
- $g_3 : \mathbb{R}^- \times \mathbb{R}; (x,y) \mapsto \arctan(y/x)$
- $g_4 : \mathbb{R} \times \mathbb{R}^-; (x,y) \mapsto \arctan(-x/y) = \arctan(y/x) - \pi/2$

Trigonometric functions

α	0	30°	45°	60°	90°	120°	150°	180°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$\sin\alpha$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	1/2	0
$\cos\alpha$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	−1/2	$-\sqrt{3}/2$	-1
$\tan\alpha$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	N/A	$-\sqrt{3}$	$-\sqrt{3}/3$	0

Proposition Trigonometric properties (S3.42 + K3.43 Ana I)

(1) $\exp(iz) = \cos z + i \sin z$

(2) $\cos z = \cos(-z)$, $\sin(-z) = -\sin(z)$

(3) $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + -e^{-iz}}{2}$

(4) $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
$\implies \sin(2z) = 2\sin z \cos z$,
$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
$\implies \cos(2z) = \cos^2 z - \sin^2 z$,

(5) $\cos^2 z + \sin^2 z = 1$