Numerical analysis of Taylor Series

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1 Theory

Any one-variable infinitely differentiable real-valued function $f(x) : A \to B$ where $A, B \subseteq \mathbb{R}$ might be expanded as an infinite power series function with parameter $x_0 \in A$, This series function is also termed as **Taylor series** representation of f because of its procurement from the **Taylor's Theorem.**

$$\begin{aligned} f(x) = T(x, x_0) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned} \tag{1}$$

Taylor series representation of a function with the parameter $x_0 = 0$ is called the **Maclaurin series**.

$$f(x) = T(x, 0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n \tag{2}$$

The point on the line $x = x_0$ is called the center of taylor series. The value of the function and its derivatives must be known at the center and the radius of convergence of the series is determined about this point.

Radius of Convergence of a power series: Every power series has a radius of convergence R which is the distance of its center from the nearest singularity(point of divergence). If $R > 0$, then the power series $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ converges for all $|x - a| \leq R$ and diverges for $|x - a| > R$. If the series converges for all x, then we write $R = \infty$.

Taylor series representation for a function of two variables $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ about $(x, y) = (x_0, y_0)$ is given by the Taylor theorem as follows,

$$\begin{aligned} f(x, y) = T((x, y), (x, y_0)) = f(x_0, y_0) + f_x|_{x_0,y_0}(x - x_0) + f_y|_{x_0,y_0}(y - y_0) + +f_x x|_{x_0,y_0}(x - x_0)^2 \\ +f_{yy}|_{x_0,y_0}(y - y_0)^2 + f_{xy}|_{x_0,y_0}(x - x_0)(y - y_0) + \cdots \end{aligned} \tag{3}$$

The functions $\exp(x), \sin(x), \cos(x) : \mathbb{R} \to \mathbb{R}$ are defined by the following Maclaurin series expansions.

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x \tag{4}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \text{for all } x \tag{5}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \text{for all } x \tag{6}$$

2 Algorithm

Algorithm 1 Use Composite Trapezoidal rule to find fixed-tolerance numerical approximation for the given definite integral.

procedure MyTrap(f,a,b,max_subs,d)

Input: f is the integrand, a is the lower limit and b is the upper-limit of integration, max_subs is the number of sub-intervals that the algorithm cannot exceed and d is the number of significant digits required in the numerical approximation of the integral.

Output: Returns $I_{num}(f)$, the numerical approximation of I(f)

▷ We calculate the fixed-tolerance approximation integral by continually calculating the integral for double the number of subintervals than done in the prior iteration in order to avoid calculation of already calculated values of f(x).

$m \leftarrow$ A vector of number of subintervals to calculate the integrals for, A geometric progression with commmon ratio 2.

$I \leftarrow$ A vector that stores the value of the integral obtained after each iteration.

$X^0 \leftarrow$ A vector of equally spaced nodes in closed interval $[a, b]$ obtained for m_0

$l \leftarrow length(m_0)$

$n \leftarrow length(X)$

$$I_0 \leftarrow \frac{b-a}{3m_0}\left[f(X_0) + 2\sum_{j=1}^{n-1} f(X_j) + f(X_n)\right]$$

for $k = 1, 2, 3 \ldots l$ **do**

$X^k \leftarrow$ A vector of equally spaced nodes in closed interval $[a, b]$ obtained for m_k

$n \leftarrow length(X^k)$

$$I_k = \frac{I_{k-1}}{2} + \frac{b-a}{m_k}\left[\sum_{j=1}^{n-1} f(X_j)\right]$$

if $|I_k - I_{k-1}| \leq 0.5 \times 10^{-d} \times |I_k|$ **then**

Return I_k, m_k

EXIT

end if

end for

Could not reach tolerance.

Return I_l, m_l

EXIT

end procedure

Algorithm 2 Use composite Trapezoidal rule to find fixed-tolerance numerical approximation for the given definite integral.

procedure MYSIMP(f,a,b,max_subs,d)

Input: f is the integrand, a is the lower limit and b is the upper-limit of integration, max_subs is the number of sub-intervals that the algorithm cannot exceed and d is the number of significant digits required in the numerical approximation of the integral.

Output: Returns $I_{num}(f)$, the numerical approximation of I(f)

▷ We calculate the fixed-tolerance approximation integral by continually calculating the integral for double the number of subintervals than done in the prior iteration in order to avoid calculation of already calculated values of f(x).

$m \leftarrow$ A vector of number of subintervals to calculate the integrals for, A geometric progression with commmon ratio 2.

$I \leftarrow$ A vector that stores the value of the integral obtained after each iteration.

$X \leftarrow$ A vector of equally spaced nodes in closed interval $[a, b]$ obtained for m_0

$l \leftarrow length(m_0)$

$n \leftarrow length(X)$

$$I_0 \leftarrow \frac{b-a}{3m_0}\left[f(X_0) + 2\sum_{j=1}^{n/2-1} f(X_{2j}) + 4\sum_{j=1}^{n/2} f(X_{2j-1}) + f(X_n)\right]$$

for $k = 1, 2, 3 \dots l$ **do**

$X^k \leftarrow$ A vector of equally spaced nodes in closed interval $[a, b]$ obtained for m_k

$n \leftarrow length(X^k)$

$$I_k = \frac{I_{k-1}}{2} + \frac{b-a}{3m_k}\left[4\sum_{j=1}^{n/2} f(X_{2j-1}) - 2\sum_{j=1}^{n/2-1} f(X_{2j})\right]$$

if $|I_k - I_{k-1}| \leq 0.5 \times 10^{-d} \times |I_k|$ **then**

Return I_k, m_k

EXIT

end if

end for

Could not reach tolerance.

Return I_l, m_l

EXIT

end procedure